# SRB MEASURES FOR C $\infty$ SURFACE DIFFEOMORPHISMS

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ABSTRACT. A $C^\infty$ surface diffeomorphism admits a SRB measure if and only if the set $\{x,\ \limsup_n \frac{1}{n}\log\|d_x f^n\| > 0\}$ has positive Lebesgue measure. Moreover the basins of the ergodic SRB measures are covering this set Lebesgue almost everywhere.

## 1. INTRODUCTION

One fundamental problem in dynamics consists in understanding the statistical behaviour of the system. Given a topological system $(X, f)$ we are more precisely interesting in the asymptotic distribution of the empirical measures $\left(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k x}\right)_n$ for typical points $x$ with respect to a reference measure. In the setting of differentiable dynamical systems the natural reference measure to consider is the Lebesgue measure on the manifold.

The basin of a $f$-invariant measure $\mu$ is the set $\mathcal{B}(\mu)$ of points whose empirical measures are converging to $\mu$ in the weak-$*$ topology. By Birkhoff's ergodic theorem the basin of an ergodic measure $\mu$ has full $\mu$-measure. An invariant measure is said physical when its basin has positive Lebesgue measure. We may wonder when such measures exist and then study their basins.

In the works of Y. Sinai, D. Ruelle and R. Bowen [38, 11, 34] these questions have been successfully solved for uniformly hyperbolic systems. A SRB measure of a $C^{1+}$ system is an invariant probability measure with at least one positive Lyapunov exponent almost everywhere, which has absolutely continuous conditional measures on unstable manifolds [41]. Physical measures may neither be SRB measures nor sinks (as in the famous figure-eight attractor), however hyperbolic ergodic SRB measures are physical measures. For uniformly hyperbolic systems, there is a finite number of such measures and their basins cover a full Lebesgue subset of the manifold. Beyond the uniformly hyperbolic case such a picture is also known for large classes of partially hyperbolic systems [10, 2, 1]. Corresponding results have been established for unimodal maps with negative Schwartzian derivative [24]. SRB measures have been also deeply investigated for parameter families such as the quadratic family and Henon maps [23, 5, 6, 7]. In his celebrated ICM's talk, M. Viana conjectured that a surface diffeomorphism admits a SRB measure, whenever the set of points with positive Lyapunov exponent has positive Lebesgue measure. In recent works some weaker versions of the

*Date*: October 2021.
2010 *Mathematics Subject Classification.* Primary 37C40, 37D25.

conjecture (with some additional assumptions of recurrence and Lyapunov regularity) have been proved [19, 8]. Finally we mention that in the present context of $C^\infty$ surface diffeomorphims J. Buzzi, S. Crovisier, O. Sarig have recently shown the existence of a SRB measure once there are ergodic measures with lower bounded entropy and with unstable dimension arbitrarily close to one [17].

In this paper we define a general entropic approach to build SRB measures by using hyperbolic times, which we apply to prove Viana's conjecture for $C^\infty$ surface diffeomorphisms. We strongly believe the same approach may be used to recover the existence of SRB measures for weakly mostly expanding partially hyperbolic systems [1] and to give another proof of Ben Ovadia's criterion for $C^{1+}$ diffeomorphisms in any dimension [8].

We state now the main result of our paper. Let $(M, \|\cdot\|)$ be a compact Riemannian surface and let Leb be a volume form on $M$, called Lebesgue measure. We consider a $C^\infty$ surface diffeomorphism $f : M \circlearrowleft$. The maximal Lyapunov exponent at $x \in M$ is given by $\chi(x) = \limsup_n \frac{1}{n} \log \|d_x f^n\|$. When $\mu$ is a $f$-invariant probability measure, we let $\chi(\mu) = \int \chi(x)\, d\mu(x)$. For two Borel subsets $A$ and $B$ of $M$ we write $A \overset{o}{\subset} B$ (resp. $A \overset{o}{=} B$) when we have $\mathrm{Leb}(A \setminus B) = 0$ (resp. $\mathrm{Leb}(A \Delta B) = 0$).

**Main Theorem.** *Let $f : M \circlearrowleft$ be a $C^\infty$ surface diffeomorphism. There are countably many ergodic SRB measures $(\mu_i)_{i\in I}$, such that we have with $\Lambda = \{\chi(\mu_i),\ i \in I\} \subset \mathbb{R}_{>0}$:*

- $\{\chi > 0\} \overset{o}{=} \{\chi \in \Lambda\}$,
- $\{\chi = \lambda\} \overset{o}{\subset} \bigcup_{i,\chi(\mu_i)=\lambda} \mathcal{B}(\mu_i)$ *for all* $\lambda \in \Lambda$.

**Corollary 1.** *Let $f : M \circlearrowleft$ be a $C^\infty$ surface diffeomorphism. Then*

$$\{\chi > 0\} \overset{o}{\subset} \bigcup_{\mu \text{ SRB ergodic}} \mathcal{B}(\mu).$$

**Corollary 2.** *Let $f : M \circlearrowleft$ be a $C^\infty$ surface diffeomorphism.*
*If $\mathrm{Leb}(\chi > 0) > 0$, then there exists a SRB measure.*

When $f$ is a $C^{1+}$ topologically transitive surface diffeomorphism, there is at most one SRB measure, i.e. $\sharp I \leq 1$ [22]. If moreover the system is topologically mixing, then the SRB measure when it exists is Bernoulli [16]. By the spectral decomposition of $C^\infty$ surface diffeomorphisms [16] there are at most finitely many ergodic SRB measures with entropy and thus maximal exponent larger than a given positive constant. Therefore, in the Main Theorem, the set $\Lambda = \{\chi(\mu_i),\ i \in I\}$ is either finite or a sequence decreasing to zero.

Finally we emphasize the necessity of $C^\infty$ smoothness. We prove in a forthcoming paper [12] that the Main Theorem is false in finite smoothness

by building for any finite $r > 1$ a $C^r$ surface diffeomorphism $(f, M)$ with a periodic saddle hyperbolic point $p$ such that $\chi(x) > 0$ for all $x \in U$ for some set $U \subset \mathcal{B}(\mu_p)$ with $\mathrm{Leb}(U) > 0$, where $\mu_p$ denotes the periodic measure associated to $p$ (see [14] for such an example of interval maps). However, as in the examples of $C^r$ surface diffeomorphisms without measure of maximal entropy [15], the construction is local. In particular, in our example, the exponent satisfies $\chi(x) \leq \frac{R(f)}{r}$ for all $x \in U$. Hopefully we could have with $R(f) := \lim_n \frac{1}{n} \log \sup_{x \in M} \|d_x f^n\|$:

**Conjecture 1.** *Let $f : M \circlearrowleft$ be a $C^r$ surface diffeomorphism with $r > 1$.*
*If $\mathrm{Leb}\left(\chi > \frac{R(f)}{r}\right) > 0$, then there exists a SRB measure.*

In higher dimensions we let $\Sigma^k \chi(x) := \limsup_n \frac{1}{n} \|\Lambda^k d_x f^n\|$ where $\Lambda^k df$ denotes the action induced by $f$ on the $k^{th}$ exterior power of $TM$ for $k = 1, \cdots, d$ with $d$ being the dimension of $M$. By convention we also let $\Sigma^0 \chi = 0$. For any $C^1$ diffeomorphism $(M, f)$ we have $\mathrm{Leb}(\Sigma^d \chi > 0) = 0$ (see [3]). The product of a figure-eight attractor with a surface Anosov diffeomorphism does not admit any SRB measure whereas $\chi$ is positive on a set of positive Lebesgue measure. However we conjecture :

**Conjecture 2.** *Let $f : M \circlearrowleft$ be a $C^\infty$ diffeomorphism on a compact manifold (of any dimension).*
*If $\mathrm{Leb}\left(\Sigma^k \chi > \Sigma^{k-1} \chi \geq 0\right) > 0$, then there exists an ergodic measure with at least $k$ positive Lyapunov exponents, such that its entropy is larger than or equal to the sum of its $k$ smallest positive Lyapunov exponents.*

In the present two-dimensional case the semi-algebraic tools used to bound the distorsion and the local volume growth of $C^\infty$ curves are elementary. This is a challenging problem to adapt this technology in higher dimensions.

When the empirical measures from $x \in M$ are not converging, the point $x$ is said to have historic behaviour [35]. A set $U$ is contracting when the diameter of $f^n U$ goes to zero when $n \in \mathbb{N}$ goes to infinity. In a contracting set the empirical measures of all points have the same limit set, however they may not converge. P. Berger and S. Biebler have shown that $C^\infty$ densely inside the Newhouse domains [9] there are contracting domains with historic behaviour. As a consequence of the Main Theorem, Lebesgue almost every point $x$ with historic behaviour satisfies $\chi(x) \leq 0$ for $C^\infty$ surface diffeomorphisms. We also show the following statement.

**Theorem 1.** *Let $f$ be a $C^\infty$ diffeomorphism on a compact manifold (of any dimension). Then Lebesgue a.e. point $x$ in a contracting set satisfies $\chi(x) \leq 0$.*

**Question.** Let $f$ be a $C^\infty$ surface diffeomorphism. Assume the set $H$ of points with historic behaviour has positive Lebesgue measure. Does every

Lebesgue density point of $H$ belongs to a almost contracting* set with positive Lebesgue measure?

We explain now in few lines the main ideas to build a SRB measure under the assumptions of the Main Theorem. The geometric approach for uniformly hyperbolic systems consists in considering a weak limit of $\left(\frac{1}{n}\sum_{k=0}^{n-1} f_*^k \operatorname{Leb}_{D_u}\right)_n$, where $D_u$ is a local unstable disc and $\operatorname{Leb}_{D_u}$ denotes the normalized Lebesgue measure on $D_u$ induced by its inherited Riemannian structure as a submanifold of $M$. Here we take a smooth $C^\infty$ embedded curve $D$ such that

$$\chi(x, v_x) := \limsup_n \frac{1}{n} \log \|d_x f^n(v_x)\| > a > 0$$

for $(x, v_x)$ in the tangent space $TD$ of $D$ with $x$ in a subset $A$ of $D$ with positive $\operatorname{Leb}_D$-measure. Let $F : \mathbb{P}TM \circlearrowleft$ be the map induced by $f$ on the projective tangent bundle $\mathbb{P}TM$ and let $\phi(x, v) = \log \|d_x f(v)\|$ for any $(x, v) \in \mathbb{P}TM$. By Pliss Lemma, for any point $x \in A$, the set $E(x)$ of $b$-hyperbolic times of the sequence $\left(\sum_{k=0}^{n} \phi\left(F^k(x, v_x)\right)\right)_n$, with $b < a$ has positive asymptotic density. We build in this context a SRB measure by considering a weak limit $\mu$ of a sequence of the form $\left(\frac{1}{\sharp F_n}\sum_{k\in F_n} F_*^k \mu_n\right)_n$ such that :

- $(F_n)_n$ is a Fölner sequence, so that the weak limit $\mu$ will be invariant by $F$,
- for all $n$, the measure $\mu_n$ is the probability measure induced by $\operatorname{Leb}_D$ on $A_n$, the $\operatorname{Leb}_D$-measure of $A_n$ being not exponentially small,
- the sets $(F_n)_n$ are in some sense *filled with* the set of hyperbolic times $E(x)$ for $x \in A_n$. Then the measure $\mu$ on $\mathbb{P}TM$ will be supported on the unstable Oseledec's bundle.

Finally we check that the limit empirical measure $\mu$ projects to a SRB measure on $M$ by computing the entropy of $\mu$ as in [13]. For this last step we use the fact that for $n \in E(x)$ the following properties hold with $r = r(b)$:

- the geometry of $f^n D$ around $f^n x$ is *bounded* meaning that for some uniform $\epsilon = \epsilon(b) > 0$, the connected component $D_n^\epsilon(x)$ of $f^n D$ with the ball at $f^n x$ of radius $\epsilon > 0$ is a curve with bounded $s$-derivative for $s \leq r$,
- the distorsion of $df^{-n}$ on the tangent space of $D_n^\epsilon(x)$ is controlled.

Together with Yomdin's estimate on local volume growth, we conclude the projection of $\mu$ on $M$ satisfies the entropy formula, and therefore is a SRB measure.

The paper is organized as follows. In Section 2 we recall for general sequences of integers the notion of asymptotical density and we build for any

*See Section 7 for the definition of almost contracting set.

sequence $E$ with positive upper density a Fölner set $F$ filled with $E$. Then we use a Borel-Cantelli argument to define our sets $(A_n)_n$ and the Fölner sequence $(F_n)_n$. In Section 3, we study the maximal Lyapunov exponent and the entropy of the generalized empirical measure $\mu$. We conclude the entropy computation in Section 4, by assuming the distorsion property at hyperbolic times which is proved in Section 5. Then we prove the covering property of the basins in Section 6 by using the standard argument of absolute continuity of the stable foliation. Section 7 is devoted to the proof of Theorem 1.

## 2. Some asymptotic properties of integers

2.1. **Asymptotic density.** We first introduce some notations. In the following we let $\mathcal{P}_{\mathbb{N}}$ and $\mathcal{P}_n$ be respectively the power sets of $\mathbb{N}$ and $\{1, 2, \cdots, n\}$, $n \in \mathbb{N}$. The **boundary** $\partial E$ of $E \in \mathcal{P}_{\mathbb{N}}$ is the subset of $\mathbb{N}$ consisting in the integers $n \in E$ with $n-1 \notin E$ or $n+1 \notin E$. We also let $E^- := \{n \in E,\ n+1 \in E\}$. For $a, b \in \mathbb{N}$ we write $[\![a, b]\!]$ (resp. $[\![a, b[\![$, $]\!a, b]\!]$) the interval of integers $k$ with $a \leq k \leq b$ (resp $a \leq k < b$, $a < k \leq b$). The **connected components** of $E$ are the maximal intervals of integers contained in $E$. An interval of integers $[\![a, b[\![$ is said $E$-**irreducible** when we have $a, b \in E$ and $[\![a, b[\![ \cap E = \{a\}$. For $E \in \mathcal{P}_{\mathbb{N}}$ we let $E_{(n)} := E \cap [\![1, n]\!] \in \mathcal{P}_n$ for all $n \in \mathbb{N}$. For $M \in \mathbb{N}$, we denote by $E_M$ the union of the intervals $[\![a, b]\!]$ with $a, b \in E$ and $|a - b| \leq M$.

