Generalized range of slow random walks on trees

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Abstract: In this work we are interested in the set of visited vertices of a tree $\mathbb{T}$ by a randomly biased random walk $\mathbb{X} := (X_n, n \in \mathbb{N})$. The aim is to study a generalized range, that is to say the volume of the trace of $\mathbb{X}$ with both constraints on the trajectories of $\mathbb{X}$ and on the trajectories of the underlying branching random potential $\mathbb{V} := (V(x), x \in \mathbb{T})$. Focusing on slow regime's random walks (see [HS16b], [AC18]) we prove a general result and detail examples. These examples exhibit many different behaviors for a wide variety of ranges, showing the interactions between trajectories of $\mathbb{X}$ and the ones of $\mathbb{V}$.

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1. Introduction

The construction of the process we are interested in starts with a supercritical Galton-Watson tree $\mathbb{T}$ with offspring distributed as a random variable ν such that $\mathbb{E}[\nu] > 1$. We adopt the following usual notations for tree-related quantities: the root of $\mathbb{T}$ is denoted by e, for any $x \in \mathbb{T}$, ν_x denotes the number of descendants of x, the parent of a vertex x is denoted by x^* and its children by $\{x^i,\ 1 \le i \le \nu_x\}$. For technical reasons, we add to the root e, a parent e^* which is not considered as a vertex of the tree. We denote $|x|$ the generation of x, that is the length of the path from e to x and we write $x < y$ when y is a descendent of x, also $x \le y$ signifying that x can also be equal to y. Finally, we write $\mathbb{T}_n$ for the tree truncated at generation n. We then introduce a real-valued branching random walk indexed by $\mathbb{T}$: $(V(x), x \in \mathbb{T})$. We suppose that $V(e) = 0$ and for any generation n, conditionally to $\mathscr{E}_n = \{\mathbb{T}_n, (V(x), x \in \mathbb{T}_n)\}$, the vectors of increments $((V(x^i) - V(x), i \le \nu_x), |x| = n)$ are assumed to be i.i.d. Finally we denote $\mathbf{P}$ the distribution of $\mathscr{E} = \{\mathbb{T}, (V(x), x \in \mathbb{T})\}$ and $\mathbf{P}^*$, the probability conditioned on the survival set of the tree $\mathbb{T}$.

We can now introduce the main process of this work which is a random walk $(X_n)_{n\in\mathbb{N}}$ on $\mathbb{T} \cup \{e^*\}$: for a given realization of the environment $\mathscr{E}$, $(X_n)_{n\in\mathbb{N}}$ is a Markov chain with transition probabilities given by

$$\begin{aligned}
&\mathbb{P}^{\mathscr{E}}(X_{n+1} = e | X_n = e^*) = 1\ , \\
\forall x \in \mathbb{T} \setminus \{e^*\},\quad &\mathbb{P}^{\mathscr{E}}(X_{n+1} = x^* | X_n = x) = \frac{e^{-V(x)}}{e^{-V(x)} + \sum_{i=1}^{\nu_x} e^{-V(x^i)}}, \\
\forall j \le \nu_x,\quad &\mathbb{P}^{\mathscr{E}}\big(X_{n+1} = x^j | X_n = x\big) = \frac{e^{-V(x^j)}}{e^{-V(x)} + \sum_{i=1}^{\nu_x} e^{-V(x^i)}}\ .
\end{aligned}$$

The measure $\mathbb{P}^{\mathscr{E}}$ is usually referred to as the quenched distribution of the walk $(X_n)_{n\in\mathbb{N}}$ in contrast to the annealed distribution $\mathbb{P}$ which is the measure $\mathbb{P}^{\mathscr{E}}$ integrated with respect to the law of $\mathscr{E}$:

$$\mathbb{P}(\,\cdot\,) = \int \mathbb{P}^{\mathscr{E}}(\,\cdot\,)\,\mathbf{P}(\,\mathrm{d}\mathscr{E}\,)\,.$$

Similarly $\mathbb{P}^*$ is the annealed probability conditioned on the survival set of the tree $\mathbb{T}$ (defined by replacing $\mathbf{P}$ by $\mathbf{P}^*$ in the above probability). For $x \in \mathbb{T}\cup\{e^*\}$, we use the notation $\mathbb{P}^{\mathscr{E}}_x$ for the conditional probability $\mathbb{P}^{\mathscr{E}}(\cdot|X_0 = x)$; when there is no subscript, the walk is supposed to start at the root e. Recurrent criteria for these walks is determined from the fluctuations of log-Laplace transform

$$\psi(s) := \log \mathbf{E}\Big(\sum_{|z|=1} e^{-sV(z)}\Big),$$

for $s > 0$. If $\inf_{0\le s\le 1}\psi(s) > 0$ then (X_n, n) is $\mathbf{P}$ almost surely transient and recurrent otherwise. It turns out that recurrent cases can be themselves classified, this can be found in the works of G. Faraud [Far11] and equivalently for transient cases in E. Aidekon [Aïd08].

Here we consider recurrent cases and more particularly in the regime where the random walk is particularly slow (see [HS16b]) that is to say we put ourselves in the boundary case for which

$$\psi(1) = \psi'(1) = 0. \tag{1}$$

We focus on a generalization of the range defined as follows : for any n, let $\mathbf{f}^n = \{f^{n,k} : \mathbb{R}^k \to \mathbb{R}_+;\ k \in \mathbb{N}^*\}$ be a collection of bounded functions. Also let $g_n : \mathbb{R}_+ \to \mathbb{R}$ a positive function. Then generalized range $\mathscr{R}_n(g_n, \mathbf{f}^n)$ is given by

$$\mathscr{R}_n(g_n, \mathbf{f}^n) := \sum_{x\in\mathbb{T}} g_n(\mathscr{L}^n_x) f^{n,|x|}(V(x_1), V(x_2), \cdots, V(x)),\ \text{with} \tag{2}$$

$$\mathscr{L}^n_x := \sum_{k=1}^{n} \mathbb{1}_{\{X_k = x\}},$$

$(x_i, i \le |x|)$ being the sequence of vertices of the unique path from the root (excluded) to vertex x and $\mathscr{L}^n_x$ is the usual local time of the walk at x before the instant n. As we may see $\mathscr{R}_n(g_n, \mathbf{f}^n)$ is quite general and can not be treated in this form at once for any of these functions g_n and $\mathbf{f}^n$ so additional assumptions (involving $\mathbf{f}^n$, g_n and distribution $\mathbf{P}$) will be introduced in Section 1.2.
The aim of studying this extended range is twofold, first it allows to understand the interactions between trajectories of the main process $\mathbb{X}$ and of the underlying branching potential $\mathbb{V}$, second we develop a general tool allowing to treat many examples (for chosen $\mathbf{f}^n$ and g_n). Note, for example, that if we take for any n, trivial $\mathbf{f}^n = 1$ and for any u, $g_n(u) = \mathbb{1}_{u\ge 1}$ then we get the regular range (treated in [AC18]), and if $g_n(u) = \mathbb{1}_{u\ge n^b}$ with $0 < b < 1$ we get the heavy-range (see [AD20] and [Che20]).
The presentation of the results is divided into two subsections, in the first one below we detail and comment particular examples showing a large variety of behaviors for different $\mathbf{f}^n$ and g_n. In a second subsection we present a general result including all previous examples, its statement will need additional technical assumptions.

1.1. First results : examples

The first two theorems (Theorems 1.1 and 1.2), we present in this section, derive from three other works : in the first one [HS16a] it is proved that, during its first n steps, the walk can reach height of

potential of order $(\log n)^2$. More precisely it is proved that random variable ${}^{\max_{1\leq k\leq n} V(X_k)}/_{(\log n)^2}$ converges almost surely to one half. Note that this behavior can be quite disappointing if we have in mind the intuitive behavior of Sinai's one dimensional random walk in random environment [Sin82] for which the highest height of potential reached by the walk is of order $\log n$. Of course the fact that the walk evolves on a tree instead of a one dimensional lattice changes the deal but at the same time it is also proved in [HS16b] that this walk has a similar behavior than Sinai's one (they are both at a distance of order $(\log n)^2$ from the origin at a given instant n). In both cases, the potential plays a crucial role. In the two other papers ([AC18] and [AD20]) the range is studied : in [AC18] it is proved that regular range (the number of visited vertices before the instant n) is of order ${}^{n}/_{\log n}$, whereas in [AD20] it is proved that the number of edges visited more than n^b (with $0 < b < 1$) times is typically of order n^{1-b} (this particular range is called "heavy range" in that paper, see also [Che20] for a refinement of this work).

Our first theorem below mixes the two approaches showing the influence of a strong constraint on V on both regular or heavy range. What we mean by strong constraint here is a condition of the form $V \geq (\log n)^\alpha$ with $1 < \alpha < 2$, that is to say when the potential is larger than what we can call regular height of potential for this walks (that is height of order $\log n$, as it can be proved that ${}^{V(X_n)}/_{\log n}$ converges weakly, see [HS16b]) but smaller than the extreme value $(\log n)^2$ of [HS16a]. Before stating this result, let us introduce the following hypothesis on distribution of branching random walk : there exists $\theta > 0$ and $\delta_1 \in (0, 1/2]$ such that

$$\mathbf{E}\Big[\sum_{|z|=1} e^{-(1+\theta)V(z)}\Big] + \mathbf{E}\Big[\sum_{|z|=1} e^{\theta V(z)}\Big] < \infty, \tag{3}$$

$$\mathbf{E}\Big[\Big(\sum_{|z|=1} (1+|V(u)|)e^{-V(u)}\Big)^2\Big] + \mathbf{E}\Big[\Big(\sum_{|z|=1} e^{-(1-\delta_1)V(x)}\Big)^2\Big] < \infty, \tag{4}$$

these are common hypothesis used for example in [AC18].

Theorem 1.1. *Assume (1), (3) and (4) hold. If for any n and k, $f^{n,k}(t_1, t_2, \cdots, t_k) = \mathbb{1}_{\{t_k \geq (\log n)^\alpha\}}$ with $\alpha \in (1, 2)$ and if $g_n(t) = \mathbb{1}_{\{t\geq n^b\}}$ with $b \in [0, 1)$, then*

$$\tfrac{\log^+ \mathscr{R}_n(g_n, \mathbf{f}^n) - (1-b)\log n}{(\log n)^{\alpha-1}} \text{ converges in } \mathbb{P}^*\text{-probability to } -1,$$

where $\log^+ x = \log(\max(1, x))$.

This result shows that the number of vertices with high potential visited at least once (resp. strongly visited, with $b > 0$) is of the same order, though smaller, than the regular range (resp. heavy-range). So visiting hight potential is not just an accident appearing a couple of times on very specific paths of the tree. Far from that in fact as the constraint of high potential creates a decrease of order $e^{-(\log n)^{\alpha-1}+o(1)}$ and therefore appears as a second order correction comparing to ranges without constraint on the environment.

In the second theorem below we add a slight different constraint which force the random walk to reach a high level of potential far from the ultimate visited vertices of given paths :

Theorem 1.2. *Assume (1), (3) and (4) hold. If for any n and k, $f^{n,k}(t_1, t_2, \cdots, t_k) = \mathbb{1}_{\{t_{\lfloor k/\beta \rfloor} \geq (\log n)^\alpha\}}$ with $\beta > 1$, $\alpha \in (1, 2)$ and for any $b \in [0, 1)$, $g_n(t) = \mathbb{1}_{\{t\geq n^b\}}$ then*

$$\tfrac{\log^+ \mathscr{R}_n(g_n, \mathbf{f}^n) - (1-b)\log n}{(\log n)^{\alpha-1}} \text{ converges in } \mathbb{P}^*\text{-probability to } -1 - \tfrac{\pi}{2}\sqrt{\beta-1} + \rho\Big((\beta-1)\tfrac{\pi^2}{4}\Big),$$

where for any $c > 0$,

$$\rho(c) := \frac{c\sigma}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\frac{c\sigma^2}{2}u}\Big[\frac{2}{u^{1/2}}\mathbb{P}(\overline{\mathfrak{m}}_1 > 1/\sqrt{u\sigma^2}) - \frac{1}{2}\int_u^{+\infty} \frac{1}{y^{3/2}}\mathbb{P}(\overline{\mathfrak{m}}_1 > 1/\sqrt{y\sigma^2})dy\Big]du,$$

where $\mathfrak{m}$ *is the Brownian meander,* $\overline{\mathfrak{m}}_1 := \sup_{s\leq 1}\mathfrak{m}_s$ *and* $\sigma^2 := \mathbf{E}[\sum_{|x|=1} V^2(x)e^{-V(x)}]$. *Also* $\lfloor x\rfloor$ *stands for the integer part of* x.

As we may see, a slight change in function $\mathbf{f}^n$ (comparing to previous theorem), makes appear something new as the constant in the limit is very different than in Theorem 1.1. Note that ρ can be explicitly calculated : for any $c>0$

$$\rho(c) = 2\sqrt{c}\Big(\frac{1-e^{-\sqrt{c}}}{\sinh(\sqrt{c})}\Big) - 2\Big(\sqrt{c} - \log((e^{\sqrt{c}}+1)/2)\Big), \tag{5}$$

so clearly we obtain continuity when β converges to 1 getting back previous theorem. At this point we would like also to discuss the appearance of Brownian meander distribution in ρ. First note that a Brownian meander appears in the asymptotic distribution of the (correctly normalized) generation of X_n (see [HS16b]) which is the consequence of an induced constraint on the largest downfall of V (we call here maximal downfall, for a given $x\in\mathbb{T}$, the quantity $\max_{y\leq x}(\overline{V}(y)-V(y)$, where $\overline{V}(y) := \max_{z\leq y} V(z)$) visited by the walk before instant n. Also in [AC18] the distribution of two independent Brownian meanders ($\mathfrak{m}^1$ and $\mathfrak{m}^1$) appears in the result for the regular range $\mathscr{R}_n$ (that is when $\mathbf{f}^n=1$ and for any u, $g_n(u)=\mathbb{1}_{u\geq 1}$) : in $\mathbb{P}^*$-probability

$$\lim_{n\to+\infty} \mathscr{R}_n \frac{\log n}{n} = C(\mathscr{D}_{\mathfrak{m}^1}, \mathscr{D}_{\mathfrak{m}^2}), \tag{6}$$

one of these Brownian meander coming from the positivity of V (see Fact 4 below) and the other one coming from the fact that for a given visited vertex x the maximum of V (on the unique path from the root to x) is attained pretty near the generation of x.
Here the brownian meander appears as we ask a visited vertex x to have reached a high level of potential in an early generation before the one of x, and it turns out that the constraint of low downfall of V appearing in [HS16b] ($\max_{y\leq x}(\overline{V}(y)-V(y)\leq \log n$) along this kind of path produces this appearance of the Brownian meander. However contrarily to (6), Brownian meander is involved in the correction of the main fluctuation ($e^{-C(\mathscr{D}_{\mathfrak{m}})(\log n)^{\alpha-1}}$) and not just in the constant of the limit ($C(\mathscr{D}_{\mathfrak{m}^1}, \mathscr{D}_{\mathfrak{m}^2})$).

In the third example below we choose f in such a way that an interaction appears between the trajectory of $\mathbb{X}$ and the downfalls of V which have an important role in the behavior of these walks. More particularly let us introduce, for given $\mathbf{t}=(t_1,t_2,\cdots,t_k)$ with k a positive integer, following quantity,

$$H_k(\mathbf{t}) := \sum_{j=1}^{k} e^{t_j - t_k},$$

then we call *sum of exponential downfalls* of V at $x\in\mathbb{T}$ with $|x|=k$ the quantity

$$H_{|x|}(\mathbf{V}_x) := H_{|x|}(V(x_1),\cdots,V(x_k)) = \sum_{i=1}^{|x|} e^{V(x_i)-V(x_k)}, \tag{7}$$

to simplify notation and when there is no possible confusion we will simply write H_x instead of $H_{|x|}(\mathbf{V}_x)$ in the sequel.

Theorem 1.3. *Assume* (1), (3) *and* (4) *hold.*
For any n *and* k *let* $f^{n,k}(t_1,t_2,\cdots,t_k) = \mathbb{1}_{\{t_k\geq a(\log n)^\alpha\}}(\sum_{j=1}^k H_j(\mathbf{t}))^{-d}$ *with* $\alpha\in[1,2)$, $a\in\mathbb{R}$, $d\in\{0,1\}$ *and* $g_n(t)=\mathbb{1}_{\{t\geq n^b\}}$ *for* $b\geq 0$, *then :*
If $b\in[0,1/(1+d))$ *and* $\alpha=1$ *(with* $a>1/\delta_1$ *when* $d=1$*) then*

$$\frac{\log^+ \mathscr{R}_n(g_n,\mathbf{f}^n)}{\log n} \text{ converges in } \mathbb{P}^*\text{-probability to } 1-(1+d)b,$$

otherwise if $a=1$, $b=0$, $d=1$ *and* $1<\alpha<2$

$$\frac{\log^+ \mathscr{R}_n(g_n,\mathbf{f}^n)-\log n}{(\log n)^{\alpha/2}} \text{ converges in } \mathbb{P}^*\text{-probability to } -2,$$

finally if $a=1$, $0<b<1/2$, $d=1$ *and* $1<\alpha<2$

$$\frac{\log^+ \mathscr{R}_n(g_n,\mathbf{f}^n)-(1-2b)\log n}{(\log n)^{\alpha-1}} \text{ converges in } \mathbb{P}^*\text{-probability to } -1/b.$$

For the first limit (when $\alpha=1$, implying that we have set a common height of potential - see Fact 1) by taking $d=0$ we obtain the limit $(1-b)$ of the usual heavy range of [AD20] otherwise if we add the penalization with the cumulative exponential downfalls ($\sum_{y\le x} H_y$) that is when ($d=1$) then an extra cost $d*b=b$ appears. This example can be used as a point of comparison to the two other more interesting cases that follow.
The second case (with $b=0$ but $1<\alpha<2$), has two constraints on the environment so the normalization $(\log n)^{\alpha/2}$ appears as a compromise between the fact that high level of potential is asked ($\mathbb{1}_{\{t_k\ge(\log n)^\alpha\}}$), which alone yields by Theorem 1.1 a normalization $(\log n)^{\alpha-1}$, but at the same time cumulative exponential downfall fluctuations ($\sum_{m\le k} H_m(\mathbf{t})$) can not be two large as it appears in the denominator of the range, this yields the $(\log n)^{\alpha/2}$ (note that as $\alpha<2$, $\alpha/2>\alpha-1$).
For the last case ($0<b<1/2$ and $1<\alpha<2$), the range is of order $n^{1-2b}e^{-(\log n)^{\alpha-1}/b}$ comparing to $ne^{-2(\log n)^{\alpha/2}}$ when $b=0$ of the previous case. In particular the parameter b of the heavy range appears in both the main normalization n^{1-2b} and in the correction $e^{-(\log n)^{\alpha-1}/b}$. This can be intuitively understand as follows : first $n^{1-2b}=n*n^{-b}*n^{-b}$, one n^{-b} is classical from the heavy range when asking for a local time to be larger than n^b (which already appears in the first part of the Theorem), the second n^{-b} comes from the fact that a local time at a given vertex x can be larger than n^b only if $\sum_{j=1}^{|x|} e^{V(x_j)-V(x)}\ge n^b$ and as this appears in the normalization of the range (via $f^{n,k}(t_1,t_2,\cdots,t_k)$) this produced this second n^{-b}. So this part (n^{1-2b}) appears as a first interaction between the constraints on the trajectory of $\mathbb{X}$ and the one of $\mathbb{V}$. Let us now discuss about $e^{-(\log n)^{\alpha-1}/b}=e^{-\frac{(\log n)^\alpha}{b\log n}}$ for this term we see, intuitively, the constrains for the walk to reach height of potential of order $(\log n)^\alpha$ but a the same time, in order to keep the denominator $\sum_{j\le} H_j(\mathbf{t})$ as low as possible, the maximal downfalls has to remain smaller than $b\log n$ this produces the ratio $(\log n)^\alpha/(b\log n)$.

In the ultimate example below we ask similar constraints for the environment than above but only in the early visited generations :

Theorem 1.4. *Assume* (1), (3) *and* (4) *hold. Let* $\beta>1$. *For any* n *and* k *let* $f^{n,k}(t_1,t_2,\cdots,t_k)=\mathbb{1}_{\{t_{\lfloor k/\beta\rfloor}\ge(\log n)^\alpha\}}(\sum_{j=1}^{\lfloor k/\beta\rfloor} e^{t_j-t_k})^{-1}$, $\alpha\in(1,2)$ *and if* $g_n(t)=\mathbb{1}_{\{t\ge n^b\}}$ *with* $b\in(0,1)$,

$$\frac{\log^+ \mathscr{R}_n(g_n,\mathbf{f}^n)-(1-b)\log n}{(\log n)^{\alpha/2}} \text{ converges in } \mathbb{P}^*\text{-probability to } -2.$$

This last theorem just prove that if the factor $((\sum_{j=1}^{\lfloor k/\beta\rfloor} e^{t_j-t_k})^{-1})$ only concerns the beginning of the trajectory, that is the sites at a distance $\lfloor|x|/\beta\rfloor$ of the root (if x is a visited vertex), then things go back to normal: there is no more multiple interactions between $\mathbb{X}$ and $\mathbb{V}$.

We can imagine more examples like the ones we present above (for example we did not act very much on function g_n), but for now let us introduce a more general result with general hypothesis on g_n and $\mathbf{f}^n$.

1.2. A general result

In this section we present a result (see Theorem 1.5) for $\mathscr{R}_n(g_n, \mathbf{f}^n)$ with g_n and $\mathbf{f}^n$ as general as possible including, in particular, previous results. Comparing to previous theorems it is less readable both in its statement and on the complexity of additional assumptions needed to introduce it. These assumptions that we describe below mix in particular $\mathbf{f}^n$ and distribution of V.

First recall the expression of generalized range (2),

$$\mathscr{R}_n(g_n, \mathbf{f}^n) := \sum_{x\in\mathbb{T}} g_n(\mathscr{L}_x^n) f^{n,|x|}(V(x_1), V(x_2), \cdots, V(x)),$$

with $\mathscr{L}_x^n$ the local time of $\mathbb{X}$ at x before the instant n.
We assume that g_n can be written as the product of an indicator function and a function φ which is positive increasing: for any $b \geq 0$ and m, $g_n(m) := \mathbb{1}_{\{m\geq n^b\}}\varphi(m)$. The indicator function is here to include all types of range (regular or heavy). Also we ask the function $t \to \varphi(t)/t$ to be decreasing, so that $\varphi(\mathscr{L}_x^n)$ remains reasonable (at most of the order of the local time itself).
We now introduce more complex assumptions, for that let us first define branching object $\Psi(\cdot)$. Let $0 \vee \lambda' < \lambda$ be real numbers and $k \in \mathbb{N}^*$ an integer, also let $\phi : \mathbb{R}^k \longrightarrow \mathbb{R}$ a bounded function then $\Psi^k_{\lambda,\lambda'}(\phi)$ is defined as a mean of ϕ along the trajectory of V (with constraints) until generation k, that is

$$\Psi^k_{\lambda,\lambda'}(\phi) := \mathbf{E}\Big[\sum_{|x|=k} e^{-V(x)}\phi\left(V\left(x_1\right), \ldots, V(x)\right)\mathbb{1}_{\mathscr{O}_{\lambda,\lambda'}}(x)\Big], \tag{8}$$

where $\mathscr{O}_{\lambda,\lambda'}$ is the set of (λ,λ')-regular lines

$$\mathscr{O}_{\lambda,\lambda'} := \big\{x \in \mathbb{T};\ \max_{j\leq|x|} H_{x_j} \leq \lambda,\ H_x > \lambda'\big\},\quad \text{with } H_{x_j} = \sum_{i=1}^{j} e^{V(x_i)-V(x_j)}, \tag{9}$$

also we denote

$$\mathscr{O}_\lambda := \big\{x \in \mathbb{T};\ \max_{j\leq|x|} H_{x_j} \leq \lambda\big\},\quad \text{and } \Psi^k_\lambda(\phi) := \mathbf{E}\Big[\sum_{|x|=k} e^{-V(x)}\phi\left(V\left(x_1\right), \ldots, V(x)\right)\mathbb{1}_{\mathscr{O}_\lambda}(x)\Big].$$

Note that since $H_x > 1$, for all $\lambda' \leq 1$, $\mathscr{O}_\lambda = \mathscr{O}_{\lambda,\lambda'}$ and $\Psi^k_\lambda(\phi) = \Psi^k_{\lambda,\lambda'}(\phi)$.

The appearance of this set of regular lines $\mathscr{O}_{\lambda,\lambda'}$ is partly inspired from the works of [HS16b] (λ representing extreme exponential downfalls of V related to a reflecting barrier for walk $(X_k, k \leq n)$), and also (for λ') from the constraint on the local time appearing on function g_n. It turns out indeed that constraints on the value of the local time at some site x implies constraints on H_x (this actually appears clearly in Theorem 1.3).
In all the paper, $C_\infty := C_\infty(\{\mathbf{f}^n; n \geq 1\})$ stands for the supremum of $\{\mathbf{f}^n; n \geq 1\}$ that is

$$C_\infty := \sup_{m,\ell} \|f^{m,\ell}\|_\infty.$$

Then, introduce the set

$$\mathscr{U}_b := \big\{\kappa \geq 0;\ \text{for all } k \geq 1, \mathbf{t} \in \mathbb{R}^k, n \geq 1 : \mathbb{1}_{\{H_k(\mathbf{t})>n^b\}} f^{n,k}(\mathbf{t}) \leq C_\infty n^{-\kappa}\big\}, \tag{10}$$

note that $\mathscr{U}_b \neq \emptyset$ because $0 \in \mathscr{U}_b$ and as the supremum is attained let

$$\kappa_b =: \max \mathscr{U}_b. \tag{11}$$

Finally introduce sequence (h_n, n) which is an important quantity appearing on the asymptotic of the logarithm of $(g_n, \mathbf{f}^n)$-range : for any $n \geq 2$

$$h_n := \begin{cases} \left|\log\left(n^{\kappa_b}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k})\right)\right| & \text{if} \quad \exists\, \gamma \in (0,1) : \ \frac{(\log n)^\gamma}{\log\left(n^{\kappa_b}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{k,n})\right)} \to 0 \\ \log n \qquad \text{otherwise} \end{cases} . \tag{12}$$

We discuss about the order of h_n below (see Remark 1), before that let us introduce our first main assumption which is a lower bound involving Ψ.

Assumption 1 (lower bound).
For all $b \in [0,1)$, $\varepsilon > 0$ and n large enough

$$\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k}) \geq \frac{1}{n^{(\kappa_b+\varepsilon)\wedge 1}}, \tag{A1}$$

Remark 1. *By definition of κ_b,*

$$n^{\kappa_b}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k}) \leq C_\infty \sum_{k\geq 1}\Psi^k_n(1) = C_\infty \mathbf{E}\Big[\sum_{x\in\mathbb{T}} e^{-V(x)}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big] \leq C_\infty \ell_n,$$

where the last inequality is a quite elementary fact that will be proved later (see Remark 2). This implies, in particular, that if there exists $0 < \gamma < 0$ such that $(\log n)^\gamma/\log\left(n^{\kappa_b}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{k,n})\right) \to 0$ necessarily $\log(n^{\kappa_b}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k})) < 0$ and $\lim_{n\to+\infty}\log(n^{\kappa_b}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k})) = -\infty$. Moreover in this case, there exists $0 < \gamma < 1$ such that $h_n \geq (\log n)^\gamma$. Also assumption (A1) above ensures, that

$$\log(n^{\kappa_b}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k})) \geq \log\left(\frac{n^{\kappa_b}}{n^{(\kappa_b+\varepsilon)\wedge 1}}\right) \geq -((\kappa_b+\varepsilon)\wedge 1-\kappa_b)\log n \geq -\varepsilon\log n,$$

overall, definition of h_n implies, under (A1), that

$$(\log n)^\gamma \leq h_n \leq \log n.$$

Before going any further let us give an heuristic about the way $\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k})$ appears in the asymptotic of the range.
First introduce for any $k \geq 1$ the k^{th} return time to e^*, $T^k := \inf\{k > T^{k-1}, X_k = e^*\}$ and take $T^0 = 0$, then let $\overline{\mathscr{R}}_{T^n}(g_n, \mathbf{f}^n) := \sum_{x\in\mathbb{T}}\varphi(\mathscr{L}_x^{T^n})\mathbb{1}_{\{\mathscr{L}_x^{T^n}\geq n^b\}} f^{n,|x|}(V_x)\mathbb{1}_{\{\overline{V}(x)\geq A\log n\}}$ with $A > 0$. $\overline{\mathscr{R}}_{T^n}(g_n, \mathbf{f}^n)$ is a version of generalized range where we have replaced the instant n by T^n and we have made appear the additional constraint $\overline{V}(x) \geq A\log n$. Note that it is known (following Lemma 2.1 in [AC18] and its proof at the beginning of Section 4.2) that this additional condition $\mathbb{1}_{\{\overline{V}(x)\geq A\log n\}}$ has no effect on the normalization of the regular range $\mathscr{R}_{T^n}$, that is

Fact 1 : There exists $0 < c_1 = c_1(A) \leq 1$ such that $\lim_{n\to+\infty}\mathbb{P}^*\left(\frac{\mathscr{R}_{T^n}}{\overline{\mathscr{R}}_{T^n}} = c_1\right) = 1$.

So here we typically consider collections of functions $\mathbf{f}^n$ such that $\mathscr{R}_{T^n}(g_n, \mathbf{f}^n)/\overline{\mathscr{R}}_{T^n}(g_n, \mathbf{f}^n) \to Cte > 0$. One of the main gain of this is the fact that relatively high potential yields interesting quasi-independence in the trajectory of (X_n, n).
With this fact we have (see Section 3.1) something like, in probability, $\overline{\mathscr{R}}_{T^n}(g_n, \mathbf{f}^n) \gtrsim n\mathbb{E}^{\mathscr{E}}[\overline{\mathscr{R}}_{T_e}(g_n, \mathbf{f}^n)]$ and thanks to the fact that φ is increasing and to the expression of quenched mean of $\overline{\mathscr{R}}_{T_e}(g_n, \mathbf{f}^n)$, in probability for large n

$$\overline{\mathscr{R}}_{T^n}(g_n, \mathbf{f}^n) \gtrsim n\mathbb{E}^{\mathscr{E}}[\overline{\mathscr{R}}_{T_e}(g_n, \mathbf{f}^n)] \gtrsim \frac{\varphi(n^b)}{n^b} n\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k})$$

which makes appear $\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k})$. This also tells that if this sum is very small (for example by asking excessive constraints on $\mathbb{V}$), this should lead to a very small range. Let us now focus on a second assumption.

Assumption 2 (upper bound).

This assumption ((**A2**) below) is an upper bound for a conditional version of $\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k})$, to introduce it we actually need two facts and additional notations.

Fact 2 : ([AD20], Lemma 2.3) there exists a real number $r_2>0$ such that for any $h>0$

$$\mathbf{P}^*\big(\max_{|w|\leq\lceil h/r_2\rceil}|V(w)|>h\big)\leq he^{-h}. \tag{13}$$

This fact, that will be useful when cutting on early generations of the tree, justifies the introduction of following notation : for any n and k, $f^{n,k}_h$ is the function defined by

$$f^{n,k}_h(t_1,\dots,t_k):=\inf_{\mathbf{s}\in[-h,h]^m}f^{n,k+m}\left(s_1,\dots,s_m,t_1+s_m,\dots,t_k+s_m\right), \tag{14}$$

with $m=\lceil h/r_2\rceil$ and $\mathbf{s}=(s_1,\dots,s_m)\in\mathbb{R}^m$.

The second fact is about the largest generation visited by the walk before time n or after n excursions to vertex e^*.

Fact 3 : ([AD20], Lemma 3.2) let $(\ell_n:=(\log n)^3, n\geq 2)$, there exists $A>0$ such that :

$$\lim_{n\to+\infty}\mathbb{P}(\max_{k\leq T^n}|X_k|\leq A\ell_n)=1.$$

This fact is here essentially to justify the introduction of the sequence (ℓ_n,n) which appears in our second assumption. Note that a very precise result on the largest generation visited by the walk before the instant n can be found in [FHS11] .

A last notation we need to introduce, is a conditional and translated version of $\Psi^k_\lambda(F)$ for a given bounded function F. Let $k\in\mathbb{N}^*$, if $l\in\mathbb{N}^*$, $F:\mathbb{R}^{l+k}\longrightarrow\mathbb{R}$, for all $\mathbf{t}=(t_1,\dots,t_l)\in\mathbb{R}^l$

$$\Psi^k_{\lambda,\lambda'}(F|\mathbf{t}):=\mathbf{E}\Big[\sum_{|x|=k}e^{-V(x)}F(t_1,\dots,t_l,V(x_1)+t_l,\dots,V(x)+t_l)\mathbb{1}_{\mathscr{O}_{\lambda,\lambda'}}(x)\Big] \tag{15}$$

otherwise if $l=0$, $\Psi^k_{\lambda,\lambda'}(F|\mathbf{t}):=\Psi^k_{\lambda,\lambda'}(F)$.

We are now ready to introduce the second assumption : for all $\delta,\varepsilon,A,B>0$ and $b\in[0,1)$ there exists $n_0\in\mathbb{N}^*$ such that for any $n\geq n_0$, $l\leq\lfloor A\ell_n\rfloor$ and any $\mathbf{t}=(t_1,\dots,t_l)\in\mathbb{R}^l$ with $t_l\geq -B$ and $H_l(\mathbf{t})\leq n$

$$\sum_{k\geq 1}\Psi^k_{n,n^b-H_l(\mathbf{t})}\big(f^{n,l+k}_{\varepsilon h_n}|\mathbf{t}\big)\leq e^{\delta t_l+\frac{\varepsilon}{A}h_n}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k}). \tag{**A2**}$$

Let us comment this inequality which plays two roles. A first one ensures that fluctuations of V in the early generations of the tree have minor influence, this yields the presence of $e^{\frac{\varepsilon}{A}h_n}$.

Second point is technical and aims to show that $\mathbb{E}^{\mathscr{E}}[\overline{\mathscr{R}}_{T_e}(g_n,\mathbf{f}^n)]\gtrsim n^{-b}\varphi(n^b)\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k})$ in probability. For that, the second moment of

$$Z_n:=\sum_{x\in\mathscr{O}_{n,n^b}}e^{-V(x)}f^{n,k}_{\varepsilon h_n}(\boldsymbol{V}_x)\mathbb{1}_{\{\overline{V}(x)\geq A\log n,\underline{V}(x)\geq -B,\overline{V}(x)=V(x)\}}$$

has to be controlled, with $\boldsymbol{V}_x:=(V(x_1),\dots,V(x))$ and $\underline{V}(x):=\min_{v\leq x}V(v)$. Observe that

$$Z_n^2\approx\sum_{z\in\mathbb{T}}\sum_{x,y>z}\prod_{u\in\{x,y\}}\mathbb{1}_{\{u\in\mathscr{O}_{n,n^b}\}}e^{-V(u)}f^{n,|u|}_{\varepsilon h_n}(\boldsymbol{V}_u)\mathbb{1}_{\{V(u)\geq A\log n,\underline{V}(u)\geq -B,\overline{V}(u)=V(u)\}}.$$

$\sum_{k\geq 1}\Psi^k_{n,n^b-H_l(\mathbf{t})}(f^{n,l+k}_{\varepsilon h_n}|\mathbf{t})$ in (**A2**) actually appears as the conditionnal expectation of a well chosen function of translated potential $(V_z(u):=V(u)-V(z))_{u>z}$ in the previous expression of Z_n^2. Indeed, note that $u\in\mathscr{O}_{n,n^b}$ together with $\overline{V}(u)=V(u)$ implies that $u\in\mathscr{O}^z_{n,n^b-H_z}:=\{u>z:\max_{z<v\leq u}H_{z,v}\leq n, H_{z,u}>n^b-H_z\}$ with $H_{z,v}:=\sum_{z<w\leq v}e^{V_z(w)-V_z(v)}$. Hence, for all $\delta\in(0,1/2]$, by independence of the increments of branching random walk $(\mathbb{T},V(u);u\in\mathbb{T})$

$$\begin{aligned}\mathbf{E}[Z_n^2]&\lesssim e^{(1-2\delta)B}\mathbf{E}\Big[\sum_{z\in\mathscr{O}_n}e^{-V(z)}\sum_{x,y>z}\prod_{u\in\{x,y\}}\mathbb{1}_{\{u\in\mathscr{O}^z_{n,n^b-H_z}\}}e^{-V_z(u)}F^{n,|u|}_{\boldsymbol{V}_z}(V_z(u_{|z|+1}),\dots,V_z(u))\Big]\\&\approx e^{(1-2\delta)B}\mathbf{E}\Big[\sum_{z\in\mathscr{O}_n}e^{-V(z)}\Big(e^{\delta V(z)}\sum_{k\geq 1}\Psi^k_{n,n^b-H_z}(f^{n,l+k}|\boldsymbol{V}_z)\Big)^2\Big],\end{aligned}$$

where, for $|z|=l$ and any $\mathbf{t}=(t_1,\dots,t_l)\in\mathbb{R}^l$

$$F^{n,|u|}_{\mathbf{t}}(V_z(u_{l+1}),\dots,V_z(u)):=e^{\delta t_l}f^{n,|u|}_{\varepsilon h_n}(t_1,\dots,t_l,V_z(u_{l+1})+t_l,\dots,V_z(u)+t_l).$$

Assumtions (**A2**) finally allows to say that $\mathbf{E}[Z_n^2]\lesssim e^{\varepsilon h_n}(\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k}))^2$ for all $\varepsilon>0$ and n large enough.

