HAL open science

Metric dimension on sparse graphs and its applications to zero forcing sets

Nicolas Bousquet, Quentin Deschamps, Aline Parreau, Ignacio M Pelayo

To cite this version:

Nicolas Bousquet, Quentin Deschamps, Aline Parreau, Ignacio M Pelayo. Metric dimension on sparse graphs and its applications to zero forcing sets. 2021. hal-03421594

HAL Id: hal-03421594

https://hal.science/hal-03421594v1

Preprint submitted on 9 Nov 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Metric dimension on sparse graphs and its applications to zero forcing sets*†

Nicolas Bousquet[1], Quentin Deschamps[1], Aline Parreau[1], and Ignacio M. Pelayo[2]

[1]Université de Lyon, Université Lyon 1, LIRIS UMR CNRS 5205, F-69621, Lyon, France
[2]Universitat Politècnica de Catalunya, Barcelona, Spain

November 9, 2021

Abstract

The *metric dimension dim(G)* of a graph G is the minimum cardinality of a subset S of vertices of G such that each vertex of G is uniquely determined by its distances to S. It is well-known that the metric dimension of a graph can be drastically increased by the modification of a single edge. Our main result consists in proving that the increase of the metric dimension of an edge addition can be amortized in the sense that if the graph consists of a spanning tree T plus c edges, then the metric dimension of G is at most the metric dimension of T plus $6c$.

We then use this result to prove a weakening of a conjecture of Eroh et al. The *zero forcing number* $Z(G)$ of G is the minimum cardinality of a subset S of *black* vertices (whereas the other vertices are colored white) of G such that all the vertices will turned black after applying finitely many times the following rule: *a white vertex is turned black if it is the only white neighbor of a black vertex.*

Eroh et al. conjectured that, for any graph G, $dim(G) \leq Z(G) + c(G)$, where $c(G)$ is the number of edges that have to be removed from G to get a forest. They proved the conjecture is true for trees and unicyclic graphs. We prove a weaker version of the conjecture: $dim(G) \leq Z(G) + 6c(G)$ holds for any graph. We also prove that the conjecture is true for graphs with edge disjoint cycles, widely generalizing the unicyclic result of Eroh et al.

1 Introduction

A *resolving set* of a graph G is a subset of vertices of G such that any vertex in the graph is identified by its distances to the vertices of the resolving set. In the example of Figure 1 the set $\{u, v\}$ is a resolving set of the graph because all the vertices have a different *distance vector* to $\{u, v\}$ so the knowledge of the distance to u and v uniquely identifies a vertex. This notion has been introduced in 1975 by Slater [2] for trees and by Harary and Melter [3] for graphs. Determining the minimum size of a resolving set, called *metric dimension* and denoted by $\dim(G)$, is an NP-complete problem [4] even restricted to planar graphs [5]. Applications of metric dimension goes from piloting a sonar [3] to the navigation of a robot in an Euclidean space [6].

One of the main issues to compute the metric dimension of a graph comes from the fact that it is unstable when the graph is modified. When a vertex is added, the metric dimension can be drastically modified. Indeed, while a path admits a constant size resolving set, a path plus a universal vertex only admits resolving sets of linear size (in the number of vertices). However, in this example even if only one vertex was added, a linear number of edges were also added which

*An extended abstract of this work has been published in EuroComb 2021 [1].

†The first three authors of this work are supported by ANR project GrR (ANR-18-CE40-0032).
The Ignacio M. Pelayo work is supported by project PGC2018-095471-B-100.

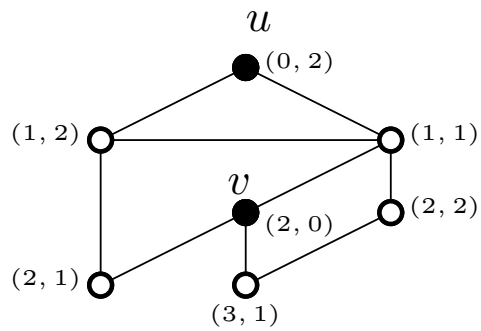


Figure 1: The black vertices form a resolving set. For each vertex x the vector next to x, $(d(u,x), d(v,x))$ is unique.

had permitted to put all the vertices at distance 2. One can then wonder if the situation is better when only one edge is modified in the graph. Unfortunately again, the metric dimension of a graph can be drastically modified by the modification of a single edge. In Figure 2 (first proposed in [7]), the metric dimension of the left graph is $2k$ where k is the number of layers in the graph while the right graph has metric dimension $k+1$. So the addition of one edge can modify the metric dimension by $\Omega(n)$.

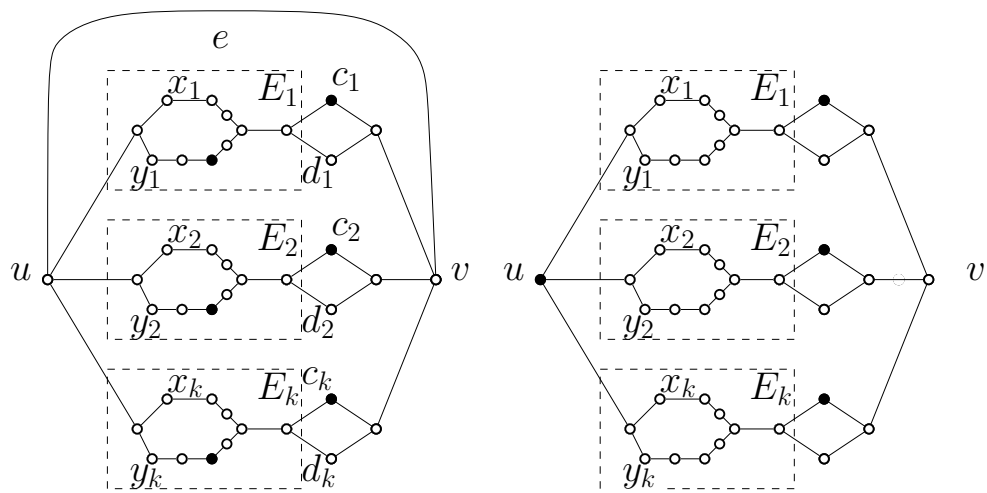


Figure 2: On the left graph, all the pairs (c_i, d_i) are twins (both vertices have the same neighbourhood) so for all $i \leq k$ any resolving set should contain c_i or d_i. Then, for $i \leq k$, the pair (x_i, y_i) can only be resolved by a vertex of E_i so any resolving set should contain a vertex of E_i for all $i \leq k$. So any resolving set contains at least $2k$ vertices. The set of black vertices is a resolving set so the dimension is $2k$. On the right graph the pairs (c_i, d_i) are twins so for all $i \leq k$ any resolving set should contain c_i or d_i. One can easily check that a set containing only vertices on $\{c_i; d_i, 1 \leq i \leq k\}$ is not resolving so the dimension is higher than k. The black vertices form a resolving set so the metric dimension is $k+1$.

Metric dimension and cycle rank. Eroh et al. [8] proved that, if G is a unicyclic graph, then the metric dimension of G is at most the metric dimension of any spanning tree plus one. Moreover, all the existing examples where the metric dimension is drastically modified with an edge modification already contain many cycles. One can then wonder if the metric dimension of a graph which does not contain many cycles is close to the metric dimension of any of its spanning trees. The goal of this paper is to answer this question positively.

One can easily remark that in the star $K_{1,n}$, any resolving set contains at least $n-1$ vertices. Indeed otherwise, two leaves are not in the resolving set and then these two vertices cannot be identified. This argument can be generalized to any graph as follows: if r pending paths are attached to a vertex v, any resolving set contains a vertex in at least $r-1$ of them. Let $L(G)$ be the sum over all the vertices v on which there are attached pending paths, of the number of paths attached to v minus one. Chartrand et al. [9] remarked that, for every connected graph G, $\dim(G) \geq L(G)$ with equality for trees. However, this bound has no reason to be closed from the optimal value (a graph with no leaf can have an arbitrarily large metric dimension). A wheel (induced cycle plus a universal vertex), for instance, has no leaf and its metric dimension is linear in n (see Figure 5).

The *cycle rank* of a connected graph, denoted by $c(G)$ is the number of edges that has to be removed from G to obtain a spanning tree. We prove that the following holds:

Theorem 1. *For any graph G, $L(G) \leq \dim(G) \leq L(G) + 6c(G)$.*

Since the value of $L(G)$ cannot decrease by removing edges in G that are not bridges, it implies the following:

Corollary 2. *Let G be a graph and T be any spanning tree of G. We have*

$$\dim(G) \leq \dim(T) + 6c(G)$$

Informally, Corollary 2 ensures that, even if the metric dimension can be widely modified when we add a single edge, the "amortized cost" of an edge addition is at most 6 with respect to any spanning tree of G. As far as we know it is the first bound of the metric dimension in terms of the natural lower bound $L(G)$ or of the metric dimension of a spanning tree of G.

Before explaining briefly the outline of the proof, let us discuss a bit the tightness of these results. Let $k \in \mathbb{N}$. Consider the graph G_k which is a collection of k C_4 glued on the central vertex of a path of length 3 like in Figure 4. The metric dimension of G_k is equal to $2k+1$ (we need to select exactly two vertices per C_4 distinct from the center and one extremity of the path), $L(G_k) = 1$, and $c = k$. Since $L(G_k) = 1$ for every k, there exist graphs G such that $\dim(G) = L(G) + 2c(G)$. We ask the following question:

Question 3. *Is it true that for any graph G, $\dim(G) \leq L(G) + 2c(G)$?*

Note that Sedlar and Skrekovski independently ask the same question in [10]. The same authors also prove in [11] that Question 3 is true for cacti (and is tight since G_k is a cactus).

Let us now discuss the tightness of Corollary 2. If, in the graph G_k, we remove one edge of each C_4 incident to the central vertex, the resulting spanning tree has metric dimension of order $k+1$ as long as $k \geq 2$. So there exist graphs G for which there exists a spanning tree T satisfying $\dim(G) = \dim(T) + c$. We actually ask the following question:

Question 4. *Is it true that for any graph G and for every spanning tree T of G, we have $\dim(G) \leq \dim(T) + c$?*

One can then wonder what happens if we select the best possible spanning tree to start with, i.e. the spanning that maximizes the metric dimension. In G_k, one can note that if we break the edge of the C_4s that are not incident to the center and denote by T_k the resulting tree, then $\dim(G_k) = \dim(T_k) + 1$. Surprisingly, we did not find any graph G where the metric dimension is a function of c larger than *any* spanning tree of G. We left the existence of such a graph as an open problem.

Let us now briefly discuss the main ingredients of the proof of Theorem 1. First, it consists in finding a small feedback vertex set X of the graph such that every connected component is attached to at most two vertices of X. We then prove that, if we add to a resolving set of every connected component of $G \setminus X$ few vertices (in terms of c) we can "detect" shortcuts passing through the rest of the graph and then obtain a resolving set of the whole graph G. The proof of Theorem 1 is given in Section 3.

The second part of the paper consists in applying this result in order to prove a weak version of a conjecture linking metric dimension and zero forcing sets in graphs.

Zero forcing sets. A *zero forcing set* is a subset of vertices colored in black which colors the whole vertex set in black when we apply the following rule: *A vertex is colored black if it is the unique non-black neighbor of a black vertex.* See Figure 3 where the initial set contains three vertices. The *zero forcing number* of a graph is the minimal size of a zero forcing set, denoted by $Z(G)$. The zero forcing number has been introduced in 2008 to bound the rank of some families of adjacency matrices [12]. Deciding if the zero forcing number of a graph is at most k is NP-complete [13].