We let $\mathfrak{N}$ be the set of increasing sequences of natural integers, which may be identified with the subset of $\mathcal{P}(\mathbb{N})$ given by infinite subsets of $\mathbb{N}$. For $\mathfrak{n} \in \mathfrak{N}$ we define the **generalized power set of** $\mathfrak{n}$ as $\mathcal{Q}_{\mathfrak{n}} := \prod_{n \in \mathfrak{n}} \mathcal{P}_n$.

We recall now the classical notion of upper and lower asymptotic densities. For $n \in \mathbb{N}^*$ and $F_n \in \mathcal{P}_n$ we let $d_n(F_n)$ be the frequency of $F_n$ in $[\![1, n]\!]$:

$$d_n(F_n) = \frac{\sharp F_n}{n}.$$

The **upper and lower asymptotic densities** $\overline{d}(E)$ and $\underline{d}(E)$ of $E \in \mathcal{P}(\mathbb{N})$ are respectively defined by

$$\overline{d}(E) := \limsup_{n \in \mathbb{N}} d_n(E_{(n)}) \text{ and}$$

$$\underline{d}(E) := \liminf_{n \in \mathbb{N}} d_n(E_{(n)}).$$

We just write $d(E)$ for the limit, when the frequencies $d_n(E_{(n)})$ are converging. For any $\mathfrak{n} \in \mathfrak{N}$ we let similarly $\overline{d}^{\mathfrak{n}}(E) := \limsup_{n \in \mathfrak{n}} d_n(E_{(n)})$ and $\underline{d}^{\mathfrak{n}}(E) := \liminf_{n \in \mathfrak{n}} d_n(E_{(n)})$. The concept of upper and lower asymptotic densities of $E \in \mathcal{P}(\mathbb{N})$ may be extended to generalized power sets as follows. For $\mathfrak{n} \in \mathfrak{N}$ and $\mathcal{F} = (F_n)_{n \in \mathfrak{n}} \in \mathcal{Q}_{\mathfrak{n}}$ we let

$$\overline{d}^{\mathfrak{n}}(\mathcal{F}) := \limsup_{n \in \mathfrak{n}} d_n(F_n) \text{ and}$$

$$\underline{d}^{\mathfrak{n}}(\mathcal{F}) := \liminf_{n \in \mathfrak{n}} d_n(F_n).$$

Again we just write $d^{\mathfrak{n}}(E)$ and $d^{\mathfrak{n}}(\mathcal{F})$ when the corresponding frequencies are converging.

2.2. **Fölner sequence and density along subsequences.** We say that $E \in \mathcal{P}_{\mathbb{N}}$ is **Fölner along** $\mathfrak{n} \in \mathfrak{N}$ when its boundary $\partial E$ has zero upper asymptotic density with respect to $\mathfrak{n}$, i.e; $\overline{d}^{\mathfrak{n}}(\partial E) = 0$. More generally $\mathcal{F} = (F_n)_{n\in\mathfrak{n}} \in \mathcal{Q}_{\mathfrak{n}}$ with $\mathfrak{n} \in \mathfrak{N}$ is Fölner when we have $\overline{d}^{\mathfrak{n}}(\partial\mathcal{F}) = 0$ with $\partial\mathcal{F} = (\partial F_n)_{n\in\mathfrak{n}}$. In general this property seems to be weaker than the usual Fölner property $\limsup_{n\in\mathfrak{n}} \frac{\sharp\partial F_n}{\sharp F_n} = 0$. But in the following we will work with sequences $\mathcal{F}$ with $\underline{d}^{\mathfrak{n}}(\mathcal{F}) > 0$. In this case our definition coincides with the standard one.

Let $E, F \in \mathcal{P}_{\mathbb{N}}$ and $\mathfrak{n} \in \mathfrak{N}$. We say that $F$ is **$\mathfrak{n}$-filled with** $E$ or $E$ is **dense in** $F$ **along** $\mathfrak{n}$ when we have

$$\overline{d}^{\mathfrak{n}}(F \setminus E_M) \xrightarrow{M\to+\infty} 0.$$

Observe that $\big(\overline{d}(E_M)\big)_M$ is converging nondecreasingly to some $a \geq \overline{d}(E)$ when $M$ goes to infinity. The limit $a$ is in general strictly less than 1. For example if $E := \bigcup_n [\![2^{2n}, 2^{2n+1}]\!]$ one easily computes $\overline{d}(E_M) = \overline{d}(E) = 2/3$ for all $M$. In this case, the set $E$ is moreover a Fölner set.

Also $\mathcal{F} = (F_n)_{n\in\mathfrak{n}} \in \mathcal{Q}_{\mathfrak{n}}$ is said filled with $E$ when we have with $\mathcal{F}\setminus E_M := (F_n \setminus E_M)_{n\in\mathfrak{n}}$ :

$$\overline{d}^{\mathfrak{n}}(\mathcal{F} \setminus E_M) \xrightarrow{M\to+\infty} 0.$$

2.3. **Fölner set $F$ filled with a given $E$ with $\overline{d}(E) > 0$.** Given a set $E$ with positive upper asymptotic density we build a Fölner set $F$ filled with $E$ by using a diagonal argument. More precisely we will build $F$ by filling the holes in $E$ of larger and larger size when going to infinity.

**Lemma 1.** *For any $E$ with $\overline{d}(E) > 0$ there is $\mathfrak{n} \in \mathfrak{N}$ and $F \in \mathcal{P}_{\mathbb{N}}$ with $\partial F \subset E$ such that*

- $d^{\mathfrak{n}}(F) \geq d^{\mathfrak{n}}(E \cap F) = \overline{d}(E)$*;*
- *$F$ is Fölner along $\mathfrak{n}$;*
- *$E$ is dense in $F$ along $\mathfrak{n}$.*

*Proof.* We first consider an increasing sequence of integers $\mathfrak{n}^0 = (\mathfrak{n}^0_k)_k$ satisfying $d^{\mathfrak{n}^0}(E) = \overline{d}(E)$. We can ensure that $\mathfrak{n}^0_k$ belong to $E$ for all $k$. Observe that $\overline{d}(E \setminus E_M) \leq 1/M$ for all $M \in \mathbb{N}^*$. Therefore for $M > 2/\overline{d}(E)$, we have $\overline{d}^{\mathfrak{n}^0}(E_M) \geq \overline{d}^{\mathfrak{n}^0}(E_M \cap E) > \overline{d}(E)/2 > 0$. We fix such an integer $M$ and we extract again a subsequence $\mathfrak{n}^M = (\mathfrak{n}^M_k)_k$ of $\mathfrak{n}^0$ such that $d^{\mathfrak{n}^M}(E_M)$ is a limit equal to $\Delta_M := \overline{d}^{\mathfrak{n}^0}(E_M)$. Then we put $\Delta_{M+1} = \overline{d}^{\mathfrak{n}^M}(E_{M+1})$ and we consider a subsequence $\mathfrak{n}^{M+1}$ of $\mathfrak{n}^M$ such that $d^{\mathfrak{n}^{M+1}}(E_{M+1})$ is a limit equal to $\Delta_{M+1} \geq \Delta_M$ and $d_l(E_{M+1}) \leq \Delta_{M+1} + 1/2^{M+1}$ for all $l \in \mathfrak{n}^{M+1}$. We

define by induction in this way nested sequences $\mathfrak{n}^k$ for $k > M$ such that $d^{\mathfrak{n}^k}(E_k)$ is a limit equal to $\Delta_k = \overline{d}^{\mathfrak{n}^{k-1}}(E_k)$ and $d_l(E_k) \leq \Delta_k + 1/2^k$ for all $l \in \mathfrak{n}^k$. We let $\Delta_\infty > \overline{d}(E)/2 > 0$ be the limit of the nondecreasing sequence $(\Delta_k)_k$. We consider the diagonal sequence $\mathfrak{n} = (\mathfrak{n}_k)_{k\geq M} = (\mathfrak{n}_k^k)_{k\geq M}$ and we let

$$F = \bigcup_{k>M} [\![\mathfrak{n}_{k-1}, \mathfrak{n}_k]\!] \cap E_k.$$

Clearly we have $\partial F \subset \mathfrak{n}^0 \cup E \subset E$.

On the one hand, $F \cap [\![1, \mathfrak{n}_k]\!]$ is contained in $E_k \cap [\![1, \mathfrak{n}_k]\!]$ so that

$$\begin{aligned} d_{\mathfrak{n}_k}(F) &\leq d_{\mathfrak{n}_k^k}(E_k), \\ &\leq \Delta_k + 1/2^k, \\ \overline{d}^{\mathfrak{n}}(F) &\leq \lim_k \Delta_k = \Delta_\infty, \end{aligned}$$

On the other hand, $F \cap [\![1, \mathfrak{n}_k]\!]$ contains $E_l \cap [\![\mathfrak{n}_{l-1}, \mathfrak{n}_k]\!]$ for all $M < l < k$. Therefore

$$\begin{aligned} d_{\mathfrak{n}_k^k}(E_l) - \frac{\mathfrak{n}_{l-1}}{\mathfrak{n}_k^k} &\leq d_{\mathfrak{n}_k}(F), \\ \Delta_l &\leq \underline{d}^{\mathfrak{n}}(F), \\ \Delta_\infty &\leq \underline{d}^{\mathfrak{n}}(F). \end{aligned}$$

We conclude $d^{\mathfrak{n}}(F) = \Delta_\infty$.

Similarly we have for all $l > M$ :

$$\begin{aligned} \underline{d}^{\mathfrak{n}}(E \cap F) &\geq \underline{d}^{\mathfrak{n}}(E \cap E_l), \\ &\geq d^{\mathfrak{n}}(E) - 1/l, \\ &\geq \overline{d}(E) - 1/l, \end{aligned}$$

therefore $\underline{d}^{\mathfrak{n}}(E \cap F) \geq \overline{d}(E)$. Also $\overline{d}^{\mathfrak{n}}(E \cap F) \leq d^{\mathfrak{n}}(E) = \overline{d}(E)$. Consequently we get $d^{\mathfrak{n}}(E \cap F) = \overline{d}(E)$.

We check now that $E$ is dense in $F$. For $l$ fixed and for all $k \geq l$ we have

$$\begin{aligned} d_{\mathfrak{n}_k}(F \setminus E_l) &= d_{\mathfrak{n}_k^k}(E_k \setminus E_l), \\ &\leq d_{\mathfrak{n}_k^k}(E_k) - d_{\mathfrak{n}_k^k}(E_l), \\ &\leq \Delta_k + 1/2^k - d_{\mathfrak{n}_k^k}(E_l). \end{aligned}$$

By taking the limit in $k$, we get $\overline{d}^{\mathfrak{n}}(F \setminus E_l) \leq \Delta_\infty - \Delta_l \xrightarrow{l} 0$.

Let us prove finally the Fölner property of the set $F$. For $\mathfrak{n}_k < K \in \partial F$ either $[K-k, K[$ or $]K, K+k]$ lies in the complement of $E$. Therefore $\overline{d}^{\mathfrak{n}}(\partial F) \leq 2/k$. As it holds for all $k$, the set $F$ is Fölner along $\mathfrak{n}$. □

2.4. **Borel-Cantelli argument.** Let $(X, \mathcal{A}, \lambda)$ be a measure space with $\lambda$ being a finite measure. A map $E : X \to \mathcal{P}_{\mathbb{N}}$ is said measurable, when for all $n \in \mathbb{N}$ the set $\{x,\ n \in E(x)\}$ belongs to $\mathcal{A}$ (equivalently writing $E$ as an increasing sequence $(n_i)_{i\in\mathbb{N}}$ the integers valued functions $n_i$ are measurable). For such measurable maps $E$ and $\mathfrak{n}$, the upper asymptotic density $\overline{d}^{\mathfrak{n}}(E)$ defines a measurable function.

**Lemma 2.** *Assume $E$ is a measurable sequence of integers such that $\overline{d}(E(x)) > \beta > 0$ for $x$ in a measurable set $A$ of a positive $\lambda$-measure. Then there exist $\mathfrak{n} \in \mathfrak{N}$, measurable subsets $(A_n)_{n\in\mathfrak{n}}$ of $X$ and $\mathcal{F} = (F_n)_{n\in\mathfrak{n}} \in \mathcal{Q}_{\mathfrak{n}}$ with $\partial F_n \subset E(x)$ for all $x \in A_n$, $n \in \mathfrak{n}$ such that :*

- $\underline{d}^{\mathfrak{n}}(\mathcal{F}) \geq \beta$;
- $\lambda(A_n) \geq \frac{e^{-n\delta_n}}{n^2}$ *for all* $n \in \mathfrak{n}$ *with* $\delta_n \xrightarrow{\mathfrak{n}\ni n\to+\infty} 0$;
- $\mathcal{F}$ *is a Følner sequence;*
- $E$ *is dense in* $\mathcal{F}$ *uniformly on* $A_n$, *i.e.*

$$\limsup_{n\in\mathfrak{n}} \sup_{x\in A_n} d_n\left(F_n \setminus E_M(x)\right) \xrightarrow{M} 0.$$

- $$\liminf_{n\in\mathfrak{n}} \inf_{x\in A_n} d_n\left(E(x) \cap F_n\right) \geq \beta.$$

*Proof.* The sequences $\mathfrak{n}$ and $F$ built in the previous lemma define measurable sequences on $A$. By taking a smaller subset $A$ we may assume

- $\mathfrak{n}_k(x)$ is bounded on $A$ for all $k$,
- $d_{\mathfrak{n}_k(x)}(\partial F(x)) \xrightarrow{k} 0$ uniformly in $x \in A$,
- $\limsup_k \sup_{x\in A} d_{\mathfrak{n}_k(x)}(F(x) \setminus E_M(x)) \xrightarrow{M} 0$,
- $d_{\mathfrak{n}_k(x)}(E(x) \cap F(x)) \xrightarrow{k} d^{\mathfrak{n}(x)}(E(x) \cap F(x)) \geq \beta$ uniformly in $x \in A$.