We are now almost ready to state a first result which is a proposition giving a lower and upper bound for the generalized range stopped at T^n, this proposition is followed by a theorem more easy to read but that needs extra assumptions. First, let us introduce two ultimate notations, the first one is a positive sequence (v_n,n), defined as follows, for any $n\geq 2$:

$$v_n:=\frac{1}{\delta_1}\log(n\ell_n),\tag{16}$$

(see (4) for the definition of δ_1) and for $v>0$, introduce

$$\mathscr{H}^k_{v_n}=\left\{(t_1,\dots,t_k)\in\mathbb{R}^k;\ t_k\geq v\right\},\ \mathscr{H}^k_{B,v_n}=\{(t_1,\dots,t_k)\in\mathbb{R}^k;\ t_k\geq v_n,\min_{i\leq k}t_i\geq -B\}\tag{17}$$

respectively the set of vectors such that its last coordinate is larger than v_n and additionally with any coordinates larger than $-B$. The introduction of these last two objects are justified by

Fact 4 : for any $\epsilon>0$ there exists $a>0$ such that (see [Aïd13])

$$\mathbf{P}(\inf_{u\in\mathbb{T}}V(u)<-a)\leq\epsilon,$$

and Fact 1 we have already talked about saying that, in $\mathbb{P}^*$-probability, v_n is a height of potential usually reach by the walk.

Proposition 1. *Recall* (11), *let* $\varepsilon_b:=\min(b+\mathbb{1}_{\{b=0\}},1-b)/13$ *and* $W:=\sum_{|z|=1}e^{-V(z)}$. *Assume* (1), (3) *and* (4) *hold as well as* (**A1**) *and* (**A2**).
Lower bound: there exists $c_5>0$ *such that for all* $b\in[0,1)$, $\varepsilon\in(0,\varepsilon_b)$, $B>0$ *and* n *large enough*

$$\mathbb{P}\Big(\frac{\mathscr{R}_{T^n}(g_n,\mathbf{f}^n)}{n^{1-b}\varphi(n^b)u_{1,n}}<e^{-5\varepsilon h_n}\Big)\leq\frac{e^{-c_5\varepsilon h_n}}{(u_{1,n})^2}\Big(\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k}_{\varepsilon h_n})\Big)^2+h_ne^{-\varepsilon h_n}+\frac{e^{-\min(\varepsilon\log n,3h_n)}}{(n^{\kappa_b}u_{1,n})^2},\tag{18}$$

with

$$u_{1,n}=u_{1,n}(\varepsilon):=\sum_{k\geq 1}\Psi^k_{\lambda_n/2,n^b}\big(f^{n,k}_{\varepsilon h_n}\mathbb{1}_{\Upsilon^k_n}\big),$$

$\Upsilon_n^k := \{\mathbf{t} \in \mathbb{R}^k;\ H_k(\mathbf{t}) \le n^b e^{\varepsilon h_n}\} \cap \mathscr{H}^k_{B,v'_n}$, $v'_n = v_n + \varepsilon h_n$ *and* $\lambda_n = ne^{-\min(10\varepsilon \log n, 5h_n)}$.
Upper bound: for any $\varepsilon > 0$ *and* n *large enough*

$$\mathbb{P}\Big(\frac{\mathscr{R}_{T^n}(g_n, \mathbf{f}^n)}{n^{1-b}\varphi(n^b)u_{2,n}} > e^{\varepsilon h_n}\Big) \le e^{-\frac{\varepsilon}{2}h_n} + o(1) \tag{19}$$

with

$$u_{2,n} := \sum_{k\ge 1}\big(\Psi^k_{n,n^b}(f^{n,k}) + \Psi^k_n\big(f^{n,k}\mathbb{1}_{\mathbb{R}^k\setminus\mathscr{H}^k_{v_n}}\big) + \Psi^k_{n,\lambda'_n}(f^{n,k}) + \mathbf{E}\big[W\Psi^k_{n,\lambda'_n/W}(f^{n,k})\big]\big)$$

and $\lambda'_n := n^b(\log n)^{-2}$.
(18) *and* (19) *remain true replacing* $\mathscr{R}_{T^n}(g_n, \mathbf{f}^n)$ *by* $\mathscr{R}_{T^{k_n}}(g_n, \mathbf{f}^n)$ *with* $k_n = \lfloor n/(\log n)^p \rfloor$, $p > 0$.

This proposition is technical and difficult to read, we present it here however because it shows that all the estimations depend deeply on $\Psi^{\cdot}_{\cdot,\cdot}(f)$ and g_n, recall indeed that key sequence h_n defined in (12) depends both on $\Psi^{\cdot}_{\cdot,\cdot}(f)$ and κ_b (b coming from function g_n). This also means that without any more informations on $\Psi^{\cdot}_{\cdot,\cdot}(f)$ it is difficult to state a more explicit result. Finally, note that the exact role of (**A1**) and (**A2**) will appear clearly in the proof of the lower bound.
We now present a more readable result involving two additional assumptions (**A3**) and (**A4**). They tell essentially that quantities $u_{1,n}$ and $u_{2,n}$, which appear as the important quantities in the proposition, are actually very similar. Before that, introduce two additional values : L (with $L = \pm\infty$ possibly) and $\xi \in \{-1, 0\}$ defined as follows

$$L := \liminf_{n\to\infty} h_n^{-1}\log\big(n^{1-b-\kappa_b}\varphi(n^b)\big), \text{ and} \tag{20}$$

$$\xi := \lim_{n\to\infty} h_n^{-1}\log\big(n^{\kappa_b}\sum_{k\ge 1}\Psi^k_{n,n^b}(f^{k,n})\big). \tag{21}$$

First note that, following Remark 1, ξ necessarily exists. Now introduce two additional assumptions :

Assumption 3 For all $b \in [0,1)$, $\varepsilon \in (0, \varepsilon_b)$, $\varepsilon_1 \in (0, \epsilon)$ and n large enough

$$u_{1,n} \ge e^{-\varepsilon_1 h_n}\sum_{k\ge 1}\Psi^k_{n,n^b}(f^{n,k}). \tag{**A3**}$$

Assumption 4 For all $\varepsilon_1 > 0$, $b \in [0,1)$ and n large enough

$$u_{2,n} \le e^{\varepsilon_1 h_n}\sum_{k\ge 1}\Psi^k_{n,n^b}(f^{n,k}). \tag{**A4**}$$

Theorem 1.5. *Assume* (1), (3) *and* (4) *hold,* $b \in [0,1)$ *and* (**A1**), (**A2**), (**A3**) *and* (**A4**) *are satisfied, then if* $L \in (-\xi, +\infty]$ *in* $\mathbb{P}^*$*-probability*

$$h_n^{-1}\big(\log^+ \mathscr{R}_n(g_n, \mathbf{f}^n) - \log(n^{1-b-\kappa_b}\varphi(n^b))\big) \underset{n\to\infty}{\longrightarrow} \xi,$$

if $L = -\xi$, *with* $\underline{\Delta}_n := h_n^{-1}\log(n^{1-b-\kappa_b}\varphi(n^b)) - \inf_{\ell\ge n} h_\ell^{-1}\log(\ell^{1-b-\kappa_b}\varphi(\ell^b))$, *in* $\mathbb{P}^*$*-probability*

$$h_n^{-1}\log^+ \mathscr{R}_n(g_n, \mathbf{f}^n) - \underline{\Delta}_n \underset{n\to\infty}{\longrightarrow} 0$$

Otherwise $L \in [-\infty, -\xi[$ *and in* $\mathbb{P}^*$*-probability*

$$\mathscr{R}_{n_\ell}(g_{n_\ell}, \mathbf{f}^{n_\ell}) \underset{\ell\to\infty}{\longrightarrow} 0,$$

for some increasing sequence $(n_\ell)_\ell$ *of positive integers. Note that when* $\lim h_n^{-1}\log(n^{1-b-\kappa_b}\varphi(n^b)) = L$, $n_\ell = \ell$.

We now present examples which lead to different values of L. First note that all theorems presented in the previous section satisfy condition $L = +\infty$ and $\xi = -1$, for our point of view it is the most interesting case.
Let us take, for example, $g_n(x) = \mathbb{1}_{\{x \geq n^b\}}$ and $f^{n,k}(t_1, t_2, \cdots, t_k) = \mathbb{1}_{\{t_k \geq (\log n)^\alpha\}} (\sum_{l \leq k} H_l(\mathbf{t}))^{-1}$, $\alpha \in [1, 2)$. If $\alpha > 1$ and $b \in (1/2, 1)$ we can prove that that $h_n \sim (\log n)^{\alpha-1}/b$ (with usual notation $t_n \sim s_n$ if and only if $t_n/s_n \to 1$) and $n^{1-b-\kappa_b}\varphi(n^b) = n^{1-2b}$ so we obtain $\lim h_n^{-1} \log(n^{1-b-\kappa_b}\varphi(n^b)) = L = -\infty$. When $\alpha > 1$ and $b = 1/2$ then we can prove that $h_n \sim (\log n)^{\alpha-1}/b$, $\xi = -1$ and $n^{1-b-\kappa_b}\varphi(n^b) = 1$ so $\lim h_n^{-1} \log(n^{1-b-\kappa_b}\varphi(n^b)) = L = 0$ and we are in the case $L \in (-\infty, -\xi)$. Otherwise when $\alpha = 1$ and $b = 1/2$, we can prove that $h_n = \log n$, $\xi = 0$ and we are in the case $L = -\xi$.
Let us finally take the simple example $g_n(x) = x\mathbb{1}_{\{x \geq n^b\}}$ and $f^{n,k} = 1$. We can prove that for all $b \in (0, 1)$, $h_n = \log n$, $\xi = 0$ and $n^{1-b-\kappa_b}\varphi(n^b) = n$ so $\lim h_n^{-1} \log(n^{1-b-\kappa_b}\varphi(n^b)) = L = 1$ and we are in the case $L \in (-\xi, +\infty)$.

The rest of the paper is decomposed as follows, in Section 2 after short preliminaries (Section 2.1) we prove theorems of Section 1.1. For these proofs (Section 2.2) we check that the four assumptions **A1**-**A4** of Theorem 1.5 are realized and obtain simultaneously the asymptotic of h_n. In section 2.3 we prove Theorem 1.5 : essentially Proposition 1 is assumed to be true and we only check that if Assumptions **A3** and **A4** are true then the theorem comes.
In Section 3 we prove Proposition 1, this is the most technical part of the paper which can be read independently of the other parts : in Section 3.1 we summarize usual facts, in a second sub-section we prove a lower bound for stopped generalized range $\mathscr{R}_{T^n}(g_n, \mathbf{f}_n)$ and finally in a last one an upper bound.
Finally in the last section we present some estimates about sums of i.i.d. random variables useful for the proof of the examples of Section 1.1.

2. Proof of the theorems

This section is decomposed in three parts, in the first section below one can find preliminaries that are useful all along the rest of the paper. In the second sub-section we prove the four theorems presented as examples, finally last section is devoted to the proof of Theorem 1.5.

2.1. Preliminary material

We recall the many-to-one formula (see [Shi15] Chapter 1, and [FHS11] equation 2.1) which will be used several times in the paper to compute expectations related to the environment. Note that the identity below comes from a change of probability measure (see references above), however we still keep notations $\mathbf{P}$ and $\mathbf{E}$ for simplicity.

Lemma 2.1 (Many-to-one Lemma)**.** *For any $t > 0$,*

$$\mathbf{E}\Big[\sum_{|x|=m} f(V(x_i), 1 \leq i \leq m) \Big] = \mathbf{E}(e^{tS_m + \psi(t)m} f(S_i, 1 \leq i \leq m)).$$

where $(S_n)_{n\in\mathbb{N}}$ is the random walk starting at 0, such that the increments $(S_{n+1} - S_n)_{n\in\mathbb{N}}$ are i.i.d. and for any measurable function $h : \mathbb{R}^m \to [0, \infty)$,

$$\mathbf{E}(h(S_1)) = e^{-\psi(t)} \mathbf{E}\Big(\sum_{|x|=1} e^{-tV(x)} h(V(x)) \Big).$$

A second very useful fact is contained in the following remark, it tells essentially that, in probability, the $e^{-V(x)}$-weighted number of vertices x such that $x \in \mathscr{O}_n$ (recall (9)) can be found in a quite small quantity when $|x| \leq A\ell_n$ and can not be found when $|x| > A\ell_n$. This remark is not precise at all but will be enough for our purpose.

Remark 2. *There exists $c_1 \in (0,1)$ such that for any $A > 0$ and n large enough*

$$\mathbf{E}\Big[\sum_{|x|>\lfloor A\ell_n\rfloor} e^{-V(x)}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big] \leq n^{-Ac_1} \text{ and } \mathbf{E}\Big[\sum_{|x|\leq\lfloor A\ell_n\rfloor} e^{-V(x)}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big] \leq \ell_n/2,$$

which implies $\mathbf{E}\Big[\sum_{x\in\mathbb{T}} e^{-V(x)}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big] \leq \ell_n$.

Proof. We give a proof here which essentially use technical Lemma 4.6 (for the second inequality below), indeed by Lemma 2.1 above

$$\begin{aligned}\mathbf{E}\Big[\sum_{|x|>\lfloor A\ell_n\rfloor} e^{-V(x)}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big] &\leq \sum_{k>\lfloor A\ell_n\rfloor} \mathbb{P}(\sup_{i\leq k}(\overline{S}_i - S_i) \leq \log n)\\ &\leq \sum_{k>\lfloor A\ell_n\rfloor} \exp(-\frac{k\pi^2\sigma^2(1-\epsilon)}{8\log n}) \leq n^{-Ac_1}.\end{aligned}$$

A similar computation gives the second fact and both of them the last one. □

2.2. Proofs of Theorems 1.1 to 1.4

The pattern of the proofs of each theorem is the following : we first prove two facts (an upper and a lower bound) about sum $\sum_{k\geq 1}\Psi^k_{\cdot,\cdot}(F)$ with specific F depending on the considered function $f^{n,k}(t_1,t_2,\cdots,t_k)$ and also slightly different wether we are looking for an upper or a lower bound. Then we use this two facts to prove that (**A1**), (**A2**), (**A3**) and (**A4**) are satisfied.
In these proof we use several time notation $\varepsilon_b = \min(b + \mathbb{1}_{\{b=0\}}, 1-b)/13$ which was introduced in Proposition 1.

Proof of Theorem 1.1. Recall that $f^{n,k}(t_1,t_2,\cdots,t_k) = \mathbb{1}_{\{t_k\geq(\log n)^\alpha\}}$, $\alpha \in (1,2)$ and see (9) for the definition of $\mathscr{O}_{\lambda,\lambda'}$. All along the proof we assume that $B,\delta > 0$, $\varepsilon \in (0,\varepsilon_b)$, n is large enough and $t \geq -B$. Let us start with the proof of the following two facts:

$$\mathbf{E}\Big[\sum_{x\in\mathscr{O}_n} e^{-V(x)}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha - t\}}\Big] \leq e^{\delta t-(\log n)^{\alpha-1}(1-\varepsilon)}, \tag{22}$$

and for any $0 \leq m \leq \log n$

$$\mathbf{E}\Big[\sum_{x\in\mathscr{O}_{\boldsymbol{\lambda}_n,n^b}} e^{-V(x)}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha+m,\, H_x\leq n^b e^{\varepsilon(\log n)^{\alpha-1}},\, \underline{V}(x)\geq -B\}}\Big] \geq e^{-(\log n)^{\alpha-1}(1+\varepsilon)}, \tag{23}$$

with $\boldsymbol{\lambda}_n = ne^{-6(\log n)^{\alpha-1}}$ and recall $\underline{V}(x) = \min_{u\leq x} V(u)$. We first deal with the upper bound (22). Recall $\ell_n = (\log n)^3$,

$$\begin{aligned}\mathbf{E}\Big[\sum_{x\in\mathscr{O}_n} e^{-V(x)}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha - t\}}\Big] \leq &\sum_{k\leq\lfloor A\ell_n\rfloor} \mathbf{E}\Big[\sum_{|x|=k} e^{-V(x)}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha - t\}}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big]\\ &+ \mathbf{E}\Big[\sum_{|x|>\lfloor A\ell_n\rfloor} e^{-V(x)}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big],\end{aligned}$$

where $A>0$ is chosen such that $\mathbf{E}[\sum_{|x|>\lfloor A\ell_n\rfloor} e^{-V(x)}\mathbb{1}_{\{x\in\mathscr{O}_n\}}]\le 1/n$ (see Remark 2). This yields, as $\alpha\in(1,2)$, $\mathbf{E}[\sum_{|x|>\lfloor A\ell_n\rfloor} e^{-V(x)}\mathbb{1}_{\{x\in\mathscr{O}_n\}}]\le e^{\delta(t+B)}\frac{1}{n}\le\frac{1}{2}e^{\delta t-(\log n)^{\alpha-1}(1-\varepsilon)}$ for n large enough and any $t\ge -B$. Thanks to many-to-one Lemma 2.1, the first sum in the above inequality is smaller than

$$\sum_{k\le\lfloor A\ell_n\rfloor}\mathbf{P}\big(S_k\ge(\log n)^\alpha-t,\ \max_{j\le k}\overline{S}_j-S_j\le\log n\big)\le\lfloor A\ell_n\rfloor\mathbf{P}\big(\max_{j\le\tau_{(\log n)^\alpha-t}}\overline{S}_j-S_j\le\log n\big),$$

with $\tau_r=\inf\{i\ge1;S_i\ge r\}$. Then, thanks to Lemma A.3 in [HS16a], and as $t\ge -B$

$$\begin{aligned}\lfloor A\ell_n\rfloor\mathbf{P}\big(\max_{j\le\tau_{(\log n)^\alpha-t}}\overline{S}_j-S_j\le\log n\big)\le\lfloor A\ell_n\rfloor e^{\frac{t}{\log n}-(\log n)^{\alpha-1}(1-\frac{\varepsilon}{2})}&\le\lfloor A\ell_n\rfloor e^{\frac{t+B}{\log n}-(\log n)^{\alpha-1}(1-\frac{\varepsilon}{2})}\\&\le e^{\delta t+\delta B-(\log n)^{\alpha-1}(1-\frac{\varepsilon}{2})}\\&\le\frac{1}{2}e^{\delta t-(\log n)^{\alpha-1}(1-\varepsilon)},\end{aligned}$$

so we get exactly (22).

We now turn to the lower bound (23). Let $\ell_n'=(\log n)^4$ and $\alpha_n=(\log n)^\alpha+\log n$. By many-to-one Lemma, for any $m\le\log n$, expectation in (23) is larger than

$$\sum_{k\ge1}\mathbf{P}\big(S_k\ge\alpha_n,\ \max_{j\le k}H_j^S\le\boldsymbol{\lambda}_n,n^b<H_k^S\le n^be^{\varepsilon(\log n)^{\alpha-1}},\underline{S}_k\ge -B\big),$$

with $H_j^S:=\sum_{i=1}^j e^{S_i-S_j}$. For any $b\in(0,1)$, by Lemma 4.3 (76) (with $\ell=(\log n)^2$, $t_\ell=\alpha_n$, $q=1$, $a_b=a=6$, $d=(\alpha-1)/2$ and $c=\varepsilon$) above sum is larger than $e^{-(\log n)^{\alpha-1}(1+\varepsilon)}$. Otherwise if $b=0$, observe that for all $k\le\ell_n'$, $\overline{S}_k=S_k$ implies $H_k^S\le k\le\ell_n'$ so the sum is larger than $\sum_{k\le\ell_n'}\mathbf{P}\big(S_k\ge\alpha_n,\ \max_{j\le k}H_j^S\le\boldsymbol{\lambda}_n,\overline{S}_k=S_k,\underline{S}_k\ge -B\big)$. Lemma 4.5 (with $\ell=(\log n)^2$, $t_\ell=\alpha_n$, $d=1/2$, $a=6$ and $d'=(\alpha-1)/2$) leads to (23) also for $b=0$.

We are now ready to prove that $\mathbf{f}^n$ satisfies assumptions (**A1**), (**A2**), (**A3**) and (**A4**). Recall that $\Psi^k_{n,n^b}(f^{n,k})=\mathbf{E}\big[\sum_{|x|=k}e^{-V(x)}f^{n,k}(V(x_1),\dots,V(x))\mathbb{1}_{\{x\in\mathscr{O}_{n,n^b}\}}\big]$ where $x\in\mathscr{O}_{n,n^b}$ if and only if $\max_{j\le|x|}H_{x_j}\le n$ and $H_x>n^b$, also $\mathscr{U}_b=\{\kappa\ge0;\text{ for all }k\ge1,\mathbf{t}\in\mathbb{R}^k,n\ge1:\mathbb{1}_{\{H_k(\mathbf{t})>n^b\}}f^{n,k}(\mathbf{t})\le C_\infty n^{-\kappa}\}$ with $C_\infty=\sup_{n,\ell}\|f^{n,\ell}\|_\infty$.

• Check of (**A1**) and asymptotic of h_n. We obtain from (23) with $m=0$ that for any $\varepsilon\in(0,\varepsilon_b)$ and n large enough $\mathbf{E}[\sum_{x\in\mathscr{O}_{n,n^b}}e^{-V(x)}\mathbb{1}_{\{V(x)\ge(\log n)^\alpha\}}]$ is larger than (as $\boldsymbol{\lambda}_n\le n$)

$$\mathbf{E}\Big[\sum_{x\in\mathscr{O}_{\boldsymbol{\lambda}_n,n^b}}e^{-V(x)}\mathbb{1}_{\{V(x)\ge(\log n)^\alpha\}}\mathbb{1}_{\{H_x\le n^be^{\varepsilon(\log n)^{\alpha-1}},\underline{V}(x)\ge -B\}}\Big]\ge e^{-(\log n)^{\alpha-1}(1+\varepsilon)}.$$

Note that, above inequality, implies that for all $b\in[0,1)$, $\kappa_b=\max\mathscr{U}_b=0$, indeed if we had $\kappa_b>0$ then this should imply that for any $x\in\mathbb{T}$

$$e^{-V(x)}\mathbb{1}_{\{x\in\mathscr{O}_{n,n^b}\}}f^{n,k}(V(x_1),\dots,V(x))\le C_\infty n^{-\kappa_b}e^{-V(x)}\mathbb{1}_{\{x\in\mathscr{O}_{n,n^b}\}}$$

which implies that $\mathbf{E}[\sum_{x\in\mathscr{O}_{n,n^b}}e^{-V(x)}\mathbb{1}_{\{V(x)\ge(\log n)^\alpha\}}\mathbb{1}_{\{H_x\le n^be^{\varepsilon(\log n)^{\alpha-1}},\underline{V}(x)\ge -B\}}]$ is smaller than $C_\infty n^{-\kappa_b}\mathbf{E}[\sum_{x\in\mathscr{O}_n}e^{-V(x)}]\le C_\infty\ell_n n^{-\kappa_b}$ by Remark 2, but this contradicts the above lower bound (23) as $\alpha\in(1,2)$.

Then, by definition of Ψ^k_{n,n^b},

$$\sum_{k\ge1}\Psi^k_{n,n^b}(f^{n,k})=\mathbf{E}\Big[\sum_{x\in\mathscr{O}_{n,n^b}}e^{-V(x)}\mathbb{1}_{\{V(x)\ge(\log n)^\alpha\}}\Big]\ge e^{-(\log n)^{\alpha-1}(1+\varepsilon)},$$

and additionally with (22) (taking $t=0$) asymptotic of h_n is given by

$$h_n = \Big| n^{\kappa_b} \log\big(\sum_{k\geq 1} \Psi^k_{n,n^b}(f^{n,k})\big)\Big| = \Big|\log \mathbf{E}\Big[\sum_{x\in\mathscr{O}_{n,n^b}} e^{-V(x)}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha\}}\Big]\Big| \sim (\log n)^{\alpha-1}.$$

We also deduce from the previous lower bound that (**A1**) is satisfied, indeed, as $\alpha\in(1,2)$, $\sum_{k\geq1}\Psi^k_{n,n^b}(f^{n,k}) \geq n^{-(\kappa_b+\varepsilon_1)\wedge 1}$ for any $\varepsilon_1>0$ and n large enough.

• For (**A2**), recalling $m_n = \lceil \varepsilon h_n/r_2\rceil$ (see (13)), then by definition

$$\begin{aligned} f^{n,j}_{\varepsilon h_n}(t_1,\dots,t_j) &= \inf_{\mathbf{s}\in[-\varepsilon h_n,\varepsilon h_n]^{m_n}} f^{n,m_n+j}(s_1,\dots,s_{m_n},t_1+s_{m_n},\dots,t_j+s_{m_n}) \\ &= \inf_{s_{m_n}\in[-\varepsilon h_n,\varepsilon h_n]} \mathbb{1}_{\{t_j+s_m\geq(\log n)^\alpha\}} = \mathbb{1}_{\{t_j\geq(\log n)^\alpha+\varepsilon h_n\}}. \end{aligned}$$

Observe that for $A>0$, n large enough, any $l\in\mathbb{N}$ and $\mathbf{t}=(t_1,\dots,t_l)$, by definition of $\Psi^k_n(F|\mathbf{t})$ (see (15)) and (22) with $\varepsilon/3A$ instead of ε

$$\begin{aligned} \sum_{k\geq1}\Psi^k_{n,n^b-H_l(\mathbf{t})}\big(f^{n,l+k}_{\varepsilon h_n}|\mathbf{t}\big) &= \mathbf{E}\Big[\sum_{x\in\mathscr{O}_n} e^{-V(x)}\mathbb{1}_{\{V(x)+t_l\geq(\log n)^\alpha+\varepsilon h_n\}}\mathbb{1}_{\{H_x>n^b-H_l(\mathbf{t})\}}\Big] \\ &\leq \mathbf{E}\Big[\sum_{x\in\mathscr{O}_n} e^{-V(x)}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha-t_l\}}\Big] \leq e^{\delta t_l-(\log n)^{\alpha-1}(1-\frac{\varepsilon}{3A})}. \end{aligned}$$

Moreover, $e^{-(\log n)^{\alpha-1}(1-\frac{\varepsilon}{3A})} = e^{\frac{2\varepsilon}{3A}(\log n)^{\alpha-1}}e^{-(\log n)^{\alpha-1}(1+\frac{\varepsilon}{3A})} \leq e^{\frac{\varepsilon}{A}h_n}\sum_{k\geq1}\Psi^k_{n,n^b}(f^{n,k})$, the last inequality coming from the fact that $h_n\sim(\log n)^{\alpha-1}$ and (23) with $m=0$ and as above $\frac{\varepsilon}{3A}$ instead of ε. So (**A2**) is satisfied.

We are left to prove that technical assumptions (**A3**) and (**A4**) are realised.

• For (**A3**), recall first, from Proposition 1, that for all $b\in[0,1)$, Υ^k_n is the set

$$\{\mathbf{t}=(t_1,\dots,t_k)\in\mathbb{R}^k; H_k(\mathbf{t})\leq n^b e^{\varepsilon h_n}, t_k\geq v'_n, \min_{j\leq k} t_j\geq -B\},$$

with $v'_n = \log(n\ell_n)/\delta_1+\varepsilon h_n$ and $\lambda_n = ne^{\min(10\epsilon\log n,-5h_n)} = ne^{-5h_n}$ for large n. Let $0<\varepsilon_1<\epsilon$, note that $\lambda_n/2\geq \boldsymbol{\lambda}_n = ne^{-6(\log n)^{\alpha-1}}$ so for n large enough

$$\begin{aligned} u_{1,n} = \sum_{k\geq1}\Psi^k_{\lambda_n/2,n^b}(f^{n,k}_{\varepsilon h_n}\mathbb{1}_{\Upsilon^k_n}) &= \mathbf{E}\Big[\sum_{x\in\mathscr{O}_{\lambda_n/2,n^b}} e^{-V(x)}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha+\varepsilon h_n, H_x\leq n^b e^{\varepsilon h_n}, \underline{V}(x)\geq -B\}}\Big] \\ &\geq \mathbf{E}\Big[\sum_{x\in\mathscr{O}_{\boldsymbol{\lambda}_n,n^b}} e^{-V(x)}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha+h_n, H_x\leq n^b e^{\frac{\varepsilon_1}{3}(\log n)^{\alpha-1}}, \underline{V}(x)\geq -B\}}\Big] \\ &\geq e^{-(\log n)^{\alpha-1}(1+\frac{\varepsilon_1}{3})}. \end{aligned}$$

where we use that $(\log n)^\alpha > \log(n\ell_n)/\delta_1$ for the second equality and the last inequality comes from (23), with $m=h_n$ and $\varepsilon_1/3$ instead of ε.

Moreover $e^{-(\log n)^{\alpha-1}(1+\frac{\varepsilon_1}{3})} = e^{-\frac{2\varepsilon_1}{3}(\log n)^{\alpha-1}}e^{-(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{3})} \geq e^{-\varepsilon_1 h_n}\sum_{k\geq1}\Psi^k_{n,n^b}(f^{n,k})$ which comes from the fact that $h_n\sim(\log n)^{\alpha-1}$ and (22) with $t=0$, $\frac{\varepsilon_1}{3}$ instead of ε.

• Finally for (**A4**), recall definition of $u_{2,n}$ just below (19). First observe that as $\alpha\in(1,2)$, for n large enough, $(\log n)^\alpha > \log(n\ell_n)/\delta_1$ so for any k

$$\Psi^k_n(f^{n,k}\mathbb{1}_{\mathbb{R}\setminus\mathscr{H}^k_{v_n}}) = \mathbf{E}\Big[\sum_{|x|=k} e^{-V(x)}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha, V(x)<\log(n\ell_n)/\delta_1\}}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big] = 0.$$

Recall that $\mathbf{E}[W] = e^{\psi(1)} = 1$ and $\lambda'_n = n^b(\log n)^{-2}$ so

$$\sum_{k\geq 1}\big(\Psi^k_{n,\lambda'_n}(f^{n,k}) + \mathbf{E}\big[W\Psi^k_{n,\lambda'_n/W}(f^{n,k})\big]\big) \leq \sum_{k\geq 1}\big(\Psi^k_n(f^{n,k}) + \mathbf{E}\big[W\Psi^k_n(f^{n,k})\big]\big) = 2\sum_{k\geq 1}\Psi^k_n(f^{n,k}),$$

$2\sum_{k\geq 1}\Psi^k_n(f^{n,k}) = 2\mathbf{E}[\sum_{x\in\mathscr{O}_n} e^{-V(x)}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha\}}] \leq 2e^{-(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{6})} \leq e^{-(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{3})}$ thanks to (22) with $t=0$, $\varepsilon_1/6$ instead of ε.
Moreover $e^{-(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{3})} = e^{\frac{2\varepsilon_1}{3}(\log n)^{\alpha-1}}e^{-(\log n)^{\alpha-1}(1+\frac{\varepsilon_1}{3})} \leq e^{\varepsilon_1 h_n}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k})$. The last inequality comes from the fact that $h_n \sim (\log n)^{\alpha-1}$ and (23) with $m=0$, $\frac{\varepsilon_1}{3}$ instead of ε. □

Proof of Theorem 1.2. Here $f^{n,k}(t_1,t_2,\cdots,t_k) = \mathbb{1}_{\{t_{\lfloor k/\beta\rfloor}\geq(\log n)^\alpha\}}$ with $\beta>1$ and $\alpha\in(1,2)$, let us start with the proof of the two following facts, for all $B,\delta>0$, $\varepsilon\in(0,\varepsilon_b)$, n large enough, any $t\geq -B$ and $i\in\mathbb{N}$

$$\mathbf{E}\Big[\sum_{x,\lfloor(|x|+i)/\beta\rfloor>i} e^{-V(x)}\mathbb{1}_{\{V(x_{\lfloor(|x|+i)/\beta\rfloor-i})\geq(\log n)^\alpha - t\}}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big] \leq e^{\delta t - c_\beta(\log n)^{\alpha-1}(1-\varepsilon)}, \tag{24}$$

and for any $m\leq \log n$

$$\mathbf{E}\Big[\sum_{x,\lfloor(|x|+i)/\beta\rfloor>i} e^{-V(x)}\mathbb{1}_{\{V(x_{\lfloor(|x|+i)/\beta\rfloor-i})\geq(\log n)^\alpha + m\}}\mathbb{1}_{\{x\in\Upsilon_n\cap\mathscr{O}_{\boldsymbol{\lambda}_n,n^b}\}}\Big] \geq e^{-c_\beta(\log n)^{\alpha-1}(1+\varepsilon)}, \tag{25}$$

with $\boldsymbol{\lambda}_n = ne^{-6c_\beta(\log n)^{\alpha-1}}$, for any $a>\frac{1}{\delta_1}$

$$\Upsilon_n = \Upsilon_n(\varepsilon) := \{x\in\mathbb{T}; H_x \leq n^b e^{\varepsilon c_\beta(\log n)^{\alpha-1}}, V(x)\geq a\log n, \underline{V}(x)\geq -B\},$$

and $c_\beta = -1-\pi\sqrt{\beta-1}/2 + \rho((\beta-1)\pi^2/4)$ (for ρ see (5)). Recall $\ell_n = (\log n)^3$ and introduce $L_n := \lfloor(\log n)^{2+\varepsilon_\alpha}\rfloor$ with $\varepsilon_\alpha\in(0,\alpha-1)$.
Proof of (24) : first note that if $t>(\log n)^\alpha/2$, (24) is obviously satisfied, indeed

$$\mathbf{E}\Big[\sum_{x,\lfloor(|x|+i)/\beta\rfloor>i} e^{-V(x)}\mathbb{1}_{\{V(x_{(\lfloor(|x|+i)/\beta\rfloor-i)})\geq(\log n)^\alpha - t\}}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big] \leq \mathbf{E}\Big[\sum_{x\in\mathscr{O}_n} e^{-V(x)}\Big],$$

and by Remark 2, $\mathbf{E}[\sum_{x\in\mathscr{O}_n} e^{-V(x)}] = \mathbf{E}[\sum_{x\in\mathscr{O}_n} e^{-V(x)}]e^{\delta t-\delta t} \leq \ell_n e^{\delta t - \frac{\delta}{2}(\log n)^\alpha} \leq e^{\delta t - c_\beta(\log n)^{\alpha-1}}$ for n large enough. Now assume $t\leq(\log n)^\alpha/2$. Expectation in (24) is smaller than

$$\sum_{k\leq\lfloor A\ell_n\rfloor}\sum_{p\geq 1}\mathbb{1}_{\{p=\lfloor\frac{k+i}{\beta}\rfloor-i\}}\mathbf{E}\Big[\sum_{|x|=k} e^{-V(x)}\mathbb{1}_{\{V(x_p)\geq(\log n)^\alpha - t\}}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big] + \mathbf{E}\Big[\sum_{|x|>\lfloor A\ell_n\rfloor} e^{-V(x)}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big]$$

with $A>0$ such that the last term is smaller than $1/n$ (Remark 2). Note that $p = \lfloor\frac{k+i}{\beta}\rfloor - i$ implies $k\geq\lceil\beta p\rceil$ and as $\sum_{k\leq\lfloor A\ell_n\rfloor}\mathbb{1}_{\{p=\lfloor\frac{k+i}{\beta}\rfloor-i\}} \leq \beta$ for any $p\geq 1$, the sum above is smaller, by many-to-one Lemma, than

$$\begin{aligned}\beta\sum_{p\leq\lfloor A\ell_n\rfloor}\mathbf{P}\big(S_p\geq(\log n)^\alpha - t, \max_{j\leq\lceil p\beta\rceil} H^S_j\leq n\big) + \frac{1}{n} \leq& \beta\sum_{p=L_n}^{\lfloor A\ell_n\rfloor}\mathbf{P}\big(S_p\geq(\log n)^\alpha - t, \max_{j\leq\lceil p\beta\rceil} H^S_j\leq n\big)\\ &+\beta\sum_{p<L_n}\mathbf{P}\big(S_p\geq(\log n)^\alpha - t\big) + \frac{1}{n}.\end{aligned} \tag{26}$$