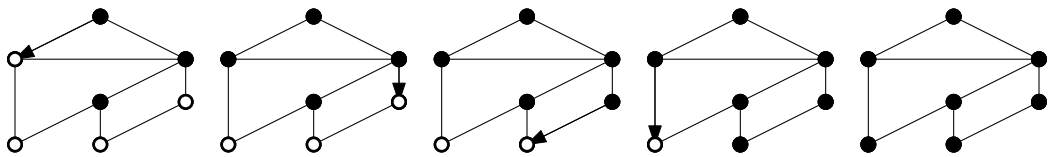
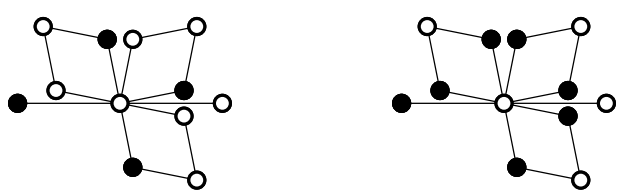


Figure 3: Iterations of the change color rule. On the graph on the left, the three black vertices form a zero forcing set.

In general, the gap between metric dimension and zero-forcing number can be arbitrarily large. But for some restricted sparse graph classes like paths or cycles, both the optimal parameters and optimal sets are the same. Eroh, Kang and Yi then started a systematic comparison between them [8]. They proved that $\dim(G) \leq Z(G)$ when G is a tree and that $\dim(G) \leq Z(G) + 1$ when G has one cycle (in other words G is a tree plus an edge). On the other hand, $\dim(G)$ can be arbitrarily larger than the zero forcing number when the number of cycles increases. They conjectured the following:

Conjecture 5 (Cycle-rank conjecture [8])**.** *For a connected graph* $\dim(G) \leq Z(G) + c(G)$.

Conjecture 5 is tight for an infinite family of graphs: The graph G_k contains a path of 3 vertices and k cycles of size 4 with the central vertex of the path in common. Figure 4 shows the graph G_3 with $c(G_3) = 3$.



$Z(G) = c + 1$ $\qquad$ $\dim(G) = 2c + 1$

Figure 4: Tightness of Conjecture 5

Eroh et al. proved in [8] that $\dim(G) \leq Z(G) + 2c(G)$ if G contains no even induced cycles. Our main contribution to this question is to prove a weaker version of Conjecture 5 in Section 4.1, whose proof is mainly based on an application of Theorem 1.

Theorem 6. *For every graph G, we have*

$$\dim(G) \leq Z(G) + 6c(G).$$

As far as we know, it is the first upper bound of $\dim(G)$ of the form $Z(G) + f(c(G))$.

Note that the dependency on $c(G)$ cannot be removed, i.e. $\dim(G)$ cannot be upper bounded by a function of $Z(G)$ only. For the wheel of n vertices (a cycle plus a universal vertex, see Figure 5), the zero forcing number is 3 for any $n \geq 4$ but the metric dimension is a linear function in n.

We also prove Conjecture 5 in several particular cases. We first focus on unicyclic graphs. We give an alternative proof of Conjecture 5 for unicyclic graphs with a much shorter and simpler

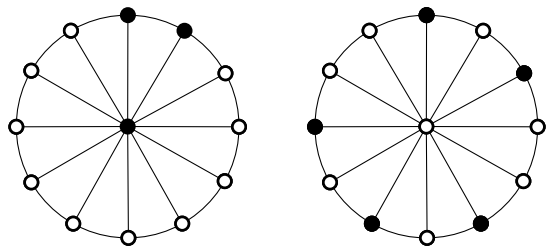


Figure 5: The zero forcing number of a wheel is 3 (while $n \geq 4$) but its metric dimension is linear in n.

proof than the one of [8]. We then extend our results to prove Conjecture 5 for cactus graphs[1] (graphs with edge-disjoint cycles). It generalizes the result on unicyclic graphs and is based on a very simple induction whose base case is the case of unicyclic graphs. Since cactus graphs contain the class of graphs with no even cycles, it improves the result of [8] on even cycle-free graphs.

We finally show that $\dim(G) \leq Z(G)$ when the unique cycle of G has odd length. This result is tight and cannot be extended to unicyclic graphs with an even cycle as shown in Figure 6. All the results related to zero forcing sets are proved in Section 4.

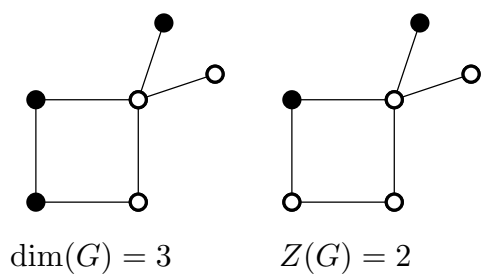


Figure 6: Black vertices form respectively a metric basis and a minimal zero forcing set.

2 Preliminaries

2.1 Definitions and notations

Unless otherwise stated, all the graphs considered in this paper are undirected, simple, finite and connected. For standard terminology and notations on graphs, we refer the reader to [14].

Let $G = (V, E)$ be a graph. The *distance* between two vertices $u, v \in V$, denoted by $d_G(u, v)$ (or simply $d(u, v)$ when G is clear from context), is the length of a shortest path from u to v in G. When no such path exists, we state $d_G(u, v) = +\infty$. For $v \in V$, let $N(v)$ be the (open) neighborhood of v defined as $N(v) = \{u \in V,\ uv \in E\}$. We say that two vertices v and w are *twins* if $N(v) \setminus \{w\} = N(w) \setminus \{v\}$. For $X \subseteq V$, let $G[X]$ be the subgraph of G induced by X. In other words, $G[X]$ is the graph with vertex set X where xy is an edge if and only if it is an edge of G. We denote by $G \setminus X$ the subgraph of G induced by $V \setminus X$. The *border of* X, denoted by ∂X, is $\{u \in G \setminus X |\ \exists v \in X, uv \in E\}$.

A vertex $w \in V$ *resolves* a pair of vertices (u, v) if $d(w, u) \neq d(w, v)$. Let $S \subseteq V$. The set S *resolves* the pair (u, v) if at least one vertex in S resolves the pair (u, v) and S resolves a set $W \subseteq V$ if S resolves all the pairs of W. A set S is a *resolving set* of G if S resolves V. *The metric dimension* $\dim(G)$ of G is the minimum cardinality of a resolving set in G. A resolving set of minimum size is called a *metric basis*.

[1] This result is proved independently in [10] with a different method.

Let $Z \subseteq V$ be a set of vertices. The vertices in Z are colored in black whereas the other vertices are white. The *color change rule* converts a white vertex u into a black vertex if u is the only white neighbor of a black vertex. The set Z is a *zero forcing set* of G if all the vertices of G can be turned black after finitely many applications of the color change rule. For u and v two vertices in V and a sequence of color change rule, we say that u *forces* v if at some step u is turned black with the color change rule because of v. We say that the edge uv is used to force u. *The zero forcing number* $Z(G)$ of G is the minimal cardinality of a zero forcing set in G.

The *cycle rank* of G (or *feedback edge set*), denoted by $c(G)$ (or c if the context is clear enough), is the minimum number of edges that should be deleted to G to get a forest. Note that we have $c(G) = |E| - |V| + cc(G)$ where $cc(G)$ is the number of connected components of G. A graph G is *unicyclic* if G is connected with $c(G) = 1$. A *feedback vertex set* of G is a subset of vertices such that $G \setminus X$ is a forest. We denote by $\tau(G)$ (or τ if the context is clear enough) the minimum size of a feedback vertex set of G. Note that if X has minimum size, then $\tau(G) \leq c(G)$.

2.2 Resolving sets and zero forcing sets on trees

Chartrand et al. [9] introduced the following terminology to study resolving sets in trees. We extend this terminology to general graphs (see Figure 7 for an illustration).
A vertex of degree 1 is called a *terminal vertex*.
A vertex of degree at least 3 is a *major vertex*. A terminal vertex u is called *a terminal vertex of a major vertex* v if $d(u, v) < d(u, w)$ for every other major vertex w. In other words u and v are linked by a path of degree 2 vertices. The *terminal degree* of a major vertex v is the number of terminal vertices of v, denoted by $ter(v)$. A major vertex is *exterior* if its terminal degree is positive, and *interior* otherwise.
A degree-2 vertex is an *exterior degree-2 vertex* if it lies on a path between a terminal vertex and its major vertex. It is an *interior degree-2 vertex* otherwise.

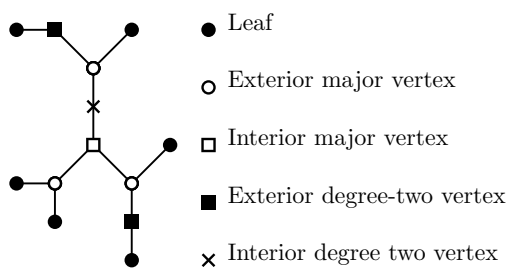


Figure 7: Vertices denomination in a graph

Let $\sigma(G)$ be the sum of the terminal degrees over all the major vertices in G and $ex(G)$ be the number of exterior major vertices in G. Let $L(G) = \sigma(G) - ex(G)$. We can bound $\dim(G)$ and $Z(G)$ with this parameter:

Lemma 7. *[8] For any connected graph G, $\dim(G) \geq \sigma(G) - ex(G)$ and $Z(G) \geq L(G)$.*

Lemma 8. *[7] Let T be a tree which is not a path. Then, $dim(T) = L(T)$. Moreover, any set containing all but exactly one terminal vertices of every major vertex is a resolving set of T.*

There is no similar result to compute the zero forcing number of a tree. The gap between the zero forcing number and the metric dimension can be arbitrarily large on trees.

Lemmas 7 and 8 imply that trees satisfy Conjecture 5. Moreover, the equality case can be characterized:

Lemma 9. *[8] For every tree T, $\dim(T) \leq Z(T)$. The equality holds if and only if T has no interior degree-2 vertices and each major vertex has terminal degree at least two.*

2.3 Elementary results on metric dimension

This section is devoted to some elementary results about metric dimension.

Lemma 10. *Let G be a graph and u, v be two vertices of G. Let s and t be two different vertices on a shortest path between u and v. Then, $d(u,s) \neq d(v,s)$ or $d(u,t) \neq d(v,t)$.*

Proof. Let P be a shortest path from u to v containing s and t. Up to symmetry, we can assume that u, s, t, v appear in that order in P. Since P is a shortest path, $d(s,v) = d(s,t)+d(t,v) > d(t,v)$ and $d(t,u) = d(s,u) + d(s,t) > d(s,u)$. Assume that $d(u,s) = d(v,s)$. Then, $d(u,t) = d(u,s) + d(t,s)$ and $d(v,s) = d(v,t) + d(t,s)$. So, $d(u,t) = d(v,t) + 2d(t,s) \neq d(v,t)$ as $t \neq s$. □

Lemma 11. *Let G be a unicyclic graph with a cycle C of odd length. Then, every pair of vertices of C resolves C.*

Proof. Let u and v be two vertices of C. There are two paths between u and v on C, one of odd length and the other of even length. There exists a unique vertex w of C at the same distance from u and v in C and then in G since G is unicyclic which is the middle of the path of even length. The vertex w is the unique vertex of C which does not resolve the pair (u,v). So any pair of vertices of C resolves C. □

Lemma 12. *Let $G = (V,E)$ be a graph and C be a cycle of G. If $V(C) = \{v_0, v_1, ...v_k\}$ such that for any $i \leq j$ $d(v_i, v_j) = \min(j-i, k-j+i+1)$ [2], then, for any set $S \subseteq C$ of size at least 3, S resolves C.*

Proof. Let $S = \{v_a, v_b, v_c\}$ be any set of three vertices of C and $v_x \neq v_y$ be two vertices of C. Assume by contradiction that S does not resolve the pair (v_x, v_y).