By Borel-Cantelli Lemma, the subset $A_n := \{x \in A,\ n \in \mathfrak{n}(x)\}$ has $\lambda$-measure larger than $1/n^2$ for infinitely many $n \in \mathbb{N}$. We let $\mathfrak{n}$ be this infinite subset of integers. By the (uniform in $x$) Følner property of $F(x)$, the cardinality of the boundary of $(F(x))_{(n)} = F(x) \cap [\![1, n]\!]$ for $x \in A_n$ and $n \in \mathfrak{n}$ is less than $n\alpha_n$ for some sequence $(\alpha_n)_{n\in\mathfrak{n}}$ (independent of $x$) going to 0. Therefore there are at most $2\sum_{k=1}^{[n\alpha_n]} \binom{n}{k}$ choices for $(F(x))_{(n)}$ and thus it may be fixed by dividing the measure of $A_n$ by $2\sum_{k=1}^{[n\alpha_n]} \binom{n}{k} = e^{n\delta_n}$ for some $\delta_n \xrightarrow{n} 0$. □

## 3. Empirical measures associated to Følner sequences

Let $(X, T)$ be a topological system, i.e. $X$ is a compact metrizable space and $T : X \circlearrowleft$ is continuous. We denote by $\mathcal{M}(X)$ the set of Borel probability measures on $X$ endowed with the weak-$*$ topology and by $\mathcal{M}(X, T)$ the compact subset of invariant measures. We will write $\delta_x$ for the Dirac measure at $x \in X$. We let $T_*$ be the induced (continuous) action on $\mathcal{M}(X)$. For

$\mu \in \mathcal{M}(X)$ and a finite subset $F$ of $\mathbb{N}$, we let $\mu^F$ be the empirical measure $\mu^F := \frac{1}{\sharp F} \sum_{k\in F} T_*^k \mu$.

3.1. **Invariant measures.** The following lemma is standard, but we give a proof for the sake of completeness. We fix $\mathfrak{n} \in \mathfrak{N}$ and $\mathcal{F} = (F_n)_{n\in\mathfrak{N}} \in \mathcal{Q}_\mathfrak{n}$.

**Lemma 3.** *Assume $\mathcal{F}$ is a Følner sequence and $\underline{d}^\mathfrak{n}(\mathcal{F}) > 0$. Let $(\mu_n)_{n\in\mathfrak{n}}$ be a family in $\mathcal{M}(X)$ indexed by $\mathfrak{n}$. Then any limit of $\left(\mu_n^{F_n}\right)_{n\in\mathfrak{n}}$ is a $T$-invariant Borel probability measure.*

*Proof.* Let $\mathfrak{n}'$ be a subsequence of $\mathfrak{n}$ such that $\left(\mu_n^{F_n}\right)_{n\in\mathfrak{n}'}$ is converging to some $\mu'$. It is enough to check that $\left|\int \phi \, d\mu_n^{F_n} - \int \phi \circ T \, d\mu_n^{F_n}\right|$ goes to zero for when $\mathfrak{n}' \ni n \to +\infty$ for any $\phi : X \to \mathbb{R}$ continuous.

This follows from

$$\int \phi \, d\mu_n^{F_n} - \int \phi \circ T \, d\mu_n^{F_n} = \frac{1}{\sharp F_n} \int \left( \sum_{\substack{k+1\in F_n \\ k\notin F_n}} \phi \circ T^k - \sum_{\substack{k+1\notin F_n \\ k\in F_n}} \phi \circ T^k \right) d\,\mu_n,$$

$$\left|\int \phi \, d\mu_n^{F_n} - \int \phi \circ T \, d\mu_n^{F_n}\right| \leq \sup_{x\in X} |\phi(x)| \frac{\sharp \partial F_n}{\sharp F_n},$$

$$\liminf_{n\in\mathfrak{n}} \left|\int \phi \, d\mu_n^{F_n} - \int \phi \circ T \, d\mu_n^{F_n}\right| \leq \liminf_{n\in\mathfrak{n}} \frac{\sharp \partial F_n}{\sharp F_n},$$

$$\leq \frac{d^\mathfrak{n}(\partial\mathcal{F})}{\underline{d}^\mathfrak{n}(\mathcal{F})} = 0.$$

□

3.2. **Subadditive cocycles.** We fix a general continuous subadditive process $\Phi = (\phi_n)_{n\in\mathbb{N}}$ with respect to $(X, T)$, i.e. $\phi_0 = 0$, $\phi_n : X \to \mathbb{R}$ is a continuous function for all $n$ and $\phi_{n+m} \leq \phi_n + \phi_m \circ T^n$ for all $m, n$. In the proof of the main theorem we will only consider additive cocycles, but we think it could be interesting to consider general subadditive cocycles in other contexts.

Observe that $\Phi^+ = (\phi_n^+)_n$, with $\phi_n^+ = \max(\phi_n, 0)$ for all $n$, is also subadditive. For any $\mu \in \mathcal{M}(X, T)$, we let $\phi^+(\mu) = \lim_n \frac{1}{n} \int \phi_n^+ \, d\mu = \inf_n \frac{1}{n} \int \phi_n^+ \, d\mu$ (the existence of the limit follows from the subadditivity property). Recall also that by the subadditive ergodic theorem [25], the limit $\phi_*(x) = \lim_n \frac{\phi_n(x)}{n}$ exists for $x$ in a set of full measure with respect to any invariant measure $\mu$. When $\phi_*(x) \geq 0$ for $\mu$-almost every point $x$ with $\mu \in \mathcal{M}(X, T)$, then we have $\phi^+(\mu) = \int \phi_*(x) \, d\mu(x)$. If $\mu$ is moreover ergodic, then $\phi_*(x) = \phi^+(\mu)$ for $\mu$ almost every $x$.

Let $E : Y \to \mathcal{P}_\mathbb{N}$ be a measurable sequence of integers defined on a Borel subset $Y$ of $X$. For a set $F_n \in \mathcal{P}_n$ with $\partial F_n \subset E(x)$ for some $x \in X$, we may write $F_n^-$ uniquely as the finite union of $E(x)$-irreducible intervals

$F_n^- = \bigcup_{\mathsf{k}\in\mathsf{K}} [\![a_\mathsf{k}, b_\mathsf{k}[\![$. Let $n_\mathsf{k} = b_\mathsf{k} - a_\mathsf{k}$ for any $\mathsf{k} \in \mathsf{K}$. Then we define

$$\forall x \in X,\ \phi_E^{F_n}(x) := \sum_{\mathsf{k}\in\mathsf{K}} \phi_{n_\mathsf{k}}(T^{a_\mathsf{k}} x)$$

When $\Phi$ is additive, i.e. $\Phi_n = \sum_{0\le k<n} \phi \circ T^k$ for some continuous function $\phi : X \to \mathbb{R}$, we always have $\phi_E^{F_n}(x) = \phi^{F_n}(x) := \sum_{k\in F_n^-} \phi(T^k x)$.

The set valued map $E$ is said **$a$-large** with respect to $\Phi$ for some $a \ge 0$ when we have $\phi_{l-k}(T^k x) \ge (l-k)a$ for all consecutive integers $l > k$ in $E(x)$.

**Lemma 4.** *Let $\Phi$, $F_n$ and $E$ as above. Assume $E$ is 0-large. Then for all $x$ and $n \ge N \ge M$ we have*

$$\frac{\phi_E^{F_n}(x)}{\sharp F_n} \ge \int \frac{\phi_N^+}{N}\, d\delta_x^{F_n} - \frac{d_n\left(F_n \setminus E_M(x)\right) + N d_n(\partial F_n) + 4M/N}{d_n(F_n)} \sup_y |\phi_1(y)|$$

*Proof.* Let $k \in \{0, \cdots, N-1\}$ and $l \in \mathbb{N}$. The interval of integers $J_{k,l} = [\![k + lN, k + (l+1)N[\![$ may be written as

$$J_{k,l} = I_1 \coprod I_2 \coprod I_3 \coprod I_4$$

where $I_1$ is the union of disjoint $E$-irreducible intervals of length less than $M$ contained in $J_{k,l}$, $I_2 \subset \mathbb{N} \setminus F_n^-$, $I_3 \subset F_n^- \setminus E_M(x)$ and $I_4$ is the union of at most two subintervals of $E$-irreducible intervals of length less than $M$ containing an extremal point of $J_{k,l}$.

Therefore for a fixed $k$, by summing over all $l$ with $k + lN \in F_n$ we get as $E$ is 0-large and $\Phi$ is subadditive:

$$\begin{aligned}
&\sum_{l,\ k+lN\in F_n} \phi_N^+(T^k x) \\
&\le \sum_{\mathsf{k}\in\mathsf{K},\ [\![a_\mathsf{k},b_\mathsf{k}[\![\cap(k+N\mathbb{N})=\emptyset} \phi_{n_\mathsf{k}}(T^{a_\mathsf{k}} x) + \sup_y |\phi_1(y)| \left(N\sharp\partial F_n + \sharp\left(F_n \setminus E_M(x)\right) + 2M([n/N]+1)\right) \\
&\le \phi_E^{F_n}(x) + \sup_y |\phi_1(y)| \left(N\sharp\partial F_n + \sharp\left(F_n \setminus E_M(x)\right) + 2M([n/N]+1)\right).
\end{aligned}$$

Then by summing over all $k \in \{0, \cdots, N-1\}$ and dividing by $N$, we conclude that

$$\sharp F_n \int \frac{\phi_N^+}{N}\, d\delta_x^{F_n} \le \phi_E^{F_n}(x) + \sup_y |\phi_1(y)| \left(N\sharp\partial F_n + \sharp F_n \setminus E_M(x) + 2M([n/N]+1)\right).$$

□

3.3. **Positive exponent of empirical measures for additive cocycles.** We consider here an additive cocycle $\Phi$ associated to a continuous function $\phi : X \to \mathbb{R}$. With the notations of Lemma 2 and Lemma 3, we have :

**Lemma 5.** *Let $(\mu_n)_{n\in\mathfrak{n}}$ with $\mu_n(A_n)=1$ for all $n\in\mathfrak{n}$. Assume $E$ is $a$-large with $a>0$. Then for any weak-$*$ limit $\mu$ of $\mu_n^{F_n}$ we have*

$$\phi_*(x)\geq a \text{ for } \mu \text{ a.e. } x.$$

*Proof.* We claim that for any $0<\alpha<1$ and any $\epsilon>0$, there is arbitrarily large $N_0$ such that

$$\limsup_n \mu_n^{F_n}(\phi_{N_0}/N_0\geq \alpha a)\geq 1-\epsilon. \tag{3.1}$$

By weak-$*$ convergence of $\mu_n$ to $\mu$, it will imply, the set $\{\phi_{N_0}/N_0\geq\alpha a\}$ being closed :

$$\mu(\phi_{N_0}/N_0\geq\alpha a)\geq 1-\epsilon.$$

Then we may consider a sequence $(N_k)_k$ going to infinity such that

$$\mu(\phi_{N_k}/N_k\geq\alpha a)\geq 1-\epsilon/2^k.$$

Therefore $\mu\left(\bigcap_k\{\phi_{N_k}/N_k\geq\alpha a\}\right)\geq 1-2\epsilon$. We conclude $\limsup_n \frac{\phi_n(x)}{n}\geq\alpha a$ for $\mu$ a.e. $x$ by letting $\epsilon$ go to zero.

Let us show now our first claim (3.1). It is enough to show the inequality for $\mu_n=\delta_x$ uniformly in $x\in A_n$. We use the same notations as in the proof of Lemma 4. Fix $x\in A_n$. For $k,l$ with $k+lN\in F_n$, the interval $J_{k,l}$ is said admissible, when $\phi_N(f^{k+lN}x)/N\geq\alpha a$. If $J_{k,l}$ is not admissible we have

$$\begin{aligned}
\phi_N(f^{k+lN}x)&\geq \sum_{i\in I_1}\phi(f^ix)-\sup_y|\phi(y)|\sharp(I_2\cup I_3\cup I_4),\\
\alpha aN&\geq\ a\sharp I_1-\sup_y|\phi(y)|\sharp(I_2\cup I_3\cup I_4),\\
&\geq aN-(a+\sup_y|\phi(y)|)\sharp(I_2\cup I_3\cup I_4),\\
\sharp(I_2\cup I_3\cup I_4)&\geq\frac{(1-\alpha)aN}{a+\sup_y|\phi(y)|}.
\end{aligned}$$

If we sum over all $l$ with $k+lN\in F_n$ and then over $k\in\{0,\cdots,N-1\}$, we get by arguing as in the proof of Lemma 4 :

$$N\Big(N\sharp\partial F_n+\sharp\left(F_n\setminus E_M(x)\right)+2M([n/N]+1)\Big)\geq$$
$$\sharp\{J_{k,l}\text{ not admissible, }k+lN\in F_n\}\times\frac{(1-\alpha)aN}{a+\sup_y|\phi(y)|}.$$

Therefore by Lemma 2 (third and fourth items) we have for $\mathfrak{n}\ni n\gg N\gg M$ uniformly in $x\in A_n$,

$$\sharp\{J_{k,l}\text{ not admissible, }k+lN\in F_n\}\leq\epsilon\sharp F_n.$$

By definition of admissible intervals we conclude that

$$\limsup_n\delta_x^{F_n}(\phi_N/N\geq\alpha a)\geq 1-\epsilon.$$

□

3.4. **Entropy of empirical measures.** Following Misiurewicz's proof of the variational principle, we estimate the entropy of empirical measures from below. For a finite partition $P$ of $X$ and a finite subset $F$ of $\mathbb{N}$, we let $P^F$ be the iterated partition $P^F = \bigvee_{k\in F} f^{-k}P$. When $F = [\![0, n-1]\!]$, $n \in \mathbb{N}$, we just let $P^F = P^n$. We denote by $P(x)$ the element of $P$ containing $x \in X$.