For the second sum in (26), by exponential Markov inequality, for n large enough, all $p < L_n$ and $t \geq -B$

$$\mathbf{P}\big(S_p \geq (\log n)^\alpha - t\big) \leq e^{\delta_n t - \delta_n (\log n)^\alpha + p\psi(1-\delta_n)} \leq e^{\delta_n (t+B) - \frac{(\log n)^{2\alpha}}{2\sigma^2 L_n} + L_n \psi(1-\delta_n)} \leq e^{\delta t - (1-\varepsilon)\frac{(\log n)^{2\alpha}}{2\sigma^2 L_n}},$$

with $\delta_n := (\log n)^\alpha / \sigma^2 L_n$, and we have used that $\psi(1-\delta_n) \in \mathbb{R}^+$ for the second inequality and that $\delta_n \to 0$ ($\alpha \in (1,2)$) together with $\psi(1) = \psi'(1) = 0$ and $\psi''(1) = \sigma^2$ for the last one.
For the first sum in (26), which gives the main contribution, by Markov property at time p, $\mathbf{P}\big(S_p \geq (\log n)^\alpha - t, \max_{j \leq \lceil \beta p \rceil} H_j^S \leq n\big)$ is smaller than $\mathbf{P}\big(S_p \geq (\log n)^\alpha - t, \max_{j \leq p} H_j^S \leq n\big)\mathbf{P}\big(\max_{j \leq \lceil (\beta-1)p \rceil} H_j^S \leq n\big)$. Then thanks to Lemma 4.6 (78) (with $\ell = (\log n)^2$, $\lceil (\beta-1)p \rceil$ and $\varepsilon/2$ instead respectively of k and ε), for n large enough and any $p \in \{L_n, \dots, \lfloor A\ell_n \rfloor\}$

$$\mathbf{P}\Big(\max_{j \leq \lceil (\beta-1)p \rceil} H_j^S \leq n\Big) \leq e^{-p\frac{\pi^2\sigma^2(\beta-1)}{8(\log n)^2}(1-\frac{\varepsilon}{2})} = e^{-p\frac{\pi^2\sigma^2(\beta-1)}{8((1-\varepsilon/2)^{-1/2}\log n)^2}}.$$

Hence, as $\log n \leq (1-\varepsilon/2)^{-1/2}\log n$, $\sum_{p=L_n}^{\lfloor A\ell_n \rfloor} \mathbf{P}\big(S_p \geq (\log n)^\alpha - t, \max_{j \leq \lceil p\beta \rceil} H_j^S \leq n\big)$ is smaller than

$$\begin{aligned}
&\sum_{p=L_n}^{\lfloor A\ell_n \rfloor} \mathbf{E}\Big[\mathbb{1}_{\{\tau_{(\log n)^\alpha - t} \leq p,\ \max_{j \leq k} \overline{S}_j - S_j \leq (1-\varepsilon/2)^{-1/2}\log n\}} e^{-p\frac{\pi^2\sigma^2(\beta-1)}{8((1-\varepsilon/2)^{-1/2}\log n)^2}}\Big] \\
&\leq A\ell_n \mathbf{E}\Big[\mathbb{1}_{\{\max_{j \leq \tau_{(\log n)^\alpha - t}} \overline{S}_j - S_j \leq (1-\varepsilon/2)^{-1/2}\log n\}} e^{-\tau_{(\log n)^\alpha - t}\frac{\pi^2\sigma^2(\beta-1)}{8((1-\varepsilon/2)^{-1/2}\log n)^2}}\Big] \\
&\leq A\ell_n e^{\sqrt{1-\frac{\varepsilon}{2}}\frac{c_\beta t}{\log n} - c_\beta(\log n)^{\alpha-1}(1-\frac{\varepsilon}{2})} \leq A\ell_n e^{\frac{c_\beta(t+B)}{\log n} - c_\beta(\log n)^{\alpha-1}(1-\frac{\varepsilon}{2})} \leq \frac{1}{3}e^{\delta t - c_\beta(\log n)^{\alpha-1}(1-\varepsilon)},
\end{aligned}$$

where Lemma 4.1 (with $\ell = ((1-\varepsilon/2)^{-1/2}\log n)^2$, $r(\ell) = (\log n)^\alpha - t$, $c = \pi^2(\beta-1)/4$ and $1 - \sqrt{1-\varepsilon/2}$ instead of ε) provides the second inequality. Finally collecting all upper bounds of the three sums in (26), for n large enough

$$\begin{aligned}
&\mathbf{E}\Big[\sum_{x;\lfloor (|x|+i)/\beta \rfloor > i} e^{-V(x)} \mathbb{1}_{\{V(x_{(\lfloor (|x|+i)/\beta \rfloor - i)}) \geq (\log n)^\alpha - t\}} \mathbb{1}_{\{x \in \mathscr{O}_n\}}\Big] \\
&\leq \frac{1}{3}e^{\delta t - c_\beta(\log n)^{\alpha-1}(1-\varepsilon)} + \beta e^{\delta t - (1-\varepsilon)\frac{(\log n)^{2\alpha}}{2\sigma^2 L_n}} + \frac{1}{n} \leq \frac{2}{3}e^{\delta t - c_\beta(\log n)^{\alpha-1}(1-\varepsilon)} + \frac{e^{\delta(t+B)}}{n},
\end{aligned}$$

which is smaller than $e^{\delta t - c_\beta(\log n)^{\alpha-1}(1-\varepsilon)}$ (we have used that $(\log n)^{2\alpha}/L_n \geq (\log n)^{2(\alpha-1)-\varepsilon_\alpha}$ and $(\log n)^{\alpha-1} = o((\log n)^{2(\alpha-1)-\varepsilon_\alpha})$). This yields the upper bound in (24).
Proof of (25). Let $\alpha_n := (\log n)^\alpha + \log n$. For all $m \leq \log n$, by many-to-one Lemma expectation in (25) is larger than

$$\sum_{p,k \geq 1} \mathbb{1}_{\{p = \lfloor (k+i)/\beta \rfloor - i\}} \mathbf{P}\big(S_p \geq \alpha_n, n^b < H_k^S \leq n^b e^{\varepsilon c_\beta (\log n)^{\alpha-1}}, \max_{j \leq k} H_j^S \leq \boldsymbol{\lambda}_n, S_k \geq v'_n, \underline{S}_k \geq -B\big).$$

The probability above is larger than (as $\alpha_n > a\log n$ for all $a > \frac{1}{\delta_1}$)

$$\mathbf{P}\big(S_p \geq \alpha_n, \underline{S}_p \geq -B, \overline{S}_p = S_p, n^b < H_k^S \leq n^b e^{\varepsilon c_\beta (\log n)^{\alpha-1}}, \max_{j \leq k} H_j^S \leq \boldsymbol{\lambda}_n, \min_{p < j \leq k} S_j \geq S_p\big).$$

Recall that $H_j^S = \sum_{i=1}^j e^{S_i - S_j}$ so we have, for any $p < j \leq k$, $H_j^S = e^{S_p - S_j} H_p^S + H_{p,j}^S$ where $H_{p,j}^S = \sum_{i=p+1}^j e^{S_i - S_j}$. Note that $\overline{S}_p = S_p$ and $\min_{p<j\leq k} S_j \geq S_p$ implies $H_j^S \leq p + H_{p,j}^S$ so the

previous probability is larger than

$$\mathbf{P}\big(S_p \geq \alpha_n, \underline{S}_p \geq -B, \overline{S}_p = S_p, \max_{j\leq p} H_j^S \leq \boldsymbol{\lambda}_n, n^b < H_{p,k}^S \leq n^b e^{\varepsilon c_\beta (\log n)^{\alpha-1}} - p \\ , \max_{p<j\leq k} H_{p,j}^S \leq \boldsymbol{\lambda}_n - p, \min_{p<j\leq k} S_j \geq S_p\big),$$

which, thanks to Markov property at time p, is nothing but the product of $\mathbf{P}\big(S_p \geq \alpha_n, \underline{S}_p \geq -B, \overline{S}_p = S_p, \max_{j\leq p} H_j^S \leq \boldsymbol{\lambda}_n\big)$ and $\mathbf{P}\big(n^b < H_{k-p}^S \leq n^b e^{\varepsilon c_\beta (\log n)^{\alpha-1}} - p, \max_{j\leq k-p} H_j^S \leq \boldsymbol{\lambda}_n - p = ne^{-6(\log n)^{\alpha-1}} - p, \underline{S}_{k-p} \geq 0\big)$. From now, let $p \in \{L_n, \dots, \ell'_n = (\log n)^4\}$. We first deal with the second probability. Observe that for all $i \geq 0$, $p = \lfloor (k+i)/\beta \rfloor - i$ implies $k - p \geq \lceil (\beta-1)L_n \rceil$. It follows that for all $\varepsilon \in (0, \varepsilon_b)$, n large enough, for all $L_n \leq p \leq \ell'_n$, $k \geq 1$, $i \geq 0$ such that $p = \lfloor (k+i)/\beta \rfloor - i$, $\mathbf{P}\big(n^b < H_{k-p}^S \leq n^b e^{\varepsilon c_\beta (\log n)^{\alpha-1}} - p, \max_{j\leq k-p} H_j^S \leq \boldsymbol{\lambda}_n - p, \underline{S}_{k-p} \geq 0\big)$ is larger than (as $\boldsymbol{\lambda}_n - p \geq \boldsymbol{\lambda}_n - \ell'_n \geq ne^{-7c_\beta (\log n)^{\alpha-1}}$)

$$\mathbf{P}\big(n^b < H_{k-p}^S \leq n^b e^{\frac{\varepsilon}{2} c_\beta (\log n)^{\alpha-1}}, \max_{j\leq k-p} H_j^S \leq ne^{-7c_\beta (\log n)^{\alpha-1}}, \underline{S}_{k-p} \geq 0\big) \geq e^{-\frac{\pi^2\sigma^2}{8}\frac{(k-p)}{(\log \boldsymbol{\lambda}'_n)^2}},$$

with $\boldsymbol{\lambda}'_n := n^{(1+\varepsilon/2)^{-1/2}}$. Last inequality comes from Lemma 4.6 (79) (with $\ell = (\log n)^2$, $a = 7$, $c = \frac{\varepsilon c_\beta}{2}$, $d = \frac{\alpha-1}{2}$, $k-p$ and $\varepsilon/2$ respectively instead of k and ε). Equality $p = \lfloor (k+i)/\beta \rfloor - i$ also implies, for any $0 \leq i \leq \log n$ that $k - p \leq (p + \log n)(\beta - 1) + \beta$ so it follows that above probability is larger than $C \exp(\frac{\pi^2\sigma^2(\beta-1)}{8(\log \boldsymbol{\lambda}'_n)^2} p)$ for some positive constant $C \in (0,1)$. Collecting the previous inequalities together with Lemma 4.4 give, as $\sum_{k\geq 1} \mathbb{1}_{\{p = \lfloor (k+i)/\beta \rfloor - i\}} \geq 1$, that for n large enough mean in (25) is larger than

$$C \sum_{p=L_n}^{\ell'_n} \mathbf{E}\Big[e^{-\frac{\pi^2\sigma^2(\beta-1)}{8(\log \boldsymbol{\lambda}'_n)^2} p} \mathbb{1}_{\{S_p \geq \alpha_n, \underline{S}_p \geq -B, \overline{S}_p = S_p, \max_{j\leq p} H_j^S \leq ne^{-7c_\beta (\log n)^{\alpha-1}}\}}\Big] \sum_{k\geq 1} \mathbb{1}_{\{p = \lfloor (k+i)/\beta \rfloor - i\}}$$
$$\geq C\mathbf{P}(\underline{S}_{\ell'_n} \geq 0)^2 \mathbf{E}\Big[e^{-\frac{\pi^2\sigma^2(\beta-1)}{8(\log \boldsymbol{\lambda}'_n)^2} \tau_{\alpha_n}} \mathbb{1}_{\{L_n \leq \tau_{\alpha_n} \leq \ell'_n, \forall j \leq \tau_{\alpha_n} : \overline{S}_j - S_j \leq \log \boldsymbol{\lambda}'_n\}}\Big]$$
$$\geq C\mathbf{P}(\underline{S}_{\ell'_n} \geq 0)^2 \mathbf{P}(\overline{S}_{\ell'_n} \geq \alpha_n) \mathbf{E}\Big[e^{-\frac{\pi^2\sigma^2(\beta-1)}{8(\log \boldsymbol{\lambda}'_n)^2} \tau_{\alpha_n}} \mathbb{1}_{\{\forall j \leq \tau_{\alpha_n} : \overline{S}_j - S_j \leq \log \boldsymbol{\lambda}'_n\}}\Big] - \mathbf{P}(\overline{S}_{L_n} \geq \alpha_n).$$

Note that thanks to (68) and the fact that $\alpha \in (1,2)$, we can find constant $c_{(1.2)} > 0$ such that $C\mathbf{P}\big(\underline{S}_{\ell'_n} \geq 0\big)^2 \mathbf{P}\big(\overline{S}_{\ell'_n} \geq \alpha_n\big) \geq c_{(1.2)} (\ell'_n)^{-1} \geq 2e^{-\frac{\varepsilon}{2}(\log n)^{\alpha-1}}$. Then applying Lemma 4.1 (with $\ell = \log \boldsymbol{\lambda}'_n$, $r = \alpha_n$, $c = \pi^2(\beta-1)/4$ and $\sqrt{1+\varepsilon/2} - 1$ instead of ε), for n large enough

$$\mathbf{E}\Big[e^{-\frac{\pi^2\sigma^2(\beta-1)}{8(\log \boldsymbol{\lambda}'_n)^2} \tau_{\alpha_n}} \mathbb{1}_{\{\forall j \leq \tau_{\alpha_n} : \overline{S}_j - S_j \leq \log \boldsymbol{\lambda}'_n\}}\Big] \geq e^{-c_\beta (\log n)^{\alpha-1}(1+\frac{\varepsilon}{2})}.$$

Finally, by Markov inequality, $\mathbf{P}(\overline{S}_{L_n} \geq \alpha_n) \leq L_n e^{-c'_{(1.2)} \alpha_n^2 / L_n}$ for some constant $c'_{(1.2)} > 0$. Since $\alpha_n^2 / L_n \geq (\log n)^{2(\alpha-1)-\varepsilon_\alpha}$ and $(\log n)^{\alpha-1} = o((\log n)^{2(\alpha-1)-\varepsilon_\alpha})$ we get that $\mathbf{P}(\overline{S}_{L_n} \geq \alpha_n) \leq e^{-c_\beta (\log n)^{\alpha-1}(1+\varepsilon)}$. Collecting the different estimates yields (25).

We are ready to prove that $\mathbf{f}^n$ satisfies assumptions (**A1**), (**A2**), (**A3**) and (**A4**). Recall that $\Psi_{n,n^b}^k(f^{n,k}) = \mathbf{E}\big[\sum_{|x|=k} e^{-V(x)} f^{n,k}(V(x_1), \dots, V(x)) \mathbb{1}_{\{x \in \mathscr{O}_{n,n^b}\}}\big]$ where $x \in \mathscr{O}_{n,n^b}$ if and only if $\max_{j\leq |x|} H_{x_j} \leq n$ and $H_x > n^b$, $\mathscr{U}_b = \{\kappa \geq 0;$ for all $k \geq 1, \mathbf{t} \in \mathbb{R}^k, n \geq 1 : \mathbb{1}_{\{H_k(\mathbf{t}) > n^b\}} f^{n,k}(\mathbf{t}) \leq C_\infty n^{-\kappa}\}$ with $C_\infty = \sup_{n,\ell} \|f^{n,\ell}\|_\infty$.

• Check of (**A1**) and asymptotic of h_n. We obtain from (25) with $i=m=0$ and n large enough

$$\mathbf{E}\Big[\sum_{x\in\mathscr{O}_{n,n^b}} e^{-V(x)}\mathbb{1}_{\{V(x_{\lfloor|x|/\beta\rfloor})\geq(\log n)^\alpha\}}\Big] \geq \mathbf{E}\Big[\sum_{x\in\mathbb{T}} e^{-V(x)}\mathbb{1}_{\{V(x_{\lfloor|x|/\beta\rfloor})\geq(\log n)^\alpha\}}\mathbb{1}_{\{x\in\Upsilon_n\cap\mathscr{O}_{\boldsymbol{\lambda}_{n,n^b}}\}}\Big]$$
$$\geq e^{-c_\beta(\log n)^{\alpha-1}(1+\varepsilon)}.$$

This implies that for all $b\in[0,1)$, $\kappa_b=\max\mathscr{U}_b=0$ (we use a similar argument than in the proof of Theorem 1.1) and additionaly with (24), gives, taking $i=t=0$

$$h_n=\Big|n^{\kappa_b}\log\big(\sum_{k\geq1}\Psi^k_{n,n^b}(f^{n,k})\big)\Big|=\Big|\log\mathbf{E}\Big[\sum_{x\in\mathscr{O}_{n,n^b}} e^{-V(x)}\mathbb{1}_{\{V(x_{\lfloor|x|/\beta\rfloor})\geq(\log n)^\alpha\}}\Big]\Big|\sim c_\beta(\log n)^{\alpha-1}.$$

We also deduce from the previous lower bound that (**A1**) is satisfied.

• For (**A2**), recalling $m_n=\lceil\varepsilon h_n/r_2\rceil$ (r_2 is defined in (13)), by definition, for any $j>0$

$$\begin{aligned} f^{n,j}_{\varepsilon h_n}(t_1,\dots,t_j) &= \inf_{\mathbf{s}\in[-\varepsilon h_n,\varepsilon h_n]^{m_n}} f^{n,m_n+j}(s_1,\dots,s_{m_n},t_1+s_{m_n},\dots,t_j+s_{m_n}) \\ &= \inf_{s_{m_n}\in[-\varepsilon h_n,\varepsilon h_n]} \mathbb{1}_{\{t_{\lfloor(m_n+j)/\beta\rfloor-m_n}\geq(\log n)^\alpha-s_{m_n}\}} \\ &= \mathbb{1}_{\{\lfloor(m_n+j)/\beta\rfloor>m_n\}}\mathbb{1}_{\left\{t_{\lfloor(m_n+j)/\beta\rfloor-m_n}\geq(\log n)^\alpha+\varepsilon h_n\right\}}. \end{aligned}$$

Then for any $l\in\mathbb{N}^*$ and all $\mathbf{t}=(t_1,\dots,t_l)\in\mathbb{R}^l$, $f^{n,l+k}_{\varepsilon h_n}(t_1,\dots,t_l,V(x_1)+t_l,\dots,V(x)+t_l)$, with $|x|=k$, is equal to

$$\mathbb{1}_{\{m_n<\lfloor(k+i)/\beta\rfloor\leq i\}}\mathbb{1}_{\{t_{\lfloor(k+i)/\beta\rfloor-m_n}\geq(\log n)^\alpha+\varepsilon h_n\}}+\mathbb{1}_{\{\lfloor(k+i)/\beta\rfloor>i\}}\mathbb{1}_{\{V(x_{(\lfloor(k+i)/\beta\rfloor-i)})+t_l\geq(\log n)^\alpha+\varepsilon h_n\}},$$

with $i=m_n+l$. Recall definition of $\Psi_{\cdot,\cdot,\cdot}(F|\mathbf{t})$ in (15), we have

$$\begin{aligned} \sum_{k\geq1}\Psi^k_{n,n^b-H_l(\mathbf{t})}\big(f^{n,l+k}_{\varepsilon h_n}|\mathbf{t}\big) &\leq \mathbf{E}\Big[\sum_{x\in\mathscr{O}_n} e^{-V(x)}f^{n,l+k}_{\varepsilon h_n}(t_1,\dots,t_l,V(x_1)+t_l,\dots,V(x)+t_l)\Big] \\ &\leq \sum_{k\geq1}\mathbb{1}_{\{m_n<\lfloor(i+k)/\beta\rfloor\leq i\}}\mathbb{1}_{\{t_{\lfloor(i+k)/\beta\rfloor-m_n}\geq(\log n)^\alpha\}}\Psi^k_n(1) \\ &\quad+\mathbf{E}\Big[\sum_{x;\lfloor(|x|+i)/\beta\rfloor>i} e^{-V(x)}\mathbb{1}_{\{V(x_{(\lfloor(|x|+i)/\beta\rfloor-i)})\geq(\log n)^\alpha-t_l\}}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big]. \end{aligned}$$

$\sum_{k\geq1}\mathbb{1}_{\{m_n<\lfloor(i+k)/\beta\rfloor\leq i\}}\mathbb{1}_{\{t_{\lfloor(i+k)/\beta\rfloor-m_n}\geq(\log n)^\alpha\}}\Psi^k_n(1)$ is equal to

$$\sum_{p=1}^{l}\mathbb{1}_{\{t_p\geq(\log n)^\alpha\}}\sum_{k\geq1}\Psi^k_n(1)\mathbb{1}_{\{p=\lfloor\frac{i+k}{\beta}\rfloor-m_n\}}\leq\beta\sum_{p=1}^{l}\mathbb{1}_{\{t_p\geq(\log n)^\alpha\}},$$

where we used that $\sum_{k\geq1}\Psi^k_n(1)\mathbb{1}_{\{p=\lfloor\frac{i+k}{\beta}\rfloor-m_n\}}\leq e^{\psi(1)}\sum_{k\geq1}\mathbb{1}_{\{p=\lfloor\frac{i+k}{\beta}\rfloor-m_n\}}\leq\beta$. Also by (24) with $i=m_n+l$, $t=t_l$, $\frac{\varepsilon}{4A}$ instead of ε,

$$\mathbf{E}\Big[\sum_{x;\lfloor(|x|+i)/\beta\rfloor>i} e^{-V(x)}\mathbb{1}_{\{V(x_{(\lfloor(|x|+i)/\beta\rfloor-i)})\geq(\log n)^\alpha-t_l\}}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big]\leq e^{\delta t_l-c_\beta(\log n)^{\alpha-1}(1-\frac{\varepsilon}{4A})},$$

so

$$\sum_{k\geq1}\Psi^k_{n,n^b-H_l(\mathbf{t})}\big(f^{n,l+k}_{\varepsilon h_n}|\mathbf{t}\big)\leq\beta\sum_{p=1}^{l}\mathbb{1}_{\{t_p\geq(\log n)^\alpha\}}+\frac{1}{2}e^{\delta t_l-c_\beta(\log n)^{\alpha-1}(1-\frac{\varepsilon}{3A})}.$$

Note that $\beta\sum_{p=1}^{l}\mathbb{1}_{\{t_p\geq(\log n)^\alpha\}}$ is very small for n large enough, any $l<\lfloor A\ell_n\rfloor$ and $H_l(\mathbf{t})\leq n$. Indeed $\sum_{p=1}^{l}e^{\delta(t_p-t_l)}\leq lH_l(\mathbf{t})^\delta\leq A\ell_n n^\delta$ so

$$\beta\sum_{p=1}^{l}\mathbb{1}_{\{t_p\geq(\log n)^\alpha\}}=e^{\delta t_l}\beta\sum_{p=1}^{l}e^{\delta(t_p-t_l)}e^{-\delta t_p}\mathbb{1}_{\{t_p\geq(\log n)^\alpha\}}\leq e^{\delta t_l}\beta A\ell_n n^\delta e^{-\delta(\log n)^\alpha},$$

which, as $\alpha\in(1,2)$, is smaller than $\frac{1}{2}e^{\delta t_l-c_\beta(\log n)^{\alpha-1}(1-\frac{\varepsilon}{3A})}$. Finally observe that

$$e^{-c_\beta(\log n)^{\alpha-1}(1-\frac{\varepsilon}{3A})}=e^{c_\beta(\log n)^{\alpha-1}\frac{2\varepsilon}{3A}}e^{-c_\beta(\log n)^{\alpha-1}(1+\frac{\varepsilon}{3A})}\leq e^{\frac{\varepsilon}{A}h_n}\sum_{k\geq1}\Psi^k_{n,n^b}(f^{n,k}),$$

where we have used that $h_n\sim c_\beta(\log n)^{\alpha-1}$ and (25) with $i=m=0$.

We are left to prove that technical assumptions (**A3**) and (**A4**) are realised, the ideas are very similar than for the same assumptions of previous theorem we detail them here however to keep the proofs independent the one from others.

• For (**A3**), recall that Υ^k_n is the set

$$\{\mathbf{t}=(t_1,\dots,t_k)\in\mathbb{R}^k;H_k(\mathbf{t})\leq n^be^{\varepsilon h_n},t_k\geq v'_n,\min_{j\leq k}t_j\geq-B\},$$

with $v'_n=\frac{1}{\delta_1}\log(n\ell_n)+\varepsilon h_n$. Let $0<\varepsilon_1<\epsilon$ and recall that $\lambda_n=ne^{-5h_n}$. Note that $\lambda_n/2\geq\boldsymbol{\lambda}_n=ne^{-6(\log n)^{\alpha-1}}$ so $\sum_{k\geq1}\Psi^k_{\lambda_n/2,n^b}(f^{n,k}_{\varepsilon h_n}\mathbb{1}_{\Upsilon^k_n})\geq\sum_{k\geq1}\Psi^k_{\boldsymbol{\lambda}_n,n^b}(f^{n,k}_{\varepsilon h_n}\mathbb{1}_{\Upsilon^k_n})$ which is nothing but

$$\begin{aligned}&\mathbf{E}\Big[\sum_{x\in\mathscr{O}_{\boldsymbol{\lambda}_n,n^b}}e^{-V(x)}\mathbb{1}_{\{\lfloor\frac{|x|+m_n}{\beta}\rfloor>m_n\}}\mathbb{1}_{\{V(x_{\lfloor(|x|+m_n)/\beta\rfloor-m_n})\geq(\log n)^\alpha+\varepsilon h_n,H_x\leq n^be^{\varepsilon h_n},V(x)\geq v'_n,\underline{V}(x)\geq-B\}}\Big]\\&\geq\mathbf{E}\Big[\sum_{x,\lfloor(|x|+m_n)/\beta\rfloor>m_n}e^{-V(x)}\mathbb{1}_{\{V(x_{\lfloor(|x|+m_n)/\beta\rfloor-m_n})\geq(\log n)^\alpha+h_n\}}\mathbb{1}_{\{x\in\Upsilon_n(\frac{\varepsilon_1}{3})\cap\mathscr{O}_{\boldsymbol{\lambda}_n,n^b}\}}\Big]\\&\geq e^{-c_\beta(\log n)^{\alpha-1}(1+\frac{\varepsilon_1}{3})}.\end{aligned}$$

where that last inequality comes from (25) with $i=m=m_n$ and $\varepsilon_1/3$ instead of ε. Moreover $e^{-c_\beta(\log n)^{\alpha-1}(1+\frac{\varepsilon_1}{3})}=e^{-\frac{2\varepsilon_1}{3}c_\beta(\log n)^{\alpha-1}}e^{-c_\beta(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{3})}\geq e^{-\varepsilon_1h_n}\sum_{k\geq1}\Psi^k_{n,n^b}(f^{n,k})$, the last inequality comes from the fact that $h_n\sim c_\beta(\log n)^{\alpha-1}$ and (24) with $i=t=0$.

• For (**A4**), recall that $\lambda'_n=n^b(\log n)^{-2}$. First, observe that for all $k\in\mathbb{N}^*$, $\alpha\in(1,2)$, $(\log n)^\alpha-\log(n\ell_n)/\delta_1>\log n$ for n large enough so

$$\begin{aligned}\Psi^k_n(f^{n,k}\mathbb{1}_{\mathbb{R}\setminus\mathscr{H}^k_{v_n}})&=\mathbf{E}\Big[\sum_{|x|=k}e^{-V(x)}\mathbb{1}_{\{V(x_{\lfloor|x|/\beta\rfloor})\geq(\log n)^\alpha,V(x)<\log(n\ell_n)/\delta_1\}}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big]\\&\leq\mathbf{E}\Big[\sum_{|x|=k}e^{-V(x)}\mathbb{1}_{\{\overline{V}(x)\geq(\log n)^\alpha,V(x)<\log(n\ell_n)/\delta_1\}}\mathbb{1}_{\{\overline{V}(x)-V(x)\leq\log n\}}\Big]=0.\end{aligned}$$

Recall that, $W=\sum_{|z|=1}e^{-V(z)}$ and $\mathbf{E}[W]=e^{\psi(1)}=1$ so

$$\sum_{k\geq1}\big(\Psi^k_{n,\lambda'_n}(f^{n,k})+\mathbf{E}\big[W\Psi^k_{n,\lambda'_n/W}(f^{n,k})\big]\big)\leq\sum_{k\geq1}\big(\Psi^k_n(f^{n,k})+\mathbf{E}\big[W\Psi^k_n(f^{n,k})\big]\big)=2\sum_{k\geq1}\Psi^k_n(f^{n,k})$$

and thanks to (24) with $i=t=0$ and $\frac{\varepsilon_1}{4}$ instead of ε

$$\begin{aligned}2\sum_{k\geq1}\Psi^k_n(f^{n,k})=2\mathbf{E}\Big[\sum_{x\in\mathscr{O}_n}e^{-V(x)}\mathbb{1}_{\{V(x_{|x|/\beta})\geq(\log n)^\alpha\}}\Big]&\leq2e^{-c_\beta(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{4})}\\&\leq e^{-c_\beta(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{3})}.\end{aligned}$$

Moreover $e^{-c_\beta(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{3})} = e^{\frac{2\varepsilon_1}{3}c_\beta(\log n)^{\alpha-1}} e^{-c_\beta(\log n)^{\alpha-1}(1+\frac{\varepsilon_1}{3})} \leq e^{\varepsilon_1 h_n} \sum_{k\geq 1} \Psi^k_{n,n^b}(f^{n,k})$, the last inequality comes from the fact that $h_n \sim c_\beta(\log n)^{\alpha-1}$ and (25) with $i = m = 0$. □

Proof of Theorem 1.3.
Assume first that $a = d = 1$ **and** $\alpha \in (1,2)$ which corresponds to the second and third case of the theorem. Let us start with the proof of the two facts, note that we distinguish wether $b = 0$ or $b \in (0, 1/2)$.
Facts for the case $b = 0$: for all $B, \delta > 0$, $\varepsilon \in (0, \varepsilon_b)$ and n large enough, for any $t \geq -B$,

$$\mathbf{E}\Big[\sum_{x\in\mathscr{O}_n} \frac{e^{-V(x)}}{\sum_{j\leq|x|} H_{x_j}} \mathbb{1}_{\{V(x)\geq(\log n)^\alpha - t\}}\Big] \leq e^{\delta t - 2(\log n)^{\alpha/2}(1-\varepsilon)}, \tag{27}$$

and for all $0 \leq m \leq \log n$, $0 \leq M \leq e^{(\log n)^{\alpha/2}}$

$$\mathbf{E}\Big[\sum_{x\in\mathbb{T}} \frac{e^{-V(x)}\mathbb{1}_{\{x\in\Upsilon_{n,1}\cap\mathscr{O}_{\boldsymbol{\lambda}_{n,1}}\}}}{M|x| + \sum_{j\leq|x|} H_{x_j}} \mathbb{1}_{\{V(x)\geq(\log n)^\alpha + m\}}\Big] \geq e^{-2(\log n)^{\alpha/2}(1+\varepsilon)}, \tag{28}$$

with $\boldsymbol{\lambda}_{n,1} = ne^{-12(\log n)^{\alpha/2}}$ and

$$\Upsilon_{n,1} = \Upsilon_{n,1}(\varepsilon) := \{x \in \mathbb{T}; H_x \leq e^{2\varepsilon(\log n)^{\alpha/2}}, \underline{V}(x) \geq -B\}.$$

We first deal with the upper bound (27). Note that if $t > (\log n)^\alpha/2$, (27) is obviously satisfied, indeed, $(\sum_{j\leq|x|} H_{x_j})^{-1}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha - t\}} \leq 1$ so for n large enough

$$\begin{aligned}\mathbf{E}\Big[\sum_{x\in\mathscr{O}_n} \frac{e^{-V(x)}}{\sum_{j\leq|x|} H_{x_j}} \mathbb{1}_{\{V(x)\geq(\log n)^\alpha - t\}}\Big] &\leq \mathbf{E}\Big[\sum_{x\in\mathscr{O}_n} e^{-V(x)}\Big] e^{-\delta t} e^{\delta t} \leq \ell_n e^{\delta t - \frac{\delta}{2}(\log n)^\alpha} \\ &\leq e^{\delta t - 2(\log n)^{\alpha/2}(1-\varepsilon)},\end{aligned}$$

where we have used Remark 2. Now assume $t \leq (\log n)^\alpha/2$, by many-to-one Lemma, expectation in (27) is smaller than

$$\sum_{k\leq\lfloor A\ell_n\rfloor} \mathbf{E}\Big[\frac{1}{\sum_{j=1}^k H^S_j} \mathbb{1}_{\{\tau_{(\log n)^\alpha - t}\leq k,\, \max_{j\leq k} H^S_j \leq n\}}\Big] + \mathbf{E}\Big[\sum_{|x|>\lfloor A\ell_n\rfloor} e^{-V(x)}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big], \tag{29}$$

the second sum is treated as usual : Remark 2 with a choosen A, together with the fact that $\alpha \in (1,2)$ and $t \geq -B$ implies that $\mathbf{E}\Big[\sum_{|x|>\lfloor A\ell_n\rfloor} e^{-V(x)}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big] \leq 1/n \leq \frac{1}{2}e^{\delta t - 2(\log n)^{\alpha/2}(1-\varepsilon)}$. Also using that $(\sum_{j=1}^k H^S_j)^{-1} \leq e^{-\max_{j\leq k}\overline{S}_j - S_j}$ leads to

$$\sum_{k\leq\lfloor A\ell_n\rfloor} \mathbf{E}\Big[\frac{1}{\sum_{j=1}^k H^S_j} \mathbb{1}_{\{\tau_{(\log n)^\alpha - t}\leq k,\, \max_{j\leq k} H^S_j \leq n\}}\Big] \leq \lfloor A\ell_n\rfloor \mathbf{E}\Big[e^{-\max_{j\leq\tau_{(\log n)^\alpha - t}}\overline{S}_j - S_j}\Big].$$

Since $t < (\log n)^\alpha/2$, $(\log n)^\alpha - t > (\log n)^\alpha/2$ so by Lemma 4.2 with $\frac{\varepsilon}{2}$ instead of ε and any $t \geq -B$

$$\mathbf{E}\Big[e^{-\max_{j\leq\tau_{(\log n)^\alpha - t}}\overline{S}_j - S_j}\Big] \leq e^{-2(1-\frac{\varepsilon}{2})\sqrt{(\log n)^\alpha - t}} \leq e^{-2(1-\frac{\varepsilon}{2})\frac{(\log n)^\alpha - (t+B)}{\sqrt{(\log n)^\alpha + B}}} \leq \frac{1}{2}e^{\delta t - 2(\log n)^{\alpha/2}(1-\varepsilon)}.$$

this treats the first sum in (29) and yields (27).
We now turn to the lower bound (28). Recall $\ell'_n = (\log n)^4$, using that $\sum_{j=1}^k H^S_j \leq k \max_{j\leq k} H^S_j$

and the fact that $m \leq \log n$, $0 \leq M \leq e^{(\log n)^{\alpha/2}}$ and $\boldsymbol{\lambda}_{n,1} > e^{(\log n)^{\alpha/2}}$, we obtain thanks to many-to-one Lemma

$$\begin{aligned}
&\mathbf{E}\Big[\sum_{x\in\mathbb{T}} \frac{e^{-V(x)}\mathbb{1}_{\{x\in\Upsilon_{n,1}\cap\mathscr{O}_{\boldsymbol{\lambda}_{n,1}}\}}}{M|x|+\sum_{j\leq|x|}H_{x_j}}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha+m\}}\Big]\\
&\geq \sum_{k\leq\ell'_n}\mathbf{E}\Big[\frac{1}{2ke^{(\log n)^{\alpha/2}}}\mathbb{1}_{\{S_k\geq\alpha_n,\,\max_{j\leq k}H^S_j\leq e^{(\log n)^{\alpha/2}},\,\underline{S}_k\geq -B,\,\overline{S}_k=S_k\}}\Big]\\
&\geq \frac{e^{-(\log n)^{\alpha/2}}}{2\ell'_n}\sum_{k\leq\ell'_n}\mathbf{P}\big(S_k\geq\alpha_n,\,\max_{j\leq k}H^S_j\leq e^{(\log n)^{\alpha/2}},\underline{S}_k\geq -B,\overline{S}_k=S_k\big),
\end{aligned}$$

where $\alpha_n=(\log n)^\alpha+\log n$. By Lemma 4.5 (with $\ell=(\log n)^2$, $t_\ell=\alpha_n$, $d=\alpha/4$ and $a=0$) the previous probability is larger than $e^{-(\log n)^{\alpha/2}(1+\frac{\varepsilon}{2})}$. Finally collecting the inequalities we get (28).