Note that neither v_x nor v_y belongs to $\{v_a, v_b, v_c\}$ since otherwise (v_x, v_y) would be resolved. Without loss of generality, we can assume that $v_x = v_0$ and $a < b < c$.

Assume first that $y < a$. The shortest path on G between v_0 and v_a cannot contain v_y otherwise $d(v_x, v_a) > d(v_y, v_a)$. Thus, $d(v_0, v_a) = k-a+1$ and similarly $d(v_0, v_b) = k-b+1$ so in particular $d(v_0, b) < d(v_0, a)$. Consider now the path between v_y and v_b. If this path passes through v_a, then $d(v_y, a) < d(v_y, b)$ and if this path passes through v_0 then $d(v_y, b) > d(v_0, b)$. Both cases give a contradiction with the assumption that S does not resolve the pair (v_0, v_y).

Assume now that $a < y < b$. If the shortest path between v_0 and v_b passes through v_y then $d(v_0, v_b) > d(v_y, v_b)$ gives a contradiction. Thus, $d(v_0, v_b) = k-b+1$ and so $d(v_0, v_c) = k-c+1$. Similarly if the path between v_y and v_b passes through v_0 then $d(v_y, v_b) > d(v_0, v_b)$. So $d(v_y, v_b) = b-y$ and $d(v_y, v_c) = c-y$. We get $b-y = k-b+1$ and $c-y = k-c+1$ which is impossible since $b \neq c$.

The two last cases, $b < y < c$ and $y > c$, are respectively symmetric to the cases $a < y < b$ and $y < a$. □

The following result has been stated in [8] but the proof contains a flaw. We provide a corrected version of the proof in Appendix A. It bounds the variation of the metric dimension when an edge is deleted in some conditions.

Lemma 13. *Let $G = (V,E)$ be a graph and C be a cycle of G. Let $V(C) = \{v_0, v_1, ...v_k\}$ be the vertices of C. Denote by $G_i = (V_i, E_i)$ the connected components of the vertex u_i in $G \setminus E(C)$. If, for every $i \neq j$, $V_i \cap V_j = \emptyset$, then, for any $e \in E(C)$, $\dim(G) \leq \dim(G-e) + 1$.*

The following lemma is a well-known fact about twins and resolving sets.

Lemma 14. *Let u and v be two twins of a graph G. Any resolving set S of G verifies $S \cap \{u,v\} \neq \emptyset$.*

Lemma 15 is a crucial observation for studying resolving sets in particular in trees.

Lemma 15. *Let $G = (V,E)$ be a connected graph, u be a vertex of G and S be a resolving set of G. At most one connected component of $G \setminus \{u\}$ does not contain any vertex of S.*

[2] This condition ensures that there is no shortcut between the vertices of C.

Proof. Assume by contradiction that two connected components G_i and G_j of $G \setminus \{x\}$ do not contain any vertex of S. Let $v \in V$ and $w \in V(G_j)$ be two vertices incident to u. Then, no vertex in S resolves the pair (v, w) since, for every $s \in S$, $d(s, v) = d(s, w) = d(s, u) + 1$. □

3 Bounds for metric dimension

Definition 16. *Let G be a graph. Recall that $L(G) = \sigma(G) - ex(G)$. If G is a path P_n for some $n \geq 1$, let $L(G) = 1$ (so $L(T) = \dim(T)$ for all trees).*

The goal is to prove Theorem 1 we recall here:

Theorem 1. *For any graph G, $L(G) \leq \dim(G) \leq L(G) + 6c(G)$.*

This result implies the following one:

Corollary 2. *Let G be a graph and T be any spanning tree of G. We have*

$$\dim(G) \leq \dim(T) + 6c(G)$$

Proof. For any graph $G = (V, E)$ that is not a tree and any edge $e \in E$ such that $G - e$ is connected, $L(G) \leq L(G - e)$. Indeed if a major vertex v has terminal degree $d \geq 2$ in G, then v is still a major vertex in $G - e$ of terminal degree at least d. So $L(G) \leq L(G - e)$ and then, for T a spanning tree of G, $L(G) \leq L(T)$. As $\dim(G) \leq L(G) + 6c(G)$ and $\dim(T) = L(T)$ we get $\dim(G) \leq \dim(T) + 6c(G)$. □

The rest of the section is devoted to prove Theorem 1.

3.1 Construction of the resolving set

If $c(G) = 0$ then G is a tree and $\dim(G) = L(G)$ by Lemma 8. If $c(G) = 1$ let us prove a stronger result.

Lemma 17. *Let $G = (V, E)$ be a connected unicyclic graph. Then $\dim(G) \leq L(G) + 3$.*

Proof. Let uv be an edge of the cycle. Let $T = G - e$, then $\dim(T) = L(T)$ by Lemma 8 and $\dim(G) \leq \dim(T) + 1$ by Lemma 13. As $L(T) \leq L(G) + 2$ we get the inequality. □

We now focus on the case $c(G) \geq 2$. The first part of the proof will consist in defining a subset S of vertices. We then prove in the second part of the proof that it is, indeed, a resolving set. In order to build this set S, we first find a small subset of vertices M such that $G \setminus M$ is a forest such that each connected component has at most two edges incident to M. We then construct the set S.

Let us start with a simple lemma.

Lemma 18. *Let G be a connected graph with no vertex of degree 1 that is not an induced cycle. There exists a feedback vertex set X of size $\tau(G)$ only containing vertices of degree at least 3.*

Proof. Let X be a minimum feedback vertex set with the minimum number of vertices of degree less than 3. Note that X does not contain vertices of degree 1. Assume by contradiction that X contains a vertex x of degree 2. Let P be the maximal path of vertices of degree 2 containing x. Since G is not a cycle (and is not acyclic otherwise X would be empty), P does not contain the whole graph. Let y be an endpoint of P adjacent to a vertex z of $V \setminus P$. Let $X' = X \setminus \{x\} \cup \{z\}$. The set X' is still a feedback vertex set, a contradiction with the minimality of X. □

Let X be a feedback vertex set of G only containing vertices of degree at least 3 in the graph where all the vertices of degree one have been iteratively removed[3]. Let $G_1, G_2, ..., G_k$ be the

[3]Note that we can assume that the resulting graph is not a cycle since otherwise the graph is unicyclic and the conclusion follows by Lemma 17.

connected components of $G \setminus X$. Note that each G_i induces a tree. For each G_i, let $X_i \subseteq X$ be the set of vertices of X connected to at least one vertex of G_i. Let $N_i \subseteq V(G_i)$ be the set of vertices in G_i adjacent to (at least) one vertex of X_i.

Let T_i be the minimal subtree of G_i containing the vertices of N_i. In other words, T_i is the subtree of G_i restricted to the union of the paths between a and b for any pair $a, b \in N_i$. Let T_i' be the tree built from T_i by adding to each vertex $u \in N_i$, $|N(u) \cap X|$ pending leaves. Let M_i be the set of vertices in T_i' of degree at least 3 and $M := X \cup (\bigcup_{i=1}^{k} M_i)$. Figure 8 illustrates these notations.

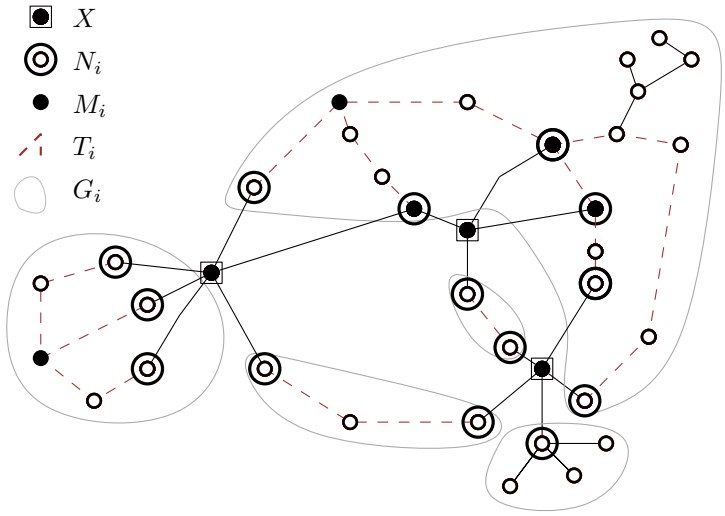


Figure 8: Illustration of the notations X, N_i, M_i, T_i and G_i.

Lemma 19. *For each connected component H of $G \setminus M$ there are at most two edges in G between H and $G \setminus H$.*

Proof. Let U_H be the minimal subtree of H containing all the vertices incident to an edge between H and $G \setminus H$. Then, for each edge between a vertex $v \in H$ and a vertex in $G \setminus H$, add one new vertex in U_H adjacent to v. Let us still denote by U_H the resulting graph. Note that U_H has as many leaves as edges leaving H. So, there are at most two edges with exactly one endpoint in H if and only if U_H has no vertex of degree three.

Let us prove by contradiction that if a vertex v has degree at least 3 in U_H then this vertex should have been added in M.

Let i be such that H is a subgraph of G_i and consider T_i'. Let P_1, P_2, P_3 be the three paths from v to M in U_H which are internally disjoint, i.e. only v and possibly the endpoint in M are common. We claim that every vertex of M_i can appear at most once since otherwise G_i would contain a cycle. So for every path P_i, we can complete P_i into a path P_i' u_1, u_2, u_3 of H_i from v to X such that the paths are internally disjoint. So v has degree at least 3 in T_i', a contradiction. □

Lemma 19 indeed implies the following.

Corollary 20. *Every connected component of $G \setminus M$ is connected to at most two vertices of M.*

A connected component of $G \setminus M$ can be attached to M in three different ways, called *Types*, illustrated in Figure 9. A connected component of $G \setminus M$ has *Type A* (respectively *Type B*) if there are exactly two edges between H and M with distinct endpoints in H and such that their endpoints in M are distinct (resp. the same). A component H has *Type C* if all the edges of G between H and M have the same endpoint in H (but possibly distinct endpoints in M).

Let H be a connected component of $G \setminus M$ of Type A or B and let x and y be the two endpoints in M of the edges between H and M. Let ρ_H be one vertex on the path in H between x and y such that $|d_H(x, \rho_H) - d_H(y, \rho_H)| \leq 1$. In other words, ρ_H is one of the vertices in the middle of the path between x and y in H. Let P be the union of the vertices ρ_H for all the connected components H of *Type A* and B.

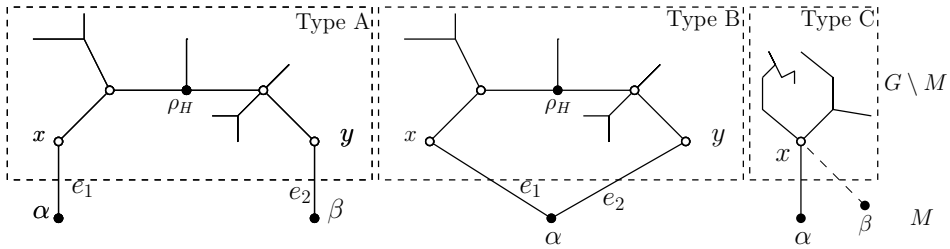


Figure 9: The three *Types* of connected components of $G \setminus M$.