For a Borel probability measure $\mu$ on $X$, the static entropy $H_\mu(P)$ of $\mu$ with respect to a (finite measurable) partition $P$ is defined as follows:

$$\begin{aligned} H_\mu(P) &= -\sum_{A\in P} \mu(A)\log\mu(A), \\ &= -\int \log\mu\left(P(x)\right)\, d\mu(x). \end{aligned}$$

When $\mu$ is $T$-invariant, we recall that the measure theoretical entropy of $\mu$ with respect to $P$ is then

$$h_\mu(P) = \lim_n \frac{1}{n} H_\mu(P^n)$$

and the entropy $h(\mu)$ of $\mu$ is

$$h(\mu) = \sup_P h_\mu(P).$$

We will use the two following standard properties of the static entropy[20]:

- for a fixed partition $P$, the map $\mu \mapsto H_\mu(P)$ is concave on $\mathcal{M}(X)$,
- for two partitions $P$ and $Q$, the joined partition $P \vee Q$ satisfies

$$H_\mu(P \vee Q) \leq H_\mu(P) + H_\mu(Q). \tag{3.2}$$

**Lemma 6.** *Let $\mathcal{F} = (F_n)_{n\in\mathfrak{n}}$ be a Følner sequence with $\underline{d}^{\mathfrak{n}}(\mathcal{F}) > 0$. For any measurable finite partition $P$ and $m \in \mathbb{N}^*$, there exist a sequence $(\epsilon_n)_{n\in\mathfrak{n}}$ converging to 0 such that*

$$\forall n \in \mathfrak{n},\ \frac{1}{m} H_{\mu_n^{F_n}}(P^m) \geq \frac{1}{\sharp F_n} H_{\mu_n}(P^{F_n}) + \epsilon_n.$$

*Proof.* When $F_n$ is an interval of integers, we have [29] :

$$\frac{1}{m} H_{\mu_n^{F_n}}(P^m) \geq \frac{1}{\sharp F_n} H_{\mu_n}(P^{F_n}) - \frac{3m\log\sharp P}{\sharp F_n}. \tag{3.3}$$

Consider a general set $F_n \in \mathcal{P}_n$. We decompose $F_n$ into connected components $F_n = \coprod_{k=1,\cdots,K} F_n^k$. Observe $K \leq \sharp\partial F_n$. Then we get :

$$\begin{aligned} \frac{1}{m} H_{\mu_n^{F_n}}(P^m) &\geq \sum_{k=1}^{K} \frac{\sharp F_n^k}{m\sharp F_n} H_{\mu_n^{F_n^k}}(P^m), \text{by concavity of } \mu \mapsto H_\mu(P^m) \\ &\geq \frac{1}{\sharp F_n} \sum_{k=1}^{K} H_{\mu_n}(P^{F_n^k}) - \frac{3mK\log\sharp P}{\sharp F_n}, \text{ by applying (3.3) to each } F_n^k, \\ &\geq \frac{1}{\sharp F_n} H_{\mu_n}(P^{F_n}) - 3m\log\sharp P \frac{\sharp\partial F_n}{\sharp F_n}, \text{ according to (3.2).} \end{aligned}$$

This concludes the proof with $\epsilon_n = 3m\frac{\sharp\partial F_n}{\sharp F_n}\log\sharp P$, because $\mathcal{F}$ is a Følner sequence with $\underline{d}^{\mathfrak{n}}(\mathcal{F}) > 0$. □

With the notations of Lemma 2 we let $\mu_n$ be the probability measure induced by $\lambda$ on $A_n$, i.e. $\mu_n = \frac{\lambda(A_n\cap\cdot)}{\lambda(A_n)}$. We assume with the notations of Lemma 4:

(H) For any $\delta > 0$, there exists $\epsilon > 0$ such that we have for any partition $P$ with diameter less than $\epsilon$ :

$$\exists N\ \forall x \in A_n \text{ with } N < n \in \mathfrak{n}, \quad \frac{1}{\lambda\left(P^{F_n}(x)\cap A_n\right)} \geq e^{-\delta n}e^{\phi_E^{F_n}(x)}.$$

**Proposition 2.** *Under the above hypothesis (H), any weak-∗ limit $\mu$ of $(\mu_n^{F_n})_{n\in\mathfrak{n}}$ satisfies*

$$h(\mu) \geq \phi^+(\mu).$$

*Proof.* Without loss of generality we may assume $(\mu_n^{F_n})_{n\in\mathfrak{n}}$ is converging to $\mu$. Fix $\delta > 0$ and take a partition $P$ with $\mu(\partial P) = 0$ and with diameter less than $\epsilon$. In particular we have for all fixed $m \in \mathbb{N}$:

$$\frac{1}{m}H_\mu(P^m) = \lim_n \frac{1}{m}H_{\mu_n^{F_n}}(P^m).$$

Then we get for $n \gg N \gg M \gg m$

$$\begin{aligned}
\frac{1}{m}H_\mu(P^m) &\geq \limsup_{n\in\mathfrak{n}} \frac{1}{\sharp F_n}H_{\mu_n}(P^{F_n}), \text{ by Lemma 6,}\\
&\geq \limsup_{n\in\mathfrak{n}} \frac{1}{\sharp F_n}\int \left(-\log\lambda\left(P^{F_n}(x)\cap A_n\right) + \log\lambda(A_n)\right)\, d\mu_n(x),\\
&\geq \limsup_{n\in\mathfrak{n}} \int \frac{\phi_E^{F_n} - \delta n}{\sharp F_n}\, d\mu_n(x), \text{ by Hypothesis (H),}\\
&\geq \limsup_{n\in\mathfrak{n}} \left(\int \frac{\phi_N^+}{N}\, d\mu_n^{F_n}\right.\\
&\quad \left. - \frac{\sup_y|\phi(y)|\left(\sup_{x\in A_n} d_n(F_n\setminus E_M(x)) + Nd_n(\partial F_n) + 4M/N\right) + \delta}{d_n(F_n)}\right), \text{ by Lemma 4,}\\
&\geq \int \frac{\phi_N^+}{N}\, d\mu - \frac{1}{\underline{d}^{\mathfrak{n}}(\mathcal{F})}\left(\sup_y|\phi(y)|\left(\limsup_{n\in\mathfrak{n}}\sup_{x\in A_n} d_n(F_n\setminus E_M(x)) + 4M/N\right) + \delta\right),\\
&\geq \phi^+(\mu) - \frac{1}{\underline{d}^{\mathfrak{n}}(\mathcal{F})}\left(\sup_y|\phi(y)|\left(\limsup_{n\in\mathfrak{n}}\sup_{x\in A_n} d_n(F_n\setminus E_M(x)) + 4M/N\right) + \delta\right).
\end{aligned}$$

Letting $N$, then $M$, then $m$ go to infinity, we conclude that

$$h(\mu) \geq h_\mu(P) \geq \phi^+(\mu) - \frac{\delta}{\underline{d}^{\mathfrak{n}}(\mathcal{F})}.$$

We conclude by taking $\delta$ arbitrarily small. □

## 4. Existence of SRB measures for $C^\infty$ surface diffeomorphisms

4.1. **Entropy formula.** By Ruelle's inequality [36], for any $C^1$ system, the entropy of an invariant measure is less than or equal to the integral of the sum of its positive Lyapunov exponents. For $C^{1+}$ systems, the following entropy characterization of SRB measures was obtained by Ledrappier and Young :

**Theorem 3.** [28] *An invariant measure of a $C^{1+}$ diffeomorphism on a compact manifold is a SRB measure if and only it has a positive Lyapunov exponent almost everywhere and the entropy is equal to the integral of the sum of its positive Lyapunov exponents.*

As the entropy is harmonic (i.e. preserves the ergodic decomposition), the ergodic components of a SRB measures are also SRB measures.

4.2. **Settings.** We consider from now a $C^1$ diffeomorphism $f : M \circlearrowleft$. Let $\|\|$ be a Riemaninan structure on $M$ and let $d$ be the induced metric on $M$. The (forward upper) Lyapunov exponent of $(x, v)$ for $x \in M$ and $v \in T_xM$ is defined as follows (see [31] for an introduction to Lyapunov exponents):

$$\chi(x, v) := \limsup_{n\to+\infty} \frac{1}{n} \log \|d_x f^n(v)\|.$$

The function $\chi(x, \cdot)$ admits only finitely many values $\chi_1(x) > ... > \chi_{p(x)}(x)$ on $T_xM \setminus \{0\}$ and generates a flag $0 \subsetneq V_{p(x)}(x) \subsetneq \cdots \subsetneq V_1 = T_xM$ with $V_i(x) = \{v \in T_xM,\ \chi(x, v) \leq \chi_i(x)\}$. In particular, $\chi(x, v) = \chi_i(x)$ for $v \in V_i(x) \setminus V_{i+1}(x)$. The function $p$ as well the functions $\chi_i$ and the vector spaces $V_i(x)$, $i = 1, ..., p(x)$ are invariant and depend Borel measurably on $x$. One can show the maximal Lyapunov exponent $\chi$ introduced in the introduction coincides with $\chi_1$ (see Appendix A).

A point $x$ is said **regular** when there exists a decomposition

$$T_xM = \bigoplus_{i=1}^{p(x)} H_i(x)$$

such that

$$\forall v \in H_i(x) \setminus \{0\},\ \lim_{n\to\pm\infty} \frac{1}{|n|} \log \|d_x f^n(v)\| = \chi_i(x)$$

with uniform convergence in $\{v \in H_i(x),\ \|v\| = 1\}$ and

$$\lim_{n\to\pm\infty} \frac{1}{|n|} \log \mathrm{Jac}\,(d_x f^n) = \sum_i \dim(H_i(x))\chi_i(x).$$

In particular we have $V_i(x) = \bigoplus_{j=1}^{i} H_j(x)$ for all $i$. The set $\mathcal{R}$ of regular points is an invariant measurable set of full measure for any invariant measure [27]. The invariant subbundles $H_i$ are called the Oseledec's bundles. We also let $\mathcal{R}^* := \{x \in \mathcal{R},\ \forall i\ \chi_i(x) \neq 0\}$. For $x \in \mathcal{R}^*$ we put

$E_u(x) = \bigoplus_{i,\ \chi_i(x)>0} H_i(x)$ and $E_s(x) = \bigoplus_{i,\ \chi_i(x)<0} H_i(x)$.

In the following we only consider surface diffeomorphisms. Therefore we always have $p(x) \leq 2$ and when $p(x)$ is equal to 1, we let $\chi_2(x) = \chi_1(x)$. Let $\mathcal{R}_1$ (resp. $\mathcal{R}_2$) be the invariant subset of regular points $x$ of $(M, f)$ with $\chi_1(x) > 0 > \chi_2(x)$ (resp. with $\chi_1(x) \geq \chi_2(x) \geq 0$). When $\nu$ is $f$-invariant we let $\chi_i(\nu) = \int \chi_i \, d\nu$. By Ruelle's inequality, when an ergodic measure $\nu$ has positive entropy, then $\chi_1(\nu) = \chi_f(\nu) \geq h_f(\nu) > 0$ and $-\chi_2(\nu) = \chi_{f^{-1}}(\nu) \geq h_{f^{-1}}(\nu) = h_f(\nu) > 0$ and therefore $\nu(\mathcal{R}_1) = 1$.

For a $C^1$ curve $\sigma : I \to M$, $I$ being a compact interval of $\mathbb{R}$, we let $\sigma_* = \sigma(I)$. The length of $\sigma_*$ for the induced Riemanian metric is denoted by $|\sigma_*|$. We also let $v_x \in \mathbb{P}T^1M$ be the projective vector tangent to $\sigma_*$ at $x$ . We write $\hat{x} = (x, v_x)$ and $\hat{\sigma_*} = \{\hat{x},\ x \in \sigma_*\}$. Assume $\limsup_n \frac{1}{n} \log \|d_x f^n\| > 0$ on a set of positive Lebesgue measure as in the Main Theorem or Corollary 2. Then by using Fubini's theorem as in [14] there is a $C^\infty$ smooth embedded curve $\sigma : I \to M$ and a subset $A$ of $\sigma_*$ with $\mathrm{Leb}_{\sigma_*}(A) > 0$, such that we have $\limsup_n \frac{1}{n} \log \|d_x f^n(v_x)\| > a > 0$ for all $x \in A$. Here $\mathrm{Leb}_{\sigma_*}$ denotes the Lebesgue measure on $\sigma_*$ induced by its inherited Riemannian structure as a submanifold of $M$. This a finite measure with $\mathrm{Leb}_{\sigma_*}(M) = |\sigma_*|$. Fix $0 < b < a$.

We will apply the results of the previous section to the projective action $F : \mathbb{P}T^1M \circlearrowleft$ induced by $f$, where we consider:

- the additive derivative cocycle $\Phi = (\phi_k)_k$ for $F$ on $\mathbb{P}T^1M$ given by $\phi(x, v) = \phi_1(x, v) = \log \|d_x f(v)\|$,
- the measure $\lambda$ on $\mathbb{P}T^1M$ given by $\mathfrak{s}^* \mathrm{Leb}_{\sigma_*}$ with $\mathfrak{s} : x \in \sigma_* \mapsto (x, v_x)$,
- the $b$-large set $E(x)$, $x \in A$, of $b$-**hyperbolic times** for $\phi$ at $x$, i.e. the set of integers $n > 0$ with $\phi_k(F^{n-k}(x, v_x)) \geq kb$ for all $0 \leq k \leq n$.

By Pliss lemma [32], we have for some constant $\beta > 0$ depending only on $a$, $b$ and $\|df\|$:

$$\forall x \in A, \quad \overline{d}(E(x)) > \beta. \tag{4.1}$$

Let $\pi : \mathbb{P}TM \to M$ be the natural projection. The topological extension $\pi : (\mathbb{P}TM, F) \to (M, f)$ is principal[†] by a straightforward application of Ledappier-Walters variational principle [28] and Lemma 3.3 in [37]. In fact this holds in any dimension and more generally for any finite dimensional vector bundle morphism instead of $df : TM \circlearrowleft$.

4.3. **Reduction to property** $(H)$**.** We prove now Corollary 2 assuming the following Proposition 4, whose proof is given in the next section. This is also a first step in the proof of the Main Theorem.

**Proposition 4.** *The $b$-large measurable sequence $E : \mathbb{P}TM \to \mathcal{P}_\mathbb{N}$ satisfies the property $(H)$.*

[†]i.e. $h_f(\pi_*\mu) = h_F(\mu)$ for all $F$-invariant measure $\mu$.

*Proof of Corollary 2.* Let $\mathcal{F} = (F_n)_{n\in\mathfrak{n}}$ and $(A_n)_{n\in\mathfrak{n}}$ be the sequences associated to $E$ given by Lemma 2. Any weak-$*$ limit $\mu$ of $\mu_n^{F_n}$ is invariant under $F$ and thus supported by Oseledec's bundles. By Lemma 5, $\mu$ is supported by the unstable bundle $E_u$ and $\phi_*(\hat{x}) \geq b$ for $\mu$ a.e. $\hat{x} \in \mathbb{P}TM$. By the subadditive ergodic theorem, recall that :

$$\phi^+(\mu) = \int \phi_* \, d\mu \geq b.$$

Write $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ with $\mu_i(\pi^{-1}\mathcal{R}_i) = 1$ for $i = 1, 2$. Observe that the measures $\mu_i$ are $F$-invariant. Moreover $h(\mu_2) = h(\pi\mu_2) = 0$. Therefore by Proposition 2 we obtain:

$$\begin{aligned} h(\mu) &\geq \int \phi_* \, d\mu, \\ &\geq \alpha \int \phi_* \, d\mu_1 + (1-\alpha) \int \phi_* \, d\mu_2, \\ &\geq \alpha \int \phi_* \, d\mu_1 = \alpha\chi_1(\pi\mu_1). \end{aligned} \tag{4.2}$$

But recall $h(\mu) = h(\pi\mu) = \alpha h(\pi\mu_1) \geq b$, so that we have $\alpha > 0$ and then

$$h(\pi\mu_1) \geq \chi_1(\pi\mu_1) \geq b.$$

The converse inequality $h(\pi\mu_1) \leq \chi_1(\pi\mu_1)$ follows from Ruelle's inequality. Therefore $h(\pi\mu_1) = \chi_1(\pi\mu_1) > 0$ and by Ledrappier-Young characterization (Theorem 3), $\nu = \pi\mu_1$ is a SRB measure of $(M, f)$. In fact from (4.2), $h(\mu_2) = 0$ and $\phi_* \geq b$ $\mu$ a.e. we get $\alpha = 1$ and thus $\pi\mu = \pi\mu_1$ is a SRB measure. □

## 5. Proof of Proposition 4

In this section we use tools introduced firstly in [14] to bound the distorsion of unstable manifolds. As these manifolds are one-dimensional, the reparametrizations in Yomdin's theory are just affine. In fact we can in this case completely avoid the use of the classical Yomdin's Reparametrization Lemma by applying the Landau-Kolmogorov inequality as in [14].