Facts for the case $b\in(0,1/2)$: for any $t\geq -B$, $r\geq 0$ and $w>0$

$$\mathbf{E}\Big[\sum_{x\in\mathscr{O}_n}\frac{e^{-V(x)}\mathbb{1}_{\{r+H_x>\lambda'_n/w\}}}{r+\sum_{j\leq|x|}H_{x_j}}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha-t\}}\Big]\leq(w+1)n^{-b}e^{\delta t-\frac{1-\varepsilon}{b}(\log n)^{\alpha-1}} \tag{30}$$

where we recall that $\lambda'_n=n^b(\log n)^{-2}$. Also for all $0\leq m\leq\log n$, $0\leq M\leq n^b$

$$\mathbf{E}\Big[\sum_{x\in\mathbb{T}}\frac{e^{-V(x)}\mathbb{1}_{\{x\in\Upsilon_{n,2}\cap\mathscr{O}_{\boldsymbol{\lambda}_{n,2},n^b}\}}}{M|x|+\sum_{j\leq|x|}H_{x_j}}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha+m\}}\Big]\geq n^{-b}e^{-\frac{1+\varepsilon}{b}(\log n)^{\alpha-1}} \tag{31}$$

with $\boldsymbol{\lambda}_{n,2}=ne^{-\frac{6}{b}(\log n)^{\alpha-1}}$ and

$$\Upsilon_{n,2}=\Upsilon_{n,2}(\varepsilon):=\{x\in\mathbb{T};H_x\leq n^be^{\frac{\varepsilon}{b}(\log n)^{\alpha-1}},\underline{V}(x)\geq -B\}.$$

We first deal with the upper bound (30). We split the sum according to the generation of x: when $|x|>\lfloor A\ell_n\rfloor$, we use that $\mathbb{1}_{\{r+H_x>\lambda'_n/w,V(x)\geq(\log n)^\alpha-t\}}(r+\sum_{j\leq|x|}H_{x_j})^{-1}\leq 1$ so expectation in (30) is smaller than

$$\mathbf{E}\Big[\sum_{|x|>\lfloor A\ell_n\rfloor}e^{-V(x)}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big]+\mathbf{E}\Big[\sum_{|x|\leq\lfloor A\ell_n\rfloor}\frac{e^{-V(x)}\mathbb{1}_{\{r+H_x>\lambda'_n/w\}}}{r+\sum_{j\leq|x|}H_{x_j}}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha-t\}}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big].$$

Then, when $|x|\leq\lfloor A\ell_n\rfloor$, we again split the sum but this time according to $\max_{j\leq|x|}H_{x_j}$: when $\max_{j\leq|x|}H_{x_j}>n^be^{\frac{1}{b}(\log n)^{\alpha-1}}$, we use that $\mathbb{1}_{\{r+H_x>\lambda'_n/w,V(x)\geq(\log n)^\alpha-t\}}(r+\sum_{j\leq|x|}H_{x_j})^{-1}\leq(\max_{j\leq|x|}H_{x_j})^{-1}\leq n^{-b}e^{-\frac{1}{b}(\log n)^{\alpha-1}}$. Otherwise, observe that $\mathbb{1}_{\{r+H_x>\lambda'_n/w\}}(r+\sum_{j\leq|x|}H_{x_j})^{-1}\leq\mathbb{1}_{\{r+H_x>\lambda'_n/w\}}(r+H_x)^{-1}\leq w/\lambda'_n$. Therefore, expectation in (30) is smaller than

$$\begin{aligned}
&\mathbf{E}\Big[\sum_{|x|>\lfloor A\ell_n\rfloor}e^{-V(x)}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big]+\mathbf{E}\Big[\sum_{|x|\leq\lfloor A\ell_n\rfloor}e^{-V(x)}\mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big]n^{-b}e^{-\frac{1}{b}(\log n)^{\alpha-1}}\\
&+\frac{w}{\lambda'_n}\mathbf{E}\Big[\sum_{|x|\leq\lfloor A\ell_n\rfloor}e^{-V(x)}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha-t,\,\max_{j\leq|x|}H_{x_j}\leq n^be^{\frac{1}{b}(\log n)^{\alpha-1}}\}}\Big],
\end{aligned}$$

which, by Remark 2 and many-to-one Lemma, is smaller, for n large enough, than

$$\frac{1}{n}+\ell_nn^{-b}e^{-\frac{1}{b}(\log n)^{\alpha-1}}+\frac{w}{\lambda'_n}\sum_{k\leq\lfloor A\ell_n\rfloor}\mathbf{P}\big(S_k\geq(\log n)^\alpha-t,\max_{j\leq k}H^S_j\leq n^be^{\frac{1}{b}(\log n)^{\alpha-1}}\big).$$

Also $\sum_{k\le\lfloor A\ell_n\rfloor}\mathbf{P}\big(S_k\ge(\log n)^\alpha-t,\max_{j\le k}H^S_j\le n^be^{-\frac1b(\log n)^{\alpha-1}}\big)$ is smaller than

$$\lfloor A\ell_n\rfloor\mathbf{P}\big(\max_{j\le\tau_{(\log n)^\alpha-t}}\overline{S}_j-S_j\le b\log n+\frac1b(\log n)^{\alpha-1}\big)\le e^{\frac{t}{\log n}-\frac1b(\log n)^{\alpha-1}(1-\frac{\varepsilon}{2})},$$

where Lemma A.3 in [HS16a] provides us the last inequality for n large enough and any t. Finally, note that for any $\delta>0$, n large, $w>0$ and $t\ge -B$, $1/n\le\frac13 n^{-b}e^{-\delta B-\frac{1-\varepsilon}{b}(\log n)^{\alpha-1}}\le\frac{w+1}{3}n^{-b}e^{\delta t-\frac{1-\varepsilon}{b}(\log n)^{\alpha-1}}$, $\ell_n n^{-b}e^{-\frac1b(\log n)^{\alpha-1}}\le\frac13 n^{-b}e^{-\delta B-\frac{1-\varepsilon}{b}(\log n)^{\alpha-1}}\le\frac{w+1}{3}n^{-b}e^{\delta t-\frac{1-\varepsilon}{b}(\log n)^{\alpha-1}}$, $\frac{w}{\lambda'_n}e^{\frac{t}{\log n}-\frac1b(\log n)^{\alpha-1}(1-\frac{\varepsilon}{2})}\le(w+1)n^{-b}(\log n)^2e^{\frac{t+B}{\log n}-\frac1b(\log n)^{\alpha-1}(1-\frac{\varepsilon}{2})}\le\frac{w+1}{3}n^{-b}e^{\delta t-\frac{1-\varepsilon}{b}(\log n)^{\alpha-1}}$ and this finish the proof of the first fact. We now turn to the lower bound (31). By many-to-one Lemma for any $m\le\log n$, $0\le M\le n^b$ and $A>0$, the mean in (31) is larger than (as $\boldsymbol{\lambda}_{n,2}>n^be^{\frac{\varepsilon}{3b}(\log n)^{\alpha-1}}$)

$$\sum_{k\le\lfloor A\ell_n\rfloor}\mathbf{E}\Big[\frac{1}{kn^b+\sum_{j=1}^k H^S_j}\mathbb{1}_{\{S_k\ge\alpha_n,\max_{1\le j\le k}H^S_j\le n^be^{\frac{\varepsilon}{3b}(\log n)^{\alpha-1}},H_k>n^b,\underline{S}_k\ge -B\}}\Big]$$
$$\ge\frac{n^{-b}}{2A\ell_n}e^{-\frac{\varepsilon}{3b}(\log n)^{\alpha-1}}\sum_{k\le\lfloor A\ell_n\rfloor}\mathbf{P}\big(S_k\ge\alpha_n,\max_{1\le j\le k}H^S_j\le n^be^{\frac{\varepsilon}{3b}(\log n)^{\alpha-1}},H_k>n^b,\underline{S}_k\ge -B\big),$$

with $\alpha_n:=(\log n)^\alpha+\log n$. By Lemma 4.3 (76) (with $\ell=(\log n)^2$, $t_\ell=\alpha_n$, $q=b$, $a_b=-a=-\frac{\varepsilon}{3b}$, $d=\frac{\alpha-1}{2}$ and $c=\frac{\varepsilon}{3b}$.) the above sum is larger, for n large enough, than $e^{-\frac1b(\log n)^{\alpha-1}(1+\frac{\varepsilon}{2})}\ge 2A\ell_n e^{-\frac1b(\log n)^{\alpha-1}(1+\varepsilon)}$, which completes the proof of the upper bound.

We are ready to prove that $\mathbf{f}^n$ satisfies assumptions (**A1**), (**A2**), (**A3**) and (**A4**).
• Check of (**A1**) and asymptotic of h_n. (28) with $m=M=0$ implies for $b=0$ and n large enough

$$\sum_{k\ge1}\Psi^k_n(f^{n,k})\ge\mathbf{E}\Big[\sum_{x\in\mathbb{T}}\frac{e^{-V(x)}}{\sum_{j\le|x|}H_{x_j}}\mathbb{1}_{\{V(x)\ge(\log n)^\alpha\}}\mathbb{1}_{\{x\in\Upsilon_{n,1}\cap\mathscr{O}_{\boldsymbol{\lambda}_{n,1}}\}}\Big]\ge e^{-2(\log n)^{\alpha/2}(1+\varepsilon)}.$$

This implies that $\kappa_0=\max\mathscr{U}_0=0$ (see the part concerning κ_b of the proof of Theorem 1.1 for details) and additionaly with (27) and $t=0$

$$h_n=\Big|n^{\kappa_b}\log\big(\sum_{k\ge1}\Psi^k_{n,n^b}(f^{n,k})\big)\Big|\sim 2(\log n)^{\alpha/2}.$$

We also deduce from the previous lower bound that (**A1**) is satisfied.
From (30) with $r=t=0$, $w=1$ and $\frac{\varepsilon}{2}$ instead of ε we get for all $b\in(0,1)$ and n large enough

$$\sum_{k\ge1}\Psi^k_{n,n^b}(f^{n,k})\le\mathbf{E}\Big[\sum_{x\in\mathscr{O}_n}\frac{e^{-V(x)}\mathbb{1}_{\{H_x>\lambda'_n\}}}{\sum_{j\le|x|}H_{x_j}}\mathbb{1}_{\{V(x)\ge(\log n)^\alpha\}}\Big]\le n^{-b}e^{-\frac{1-\varepsilon}{b}(\log n)^{\alpha-1}}.$$

This implies that for all $b\in(0,1)$, $\kappa_b\ge b$. From (31) with $m=M=0$, we get that for all $b\in(0,1)$

$$\sum_{k\ge1}\Psi^k_{n,n^b}(f^{n,k})\ge\mathbf{E}\Big[\sum_{x\in\mathbb{T}}\frac{e^{-V(x)}}{\sum_{j\le|x|}H_{x_j}}\mathbb{1}_{\{V(x)\ge(\log n)^\alpha\}}\mathbb{1}_{\{x\in\Upsilon_{n,2}\cap\mathscr{O}_{\boldsymbol{\lambda}_{n,2},n^b}\}}\Big]\ge n^{-b}e^{-\frac{1+\varepsilon}{b}(\log n)^{\alpha-1}}.$$

This implies that for all $b\in(0,1/2)$, $\kappa_b\le b$. Finally, for any $b\in(0,1/2)$, $\kappa_b=b$ and

$$h_n=\Big|n^{\kappa_b}\log\big(\sum_{k\ge1}\Psi^k_{n,n^b}(f^{n,k})\big)\Big|\sim\frac1b(\log n)^{\alpha-1}.$$

We also deduce from the previous lower bound that (**A1**) is satisfied.
• For (**A2**), recalling $m_n = \lceil \varepsilon h_n / r_2 \rceil$ (see (13)) and for all $\mathbf{s} = (s_1, \dots, s_{m_n}) \in \mathbb{R}^{m_n}$, $\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k$, with $\mathbf{u} = (s_1, \dots, s_{m_n}, t_1 + s_{m_n}, \dots, t_k + s_{m_n})$

$$f^{n,m_n+k}(s_1, \dots, s_{m_n}, t_1 + s_{m_n}, \dots, t_k + s_{m_n}) = \mathbb{1}_{\{t_k + s_{m_n} \geq (\log n)^\alpha\}} \frac{1}{\sum_{j=1}^{m_n+k} H_j(\mathbf{u})} \tag{32}$$

Note that $\sum_{j=1}^{m_n+k} H_j(\mathbf{u}) = \sum_{j=1}^{m_n} H_j(\mathbf{s}) + \sum_{j=1}^{k} \left(e^{-t_j} H_{m_n}(\mathbf{s}) + H_j(\mathbf{t})\right) \geq \sum_{j=1}^{k} H_j(\mathbf{t})$ so

$$\begin{aligned} f^{n,k}_{\varepsilon h_n}(t_1, \dots, t_k) &= \inf_{\mathbf{s} \in [-\varepsilon h_n, \varepsilon h_n]^{m_n}} f^{n,m_n+k}(s_1, \dots, s_{m_n}, t_1 + s_{m_n}, \dots, t_k + s_{m_n}) \\ &\leq \inf_{s_{m_n} \in [-\varepsilon h_n, \varepsilon h_n]} \mathbb{1}_{\{t_k + s_{m_n} \geq (\log n)^\alpha\}} \frac{1}{\sum_{j=1}^{k} H_j(\mathbf{t})} = \frac{\mathbb{1}_{\{t_k \geq (\log n)^\alpha + \varepsilon h_n\}}}{\sum_{j=1}^{k} H_j(\mathbf{t})}. \end{aligned}$$

It follows that $f^{n,k}_{\varepsilon h_n}(t_1, \dots, t_k) \leq \mathbb{1}_{\{t_k \geq (\log n)^\alpha\}} \left(\sum_{j=1}^{k} H_j(\mathbf{t}_j)\right)^{-1}$ and for $|x| = k$ with $\mathbf{u}_x = (t_1, \dots, t_l, V(x_1) + t_l, \dots, V(x) + t_l)$

$$f^{n,l+k}_{\varepsilon h_n}(t_1, \dots, t_l, V(x_1) + t_l, \dots, V(x) + t_l) \leq \mathbb{1}_{\{V(x) \geq (\log n)^\alpha - t_l\}} \frac{1}{\sum_{j=1}^{l+k}(\mathbf{u}_x)}.$$

Assume $b = 0$. Observe again that $\sum_{j=1}^{l+k}(\mathbf{u}_x) = \sum_{j=1}^{l} H_j(\mathbf{t}) + \sum_{j=1}^{k} \left(e^{-V(x_j)} H_l(\mathbf{t}) + H_{x_j}\right) \geq \sum_{j \leq k} H_{x_j}$. Then by definition of $\Psi^k_n(F|\mathbf{t})$ (see (15)), for all $A, B, \varepsilon, \delta > 0$, n large enough, for any $l \in \mathbb{N}^*$ and all $\mathbf{t} = (t_1, \dots, t_l) \in \mathbb{R}^l$ with $t_l \geq -B$

$$\begin{aligned} \sum_{k \geq 1} \Psi^k_{n, n^b - H_l(\mathbf{t})}\left(f^{n,l+k}_{\varepsilon h_n} | \mathbf{t}\right) &\leq \mathbf{E}\Big[\sum_{x \in \mathscr{O}_n} e^{-V(x)} f^{n,l+k}_{\varepsilon h_n}(t_1, \dots, t_l, V(x_1) + t_l, \dots, V(x) + t_l) \Big] \\ &\leq \mathbf{E}\Big[\sum_{x \in \mathscr{O}_n} e^{-V(x)} \mathbb{1}_{\{V(x) \geq (\log n)^\alpha - t_l\}} \frac{1}{\sum_{j \leq k} H_{x_j}} \Big] \leq e^{\delta t_l - 2(\log n)^{\alpha/2}(1 - \frac{\varepsilon}{3A})}, \end{aligned}$$

where we have used (27) with $t = t_l$ and replaced ε by $\frac{\varepsilon}{3A}$ for the last inequality. Finally, observe that

$$e^{-2(\log n)^{\alpha/2}(1 - \frac{\varepsilon}{3A})} = e^{\frac{4\varepsilon}{3A}(\log n)^{\alpha/2}} e^{-2(\log n)^{\alpha/2}(1 + \frac{\varepsilon}{3A})} \leq e^{\frac{\varepsilon}{A} h_n} \sum_{k \geq 1} \Psi^k_{n, n^b}(f^{n,k}),$$

where we used that $h_n \sim 2(\log n)^{\alpha/2}$ and (28) with $m = M = 0$.
Assume $b \in (0, 1/2)$. Note that $\sum_{j=1}^{l+k} H_j(\mathbf{u}_x) \geq H_l(\mathbf{t}) + \sum_{j \leq k} H_{x_j}$. Then for all $A, B, \varepsilon, \delta > 0$, n large enough, for any $l \in \mathbb{N}^*$ and all $\mathbf{t} = (t_1, \dots, t_l) \in \mathbb{R}^l$ with $t_l \geq -B$

$$\begin{aligned} \sum_{k \geq 1} \Psi^k_{n, n^b - H_l(\mathbf{t})}\left(f^{n,l+k}_{\varepsilon h_n} | \mathbf{t}\right) &\leq \mathbf{E}\Big[\sum_{x \in \mathscr{O}_n} e^{-V(x)} \frac{\mathbb{1}_{\{V(x) \geq (\log n)^\alpha - t_l\}}}{H_l(\mathbf{t}) + \sum_{j \leq |x|} H_{x_j}} \mathbb{1}_{\{H_l(\mathbf{t}) + H_x > \lambda'_n\}} \Big] \\ &\leq 2n^{-b} e^{\delta t_l - \frac{1}{b}(\log n)^{\alpha - 1}(1 - \frac{\varepsilon}{4A})} \leq n^{-b} e^{\delta t_l - \frac{1}{b}(\log n)^{\alpha-1}(1 - \frac{\varepsilon}{3A})}, \end{aligned}$$

where we used (30) with $r = H_l(\mathbf{t})$, $w = 1$, $t = t_l$ and $\frac{\varepsilon}{4A}$ instead of ε for the last inequality. Finally, observe that

$$n^{-b} e^{\delta t_l - \frac{1}{b}(\log n)^{\alpha-1}(1 - \frac{\varepsilon}{3A})} = e^{\frac{2\varepsilon}{3bA}(\log n)^{\alpha-1}} n^{-b} e^{-\frac{1}{b}(\log n)^{\alpha-1}(1 + \frac{\varepsilon}{3A})} \leq e^{\frac{\varepsilon}{A} h_n} \sum_{k \geq 1} \Psi^k_{n, n^b}(f^{n,k}),$$

where we used that $h_n \sim \frac{1}{b}(\log n)^{\alpha-1}$ and (31) with $m = M = 0$.

We are left to prove that technical assumptions (**A3**) and (**A4**) are realised.
• For (**A3**), recall that $\Upsilon_n^k = \{\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k; H_k(\mathbf{t}) \le n^b e^{\varepsilon h_n}, V(x) \ge v_n', t_k \ge -B\}$, whith $v_n' = \frac{1}{\delta_1}\log(n\ell_n) + \varepsilon h_n$. By (32), for $|x| = k$ with $\mathbf{v}_x = (s_1, \dots, s_{m_n}, V(x_1) + s_{m_n}, \dots, V(x) + s_{m_n})$

$$f_{\varepsilon h_n}^{n,k}(V(x_1), \dots, V(x)) = \inf_{\mathbf{s}\in[-\varepsilon h_n, \varepsilon h_n]^{m_n}} \mathbb{1}_{\{V(x)+s_{m_n} \ge (\log n)^\alpha\}} \frac{1}{\sum_{j=1}^{m_n+k} H_j(\mathbf{v}_x)}.$$

and recall that $\sum_{j=1}^{m_n+k} H_j(\mathbf{v}_x) = \sum_{j=1}^{m_n} H_j(\mathbf{s}) + \sum_{j=1}^{k} \left(e^{-V(x_j)} H_{m_n}(\mathbf{s}) + H_{x_j}\right)$. For $|x| = k$ such that $\underline{V}(x) \ge -B$, observe as $\mathbf{s} \in [-\varepsilon h_n, \varepsilon h_n]^{m_n}$, that $\sum_{j=1}^{m_n+k} H_j(\mathbf{v}_x) \le m_n e^{2\varepsilon h_n} + k m_n^2 e^{2\varepsilon h_n + B} + \sum_{j=1}^{k} H_{x_j}$. Also recall, by definition, that $h_n \ge (\log n)^\gamma$ for $\gamma \in (0,1)$ so $\sum_{j=1}^{m_n+k} H_j(\mathbf{v}_x) \le 2k m_n^2 e^{2\varepsilon h_n + B} + \sum_{j=1}^{k} H_{x_j} \le k e^{3\varepsilon h_n} + \sum_{j=1}^{k} H_{x_j}$. It follows that

$$f_{\varepsilon h_n}^{n,k}(V(x_1), \dots, V(x)) \ge \mathbb{1}_{\{V(x) \ge (\log n)^\alpha + \varepsilon h_n\}} \Big(k e^{3\varepsilon h_n} + \sum_{j=1}^{k} H_{x_j}\Big)^{-1}.$$

Let $0 < \varepsilon_1 < \epsilon$ and recall $\lambda_n = n e^{-5h_n} \ge 2\boldsymbol{\lambda}_{n,i}$, $i \in \{1, 2\}$. Thanks to the previous inequality and the fact that $(\log n)^\alpha > \frac{1}{\delta_1}\log(n\ell_n)$, we have

$$\sum_{k\ge1} \Psi_{\lambda_n/2, n^b}^k (f_{\varepsilon h_n}^{n,k} \mathbb{1}_{\Upsilon_n^k}) \ge \mathbf{E}\Big[\sum_{x\in\mathscr{O}_{\boldsymbol{\lambda}_{n,i}, n^b}} \frac{e^{-V(x)} \mathbb{1}_{\{V(x)\ge(\log n)^\alpha + \varepsilon h_n\}}}{|x| e^{3\varepsilon h_n} + \sum_{j\le|x|} H_{x_j}} \mathbb{1}_{\{H_x \le n^b e^{\varepsilon h_n}, \underline{V}(x) \ge -B\}}\Big].$$

Assume $b = 0$. By (28) with $m = h_n$, $M = e^{(\log n)^{\alpha/2}}$ and $\frac{\varepsilon_1}{3}$ instead of ε, together with the fact that $h_n \sim 2(\log n)^{\alpha/2}$, for n large enough

$$\begin{aligned}\sum_{k\ge1} \Psi_{\lambda_n/2}^k (f_{\varepsilon h_n}^{n,k} \mathbb{1}_{\Upsilon_n^k}) &\ge \mathbf{E}\Big[\sum_{x\in\mathbb{T}} \frac{e^{-V(x)} \mathbb{1}_{\{V(x)\ge(\log n)^\alpha + h_n\}}}{|x| e^{(\log n)^{\alpha/2}} + \sum_{j\le|x|} H_{x_j}} \mathbb{1}_{\{x\in\Upsilon_{n,1}(\frac{\varepsilon_1}{3})\cap\mathscr{O}_{\boldsymbol{\lambda}_{n,1}}\}}\Big] \\ &\ge e^{-2(\log n)^{\alpha/2}(1+\frac{\varepsilon_1}{3})}.\end{aligned}$$

Moreover $e^{-2(\log n)^{\alpha/2}(1+\frac{\varepsilon_1}{3})} = e^{-\frac{4\varepsilon_1}{3}(\log n)^{\alpha/2}} e^{-2(\log n)^{\alpha/2}(1-\frac{\varepsilon_1}{3})} \ge e^{-\varepsilon_1 h_n} \sum_{k\ge1} \Psi_{n,n^b}^k(f^{n,k})$, the last inequality comes from the fact that $h_n \sim 2(\log n)^{\alpha/2}$ and (27) with $t = 0$.
Assume $b \in (0, 1/2)$. By (31) with $m = h_n$ and $M = n^b$, together with the fact that $h_n \sim \frac{1}{b}(\log n)^{\alpha-1}$, implies for n large enough

$$\begin{aligned}\sum_{k\ge1} \Psi_{\lambda_n/2, n^b}^k (f_{\varepsilon h_n}^{n,k} \mathbb{1}_{\Upsilon_n^k}) &\ge \mathbf{E}\Big[\sum_{x\in\mathbb{T}} \frac{e^{-V(x)} \mathbb{1}_{\{V(x)\ge(\log n)^\alpha + h_n\}}}{|x| n^b + \sum_{j\le|x|} H_{x_j}} \mathbb{1}_{\{x\in\Upsilon_{n,2}(\frac{\varepsilon_1}{3})\cap\mathscr{O}_{\boldsymbol{\lambda}_{n,2}}\}}\Big] \\ &\ge n^{-b} e^{-\frac{1}{b}(\log n)^{\alpha-1}(1+\frac{\varepsilon_1}{3})}.\end{aligned}$$

Moreover $e^{-\frac{1}{b}(\log n)^{\alpha-1}(1+\frac{\varepsilon_1}{3})} = e^{-\frac{2\varepsilon_1}{3b}(\log n)^{\alpha-1}} e^{-\frac{1}{b}(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{3})} \ge n^b e^{-\varepsilon_1 h_n} \sum_{k\ge1} \Psi_{n,n^b}^k(f^{n,k})$, the last inequality comes from the fact that $h_n \sim \frac{1}{b}(\log n)^{\alpha-1}$ and (30) with $r = t = 0$, $w = 1$ and we have used that $\lambda_n' = n^b(\log n)^{-2} < n^b$.
• Finally for (**A4**), we first observe that for all $k \in \mathbb{N}^*$, $\alpha \in (1, 2)$, $(\log n)^\alpha > \frac{1}{\delta_1}\log(n\ell_n)$ for n large enough so

$$\Psi_n^k(f^{n,k} \mathbb{1}_{\mathbb{R}\setminus\mathscr{H}_{v_n}^k}) = \mathbf{E}\Big[\sum_{|x|=k} \frac{e^{-V(x)}}{\sum_{j=1}^k H_{x_j}} \mathbb{1}_{\{V(x)\ge(\log n)^\alpha, V(x) < \log(n\ell_n)/\delta_1\}} \mathbb{1}_{\{x\in\mathscr{O}_n\}}\Big] = 0.$$

Recall that $W=\sum_{|z|=1}e^{-V(z)}$ and $\mathbf{E}[W]=e^{\psi(1)}=1$ so when $b=0$

$$\sum_{k\geq 1}\big(\Psi^k_{n,\lambda'_n}(f^{n,k})+\mathbf{E}\big[W\Psi^k_{n,\lambda'_n/W}(f^{n,k})\big]\big)\leq\sum_{k\geq 1}\big(\Psi^k_{n}(f^{n,k})+\mathbf{E}\big[W\Psi^k_{n}(f^{n,k})\big]\big)=2\sum_{k\geq 1}\Psi^k_{n}(f^{n,k})$$

and thanks to (27) for n large enough with $t=0$

$$2\sum_{k\geq 1}\Psi^k_{n}(f^{n,k})=2\mathbf{E}\Big[\sum_{x\in\mathscr{O}_n}\frac{e^{-V(x)}}{\sum_{j\leq|x|}H_{x_j}}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha\}}\Big]\leq 2e^{-2(\log n)^{\alpha/2}(1-\frac{\varepsilon_1}{4})}\leq e^{-2(\log n)^{\alpha/2}(1-\frac{\varepsilon_1}{3})}.$$

Moreover $e^{-2(\log n)^{\alpha/2}(1-\frac{\varepsilon_1}{3})}=e^{\frac{4\varepsilon_1}{3}(\log n)^{\alpha/2}}e^{-2(\log n)^{\alpha/2}(1+\frac{\varepsilon_1}{3})}\leq e^{\varepsilon_1 h_n}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k})$, the last inequality comes from the fact that $h_n\sim 2(\log n)^{\alpha/2}$ and (28) with $m=M=0$.
Otherwise, $b\in(0,1/2)$ and thanks to (30) for n large enough with $r=t=0$, $w=1$ and $\frac{\varepsilon_1}{4}$ instead of ε

$$\sum_{k\geq 1}\Psi^k_{n,\lambda'_n}(f^{n,k})=\Big[\sum_{x\in\mathscr{O}_n}\frac{e^{-V(x)}\mathbb{1}_{\{H_x>\lambda'_n\}}}{r+\sum_{j\leq|x|}H_{x_j}}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha\}}\Big]\leq\frac{1}{n^b}e^{-\frac{1}{b}(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{3})},$$

and we also get from (30) with $r=t=0$ and $w=W$ that for n large enough

$$\Psi^k_{n,\lambda'_n/W}(f^{n,k})=\mathbf{E}\Big[\sum_{x\in\mathscr{O}_n}\frac{e^{-V(x)}\mathbb{1}_{\{H_x>\lambda'_n/W\}}}{r+\sum_{j\leq|x|}H_{x_j}}\mathbb{1}_{\{V(x)\geq(\log n)^\alpha\}}\Big]\leq\frac{W+1}{n^b}e^{-\frac{1}{b}(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{4})}.$$

By (4), telling that $\mathbf{E}[W^2]<\infty$, we have $C_4:=\mathbf{E}[W(W+1)+1]=\mathbf{E}[W^2+2]<\infty$ and then

$$\sum_{k\geq 1}\big(\Psi^k_{n,\lambda'_n}(f^{n,k})+\mathbf{E}\big[W\Psi^k_{n,\lambda'_n/W}(f^{n,k})\big]\big)\leq\frac{C_4}{n^b}e^{-\frac{1}{b}(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{4})}\leq\frac{1}{2n^b}e^{-\frac{1}{b}(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{3})}.$$

Moreover $e^{-\frac{1}{b}(\log n)^{\alpha-1}(1-\frac{\varepsilon_1}{3})}=e^{\frac{\varepsilon_1}{3b}(\log n)^{\alpha-1}}e^{-\frac{1}{b}(\log n)^{\alpha-1}(1+\frac{\varepsilon_1}{3})}\leq n^b e^{\varepsilon_1 h_n}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k})$, the last inequality comes from the fact that $h_n\sim\frac{1}{b}(\log n)^{\alpha-1}$ and (31) with $m=M=0$. Proof is complete for these two cases.
Assume now $\alpha=1$ and $a\in\mathbb{R}$ (with $a>1/\delta_1$ when $d=1$) which corresponds to the first case of the theorem. As usual, let us first state the following two facts:
for all $b\in[0,1/(d+1))$, $B,\delta>0$, $\varepsilon\in(0,\varepsilon_b)$ and n large enough, for any $t\geq -B$, $r\geq 0$ and $w>0$

$$\mathbf{E}\Big[\sum_{x\in\mathscr{O}_n}\frac{e^{-V(x)}\mathbb{1}_{\{r+H_x>\lambda'_n/w\}}}{\big(r+\sum_{j\leq|x|}H_{x_j}\big)^d}\mathbb{1}_{\{V(x)\geq a\log n-t\}}\Big]\leq(w+1)\ell_n^2e^{\delta t}n^{-bd},\tag{33}$$

with $\lambda'_n=n^b(\log n)^{-2}$. For any $0\leq M\leq n^b$, $\varepsilon<b/3$ (when $b>0$)

$$\mathbf{E}\Big[\sum_{x\in\mathbb{T}}\frac{e^{-V(x)}\mathbb{1}_{\{x\in\Upsilon_n\cap\mathscr{O}_{\boldsymbol{\lambda}_n,n^b}\}}}{\big(M|x|+\sum_{j\leq|x|}H_{x_j}\big)^d}\mathbb{1}_{\{V(x)\geq a\log n\}}\Big]\geq\frac{1}{\ell_n^2}n^{-bd},\tag{34}$$

with $\boldsymbol{\lambda}_n=n^{1-11\varepsilon}$ and for any $a'>1/\delta_1$

$$\Upsilon_n=\Upsilon_n(\varepsilon):=\{x\in\mathbb{T};H_x\leq n^{b+\varepsilon},V(x)\geq a'\log n,\underline{V}(x)\geq -B\}.$$

These facts ensure that $\mathbf{f}^n$ satisfies assumptions (**A1**), (**A2**), (**A3**) and (**A4**) for $b \in (0, 1/(d+1))$. (**A3**) does not hold exactly when $b = 0$ so we use (38) (which appears in the proof of Therorem 1.5) together with the result when $b > 0$ to conclude this case.

• Check of (**A1**) and asymptotic of h_n. We get from (34) that $\kappa_b = \max \mathscr{U}_b \le bd$ and (33) gives $\kappa_b \ge bd$. It follows that for all $b \in [0, 1/(d+1))$, $\kappa_b = bd$ and for any $n \ge 2$, $h_n = \log n$. Indeed, on the one hand, (33) with $r = t = 0$ and $w = 1$ gives, for n large enough

$$n^{\kappa_b} \sum_{k\ge 1} \Psi^k_{n,n^b}(f^{n,k}) = n^{bd} \mathbf{E}\Big[\sum_{x\in \mathscr{O}_{n,n^b}} \frac{e^{-V(x)}}{\big(\sum_{j=1}^k H_{x_j}\big)^d} \mathbb{1}_{\{V(x)\ge a\log n\}}\Big] \le 2^d \ell_n^2,$$

and on the other hand, we get from (34), for all n large enough that

$$n^{\kappa_b} \sum_{k\ge 1} \Psi^k_{n,n^b}(f^{n,k}) \ge \frac{1}{\ell_n^2}.$$

From these inequalities, we get that for any $\gamma \in (0,1)$, $|\log(n^{\kappa_b} \sum_{k\ge1} \Psi^k_{n,n^b}(f^{n,k}))| \le 3\log \ell_n = o((\log n)^\gamma)$. Then $h_n = \log n$ and we also deduce that (**A1**) is satisfied.

• For (**A2**), let $|x| = k$ and observe that $f^{n,l+k}_{\varepsilon h_n}(t_1,\dots,t_l,V(x_1),\dots,V(x)) \le (H_l(\mathbf{t}) + H_x)^{-d}$ so it follows, for all $\varepsilon \in (0,\varepsilon_b)$, $A,\delta,B > 0$, n large enough, any $l \in \mathbb{N}^*$, $\mathbf{t} = (t_1,\dots,t_l) \in \mathbb{R}^l$ and $t_l \ge -B$, by (33) with $r = H_l(\mathbf{t})$, $t = t_l$ and $w = 1$

$$\sum_{k\ge1} \Psi^k_{n,n^b - H_l(\mathbf{t})}\big(f^{n,l+k}_{\varepsilon h_n}|\mathbf{t}\big) \le 2^d \ell_n^2 e^{\delta t_l} n^{-bd} \le e^{\delta t_l + \frac{\varepsilon}{A} h_n} \sum_{k\ge1} \Psi^k_{n,n^b}(f^{n,k})$$

where last inequality comes from (34).

• For (**A3**), recall that $\Upsilon^k_n = \{\mathbf{t} = (t_1,\dots,t_k) \in \mathbb{R}^k; H_k(\mathbf{t}) \le n^{b+\varepsilon}, V(x) \ge v'_n, t_k \ge -B\}$, where we recall $v'_n = \frac{1}{\delta_1}\log(n\ell_n) + \varepsilon h_n$. For $|x| = k$, we have

$$f^{n,k}_{\varepsilon h_n}(V(x_1),\dots,V(x)) \ge \mathbb{1}_{\{V(x)\ge(a+\varepsilon)(\log n)\}} \Big(kn^{3\varepsilon} + \sum_{j=1}^k H_{x_j}\Big)^{-d},$$

and thanks to (34) with $M = n^b$, $b \in (0, 1/(d+1))$

$$\sum_{k\ge1} \Psi^k_{\lambda_n/2,n^b}(f^{n,k}_{\varepsilon h_n}\mathbb{1}_{\Upsilon^k_n}) \ge \mathbf{E}\Big[\sum_{x\in\mathbb{T}} \frac{e^{-V(x)}\mathbb{1}_{\{\Upsilon_n\cap\mathscr{O}_{\boldsymbol{\lambda}_n,n^b}\}}}{(|x|n^b + \sum_{j\le|x|} H_{x_j})^d}\Big] \ge \frac{1}{\ell_n^2} n^{-bd} \ge e^{-\varepsilon_1 h_n} \sum_{k\ge1} \Psi^k_{n,n^b}(f^{n,k}),$$

where we recall $\lambda_n = n^{1-10\varepsilon}$.