Lemma 21. *Let H be a connected component of $G \setminus M$ of Type A or B. Let ρ be the vertex of $P \cap H$. Then, for any vertex $v \in H$, there is no shortest path between z and ρ using vertices in $G \setminus H$. Moreover, for every $z \in \partial H$, there is a shortest path using only vertices in $H \cup \{z\}$.*

Proof. Let x and y be the two vertices in H adjacent to a vertex of M, α be the vertex of M adjacent to x and β the vertex in M adjacent to y. By definition of *Types A* or B, $x \neq y$ (but α and β could be the same vertex if the *Type* is B).

By definition of ρ, we have $|d_H(x, \rho) - d_H(y, \rho)| \leq 1$. Let $v \in V(H)$. Assume by contradiction that a shortest path between v and ρ in G passes through vertices in $G \setminus H$. So this shortest path between ρ and v passes through x and y. By symmetry we can assume $d_G(v, \rho) = d_H(v, x) + d(x, \alpha) + d(\alpha, \beta) + d(\beta, y) + d_H(y, \rho)$. In other words, the path from v to ρ passes through x and then y.

Assume by contradiction that $d_H(z, x) + d(x, \alpha) + d(\alpha, \beta) + d(\beta, y) + d_H(y, \rho) \leq d_G(v, \rho)$. As $x\alpha$ and $y\beta$ are edges, we have $d_H(v, x) + d(\alpha, \beta) + d_H(y, \rho) + 2 \leq d_G(v, \rho)$. By triangular inequality, $d_G(v, \rho) \leq d_H(v, \rho) \leq d_H(v, x) + d_H(x, \rho)$. So $d_H(x, \rho) \geq d_H(y, \rho) + d(\alpha, \beta) + 2$ which contradicts $|d_H(x, \rho) - d_H(y, \rho)| \leq 1$.

If z is in ∂H and $\alpha = \beta$ then $z = \alpha = \beta$ and the result is immediate. Otherwise assume that $z = \alpha$. We want to contradict $d(\alpha, \beta) + d(\beta, \rho) < d(\alpha, \rho)$. As α is adjacent to x and β adjacent to y the inequality is equivalent to $d(\alpha, \beta) + d(y, \rho) < d(x, \rho)$. As $\alpha \neq \beta$, $d(\alpha, \beta) \geq 1$. So $d(y, \rho) + 2 \leq d(x, \rho)$ which contradicts $|d_H(x, \rho) - d_H(y, \rho)| \leq 1$. □

We now have all the ingredients to define the set S that will be a resolving set based on resolving sets of each connected component H of $G \setminus M$. The union of the resolving sets of the different graphs is not a resolving set for G, so we will need to add a few vertices. Moreover, the size of the union of the resolving set of the connected components of $G \setminus M$, is not bounded by $L(G) + c$. Some vertices have to be removed of these resolving sets.

Let H be a connected component of $G \setminus M$. Let S'_H be a metric basis for H such that, for each major vertex of terminal degree at least 2 in H, all but one of its leaves are in S'_H.

To get the announced bound we divided the component of *Type B* in two parts. A component has *Type* B_1 if it is a component of *Type B* and the two endpoints in H are the roots of a path in H and the major vertex of the leaf of the rooted path has terminal degree at least 2. A component has *Type* B_2 if it is a component of *Type B* and not a component of *Type* B_1. We define the set S_H which is equal to S'_H but with the following slight modifications:

- H has *Type A*. For every $x \in H$ adjacent to M, if x is a leaf and its terminal vertex v in H has degree at least 2, then, if $x \in S'_H$, remove x from S'_H, otherwise remove from S'_H another leaf associated to the terminal vertex of x.

- H has *Type* B_1, we can assume that S'_H contains the two leaves x and y of H such that z_x and z_y are adjacent to M.
- H has *Type* C, let x be the unique vertex of H adjacent to M. If x is a leaf and its terminal vertex v in H has degree at least 2, then, if $x \in S'_H$, remove x from S'_H, otherwise remove from S'_H another leaf associated to the terminal vertex of x.
- H has *Type* C and H is a path with one extremity adjacent to M. Let w be the vertex of M adjacent to H. If there is only one component of *Type* C attached to w which is a path connected to w by an endpoint of the path, let $S_H = \emptyset$. If there are several such components, then let $S_H = \emptyset$ for one of these components and S_K be the extremity of the path not connected to w for all the other such components K (or the unique vertex of H if H is reduced to a single vertex).

The set S is defined as $S = M \cup P \cup (\bigcup_i S_i)$. The goal will consist in proving that S is a resolving S.

3.2 The set S is a resolving set

Before proving the main result of this section, we start with a definition.

Definition 22. *Let H be a connected component of $G \setminus M$ of Type A or B and let x and y be the two vertices in H adjacent to M. Let u be any vertex of H. The* projection z_u of u (on the path between x and y) *is the unique vertex in the path between x and y in H at minimum distance to u.*

We will prove several lemmas that restrict the components where pairs of unresolved vertices can belong to. Let us first prove that they must belong to the same connected component of $G \setminus M$.

Lemma 23. *Let u and v be two vertices of G. If u, v are not resolved by S then there exists a connected component of $H \in G \setminus M$ such that both u, v belong to H.*

Proof. First note that since $M \subseteq S$, $u, v \notin M$. So there exist H_u and H_v, connected components in $G \setminus M$, such that $u \in H_u$ and $v \in H_v$. Assume by contradiction that $H_u \neq H_v$.

- Assume H_u is of *Type* A or B and let $x\alpha$ and $y\beta$ be the two edges connecting H_u to M (with $x, y \in H_u$). Let ρ be the vertex in $P \cap H_u$. Since $v \notin H_u$, the shortest path between v and ρ passes through α or β. Up to symmetry, α is on the shortest path between v and ρ so $d(v, \rho) = d(v, \alpha) + d(\alpha, \rho)$. As α and ρ are in S we also have $d(u, \rho) = d(u, \alpha) + d(\alpha, \rho)$, a contradiction with Lemma 21.
- Assume now that both H_u and H_v are of *Type* C. Let α and β the vertices of M connected to H_u and H_v respectively. If $\alpha \neq \beta$, then the shortest path between u and v contains two distinct vertices of S. Hence, by Lemma 10 u and v are resolved. We assume now that $\alpha = \beta$.

  Since all the components H of *Type* C attached to α but at most one contain a vertex of S_H, by construction, H_u or H_v contains a vertex of S. Without loss of generality, there exists $\gamma \in S \cap H_u$. If u is on the path between γ and v then $d(\gamma, u) < d(\gamma, v)$. Otherwise, let m_u in H_u be at the intersection of the path between u and γ and between α and γ. Then, vertices m_u and α are on the shortest path between u and v. By Lemma 10, one of them must resolve u and v. By assumption, it is not α. If it is m_u, then we would have $d_G(u, \gamma) \neq d_G(v, \gamma)$. Since the shortest paths between u, v and γ go through m_u, we obtain a contradiction.

□

We now prove that, if two vertices are in the same connected component of $G \setminus M$, then they are resolved by S. We start with connected components of *Type* A.

Lemma 24. *Let H be a connected component of $G \setminus M$ of Type A. Let u and v be two vertices of H such that,for all s in S, $d(u,s) = d(v,s)$. Then, $z_u = z_v$.*

Proof. Let $u\alpha$ and $v\beta$ be the two edges between H and M with $u \in H$ and $v \in H$. The graph H is a tree with a path between the two vertices u and v. Assume $z_u \neq z_v$ and, without loss of generality, we can suppose that $z_u \neq \rho$ with ρ the vertex of $P \cap H$. We have $d(u,\alpha) = d(u,z_u) + d(z_u,\alpha)$, $d(v,\alpha) = d(v,z_v) + d(z_v,\alpha)$, $d(u,\beta) = d(u,z_u) + d(z_u,\beta)$ and $d(v,\beta) = d(v,z_v) + d(z_v,\beta)$. As $d(u,\alpha) = d(v,\alpha)$ and $d(u,\beta) = d(v,\beta)$ we get

$$d(z_u,\alpha) + d(z_v,\beta) = d(z_u,\beta) + d(z_v,\alpha).$$

The vertices z_u and ρ are distinct and both between α and β. So z_u is between α and ρ or β and ρ. Assume z_u is between α and ρ. Then $d(z_u,\alpha) \leq d(z_u,\beta)$, so $d(z_v,\alpha) \leq d(z_v,\beta)$ meaning z_v is also between α and ρ. The shortest path between α and ρ passes through z_u and z_v by Lemma 21. Assume z_u is closer than z_v to ρ. Then $d(\alpha,\rho) = d(\alpha,z_v) + d(z_v,z_u) + d(z_u,\rho)$ gives

$$d(\alpha,z_u) + d(z_v,\rho) = d(\alpha,z_v) + d(z_u,\rho) + 2d(z_u,z_v).$$

Use now the paths to ρ: $d(u,\rho) = d(u,z_u) + d(z_u,\rho)$ as $z_u \neq \rho$ and $d(v,\rho) \leq d(v,z_v) + d(z_v,\rho)$. Then $d(u,\alpha) = d(u,z_u) + d(z_u,\alpha)$, $d(v,\alpha) = d(v,z_v) + d(z_v,\alpha)$ gives

$$d(z_v,\alpha) + d(z_u,\rho) \leq d(z_u,\alpha) + d(z_v,\rho).$$

A combination of the previous equality gives $d(z_u,z_v) \leq 0$ so $z_u = z_v$. □

Lemma 25. *Let H be a connected component of $G \setminus M$ of Type A. Let u and v be two vertices of H such that, for all s in S, $d(u,s) = d(v,s)$. Then $u = v$.*

Proof. Assume by contradiction $u \neq v$. Let α and β be the two vertices of M adjacent to H. By construction of S_H, $S_H \cup \{\alpha,\beta\}$ is a resolving set of $H \cup \{\alpha,\beta\}$. Let γ which resolves the pair (u,v) in $H \cup \{\alpha,\beta\}$. By Lemma 24, $z_u = z_v$ so γ still resolves (u,v) in G. Indeed if $z_\gamma = z_u$ then the distances are the same in G and in $H \cup \{\alpha,\beta\}$. If $z_\gamma \neq z_u$ then $d_G(u,\gamma) = d_H(u,z_u) + d_G(z_u,\gamma)$ and $d_G(v,\gamma) = d_H(v,z_v) + d_G(z_v,\gamma)$. As $d_H(u,\gamma) \neq d_H(v,\gamma)$ with $d_H(u,\gamma) = d_H(u,z_u) + d_H(z_u,\gamma)$ and $d_H(v,\gamma) = d_H(v,z_u) + d_H(z_u,\gamma)$ we get $d_H(u,z_u) \neq d_H(v,z_u)$ so $d_G(u,\gamma) \neq d_G(v,\gamma)$. So γ resolves (u,v) in G, a contradiction.

□

Lemma 26. *Let H be a connected component of $G \setminus M$ of Type B. If $u,v \in H$ are not resolved by S, then $z_u = z_v$.*

Proof. Let us prove it by contradiction. Let α be the vertex of M connected to H. Let x,y be the two vertices of H connected to α.

Case 1: H has *Type* B_1

By construction, $\{\alpha,\rho,y\} \subseteq S$ with y such that z_y is connected to α. Assume by contradiction $z_u \neq z_v$. We first show that (z_u,z_v) is resolved by $\{\alpha,\rho,y\}$. Indeed, $d(y,z_u) = d(y,z_y) + d(z_y,z_u)$ and $d(y,z_v) = d(y,z_y) + d(z_y,z_v)$. Lemma 12 ensures that (z_u,z_v) is resolved by a vertex of $\{\alpha,\rho,z_y\}$ and if z_y resolves (z_u,z_v), then y resolves (z_u,z_v). So (z_u,z_v) is resolved by a vertex of $\{\alpha,\rho,y\}$, let γ be such a vertex.