5.1. **Some terminology.** Following [14] a $C^r$, $r \in \mathbb{N} \setminus \{0, 1\}$, smooth curve $\sigma : [-\epsilon, \epsilon] \to \mathbb{R}^2$ with $\epsilon > 0$ is said $(\epsilon, r)$-**bounded** when

$$\max_{s=2,\cdots,r} \|d^s\sigma\|_\infty \leq \|d\sigma\|_\infty. \tag{5.1}$$

If moreover $\|d\sigma\|_\infty \leq 1$ we say that $\sigma$ is **strongly** $(\epsilon, r)$-**bounded**. In particular such a map satisfies $\|\sigma\|_r := \max_{1\leq s\leq r} \|d^s\sigma\| \leq 1$, which is the standard $C^r$ upper bound required for the reparametrizations in the usual Yomdin's theory. Our stronger condition (5.1) will allow to control the distorsion along the curve. The angle between two lines $e$ and $f$ will be denoted by $\angle e, f$.

**Lemma 7.** *Let $\sigma$ be a $(\epsilon, r)$-bounded curve. Then we have for $\epsilon$ small enough*

- $\forall t, s \in [-\epsilon, \epsilon],\ \frac{\|\sigma'(t)\|}{\|\sigma'(s)\|} \leq 1 + 5\epsilon$,
- $\forall x, y \in \sigma_*,\ \angle v_x, v_y < 10\epsilon$.

*Proof.* We have

$$\forall t, s \in [-\epsilon, \epsilon],\ \|\sigma'(t) - \sigma'(s)\| \leq 2\epsilon \|d^2\sigma\| \leq 2\epsilon \|d\sigma\|, \tag{5.2}$$

$$\text{therefore } \frac{\|\sigma'(t)\|}{\|\sigma'(s)\|} \leq \frac{1+2\epsilon}{1-2\epsilon} \sim 1 + 4\epsilon.$$

Then let $s_0$ with $\|\sigma'(s_0)\| = \|\sigma'\|_\infty$ and put $x_0 = \sigma(s_0)$. Together with the above inequality (5.2) we get :

$$\forall t \in [-\epsilon, \epsilon],\ \left\| \frac{\sigma'(t)}{\|\sigma'(s_0)\|} - v_{x_0} \right\| \leq 2\epsilon,$$

and then with $x = \sigma(t)$ :

$$\|v_x - v_{x_0}\| \leq \left\| \frac{\sigma'(t)}{\|\sigma'(s_0)\|} - v_{x_0} \right\| + \left| 1 - \|\sigma'(t)\| / \|\sigma'(s_0)\| \right| \leq 9\epsilon/2.$$

□

To study the local properties of $f^n\sigma_*$ for a smooth curve $\sigma$ in $M$ we work in $\mathbb{R}^2$ by looking through charts of the manifold, but the dynamics is then nonautonomous.

We consider a family $\mathcal{F} = (\mathtt{f}_n)_{n\in\mathbb{N}}$ of $C^r$ maps $\mathtt{f}_n : B(R) \to \mathbb{R}^2$ (with $\mathtt{f}_0$ being the inclusion map), where $B(R)$ denotes the Euclidean ball in $\mathbb{R}^2$ of radius $R$ centered at 0 . For all $n \in \mathbb{N}$ we let $\mathtt{f}^n$ be the composition $\mathtt{f}_n \circ \cdots \circ \mathtt{f}_0$ defined on $B_n(R) := \bigcap_{0\leq k<n} (\mathtt{f}^k)^{-1} B(R)$. Then a curve $\sigma$ in $\mathbb{R}^2$ is said **strongly $(\epsilon, n, r)$-bounded with respect to $\mathcal{F} = (\mathtt{f}_n)_{n\in\mathbb{N}}$** when $\sigma_*$ is contained in $B_n(R)$ and the curves $\mathtt{f}^k \circ \sigma : [-\epsilon, \epsilon] \to \mathbb{R}^2$ are strongly $(\epsilon, r)$-bounded for $k = 0, \cdots, n-1$. Let $\hat{\mathcal{F}} = (F_n)_n$ be the projective derivative maps of $\mathcal{F}$ and we put again $F^n = F_n \circ \cdots \circ F_0$.

Let $\mathtt{d}$ be the distance on $\mathbb{P}TB(R) \sim B(0,R) \times \mathbb{PR}^2$ given by the sum of the Euclidean distance of the base points and the angles of the two lines. We define then the $\hat{\mathcal{F}}$-dynamical ball at $\hat{x} \in \mathbb{P}TB(R)$ by :

$$B^{\hat{\mathcal{F}}}_{\mathtt{d}}(\hat{x}, n, \epsilon) := \bigcap_{k=0}^{n-1} F^{-k} B_{\mathtt{d}}(\hat{x}, \epsilon)$$

and the additive cocycle (for the nonautonomous system $\mathcal{F}$) by :

$$\forall \hat{x} = (x, v) \in \mathbb{P}TB_n(R),\ \phi_n(\hat{x}) = \log \|d_x \mathtt{f}^n(v)\|.$$

From Lemma 7 we get by an immediate induction :

**Lemma 8.** *Let $\sigma$ be a strongly $(\epsilon, n, r)$-bounded curve with respect to $\mathcal{F}$. Then for any $1 \leq k \leq n$ and for all $\hat{y}, \hat{z} \in \hat{\sigma}_*$ we have :*

- $\hat{\sigma_*} \subset B^F_{\mathtt{d}}(\hat{x}, n, 12\epsilon)$ *with* $x = \sigma(0)$ *and* $\hat{x} = (x, v_x)$,
- $\frac{e^{\phi_k(\hat{y})}}{e^{\phi_k(\hat{z})}} \leq (1 + 5\epsilon)^2 \leq 1 + 100\epsilon$.

5.2. **Large $r$-bounded curves at hyperbolic times.** When $n$ is a $b$-hyperbolic time for the additive cocycle $(\phi_n)_n$ at $x$ we will show that the geometry of the curve $f^n\sigma_*$ around $f^n x$ is *nice*.

5.2.1. *Bounding $C^r$ norms.* Let $\mathtt{f} : B(R) \to \mathbb{R}^2$ be a $C^r$ smooth map. There is $\epsilon > 0$ depending only on $\|\mathtt{f}\|_r$ so small that for any $x \in M$ the map $\mathtt{f}_\epsilon = \mathtt{f}(\epsilon\cdot)$ satisfies $\|\mathtt{f}_\epsilon\|_r = \|d\mathtt{f}_\epsilon\|$. For any $1 \geq b > 0$ we let $\psi_b : [-\epsilon, \epsilon] \circlearrowleft$, be the linear map $x \mapsto bx$ and for any $\epsilon > 0$ and any $\sigma : [-\epsilon, \epsilon] \to \mathbb{R}^2$ we let $\sigma_\epsilon = \epsilon^{-1}\sigma$. Finally we write $\mathrm{m}(d_x\mathtt{f})$ for the conorm of $d_x\mathtt{f}$, i.e. $\mathrm{m}(d_x\mathtt{f}) := \min_{v\in\mathbb{R}^2,\ \|v\|=1} \|d_x\mathtt{f}(v)\|$, and we let $\mathrm{m}(d\mathtt{f}) := \min_{x\in B(R)} \mathrm{m}(d_x\mathtt{f})$.

**Lemma 9.** *Let $\sigma$ be a $(\epsilon, r)$-bounded curve with $\sigma(0) = x$. There exists a constant $C(r) > 1$ depending only on $r$ such that the map $\mathtt{f} \circ \sigma \circ \psi_a : [-\epsilon, \epsilon] \to M$ is $(\epsilon, r)$-bounded with*

$$a = a(r, \mathtt{f}) = C(r)^{-1} \left(\|d\mathtt{f}\|/\,\mathrm{m}(d\mathtt{f})\right)^{-1/r-1}.$$

*Proof.* We compute for some constant $C'(r)$ with $b = \left(C'(r)\|d\mathtt{f}\|/m(d\mathtt{f})\right)^{-1/r-1}$

$$\begin{aligned}
d^r(\mathtt{f} \circ \sigma \circ \psi_b) &= d^r(\mathtt{f} \circ \sigma)b^r,\\
\|d^r(\mathtt{f} \circ \sigma \circ \psi_b)\| &\leq C'(r)\|\mathtt{f}_\epsilon\|_r\|\sigma_\epsilon\|_r b^r, \text{ by Faà di Bruno's formula,}\\
&\leq C'(r)\|\mathtt{f}_\epsilon\|_r\epsilon^{-1}\|\sigma\|_r b^r,\\
&\leq C'(r)\|d\mathtt{f}_\epsilon\|\epsilon^{-1}\|\sigma\|_r b^r, \text{ by the choice of } \epsilon,\\
&\leq C'(r)\|d\mathtt{f}_\epsilon\|\epsilon^{-1}\|d\sigma\| b^r, \text{ since } \sigma \text{ is } (\epsilon, r)\text{-bounded,}\\
&\leq C'(r)\|d\mathtt{f}\|\|d\sigma\| b^r,\\
&\leq (C'(r)b^{r-1}\|d\mathtt{f}\|)\|d\sigma\| b,\\
&\leq \mathrm{m}(d\mathtt{f})\|d\sigma\| b, \text{ by definition of } b,\\
&\leq \|d(\mathtt{f} \circ \sigma \circ \psi_b)\|.
\end{aligned}$$

Then by the Landau-Kolmogorov inequality (see [14]), we have for some other constant $C''(r) \geq 1$ :

$$\|\mathtt{f} \circ \sigma \circ \psi_b\|_r \leq C''(r)\|d(\mathtt{f} \circ \sigma \circ \psi_b)\|.$$

By taking $a = b/C''(r)$ we get the desired result. □

5.2.2. *Non autonomous system associated to $(M, f)$.* We let exp and $R = R_{inj}$ be respectively the exponential map and the radius of injectivity of the Riemanian manifold $(M, \|\cdot\|)$. For $x \in M$ we let $\mathcal{F}^x = \mathcal{F}^x(f) = (\mathtt{f}_n^x)_n$ be the non autonomous systems defined for all $n$ by

$$\mathtt{f}_n^x = \exp_{f^n x}^{-1} \circ f \circ \exp_{f^{n-1}x} : B(0, R) \subset T_xM \sim \mathbb{R}^2 \to T_{fx}M \sim \mathbb{R}^2.$$

For a $C^r$ map $g : U \to \mathbb{R}^2$ with $U$ an open subset of $\mathbb{R}^2$ we write again $\|g\|_r = \max_{1\leq s\leq r} \|d^s g\|$. Observe that $\sup_{n,x} \|\mathtt{f}_n^x\|_r < +\infty$. A curve $\sigma : [-\epsilon, \epsilon] \to M$ with $x = \sigma(0)$ is said strongly $(\epsilon, n, r)$-bounded when $\exp_x^{-1} \circ \sigma$ is strongly $(\epsilon, n, r)$-bounded with respect to $\mathcal{F}^x$.

Let $\sigma : I \to M$ be a $C^r$ embedded smooth curve. For $x \in \sigma_*$ and $n \in \mathbb{N}$ we let $I_n^x = I_n^x(\epsilon, \sigma, r)$ be the largest subinterval of $I$ such that $\sigma \circ \psi_n^x$ is strongly $(\epsilon, n, r)$-bounded, where $\psi_n^x$ denotes the affine reparametrization $\psi_n^x = \psi_{n,r}^x : [-\epsilon, \epsilon] \to I_n^x$ with $\sigma \circ \psi_n^x(0) = x$. For $y \in \sigma_*$ we let $D_k(y) (= D_{k,\sigma}^{r,\epsilon}(y)) := (f^k \circ \sigma \circ \psi_k^y)_*$ and $E_k(y) := (\sigma \circ \psi_k^y)_*$. Finally we let $a(r, f) = \inf_{\mathtt{f} \in \bigcup_{x \in M} \mathcal{F}_x(f)} a(r, \mathtt{f}) > 0$.

5.2.3. *The length of $D_k(y)$ at hyperbolic times $k$.* In standard Yomdin's theory one reparametrizes the intersection of a disc $\sigma$ with a dynamical ball by maps $\psi$ satisfying $\|f^n \circ \sigma \circ \psi\|_r \leq 1$. However the method does not allow to control neither the distorsion on $\sigma_*$ nor the length of $(f^n \circ \sigma \circ \psi)_*$. For curves, by the simple modified approach developed in the previous subsections, we can always control the distorsion. Moreover the length of $D_n(y)$ is bounded away from zero at hyperbolic times $n$.

A $(\epsilon, r)$-bounded curve $\sigma$ with $\sigma(0) = x$ is said **ample** when $\|d\left(\exp_x^{-1} \circ \sigma\right)\| > 1/2$. By the distorsion property of Lemma 7, the length of $\sigma_*$ is then larger $\frac{\epsilon}{2}$. Without loss of generality the starting curve $\sigma$ in Subsection 4.2 may be chosen to be an ample $(\epsilon, r)$-bounded curve.