• Finally for (**A4**) with $d = 1$ (and then $a > 1/\delta_1$)

$$\Psi^k_n(f^{n,k}\mathbb{1}_{\mathbb{R}\setminus\mathscr{H}^k_{v_n}}) = \mathbf{E}\Big[\sum_{|x|=k} \frac{e^{-V(x)}\mathbb{1}_{\{x\in\mathscr{O}_n\}}}{\big(\sum_{j=1}^k H_{x_j}\big)^d} \mathbb{1}_{\{V(x)\ge a\log n, V(x)<\log(n\ell_n)/\delta_1\}}\Big] = 0,$$

otherwise, $d = 0$ and for any $a \in \mathbb{R}$, thanks to Remark 2

$$\sum_{k\ge1} \Psi^k_n(f^{n,k}\mathbb{1}_{\mathbb{R}\setminus\mathscr{H}^k_{v_n}}) = \mathbf{E}\Big[\sum_{x\in\mathscr{O}_n} e^{-V(x)}\mathbb{1}_{\{V(x)\ge a\log n, V(x)<\log(n\ell_n)/\delta_1\}}\Big] \le \mathbf{E}\Big[\sum_{x\in\mathscr{O}_n} e^{-V(x)}\Big] \le \ell_n,$$

which, thanks to (34) is smaller than $e^{\varepsilon_1 h_n}\sum_{k\ge1}\Psi^k_{n,n^b}(f^{n,k})$ for all $\varepsilon_1 > 0$. We get from (33) with $r = t = 0$ and $w = W$ that for n large enough

$$\Psi^k_{n,\lambda'_n/W}(f^{n,k}) = \mathbf{E}\Big[\sum_{x\in\mathscr{O}_n} \frac{e^{-V(x)}\mathbb{1}_{\{H_x>\lambda'_n/W\}}}{(\sum_{j\le|x|} H_{x_j})^d} \mathbb{1}_{\{V(x)\ge a\log n\}}\Big] \le (W+1)\ell_n^2 n^{-bd}$$

By (4), telling that $\mathbf{E}[W^2]<\infty$, we have $C_4:=\mathbf{E}[W(W+1)+1]=\mathbf{E}[W^2+2]<\infty$ and then

$$\sum_{k\geq 1}\big(\Psi^k_{n,\lambda'_n}(f^{n,k})+\mathbf{E}\big[W\Psi^k_{n,\lambda'_n/W}(f^{n,k})\big]\big)\leq 2C_4\ell n^{-bd}\leq e^{\varepsilon_1 h_n}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k}),$$

where, again, last inequality comes from (34). This finishes the proof of the result of the theorem for $b\in(0,1/(d+1))$.
Now assume $b=0$ and let $\varepsilon>0$. Using the result of the theorem with $b_\varepsilon=\varepsilon/(2+d)$ and the fact that $\mathscr{R}_n(\mathbb{1}_{[n^{b_\varepsilon},\infty)},\mathbf{f}^n)\leq\mathscr{R}_n(\mathbb{1}_{[1,\infty)},\mathbf{f}^n)$, we get the following lower bound for $\mathscr{R}_n(\mathbb{1}_{[1,\infty)},\mathbf{f}^n)$: $\mathbb{P}(\log^+\mathscr{R}_n(\mathbb{1}_{[1,\infty)},\mathbf{f}^n)<(1-\varepsilon)\log n)$ is smaller than

$$\mathbb{P}\big(\log^+\mathscr{R}_n(\mathbb{1}_{[n^{b_\varepsilon},\infty)},\mathbf{f}^n)<(1-(1+d)b_\varepsilon-\varepsilon/(2+d))\log n\big)\to 0,$$

where we have used the case $b>0$. For the upper bound, we use an intermediate result in the proof of Theorem 1.5: recall that $\kappa_0=0$ and $h_n=\log n$.
Also recall $\xi=\lim_{n\to\infty}h_n^{-1}\log(n^{\kappa_b}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{k,n}))$. It's easy to see that $\xi=0$ and by (38)

$$\mathbb{P}(\log^+\mathscr{R}_n(\mathbb{1}_{[1,\infty)},\mathbf{f}^n)>(1+\varepsilon)\log n)\leq\mathbb{P}\Big(\frac{\mathscr{R}_n(g_n,\mathbf{f}^n)}{n}>e^{\varepsilon h_n}\Big)\to 0,$$

this ends the proof of the theorem for all $b\in[0,1/(d+1))$. □

Proof of Theorem 1.4. Here $f^{n,k}(t_1,t_2,\cdots,t_k)=\mathbb{1}_{\{t_{\lfloor k/\beta\rfloor}\geq(\log n)^\alpha\}}(\sum_{j=1}^{\lfloor k/\beta\rfloor}H_j(\mathbf{t}))^{-1}$ with $\beta>1$ and $\alpha\in(1,2)$. We state the following facts: for all $B,\delta>0$, $\varepsilon\in(0,\varepsilon_b)$, n large enough, any $t\geq -B$ and $i\in\mathbb{N}$

$$\mathbf{E}\Big[\sum_{x;\lfloor|x|+i/\beta\rfloor>i}\frac{e^{-V(x)}\mathbb{1}_{\{x\in\mathscr{O}_n\}}}{\sum_{j=1}^{\lfloor|x|/\beta\rfloor}H_{x_j}}\mathbb{1}_{\{V(x_{\lfloor(|x|+i)/\beta\rfloor-i})\geq(\log n)^\alpha-t\}}\Big]\leq e^{\delta t-2(\log n)^{\alpha/2}(1-\varepsilon)},\tag{35}$$

and for all $0\leq i,m\leq\log n$, $0\leq M\leq e^{(\log n)^{\alpha/2}}$

$$\mathbf{E}\Big[\sum_{x;\lfloor|x|+i/\beta\rfloor>i}\frac{e^{-V(x)}\mathbb{1}_{\{x\in\Upsilon_n\cap\mathscr{O}_{\boldsymbol{\lambda}_n}\}}}{M|x|+\sum_{j=1}^{\lfloor|x|/\beta\rfloor}H_{x_j}}\mathbb{1}_{\{V(x_{\lfloor(|x|+i)/\beta\rfloor-i})\geq(\log n)^\alpha+m\}}\Big]\geq e^{-2(\log n)^{\alpha/2}(1+\varepsilon)},\tag{36}$$

with $\boldsymbol{\lambda}_n=ne^{-12(\log n)^{\alpha/2}}$ and for any $a>\frac{1}{\delta_1}$

$$\Upsilon_n=\Upsilon_n(\varepsilon):=\{x\in\mathbb{T};H_x\leq e^{2\varepsilon(\log n)^{\alpha/2}},V(x)\geq a\log n,\underline{V}(x)\geq -B\}.$$

From these two facts we follows the same lines as in the previous theorem to prove that $h_n\sim 2(\log n)^{\alpha/2}$ and that Assumptions (**A1**) to (**A4**) are satisfied. □

2.3. Proof of Theorem 1.5

First, note that Remark 1 implies that $\xi=\lim_{n\to\infty}h_n^{-1}\log(n^{\kappa_b}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{k,n}))$ well exists. To prove Theorem we first show that assumptions (**A3**) and (**A4**) yield a simpler statement for both lower and upper bound of Proposition 1. This implies a convergence in probability for stopped ranges $\mathscr{R}_{T^{k_n}}$ with $k_n=\lceil n/(\log n)^{3/2}\rceil$ and $\mathscr{R}_{T^n}$. Then we use result of [HS16b] (Proposition 2.4) implying that $T_n/(n\log n)$ converges in probability to a positive limit to obtain the result for $\mathscr{R}_n$. Let us start with the

Lower bound : Recalling the expression of $u_{1,n} = \sum_{k\geq 1} \Psi^k_{\lambda_n/2,n^b}(f^{n,k}_{\varepsilon h_n}\mathbb{1}_{\Upsilon^k_n})$ (see below (18)) togeher with (**A3**) choosing $\varepsilon_1 = \min(1,c_5)\frac{\varepsilon}{4}$ (see Proposition 1 for c_5) we get

$$u_{1,n} \geq e^{-\min(1,c_5)\frac{\varepsilon}{4}} \sum_{k\geq 1} \Psi^k_{n,n^b}\big(f^{k,n}\big).$$

This together with the fact that, by definition of ξ, for n large enough $n^{\kappa_b}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k}) > e^{(\xi-\varepsilon)h_n}$, implies

$$\begin{aligned}\mathbb{P}\Big(\frac{\mathscr{R}_{T^{k_n}}(g_n,\mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} < e^{(\xi-7\varepsilon)h_n}\Big) &\leq \mathbb{P}\Big(\frac{\mathscr{R}_{T^{k_n}}(g_n,\mathbf{f}^n)}{n^{1-b}\varphi(n^b)\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{k,n})} < e^{-6\varepsilon h_n}\Big)\\ &\leq \mathbb{P}\Big(\frac{\mathscr{R}_{T^{k_n}}(g_n,\mathbf{f}^n)}{n^{1-b}\varphi(n^b)u_{1,n}} < e^{-5\varepsilon h_n}\Big).\end{aligned}$$

Also considering (18), $\mathbb{P}\big(\frac{\mathscr{R}_{T^{k_n}}(g_n,\mathbf{f}^n)}{n^{1-b}\varphi(n^b)u_{1,n}} < e^{-5\varepsilon h_n}\big)$ is smaller than

$$\begin{aligned}&e^{(-c_5+\frac{\min(1,c_5)}{2})\varepsilon h_n} + h_n e^{-\varepsilon h_n} + \frac{e^{-\min(\varepsilon\log n,3h_n)+\min(1,c_5)\frac{\varepsilon}{2}h_n}}{\big(n^{\kappa_b}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{k,n})\big)^2}\\ &\leq e^{-\frac{\varepsilon c_5}{2}h_n} + h_n e^{-\varepsilon h_n} + e^{-\min(\varepsilon\log n,3h_n)+\frac{\varepsilon}{2}h_n+2|\log(n^{\kappa_b}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{k,n}))|}.\end{aligned}$$

Now, thanks to Remark 1, for n large enough $|\log(n^{\kappa_b}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{k,n}))|$ is smaller than $\frac{\varepsilon}{8}\log n \leq -\min(-\frac{\varepsilon}{8}\log n, -h_n)$ and $\frac{\varepsilon}{2}h_n$ is smaller than $\leq -\frac{1}{2}\min(-\varepsilon\log n, -h_n)$ so $-\min(\varepsilon\log n, 3h_n) + \frac{\varepsilon}{2}h_n + 2|\log(n^{\kappa_b}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{k,n}))|$ is smaller than $-\frac{1}{2}\min(\varepsilon\log n, h_n)$. Finally, for all $\varepsilon\in(0,\varepsilon_b)$ and n large enough

$$\mathbb{P}\Big(\frac{\mathscr{R}_{T^{k_n}}(g_n,\mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} < e^{(\xi-7\varepsilon)h_n}\Big) \leq e^{-\frac{\varepsilon c_5}{2}h_n} + h_n e^{-\varepsilon h_n} + e^{-\frac{1}{4}\min(\varepsilon\log n, h_n)},$$

then switching ε by $\varepsilon/7$ in the above probability, we obtain as $h_n\to+\infty$, the desired expression : for all $\varepsilon\in(0,7\varepsilon_b)$

$$\lim_{n\to\infty}\mathbb{P}\Big(\frac{\mathscr{R}_{T^{k_n}}(g_n,\mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} < e^{(\xi-\varepsilon)h_n}\Big) = 0.$$

We are now ready to move from $\mathscr{R}_{T^{k_n}}$ to $\mathscr{R}_n$, first note that :

$$\mathbb{P}\Big(\frac{\mathscr{R}_n(g_n,\mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} < e^{(\xi-\varepsilon)h_n}\Big) \leq \mathbb{P}\Big(\frac{\mathscr{R}_n(g_n,\mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} < e^{(\xi-\varepsilon)h_n}, T^{k_n}\leq n\Big) + \mathbb{P}(T^{k_n}>n),$$

with recall $\mathscr{R}_n(g_n,\mathbf{f}^n) := \sum_{x\in\mathbb{T}} g_n(\mathscr{L}^n_x) f^{n,|x|}(V(x_1),V(x_2),\cdots,V(x))$ and $g_n(t) = \varphi(t)\mathbb{1}_{\{t\geq n^b\}}$ and $b\in[0,1)$. Then as φ is increasing and positive so is g_n, hence $T^{k_n}\leq n$ implies $g_n(\mathscr{L}^{T^{k_n}}_x)\leq g_n(\mathscr{L}^n_x)$ and therefore $\mathscr{R}_{T^{k_n}}(g_n,\mathbf{f}^n)\leq\mathscr{R}_n(g_n,\mathbf{f}^n)$ since $f^{n,k}\geq 0$. It follows that

$$\mathbb{P}\Big(\frac{\mathscr{R}_n(g_n,\mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} < e^{(\xi-\varepsilon)h_n}\Big) \leq \mathbb{P}\Big(\frac{\mathscr{R}_{T^{k_n}}(g_n,\mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} < e^{(\xi-\varepsilon)h_n}\Big) + \mathbb{P}(T^{k_n}>n),$$

and thanks to the above convergence together with the fact that $(T^n/(n\log n))_n$ convergences in $\mathbb{P}$-probability to an almost surely finite and positive random variable we obtain the desired expression: for all $\varepsilon\in(0,7\varepsilon_b)$:

$$\lim_{n\to\infty}\mathbb{P}\Big(\frac{\mathscr{R}_n(g_n,\mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} < e^{(\xi-\varepsilon)h_n}\Big) = 0. \tag{37}$$

Upper bound: we prove the following statement, for all $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\Big(\frac{\mathscr{R}_n(g_n,\mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} > e^{(\xi+\varepsilon)h_n}\Big) = 0. \tag{38}$$

Recall that $u_{2,n} = \sum_{k\geq 1}(\Psi^k_{n,n^b}(f^{n,k}) + \Psi^k_n(f^{n,k}\mathbb{1}_{\mathbb{R}^k\setminus\mathscr{H}^k_{v_n}}) + \Psi^k_{n,\lambda'_n}(f^{n,k}) + \mathbf{E}[W\Psi^k_{n,\lambda'_n/W}(f^{n,k})])$. Assumption (**A4**) with $\varepsilon_1 = \frac{\varepsilon}{4}$ gives that

$$u_{2,n} \leq e^{\frac{\varepsilon}{4}h_n}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k}),$$

so for n large enough, as $n^{\kappa_b}\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k}) \leq e^{(\xi+\frac{\varepsilon}{2})h_n}$ and $T^n \geq n$

$$\begin{aligned}\mathbb{P}\Big(\frac{\mathscr{R}_n(g_n,\mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} > e^{(\xi+\varepsilon)h_n}\Big) &\leq \mathbb{P}\Big(\frac{\mathscr{R}_{T^n}(g_n,\mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} > e^{(\xi+\varepsilon)h_n}\Big)\\ &\leq \mathbb{P}\Big(\frac{\mathscr{R}_{T^n}(g_n,\mathbf{f}^n)}{n^{1-b}\varphi(n^b)\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{k,n})} > e^{\frac{\varepsilon}{2}h_n}\Big)\\ &\leq \mathbb{P}\Big(\frac{\mathscr{R}_{T^n}(g_n,\mathbf{f}^n)}{n^{1-b}\varphi(n^b)u_{2,n}} > e^{\frac{\varepsilon}{4}h_n}\Big) \leq e^{-\frac{\varepsilon}{8}h_n} + o(1),\end{aligned}$$

where the last inequality comes from (19) replacing ε by $\frac{\varepsilon}{4}$. Then taking the limit we get (38).
We are now ready to prove the theorem, we split this proof in three parts depending on the values of (recall) $L = \liminf_{n\to\infty} h_n^{-1}\log\big(n^{1-b-\kappa_b}\varphi(n^b)\big)$.

• Assume $L \in (-\xi, +\infty]$. For any $t \in \mathbb{R}$, $e^{\log^+ t} = e^{\log(t\vee 1)} \geq t$ so for any $\varepsilon \in (0,\varepsilon_b)$ and n large enough, $\mathbb{P}\big(\log^+\mathscr{R}_n(g_n,\mathbf{f}^n) - \log(n^{1-b-\kappa_b}\varphi(n^b)) < (\xi-\varepsilon)h_n\big)$ is smaller than

$$\mathbb{P}\big(e^{\log^+\mathscr{R}_n(g_n,\mathbf{f}^n)} < n^{1-b-\kappa_b}\varphi(n^b)e^{(\xi-\varepsilon)h_n}\big) \leq \mathbb{P}\Big(\frac{\mathscr{R}_n(g_n,\mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} < e^{(\xi-\varepsilon)h_n}\Big) \to 0,$$

where the limit comes from (37). Note that this lower bound remains true even when $L \notin (-\xi,+\infty]$. However, we need that $L \in (-\xi,+\infty]$ for the upper bound. Indeed, in this case, for n large enough, $n^{1-b-\kappa_b}\varphi(n^b) > e^{-\xi h_n}$ and for any $\varepsilon > 0$, $n^{1-b-\kappa_b}\varphi(n^b)e^{(\xi+\varepsilon)h_n} > e^{\varepsilon h_n} > 1$ so for n large enough $\mathbb{P}(\log^+\mathscr{R}_n(g_n,\mathbf{f}^n) - \log(n^{1-b-\kappa_b}\varphi(n^b)) > (\xi+\varepsilon)h_n) = \mathbb{P}(\log^+\mathscr{R}_n(g_n,\mathbf{f}^n) > \log(n^{1-b-\kappa_b}\varphi(n^b)e^{(\xi+\varepsilon)h_n}), \mathscr{R}_n(g_n,\mathbf{f}^n) > 1)$. Moreover, when $\mathscr{R}_n(g_n,\mathbf{f}^n) > 1$, $\log^+\mathscr{R}_n(g_n,\mathbf{f}^n) = \log\mathscr{R}_n(g_n,\mathbf{f}^n)$ so the previous probability is equal to

$$\mathbb{P}\big(\log\mathscr{R}_n(g_n,\mathbf{f}^n) > \log(n^{1-b-\kappa_b}\varphi(n^b)e^{(\xi+\varepsilon)h_n}), \mathscr{R}_n(g_n,\mathbf{f}^n) > 1\big) \leq \mathbb{P}\Big(\frac{\mathscr{R}_n(g_n,\mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} > e^{(\xi+\varepsilon)h_n}\Big).$$

Then taking the limit we get the result thanks to (38).

• Assume $L = -\xi$. Recall that $\underline{\Delta}_n = h_n^{-1}\log(n^{1-b-\kappa_b}\varphi(n^b)) - \inf_{\ell\geq n} h_\ell^{-1}\log(\ell^{1-b-\kappa_b}\varphi(\ell^b))$. $L = -\xi$ implies that for any $\varepsilon \in (0,\varepsilon_b)$ and n large enough, $\inf_{\ell\geq n} h_\ell^{-1}\log(\ell^{1-b-\kappa_b}\varphi(\ell^b)) > -\xi - \frac{\varepsilon}{2}$ so $h_n\underline{\Delta}_n < \log(n^{1-b-\kappa_b}\varphi(n^b)) + (\xi+\frac{\varepsilon}{2})h_n$ and as $e^{\log^+ t} \geq t$

$$\begin{aligned}\mathbb{P}\big(h_n^{-1}\log^+\mathscr{R}_n(g_n,\mathbf{f}^n) < -\varepsilon + \underline{\Delta}_n\big) &\leq \mathbb{P}\big(\mathscr{R}_n(g_n,\mathbf{f}^n) < e^{-\varepsilon h_n + h_n\underline{\Delta}_n}\big)\\ &\leq \mathbb{P}\Big(\frac{\mathscr{R}_n(g_n,\mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)} < e^{(\xi-\frac{\varepsilon}{2})h_n}\Big) \to 0,\end{aligned}$$

where the limit comes from (37). Also, $L = -\xi$ implies that for any $\varepsilon \in (0,\varepsilon_b)$ and n large enough, $\inf_{\ell\geq n} h_\ell^{-1}\log(\ell^{1-b-\kappa_b}\varphi(\ell^b)) < -\xi + \frac{\varepsilon}{2}$ so $h_n\underline{\Delta}_n > \log(n^{1-b-\kappa_b}\varphi(n^b)) + (\xi-\frac{\varepsilon}{2})h_n$ and as

$h_n(\varepsilon+\underline{\Delta}_n)>0, \mathbb{P}(h_n^{-1}\log^+\mathscr{R}_n(g_n,\mathbf{f}^n)>\varepsilon+\underline{\Delta}_n)=\mathbb{P}(\log\mathscr{R}_n(g_n,\mathbf{f}^n)>h_n(\varepsilon+\underline{\Delta}_n),\mathscr{R}_n(g_n,\mathbf{f}^n)>1)$ which is smaller than

$$\mathbb{P}\big(\mathscr{R}_n(g_n,\mathbf{f}^n)>e^{\varepsilon h_n+h_n\underline{\Delta}_n}\big)\le\mathbb{P}\Big(\frac{\mathscr{R}_n(g_n,\mathbf{f}^n)}{n^{1-b-\kappa_b}\varphi(n^b)}>e^{(\xi+\frac{\varepsilon}{2})h_n}\Big)\to 0,$$

where the limit comes from (38).

• Assume $L\in[-\infty,-\xi)$. In this case, there exists an increasing sequence $(n_\ell)_\ell$ of postive integers (with $n_\ell=\ell$ when $\lim h_n^{-1}\log(n^{1-b-\kappa_b}\varphi(n^b))=L$) and $\varepsilon_L>0$ such that for any $\ell\in\mathbb{N}^*$, $n_\ell^{1-b-\kappa_b}\varphi(n_\ell^b)<e^{-(\xi+2\varepsilon_L)h_n}$ and for any $\varepsilon'>0$

$$\mathbb{P}\big(\mathscr{R}_{n_\ell}(g_{n_\ell},\mathbf{f}^{n_\ell})>\varepsilon'\big)\le\mathbb{P}\big(\mathscr{R}_{n_\ell}(g_{n_\ell},\mathbf{f}^{n_\ell})>e^{-\varepsilon_L h_n}\big)\le\mathbb{P}\Big(\frac{\mathscr{R}_{n_\ell}(g_{n_\ell},\mathbf{f}^{n_\ell})}{n_\ell^{1-b-\kappa_b}\varphi(n_\ell^b)}>e^{(\xi+\varepsilon_L)h_n}\Big)\to 0,$$

where the limit comes from (38) with $\varepsilon=\varepsilon_L$ which ends the proof. □

3. Proof of Proposition 1

The proof of the Proposition 1 is decomposed as follows. In the first short section below we present the expression of the generating function with constraint of edge local time. In a second sub-section we prove the lower bound, this section is itself decomposed in different steps treating successively the random walk at fixed environment and then an important quantity of the environment. Finally in a third section we obtain the upper bound of the Proposition.The fact that the proposition is robust when replacing T^n by T^{k_n} with $k_n=\lfloor n/(\log n)^p\rfloor$, with $p>0$ does not need extra arguments than the ones that follow.

3.1. Preliminary

We first introduce the edge local time N_x^n of a vertex $x\in\mathbb{T}$ that is the number of times the random walk X visits the edge (x^*,x) before time n:

$$N_x^n:=\sum_{i=1}^n\mathbb{1}_{\{X_{i-1}=x^*,\,X_i=x\}},$$

the law of $N_x^{T_e}$ and $\sum_{y;y^*=x}N_y^{T_e}$ at fixed environment are given by

Lemma 3.1. *Let $x\in\mathbb{T}$, then $\mathbb{P}^{\mathscr{E}}(T_x<T_e)=e^{-V(x)}/H_x$ and for any $i\in\mathbb{N}^*$, $s\in[0,1]$ and $\nu\ge 0$,*

i) The distribution of $N_x^{T_e}$ under $\mathbb{P}_x^{\mathscr{E}}$ is geometrical on $\mathbb{N}$ with mean $H_x-1=\sum_{j<|x|}e^{V(x_j)-V(x)}$. In particular

$$\mathbb{E}^{\mathscr{E}}\big[s^{\nu N_x^{T_e}}\mathbb{1}_{\{N_x^{T_e}\ge i\}}\big]=\frac{e^{-V(x)}}{H_x^2}\Big(1-\frac{1}{H_x}\Big)^{i-1}\frac{s^{i\nu}}{1-s^\nu(1-\frac{1}{H_x})}.$$

ii) For any $z\in\mathbb{T}$ such that $z^*=x$, the distribution of $\sum_{y;y^*=x}N_y^{T_e}$ under $\mathbb{P}_z^{\mathscr{E}}$ is geometrical on $\mathbb{N}$ with mean $\tilde{H}_x:=H_x\sum_{y;y*=x}e^{-V_x(y)}$ with $V_x(y)=V(y)-V(x)$. In particular

$$\mathbb{E}^{\mathscr{E}}\big[s^{\nu\sum_{y;y*=x}N_y^{T_e}}\mathbb{1}_{\{\sum_{y;y^*=x}N_y^{T_e}\ge i\}}\big]=\frac{e^{-V(x)}}{H_x}\frac{\tilde{H}_x}{(1+\tilde{H}_x)^2}\Big(1-\frac{1}{1+\tilde{H}_x}\Big)^{i-1}\frac{s^{i\nu}}{1-s^\nu(1-\frac{1}{1+\tilde{H}_x})}.$$

Proof. The fact that $\mathbb{P}^{\mathscr{E}}(T_x < T_e) = e^{-V(x)}/H_x$, comes from a standard result for one-dimensional random walks in random environment, see for example [Gol84]. The proofs of points i) and ii) are very similar and elements for the first one can be found in [AD20] so we will only deal with the second one.
For any $x \in \mathbb{T}$, let $\mathscr{C}_x := \{y \in \mathbb{T};\ y^* = x\}$ and $\beta_x := \mathbb{P}^{\mathscr{E}}_x(T_{\mathscr{C}_x} < T_e)$ be respectively be the set of children of x and the queuched probability, starting from x, to reach $\mathscr{C}_x$ before hitting the root e, $T_{\mathscr{C}_x} = \min_{y\in\mathscr{C}_x} T_y$ and $T_y = \min\{j \geq 1;\ X_j = y\}$. Hence, $\sum_{y;y^*=x} N_y^{T_e}$ is nothing but the number of times the random walk X visits the "edge" $(x, \mathscr{C}_x)$ before time T_e. It follows, thanks to strong Markov property that for all $z \in \mathbb{T}$ such that $x^* = z$ and $k \in \mathbb{N}$

$$\mathbb{P}^{\mathscr{E}}_z\Big(\sum_{y;y^*=x} N_y^{T_e} = k\Big) = \beta_x^k(1-\beta_x). \tag{39}$$

Note that the right part above doesn't depend on z. We now compute β_x. On the one hand, thanks to (39), we have $\mathbb{E}^{\mathscr{E}}_z[\sum_{y;y^*=x} N_y^{T_e}] = \beta_x/(1-\beta_x)$ and on the other hand, thanks to the first point, $\mathbb{E}^{\mathscr{E}}_z[\sum_{y;y^*=x} N_y^{T_e}] = \sum_{y;y^*=x} \mathbb{E}^{\mathscr{E}}_z[N_y^{T_e}] = \sum_{y;y^*=x}(H_y - 1) = H_x \sum_{y;y^*=x} e^{-V_x(y)} = \tilde{H}_x$. $\sum_{y;y^*=x} N_y^{T_e}$ is finally geometrical on $\mathbb{N}$ under $\mathbb{P}^{\mathscr{E}}_z$ with mean $\tilde{H}_x$ and $\beta_x = \tilde{H}_x/(1+\tilde{H}_x)$.
We define $\alpha_x := \mathbb{P}^{\mathscr{E}}(T_{\mathscr{C}_x} < T_e)$ the quenched probability to reach $\mathscr{C}_x$ during the first excursion. Thanks to (39) we have for all $k \in \mathbb{N}^*$. It follows that

$$\mathbb{P}^{\mathscr{E}}\Big(\sum_{y;y^*=x} N_y^{T_e} = k\Big) = \alpha_x\beta_x^{k-1}(1-\beta_x) \quad \text{and} \quad \mathbb{P}^{\mathscr{E}}\Big(\sum_{y;y^*=x} N_y^{T_e} = 0\Big) = 1-\alpha_x,$$

so on the one hand, $\mathbb{E}^{\mathscr{E}}[\sum_{y;y^*=x} N_y^{T_e}] = \alpha_x/(1-\beta_x)$ and on the other hand, thanks to the first point, $\mathbb{E}^{\mathscr{E}}[\sum_{y;y^*=x} N_y^{T_e}] = \sum_{y;y^*=x} \mathbb{E}^{\mathscr{E}}[N_y^{T_e}] = \sum_{y;y^*=x} e^{-V(y)}$. It follows that $\alpha_x = \sum_{y;y^*=x} e^{-V(y)}/(1+\tilde{H}_x)$ and the result is proved.

□

3.2. Lower bound for $\mathscr{R}_{T^n}(g_n, \mathbf{f}_n)$

To obtain the lower bound of the range, let us first introduce two key random variables denoted $\mathscr{R}_{T^n}(\mathbf{f}^n)$ and $R(\mathbf{f}^n)$. $\mathscr{R}_{T^n}(\mathbf{f}^n)$ is simplified version of $\mathscr{R}_{T^n}(g_n, \mathbf{f}^n)$ which does not depend on function g_n and with a constraint to V : recall $\mathscr{H}^k_{v_n} = \big\{(t_1, \dots, t_k) \in \mathbb{R}^k;\ t_k \geq v_n\big\}$ with $v_n = \log(n\ell_n)/\delta_1$, $\delta_1 \in (0, 1/2]$, $\ell_n = (\log n)^3$ and $\lambda_n = ne^{-\min(10\varepsilon \log n, 5h_n)}$, then

$$\mathscr{R}_{T^n}(\mathbf{f}^n) := \sum_{i=1}^{n} \sum_{x\in\mathscr{O}_{\lambda_n,n^b}} \mathbb{1}_{\{N_x^{T^i} - N_x^{T^{i-1}} \geq n^b\}} \mathbb{1}_{\{\forall j\neq i: N_x^{T^j} - N_x^{T^{j-1}} = 0\}} f^{n,|x|} \mathbb{1}_{\mathscr{H}^{|x|}_{v_n}}(\mathbf{V}_x),$$

where we use notation $F(\mathbf{V}_x) := F(V(x_1), \cdots, V(x))$. Note that the local time until T^n which appears in $\mathscr{R}_{T^n}(g_n, \mathbf{f}^n)$ is replaced in $\mathscr{R}_{T^n}(\mathbf{f}^n)$ by edge local times excursion by excursion, also visited vertices are restricted to some V-regular lines $\mathscr{O}_{\lambda_n,n^b}$. $\mathscr{R}_{T^n}(g_n, \mathbf{f}^n)$ and $\mathscr{R}_{T^n}(\mathbf{f}^n)$ are related as follows, first since φ is increasing

$$\mathscr{R}_{T^n}(g_n, \mathbf{f}^n) \geq \varphi(n^b) \sum_{x\in\mathbb{T}} \mathbb{1}_{\{\mathscr{L}_x^{T^n} \geq n^b\}} f^{n,|x|} \mathbb{1}_{\mathscr{H}^{|x|}_{v_n}}(\mathbf{V}_x).$$

Then introduce, $E_x^n = \sum_{i=1}^n \mathbb{1}_{\{\mathscr{L}_x^{T^i} - \mathscr{L}_x^{T^{i-1}} \geq 1\}}$ the number of excursions to the root where the walk hits vertex x. Notice that $E_x^n = 1$ if and only if there exists $i \in \{1, \dots, n\}$ such that $\mathscr{L}_x^{T^i} - \mathscr{L}_x^{T^{i-1}} \geq$

1 and for any $j \in \{1,\dots,n\}, j \neq i$, $\mathscr{L}_x^{T^j} - \mathscr{L}_x^{T^{j-1}} = 0$ that is $N_x^{T^j} - N_x^{T^{j-1}} = 0$. Thus

$$\sum_{x\in\mathbb{T}} \mathbb{1}_{\{\mathscr{L}_x^{T^n}\geq n^b\}} f^{n,|x|}\mathbb{1}_{\mathscr{H}_{v_n}^{|x|}}(\mathbf{V}_x) \geq \sum_{x\in\mathscr{O}_{\lambda_n,n^b}} \mathbb{1}_{\{\mathscr{L}_x^{T^n}\geq n^b, E_x^n=1\}} f^{n,|x|}\mathbb{1}_{\mathscr{H}_{v_n}^{|x|}}(\mathbf{V}_x)$$

$$\geq \sum_{i=1}^{n} \sum_{x\in\mathscr{O}_{\lambda_n,n^b}} \mathbb{1}_{\{\mathscr{L}_x^{T^i} - \mathscr{L}_x^{T^{i-1}}\geq n^b\}} \mathbb{1}_{\{\forall j\neq i:\, N_x^{T^i} - N_x^{T^{i-1}}=0\}} f^{n,|x|}\mathbb{1}_{\mathscr{H}_{v_n}^{|x|}}(\mathbf{V}_x),$$

so finally as $\mathscr{L}_x^{T^i} - \mathscr{L}_x^{T^{i-1}} \geq N_x^{T^i} - N_x^{T^{i-1}}$ we have the following relation :

$$\mathscr{R}_{T^n}(g_n, \mathbf{f}^n) \geq \varphi(n^b)\mathscr{R}_{T^n}(\mathbf{f}^n). \tag{40}$$

Second random variable $R(\mathbf{f}^n)$ depends only on the environment :

$$R(\mathbf{f}^n) := \sum_{x\in\mathscr{O}_{\lambda_n,n^b}} e^{-V(x)} \frac{1}{H_x}\Big(1 - \frac{1}{H_x}\Big)^{n^b-1} f^{n,|x|}\mathbb{1}_{\mathscr{H}_{v_n}^{|x|}}(\mathbf{V}_x),$$

it can be related to the quenched mean of $\mathscr{R}_{T^n}(\mathbf{f}^n)$ as follows :

$$1 \leq \frac{nR(\mathbf{f}^n)}{\mathbb{E}^{\mathscr{E}}[\mathscr{R}_{T^n}(\mathbf{f}^n)]} \leq (1 - e^{-v_n})^{-(n-1)}. \tag{41}$$

Indeed, random variables $N_x^{T^i} - N_x^{T^{i-1}}, i \in \{1,\dots,n\}$ are i.i.d under $\mathbb{P}^{\mathscr{E}}$ so,

$$\mathbb{E}^{\mathscr{E}}\left[\mathscr{R}_{T^n}(\mathbf{f}_n)\right] = n \sum_{x\in\mathscr{O}_{\lambda_n,n^b}} \mathbb{P}^{\mathscr{E}}(N_x^{T_e} \geq n^b)\mathbb{P}^{\mathscr{E}}(N_x^{T_e} = 0)^{n-1} f^{n,|x|}\mathbb{1}_{\mathscr{H}_{v_n}^{|x|}}(\mathbf{V}_x).$$

Moreover, on the event $\{V(x) \geq v_n\}$ thanks to iii) of Lemma 3.1, $\mathbb{P}^{\mathscr{E}}(N_x^{T_e} = 0)^{n-1} = \mathbb{P}^{\mathscr{E}}(T_x > T_e)^{n-1} = (1 - e^{-V(x)}/H_x)^{n-1} \geq (1 - e^{-V(x)})^{n-1} \geq (1 - e^{-v_n})^{n-1}$ since $H_x > 1$, and thanks to Lemma 3.1 i) with $\nu = 0$, $\mathbb{P}^{\mathscr{E}}(N_x^{T_e} \geq n^b) = e^{-V(x)}(1-1/H_x)^{n^b-1}/H_x$ which gives (41). We are now ready to obtain a relation between a lower bound for $\mathscr{R}_{T^n}(g_n, \mathbf{f}^n)$ and a lower bound for $R(\mathbf{f}_n)$.