If $z_v = \rho$, then $d(\gamma,u) = d(\gamma,z_u) + d(z_u,u)$ and $d(\gamma,v) = d(\gamma,\rho) + d(\rho,v)$, so $d(z_u,u) \neq d(\rho,v)$. As $\rho \in S$, $d(u,\rho) = d(v,\rho)$ so $d(z_u,u) < d(\rho,v)$. We exploit now the equalities $d(\alpha,u) = d(\alpha,v)$ and $d(\alpha,u) = d(\alpha,z_u) + d(z_u,u)$. By definition of ρ, $d(\alpha,z_u) \leq d(\alpha,\rho)$ and $d(z_u,u) < d(\rho,v)$. So $d(\alpha,v) = d(\alpha,\rho) + d(\rho,v) > d(\alpha,u)$, a contradiction.

If $z_v \neq \rho$, then $d(\gamma,u) = d(\gamma,z_u) + d(u,z_u)$ and $d(\gamma,v) = d(\gamma,z_v) + d(v,z_v)$. By hypothesis $d(\gamma,u) = d(\gamma,v)$, so $d(u,z_u) \neq d(v,z_v)$. We can assume by symmetry $d(z_u,u) < d(z_v,v)$. Let $\beta \in \{\alpha,\rho\}$, such that $d(z_u,\beta) \leq d(z_v,\beta)$. Such a vertex exists since the distances $d(\alpha,z_u) + d(z_u,\rho)$ and $d(\alpha,z_v) + d(z_v,\rho)$ are the same if z_u, z_v are both on the same side of the xy-path with respect to ρ and differ by at most one otherwise. Then, $d(\beta,u) = d(\beta,v)$ and $d(\beta,u) = d(\beta,z_u) + d(z_u,u)$.

But $d(\beta, z_u) \leq d(\beta, z_v)$ and $d(z_u, u) < d(z_v, v)$. So $d(\beta, v) = d(\beta, z_v) + d(z_v, v) > d(\beta, u)$, a contradiction.

Case 2: H has *Type* B_2

As S contains S_H which is a resolving set of H, there exists $\gamma \in H$ such that $d_H(u, \gamma) \neq d_H(v, \gamma)$. By hypothesis $d_G(u, \gamma) = d_G(v, \gamma)$.

Assume first $z_\gamma = \rho$. Let us prove that $d_G(u, \gamma) = d_H(u, \gamma)$ and $d_G(v, \gamma) = d_H(v, \gamma)$, which gives a contradiction. By symmetry it is enough to prove that $d_G(u, \gamma) = d_H(u, \gamma)$. If $z_u = z_\gamma = \rho$ then $d_G(u, \gamma) = d_G(u, \rho) + d_G(\rho, \gamma) = d_H(u, \rho) + d_H(\rho, \gamma) = d_H(u, \gamma)$ and the conclusion follows. If $z_u \neq z_\gamma$, then

$$d_G(u, \gamma) = d_G(u, z_u) + d_G(z_u, \rho) + d_G(\rho, \gamma).$$

By Lemma 21, $d_G(z_u, \rho) = d_H(z_u, \rho)$. We have $d_G(u, z_u) = d_H(u, z_u)$ and $d_G(\rho, \gamma) = d_H(\rho, \gamma)$ since the paths between these vertices are unique. So $d_G(u, \gamma) = d_H(u, \gamma)$.

So, from now on, we can assume that $z_\gamma \neq \rho$. Since $\{\rho, \alpha\}$ does not resolve (u, v), we have $d(u, z_u) + d(z_u, \rho) = d(v, z_v) + d(z_v, \rho)$, and $d(u, z_u) + d(z_u, \alpha) = d(v, z_v) + d(z_v, \alpha)$. Thus,

$$d(z_u, \rho) + d(z_v, \alpha) = d(z_v, \rho) + d(z_u, \alpha).$$

Since ρ and α are almost opposed on the smallest cycle containing them, we also have

$$d(z_u, \rho) + d(z_u, \alpha) = d(z_v, \rho) + d(z_v, \alpha) + \epsilon$$

with $\epsilon \in \{-1, 0, 1\}$. Summing the two equalities gives $2d(z_u, \rho) = 2d(z_v, \rho) + \epsilon$. So by parity $\epsilon = 0$. Then, $d(z_u, \rho) = d(z_v, \rho)$ and finally $d(z_u, \alpha) = d(z_v, \alpha)$. Since $d(u, \alpha) = d(v, \alpha)$, we obtain $d(u, z_u) = d(v, z_v)$.

If $z_\gamma \notin \{z_u, z_v\}$, then z_u and z_v are at the same distance to α, ρ and z_γ, so, by Lemma 12, $z_u = z_v$. If $z_\gamma \in \{z_u, z_v\}$, up to symmetry we can assume $z_\gamma = z_u$. Then $d(u, \gamma) \leq d(u, z_u) + d(z_u, \gamma)$ and $d(v, \gamma) = d(v, z_v) + d(z_v, z_u) + d(z_u, \gamma)$. As $d(u, \gamma) = d(v, \gamma)$ and $d(u, z_u) = d(v, z_v)$ we get $d(z_v, z_u) \leq 0$ so $z_u = z_v$. □

Lemma 27. *Let H be a connected component of $G \setminus M$ of Type B. The set S resolves all the pairs of vertices of H.*

Proof. Let u, v be two vertices of H which are not resolved by S. By Lemma 26, $z_u = z_v$. Let $z = z_u = z_v$, if $\deg(z) = 2$ then $u = v = z$ and the result is proven. We can assume from now on that $\deg(z) \geq 3$.

Case 1: There exists $\gamma \in S_H$ which resolves the pair (u, v) in H.

If $z_\gamma = z_u = z_v$ then the distances between u (resp. v) and γ are the same in H and G, a contradiction.

So we can assume that $z_\gamma \neq z_u$. We have $d_H(\gamma, u) = d_H(u, z) + d_H(z, \gamma)$ and $d_H(\gamma, v) = d_H(v, z) + d_H(z, \gamma)$. Since $d_H(\gamma, u) \neq d_H(\gamma, v)$, we have $d_H(u, z) \neq d_H(v, z)$. Now, since by Lemma 21, for $w \in \{u, v\}$, $d_H(w, \rho) = d_G(w, \rho)$ and $d(w, \rho) = d(w, z) + d(z, \rho)$, ρ resolves (u, v), a contradiction.

Case 2: The pair (u, v) is not resolved by S_H in H.

This case can only happen if H has *Type* B_1 (since otherwise no vertex of S'_H is removed). Then there exists a vertex x such that z_x is adjacent to α which resolves the pair (u, v) in H. If $z = z_x$, then, $d(z, u) = d(z, x) - d(u, x)$ and $d(z, v) = d(z, x) - d(v, x)$ so $d(z, u) \neq d(z, v)$. If $z \neq x$ then $d(x, u) = d(x, z) + d(z, u)$ and $d(x, v) = d(x, z) + d(z, v)$. So $d(z, u) \neq d(z, v)$ in both cases. Hence α resolves the pair (u, v) in G. As $z \neq \alpha$, $d(\alpha, u) = d(\alpha, z) + d(z, u) \neq d(\alpha, z) + d(z, v) = d(\alpha, v)$. □

Lemma 28. *Let H be a connected component of $G \setminus M$ of Type C. Then S resolves any pair of vertices in H.*

Proof. Assume by contradiction two vertices $u, v \in H$ with $u \neq v$ are not resolved by S. Let x be the unique vertex of H adjacent to M and m be a vertex of M adjacent to x. Let H' be the subgraph of G with vertex set $V(H) \cup \{m\}$.

If S contains a resolving set of H, then, since x is a cut-vertex, S resolves the pair (u, v). So we can assume that at least one vertex of S'_H has been removed during the construction of S.

Note that since m does not resolve (u, v), $d(u, x) = d(v, x)$ in H. So in particular H cannot be a path with endpoint x. So by construction of S, we can assume that x is a leaf in H and its major vertex has terminal degree at least two in H.

By construction of S_H, $S \cup \{x\}$ is a resolving set for H. Then $S \cup \{m\}$ is a resolving set for H'. Let $\gamma \in S_H \cup \{m\}$ that resolves u and v in H'. The distances between u, v and γ in G and H' are the same so γ resolves u and v, a contradiction. □

Lemma 29. *The set S is a resolving set of G.*

Proof. Let (u, v) be a pair of vertices that is not resolved by S. Assume by contradiction $u \neq v$. By Lemma 23, there exists a connected component H of $G \setminus M$ such that $u \in H$ and $v \in H$. Then, if H has *Type* A, by Lemma 25, $u = v$. If H has *Type* B, by Lemma 27, $u = v$. If H has *Type* C then, by Lemma 28, $u = v$. □

3.3 Upper bound on the size of S

Lemma 29 ensures that $\dim(G) \leq |S|$. So Theorem 1 holds if $|S| \leq L(G) + 6c$. The set S is a union of three sets that we will bound the size separately. We use the following result on minors to get the bounds.

Let G be a multigraph. The graph H is a *minor* of G if H can be obtained from G via a sequence of edge deletions, vertex deletions and edge contractions (the edge contraction operation can create parallel edges between two vertices or loops). One can easily check that the minor operation can only decrease the cycle rank.

Lemma 30. $|M| \leq 2c(G) - 2$.

Proof. If $|M| \leq 2$ then the inequality holds since we can assume $c \geq 2$. So we can assume that $|M| \geq 3$. Let K be the multigraph (with possible loops) with vertex set M where, for each connected component H of $G \setminus M$ of *Type A* or *B* with endpoints x and y in M (that might be identical), we create an edge between x and y in K. Note that K is a minor of G as it can be obtained from G by contracting edges in components H of $G \setminus M$ into a single edge.

Every vertex in K has degree at least 3. The vertices of $M \setminus X$ have degree at least 3 by definition of M_i for every i. By construction of X, $x \in X$ has degree at least 3 in the graph starting from G and removing the degree one vertices. Three adjacent edges belong to cycles so contribute to the degree of x in K so $\deg_K(x) \geq 3$.

We have $3|V(K)| \leq \sum_{v \in V(K)} \deg(v) = 2|E(K)|$. So $3|M| \leq 2|E|$. Since the cycle rank of K is at most c, so $|E| \leq c + |M| - 1$. A combination of these inequalities gives $|M| \leq 2c(G) - 2$. □

Lemma 31. $|P| \leq c + |M| - 1$.

Proof. Let K' be the multigraph (with loops) with vertex set M and an edge between two vertices x and y if and only if there exists in $G \setminus M$ a connected component H adjacent to x and y. The graph K' is a minor of G since K' can be obtained from G by contracting edges with exactly one endpoint in M until no such edge exists. One can easily notice that in K' there is an edge xy with multiplicity k if and only if in $G \setminus M$ there are k connected components attached to x and y. Since K' is a minor of G, $c(K') \leq c(G)$. As K' contains $|M|$ vertices, K' has at most $c + |M| - 1$ edges. Thus, $G \setminus M$ has at most $c + |M| - 1$ components of *Type* A or B. Since P contains one vertex in each component of *Type* A or B, we have $|P| \leq c + |M| - 1$. □

Lemma 32.

$$\Big| \bigcup_{H \, con. \; comp. \; of \; G \setminus M} S_H \Big| \leq L(G) + c.$$

Proof. For every connected component H of $G \setminus M$, let $\ell_H = \sum_{x \in H}(ter(x) - 1)$ over all the major vertices x in H. We consider the three types of components.