**Lemma 10.** *Let $b > 0$. There is $\epsilon > 0$, $1 \geq \alpha > 0$ and $r \in \mathbb{N}$ depending on $b$ such that for any ample $(\epsilon, r)$-bounded curve $\sigma$ the length of $D_{n,\sigma}^{r,\epsilon}(x)$ is larger than $\alpha\epsilon$ for any $b$-hyperbolic time $n$ at $x$.*

*Proof.* Assume $n$ is a $b$-hyperbolic time for $x$. We may choose $r$ then $p$ with $\frac{-\log a(r, f^p)}{p} < b/2$. Finally take $\epsilon$ with $\|\mathtt{f}_\epsilon^p\|_r = \|d\mathtt{f}_\epsilon^p\|$ for any $\mathtt{f}^p \in \bigcup_x \mathcal{F}_x(f^p)$. We may also ensure that for all $z \in M$ and $y \in T_z M$ with $\|y\| < \epsilon$ we have

$$\mathrm{m}(d_y \exp_z) \geq e \geq \|d_y \exp_z\|. \tag{5.3}$$

Let $m$ be the largest integer less than or equal to $n$ such that $\exp_{f^m x}^{-1} \circ f^m \circ \sigma \circ \psi_m^x$ is ample, i.e. $\|d(\exp_{f^m x}^{-1} \circ f^m \circ \sigma \circ \psi_m^x)\| > 1/2$. For $L \in \mathbb{N}$ we let $A_L := \min_{l<L} a(r, f^l)$. We take $K \in \mathbb{N}$ with $A_p \geq 2e^{2-Kpb/2}$.

- If $m > n - Kp$ then $\exp_x^{-1} \circ \sigma \circ \psi_m^x(A_{Kp} \cdot)$ is strongly $(\epsilon, n, r)$-bounded with respect to $\mathcal{F}_x$ by Lemma 9. Indeed observe that for $j = m + 1, \cdots, n$ we have $\|d(\exp_{f^j x}^{-1} \circ f^j \circ \sigma \circ \psi_m^x(A_{Kp} \cdot))\| \leq 1/2$, otherwise this would contradict the definition of $m$. Using again Lemma 7, the length of $D_n(x)$ is larger than $\alpha\epsilon$ with

$$\alpha = A_{Kp} \min_{k<Kp} \mathrm{m}(df^k).$$

- Consider now the remaining case $k := n - m \geq Kp$. Write $n - m = qp + r$ with $0 \leq r < p$. By Lemma 9 we have then

$$\frac{|\psi_n^x|}{|\psi_m^x|} \geq a(r, f^p)^q A_p \geq 2e^{2-kb}. \tag{5.4}$$

It follows that

$$\begin{aligned}
\|d(\exp^{-1}_{f^n x}\circ f^n\circ\sigma\circ\psi^x_n)\| &\geq e^{-1}\|d_0(f^n\circ\sigma\circ\psi^x_n)\|, \text{by the choice of } \epsilon \text{ in (5.3)},\\
&\geq e^{kb-1}\|d_0(f^m\circ\sigma\circ\psi^x_n)\|, \text{ as } n \text{ is a } b\text{-hyperbolic time},\\
&\geq e^{kb-1}\|d_0(f^m\circ\sigma\circ\psi^x_m)\|\times\frac{|\psi^x_n|}{|\psi^x_m|}, \text{ as } \psi^x_m \text{ and } \psi^x_n \text{ are affine},\\
&\geq e\|d(f^m\circ\sigma\circ\psi^x_m)\|, \text{by (5.4) and Lemma 7},\\
&\geq \|d(\exp^{-1}_{f^m x}\circ f^m\circ\sigma\circ\psi^x_m)\|, \text{ again by (5.3)},\\
&\geq 1/2, \text{ by definition of } m.
\end{aligned}$$

Therefore the $(\epsilon, n, r)$-bounded curve $\exp^{-1}_{f^n x}\circ f^n\circ\sigma\circ\psi^x_n$ is ample, contradicting the definition of $m$.

Consequently, only the first case occurs and this completes the proof. □

5.3. **Proof of** $(H)$**.** Fix the error term $\delta > 0$. We take $\epsilon$ and $\alpha$ as in Lemma 10. Without loss of generality we may also assume that (recall $\phi(\hat{x}) = \log\|d_x f(v)\|$ for $\hat{x} = (x, v) \in \mathbb{P}TM$)

$$\forall \hat{x}, \hat{y} \in \mathbb{P}TM \text{ with } \mathtt{d}(\hat{x}, \hat{y}) < \epsilon, \quad |\phi(\hat{x}) - \phi(\hat{y})| < \delta/3. \tag{5.5}$$

By Yomdin's theorem on local volume growth [40] we may also guarantee by taking $r$ larger and $\epsilon$ smaller that $v^*(f, 2\epsilon, \sigma, n) := \sup_{x\in X}|f^n(\sigma_*\cap B(x, \epsilon, n)| < De^{\delta n/3}$ for some $D > 0$, for any curve $\sigma$ with $\|\sigma\|_r \leq 1$ and any $n \in \mathbb{N}$. We shall prove with the notations of Sections 3 and 4 that for any partition $P$ of $\mathbb{P}TM$ with diameter less than $\alpha\epsilon/4 < \epsilon$ there exists $N$ such that

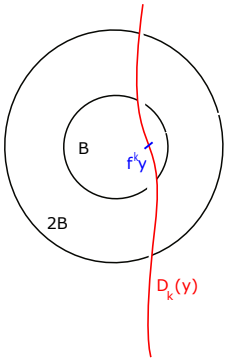
$$\forall x \in A_n \subset \sigma_*,\ \forall n \in \mathfrak{n},\ n > N, \quad \frac{1}{\lambda\left(P^{F_n}(\hat{x})\cap\pi^{-1}A_n\right)} \geq e^{-\delta n}e^{\phi^{F_n}(\hat{x})}.$$

For $G \subset \mathbb{N}$ we let $A^G$ be the set of points $x$ such that the integers in $G$ are $b$-hyperbolic times for $\phi$ at $x$. When $G = \{k\}$ or $\{k, l\}$ with $k, l \in \mathbb{N}$, we just let $A^G = A^k$ or $A^{k,l}$. In the next triplets lemmas we consider a strongly $(\epsilon, r)$-bounded curve $\sigma$ of length larger than $\alpha\epsilon$ with $\epsilon$, $\alpha$ and $r$ as in Lemma 10.

**Lemma 11.** *For any subset $E$ of $M$, any $k \in \mathbb{N}$ and any ball $B_k$ of radius less than $\alpha\epsilon/4$, there exists a finite family $(y_j)_{j\in J}$ of $\sigma_*\cap A^k\cap f^{-k}B_k\cap E$ such that :*

- $B_k\cap f^k(\sigma_*\cap A^k\cap E)\subset\bigcup_{j\in J}D_k(y_j)$,
- $B_k\cap D_k(y_j)$, $j \in J$, *are pairwise disjoint.*

*Proof.* Let $y \in \sigma_* \cap A^k \cap E$ with $f^k y \in B = B_k$. Let $2B$ be the ball of same center as $B$ with twice radius. By Lemma 7 and Lemma 10, the curve $D_k(y)$ lies in a small cone and its length is larger than $\alpha\epsilon$. Therefore $D_k(y) \cap 2B$ is a curve crossing $2B$, i.e. its two endpoints lies in the boundary of $2B$. Two such subcurves of $f^k \circ \sigma$ if not disjoint are necessarily equal. □




To simplify the notations, when $x$ belongs to $\sigma_*$ we just let $\phi(x) = \log \|d_x f(v_x)\|$ and we write $(\phi_k)_k$ for the associated cocycle. As the distorsion is bounded on $D_k(y_i)$ (by 2 for $\epsilon$ small enough by Lemma 8) and the length of $D_k(y_i)$ is larger than $\alpha\epsilon$ (because $y_i$ belongs to $A^k$ by Lemma 10), we have

$$\sum_{i\in I} \frac{1}{2} e^{-\phi_k(y_i)} |D_k(y_i)| \le \sum_{i\in I} \left| f^{-k} D_k(y_i) \right| \le |\sigma_*|,$$

$$\sum_{i\in I} \alpha\epsilon e^{-\phi_k(y_i)} \le 2\epsilon,$$

$$\sum_{i\in I} e^{-\phi_k(y_i)} \le 2/\alpha. \tag{5.6}$$

**Lemma 12.** *For any subset $E$ of $M$ and any $f$-dynamical ball $B_{[\![0,k]\!]}$ of size $\alpha\epsilon/4$, i.e. $B_{[\![0,k]\!]} = \bigcap_{m=0}^k f^{-m} B_m$ where $B_m$ is a ball of radius $\alpha\epsilon/4$ for $m = 0, \cdots, k$, there exists a finite family $(z_i)_{i\in I}$ of $\sigma_* \cap A^k \cap B_{[\![0,k]\!]} \cap E$ such that*

- $f^k \left( \sigma_* \cap A^k \cap B_{[\![0,k]\!]} \cap E \right) \subset \bigcup_{i\in I} D_k(z_i)$,
- *$B_k \cap D_k(z_i)$, $i \in I$, are pairwise disjoint,*
- *$\sharp I \le C e^{\delta k/3}$ for some constant $C$ depending only on $b$ and $f$.*

*Proof.* We argue as in the previous lemma by considering $D_k(z)$ for $z \in \sigma_* \cap A^k \cap B_{[\![0,k]\!]} \cap E$. Each of this curve is contained in a $f$-dynamical ball of size $2\epsilon$, and therefore in the same dynamical ball of size $2\epsilon$. Then $v^*(f, 2\epsilon, \sigma, k) \le D e^{\delta k/3}$ implies the upper bound on $\sharp I$ because the length of each $D_k(z)$ is at least $\alpha\epsilon$ by Lemma 10. □

**Remark 5.** *So far one could work with a $C^r$ diffeomorphisms $1 < r < +\infty$ and points $x$ with $\chi(x, v_x) > \frac{R(f)}{r}$. Then one shows as in Lemma 10 that the geometry of $D_n(x)$ at $b$-hyperbolic times $n$ with $b > \frac{R(f)}{r}$ is bounded. However we use in the proof of Lemma 12 in a crucial way that the local volume growth for $C^\infty$ maps is zero and this is false in finite smoothness* [40].

**Lemma 13.** *For any $f$-dynamical ball $B_{[\![k,l]\!]}$ of size $\alpha\epsilon/4$, i.e. $B_{[\![k,l]\!]} = \bigcap_{m=k}^{l} f^{-m}B_m$ where $B_m$ is a ball of radius $\alpha\epsilon/4$ for $m = k, \cdots, l$, there exists a finite family $(y_i)_{i\in I}$ of $\sigma_* \cap A^{k,l} \cap B_{[\![k,l]\!]}$ and a partition $I = \coprod_{j\in J} I_j$ of $I$ with $j \in I_j$ for all $j \in J \subset I$ such that*

- $f^l(\sigma_* \cap A^{k,l} \cap B_{[\![k,l]\!]}) \subset \bigcup_{i\in I} D_l(y_i)$,
- $B_{[\![k,l]\!]} \cap D_l(y_i)$, $i \in I$, *are pairwise disjoint,*
- $\forall j \in J\ \forall i, i' \in I_j,\ D_k(y_i) \cap B_k = D_k(y_j) \cap B_k$,
- $\forall j \in J,\ \sharp I_j \leq De^{\delta(l-k)/3}$ *for some constant $D$ depending only on $b$ and $f$.*

*Proof.* We first apply Lemma 11 to $\sigma$ and $E = A^{k,l} \cap B_{[\![k,l]\!]}$ to get the collection of strongly $(\epsilon, r)$-bounded $(D_k(y_j))_{j\in J}$ whose length is larger than $\alpha\epsilon$. Then we apply Lemma 12 to each $\sigma_j^k := D_k(y_j)$ for $j \in J$ and $E = f^k(B_{[\![k,l]\!]} \cap A_k \cap \sigma_*)$ to get a family $(z_i)_{i\in I_j}$ of $(\sigma_j^k)_* \cap A^{l-k} \cap f^k(B_{[\![k,l]\!]} \cap A_k \cap \sigma_*)$ satisfying:

- $f^{l-k}\left((\sigma_j^k)_* \cap A^{l-k} \cap f^k(B_{[\![k,l]\!]} \cap A_k \cap \sigma_*)\right) \subset \bigcup_{j\in J} D_k(z_i)$,
- $B_l \cap D_{l-k}(z_i)$, $i \in I_j$, are pairwise disjoint,
- $\sharp I_j \leq Ce^{\delta k/3}$ for some constant $C$ depending only on $b$ and $f$.

For all $j \in J$ we can take $j \in I_j$ and $z_j = f^k(y_j)$. We conclude the proof by letting $y_i = f^{-k}z_i \in \sigma_* \cap A^{k,l} \cap B_{[\![k,l]\!]}$ for all $i \in I := \coprod_{j\in J} I_j$.

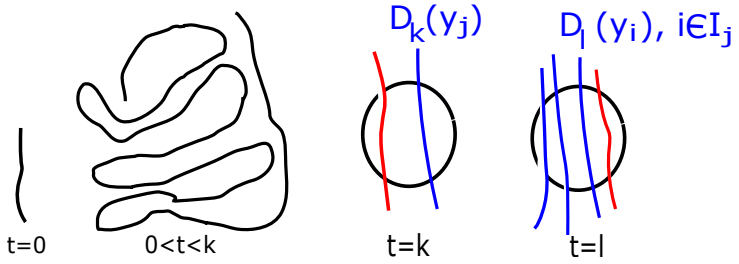



**Figure 1:** *For $0 \leq t < k$ the image of $f^t \circ \sigma$ in black may be large and the disks $D_t(y_i)$ are scattered through the surface. For $t = k$, the sets $D_k(y_j)$ for $j \in J$ are covering $(f^t \circ \sigma)_* \cap B_k$. For $t = l$, we drew in blue the sets $D_l(y_i) \subset f^{l-k}D_k(y_j)$ for $i \in I_j$.*

□

We prove now $(H)$. Recall that $\lambda = \lambda_\sigma$ is the push-forward on $\mathbb{P}TM$ of the Lebesgue measure on $\sigma_*$. As $\sharp\partial F_n = o(n)$ it is enough to show there is a constant $C$ such that for any strongly $(\epsilon, r)$-bounded curve $\sigma$ with length larger than $\alpha\epsilon$ we have

$$\lambda_\sigma\left(P^{F_n}(\hat{x}) \cap \pi^{-1}A^{\partial F_n}\right) \leq C^{\sharp\partial F_n}e^{2\delta n/3}e^{-\phi^{F_n}(x)}. \tag{5.7}$$

To prove (5.7) we argue by induction on the number of connected components of $F_n$. Let $[\![k, l]\!]$, $0 \leq k \leq l$, be the first connected component of $F_n$

and write $G_{n-l} = \mathbb{N}^* \cap (F_n - l)$. Then with the notations of Lemma 13 we get

$$\begin{aligned}\lambda_\sigma\left(P^{F_n}(\hat{x}) \cap \pi^{-1}A^{\partial F_n}\right) &\leq \lambda_\sigma\left(\coprod_{i\in I} F^{-l}\left(\pi^{-1}A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l\hat{x}) \cap D_l(y_i)\right)\right),\\ &\leq \lambda_\sigma\left(\coprod_{j\in J} F^{-k}\left(\coprod_{i\in I_j} F^{-(l-k)}\left(\pi^{-1}A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l\hat{x}) \cap D_l(y_i)\right)\right)\right)\end{aligned}$$