Lemma 3.2. *Recall $\varepsilon_b = \min(b + \mathbb{1}_{\{b=0\}}, 1-b)/13$ and let (a_n) be a sequence of positive numbers, then for all $\varepsilon \in (0, \varepsilon_b)$ and n large enough*

$$\mathbb{P}^*\big(\mathscr{R}_{T^n}(g_n, \mathbf{f}^n) < n\varphi(n^b)a_n/4n^b\big) \leq \mathbf{P}^*\left(R(\mathbf{f}^n) < a_n/n^b\right) + \frac{ne^{-\min(9\varepsilon\log n, 4h_n)}}{n^{2\kappa_b}a_n^2}. \tag{42}$$

Proof. Note that thanks to (41) for n large enough $nR(\mathbf{f}^n) \leq 2\mathbb{E}^{\mathscr{E}}[\mathscr{R}_{T^n}(\mathbf{f}^n)]$, so by (40), on the event $\{R(\mathbf{f}^n) \geq a_n/n^b\}$

$$\mathbb{P}^{\mathscr{E}}\big(\mathscr{R}_{T^n}(g_n, \mathbf{f}^n) < n\varphi(n^b)a_n/4n^b\big) \leq \mathbb{P}^{\mathscr{E}}\big(\mathscr{R}_{T^n}(\mathbf{f}^n) < \mathbb{E}^{\mathscr{E}}[\mathscr{R}_{T^n}(\mathbf{f}^n)]/2\big).$$

Using Bienaymé-Tchebychev inequality and the fact that $N_x^{T^i} - N_x^{T^{i-1}}, i \in \{1,\dots,n\}$, are i.i.d under $\mathbb{P}^{\mathscr{E}}$ implies, on the event $\{R(\mathbf{f}^n) \geq a_n/n^b\}$,

$$\mathbb{P}^{\mathscr{E}}(\mathscr{R}_{T^n}(\mathbf{f}^n) < \mathbb{E}^{\mathscr{E}}\left[\mathscr{R}_{T^n}(\mathbf{f}^n)\right]/2) \leq \frac{4}{\mathbb{E}^{\mathscr{E}}[\mathscr{R}_{T^n}(\mathbf{f}^n)]^2} n\mathbb{V}\mathrm{ar}^{\mathscr{E}}(\mathscr{R}_{T_e}(\mathbf{f}^n))$$

$$\leq \frac{16n^{2b}}{a_n^2 n} \sum_{x,y\in\mathscr{O}_{\lambda_n,n^b}} \mathbb{P}^{\mathscr{E}}(N_x^{T_e} \wedge N_y^{T_e} \geq n^b) f^{n,|x|}\mathbb{1}_{\mathscr{H}_{v_n}^{|x|}}(\mathbf{V}_x) f^{n,|y|}\mathbb{1}_{\mathscr{H}_{v_n}^{|y|}}(\mathbf{V}_y). \tag{43}$$

the last inequality coming from the fact that on $\{R(\mathbf{f}^n) \geq a_n/n^b\}$, thanks to (41) $\mathbb{E}^{\mathscr{E}}[\mathscr{R}_{T^n}(\mathbf{f}^n)]^2 \geq n^2 R(\mathbf{f}^n)^2/4 \geq n^2 a_n^2/4n^{2b}$. Markov inequality in (43) yields $\mathbb{P}^{\mathscr{E}}(N_x^{T_e} \wedge N_y^{T_e} \geq n^b) \leq \mathbb{E}^{\mathscr{E}}[N_x^{T_e} N_y^{T_e}]/n^{2b}$, so finally on the event $\{R(\mathbf{f}^n) \geq a_n/n^b\}$

$$\mathbb{P}^{\mathscr{E}}(\mathscr{R}_{T^n}(g_n, \mathbf{f}^n) < n\varphi(n^b)a_n/4n^b) \leq \frac{16}{na_n^2} \sum_{x,y \in \mathscr{O}_{\lambda_n, n^b}} \mathbb{E}^{\mathscr{E}}[N_x^{T_e} N_y^{T_e}] f^{n,|x|} \mathbb{1}_{\mathscr{H}_{v_n}^{|x|}}(\mathbf{V}_x) f^{n,|y|} \mathbb{1}_{\mathscr{H}_{v_n}^{|y|}}(\mathbf{V}_y). \tag{44}$$

To treat the above sum, we first make a simplification by using the uniform upper bound of the set $\mathscr{U}_b$ see (10),

$$\sum_{x,y \in \mathscr{O}_{\lambda_n, n^b}} \mathbb{E}^{\mathscr{E}}[N_x^{T_e} N_y^{T_e}] f^{n,|x|} \mathbb{1}_{\mathscr{H}_{v_n}^{|x|}}(\mathbf{V}_x) f^{n,|y|} \mathbb{1}_{\mathscr{H}_{v_n}^{|y|}}(\mathbf{V}_y) \leq \frac{C_\infty^2}{n^{2\kappa_b}} \sum_{x,y \in \mathscr{O}_{\lambda_n}} \mathbb{E}^{\mathscr{E}}[N_x^{T_e} N_y^{T_e}]. \tag{45}$$

We then split computations in two distinct steps: first cases $x \leq y$ or $y \leq x$ and then cases nor $x \leq y$ neither $y \leq x$. The key here is to take into account that we are only interested in vertices belonging to λ_n-regular lines $\mathscr{O}_{\lambda_n}$ with $\lambda_n = ne^{-\min(10\varepsilon \log n, 5h_n)}$ for $\varepsilon \in (0, \varepsilon_b)$.
We start with cases $x \leq y$ and $y \leq x$ and as they are symmetrical we only deal with the first one. First note that as $\mathbb{E}^{\mathscr{E}}\left[N_x^{T_e} N_y^{T_e}\right] \leq 2e^{-V(y)} H_x = 2H_x e^{-V(x)} e^{-V_x(y)}$ (see [AD20] Lemma 3.6)

$$\begin{aligned}\mathbf{E}\Big[\sum_{\substack{x \leq y \\ x,y \in \mathscr{O}_{\lambda_n}}} \mathbb{E}^{\mathscr{E}}[N_x^{T_e} N_y^{T_e}] \Big] &\leq 2\mathbf{E}\Big[\sum_{x \in \mathscr{O}_{\lambda_n}} e^{-V(x)} H_x \sum_{\substack{y \geq x \\ y \in \mathscr{O}_{\lambda_n}^x}} e^{-V_x(y)} \Big] \leq 2\mathbf{E}\Big[\sum_{x \in \mathscr{O}_{\lambda_n}} e^{-V(x)} \Big]^2 \lambda_n \\ &\leq 2\ell_n^2 \lambda_n,\end{aligned}$$

where for all $\lambda > 0$, $\mathscr{O}_\lambda^x$ is translated set of λ-regular lines

$$\mathscr{O}_\lambda^x = \big\{ y \in \mathbb{T}, y > x; \max_{|x| < j \leq |y|} H_{x,y_j} \leq \lambda \big\}, \; H_{x,y_j} = \sum_{x < w \leq y_j} e^{V_x(w) - V_x(y_j)},$$

also second inequality is obtained thanks to the regular line which yields, $H_x \mathbb{1}_{\mathscr{O}_{\lambda_n}}(x) \leq \lambda_n$, the last one comes from Remark 2.
We then move to the second case, nor $x \leq y$ neither $y \leq x$, that we denote $x \not\sim y$. In this case $\mathbb{E}^{\mathscr{E}}\left[N_x^{T_e} N_y^{T_e}\right] = 2H_{x \wedge y} e^{V(x \wedge y) - V(x) - V(y)}$ (see [AD20] Lemma 3.6). Thus

$$\mathbb{E}^{\mathscr{E}}[N_x^{T_e} N_y^{T_e}] \leq 2\lambda_n \sum_{l \geq 1} \sum_{|z| = l} e^{-V(z)} \mathbb{1}_{\{z \in \mathscr{O}_{\lambda_n}\}} \sum_{\substack{u \neq v \\ u^* = v^* = z}} e^{-V_z(u)} e^{-V_z(v)} \sum_{\substack{x \geq u \\ x \in \mathscr{O}_{\lambda_n}^u}} e^{-V_u(x)} \sum_{\substack{y \geq v \\ y \in \mathscr{O}_{\lambda_n}^v}} e^{-V_v(y)}$$

where we have used again the regular line $\mathscr{O}_{\lambda_n}$ which gives an upper bound for $H_{x \wedge y}$. Finally, independence of increments of V conditionally to $(\mathbb{T}, V(w); w \in \mathbb{T}, |w| \leq l+1)$ and Remark 2 yields

$$\begin{aligned}\mathbf{E}\Big[\sum_{\substack{x \not\sim y \\ x,y \in \mathscr{O}_{\lambda_n}}} \mathbb{E}^{\mathscr{E}}[N_x^{T_e} N_y^{T_e}] \Big] &\leq 2\lambda_n \mathbf{E}\Big[\Big(\sum_{|u|=1} e^{-V(u)} \Big)^2 \Big] \mathbf{E}\Big[\sum_{z \in \mathscr{O}_{\lambda_n}} e^{-V(z)} \Big]^3 \\ &\leq 2\lambda_n \mathbf{E}\Big[\Big(\sum_{|u|=1} e^{-V(u)} \Big)^2 \Big] (\ell_n)^3\end{aligned}$$

and thanks to (4) the second moment above is finite. Collecting the upper bounds for the two cases and moving back to (45), for n large enough we get

$$\mathbf{E}\Big[\sum_{x,y\in\mathscr{O}_{\lambda_n,n^b}}\mathbb{E}^{\mathscr{E}}[N_x^{T_e}N_y^{T_e}]f^{n,|x|}\mathbb{1}_{\mathscr{H}_{v_n}^{|x|}}(V_x)f^{n,|y|}\mathbb{1}_{\mathscr{H}_{v_n}^{|y|}}(V_y)\Big]\leq\frac{(\ell_n)^4\lambda_n}{n^{2\kappa_b}}\leq\frac{ne^{-\min(9\varepsilon\log n,4h_n)}}{n^{2\kappa_b}},\quad(46)$$

last inequality is justified by the fact (see Remark 1) that $(\ell_n)^4=o(e^{h_n})$ and $(\ell_n)^4=o(e^{\epsilon\log n})$. We are now ready to conclude the proof of the lemma : $\mathbb{P}^*\big(\mathscr{R}_{T^n}(g_n,\mathbf{f}^n)<n\varphi(n^b)a_n/4n^b\big)$ is smaller than

$$\mathbb{P}^*(R(\mathbf{f}^n)<a_n/n^b)+\mathbb{P}^*\big(\mathscr{R}_{T^n}(g_n,\mathbf{f}^n)<n\varphi(n^b)a_n/4n^b,R(\mathbf{f}^n)\geq a_n/n^b\big),$$

then as the second term in the above inequality is nothing but

$$\mathbf{E}^*\big[\mathbb{P}^{\mathscr{E}}\big(\mathscr{R}_{T^n}(g_n,\mathbf{f}^n)<n\varphi(n^b)a_n/4n^b\big)\mathbb{1}_{\{R(\mathbf{f}^n)\geq a_n/n^b\}}\big],$$

the proof ends thanks to (44) and (46). □

3.2.1. Lower bound for $R(\mathbf{f}^n)$

This is the most technical part of the proof of Proposition 1. For any $n\geq 2$ and $\varepsilon\in(0,\varepsilon_b)$ recall that $\lambda_n=ne^{-\min(10\varepsilon\log n,5h_n)}$ and $v_n=\log(n\ell_n)/\delta_1$, $\delta_1\in(0,1/2]$ (see (4)) with $\ell_n=(\log n)^3$. For any $\varepsilon>0$, let us choose (a_n) as follows

$$a_n:=e^{-4\varepsilon h_n}\sum_{k\geq 1}\Psi^k_{\lambda_n/2,n^b}\big(f^{n,k}_{\varepsilon h_n}\mathbb{1}_{\Upsilon_n^k}\big)\quad(47)$$

with $\Upsilon_n^k=\{\mathbf{t}\in\mathbb{R}^k;\ H_k(\mathbf{t})\leq n^be^{\varepsilon h_n}\}\cap\mathscr{H}^k_{B,v_n'}$. Recall that $\Psi^k_{\lambda,\lambda'}$, h_n, $\mathscr{H}^k_{B,v_n'}$ and $f^{n,k}_{\varepsilon h_n}$ can be found respectively in (8), (12), (17) and (14) and $v_n'=v_n+\varepsilon h_n$.

Lemma 3.3. *There exists $c_4>0$ such that for any $\varepsilon\in(0,\varepsilon_b)$ and n large enough*

$$\mathbf{P}^*\left(R(\mathbf{f}^n)<a_n/n^b\right)\leq\frac{e^{-\varepsilon\frac{c_4}{r_2}h_n}\mathbf{E}[Z_n^2]}{\big(\sum_{k\geq1}\Psi^k_{\lambda_n/2,n^b}\big(f^{n,k}_{\varepsilon h_n}\mathbb{1}_{\Upsilon_n^k}\big)\big)^2}+h_ne^{-\varepsilon h_n},\quad(48)$$

and recall $m_n=\lceil\varepsilon h_n/r_2\rceil$ (see (13)).

Proof. Recall the expression of $R(\mathbf{f}_n)$:

$$R(\mathbf{f}^n)=\sum_{x\in\mathscr{O}_{\lambda_n,n^b}}e^{-V(x)}\frac{1}{H_x}\Big(1-\frac{1}{H_x}\Big)^{n^b-1}f^{n,|x|}\mathbb{1}_{\mathscr{H}_{v_n}^{|x|}}(V(x_1),\cdots,V(x)),$$

with H_x and $\mathscr{H}_{v_n}^{|x|}$ respectively defined in (7) and (17) for any $z\in\mathbb{T}$. The main idea here is to cut the tree at generation m_n to introduce independence between generations.

$$R(\mathbf{f}^n)\geq\sum_{|u|=m_n}\sum_{k\geq1}\sum_{\substack{|x|=k+m_n\\ x>u;\ x\in\mathscr{O}_{\lambda_n,n^b}}}\frac{e^{-V(x)}}{H_x}\Big(1-\frac{1}{H_x}\Big)^{n^b}f^{n,k+m_n}\mathbb{1}_{\mathscr{H}_{v_n}^{k+m_n}}(V(x_1),\ldots,V(x)),$$

from here we would like to make a translation to decompose trajectories of V before and after generation m_n, to do that we have in particular to re-write H_{x_j} for $j\leq|x|$. Let $u<x$ with

$|u| = m_n$. For all $m_n < j \leq |x|$, we have $H_{x_j} = H_u e^{-V_u(x_j)} + H_{u,x_j}$ where, for any $z < v$, $H_{z,v} := \sum_{z<w\leq v} e^{V_z(w)-V_z(v)}$.

So on events $\{\max_{|w|\leq m_n} |V(w)| \leq \varepsilon h_n\}$ and $\{\underline{V}_u(x) := \min_{u<w\leq x}(V(w) - V(u)) \geq -B\}$ for any $B > 0$

$$\forall i \leq m_n : \; H_{x_i} \leq m_n e^{2\varepsilon h_n} \quad \text{and} \quad \forall\, m_n < j \leq |x| : \; H_{x_j} \leq m_n e^{2\varepsilon h_n + B} + H_{u,x_j}.$$

Assume $n^b < H_{u,x} \leq n^b e^{\varepsilon h_n}$. Then, $H_x > n^b$ and for n large enough (recall $h_n \leq \log n$ for n large enough, $h_n \to \infty$ and $\varepsilon \in (0, \varepsilon_b)$)

$$\frac{1}{H_x}\Big(1 - \frac{1}{H_x}\Big)^{n^b} \geq \frac{(1-1/n^b)^{n^b}}{m_n e^{2\varepsilon h_n + B} + H_{u,x}} \geq \frac{(1-1/n^b)^{n^b}}{m_n e^{2\varepsilon h_n + B} + n^b e^{\varepsilon h_n}} \geq \frac{e^{-3\varepsilon h_n}}{n^b}.$$

Now introduce the translated (λ, λ')-regular lines

$$\mathscr{O}^v_{\lambda,\lambda'} := \big\{y \in \mathbb{T}, y > v; \; \max_{|v|<j\leq|y|} H_{v,y_j} \leq \lambda, \; H_{v,y} > \lambda'\big\}.$$

Note that $\mathscr{O}^u_{\lambda_n/2,n^b} \subset \mathscr{O}_{\lambda_n,n^b}$ for n large enough, indeed, as we said before, for all $i \leq m_n$ we have $H_{x_i} \leq m_n e^{2\varepsilon h_n} \leq e^{3h_n} \leq \lambda_n/2$ for n large enough since $\varepsilon \in (0, 1/13)$. Moreover, if $H_{u,x_j} \leq \lambda_n/2$ for any $m_n < j \leq |x|$ it comes that $H_{x_j} \leq m_n e^{2\varepsilon h_n + B} + \lambda_n/2 \leq \lambda_n$.
For f^{n,m_n+k}, we simply write (still on the event $\{\max_{|w|\leq m_n} |V(w)| \leq \varepsilon h_n\}$),

$$f^{n,m_n+k}(V(x_1), \ldots, V(x)) \geq f^{n,k}_{\varepsilon h_n}(V_u(x_{m_n+1}), \ldots, V_u(x)),$$

where we recall $f^{n,k}_h(t_1, \ldots, t_k) = \inf_{\mathbf{s}\in[-h,h]^m} f^{n,m+k}(s_1, \ldots, s_m, t_1 + s_m, \ldots, t_k + s_m)$ with $m = \lceil h/r_2 \rceil$. In the same way $\mathbb{1}_{\{V(x)\geq v_n\}} \geq \mathbb{1}_{\{V_u(x)\geq v'_n\}}$ with $v'_n = v_n + \varepsilon h_n$ and we finally obtain, for n large enough (independently of the environment) on $\{\max_{|w|\leq m_n} |V(w)| \leq \varepsilon h_n\}$, $R(\mathbf{f}^n)$ is larger than

$$\frac{e^{-3\varepsilon h_n}}{n^b} \sum_{|u|=m_n} e^{-V(u)} \sum_{k\geq 1} \sum_{\substack{|x|=k+m_n \\ x>u;\, x\in\mathscr{O}^u_{\lambda_n/2,n^b}}} e^{-V_u(x)} \mathbb{1}_{\{H_{u,x}\leq n^b e^{\varepsilon h_n}\}} f^{n,k}_{\varepsilon h_n} \mathbb{1}_{\mathscr{H}^k_{B,v'_n}}(V_u(x_{m_n+1}), \ldots, V_u(x))$$

$$\geq \frac{e^{-4\varepsilon h_n}}{n^b} \sum_{|u|=m_n} \sum_{k\geq 1} \sum_{\substack{|x|=k+m_n \\ x>u;\, x\in\mathscr{O}^u_{\lambda_n/2,n^b}}} e^{-V_u(x)} f^{n,k}_{\varepsilon h_n} \mathbb{1}_{\Upsilon^k_n}(V_u(x_{m_n+1}), \ldots, V_u(x)). \tag{49}$$

Introduce the random variable Z^u_n

$$Z^u_n := \sum_{k\geq 1} \sum_{\substack{|x|=k+m_n \\ x>u;\, x\in\mathscr{O}^u_{\lambda_n/2,n^b}}} e^{-V_u(x)} f^{n,k}_{\varepsilon h_n} \mathbb{1}_{\Upsilon^k_n}(V_u(x_{m_n+1}), \ldots, V_u(x)),$$

we obtain

$$\mathbf{P}\big(R(\mathbf{f}^n) < e^{-4\varepsilon h_n}\mathbf{E}[Z_n]/n^b, \max_{|w|\leq m_n} |V(w)| \leq \varepsilon h_n\big) \leq \mathbf{P}\Big(\sum_{|u|=m_n} Z^u_n < \mathbf{E}[Z_n]\Big),$$

with

$$Z_n := \sum_{x\in\mathscr{O}_{\lambda_n/2,n^b}} e^{-V(x)} f^{n,|x|}_{\varepsilon h_n} \mathbb{1}_{\Upsilon^{|x|}_n}(V(x_1), \ldots, V(x)). \tag{50}$$

Hence, by Lemma 2.4 in [AD20], there exists $c_4 > 0$ such that for n large enough

$$\mathbf{P}^*\big(R(\mathbf{f}^n) < e^{-4\varepsilon h_n}\mathbf{E}[Z_n]/n^b, \max_{|w|\le m_n}|V(w)| \le \varepsilon h_n\big) \le e^{-c_4 m_n}\frac{\mathbf{E}[Z_n^2]}{\mathbf{E}[Z_n]^2} \tag{51}$$

and finally, (47) yields

$$\mathbf{P}^*\big(R(\mathbf{f}^n) < a_n/n^b, \max_{|w|\le m_n}|V(w)| \le \varepsilon h_n\big) \le \frac{e^{-\varepsilon\frac{c_4}{r_2}h_n}\mathbf{E}[Z_n^2]}{\big(\sum_{k\ge1}\Psi^k_{\lambda_n/2,n^b}\big(f^{n,k}_{\varepsilon h_n}\mathbb{1}_{\Upsilon^k_n}\big)\big)^2},$$

and we have used that $\mathbf{E}[Z_n] = \sum_{k\ge1}\Psi^k_{\lambda_n/2,n^b}\big(f^{n,k}_{\varepsilon h_n}\mathbb{1}_{\Upsilon^k_n}\big)$ and $m_n = \lceil \varepsilon h_n/r_2\rceil$. Finally, (13) finishes the proof. □

The next step is to give a lower bound for $\mathbf{E}[Z_n^2]$ we do that in the dedicated section below.

3.2.2. Control of the second moment $\mathbf{E}[Z_n^2]$

Let us prove following Lemma,

Lemma 3.4. *Assume assumptions* (**A1**) *and* (**A2**) *hold. For all* $\varepsilon \in (0,\varepsilon_b)$, $A > 2/c_1$ *and* n *large enough*

$$\mathbf{E}[Z_n^2] \le e^{\frac{6\varepsilon}{A}h_n}\Big(\sum_{k\ge1}\Psi^k_{n,n^b}(f^{n,k})\Big)^2$$

recall also that c_1 *comes from Remark 2.*

Proof. Expression of Z_n^2 is given by $\sum_{x,y\in\mathscr{O}_{\lambda_n/2,n^b}} e^{-V(x)-V(y)}f^{n,|x|}_{\varepsilon h_n}\mathbb{1}_{\Upsilon^{|x|}_n}(\boldsymbol{V}_x)f^{n,|y|}_{\varepsilon h_n}\mathbb{1}_{\Upsilon^{|y|}_n}(\boldsymbol{V}_y)$ (see (50)) and $\lambda_n \le n$ so

$$Z_n^2 \le \sum_{x,y\in\mathscr{O}_{n,n^b}} e^{-V(x)}e^{-V(y)}f^{n,|x|}_{\varepsilon h_n}\mathbb{1}_{\mathscr{H}^{|x|}_{B,v'_n}}(\mathbf{V}_x)f^{n,|y|}_{\varepsilon h_n}\mathbb{1}_{\mathscr{H}^{|y|}_{B,v'_n}}(\mathbf{V}_y) \tag{52}$$

with (recall) $F(\boldsymbol{V}_w) = F(V(w_1),\dots,V(w))$. Let us split computations of the upper bound of the mean of Z_n^2 into two main cases : the first one is when x and y in the sum (52) are directly related in the tree and the second when it is not the case:
Cases 1 ($x \le y$ or $y \le x$) : recall $v'_n = v_n + \varepsilon h_n$ with $v_n = \log(n\ell_n)/\delta_1$, $\delta_1 \in (0,1/2]$ (see (4)) for this case we simply use the fact that $f^{n,i}_{\varepsilon h_n} \le C_\infty$ and $e^{-2V(w)}\mathbb{1}_{\{V(w)\ge v'_n\}} \le e^{-V(w)}/n^2$ so by symmetry

$$\begin{aligned}\mathbf{E}\Big[\sum_{\substack{x\le y \text{ or } y\le x\\ x,y\in\mathscr{O}_{n,n^b}}} e^{-V(x)-V(y)}\mathbb{1}_{\{V(x)\ge v'_n\}}\Big] &\le 2\mathbf{E}\Big[\sum_{x\in\mathscr{O}_n} e^{-2V(x)}\mathbb{1}_{\{V(x)\ge v'_n\}}\sum_{\substack{y\ge x\\ y\in\mathscr{O}^x_n}} e^{-V_x(y)}\Big]\\ &\le \frac{2}{n^2}\mathbf{E}\Big[\sum_{x\in\mathscr{O}_n} e^{-V(x)}\sum_{\substack{y\ge x\\ y\in\mathscr{O}^x_n}} e^{-V_x(y)}\Big].\end{aligned}$$

which, by using independence and stationarity of increments of V, is $2\mathbf{E}[\sum_{x\in\mathscr{O}_n} e^{-V(x)}]^2/n^2$. Then, thanks to Remark 2 and the fact that $h_n \ge (\log n)^\gamma$ with $0<\gamma\le 1$, $2\mathbf{E}[\sum_{x\in\mathscr{O}_n} e^{-V(x)}]^2 \le \ell_n \le$

$e^{\epsilon h_n}$ in addition with assumption (**A1**), the part $\{x \le y \text{ or } y \le x\}$ in the sum (52) is smaller than $e^{\frac{\epsilon}{A}h_n}\big(\sum_{k\ge1}\Psi^k_{n,n^b}(f^{n,k})\big)^2$.

Cases 2 ($x \not\sim y$) : recall that $x \not\sim y$ iff nor $x \le y$ neither $y \le x$. First let

$$\Sigma_0(z) := \sum_{\substack{x\not\sim y\\ x,y\in\mathscr{O}_{n,n^b}}} \mathbb{1}_{\{x\wedge y=z\}} e^{-V(x)}e^{-V(y)} f^{n,|x|}_{\varepsilon h_n}\mathbb{1}_{\mathscr{H}^{|x|}_{B,v'_n}}(\mathbf{V}_x) f^{n,|y|}_{\varepsilon h_n}\mathbb{1}_{\mathscr{H}^{|y|}_{B,v'_n}}(\mathbf{V}_y).$$

We decompose $\Sigma_0(z)$ as follows: for all $A > 2/c_1$

$$\sum_{z\in\mathbb{T}}\Sigma_0(z) = \sum_{|z|\ge\lfloor A\ell_n\rfloor}\Sigma_0(z) + \sum_{|z|<\lfloor A\ell_n\rfloor}(\Sigma_1(z)+\Sigma_2(z)), \tag{53}$$

and for any $i\in\{1,2\}$,

$$\Sigma_i(z) := \sum_{\substack{x\not\sim y\\ x,y\in\mathscr{O}_{n,n^b}}} \mathbb{1}_{\{x\wedge y=z\}} e^{-V(x)}e^{-V(y)} f^{n,|x|}_{\varepsilon h_n}\mathbb{1}_{\mathscr{H}^{|x|}_{B,v'_n}}(\mathbf{V}_x) f^{n,|y|}_{\varepsilon h_n}\mathbb{1}_{\mathscr{H}^{|y|}_{B,v'_n}}(\mathbf{V}_y)\mathbb{1}_{\{(x,y)\in\mathscr{C}_{i,z}\}},$$

with $\mathscr{C}_{1,z} := \{(x,y)\in\mathbb{T}^2; x^* > z \text{ and } y^* > z\}$ and $\mathscr{C}_{2,z} := \{(x,y)\in\mathbb{T}^2; x^* = z \text{ or } y^* = z\}$.
Let us start with the easiest part: $\sum_{|z|\ge\lfloor A\ell_n\rfloor}\Sigma_0(z)$. Observe that

$$\sum_{|z|\ge\lfloor A\ell_n\rfloor}\Sigma_0(z) \le C_\infty^2 \sum_{l\ge\lfloor A\ell_n\rfloor}\sum_{|z|=l}\mathbb{1}_{\{V(z)\ge -B,\, z\in\mathscr{O}_n\}}\sum_{\substack{u\ne v\\ u^*=v^*=z}}\sum_{\substack{x\ge u\\ x\in\mathscr{O}_n}} e^{-V(x)}\sum_{\substack{y\ge v\\ y\in\mathscr{O}_n}} e^{-V(y)}.$$

By independence of increments of V conditionally to $(\mathbb{T}, V(w); w\in\mathbb{T}, |w|\le l+1)$ and Remark 2 for any n large enough

$$\begin{aligned}\mathbf{E}\Big[\sum_{|z|\ge\lfloor A\ell_n\rfloor}\Sigma_0(z)\Big] &\le C_\infty^2 e^B \mathbf{E}\Big[\Big(\sum_{|u|=1}e^{-V(u)}\Big)^2\Big]\mathbf{E}\Big[\sum_{x\in\mathscr{O}_n}e^{-V(x)}\Big]^2\sum_{l\ge\lfloor A\ell_n\rfloor}\mathbf{E}\Big[\sum_{|z|=l}e^{-V(z)}\mathbb{1}_{\{z\in\mathscr{O}_n\}}\Big]\\ &\le C_\infty^2 e^B \mathbf{E}\Big[\Big(\sum_{|u|=1}e^{-V(u)}\Big)^2\Big]\ell_n^2 n^{-2} \le \sum_{k\ge1}\Psi^k_{n,n^b}(f^{n,k}),\end{aligned} \tag{54}$$

where we have used assumption (**A1**) and (4) for the last inequality.
For $\Sigma_1(z)$, $|z| < \lfloor A\ell_n\rfloor$, we decompose according to the value of $V(w)$ with $w\in\{u,v\}$: $\Sigma_1(z) = \Sigma_{1,1}(z)+\Sigma_{1,2}(z)$ with

$$\Sigma_{1,1}(z) := \sum_{\substack{u\ne v\\ u^*=v^*=z}}\mathbb{1}_{\{V(u)\vee V(v)<v'_n\}}\sum_{\substack{x>u\\ x\in\mathscr{O}_{n,n^b}}}e^{-V(x)}f^{n,|x|}_{\varepsilon h_n}\mathbb{1}_{\mathscr{H}^{|x|}_{B,v'_n}}(\mathbf{V}_x)\sum_{\substack{y>v\\ y\in\mathscr{O}_{n,n^b}}}e^{-V(y)}f^{n,|y|}_{\varepsilon h_n}\mathbb{1}_{\mathscr{H}^{|y|}_{B,v'_n}}(\mathbf{V}_y),$$

and

$$\Sigma_{1,2}(z) := \sum_{\substack{u\ne v\\ u^*=v^*=z}}\mathbb{1}_{\{V(u)\vee V(v)\ge v'_n\}}\sum_{\substack{x>u\\ x\in\mathscr{O}_{n,n^b}}}e^{-V(x)}f^{n,|x|}_{\varepsilon h_n}\mathbb{1}_{\mathscr{H}^{|x|}_{B,v'_n}}(\mathbf{V}_x)\sum_{\substack{y>v\\ y\in\mathscr{O}_{n,n^b}}}e^{-V(y)}f^{n,|y|}_{\varepsilon h_n}\mathbb{1}_{\mathscr{H}^{|y|}_{B,v'_n}}(\mathbf{V}_y).$$

We first deal with $\Sigma_{1,1}(z)$. Observe that $x\in\mathscr{O}_{n,n^b}$ (resp. $y\in\mathscr{O}_{n,n^b}$) means $H_u\le n$ (resp. $H_v\le n$), $x\in\mathscr{O}^u_n$ (resp. $y\in\mathscr{O}^v_n$) and $n^b - H_u e^{-V_u(x)} < H_{u,x}$ (resp. $n^b - H_v e^{-V_v(y)} < H_{v,y}$). Besides,

$V(u) < v'_n$ and $V(x) > v'_n$ (resp. $V(v) < v'_n$ and $V(y) > v'_n$) implies $V_u(x) > 0$ (resp. $V_v(y) > 0$) that is $n^b - H_u < H_{u,x}$ (resp. $n^b - H_v < H_{v,y}$), so $\Sigma_{1,1}(z)$ is smaller than

$$\sum_{\substack{u\neq v\\ u^*=v^*=z}} \mathbb{1}_{\{\underline{V}(u)\wedge\underline{V}(v)\geq -B, H_u\vee H_v\leq n\}} \sum_{\substack{x>u\\ x\in\mathscr{O}^u_{n,n^b-H_u}}} e^{-V(x)} f^{n,|x|}_{\varepsilon h_n}(\mathbf{V}_x) \sum_{\substack{y>v\\ y\in\mathscr{O}^v_{n,n^b-H_v}}} e^{-V(y)} f^{n,|y|}_{\varepsilon h_n}(\mathbf{V}_y). \quad (55)$$

We now move to $\Sigma_{1,2}(z)$. Note that $\{V(u)\vee V(v)\geq v'_n\} = \{V(u)\geq v'_n, V(v)< v'_n\}\cup\{V(v)\geq v'_n, V(u)< v'_n\}\cup\{V(u)\wedge V(v)\geq v'_n\}$. By symmetry, $\Sigma_{1,2}(z)$ is equal to

$$2\sum_{\substack{u\neq v\\ u^*=v^*=z}} \mathbb{1}_{\{V(u)\geq v'_n, V(v)<v'_n\}} \sum_{\substack{x>u\\ x\in\mathscr{O}_{n,n^b}}} e^{-V(x)} f^{n,|x|}_{\varepsilon h_n} \mathbb{1}_{\mathscr{H}^{|x|}_{B,v'_n}}(\mathbf{V}_x) \sum_{\substack{y>v\\ y\in\mathscr{O}_{n,n^b}}} e^{-V(y)} f^{n,|y|}_{\varepsilon h_n} \mathbb{1}_{\mathscr{H}^{|y|}_{B,v'_n}}(\mathbf{V}_y)$$
$$+\sum_{\substack{u\neq v\\ u^*=v^*=z}} \mathbb{1}_{\{V(u)\wedge V(v)\geq v'_n\}} \sum_{\substack{x>u\\ x\in\mathscr{O}_{n,n^b}}} e^{-V(x)} f^{n,|x|}_{\varepsilon h_n} \mathbb{1}_{\mathscr{H}^{|x|}_{B,v'_n}}(\mathbf{V}_x) \sum_{\substack{y>v\\ y\in\mathscr{O}_{n,n^b}}} e^{-V(y)} f^{n,|y|}_{\varepsilon h_n} \mathbb{1}_{\mathscr{H}^{|y|}_{B,v'_n}}(\mathbf{V}_y)$$

The same decomposition of H_y we used for $\Sigma_{1,1}(z)$ also works for the part $\{V(v) < v'_n\}$ in the above sum so as in (55) and using that on $\{V(u)\geq v'_n\}$, $V(u)\geq (1-\delta_1)V(u)+\log n$, $\Sigma_{1,2}(z)$ is smaller than

$$\frac{2C_\infty}{n}\sum_{\substack{u\neq v\\ u^*=v^*=z}} e^{-(1-\delta_1)V(u)} \mathbb{1}_{\{V(u)\geq v'_n, V(v)<v'_n, \underline{V}(v)\geq -B, H_v\leq n\}} \sum_{\substack{x>u\\ x\in\mathscr{O}^u_n}} e^{-V_u(x)} \sum_{\substack{y>v\\ y\in\mathscr{O}^v_{n,n^b-H_v}}} e^{-V(y)} f^{n,|y|}_{\varepsilon h_n}(\mathbf{V}_y)$$
$$+\mathbb{1}_{\{V(z)\geq -B, z\in\mathscr{O}_n\}}\frac{C^2_\infty}{n^2}\sum_{\substack{u\neq v\\ u^*=v^*=z}} e^{-(1-\delta_1)(V(u)+V(v))} \sum_{\substack{x>u\\ x\in\mathscr{O}^u_n}} e^{-V_u(x)} \sum_{\substack{y>v\\ y\in\mathscr{O}^v_n}} e^{-V_v(y)}.$$

Note that the genealogical commun line between x and y is the commun line of individuals before u and v so for any $p\leq |z|$, $x_p = y_p = u_p = v_p$ and

$$f^{n,|x|}_{\varepsilon h_n}(\mathbf{V}_x) = f^{n,|x|}_{\varepsilon h_n}(V(u_1),\cdots,V(u), V_u(x_{|u|+1})+V(u),\cdots,V_u(x)+V(u)),$$

and

$$f^{n,|y|}_{\varepsilon h_n}(\mathbf{V}_y) = f^{n,|y|}_{\varepsilon h_n}(V(v_1),\cdots,V(v), V_v(y_{|v|+1})+V(v),\cdots,V_v(y)+V(v)).$$

Recall that for all $q\geq 1$ and $\mathbf{t}_q=(t_1,\dots,t_q)\in\mathbb{R}^q$,

$$\Psi^k_n(F|\mathbf{t}_p) = \mathbf{E}\Big[\sum_{|x|=k} e^{-V(x)} F(t_1,\dots,t_p, V(x_1)+t_p,\dots,V(x)+t_p)\mathbb{1}_{\mathscr{O}_n}(x)\Big].$$

We naturally note $\Psi^k_n(F|\boldsymbol{V}_w)$ when we evaluate the function $\Psi^k_n(F|\cdot)$ at $(V(w_1),\dots,V(w))$.
By using independence of increments of V conditionally given $(\mathbb{T}, V(w); w\in\mathbb{T}, |w|\leq l+1)$, $\mathbf{E}[\sum_{|z|=l}\Sigma_1(z)] = \mathbf{E}[\sum_{|z|=l}\Sigma_{1,1}(z)+\Sigma_{1,2}(z)]$ is smaller, for n large enough with $l<\lfloor A\ell_n\rfloor$, than

$$\mathbf{E}\Big[\sum_{|z|=l}\sum_{\substack{u\neq v\\ u^*=v^*=z}} \mathbb{1}_{\{\underline{V}(u)\wedge\underline{V}(v)\geq -B, H_u\vee H_v\leq n\}} \sum_{i,j\geq 1}\prod_{(k,w)\in\{(i,u);(j,v)\}} e^{-V(w)}\Psi^k_{n,n^b-H_w}\big(f^{n,|w|+k}_{\varepsilon h_n}|\mathbf{V}_w\big)\Big]$$
$$+\frac{2\ell_n C_\infty}{n}\mathbf{E}\Big[\sum_{|z|=l}\sum_{\substack{u\neq v\\ u^*=v^*=z}} e^{-(1-\delta_1)V(u)}\mathbb{1}_{\{\underline{V}(v)\geq -B, H_v\leq n\}}\sum_{j\geq 1} e^{-V(v)}\Psi^j_{n,n^b-H_v}\big(f^{n,|v|+j}_{\varepsilon h_n}|\mathbf{V}_v\big)\Big]$$
$$+\frac{\ell^2_n C^2_\infty}{n^2}\mathbf{E}\Big[\sum_{|z|=l}\mathbb{1}_{\{V(z)\geq -B, z\in\mathscr{O}_n\}}\sum_{\substack{u\neq v\\ u^*=v^*=z}} e^{-(1-\delta_1)(V(u)+V(v))}\Big].$$

We have used that $\mathbf{E}[\sum_{x\in\mathscr{O}_n} e^{-V(x)}] \le \ell_n$. Then, by assumption (**A2**) with $\delta = \delta_1$ (see (4) for the definition of δ_1), for all $l < \lfloor A\ell_n \rfloor$ ($|u| = |v| = l+1$) and n large enough on the event $\{V(u)\wedge V(v) \ge -B, H_u \vee H_v \le n\}$

$$\sum_{i,j\ge 1}\prod_{(k,w)\in\{(i,u);(j,v)\}} \Psi^k_{n,n^b-H_w}\big(f^{n,|w|+k}_{\varepsilon h_n}|\mathbf{V}_w\big) \le e^{\delta_1 V(u)+\delta_1 V(v)+\frac{2\varepsilon}{A}h_n}\Big(\sum_{k\ge 1}\Psi^k_{n,n^b}(f^{n,k})\Big)^2,$$

and similarly on the event $\{V(v)\ge -B, H_v \le n\}$

$$\sum_{j\ge 1}\Psi^j_{n,n^b-H_v}\big(f^{n,|v|+j}_{\varepsilon h_n}|\mathbf{V}_v\big) \le e^{\delta_1 V(v)+\frac{\varepsilon}{A}h_n}\sum_{k\ge 1}\Psi^k_{n,n^b}(f^{n,k}).$$