Let H be a connected component of *Type A*. We claim that $|S_H| = \ell_H$. Indeed let $\epsilon \in \{0, 1, 2\}$ be the number of vertices of H adjacent to M which are leaf connected to a major vertex of degree at least 2. By construction $L(H) - \epsilon = |S_H| = \dim(H) - \epsilon$.

Let H be a connected component of *Type* B_1. Then $L(H) = |S_H| + 1$ and $\ell_H = L(H) - 2$ by construction so $|S_H| \leq \ell_H + 1$.

Let H be a connected component of *Type B* not B_1. Then, $L(H) = |S_H|$ and $\ell_H \geq L(H) - 1$ because H has not *Type* B_1 so $|S_H| \leq \ell_H + 1$.

Let H be a component of *Type C* not a path. By construction of S_H, $|S_H| = \ell_H$. Indeed, let $\epsilon \in \{0, 1\}$ be the number of vertices of H adjacent to M which are leaves connected to a major vertex of degree at least 2. By construction $|S_H| = \dim(H) - \epsilon = L(H) - \epsilon$ and $\ell_H = L(H) - \epsilon$.

Let H be a component of *Type C* with H a path. If H is connected to M by a vertex x which is not an extremity of H then m is a major vertex of terminal degree 2 in G. So $|S_H| = \ell_H = 1$. If H is a path connected to a vertex $m \in M$ by an extremity, let $k \in \mathbb{N}$ be the number of such components connected to m. Denote them by $H_1, H_2, ..., H_k$. If $k \geq 2$ then m is a major vertex in G with terminal degree k and $|\cup_{i \leq k} S_{H_i}| = k - 1 = |L(G) \cap \{m\} \cup (\cup_{i \leq k} H_i)|$. If $k = 1$ then $S_H = \emptyset$ so $\ell_H = 0$.

There are at most $c(G)$ components of *Type B*: for each component H of *Type B* we can found a cycle in G by adding the vertex of M adjacent to H. By definition of $c(G)$, this gives at most $c(G)$ components of *Type B*. Summing the inequalities gives the result:

$$|\cup S_H| = \sum_{Type\ A} |S_H| + \sum_{Type\ B} |S_H| + \sum_{Type\ C} |S_H|$$

$$\leq \sum_{Type\ A} \ell_H + \sum_{Type\ B} (\ell_H + 1) + \sum_{Type\ C} \ell_H \leq L(G) + c(G)$$

□

Finally, we can prove Theorem 1:

Proof. The set S is a resolving set so $\dim(G) \leq |S|$. By definition $S = M \cup P \cup (\bigcup_H S_H)$. By Lemma 30, $|M| \leq 2c(G)$, by Lemma 31, $|P| \leq 3c$ and by Lemma 32,$|\cup S_H| \leq L(G) + c(G)$. Summing the inequalities give $\dim(G) \leq L(G) + 6c(G)$. □

One can naturally ask if this upper bound is optimal. Figure 10 gives an example of graph where Lemmas 30, 31 and 32 are tight. It ensures that our analysis of the construction is optimal but not necessarily the construction itself. Indeed, the metric dimension of the graph of Figure 10 is 8 and the square vertices form a metric basis.

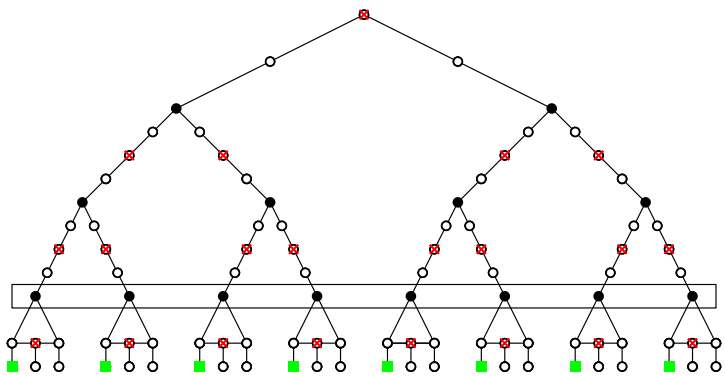
4 Metric dimension and zero-forcing sets

In this section, we study how the metric dimension and the zero forcing number can be modified when an edge is added to a graph. Then we prove a weakening of Conjecture 5, as a consequence of Theorem 1. We then give a short proof of Conjecture 5 for unicyclic graphs. We will then generalize this result to prove the conjecture for cactus graphs. We finally prove a strengthening of Conjecture 5 when the graph is unicyclic and the unique cycle has odd length.

4.1 Edge modifications and consequences for Conjecture 5

The following lemma ensures that the variations of the zero forcing number when an edge is added or deleted an edge are small [15].

Lemma 33. *[15] Let $G = (V, E)$ be a graph and $e \in E(G)$, then $Z(G) - 1 \leq Z(G - e) \leq Z(G) + 1$.*



X	$\lvert X\rvert = c$
M	$\lvert M\rvert = 2c-2$
P	$\lvert P\rvert = \lvert M\rvert + c - 1$
A minimal resolving set	

Figure 10: Tightness of Lemmas 30, 31 and 32.

We have a more precise result if $Z(G+e) < Z(G)$ which will be useful later.

Lemma 34. *Let $G = (V, E)$ be a graph and u and v two vertices of G such that $e = uv \notin E$. If $Z(G+e) < Z(G)$, then for any minimum zero forcing set of $G+e$, at some step u forces v or v forces u.*

Proof. By contradiction if a zero forcing set of minimal size for $G+e$ does not use the edge e then it is a zero forcing set of G so $Z(G) \leq Z(G+e)$. □

A similar statement does not hold for metric dimension. However, Lemma 13 gives some conditions where a similar result holds. Using these results we can get inequalities between the metric dimension and the zero-forcing number for some classes of graphs.

Corollary 35. *Let G be a connected unicyclic graph and e be an edge such that $T = G - e$ is a tree. We have* $\dim(G) \leq Z(G) + 2$.

Proof. By Lemma 9, $\dim(T) \leq Z(T)$. Lemmas 13 and 33 ensure that $\dim(G) \leq Z(G) + 2$. □

Eroh et al. [8] proved Conjecture 5 for unicyclic graphs via a very long case analysis. They start from a tree T achieving $\dim(T) = Z(T)$ and make a complete study of all the places where an edge could be added. We drastically simplify their proof by starting from a unicyclic graph G and delete a well-chosen edge.

Lemma 36. *Let $G = (V, E)$ be a graph which is not a tree and $C \subseteq V$ a cycle of G. Then, there exists an edge $e \in E(C)$ such that $Z(G - e) \leq Z(G)$.*

Proof. Let $Z \subseteq V$ be a minimum zero forcing set of G. Let $F \subseteq E$ be the *forcing edges* in a sequence starting from Z, i.e. $uv \in F$ if and only if at some stage u forces v or v forces u.

We claim that at least one edge of C is not in F. Indeed, if u forces v then u is turned black before v. So, the first vertex w of the cycle that is turned black cannot be turned black because of an edge of C (such a vertex can already be black at the beginning of the proceed). Let w_1, w_2 be the two neighbors of w on C. The vertex w can force at most one of its two neighbors. So, without loss of generality, w_2 is not forced by w and is turned black after w. So, if we remove the edge $e = ww_2$, Z is still a forcing set of $G - e$ with the same sequence of applications of the color change rule that turned G into black. Therefore $Z(G - e) \leq Z(G)$. □

We obtain as a corollary the main result of [8].

Corollary 37. *Let G be a unicyclic graph. Then,* $\dim(G) \leq Z(G) + 1$.

Proof. Let e be an edge of C such that $Z(G-e) \leq Z(G)$. Such an edge exists by Lemma 36. By Lemma 13, $\dim(G) \leq \dim(G-e)+1$. Moreover, by Lemma 9, $\dim(G-e) \leq Z(G-e)$ since $G-e$ is a tree. The combination of these three inequalities gives $\dim(G) \leq Z(G)+1$. □

All these results together with Theorem 1 permits to prove a weakening of Conjecture 5 which we restate here:

Theorem 6. *For every graph G, we have*

$$\dim(G) \leq Z(G) + 6c(G).$$

Proof. By induction with Lemma 36, there exists a spanning tree T such that $Z(T) \leq Z(G)$. By Lemma 9, $\dim(T) \leq Z(T)$ and thus, $\dim(G) \leq Z(G) + 6c(G)$ by Theorem 1. □

We generalize the proof of Conjecture 5 for unicyclic graphs to cactus graphs. Almost the same techniques can be applied. First we define the cactus graphs class.

Definition 38. *Any graph G is a cactus graph if any edge $e \in E$ is part of at most one cycle of G.*

Theorem 39. *Let $G = (V, E)$ be a cactus graph. Then,* $\dim(G) \leq c(G) + Z(G)$.

Proof. We prove Theorem 39 by induction on $c(G)$. If $c(G) = 0$ then G is a tree and $\dim(G) \leq Z(G)$. If $c(G) > 0$ let C be a cycle of G. By Lemma 36 there exists an edge $e \in C$ such that $Z(G-e) \leq Z(G)$. By induction $\dim(G-e) \leq Z(G-e)+c(G-e) \leq Z(G)+c(G-e) = Z(G)+c(G)-1$ since $c(G-e) = c(G)-1$ for any $e \in C$.
To conclude, let us prove that Lemma 13 can be applied. Let $v_1, v_2..., v_k$ be the vertices of C. Let G_i be the connected component of v_i in $G \setminus C$. Assume by contradiction that two subgraphs G_i and G_j with $1 \leq i < j \leq k$ are not disjoint. Then, there exists a path P between the vertices v_i and v_j in $G \setminus C$. Then G contains two cycles with common edges: C and a cycle containing P and a path in C between v_i and v_j, a contradiction. So, by Lemma 13, $\dim(G) \leq \dim(G-e)+1$ and then $\dim(G) \leq c(G) + Z(G)$. □

As a corollary, we obtain that cactus graphs satisfy Conjecture 5. It improves a result of [8] that ensures that $Z(G) \leq \dim(G) + 2c(G)$ if G has no even cycles. If G has no even cycles, all its cycles are edge disjoint. Indeed, if two odd cycles share at least one edge then G contains an even cycle.

4.2 Unicyclic graphs with an odd cycle

In this section, we consider the case where G is unicyclic and its cycle has odd length. In this case, we will improve the inequality of Corollary 37 to get $\dim(G) \leq Z(G)$. Such a result cannot be extended to G with an even cycle, see Figure 6 for an example. The intuitive reason why there is a difference between odd and even cycles is that, by Lemma 11, any pair of vertices resolves an odd cycle while it is false for even cycles.

Before proving the main result of this section, we need some technical lemmas. Let $k \geq 1$ and let $G = (V, E)$ be a graph containing a unique cycle C of length $2k+1$. For $u \in C$, let T_u be the connected component of u in $G' = (V, E \setminus E(C))$ rooted in u. Note that T_u is a tree. We call u the root of T_u. We say that T_u is *trivial* if $T_u = \{u\}$, is a *rooted path* if T_u is a path with u at one extremity and is a *rooted tree* otherwise. Note that a rooted path can be trivial (otherwise specified). For $u \in C$, if T_u is not trivial we denote by ℓ_u the terminal degree of u in G. Else we let $\ell_u = 0$.