For $j \in J$ we let $\sigma_j^k$ be the $(\epsilon, r)$-bounded curve $\sigma$ with length larger than $\alpha\epsilon$ given by $D_k(y_j)$. By the bounded distorsion property of Lemma 8 we get

$$\lambda_\sigma\left(P^{F_n}(\hat{x}) \cap \pi^{-1}A^{\partial F_n}\right) \leq 2\sum_{j\in J} e^{-\phi_k(y_j)}\lambda_{\sigma_j^k}\left(\coprod_{i\in I_j} F^{-(l-k)}\left(\pi^{-1}A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l\hat{x}) \cap D_l(y_i)\right)\right).$$

By using again the bounded distorsion property (now between the times $k$ and $l$) we get with $\sigma_i^l$ being the curve associated to $D_l(y_i)$ :

$$\lambda_\sigma\left(P^{F_n}(\hat{x}) \cap \pi^{-1}A^{\partial F_n}\right) \leq 4\sum_{j\in J} e^{-\phi_k(y_j)}\sum_{i\in I_j} e^{-\phi_{l-k}(f^k y_i)}\lambda_{\sigma_i^l}\left(\pi^{-1}A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l\hat{x})\right).$$

We may assume that any $(y_i, v_{y_i})$, $i \in I$, lies in $P^{F_n}(\hat{x})$ not only that $y_i$ lies in the $f$-dynamical ball at $x$. In particular we have $|\phi_{l-k}(f^k y_i) - \phi_{l-k}(f^k x)| < (l-k)\delta/3$ by (5.5). Then

$$\begin{aligned}\lambda_\sigma\left(P^{F_n}(\hat{x}) \cap \pi^{-1}A^{\partial F_n}\right) \leq& 4\left(\sum_{j\in J} e^{-\phi_k(y_j)}\right) e^{\delta(l-k)/3}e^{-\phi_{l-k}(f^k x)}\sup_j \sharp I_j\\ &\times \sup_{i\in I}\lambda_{\sigma_i^l}\left(\pi^{-1}A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l\hat{x})\right).\end{aligned}$$

By (5.6) and the last item of Lemma 13 we obtain

$$\lambda_\sigma\left(P^{F_n}(\hat{x}) \cap \pi^{-1}\partial A^{F_n}\right) \leq \frac{8De^{2\delta(l-k)/3}}{\alpha}e^{-\phi_{l-k}(f^k x)}\sup_{i\in I}\lambda_{\sigma_i^l}\left(\pi^{-1}A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l\hat{x})\right).$$

By induction hypothesis (5.7) applied to $G_{n-l}$ for each $\sigma_i^l$, we have

$$\lambda_{\sigma_i^l}\left(\pi^{-1}A^{\partial G_{n-l}} \cap P^{G_{n-l}}(F^l\hat{x})\right) \leq C^{\sharp\partial G_{n-l}}e^{2\delta(n-l)/3}e^{-\phi^{G_{n-l}(x)}}.$$

Note that $\sharp\partial F_n = \sharp\partial G_{n-l} + 2$. We conclude by taking $C = \sqrt{\frac{8D}{\alpha}}$ that

$$\begin{aligned}\lambda_\sigma\left(P^{F_n}(\hat{x}) \cap \pi^{-1}A^{\partial F_n}\right) &\leq \frac{8De^{2\delta n/3}}{\alpha}C^{\sharp\partial G_{n-l}}e^{-\phi^{F_n}(x)},\\ &\leq C^{\sharp\partial F_n}e^{2\delta n/3}e^{-\phi^{F_n(x)}}.\end{aligned}$$

This completes the proof of (H), thus also of Corollary 2.

## 6. The covering property of the basins and the values of the maximal exponent

For $x \in M$ the stable/unstable manifold $W^{s/u}(x)$ at $x$ are defined as follows :

$$W^s(x) := \{y \in M,\ \limsup_{n\to+\infty} \frac{1}{n} \log d(f^n x, f^n y) < 0\},$$

$$W^u(x) := \{y \in M,\ \limsup_{n\to+\infty} \frac{1}{n} \log d(f^{-n} x, f^{-n} y) < 0\}.$$

For a subset $\Lambda$ of $M$ we let $W^s(\Lambda) = \bigcup_{x\in\Lambda} W^s(x)$. According to Pesin's theory, there are a nondecreasing sequence of compact, a priori non-invariant, sets $(K_n)_n$ (called the Pesin blocks) with $\mathcal{R}^* = \bigcup_n K_n$ and two families of embedded $C^\infty$ discs $(W^s_{loc}(x))_{x\in K}$ and $(W^u_{loc}(x))_{x\in K}$ (called the local stable and unstable manifolds) such that :

- $W^{s/u}_{loc}(x)$ are tangent to $E_{s/u}$ at $x$,
- the splitting $E_u(x) \oplus E_s(x)$ and the discs $W^{s/u}_{loc}(x)$ are continuous on $x \in K_n$ for each $n$.

For $\gamma > 0$ and $x \in K$ we let $W^{s/u}_\gamma(x)$ be the connected component of $B(x,\gamma) \cap W^{s/u}_{loc}(x)$ containing $x$.

**Proposition 6.** *The set $\{\chi > 0\}$ is covered by the basins of ergodic SRB measures $\mu_i$, $i \in I$, up to a set of zero Lebesgue measure.*

In fact we prove a stronger statement by showing that $\{\chi > 0\}$ is contained Lebesgue a.e. in $W^s(\Lambda)$ where $\Lambda$ is any $f$-invariant subset of $\bigcup_{i\in I} \mathcal{B}(\mu_i)_{i\in I}$ with $\mu_i(\Lambda) = 1$ for all $i$.

So far we only have used the characterization of SRB measure in terms of entropy (Theorem 3). In the proof of Proposition 6 we will use the absolutely continuity property of SRB measures. Let $\mu$ be a Borel measure on $M$. We recall a measurable partition $\xi$ in the sense of Rokhlin [33] is said $\mu$-subordinate to $W^u$ when $\xi(x) \subset W^u(x)$ and $\xi(x)$ contains an open neighborhood of $x$ in the topology of $W^u(x)$ for $\mu$-almost every $x$. The measure $\mu$ is said to have **absolutely continuous conditional measures on unstable manifolds** if for every measurable partition $\xi$ $\mu$-subordinate to $W^u$, the conditional measures $\mu^\xi_x$ of $\mu$ with respect to $\xi$ satisfy $\mu^\xi_x \ll Leb_{W^u(x)}$ for $\mu$-almost every $x$.

*Proof.* We argue by contradiction. Take $\Lambda$ as above. Assume there is a Borel set $B$ with positive Lebesgue measure contained in the complement of $W^s(\Lambda)$ such that we have $\chi(x) > 0$ for all $x \in B$. Then we may consider a real number $a > 0$ and a smooth disc $\sigma$ with $\chi(x, v_x) > a > 0$ for $x \in B' \subset B$ with $\mathrm{Leb}_{\sigma_*}(B') > 0$. Let $B''$ be the subset of $B'$ given by density points of $B'$ with respect to $\mathrm{Leb}_{\sigma_*}$. In particular, we have with the notations of

Subsection 5.2.2

$$\forall x \in B'', \quad \frac{\mathrm{Leb}_{\sigma_*}(E_k(x) \cap B)}{\mathrm{Leb}_{\sigma_*}(E_k(x))} \xrightarrow{k\to+\infty} 1.$$

We choose a subset $A$ of $B''$ with $\mathrm{Leb}_{\sigma_*}(A) > 0$ such that the above convergence is uniform. Then from this set $A$ we may build $F_n$ and $\mu_n^{F_n}$ as in the previous sections. Any limit measure $\mu$ of $\mu_n^{F_n}$ is supported on the unstable bundle and projects to a SRB measure $\nu$ with $\chi(x) \geq a$ for $\nu$ a.e. $x$ according to Lemma 5. In particular $\nu$ is a barycenter of the (finitely many) ergodic SRB measures with such an exponent. Let $\beta > 0$ be the frequency of $b$-hyperbolic times for $0 < b < a$ as in (4.1). Take $P = K_N$ a Pesin block with $\nu(P) \sim 1 > 1 - \beta$. We let $\theta$ and $l$ be respectively the minimal angle between $E_u$ and $E_s$ and the minimal length of the local stable and unstable manifolds on $P$. We take $r \in \mathbb{N}$, $1 \geq \alpha > 0$ and $\epsilon > 0$ as in Lemma 10. Then $\alpha\epsilon$ is the minimal size of $D_k(y) = D_{k,\sigma}^{r,\epsilon}(y)$ when $k$ is a $b$-hyperbolic time for $\phi$ at $y$. Without loss of generality we may assume $\epsilon < \min(\theta, l)/100$. Then we let $\gamma < \alpha\epsilon$ such that for all $y, z \in P$ with $d(y, z) < \gamma$ the local stable manifold $W_\gamma^s(z)$ lies in a cone around $E_s(y)$ of width less than $\frac{\theta}{100}$.

Let $\xi$ be a measurable partition subordinate to $W^u$ with diameter less then $\gamma$. We have $\nu(\Lambda \cap P) = \int \nu_x^\xi(\Lambda \cap P)\, d\nu(x) \sim 1$ and $\nu_x^\xi \ll Leb_{W_\gamma^u(x)}$ for $\nu$ a.e. $x$. Therefore we get for some $c > 0$

$$\nu\left(x,\ \mathrm{Leb}_{W_\gamma^u(x)}(\Lambda \cap P) > c\right) \sim 1.$$

We let $F = \{x \in \Lambda \cap P,\ \mathrm{Leb}_{W_\gamma^u(x)}(\Lambda \cap P) > c\}$. Observe that we have again $\nu(F) \sim 1$. For $x \in \sigma_*$ and $y \in P$ we use the following notations :

$$\hat{x}_\sigma = (x, v_x) \in \mathbb{P}TM \qquad \hat{y}_u = (u, v_y^u) \in \mathbb{P}TM,$$

where $v_y^u$ is the element of $\mathbb{P}TM$ representing the line $E_u(y)$. Let $\hat{F}_u^\gamma$ be the open $\gamma/8$-neighborhood of $\hat{F}_u := \{\hat{y}_u,\ y \in F\}$ in $\mathbb{P}TM$. Recall $E(x)$ denotes the set of $b$-hyperbolic times for $\phi$ at $x$. We let for $n \in \mathfrak{n}$:

$$\zeta_n := \int \frac{1}{\sharp F_n} \sum_{k \in E(x) \cap F_n} \delta_{F^k \hat{x}_\sigma}\, d\mu_n(\hat{x}_\sigma).$$

Observe that $\zeta_n(\mathbb{P}TM) \geq \inf_{x \in A_n} d_n(E(x) \cap F_n)$. By the last item in Lemma 2, we have $\liminf_{n \in \mathfrak{n}} \inf_{x \in A_n} d_n(E(x) \cap F_n) \geq \beta$. Therefore there is a weak limit $\zeta = \lim_k \zeta_{p_k}$ with $\zeta \leq \mu$ and $\zeta(\mathbb{P}TM) \geq \beta$. From $\mu(\hat{F}_u^\gamma) \sim 1 > 1 - \beta$ we get $0 < \zeta(\hat{F}_u^\gamma) \leq \lim_k \zeta_{p_k}(\hat{F}_u^\gamma)$. Note also $\hat{A}_\sigma := \{\hat{y}_\sigma,\ y \in A\}$ has full $\mu_n$-measure for all $n$. In particular, for infinitely many $n \in \mathbb{N}$ there is $(x^n, v_{x^n}) = \hat{x}_\sigma^n \in \hat{A}_\sigma$ with $F^n \hat{x}_\sigma^n \in \hat{F}_u^\gamma$ and $n \in E(x^n) \cap F_n$. Let $\hat{y}_u^n = (y^n, v_{y^n}^u) \in \hat{F}_u$ which is $\gamma/8$-close to $F^n \hat{x}_\sigma^n$. Then it follows from the choices of $\gamma$ and $\epsilon$ that the curve $D_n(x^n)$ is transverse to $W^s(P \cap \Lambda \cap W_\gamma^u(y^n))$. Moreover as such a curve $D_n(x^n)$ is tangent to a cone of width $10\epsilon \leq \theta/10$, it may be written as the graph of a $C^\infty$ smooth function $\psi : E \subset E_u(y^n) \to E_s(y^n)$ with $\|d\psi\| < L$ for a universal constant $L$.

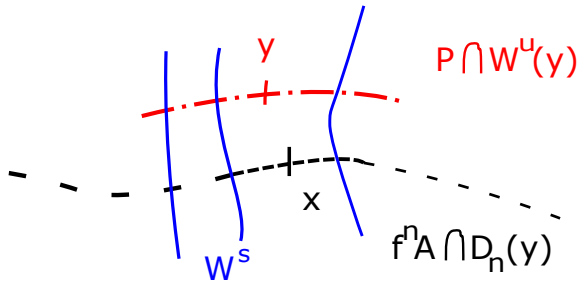



**Figure 2:** ***Holonomy of the local stable foliation between the transversals*** $D_n(x)$ ***and*** $W^u_\gamma(y)$***.***

By Theorem 8.6.1 in [31] the associated holonomy map $h : W^u_\gamma(y) \to D_n(x^n)$ is absolutely continuous and its Jacobian is bounded from below by a positive constant depending only on the Pesin block $P = K_N$ (not on $x^n$ and $y^n$). In particular $\mathrm{Leb}_{D_n(x^n)}(W^s(\Lambda \cap P)) \geq c'$ for some constant $c'$ independent of $n$. The distorsion of $df^n$ on $E_n(x^n)$ being bounded by 2, we get (recall $f^n E_n(x^n) = D_n(x^n)$ and $x^n \in A$):

$$(\alpha\epsilon)^{-1} \mathrm{Leb}\left(D_n(x^n) \setminus f^n B\right) \leq \frac{\mathrm{Leb}\left(D_n(x^n) \setminus f^n B\right)}{\mathrm{Leb}\left(D_n(x^n)\right)} \leq 4 \frac{\mathrm{Leb}\left(E_n(x^n) \setminus B\right)}{\mathrm{Leb}\left(E_n(x^n)\right)} \overset{n}{\to} 0.$$

Therefore for $n$ large enough, there exists $x \in f^n B \cap W^s(\Lambda \cap P)$, in particular $B \cap f^{-n} W^s(\Lambda) = B \cap W^s(\Lambda) \neq \emptyset$. This contradicts the definition of $B$.