Hence, $\mathbf{E}[\sum_{|z|<\lfloor A\ell_n\rfloor}\Sigma_1]$ is smaller, for n large enough, than

$$\begin{aligned}
&e^{\frac{2\varepsilon}{A}h_n}\mathbf{E}\Big[\Big(\sum_{|w|=1}e^{-(1-\delta_1)V(w)}\Big)^2\Big]\mathbf{E}\Big[\sum_{|z|<\lfloor A\ell_n\rfloor}e^{-V(z)-(1-2\delta_1)V(z)}\mathbb{1}_{\{V(z)\ge -B\}}\Big]\Big(\sum_{k\ge 1}\Psi^k_{n,n^b}(f^{n,k})\Big)^2\\
&+\frac{2\ell_n}{n}e^{\frac{\varepsilon}{A}h_n}C_\infty\mathbf{E}\Big[\Big(\sum_{|w|=1}e^{-(1-\delta_1)V(w)}\Big)^2\Big]\mathbf{E}\Big[\sum_{|z|<\lfloor A\ell_n\rfloor}e^{-V(z)-(1-2\delta_1)V(z)}\mathbb{1}_{\{V(z)\ge -B\}}\Big]\\
&\times\sum_{k\ge 1}\Psi^k_{n,n^b}(f^{n,k})+\frac{\ell_n^2C_\infty^2}{n^2}\mathbf{E}\Big[\sum_{z\in\mathscr{O}_n}e^{-V(z)-(1-2\delta_1)V(z)}\mathbb{1}_{\{V(z)\ge -B\}}\Big].
\end{aligned}$$

Finally, thanks to assumption (**A1**), (4) and by Remark 2, for n large enough

$$\mathbf{E}\Big[\sum_{|z|<\lfloor A\ell_n\rfloor}\Sigma_1(z)\Big] \le e^{\frac{5\varepsilon}{A}h_n}\Big(\sum_{k\ge 1}\Psi^k_{n,n^b}(f^{n,k})\Big)^2. \tag{56}$$

We now turn to $\Sigma_2(z)$, that is the sum

$$\sum_{\substack{x\not\sim y\\ x,y\in\mathscr{O}_{n,n^b}}}\mathbb{1}_{\{x\wedge y=z\}}e^{-V(x)}e^{-V(y)}f^{n,|x|}_{\varepsilon h_n}\mathbb{1}_{\mathscr{H}^{|x|}_{B,v'_n}}(\mathbf{V}_x)f^{n,|y|}_{\varepsilon h_n}\mathbb{1}_{\mathscr{H}^{|y|}_{B,v'_n}}(\mathbf{V}_y)\mathbb{1}_{\{(x,y)\in\mathscr{C}_{2,z}\}},$$

with $\mathscr{C}_{2,z} := \{(x,y)\in\mathbb{T}^2; x^*=z \text{ or } y^*=z\}$. The first step is to split the set $\{x^*=z \text{ or } y^*=z\}$ into three disjoint sets: $\{x^*=z \text{ and } y^*>z\}$, $\{x^*>z \text{ and } y^*=z\}$ and $\{x^*=z \text{ and } y^*=z\}$. By symmetry, the previous sum is equal to

$$\begin{aligned}
&2\sum_{\substack{x\ne v\\ x^*=v^*=z}}\mathbb{1}_{\{x\in\mathscr{O}_{n,n^b}\}}e^{-V(x)}f^{n,|x|}_{\varepsilon h_n}\mathbb{1}_{\mathscr{H}^{|x|}_{B,v'_n}}(\mathbf{V}_x)\sum_{\substack{y>v\\ y\in\mathscr{O}_{n,n^b}}}e^{-V(y)}f^{n,|y|}_{\varepsilon h_n}\mathbb{1}_{\mathscr{H}^{|y|}_{B,v'_n}}(\mathbf{V}_y)\\
&+\sum_{\substack{x\ne y\\ x^*=y^*=z}}e^{-V(x)}e^{-V(y)}\mathbb{1}_{\{x,y\in\mathscr{O}_{n,n^b}\}}f^{n,|x|}_{\varepsilon h_n}\mathbb{1}_{\mathscr{H}^{|x|}_{B,v'_n}}(\mathbf{V}_x)f^{n,|y|}_{\varepsilon h_n}\mathbb{1}_{\mathscr{H}^{|y|}_{B,v'_n}}(\mathbf{V}_y).
\end{aligned}$$

We then use a very similar approach as the one we used for $\Sigma_1(z)$: split the first above sum

according to the value of $V(v)$ then make the previous decomposition of H_y to obtain, as in (55)

$$\begin{aligned}\mathbf{E}\Big[\sum_{|z|=l}\Sigma_2(z)\Big] \le& \frac{2C_\infty}{n}\mathbf{E}\Big[\sum_{|z|=l}\sum_{\substack{x\neq v\\ x^*=v^*=z}} e^{-(1-\delta_1)V(x)}\mathbb{1}_{\{\underline{V}(v)\ge -B, H_v\le n\}}\sum_{\substack{y>v\\ y\in\mathscr{O}^v_{n,n^b-H_v}}} e^{-V(y)} f^{n,|y|}_{\varepsilon h_n}(\mathbf{V}_y)\Big]\\ &+\frac{2C_\infty^2}{n^2}\mathbf{E}\Big[\sum_{|z|=l}\mathbb{1}_{\{V(z)\ge -B\}}\sum_{\substack{x\neq v\\ x^*=v^*=z}} e^{-(1-\delta_1)(V(x)+V(v))}\sum_{\substack{y>v\\ y\in\mathscr{O}^v_n}} e^{-V_v(y)}\Big]\\ &+\frac{C_\infty^2}{n^2}\mathbf{E}\Big[\sum_{|z|=l}\mathbb{1}_{\{V(z)\ge -B\}}\sum_{\substack{x\neq y\\ x^*=y^*=z}} e^{-(1-\delta_1)(V(x)+V(y))}\Big].\end{aligned}$$

Hence, by using independence of increments of V conditionally given $(\mathbb{T}, V(w); w\in\mathbb{T}, |w|\le l+1)$, $\mathbf{E}[\sum_{|z|=l}\Sigma_2(z)]$ is smaller, for n large enough, than

$$\begin{aligned}&\frac{2C_\infty}{n}\mathbf{E}\Big[\sum_{|z|=l}\mathbb{1}_{\{V(z)\ge -B\}}\sum_{\substack{x\neq v\\ x^*=v^*=z}} e^{-(1-\delta_1)V(x)}\mathbb{1}_{\{\underline{V}(v)\ge -B, H_v\le n\}}\sum_{j\ge 1} e^{-V(v)}\Psi^j_{n,n^b-H_v}\big(f^{n,|v|+j}_{\varepsilon h_n}|\mathbf{V}_v\big)\Big]\\ &+\frac{3\ell_n C_\infty^2}{n^2}\mathbf{E}\Big[\sum_{|z|=l}\mathbb{1}_{\{V(z)\ge -B\}}\sum_{\substack{x\neq v\\ x^*=v^*=z}} e^{-(1-\delta_1)(V(x)-V(v))}\Big],\end{aligned}$$

where we used that $\mathbf{E}[\sum_{x\in\mathscr{O}_n} e^{-V(x)}]\le \ell_n$ for n large enough (see Remark 2). Then, by assumption (**A2**) with $\delta=\delta_1$ (see (4) for the definition of δ_1), for all $l<\lfloor A\ell_n\rfloor$ ($|v|=l+1$) and n large enough on the event $\{V(v)\ge -B, H_v\le n\}$

$$\sum_{j\ge 1}\Psi^j_{n,n^b-H_v}\big(f^{n,|v|+j}_{\varepsilon h_n}|\mathbf{V}_v\big)\le e^{\delta_1 V(v)+\frac{\varepsilon}{A}h_n}\sum_{k\ge 1}\Psi^k_{n,n^b}(f^{n,k}),$$

so $\mathbf{E}[\sum_{|z|=l}\Sigma_2(z)]$ is smaller than

$$\begin{aligned}&\frac{2C_\infty}{n}e^{\frac{\varepsilon}{A}h_n}\mathbf{E}\Big[\Big(\sum_{|w|=1}e^{-(1-\delta_1)V(w)}\Big)^2\Big]\mathbf{E}\Big[\sum_{|z|=l}e^{-V(z)-(1-2\delta_1)V(z)}\mathbb{1}_{\{V(z)\ge -B\}}\Big]\sum_{k\ge 1}\Psi^k_{n,n^b}(f^{n,k})\\ &+\frac{3\ell_n C_\infty^2}{n^2}\mathbf{E}\Big[\Big(\sum_{|w|=1}e^{-(1-\delta_1)V(w)}\Big)^2\Big]\mathbf{E}\Big[\sum_{|z|=l}e^{-V(z)-(1-2\delta_1)V(z)}\mathbb{1}_{\{V(z)\ge -B\}}\Big].\end{aligned}$$

Hence, thanks to assumption (**A1**), (4) and by Remark 2, for n large enough

$$\mathbf{E}\Big[\sum_{|z|<\lfloor A\ell_n\rfloor}\Sigma_2(z)\Big]\le e^{\frac{3\varepsilon}{A}h_n}\Big(\sum_{k\ge 1}\Psi^k_{n,n^b}(f^{n,k})\Big)^2. \tag{57}$$

Collecting Case 1, Case 2 ((53), inequalities (54), (56) and (57)) and considering (52) gives the lemma.

□

We are now ready to prove the lower bound of $\mathscr{R}_{T^n}(g_n,\mathbf{f}^n)$ in Proposition 1. Recall $u_{1,n}=\sum_{k\ge 1}\Psi^k_{\lambda_n/2,b_n}\big(f^{n,k}_{\varepsilon h_n}\mathbb{1}_{\Upsilon^k_n}\big)$ where $\Upsilon^k_n=\{\mathbf{t}\in\mathbb{R}^k;\ H_k(\mathbf{t})\le n^b e^{\varepsilon h_n}\}\cap\mathscr{H}^k_{B,v'_n}$, $\mathscr{H}^k_{B,v'_n}$ defined in

(17) and $v_n' = v_n + \varepsilon h_n$. Thanks to Lemmata 3.2, 3.3 and the expression of a_n (47), for n large enough as $e^{-\varepsilon h_n} \leq \frac{1}{4}$, $\mathbb{P}(\mathscr{R}_{T^n}(g_n, \mathbf{f}^n) < n^{1-b}\varphi(n^b)e^{-5\varepsilon h_n}u_{1,n})$ is smaller than

$$\mathbb{P}\big(\mathscr{R}_{T^n}(g_n, \mathbf{f}^n) < n\varphi(n^b)e^{-4\varepsilon h_n}u_{1,n}/4n^b\big) \leq e^{-\varepsilon\frac{c_4}{r_2}h_n}\frac{\mathbf{E}[Z_n^2]}{u_{1,n}^2} + h_n e^{-\varepsilon h_n} + \frac{e^{8\varepsilon h_n - \min(9\varepsilon\log n, 4h_n)}}{n^{2\kappa_b}u_{1,n}^2}.$$

Then Lemma 3.4 provides majoration of $\mathbf{E}[Z_n^2]$ so $\mathbb{P}\big(\mathscr{R}_{T^n}(g_n, \mathbf{f}^n) < n\varphi(n^b)e^{-4\varepsilon h_n}u_{1,n}/4n^b\big)$ is smaller, for n large enough than (recall that $h_n \leq \log n$)

$$e^{-(\frac{c_4}{r_2}-\frac{6}{A})\varepsilon h_n}\Big(\sum_{k\geq 1}\Psi^k_{n,n^b}(f^{n,k})/u_{1,n}\Big)^2 + h_n e^{-\varepsilon h_n} + \frac{e^{-\min(\varepsilon\log n, 3h_n)}}{n^{2\kappa_b}u_{1,n}^2}.$$

which yields the lower bound of Proposition 1.

3.3. Upper bound for $\mathscr{R}_{T^n}(g_n, \mathbf{f}^n)$

For all $n \geq 1$ and $x \in \mathbb{T}$, recall that E_x^n is the number of excursions, among the first n excursions to the root, for which the vertice x is reached, in a similar way $\tilde{E}_x^n$ is the number of excursions such that x is reaching more often from above than from below :

$$E_x^n = \sum_{i=1}^n \mathbb{1}_{\{N_x^{T^i} - N_x^{T^{i-1}} \geq 1\}} \quad \text{and} \quad \tilde{E}_x^n := \sum_{i=1}^n \mathbb{1}_{\{\sum_{y;y^*=x} N_y^{T^i} - N_y^{T^{i-1}} > N_x^{T^i} - N_x^{T^{i-1}}\}}.$$

Also introduce the event $\mathscr{A}_n$ such that every vertices of the trace of $(X_k, k \leq T^n)$ has downfall fluctuations lower than n, potential larger than v_n and which are visited during a single excursion before during the first n excursions to the root

$$\mathscr{A}_n := \Big\{\forall\, j \leq T^n, X_j \in \mathscr{O}_n, \sum_{x\in\mathscr{O}_n}(\mathbb{1}_{\{E_x^n\geq 2\}} + \mathbb{1}_{\{\tilde{E}_x^n \geq 2\}})\mathbb{1}_{\mathscr{H}_{v_n}^{|x|}}(\boldsymbol{V}_x) = 0\Big\}. \tag{58}$$

Note that $\lim_{n\to\infty}\mathbb{P}(\mathscr{A}_n) = 1$, indeed, $\tilde{E}_x^n \geq 2$ implies $E_x^n \geq 2$ so

$$1 - \mathbb{P}(\mathscr{A}_n) \leq \mathbb{P}(\exists\, j \leq T^n : X_j \notin \mathscr{O}_n) + \mathbb{P}\Big(\sum_{x\in\mathscr{O}_n}\mathbb{1}_{\{E_x^n\geq 2\}}\mathbb{1}_{\mathscr{H}_{v_n}^{|x|}}(\boldsymbol{V}_x) > 0\Big).$$

By [AC18], equation 2.2, $\mathbb{P}(\exists\, j \leq T^n : X_j \notin \mathscr{O}_n) \to 0$. Moreover, $\mathbb{P}(\sum_{x\in\mathscr{O}_n}\mathbb{1}_{\{E_x^n\geq 2\}}\mathbb{1}_{\mathscr{H}_{v_n}^{|x|}}(\boldsymbol{V}_x) > 0)$ is smaller than

$$\mathbf{E}\Big[\sum_{x\in\mathscr{O}_n}\mathbb{P}^{\mathscr{E}}(E_x^n \geq 2)\mathbb{1}_{\mathscr{H}_{v_n}^{|x|}}(\boldsymbol{V}_x)\Big] = \mathbf{E}\Big[\sum_{x\in\mathscr{O}_n}\big(\mathbb{P}^{\mathscr{E}}(E_x^n \geq 1) - \mathbb{P}^{\mathscr{E}}(E_x^n = 1)\big)\mathbb{1}_{\mathscr{H}_{v_n}^{|x|}}(\boldsymbol{V}_x)\Big].$$

Thanks to strong Markov property, $N_x^{T^i} - N_x^{T^{i-1}}, i \in \{1, \dots, n\}$, are i.i.d under $\mathbb{P}^{\mathscr{E}}$ so $\mathbb{P}^{\mathscr{E}}(E_x^n \geq 1) - \mathbb{P}^{\mathscr{E}}(E_x^n = 1) \leq \mathbb{E}^{\mathscr{E}}[E_x^n] - \mathbb{P}^{\mathscr{E}}(E_x^n = 1) = n\mathbb{P}^{\mathscr{E}}(N_x^{T_e} \geq 1)(1 - \mathbb{P}^{\mathscr{E}}(N_x^{T_e} = 0)^{n-1}) \leq n^2\mathbb{P}^{\mathscr{E}}(N_x^{T_e} \geq 1)^2$ and by Lemma 3.1, for all x with $V(x) \geq v_n$, $n^2\mathbb{P}^{\mathscr{E}}(N_x^{T_e} \geq 1)^2 \leq n^2 e^{-2V(x)} \leq n^{2-1/\delta_1}e^{-V(x)}/\ell_n^{1/\delta_1}$. $\delta_1 \in (0, 1/2]$, hence, by Remark 2

$$\mathbb{P}\Big(\sum_{x\in\mathscr{O}_n}\mathbb{1}_{\{E_x^n\geq 2\}}\mathbb{1}_{\mathscr{H}_{v_n}^{|x|}}(\boldsymbol{V}_x) > 0\Big) \leq \frac{n^{2-1/\delta_1}}{\ell_n^{1/\delta_1}}\mathbf{E}\Big[\sum_{x\in\mathscr{O}_n}e^{-V(x)}\Big] \leq \frac{n^{2-1/\delta_1}}{\ell_n^{1/\delta_1-1}} \to 0.$$

Lemma 3.5. *Let (u_n, n) be a sequence of positive numbers, then*

$$\mathbb{P}^{\mathscr{E}}(\mathscr{R}_{T^n}(g_n, \mathbf{f}^n) > u_n, \mathscr{A}_n) \leq \frac{2n^{1-b}\varphi(n^b)}{u_n}(\mathscr{X}_{1,n} + \mathscr{X}_{2,n} + \mathscr{X}_{3,n})$$

where

$$\mathscr{X}_{1,n} := \sum_{x\in\mathscr{O}_n} \mathbb{1}_{\{V(x)<v_n\}}\Big(e^{-V(x)} + \sum_{y;y^*=x} e^{-V(y)}\Big) f^{n,|x|}(\boldsymbol{V}_x), \tag{59}$$

$$\mathscr{X}_{2,n} := \sum_{x\in\mathscr{O}_n} \mathbb{1}_{\{V(x)\geq v_n\}} \frac{e^{-V(x)}}{H_x}\Big(1 - \frac{1}{H_x}\Big)^{\lceil n^b/2\rceil - 1}(n^b + H_x) f^{n,|x|}(\boldsymbol{V}_x), \tag{60}$$

and

$$\mathscr{X}_{3,n} := \sum_{x\in\mathscr{O}_n} \mathbb{1}_{\{V(x)\geq v_n\}} \frac{e^{-V(x)}}{H_x}\frac{\tilde{H}_x}{1+\tilde{H}_x}\Big(1 - \frac{1}{1+\tilde{H}_x}\Big)^{\lceil n^b/2\rceil - 1}(n^b + 1 + \tilde{H}_x) f^{n,|x|}(\boldsymbol{V}_x) \tag{61}$$

recall the definition of H_x and $\tilde{H}_x$ in Lemma 3.1.

Proof. Since $g_n(0) = 0$, we have, by Markov inequality, that $\mathbb{P}^{\mathscr{E}}(\mathscr{R}_{T^n}(g_n, \mathbf{f}^n) > u_n, \mathscr{A}_n)$ is smaller than

$$\frac{2}{u_n}\Big(\sum_{x\in\mathscr{O}_n} \mathbb{1}_{\{V(x)<v_n\}}\mathbb{E}^{\mathscr{E}}\big[g_n\big(\mathscr{L}_x^{T^n}\big)\big] f^{n,|x|}(\boldsymbol{V}_x) + \sum_{x\in\mathscr{O}_n} \mathbb{1}_{\{V(x)\geq v_n\}}\mathbb{E}^{\mathscr{E}}\big[g_n\big(\mathscr{L}_x^{T^n}\big)\mathbb{1}_{\{E_x^n, \tilde{E}_x^n\in\{0,1\}\}}\big] f^{n,|x|}(\boldsymbol{V}_x)\Big).$$

The first parts in the above sum is the easiest to deal with. Indeed, the application $t \in [1,\infty) \mapsto \varphi(t)/t$ is non increasing so $g_n(t) \leq tn^{-b}\varphi(n^b)$ and we have

$$\begin{aligned}\sum_{x\in\mathscr{O}_n} \mathbb{1}_{\{V(x)<v_n\}}\mathbb{E}^{\mathscr{E}}\big[g_n\big(\mathscr{L}_x^{T^n}\big)\big] f^{n,|x|}(\boldsymbol{V}_x) &\leq n^{1-b}\varphi(n^b)\sum_{x\in\mathscr{O}_n} \mathbb{1}_{\{V(x)<v_n\}}\mathbb{E}^{\mathscr{E}}\big[\mathscr{L}_x^{T_e}\big] f^{n,|x|}(\boldsymbol{V}_x)\\ &= n^{1-b}\varphi(n^b)\mathscr{X}_{1,n}.\end{aligned}$$

We used that for all $1 \leq i \leq n$, $\mathscr{L}_x^{T^i} - \mathscr{L}_x^{T^{i-1}}$ is distributed as $\mathscr{L}_x^{T_e}$ under $\mathbb{P}^{\mathscr{E}}$ with mean $e^{-V(x)} + \sum_{y;y^*=x} e^{-V(y)}$ by Lemma 3.1.

We then move to the high potential part. Assume $E_x^n \in \{0,1\}$ and $\tilde{E}_x^n \in \{0,1\}$. If $E_x^n = 0$, then the vertex x is never visited during any of the first n excursions and $\tilde{E}_x^n = 0$. Thus, $g_n\big(\mathscr{L}_x^{T^n}\big) = g_n(0) = 0$. If $E_x^n = 1$ and $\tilde{E}_x^n = 0$ there exists $i \in \{1,\dots,n\}$ such that $N_x^{T^i} - N_x^{T^{i-1}} \geq 1$ and $\forall j \neq i$, $N_x^{T^j} - N_x^{T^{j-1}} = 0$ and $\forall m \in \{1,\dots,n\}$, $\sum_{y;y^*=x} N_y^{T^m} - N_y^{T^{m-1}} \leq N_x^{T^m} - N_x^{T^{m-1}}$. In particular, since, starting from the root e, $\mathscr{L}_x^{T^n} = \sum_{j=1}^n \big(N_x^{T^j} - N_x^{T^{j-1}} + \sum_{y;y^*=x} N_y^{T^j} - N_y^{T^{j-1}}\big)$, we have, on $\{E_x^n = 1, \tilde{E}_x^n = 0\}$

$$\mathscr{L}_x^{T^n} = N_x^{T^i} - N_x^{T^{i-1}} + \sum_{y;y^*=x} N_y^{T^i} - N_y^{T^{i-1}} \leq 2\big(N_x^{T^i} - N_x^{T^{i-1}}\big). \tag{62}$$

Otherwise, if $E_x^n = 1$ and $\tilde{E}_x^n = 1$ there exists $i \in \{1,\dots,n\}$ such that $N_x^{T^i} - N_x^{T^{i-1}} \geq 1$ and $\forall j \neq i$, $N_x^{T^j} - N_x^{T^{j-1}} = 0$ and $\exists m' \in \{1,\dots,n\}$ such that $\sum_{y;y^*=x} N_y^{T^{m'}} - N_y^{T^{m'-1}} > N_x^{T^{m'}} - N_x^{T^{m'-1}}$

and $\forall m \neq m'$, $\sum_{y;y^*=x} N_y^{T^m} - N_y^{T^{m-1}} \leq N_x^{T^m} - N_x^{T^{m-1}}$. So we have necessarily $m' = i$ and, on $\{E_x^n = 1, \tilde{E}_x^n = 1\}$

$$\mathscr{L}_x^{T^n} = N_x^{T^i} - N_x^{T^{i-1}} + \sum_{y;y^*=x} N_y^{T^i} - N_y^{T^{i-1}} \leq 2 \sum_{y;y^*=x} N_y^{T^i} - N_y^{T^{i-1}}. \tag{63}$$

g_n is increasing so (62) and (63) give, when $E_x^n \in \{0,1\}$ and $\tilde{E}_x^n \in \{0,1\}$

$$g_n\big(\mathscr{L}_x^{T^n}\big) \leq \sum_{i=1}^n g_n\big(2\big(N_x^{T^i} - N_x^{T^{i-1}}\big)\big) + \sum_{i=1}^n g_n\big(2 \sum_{y;y^*=x} N_y^{T^i} - N_y^{T^{i-1}}\big)$$

From this inequality, it follows that $\mathbb{E}^{\mathscr{E}}\big[g_n\big(\mathscr{L}_x^{T^n}\big)\mathbb{1}_{\{E_x^n, \tilde{E}_x^n \in \{0,1\}\}}\big]$ is smaller than

$$\begin{aligned} n\mathbb{E}^{\mathscr{E}}\big[g_n\big(2N_x^{T_e}\big)\big] + n\mathbb{E}^{\mathscr{E}}\big[g_n\big(2 \sum_{y;y^*=x} N^{T_e}\big)\big] \leq & n^{1-b}\varphi(n^b)\mathbb{E}^{\mathscr{E}}\big[N_x^{T_e}\mathbb{1}_{\{N_x^{T_e} \geq \lceil n^b/2 \rceil\}}\big] \\ & + n^{1-b}\varphi(n^b)\mathbb{E}^{\mathscr{E}}\Big[\sum_{y;y^*=x} N_y^{T_e}\mathbb{1}_{\{\sum_{y;y^*=x} N_y^{T_e} \geq \lceil n^b/2 \rceil\}}\Big] \end{aligned}$$

We used that for all $1 \leq i \leq n$, $N_x^{T^i} - N_x^{T^{i-1}}$(resp. $\sum_{y;y^*=x} N_y^{T^i} - N_y^{T^{i-1}}$) is distributed as $N_x^{T_e}$ (resp. $\sum_{y;y^*=x} N_y^{T_e}$) under $\mathbb{P}^{\mathscr{E}}$ and the fact that the application $t \in [1,\infty) \mapsto \varphi(t)/t$ is deacresing. Then, by Lemma 3.1

$$\mathbb{E}^{\mathscr{E}}\big[N_x^{T_e}\mathbb{1}_{\{N_x^{T_e} \geq \lceil n^b/2 \rceil\}}\big] \leq \frac{e^{-V(x)}}{H_x}\Big(1 - \frac{1}{H_x}\Big)^{\lceil n^b/2 \rceil - 1}(n^b + H_x)$$

and

$$\mathbb{E}^{\mathscr{E}}\Big[\sum_{y;y^*=x} N_y^{T_e}\mathbb{1}_{\{\sum_{y;y^*=x} N_y^{T_e} \geq \lceil n^b/2 \rceil\}}\Big] \leq \frac{e^{-V(x)}}{H_x}\frac{\tilde{H}_x}{1+\tilde{H}_x}\Big(1 - \frac{1}{1+\tilde{H}_x}\Big)^{\lceil n^b/2 \rceil - 1}(n^b + 1 + \tilde{H}_x)$$

which ends the proof. □

Lemma 3.6. *Let $b \in [0,1)$. For n large enough*

$$\mathbf{E}[\mathscr{X}_{1,n} + \mathscr{X}_{2,n} + \mathscr{X}_{3,n}] \leq 2(\log n)^2 u_{2,n}.$$

where we recall $u_{2,n} = \sum_{k\geq 1}\big(\Psi_{n,n^b}^k(f^{n,k}) + \Psi_n^k\big(f^{n,k}\mathbb{1}_{\mathbb{R}^k \setminus \mathscr{H}_{v_n}^k}\big) + \Psi_{n,\lambda_n'}^k(f^{n,k}) + \mathbf{E}[W\Psi_{n,\lambda_n'/W}^k(f^{n,k})]\big)$, *with* $\lambda_n' = n^b(\log n)^{-2}$ *and* $W = \sum_{|z|=1} e^{-V(z)}$.

Proof. We start with the easiest part that is the expression of $\mathbf{E}[\mathscr{X}_{1,n}]$. Thanks to hypothesis (1)

$$\begin{aligned} \mathbf{E}[\mathscr{X}_{1,n}] &= \mathbf{E}\Big[\sum_{x\in\mathscr{O}_n} \mathbb{1}_{\{V(x)<v_n\}}\Big(e^{-V(x)} + e^{-V(x)}\sum_{y;y^*=x} e^{-V_x(y)}\Big)f^{n,|x|}(\boldsymbol{V}_x)\Big] \\ &= 2\mathbf{E}\Big[\sum_{x\in\mathscr{O}_n} \mathbb{1}_{\{V(x)<v_n\}}e^{-V(x)}f^{n,|x|}(\boldsymbol{V}_x)\Big] = 2\sum_{k\geq 1}\Psi_n^k\big(f^{n,k}\mathbb{1}_{\mathbb{R}^k\setminus\mathscr{H}_{v_n}^k}\big). \end{aligned}$$

Introduce

$$\tilde{\lambda}_n = \frac{\lceil n^b/2 \rceil - 1}{\log q_n} \quad \text{with} \quad q_n = \frac{4C_\infty \ell_n n^b}{\sum_{k\geq 1}\Psi_{n,n^b}^k\big(f^{n,k}\big)}.$$

We now move to $\mathbf{E}[\mathscr{X}_{2,n}]$, for that we decompose $\mathscr{X}_{2,n}$ into two parts according to the value of H_x:

$$\begin{aligned}\mathscr{X}_{2,n} &\le \sum_{x\in\mathscr{O}_n}(\mathbb{1}_{\{H_x\le\tilde\lambda_n\}}+\mathbb{1}_{\{H_x>\tilde\lambda_n\}})\frac{e^{-V(x)}}{H_x}\Big(1-\frac{1}{H_x}\Big)^{\lceil n^b/2\rceil-1}(n^b+H_x)f^{n,|x|}(\boldsymbol{V}_x)\\ &\le C_\infty\big(n^b+\tilde\lambda_n\big)\big(1-\frac{1}{\tilde\lambda_n}\big)^{\lceil n^b/2\rceil-1}\sum_{x\in\mathscr{O}_n}e^{-V(x)}+\big(1+\frac{n^b}{\tilde\lambda_n}\big)\sum_{x\in\mathscr{O}_{n,\tilde\lambda_n}}e^{-V(x)}f^{n,|x|}(\boldsymbol{V}_x).\end{aligned}$$

By definition of $\tilde\lambda_n$ and q_n (see above), $(1-1/\tilde\lambda_n)^{\lceil n^b/2\rceil-1}\le 1/q_n$. Moreover, by Remark 2, $\mathbf{E}[\sum_{x\in\mathscr{O}_n}e^{-V(x)}]\le\ell_n$ and $\mathbf{E}[\sum_{k\ge1}\Psi^k_{n,n^b}(f^{n,k})]\le C_\infty\mathbf{E}[\sum_{x\in\mathscr{O}_n}e^{-V(x)}]\le C_\infty\ell_n$ so for n large enough ($q_n\ge 4n^b$ implying $\tilde\lambda_n\le n^b$), we obtain

$$\mathbf{E}[\mathscr{X}_{2,n}]\le\frac12\sum_{k\ge1}\Psi^k_{n,n^b}(f^{n,k})+\big(1+\frac{n^b}{\tilde\lambda_n}\big)\sum_{k\ge1}\Psi^k_{n,\tilde\lambda_n}(f^{n,k}).$$

For $\mathbf{E}[\mathscr{X}_{3,n}]$, we decompose $\mathscr{X}_{3,n}$ into two parts according to the value of $\tilde H_x$: $\mathscr{X}_{3,n}$ is smaller than

$$\begin{aligned}&\sum_{x\in\mathscr{O}_n}(\mathbb{1}_{\{1+\tilde H_x\le\tilde\lambda_n\}}+\mathbb{1}_{\{1+\tilde H_x>\tilde\lambda_n\}})\frac{e^{-V(x)}}{H_x}\frac{\tilde H_x}{1+\tilde H_x}\Big(1-\frac{1}{1+\tilde H_x}\Big)^{\lceil n^b/2\rceil-1}(n^b+1+\tilde H_x)f^{n,|x|}(\boldsymbol{V}_x)\\ &\le C_\infty\big(n^b+\tilde\lambda_n\big)\big(1-\frac{1}{\tilde\lambda_n}\big)^{\lceil n^b/2\rceil-1}\sum_{x\in\mathscr{O}_n}e^{-V(x)}+\big(1+\frac{n^b}{\tilde\lambda_n}\big)\sum_{x\in\mathscr{O}_n}e^{-V(x)}\mathbb{1}_{\{1+\tilde H_x>\tilde\lambda_n\}}\\ &\times\sum_{y;y^*=x}e^{-V_x(y)}f^{n,|x|}(\boldsymbol{V}_x).\end{aligned}$$

Then as above $C_\infty\big(n^b+\tilde\lambda_n\big)\big(1-1/\tilde\lambda_n\big)^{\lceil n^b/2\rceil-1}\le\sum_{k\ge1}\Psi^k_{n,n^b}(f^{n,k})/2$, also recall that $\tilde H_x=H_x\sum_{y;y^*=x}e^{-V_x(y)}$ so by conditional independence of $\sum_{y;y^*=x}e^{-V_x(y)}$ together with the fact that this random variable has the same law as $W=\sum_{|x|=1}e^{-V(x)}$,

$$\mathbf{E}\Big[\sum_{x\in\mathscr{O}_n}e^{-V(x)}\mathbb{1}_{\{1+\tilde H_x>\tilde\lambda_n\}}\sum_{y;y^*=x}e^{-V_x(y)}f^{n,|x|}(\boldsymbol{V}_x)\Big]=\sum_{k\ge1}\mathbf{E}\Big[W\Psi^k_{n,(\tilde\lambda_n-1)/W}(f^{n,k})\Big].$$

Hence

$$\mathbf{E}[\mathscr{X}_{3,n}]\le\frac12\sum_{k\ge1}\Psi^k_{n,n^b}(f^{n,k})+\big(1+\frac{n^b}{\tilde\lambda_n}\big)\sum_{k\ge1}\mathbf{E}\big[W\Psi^k_{n,(\tilde\lambda_n-1)/W}(f^{n,k})\big].$$

Finally, by assumtpion (**A1**), $q_n\le 4C_\infty\ell_n n^{1+b}$ so for all $b\in(0,1)$ and n large enough $\tilde\lambda_n-1\ge n^b(\log n)^{-2}=\lambda'_n$. Hence, for all $b\in[0,1)$ and n large enough $(1+n^b/\tilde\lambda_n)\le 2(\log n)^2$ and $\Psi^k_{n,(\tilde\lambda_n-1)/W}(f^{n,k})$ (resp. $\Psi^k_{n,\tilde\lambda_n}(f^{n,k})$) is smaller than $\Psi^k_{n,\lambda'_n/W}(f^{n,k})$ (resp. $\Psi^k_{n,\lambda'_n}(f^{n,k})$) so we obtain the result. □

We are now ready to prove the upper bound in Proposition 1. Recall (58), and let $\varepsilon>0$

$$\mathbb{P}\Big(\frac{\mathscr{R}_{T^n}(g_n,\mathbf{f}^n)}{n^{1-b}\varphi(n^b)u_{2,n}}>e^{\varepsilon h_n}\Big)\le\mathbb{P}\Big(\frac{\mathscr{R}_{T^n}(g_n,\mathbf{f}^n)}{n^{1-b}\varphi(n^b)u_{2,n}}>e^{\varepsilon h_n},\mathscr{A}_n\Big)+1-\mathbb{P}(\mathscr{A}_n).$$

where $u_{2,n}=\sum_{k\ge1}\big(\Psi^k_{n,n^b}(f^{n,k})+\Psi^k_n\big(f^{n,k}\mathbb{1}_{\mathbb{R}^k\setminus\mathscr{H}^k_{v_n}}\big)+\Psi^k_{n,\lambda'_n}(f^{n,k})+\mathbf{E}\big[W\Psi^k_{n,\lambda'_n/W}(f^{n,k})\big]\big)$. By Lemma 3.5 with $u_n=e^{\varepsilon h_n}n^{1-b}\varphi(n^b)u_{2,n}$ and Lemma 3.6, for n large enough

$$\mathbb{P}\Big(\frac{\mathscr{R}_{T^n}(g_n,\mathbf{f}^n)}{n^{1-b}\varphi(n^b)u_{2,n}}>e^{\varepsilon h_n},\mathscr{A}_n\Big)\le\frac{2e^{-\varepsilon h_n}}{u_{2,n}}\mathbf{E}[\mathscr{X}_{1,n}+\mathscr{X}_{2,n}+\mathscr{X}_{3,n}]\le 4(\log n)^2e^{-\varepsilon h_n},$$

and then for n large enough

$$\mathbb{P}\Big(\frac{\mathscr{R}_{T^n}(g_n,\mathbf{f}^n)}{n^{1-b}\varphi(n^b)u_{2,n}} > e^{\varepsilon h_n}\Big) \le 4(\log n)^2 e^{-\varepsilon h_n} + 1 - \mathbb{P}(\mathscr{A}_n).$$

Finally, observe (see (12) for the definition of h_n) that $(\log n)^2 = o(e^{\epsilon h_n})$ and we complete the proof of the upper bound recalling (see below (58)) that $1-\mathbb{P}(\mathscr{A}_n) = o(1)$.

4. Technical estimates for one-dimensional random walk

In this section we prove some technical expressions involving sums of i.i.d. random variables, the ones introduced via the many-to-one Lemma at the beginning of Section 2. Recall that $(S_i - S_{i-1}, i \ge 1)$ is a sequence of i.i.d. random variables such that $\mathbb{E}(S_1) = 0$, there exists $\eta > 0$ for which $\mathbb{E}(e^{\eta S_1}) < +\infty$. Also we denote $\sigma^2 = \psi''(1) = \mathbb{E}(S_1^2)$. We use the following notations : for any a, $\tau_a := \inf\{k > 0,\ S_k \ge a\}$, $\tau_a^- := \inf\{k > 0,\ S_k \le a\}$ and $\tau_a^{\bar S - S} := \inf\{k > 0, \bar S_k - S_k \ge a\}$ with $\bar S_k := \max_{1\le m\le k} S_m$ and $H_j := \sum_{i=1}^j e^{S_i - S_j}$.

4.1. Two Laplace transforms

In this section we deal with Laplace transforms which appear when we study the range of high potential with the underlying constraint of the reflecting barrier and also when a penalization via cumulative downfalls of V is introduced.