Lemma 40. *Let G be an odd unicyclic graph. If there exists $u \in C$ such that $\ell_u \geq 1$ then there exists $e \in E(C)$ incident to u such that $Z(G-e) \leq Z(G)$.*

Proof. By Lemma 34, it suffices to find a minimum zero forcing set for G which does not use one of the two edges in $E(C)$ incident to u. Let Z be a minimum zero forcing set of G. If $u \in Z$ then u can force only one vertex and the result is proved since at most one edge incident to u is used. We can assume $u \notin Z$. Let P be an internal degree-two path between u and a terminal vertex l of G which exists since $\ell_u \geq 1$. If there is a vertex x in $P \cap Z$, then $(Z \setminus \{x\}) \cup \{l\}$ is still a minimum forcing set of G. Then, l iteratively forces the vertices of P until u and we are back to the previous case. Finally, if P and u are initially white, then u is the first vertex of $P \cup \{u\}$ which is turned black (eventually by one edge in $E(C)$). It then turns in black P. We cannot use the second edge of $E(C)$ since every vertex forces at most one vertex. □

Lemma 41. *Let G be an odd unicyclic graph. Let $S \subseteq V$ be such that for any u on the cycle, S is not a subset of T_u. Then, S resolves C.*

Proof. Let α and β in S such that $\alpha \in T_u$ and $\beta \in T_v$ with $u \neq v$. Assume by contradiction that x and y in C satisfies $d(x,\alpha) = d(y,\alpha)$ and $d(x,\beta) = d(y,\beta)$. Then, since $d(x,\alpha) = d(x,u) + d(u,\alpha)$ and $d(y,\alpha) = d(y,u) + d(u,\alpha)$, we have $d(x,u) = d(y,u)$. Similarly $d(x,v) = d(y,v)$, a contradiction with Lemma 11. □

Lemma 42. *Let G be an odd unicyclic graph. If, for any $u \in C$, T_u is a rooted tree then for all $e \in E(C)$, $\dim(G) \leq \dim(G-e)$.*

Proof. Let $e \in E(C)$ and S be a metric basis of $G-e$. We will prove that S is still a resolving set of G. Let us first prove that for every $u \in C$ since T_u is a rooted tree $S \cap T_u \neq \emptyset$.
By definition of rooted tree, T_u contains a vertex of degree 3 in G. Let r be such a vertex. By Lemma 15, at most one connected component of $G - e \setminus \{r\}$ does not contain element of S. The tree T_u contains at least two connected components of $G - e \setminus \{r\}$, so $S \cap T_u \neq \emptyset$.

Let (x,y) be any pair of vertices. We prove that S resolves (x,y) in G.

1. Assume first that x and y are in the same component T_u for some $u \in C$. Let $\alpha \in S$ that resolves (x,y) in $G-e$. If $\alpha \in T_u$ then $d_G(\alpha,x) = d_{G-e}(\alpha,x) \neq d_{G-e}(\alpha,y) = d_G(\alpha,y)$.

   We can assume that $\alpha \notin T_u$. Since $e \notin T_u$ and u is a cut-vertex of G and $G-e$, we have $d_{G-e}(u,w) = d_G(u,w)$ for every $w \in T_u$. For every $w \in T_u$,

   $$d_G(\alpha,w) = d_G(\alpha,u) + d_G(u,w) = d_G(\alpha,u) + d_{G-e}(u,w).$$

   Since $d_{G-e}(\alpha,w) = d_{G-e}(\alpha,u) + d_{G-e}(u,w)$ and $d_{G-e}(\alpha,x) \neq d_{G-e}(\alpha,y)$, we have $d_{G-e}(u,x) \neq d_{G-e}(u,y)$. Thus, $d_G(\alpha,x) \neq d_G(\alpha,y)$ and then α resolves (x,y) in G.

2. Assume now x and y are in different components, respectively T_u and T_v. Then, there exist $\alpha \in S \cap T_u$ and $\beta \in S \cap T_v$. Assume $d_G(\alpha,x) = d_G(\alpha,y)$ and $d_G(\beta,x) = d_G(\beta,y)$. Then, $d_G(\alpha,x) \leq d_G(\alpha,u) + d_G(u,x)$ and $d_G(\alpha,y) = d(\alpha,u) + d_G(u,v) + d_G(v,y)$ so $d_G(u,v) + d_G(v,y) \leq d_G(u,x)$. The symmetric relation is $d_G(v,u) + d_G(u,x) \leq d_G(v,y)$. Summing the two gives $2d_G(u,v) \leq 0$ which is a contradiction since $u \neq v$. □

We will prove by case distinction the following result.

Theorem 43. *Any odd unicyclic graph G satisfies $\dim(G) \leq Z(G)$.*

Proof. We make a case analysis on the structure of G.

Case 1: For every $u \in C$, T_u is a rooted tree.
By Lemma 36, there is an edge e such that $Z(G-e) \leq Z(G)$. Since the $G-e$ is a tree, Corollary 9 ensures that $\dim(G-e) \leq Z(G-e)$. By Lemma 42, $\dim(G) \leq \dim(G-e)$. The combination of these inequalities gives $\dim(G) \leq Z(G)$.

Case 2: For every $u \in C$, T_u is a rooted path.
We prove that any pair of vertices α and β of C at distance k is a resolving set of G.
Let α, β be such a pair of vertices. Let us first prove that for every $x \in G$, $d(x,\alpha) + d(x,\beta) \in \{k, k+1\}$ if and only if $x \in C$. Indeed as $d(\alpha,\beta) = k$ for any vertex x, $d(x,\alpha) + d(x,\beta) \geq k$. If $x \in C$, then either x is on the path between α and β, and $d(x,\alpha) + d(x,\beta) = k$. Or x is in the other part of the cycle and $d(x,\alpha) + d(x,\beta) = k+1$. If $x \notin C$, then let y be the vertex of C such that $x \in T_y$. Then, $d(x,\alpha) + d(x,\beta) = d(y,\alpha) + d(y,\beta) + 2d(x,y) \geq 2 + d(y,\alpha) + d(y,\beta) \geq k+2$. This implies that a vertex on C cannot have the same distance vector as a vertex in $V \setminus C$.
By Lemma 41, the set $\{\alpha, \beta\}$ resolves C. To conclude we have to show that $\{\alpha, \beta\}$ resolves $V \setminus C$. Let us prove that two vertices not in C are resolved. Assume by contradiction that two vertices x and y are not resolved by $\{\alpha, \beta\}$, with $x \in T_u$ and $y \in T_v$ for u and v in C. If $u = v$, since T_u is a rooted path, α resolves the pair (x, y). From now on, we can assume that $u \neq v$. Assume by symmetry $d(x,u) \leq d(y,v)$, and let $z \in T_v$ be the vertex on T_v such that $d(v,z) = d(x,u)$. Then, the pair $\{\alpha, \beta\}$ does not resolve the pair (z, u). But $u \in C$ and the previous cases ensures that any pair of vertices with one vertex on C is resolved by $\{\alpha, \beta\}$, a contradiction. So $\dim(G) = 2$ and G is not a path, so $Z(G) \geq 2$.

Case 3: There exists a unique $u \in C$ such that T_u is a rooted tree.
Let us prove that $\dim(G) \leq \dim(T_u) + 1$ and $Z(G) \geq Z(T_u) + 1$. Let S be a metric basis for T_u and $v \in C$ a vertex at distance k from u.
Let us show that $S \cup \{v\}$ is a resolving set of G. Let $\alpha \in S \cap T_u$, then $\{\alpha, v\}$ resolves $G \setminus T_u$. Indeed otherwise, these two vertices would have the same distances to u and v which is impossible since in $G \setminus T_u$, for every $v \in C$, T_v is a rooted path and by the claim in Case 2, $\{u, v\}$ is a resolving set for $G \setminus (T_u \setminus \{u\})$. Two vertices on T_u are resolved since S is a metric basis for T_u and u is a cut-vertex. Let $x \in T_w$ for some $w \in C$, and $y \in T_u$. By triangular inequality, $d(\alpha, y) \leq d(y,u) + d(u,\alpha)$. If $d(\alpha, y) = d(\alpha, x)$, then $d(x,u) \leq d(y,u)$ as $d(\alpha, x) = d(\alpha, u) + d(u, x)$. If $d(x,v) = d(y,v)$, then as $d(y,v) = d(y,u) + d(u,v) \geq d(x,u) + d(u,v)$. We get $d(x,v) \geq d(x,u) + d(u,v)$. Removing $d(x,w)$ on both side gives $d(w,v) \geq d(w,u) + d(u,v)$ which is impossible since $d(u,v) = k$ and $d(w,v) \leq k$. So x and y have different codes and $\dim(G) \leq \dim(T_u) + 1$.
Let Z be a minimal zero forcing set of G. If Z contains u, then it should contain at least another vertex in $G \setminus T_u$. Since the restriction of Z to T_u is a forcing set for T_u, we have $Z(G) \geq Z(T_u) + 1$. So we can assume that $u \notin Z$. Consider a sequence of color change rule that turns u into black. Either u is forced by a vertex in $G \setminus T_u$. Since $(G \setminus T_u) \cup \{u\}$ contains a cycle, there are at least two vertices in $Z \cap (G \setminus T_u)$ and $Z \cap T_u \cup \{u\}$ is a forcing set of T_u. So $Z(G) \geq Z(T_u) + 1$. Otherwise u is forced by a vertex of T_u. Then there is at least one vertex in $Z \cap (G \setminus T_u)$ and $Z \cap T_u$ is a forcing set of T_u so $Z(G) \geq Z(T_u) + 1$. So in both cases we have $\dim(G) \leq \dim(T_u) + 1 \leq Z(T_u) + 1 \leq Z(G)$.

For the other cases we use the following process: we exhibit an edge e such that $Z(G-e) \leq Z(G)$ (by Lemma 36 or 40). Then, find a vertex z in $G - e$ such that z is an interior degree-2 vertex or a major vertex with terminal degree 0 or 1. By Lemma 8, $\dim(G - e) < Z(G - e)$ and by Lemma 13, $\dim(G) \leq \dim(G - e) + 1$, so $\dim(G) \leq Z(G)$. We just give the construction of e and z.

Case 4: There exists $u, v, w, x \in C$ in this order (not necessarily adjacent) such that T_u and T_w are rooted trees and T_v and T_x are rooted paths.
Let $e \in E(C)$ such that $Z(G - e) \leq Z(G)$. Such an edge exists by Lemma 36. In $G - e$, either the vertex v or x is on the path between u and w. Let z be this vertex, z is an interior degree-2 vertex or a major vertex with terminal degree 0 or 1 in $G - e$.

Case 5: There exist u and v adjacent with T_u and T_v rooted trees and w with T_w a rooted path.

- $l_u = 0$**.** Let $e \in E(C)$ such that $Z(G - e) \leq Z(G)$. Such an edge exists by Lemma 36. In $G - e$, u is an interior degree-2 vertex or a major vertex with terminal degree 0 or 1 so $z = u$.
- $l_u = 1$**.** Let $e \in E(C)$ adjacent to u such that $Z(G - e) \leq Z(G)$. Such an edge exists by

Lemma 40. If $e = uv$ then w is an interior degree-2 vertex or a major vertex with terminal degree 0 or 1 in $G - e$ so $z = w$. If e in the other edge in $E(C)$ adjacent to u then u is an interior degree-2 vertex or a major vertex with terminal degree 0 or 1 in $G - e$ so $z = u$.