□

Recall $\mathcal{R}_1$ denotes the invariant subset of Lyapunov regular points $x$ of $(M, f)$ with $\chi_1(x) > 0 > \chi_2(x)$. Such point admits so called regular neighborhoods (or $\epsilon$-Pesin charts):

**Lemma 14.** [30] *For a fixed $\epsilon > 0$ there exists a measurable function $q = q_\epsilon : \mathcal{R}_1 \to (0, 1]$ with $e^{-\epsilon} < q(fx)/q(x) < e^\epsilon$ and a collection of embeddings $\Psi_x : B(0, q(x)) \subset T_x M = E_u(x) \oplus E_s(x) \sim \mathbb{R}^2 \to M$ with $\Psi_x(0) = x$ such that $f_x = \Psi^{-1}_{fx} \circ f \circ \Psi_x$ satisfies the following properties :*

- $$d_0 f_x = \begin{pmatrix} a^1_\epsilon(x) & 0 \\ 0 & a^2_\epsilon(x), \end{pmatrix}$$
  *with $e^{-\epsilon} e^{\chi_i(x)} < a^i_\epsilon(x) < e^\epsilon e^{\chi_i(x)}$ for $i = 1, 2$,*
- *the $C^1$ distance between $f_x$ and $d_0 f_x$ is less than $\epsilon$,*
- *there exists a constant $K$ and a measurable function $A = A_\epsilon : \mathcal{R}_1 \to \mathbb{R}$ such that for all $y, z \in B(0, q(x))$*
  $$K d(\Psi_x(y), \Psi_x(z)) \leq \|y - z\| \leq A(x) d(\Psi_x(y), \Psi_x(z)),$$
  *with $e^{-\epsilon} A(fx)/A(x) < e^\epsilon$.*

For any $i \in I$ we let

$$E_i := \{x,\ \chi(x) = \chi(\mu_i)\}.$$

The set $E_i$ has full $\mu_i$-measure by the subadditive ergodic theorem. Put $\Lambda_i = \mathcal{B}(\mu_i) \cap E_i \cap \mathcal{R}_1$ and $\Lambda = \bigcup_i \Lambda_i$. Clearly $\Lambda$ is $f$-invariant. We finally check that $\chi(x) = \chi(\mu_i)$ for $x \in W^s(\Lambda_i)$.

For uniformly hyperbolic systems, we have $\Sigma\chi(x) = \lim_n \frac{1}{n} \log \mathrm{Jac}(d_x f^n_{E_u}) = \lim_n \int \log \mathrm{Jac}(d_y f_{E_u})\, d\delta^n_x$. As the geometric potential $y \mapsto \log \mathrm{Jac}\, d_y f_{E_u}$ is continuous in this case, any point in the basin of a SRB measure $\mu$ satisfies $\Sigma\chi(x) = \int \Sigma\chi(y)\, d\mu(y)$.

**Lemma 15.** *If $y \in W^s(x)$ with $x \in \mathcal{R}_1$, then $\chi(y) = \chi(x)$.*

*Proof.* Fix $x \in \mathcal{R}_1$ and $\delta > 0$. We apply Lemma 14 with $\epsilon \ll \chi_1(x)$. For $\alpha > 0$ we let $\mathcal{C}_\alpha$ be the cone $\mathcal{C}_\alpha = \{(u, v) \in \mathbb{R}^2,\ \alpha\|u\| \geq \|v\|\}$. We may choose $\alpha > 0$ and $\epsilon > 0$ so small that for all $k \in \mathbb{N}$ we have $d_z f_{f^k x}(\mathcal{C}_\alpha) \subset \mathcal{C}_\alpha$ and $\|d_z f_{f^k x}(v)\| \geq e^{\chi_1(x)-\delta}$ for all $v \in \mathcal{C}_\alpha$ and all $z \in B(0, q_\epsilon(f^k x))$.

Let $y \in W^s(x)$. There is $C > 0$ and $\lambda$ such that $d(f^n x, f^n y) < C\lambda^n$ holds for all $n \in \mathbb{N}$. We can choose $\epsilon \ll \lambda$. In particular there is $N > 0$ such that $f^n y$ belongs to $\Psi_{f^n x} B(0, q(f^n x))$ for $n \geq N$ since we have $A(f^n x) < e^{\epsilon n} A(x)$ and $q(f^n x) > e^{\epsilon n} q(x)$. Let $z \in B(0, q(f^N x))$ with $\Psi_{f^N x}(z) = y$. Then for all $v \in \mathcal{C}_\alpha$ and for all $n \geq N$ we have $\|d_z \left(\Psi^{-1}_{f^{n-N}x} \circ f^{n-N} \circ \Psi_{f^N x}\right)(v)\| \geq e^{(n-N)(\chi_1(x)-\delta)}$. As the conorm of $d_{f^{n-N}y}\psi_{f^n x}$ is bounded from above by $A(f^n x)^{-1}$ for all $n$ we get

$$\begin{aligned}
\chi(y) &= \limsup_n \frac{1}{n} \log \|d_y f^{n-N}\|, \\
&= \limsup_n \frac{1}{n} \log \|d_z \left(f^{n-N} \circ \Psi_{f^N x}\right)\|, \\
&\geq \limsup_{n\to+\infty} \frac{1}{n} \log \left(A(f^n x)^{-1} \left\|d_z \left(\Psi^{-1}_{f^n x} \circ f^n \circ \Psi_{f^N x}\right)\right\|\right), \\
&\geq \chi_1(x) - \delta - \epsilon.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\left\|d_z \left(\Psi^{-1}_{f^n x} \circ f^n \circ \Psi_{f^N x}\right)\right\| &\leq \prod_{k=N}^{n-1} \sup_{t \in B(0, q(f^k x))} \|d_t f_{f^k x}\|, \\
&\leq \left(e^{\chi_1(x)+\epsilon} + \epsilon\right)^{n-N}, \\
&\leq e^{(n-N)(\chi_1(x)+2\epsilon)}.
\end{aligned}$$

Then it follows from $\|d_{f^{n-N}y}\psi_{f^n x}\| \leq K$:

$$\begin{aligned}
\chi(y) &\leq \limsup_{n\to+\infty} \frac{1}{n} \log \left(\left\|d_z \left(\Psi^{-1}_{f^n x} \circ f^n \circ \Psi_{f^N x}\right)\right\|\right), \\
&\leq \chi_1(x) + 2\epsilon.
\end{aligned}$$

As it holds for arbitrarily small $\epsilon$ and $\delta$ we get $\chi(y) = \chi_1(x) = \chi(x)$. □

We conclude with $\Lambda = \{\chi(\mu_i),\ i \in I\}$ that for Lebesgue a.e. point $x$, we have $\chi(x) \in \mathbb{R}^- \cup \Lambda$ and that $\{\chi = \lambda\} \overset{o}{\subset} \bigcup_{i\in I} \mathcal{B}(\mu_i)$ for all $\lambda \in \Lambda$. The proof of the Main Theorem is now complete. It follows also from Lemma 15, that the converse statement of Corollary 2 holds : if $(f, M)$ admits a SRB measure then $\mathrm{Leb}(\chi > 0) > 0$.

In the proof of Lemma 15 we can choose the cone $\mathcal{C}_\alpha$ to be contracting, so that any vector in a small cone at $y$ will converge to the unstable direction $E_u(x)$. In other terms if we endow the smooth manifold $\mathbb{P}TM$ with a smooth Riemanian structure, then the lift $\mu$ to the unstable bundle of an ergodic SRB measure $\nu$ is a physical measure of $(\mathbb{P}TM, F)$. Conversely if $\mu$ is a physical measure of $(\mathbb{P}TM, F)$ supported on the unstable bundle above $\mathcal{R}_1$, we can reproduce the scheme of the proof of the Main Theorem to show $\mu$ projects to a SRB measure $\nu$. Indeed we may consider a $C^\infty$ smooth curve $\sigma$ such that $\hat{x} = (x, v_x)$ lies in the basin of $\mu$ for $\hat{x}$ in a positive Lebesgue measure set $A$ of $\hat{\sigma}_*$. Then by following the above construction of SRB measures, we obtain that $\mu = \lim_n \frac{1}{n}\sum_{k=0}^{n-1} F_*^k \mathrm{Leb}_A$ project to a SRB measure (or one can directly use the approach of [13]). This converse statement is very similar to a result of Tsujii (Theorem A in [39]) which states in dimension two that for $C^{1+}$ surface diffeomorphism an ergodic hyperbolic measure $\nu$, such that the set of **regular** points $x \in \mathcal{B}(\nu)$ with $\chi(x) = \int \chi\, d\nu$ has positive Lebesgue measure, is a SRB measure. Indeed if $\mu$ is a physical measure of $(\mathbb{P}TM, F)$ supported on the unstable bundle above $\mathcal{R}_1$, its projection $\nu$ satisfies $\chi(\nu) = \chi(x)$ for any $x \in \pi(\mathcal{B}(\nu))$. In the present paper we are working with the stronger $C^\infty$ assumption, but, in return, points in the basin are not supposed to be regular contrarily to Tsujii's theorem.

From the above discussion, we may restate the Main Theorem as follows:

**Theorem 7.** *Let $f : M \circlearrowleft$ be a $C^\infty$ surface diffeomorphism and $F : \mathbb{P}TM$ be the induced map on the projective tangent bundle. Then the basins of the physical measures of $(\mathbb{P}TM, F)$ are covering Lebesgue almost everywhere the set $\{(x, v) \in \mathbb{P}TM,\ \chi(x, v) > 0\}$.*

## 7. Nonpositive exponent in contracting sets

In this last section we show Theorem 1. For a dynamical system $(M, f)$ a subset $U$ of $M$ is said **almost contracting** when for all $\epsilon > 0$ the set $E_\epsilon = \{k \in \mathbb{N},\ \mathrm{diam}(f^k U) > \epsilon\}$ satisfies $\overline{d}(E_\epsilon) = 0$. In [21] the authors build subsets with historic behaviour and positive Lebesgue measure which are almost contracting but not contracting. We will show Theorem 1 for almost contracting sets.

We borrow the next lemma from [13] (Lemma 4 therein), which may be stated with the notations of Section 4 as follows :

**Lemma 16.** *Let $f : M \circlearrowleft$ be a $C^\infty$ diffeomorphism admitting and let $U$ be a subset of $M$ with $\mathrm{Leb}(\{\chi > a\} \cap U) > 0$ for some $a > 0$. Then for all $\gamma > 0$ there is a $C^\infty$ smooth embedded curve $\sigma_*$ and $I \subset \mathbb{N}$ with $\sharp I = \infty$ such that*

$$\forall n \in I,\ |\{x \in U \cap \sigma_*,\ \|d_x f^n(v_x)\| > e^{na}\}| > e^{-n\gamma}.$$

We are now in a position to prove Theorem 1 for almost contracting sets.

*Proof of Theorem 1.* We argue by contradiction by assuming $\mathrm{Leb}(\{\chi > a\} \cap U) > 0$ for some $a > 0$ with $U$ being a almost contracting set. By Yomdin's Theorem on one-dimensional local volume growth for $C^\infty$ dynamical systems [40] there is $\epsilon > 0$ so small that

$$v^*(f, \epsilon) = \sup_\sigma \limsup_{n\to\infty} \frac{1}{n} \sup_{x\in M} \log |f^n(B(x, \epsilon, n) \cap \sigma_*| < a/2, \tag{7.1}$$

where the supremum holds over all $C^\infty$ smooth embedded curves $\sigma : [0, 1] \to M$. As $U$ is almost contracting, there are subsets $(C_n)_{n\in\mathbb{N}}$ of $M$ with $\lim_n \frac{\log \sharp C_n}{n} = 0$ satisfying for all $n$

$$U \subset \bigcup_{x\in C_n} B(x, \epsilon, n). \tag{7.2}$$

Fix an error term $\gamma \in ]0, \frac{a}{2}[$. Then by Lemma 16 there is a $C^\infty$-smooth curve $\sigma_* \subset U$ and an infinite subset $I$ of $\mathbb{N}$ such that for all $n \in I$

$$\begin{aligned}
\sum_{x\in C_n} |f^n(B(x, \epsilon, n) \cap \sigma_*)| &\geq |f^n(U \cap \sigma_*)|, \\
&\geq e^{na} |\{x \in U \cap \sigma_*,\ \|d_x f^n(v_x)\| > e^{na}\}|, \\
&\geq e^{n(a-\gamma)} \text{ by (7.1)}, \\
\sharp C_n \sup_{x\in M} \log |f^n(B(x, \epsilon, n) \cap \sigma_*)| &\geq e^{n(a-\gamma)} \text{ by (7.2)}.
\end{aligned}$$

Therefore we get the contradiction $v^*(f, \epsilon) > a - \gamma > a/2$. □

## Appendix A. Maximal exponent

Let $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ be a sequence in $M_d(\mathbb{R}^d)$. For any $n \in \mathbb{N}$ we let $A^n = A_{n-1} \cdots A_1 A_0$. We define the Lyapunov exponent $\chi(\mathcal{A})$ of $\mathcal{A}$ with respect to $v \in \mathbb{R}^d \setminus \{0\}$ as

$$\chi(\mathcal{A}, v) := \limsup_n \frac{1}{n} \log \|A^n(v)\|,$$

**Lemma 17.**

$$\sup_{v\in\mathbb{R}^d\setminus\{0\}} \chi(\mathcal{A}, v) = \limsup_n \frac{1}{n} \log |||A^n|||.$$

*Proof.* The inequality $\leq$ is obvious. Let us show the other inequality. Let $v_n \in \mathbb{R}^d$ with $\|v_n\| = 1$ and $\|A^n(v_n)\| = |||A^n|||$. Then take $v = \lim_k v_{n_k}$ with $\lim_k \frac{1}{n_k} \log |||A^{n_k}||| = \limsup_n \frac{1}{n} \log |||A^n|||$. We get

$$\begin{aligned} \|A^{n_k}(v)\| &\geq \|A^{n_k}(v_k)\| - \|A^{n_k}(v - v_k)\|, \\ &\geq |||A^{n_k}|||(1 - \|v - v_k\|), \\ \limsup_k \frac{1}{n_k} \log \|A^{n_k}(v)\| &\geq \limsup_n \frac{1}{n} \log |||A^n|||. \end{aligned}$$

□

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