- $l_u \geq 2$ **and** $l_v \geq 2$**.** Let $e = uv$, then $Z(G - e) \leq Z(G)$. Indeed a minimal zero-forcing set of G is also a zero-forcing set of $G - e$. Let Z be a zero-forcing set of G. The set Z contains at least one leaf of u and one leaf of v. The leaves can turn black u and v so there is a sequence of forces for G such that the edge e is not a forcing edge. So $Z(G - e) \leq Z(G)$ by Lemma 34. Then w is an interior degree-2 vertex or a major vertex with terminal degree 0 or 1 in $G - e$ so $z = w$.

□

References

[1] Nicolas Bousquet et al. "On the cycle rank conjecture about metric dimension and zero forcing number in graphs". In: *Research Perspectives CRM Barcelona*. 2021.

[2] Peter J. Slater. "Leaves of trees". In: *Congressus Numerantium* 14 (1975).

[3] F. Harary and R. A. Melter. "On the metric dimension of a graph." In: *Ars Combinatoria* 2:191-195 (1975).

[4] Garey and Johnson. *A Guide to the Theory of NP-completeness*. Journal of algorithms, 1979.

[5] Josep Díaz et al. "On the Complexity of Metric Dimension". In: *Algorithms – ESA 2012*. Ed. by Leah Epstein and Paolo Ferragina. 2012.

[6] Samir Khuller, Balaji Raghavachari, and Azriel Rosenfeld. "Landmarks in graphs". In: *Discrete Applied Mathematics* 70.3 (1996), pp. 217–229.

[7] Linda Eroh et al. "The effect of vertex or edge deletion on the metric dimension of graphs". In: *J. Comb* 6.4 (2015), pp. 433–444.

[8] Linda Eroh, Cong X Kang, and Eunjeong Yi. "A comparison between the metric dimension and zero forcing number of trees and unicyclic graphs". In: *Acta Mathematica Sinica, English Series* 33.6 (2017), pp. 731–747.

[9] Gary Chartrand et al. "Resolvability in graphs and the metric dimension of a graph". In: *Discrete Applied Mathematics* 105.1 (2000), pp. 99–113.

[10] Jelena Sedlar and Riste Škrekovski. *Vertex and edge metric dimensions of cacti*. 2021. arXiv: 2107.01397 [math.CO].

[11] Jelena Sedlar and Riste Škrekovski. "Bounds on metric dimensions of graphs with edge disjoint cycles". In: *Applied Mathematics and Computation* 396 (2021).

[12] Special Graphs Work Group AIM. "Zero forcing sets and the minimum rank of graphs". In: *Linear Algebra and its Applications* 428.7 (2008), pp. 1628–1648. ISSN: 0024-3795.

[13] Maguy Trefois and Jean-Charles Delvenne. "Zero forcing number, constrained matchings and strong structural controllability". In: *Linear Algebra and its Applications* 484 (2015), pp. 199–218.

[14] Gary Chartrand, Linda Lesniak, and Ping Zhang. *Graphs and Digraphs*. 6th. Chapman and Hall/CRC, 2015. ISBN: 1498735762.

[15] Christina J. Edholm et al. "Vertex and edge spread of zero forcing number, maximum nullity, and minimum rank of a graph". In: *Linear Algebra and its Applications* 436.12 (2012), pp. 4352–4372.

A Proof of Lemma 13

The proof of this lemma in the paper of Eroh et al. [8] contains a flaw. We present here a correction of the proof based on the same general ideas.

Lemma 13. *Let $G = (V, E)$ be a graph and C be a cycle of G. Let $V(C) = \{v_0, v_1, ...v_k\}$ be the vertices of C. Denote by $G_i = (V_i, E_i)$ the connected components of the vertex u_i in $G \setminus E(C)$. If, for every $i \neq j$, $V_i \cap V_j = \emptyset$, then, for any $e \in E(C)$, $\dim(G) \leq \dim(G - e) + 1$.*

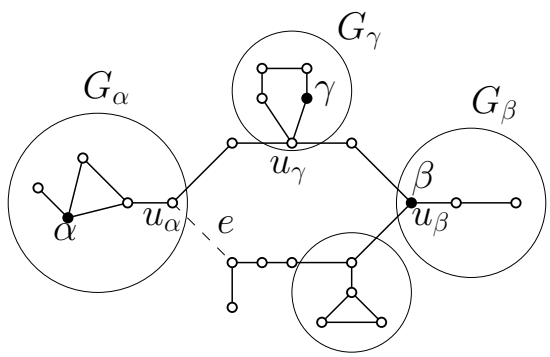


Figure 11: The set $\{\alpha, \beta, \gamma\}$ resolves any pair of vertices in two different circles

Proof. Let S be a metric basis of $G - e$ and $S_i = S \cap V_i$ for $1 \leq i \leq k$. Assume without loss of generality that $S_1 \neq \emptyset$. If $S_i = \emptyset$ for all $i \neq 1$, then $(G - e) \setminus G_1$ is a path and $S_1 \cup \{u_2\}$ is a resolving set of G. If there exists two non-empty subsets S_i and S_j such that $d_G(u_i, u_j) = \frac{k}{2}$, then let $\alpha_0 \in V(C) \setminus \{u_i, u_j\}$ and $S_0 = \{\alpha_0, \alpha_i, \alpha_j\}$ where $\alpha_i \in S_i$ and $\alpha_j \in S_j$. Otherwise, let $i \neq 1$ be such that $S_i \neq \emptyset$, let $\alpha_0 = u_{\lfloor \frac{k}{2} \rfloor}$ and $S_0 = \{\alpha_0, \alpha_i, \alpha_1\}$ where $\alpha_1 \in S_1$ and $\alpha_i \in S_i$. Then, we prove that $S \cup \{\alpha_0\}$ is a resolving set of G.
Let $v \in G_i$ and $w \in G_j$ with $i \neq j$ be two vertices in G. We show that one vertex in S_0 resolves the pair (u, v). For simplicity, rename $S_0 = \{\alpha, \beta, \gamma\}$ with $\alpha \in G_\alpha$, $\beta \in G_\beta$ and $\gamma \in G_\gamma$ with $d(u_\alpha, u_\beta) = \lfloor \frac{k}{2} \rfloor$ as in Figure 11.

Consider first the case where $u_v \notin \{u_\alpha, u_\beta, u_\gamma\}$ and $u_w \notin \{u_\alpha, u_\beta, u_\gamma\}$. Then, we have the following equalities:

$$d(v, \alpha) = d(w, \alpha) \text{ gives } d(v, u_v) + d(u_v, u_\alpha) + d(u_\alpha, \alpha) = d(w, u_w) + d(u_w, u_\alpha) + d(u_\alpha, \alpha);$$

$$d(v, \beta) = d(w, \beta) \text{ gives } d(v, u_v) + d(u_v, u_\beta) + d(u_\beta, \beta) = d(w, u_w) + d(u_w, u_\beta) + d(u_\beta, \beta);$$

$$d(v, \gamma) = d(w, \gamma) \text{ gives } d(v, u_v) + d(u_v, u_\gamma) + d(u_\gamma, \gamma) = d(w, u_w) + d(u_w, u_\gamma) + d(u_\gamma, \gamma).$$

Deleting the terms on the form $d(u_\alpha, \alpha)$ and equalizing we get

$$d(v, u_v) - d(w, u_w) = d(u_w, u_\alpha) - d(u_v, u_\alpha) = d(u_w, u_\beta) - d(u_v, u_\beta) = d(u_w, u_\gamma) - d(u_v, u_\gamma).$$

If $d(v, u_v) - d(w, u_w) = 0$, then, by Lemma 12, $u_v = u_w$ as u_v and u_w are at the same distance to three points on the cycle. Else we have

$$d(u_\alpha, u_w) + d(u_w, u_\beta) - d(u_\alpha, u_v) + d(u_v, u_\beta) = 2(d(v, u_v) - d(w, u_w)),$$

which is a contradiction as $d(u_\alpha, u_\beta) = \lfloor \frac{k}{2} \rfloor$, the difference is in $\{-1, 0, 1\}$. So we get $u_v = u_w$.

Consider now the case of one vertex u_v or u_w is equal to u_α or u_β. Assume without loss of generalities that $u_v = u_\alpha$ and $u_w \neq u_\alpha$. Then $d(v, \alpha) \leq d(v, u_v) + d(u_\alpha, \alpha)$ and as $d(w, \alpha) = d(w, u_w) + d(u_w, u_\alpha) + d(u_\alpha, \alpha)$, we get:

$$d(w, u_w) + d(u_w, u_\alpha) \leq d(v, u_v).$$

Consider now the distances to β: $d(v,\beta) = d(v,u_v) + \lfloor \frac{k}{2} \rfloor + d(u_\beta,\beta)$ and $d(w,\beta) \leq d(w,u_w) + d(u_w,u_\beta) + d(u_\beta,\beta)$. As $d(v,\beta) = d(w,\beta)$:

$$d(v,u_v) + \lfloor \frac{k}{2} \rfloor \leq d(w,u_w) + d(u_w,u_\beta).$$

Thus, $d(v,u_v) \leq d(w,u_w) + d(u_w,u_\beta) - \lfloor \frac{k}{2} \rfloor$, and so

$$d(u_w,u_\alpha) \leq d(u_w,u_\beta) - \lfloor \frac{k}{2} \rfloor.$$

We assume $u_w \neq u_\alpha$ so $d(u_w,u_\alpha) \geq 1$ and $d(u_w,u_\beta) \leq \lfloor \frac{k}{2} \rfloor$ by definition of k. We get a contradiction as $d(u_w,u_\beta) \geq 0$.
The last case is $u_v = u_\gamma$ and $u_w \notin \{u_\alpha, u_\beta, u_\gamma\}$. We have $d(v,u_v)+d(u_v,u_\alpha) = d(w,u_w)+d(u_w,u_\alpha)$ and $d(v,u_v) + d(u_v,u_\beta) = d(w,u_w) + d(u_w,u_\beta)$ and summing the two equalities gives:

$$2d(v,u_v) + d(u_v,u_\alpha) + d(u_v,u_\beta) = 2d(w,u_w) + d(u_w,u_\alpha) + d(u_w,u_\beta),$$

which implies $d(v,u_v) = d(w,u_w)$. Then, by triangular inequality $d(v,\gamma) \leq d(v,u_v) + d(u_\gamma,\gamma)$. Since $d(v,\gamma) = d(w,\gamma)$, we get $d(w,u_w) + d(u_w,u_\gamma) \leq d(v,u_v)$ and thus $d(u_w,u_\gamma) \leq 0$ which is a contradiction.
We proved that, if v and w are not in the same subgraph, then one vertex in S_0 resolves them. So, if v and w are in G_i for some i, by definition of S, there exists $\mu \in S$ which resolve (v,w) in $G-e$. Hence, μ still resolves (v,w) in G: If $\mu \in G_i$, then the distances are the same in G and in $G-e$. If $\mu \notin G_i$, then by hypothesis $d_{G-e}(v,\mu) \neq d_{G-e}(w,\mu)$. By decomposition $d_{G-e}(v,u_v) + d_{G-e}(u_v,\mu) \neq d_{G-e}(w,u_w) + d_{G-e}(u_w,\mu)$ so $d_{G-e}(v,u_v) \neq d_{G-e}(w,u_w)$. As $d_{G-e}(v,u_v) = d_G(v,u_v)$ and $d_{G-e}(w,u_w) = d_G(w,u_w)$, we get $d_G(v,u_v) + d_G(u_v,\mu) \neq d_G(w,u_w) + d_G(u_w,\mu)$. So μ resolves (v,w). Finally, $S \cup \{\alpha_0\}$ is a resolving set of G, and so $\dim(G) \leq \dim(G-e) + 1$.

□