Lemma 4.1. *Let $r := r(\ell)$ such that $\lim_{\ell\to+\infty} r(\ell)/\ell = +\infty$, then for any $\varepsilon > 0$*

$$e^{-(1+\sqrt{c}-\rho(c))\frac{r}{\ell}(1+\varepsilon)} \le \mathbb{E}\Big[e^{-\frac{c\sigma^2}{2\ell^2}\tau_r}\mathbb{1}_{\tau_r\le\tau_\ell^{\bar S - S}}\Big] \le e^{-(1+\sqrt{c}-\rho(c))\frac{r}{\ell}(1-\varepsilon)},$$

with $\rho(c) = \frac{c\sigma}{\sqrt{2\pi}}\int_0^{+\infty} e^{-\frac{c\sigma^2}{2}u} f(u)du$, and $f(u) = \frac{2}{u^{1/2}}\mathbb{P}(\overline{\mathfrak{m}}_1 > 1/\sqrt{u\sigma^2}) - \frac{1}{2}\int_u^{+\infty}\frac{1}{y^{3/2}}\mathbb{P}(\overline{\mathfrak{m}}_1 > 1/\sqrt{y\sigma^2})dy$. Note that ρ can be explicitly calculated : for any $c > 0$

$$\rho(c) = 2\sqrt{c}\Big(\frac{1-e^{-\sqrt{c}}}{\sinh(\sqrt{c})}\Big) - 2\Big(\sqrt{c} - \log((e^{\sqrt{c}}+1)/2)\Big).$$

Proof. We start with the *upper bound*.
Let us introduce the usual strict ladder epoch sequence $(T_k := \inf\{i > T_{k-1}, S_i > S_{T_{k-1}}\}, k;$ $T_0 = 0)$. Then for any k

$$\begin{aligned}\mathbb{E}\Big[e^{-\frac{c\sigma^2}{2\ell^2}\tau_r}\mathbb{1}_{\tau_r\le\tau_\ell^{\bar S-S}}\Big] &\le \mathbb{E}\Big[e^{-\frac{c\sigma^2}{2\ell^2}\tau_r}\mathbb{1}_{S_{T_k}<r}\mathbb{1}_{\tau_r\le\tau_\ell^{\bar S-S}}\Big] + \mathbb{P}(S_{T_k}\ge r)\\ &\le \Big(\mathbb{E}\Big[e^{-\frac{c\sigma^2}{2\ell^2}\tau_0}\mathbb{1}_{\tau_0\le\tau_{-\ell}^-}\Big]\Big)^k + \mathbb{P}(S_{T_k}\ge r), \end{aligned} \tag{64}$$

where the last equality comes from the strong Markov property and equality $T_1 = \tau_0 := \inf\{m > 0, S_m > 0\}$. From here we need the asymptotic in ℓ of $\mathbb{E}\Big[e^{-\frac{c\sigma^2}{2\ell^2}\tau_0}\mathbb{1}_{\tau_0\le\tau_{-\ell}^-}\Big]$. First we use following identity

$$\mathbb{E}\Big[e^{-\frac{\lambda}{\ell^2}\tau_0}\mathbb{1}_{\tau_0\le\tau_{-\ell}^-}\Big] = \mathbb{E}[e^{-\frac{\lambda}{\ell^2}\tau_0}] - \mathbb{P}(\tau_0 > \tau_{-\ell}^-) + \mathbb{E}\Big((1-e^{-\frac{\lambda}{\ell^2}\tau_0})\mathbb{1}_{\tau_0>\tau_{-\ell}}\Big), \tag{65}$$

and then give an upper bound for each of the term appearing, first Lemma 2.2 in [Aïd10] gives for m large enough

$$\mathbb{P}(\tau_0 > \tau^-_{-\ell}) = \frac{\mathbb{E}(S_{\tau_0})}{\ell} + o\left(\frac{1}{\ell}\right), \tag{66}$$

Both of the other terms can be obtained with a Tauberian theorem, we give here some details for the third one which is more delicate. Let $dH_\ell(u)$ the measure defined by $\mathbb{P}(\tau_0 > z\ell^2, \tau_0 > \tau^-_{-\ell}) = \int_z^\infty dH_\ell(u)$, integration by part gives $\mathbb{E}\left((1-e^{\frac{-\lambda}{\ell^2}\tau_0})\mathbb{1}_{\tau_0>\tau^-_{-\ell}}\right) = \int_0^{+\infty}(1-e^{-\lambda u})dH_\ell(u) = \lambda\int_0^{+\infty} e^{-\lambda u}\ \mathbb{P}(\tau_0 > u\ell^2, \tau_0 > \tau^-_{-\ell})du$. So we need an asymptotic in ℓ of the tail probabililty $\mathbb{P}(\tau_0 > u\ell^2, \tau_0 > \tau^-_{-\ell})$. Let us decompose this probability as follows

$$\begin{aligned}\mathbb{P}(\tau_0 > z\ell^2, \tau_0 < \tau^-_{-\ell}) &= \mathbb{P}(\tau_0 > \tau^-_{-\ell} > z\ell^2) + \mathbb{P}(\tau_0 > z\ell^2, \tau^-_{-\ell} \le z\ell^2)\\ &= \mathbb{P}(\boldsymbol{\tau}^-_0 > \boldsymbol{\tau}_\ell > z\ell^2) + \mathbb{P}(\boldsymbol{\tau}_0 > z\ell^2, \boldsymbol{\tau}_\ell \le z\ell^2) =: P_1 + P_2.\end{aligned} \tag{67}$$

where $\boldsymbol{\tau}^-_0 := \inf\{k>0,\ \boldsymbol{S}_k < 0\}$ with for any k, $\boldsymbol{S}_k = -S_k$ and similarly $\boldsymbol{\tau}_\ell := \inf\{k>0,\ \boldsymbol{S}_k \ge \ell\}$. For P_2, we just use Donsker's theorem for conditioned random walk to remain positive obtain in [Bol76] which gives $\lim_{\ell\to+\infty}\mathbb{P}(\boldsymbol{\tau}_\ell \le z\ell^2 | \boldsymbol{\tau}_0 > z\ell^2) = \mathbb{P}(\overline{\mathfrak{m}}_1 > 1/\sigma\sqrt{z})$, where $\mathfrak{m}$ is the Brownian meander and $\overline{\mathfrak{m}}_1 = \sup_{s\le 1}\mathfrak{m}_s$. Also we know from Feller [Fel68] (see the first equivalence page 514 of Caravenna [Car05] for the expression we use here) that for any $z>0$:

$$\lim_{\ell\to\infty}\ell\mathbb{P}(\boldsymbol{\tau}_0 > z\ell^2) = \sqrt{\frac{2}{\pi}}\frac{\mathbb{E}(S_{\tau_0})}{\sqrt{z\sigma^2}}, \tag{68}$$

so

$$\lim_{\ell\to\infty}\ell P_2 = \sqrt{\frac{2}{\pi}}\frac{\mathbb{E}(S_{\tau_0})}{\sqrt{z\sigma^2}}\mathbb{P}(\overline{\mathfrak{m}}_1 > 1/\sigma\sqrt{z}). \tag{69}$$

For P_1 we use a similar strategy, for any $A > x$, $\varepsilon > 0$ and ℓ large enough

$$\begin{aligned}P_1 &\le \mathbb{P}(z\ell^2 \le \boldsymbol{\tau}_\ell \le A\ell^2,\ \boldsymbol{\tau}_0 > \boldsymbol{\tau}_\ell) + \mathbb{P}(\boldsymbol{\tau}_0 > A\ell^2)\\ &\le \sum_{k=z\ell^2}^{A\ell^2}\mathbb{P}(\overline{\boldsymbol{S}}_{k-1} \le \ell, \boldsymbol{S}_k > \ell |\ \boldsymbol{\tau}_0 > k)\mathbb{P}(\boldsymbol{\tau}_0 > k) + \mathbb{P}(\boldsymbol{\tau}_0 > A\ell^2)\\ &\le (1+\varepsilon)\sqrt{\frac{2}{\pi}}\frac{\mathbb{E}(S_{\tau_0})}{\ell\sigma}\sum_{k=z\ell^2}^{A\ell^2}\mathbb{P}(\overline{\boldsymbol{S}}_{k-1} \le \ell, \boldsymbol{S}_k > \ell |\ \boldsymbol{\tau}_0 > k)\frac{\ell}{k^{1/2}} + \frac{C}{\ell A^{1/2}},\end{aligned}$$

where we have used (68) for the last inequality and $C>0$ is a constant. Also functional limit theorem [Bol76] implies that $\lim_{\ell\to+\infty}\sum_{k=z\ell^2}^{A\ell^2}\mathbb{P}(\overline{\boldsymbol{S}}_{k-1} \le \ell, \boldsymbol{S}_k > \ell |\ \boldsymbol{\tau}_0 > k)\frac{\ell}{k^{1/2}} = -\int_z^A \frac{1}{y^{1/2}}d\mathbb{P}(\overline{\mathfrak{m}}_1 > 1/\sqrt{y\sigma^2})$. We deduce from that, taking limits $A\to+\infty$ and $\varepsilon\to 0$,

$$\begin{aligned}&\lim_{\ell\to\infty}\ell * P_1\\ &\le -\sqrt{\frac{2}{\pi}}\frac{\mathbb{E}(S_{\tau_0})}{\sigma}\int_z^{+\infty}\frac{1}{y^{1/2}}d\mathbb{P}(\overline{\mathfrak{m}}_1 > 1/\sqrt{y\sigma^2})\\ &= \sqrt{\frac{2}{\pi}}\frac{\mathbb{E}(S_{\tau_0})}{\sigma}\Big(\frac{1}{z^{1/2}}\mathbb{P}(\overline{\mathfrak{m}}_1 > 1/\sqrt{z\sigma^2}) - \frac{1}{2}\int_z^{+\infty}\frac{1}{y^{3/2}}\mathbb{P}(\overline{\mathfrak{m}}_1 > 1/\sqrt{y\sigma^2})dy\Big).\end{aligned}$$

Note that just by noticing that $P_1 \geq \mathbb{P}(z\ell^2 \leq \boldsymbol{\tau}_\ell \leq A\ell^2,\ \boldsymbol{\tau}_0 > \boldsymbol{\tau}_\ell)$, above expression is also a lower bound for $\lim_{m\to\infty} \ell * P_1$. Considering this, (69) and (67), we obtain

$$\lim_{\ell\to\infty} \ell\mathbb{P}(\tau_0 > z\ell^2, \tau_0 > \tau^-_{-\ell}) = \sqrt{\frac{2}{\pi}}\frac{\mathbb{E}(S_{\tau_0})}{\sigma} f(z). \tag{70}$$

where f is the function given in the statement of the Lemma. Note that this convergence is uniform on any compact set in $(0,\infty)$ by monotonicity of $z \to \ell\mathbb{P}(\tau_0 > z\ell^2, \tau_0 < \tau^-_{-\ell})$, continuity of the limit and Dini's theorem. From here we follow the same lines of the proof of a Tauberian theorem (Feller [Fel68]) for completion we recall the main lines for our particular case. For any $\varepsilon > 0$, by the uniform convergence we have talked about just above,

$$\lim_{\ell\to+\infty} \ell \int_\varepsilon^{1/\varepsilon} e^{-\lambda u}\mathbb{P}(\tau_0 > u\ell^2, \tau_0 > \tau^-_{-\ell})du = \sqrt{\frac{2}{\pi}}\frac{\mathbb{E}(S_{\tau_0})}{\sigma}\int_\varepsilon^{1/\varepsilon} e^{-\lambda u} f(u)du.$$

By (68), we also have for any ℓ and $z > 0$, $\mathbb{P}(\tau_0 > z\ell^2, \tau_0 > \tau^-_{-\ell}) \leq \frac{Const}{z^{1/2}\ell}$ and as $\int_0^{+\infty} e^{-\lambda u}u^{-1/2}du < +\infty$, we get $\lim_{\varepsilon\to 0}\lim_{\ell\to+\infty}\int_0^\varepsilon e^{-\lambda u}\ell\mathbb{P}(\tau_0/\ell^2 > u) = 0$.
Similarly $\lim_{\varepsilon\to 0}\lim_{\ell\to+\infty}\int_{1/\varepsilon}^{+\infty} e^{-\lambda u}\ell\mathbb{P}(\tau_0/\ell^2 > u, \tau_0 > \tau^-_{-\ell})du = 0$. Finally

$$\begin{aligned}\lim_{\ell\to+\infty} \ell\int_0^{+\infty}(1-e^{-\lambda u})dH_\ell(u) &= \lim_{\ell\to+\infty}\ell\mathbb{E}\left((1-e^{\frac{-\lambda}{\ell^2}\tau_0})\mathbb{1}_{\tau_0>\tau_{-\ell}}\right)\\ &= \lambda\sqrt{\frac{2}{\pi}}\frac{\mathbb{E}(S_{\tau_0})}{\sigma}\int_0^{+\infty} e^{-\lambda u}f(u)du.\end{aligned} \tag{71}$$

Note also that just by using (68) we also have $\lim_{\ell\to+\infty}\ell\mathbb{E}[1-e^{-\frac{\lambda}{\ell^2}\tau_0}] = \sqrt{2\lambda}\mathbb{E}(S_{\tau_0})\sigma^{-1}$. Then collecting (65), (66) and (71) and taking $\lambda = c\sigma^2/2$ we obtain for ℓ large enough

$$\mathbb{E}\left[e^{-\frac{c\sigma^2}{2\ell^2}\tau_0}\mathbb{1}_{\tau_0\leq\tau^-_{-\ell}}\right] = 1 - \frac{\mathbb{E}(S_{\tau_0})}{\ell}\left(1+\sqrt{c} - \frac{c\sigma}{\sqrt{2\pi}}\int_0^{+\infty} e^{-\frac{c\sigma^2 u}{2}}f(u)du\right) + o\Big(\frac{1}{\ell}\Big). \tag{72}$$

To obtain an explicit expression for the above integral, we integrate by parts

$$\begin{aligned}&\int_0^{+\infty} e^{-\lambda u}f(u)du\\ &= 2\int_0^{+\infty}\frac{e^{-\lambda u}}{u^{1/2}}\mathbb{P}(\overline{\mathfrak{m}}_1 > 1/\sqrt{u\sigma^2})du - \frac{1}{2\lambda}\int_0^{+\infty}\frac{1}{u^{3/2}}(1-e^{-\lambda u})\mathbb{P}(\overline{\mathfrak{m}}_1 > 1/\sqrt{u\sigma^2})du,\end{aligned}$$

then using the expression of $\mathbb{P}(\overline{\mathfrak{m}}_1 > u) := -2\sum_{k=1}(-1)^k\exp(-(ku)^2/2),\ \forall u > 0$, and a little of computations gives :

$$\int_0^{+\infty} e^{-\lambda u}f(u)du = 2\sqrt{\frac{\pi}{\lambda}}\Big(\frac{1}{\sinh(\sqrt{2\lambda}/\sigma)} - \frac{e^{-\sqrt{2\lambda}/\sigma}}{\sinh(\sqrt{2\lambda}/\sigma)}\Big) - \frac{\sigma\sqrt{2\pi}}{\lambda}\Big(\frac{\sqrt{2\lambda}}{\sigma} - \log((e^{\sqrt{2\lambda}/\sigma}+1)/2)\Big). \tag{73}$$

Now we deal with the probability $\mathbb{P}(S_{T_k} \geq r)$ in the same way as [HS16a]. As T_k can be written as a sum of i.i.d random variables with common law given by τ_0, exponential Markov property gives for any $\delta > a > 0$, $\mathbb{P}(S_{T_k} \geq r) \leq e^{-\eta r}(\mathbb{E}(e^{\eta S_{\tau_0}}))^k$. Taking $k = (1-\varepsilon)r/\mathbb{E}(S_{\tau_0})$ we can find constants c' and c" such that $\mathbb{P}(S_{T_k} \geq r) \leq c'e^{-c"r}$ for any $r \geq 1$. So replacing this and (72) in (64), we finally get for any m large enough

$$\begin{aligned}\mathbb{E}\left[e^{-\frac{c\sigma^2}{2\ell^2}\tau_r}\mathbb{1}_{\tau_r\leq\tau^{\bar{S}-S}_\ell}\right] &\leq \left(\mathbb{E}\left[e^{-\frac{c\sigma^2}{2\ell^2}\tau_0}\mathbb{1}_{\tau_0\leq\tau_{-\ell}}\right]\right)^k + \mathbb{P}(S_{T_k}\geq r)\\ &\leq \left(1 - \frac{\mathbb{E}(S_{\tau_0})}{\ell}\Big(1+\sqrt{c}-\frac{c\sigma}{\sqrt{2\pi}}\int_0^{+\infty} e^{-\frac{c\sigma^2}{2}u}f(u)du\Big)\right)^{(1-\varepsilon)r/\mathbb{E}(S_{\tau_0})} + c'e^{-c"r},\end{aligned}$$

which gives the upper bound.
For the *lower bound* the very beginning starts with the same spirit as the proof of Lemma A.2 in [HS16a] : let $r_k = a * k$ for $0 \leq k \leq N := \frac{r}{a}$ and $a > 0$ (chosen later) then

$$\cap_{k=0}^{N} \{\inf\{i > \tau_{r_k}, S_i \geq r_{k+1}\} < \inf\{i > \tau_{r_k}, S_i \leq r_k - \ell\}\} \subset \{\tau_r \leq \tau_\ell^{\bar{S}-S}\},$$

then, strong Markov property gives

$$\begin{aligned}
\mathbb{E}\left[e^{-\frac{c\sigma^2}{2\ell^2}\tau_r}\mathbb{1}_{\tau_r \leq \tau_\ell^{\bar{S}-S}}\right] &\geq \Pi_{k=0}^{N}\mathbb{E}_{r_k}\left(e^{-\frac{c\sigma^2}{2\ell^2}\tau_{r_{k+1}}}\mathbb{1}_{\tau_{r_{k+1}} < \tau^-_{r_k-\ell}}\right) \\
&= \Pi_{k=0}^{N}\mathbb{E}\left(e^{-\frac{c\sigma^2}{2\ell^2}\tau_{r_{k+1}-r_k}}\mathbb{1}_{\tau_{r_{k+1}-r_k} < \tau^-_{-\ell}}\right) \\
&= \left(\mathbb{E}\left(e^{-\frac{c\sigma^2}{2\ell^2}\tau_a}\mathbb{1}_{\tau_a < \tau^-_{-\ell}}\right)\right)^{N+1}.
\end{aligned}$$

So we only need a lower bound for Laplace transform of the form $\mathbb{E}(e^{-h\tau_a}\mathbb{1}_{\tau_a<\tau^-_{-\ell}})$, with $h = h(\ell) \to 0$. From here we follow the same lines as for the upper bound with following differences, τ_0 (resp. $\boldsymbol{\tau}_0^-$) is replaced by τ_a (resp. by $\boldsymbol{\tau}^-_{-a}$), also estimation (68) should be replaced by following one that can be found in [AS14] : there exists $0 < \theta < +\infty$ such that uniformly in $a \in [0, a_\ell]$ with $a_\ell = o(\ell^{1/2})$

for large ℓ, where R is the usual renewal function (see (2.3) in [AS14]) with following property (see (2.6) together with Lemma 2.1 in [AS14])

$$\lim_{a\to\infty} R(a)/a = \frac{1}{\theta}\left(\frac{2}{\pi\sigma^2}\right)^{1/2}. \tag{74}$$

Now considering (67), with the change we have just talked above, as for any $a > 0$, $\lim_{\ell\to+\infty}\mathbb{P}(\boldsymbol{\tau}_\ell \leq z\ell^2 | \boldsymbol{\tau}^-_{-a} > z\ell^2) = \mathbb{P}(\overline{\mathfrak{m}}_1 > 1/\sigma\sqrt{z})$, we obtain

$$\lim_{m\to\infty} \ell P_2 = \lim_{\ell\to\infty} \ell\mathbb{P}(\boldsymbol{\tau}^-_{-a} > z\ell^2, \boldsymbol{\tau}_\ell \leq z\ell^2) = \frac{\theta R(a)}{\sqrt{z}}\mathbb{P}(\overline{\mathfrak{m}}_1 > 1/\sigma\sqrt{z}),$$

similarly for $P_1 = \mathbb{P}(\boldsymbol{\tau}^-_a > \boldsymbol{\tau}_\ell > z\ell^2)$, for ℓ large enough and then taking the limit $A \to +\infty$

$$\begin{aligned}
P_1 &\geq (1-\varepsilon)\frac{\theta R(a)}{\ell}\sum_{k=z\ell^2}^{A\ell^2}\mathbb{P}(\overline{\boldsymbol{S}}_{k-1} \leq \ell, \boldsymbol{S}_k > \ell |\ \boldsymbol{\tau}^-_a > k)\frac{\ell}{k^{1/2}} \\
&\geq (1-2\varepsilon)\frac{\theta R(a)}{\ell}\int_z^{+\infty}\frac{1}{y^{1/2}}d\mathbb{P}(\overline{\mathfrak{m}}_1 > 1/\sqrt{y\sigma^2}).
\end{aligned}$$

We then obtain the equivalent of (70), that is $\lim_{\ell\to\infty}\ell\mathbb{P}(\tau_a > z\ell^2, \tau_a > \tau^-_{-\ell}) = \theta R(a) f(z)$ from which we deduce following lower bound for associated Laplace transform :

$$\lim_{\ell\to+\infty} m\mathbb{E}\left((1-e^{\frac{-\lambda}{\ell^2}\tau_a})\mathbb{1}_{\tau_a > \tau_{-\ell}}\right) = \lambda\theta R(a)\int_0^{+\infty} e^{-\lambda u} f(u)du.$$

in the same spirit $\lim_{\ell\to+\infty}\ell\mathbb{E}[1-e^{-\frac{\lambda}{\ell^2}\tau_{-a}}] = \sqrt{\lambda\pi}\theta R(a)$, also first Lemma 2.2 in [Aïd10] gives for any $a > 0$ and any ℓ large $\mathbb{P}(\tau_{-a} > \tau^-_{-\ell}) = \mathbb{P}_{-a}(\tau_0 > \tau^-_{-\ell-a}) \sim \mathbb{E}(-S_{\tau_{-a}})/\ell$. So finally collecting

these estimates and taking $\lambda = \sigma^2 c/2$, for any $\varepsilon > 0$ and ℓ large enough

$$\mathbb{E}\left[e^{-\frac{c\sigma^2}{2\ell^2}\tau_r}\mathbb{1}_{\tau_r \leq \tau_\ell^{\overline{S}-S}}\right]$$
$$\geq \left(1 - \left(\frac{\mathbb{E}(-S_{\tau_{-a}})}{\ell} + \frac{\theta R(a)}{\ell}\left(\sqrt{\frac{\pi}{2}}\sigma\sqrt{c} - \frac{c\sigma^2}{2}\int_0^{+\infty} e^{-\frac{c\sigma^2}{2}u} f(u)du\right)\right)(1+\varepsilon)\right)^{N+1}.$$

Now recall that $N = r/a$, so let us take a large enough in such a way that (using (74)) $R(a)/a \leq \frac{1}{\theta}\left(\frac{2}{\pi\sigma^2}\right)^{1/2}(1+\varepsilon)$ also for large a, $\mathbb{E}(-S_{\tau_{-a}})/a \leq (1+\varepsilon)$ (this can be seen easily, noticing that undershoot $S_{\tau_{-a}} - a$ has a second moment). This finish the proof. □

Lemma 4.2. *For any $\varepsilon > 0$, $\beta > 0$, any r large enough uniformly in $t = t(r)$ with $\lim_{r\to+\infty} r - t = +\infty$,*

$$\mathbb{E}\left(e^{-\max_{1\leq j\leq \tau_{r-t}} \overline{S}_j - S_j}\right) \leq e^{-2\sqrt{r-t}(1-\varepsilon)}.$$

Proof. Like in the proof of Lemma 4.1 we use strict ladder epoch sequence ($T_k := \inf\{s > T_{k-1}, S_s > S_{T_{k-1}}\}, k; T_0 = 0$), also let us introduce random variable $Y_k := \max_{T_{k-1}\leq j\leq T_k} \overline{S}_j - S_j$ for any $k \geq 1$. Let m a positive integer to be chosen later, by strong Markov property

$$\mathbb{E}\left(e^{-\max_{1\leq k\leq m} Y_k}\right) = \sum_{k=1}^{m} \mathbb{E}(e^{-Y_k}\mathbb{1}_{Y_k > \max_{i\leq k-1} Y_i,\ Y_k \geq \max_{k+1\leq i\leq m} Y_i})$$
$$\leq m\mathbb{E}(e^{-Y_2}(1 - \mathbb{P}(Y_1 > Y_2|Y_2))^{m-1}).$$

At this point we need an asymptotic in y of $M(y) := \mathbb{P}(Y_1 > y) = \mathbb{P}(\max_{0\leq s\leq T_0} S_s < -y) = \mathbb{P}(\tau_0 > \tau_{-y})$, for that we use following equality (see for example [Aïd10] Lemma 2.2) : for large y, $\mathbb{P}(\tau_0 > \tau_{-y}) = \mathbb{E}(S_{\tau_0})/y + o(1/y)$. So for any large A, and $\varepsilon > 0$

$$\begin{aligned}
&e^{-Y_2}(1 - \mathbb{P}(Y_1 > Y_2|Y_2))^{m-1}\\
=&e^{-Y_2}(1 - \mathbb{P}(Y_1 > Y_2|Y_2))^{m-1}\mathbb{1}_{Y_2>A} + e^{-Y_2}(1 - \mathbb{P}(Y_1 > Y_2|Y_2))^{m-1}\mathbb{1}_{Y_2\leq A}\\
\leq&e^{-Y_2}\left(1 - \mathbb{E}(S_{\tau_0})(1-\varepsilon)(Y_2)^{-1}\right)^{m-1}\mathbb{1}_{Y_2>A} + (1 - \mathbb{P}(Y_1 > A))^{m-1},
\end{aligned}$$

For the second term above we can find constant $c = c(A)$ such that $(1 - \mathbb{P}(Y_1 > A))^{m-1} \leq e^{-cm}$. For the first term , let us introduce measure dM defined as $M(x) = \int_x^{+\infty} dM(z)dz$, then integrating by parts

$$\mathbb{E}(e^{-Y_2}\left(1 - \mathbb{E}(S_{\tau_0})(1-\varepsilon)(Y_2)^{-1}\right)^{m-1}\mathbb{1}_{Y_2>A}) = -\int_A^{+\infty} e^{-x}\left(1 - \frac{\mathbb{E}(S_{\tau_0})(1-\varepsilon)}{x}\right)^{m-1} dR(x)$$
$$\leq e^{-A}\left(1 - \frac{\mathbb{E}(S_{\tau_0})(1-\varepsilon)}{A}\right)^{m-1} - \int_A^{+\infty} e^{-x}\left(1 - \frac{\mathbb{E}(S_{\tau_0})(1-\varepsilon)}{x}\right)^{m-1} R(x)dx$$
$$- (m-1)S_{\tau_0}(1-\varepsilon)\int_A^{+\infty}\frac{e^{-x}}{x^2}\left(1 - \frac{\mathbb{E}(S_{\tau_0})(1-\varepsilon)}{x}\right)^{m-2} R(x)dx$$
$$\leq e^{-2(1-4\varepsilon)\sqrt{\mathbb{E}(S_{\tau_0})m}},$$

last inequality is definitely not optimal but enough for what we need, we can obtain it easily decomposing the interval $(A, +\infty)$ on the intervals $(A, \sqrt{\mathbb{E}(S_{\tau_0})m}(1-\varepsilon))$, $(\sqrt{\mathbb{E}(S_{\tau_0})m}(1-\varepsilon), \sqrt{\mathbb{E}(S_{\tau_0})m}(1+\varepsilon))$ and $(\sqrt{\mathbb{E}(S_{\tau_0})m}(1+\varepsilon), +\infty)$. Collecting the above inequalities, we obtain that for any $\varepsilon > 0$ and m large enough

$$\mathbb{E}\left(e^{-\max_{1\leq k\leq m} Y_k}\right) \leq 2m^{-2(1-4\varepsilon)\sqrt{\mathbb{E}(S_{\tau_0})m}}.$$

To finish the proof we follow the same lines as the end of the proof of Lemma 4.1 (below (73)), that is saying that $\mathbb{E}\left(e^{-\max_{1\le j\le \tau_{r-t}} \overline{S}_j - S_j}\right) \le \mathbb{E}\left(e^{-\max_{1\le k\le m} Y_k}\right) + \mathbb{P}(S_{T_k} \ge r-t)$ then taking $k = (1-\varepsilon)(r-t)/\mathbb{E}(S_{\tau_0})$. □

4.2. Additional technical estimates

Following Lemma and more especially inequality (76) below is used when we ask for the behavior of heavy range together with high potential.

Lemma 4.3. *Let (t_ℓ) a positive increasing sequence such that $t_\ell \ell^{-1/2} \to +\infty$ but $t_\ell \ell^{-1} \to 0$. For any $B > 0$ and ℓ large enough*

$$\mathbb{P}(\tau^{\overline{S}-S}_{\ell^{1/2}} \vee \tau^-_{-B} > \tau_{t_\ell}) \ge e^{-\frac{t_\ell}{\sqrt{\ell}}(1+o(1))}. \tag{75}$$

Let $A > 0$ large, $d \in (0,1/2)$, $a > 0$, $0 < b < 1$, $q \in [b,1]$, $a_b := a(2\mathbb{1}_{q>b} - 1)$ and $c > 0$

$$\sum_{j \le A\ell^{3/2}} \mathbb{P}\big(S_j \ge t_\ell, \sup_{m\le j} H^S_m \le e^{q\sqrt{\ell} - a_b \ell^d}, e^{b\sqrt{\ell}} \le H^S_j \le e^{b\sqrt{\ell}+c\ell^d}, \underline{S}_j \ge -B\big) \ge e^{-\frac{t_\ell}{q\sqrt{\ell}}(1+o(1))}. \tag{76}$$

Proof. The proof of (75) follows the same lines as the proof of Lemma A.2 in [HS16a]. For (76), as $j \le A\ell^{3/2}$, for any (d,e) and any $m \le j$, $A\ell^{3/2}\exp(\overline{S}_m - S_m) \le e^{d\sqrt{\ell}+e\ell^d}$ implies $H^S_m \le e^{d\sqrt{\ell}+e\ell^d}$ then

$$\begin{aligned}
&\mathbb{P}\left[S_j \ge t_\ell, e^{b\sqrt{\ell}} \le H^S_j \le e^{b\sqrt{\ell}+c\ell^d}, \sup_{m\le j} H^S_m \le e^{q*\sqrt{\ell} - a_b\ell^d}, \underline{S}_j \ge -B\right] \\
&\ge \mathbb{P}\left[S_j \ge t_\ell, b\sqrt{\ell} \le \overline{S}_j - S_j \le b\sqrt{\ell} + c'\ell^d, \sup_{m\le j} \overline{S}_m - S_m \le q\sqrt{\ell} - a'\ell^d, \underline{S}_j \ge -B\right]
\end{aligned}$$

with $c' = c/2$ and $a' = a_b + 1$. To obtain a lower bound for the above probability, the idea is to say that maximum of S is obtained at a certain instant $k \le j$ and that this maximum is larger than $t_\ell + b\sqrt{\ell} + c'\ell^d + r$ for a certain $r > 0$ to be chosen latter, then above probability is larger than :

$$\begin{aligned}
\sum_{k\le j} \mathbb{P}(&\overline{S}_{k-1} < S_k, S_k \ge t_\ell + b\sqrt{\ell} + c'\ell^d + r, \sup_{m\le k} \overline{S}_m - S_m \le \sqrt{\ell} - a'\ell^d, \underline{S}_k \ge -B; S_j - S_k \ge t_\ell - S_k, \\
&b\sqrt{\ell} \le S_k - S_j \le b\sqrt{\ell} + c'\ell^d, \forall m \ge k+1, S_m \le S_k, S_k - S_m \le \sqrt{\ell} - a'\ell^d, S_m - S_k \ge -B - S_k).
\end{aligned}$$

Now, the events $\{S_m - S_k \ge -B - x\}$, as well as $\{S_j - S_k \ge t_\ell - x\}$ increases in x and as $S_k \ge t_\ell + b\sqrt{\ell} + c'\ell^d + r$ so we can replace, in the two events of the above probability, "$-S_k$" by $-(t_\ell + b\sqrt{\ell} + c'\ell^d + r)$. This makes appear two independent events, so above probability is larger than

$$\begin{aligned}
&\mathbb{P}(\overline{S}_{k-1} < S_k, S_k \ge t_\ell + b\sqrt{\ell} + c'\ell^d + r, \sup_{m\le k} \overline{S}_m - S_m \le \sqrt{\ell} - a'\ell^d, \underline{S}_k \ge -B) \times \\
&\mathbb{P}(S_j - S_k \ge -b\sqrt{\ell} - c'\ell^d - r, b\sqrt{\ell} \le S_k - S_j \le b\sqrt{\ell} + c'\ell^d, \forall m \ge k+1, \\
&-B - t_\ell - b\sqrt{\ell} + c'\ell^d - r \le S_m - S_k \le 0, S_m - S_k \ge -\sqrt{\ell} + a'\ell^d) =: p_1(k) * p_2(k,j).
\end{aligned} \tag{77}$$

Probability p_2 can be easily simplified, indeed as $\lim_{\ell\to+\infty} t_\ell/\sqrt{\ell} = +\infty$ and ℓ large, $-B - t_\ell -$

$b\sqrt{\ell}+c'\ell^d-r\leq-\sqrt{\ell}$ and by taking $r=c'\ell^d$, p_2 is smaller than

$$\begin{aligned}&\mathbb{P}(-b\sqrt{\ell}-c'\ell^d\leq S_j-S_k\leq-b\sqrt{\ell},\forall m\geq k+1,-\sqrt{\ell}+a'\ell^d\leq S_m-S_k\leq 0)\\ =&\mathbb{P}(\forall m\leq j-k,-\sqrt{\ell}+a'\ell^d\leq S_m\leq 0,-b\sqrt{\ell}-c'\ell^d\leq S_{j-k}\leq-b\sqrt{\ell})\\ =&\mathbb{P}(\forall m\leq j-k,\ \boldsymbol{S}_m\leq\sqrt{\ell}-a'\ell^d|\underline{\boldsymbol{S}}_{j-k}\geq 0,\boldsymbol{S}_{j-k}\in[b\sqrt{\ell},b\sqrt{\ell}+c'\ell^d])\times\\ &\mathbb{P}(\underline{\boldsymbol{S}}_{j-k}\geq 0,\boldsymbol{S}_{j-k}\in[b\sqrt{\ell},b\sqrt{\ell}+c'\ell^d]),\end{aligned}$$

with $\boldsymbol{S}_m=-S_m$ for any m. For the conditional probability we can use a similar result proved by Caravenna and Chaumont [CC13] telling that the distribution $\mathbb{P}_x(\cdot|\forall m\leq n,S_m\geq 0,S_n\in[0,h))$ converges. Note that they need in their workr additional hypothesis on the distribution of S_1 (more especially absolute continuity of the distribution of S_1) which is not necessary here as the size of intervall $[b\sqrt{\ell},b\sqrt{\ell}+c'\ell^d]$ equals $c'\ell^d\to+\infty$, in particular as $a'\ell^d=o(\sqrt{\ell})$

$$\lim_{\ell\to+\infty}\mathbb{P}(\forall m\leq\ell,\boldsymbol{S}_m\leq c\sqrt{\ell}-a'\ell^d|\underline{\boldsymbol{S}}_\ell\geq 0,\boldsymbol{S}_\ell\in[b\sqrt{\ell},b\sqrt{\ell}+c'\ell^d])=Cte>0.$$

Moreover another work of Caravenna ([Car05] Theorem 1) gives for large ℓ, $\mathbb{P}(\underline{\boldsymbol{S}}_\ell\geq 0,\boldsymbol{S}_\ell\in[b\sqrt{\ell},b\sqrt{\ell}+c'\ell^d)\geq b/\ell$. So finally when $j-k$ is of the order of ℓ, there exists a constante $Cte>0$ such that $p_2(k,j)\geq Cte*\ell^{-1}$. Turning back to (77) and summing over k and j, we obtain

$$\begin{aligned}&\sum_{j\leq A\ell^{3/2}}\sum_{k\leq j}p_1(k)p_2(k,j)\\ &=\sum_{k\leq A\ell^{3/2}}p_1(k)\sum_{j\geq k}p_2(k,j)\geq\sum_k p_1(k)\sum_{j,j-k\sim\ell}p_2(k,j)\\ &\geq\frac{Cte}{\ell}\sum_{k\leq A\ell^{3/2}}\mathbb{P}(\overline{S}_{k-1}<S_k,S_k\geq t_\ell+b(1+2\varepsilon_\ell)\sqrt{\ell},\sup_{m\leq k}\overline{S}_m-S_m\leq\sqrt{\ell},\underline{S}_k\geq -B)\\ &\geq\frac{Cte}{\ell}\Big(\mathbb{P}(\tau^{\overline{S}-S}_{\sqrt{\ell}}\vee\tau_{-B}>\tau_{t_\ell+b\sqrt{\ell}+c'\ell^d})-\sum_{k>A\ell^{3/2}}\mathbb{P}(\sup_{m\leq k}\overline{S}_m-S_m\leq\sqrt{\ell})\Big)\end{aligned}$$

Now we can check that above sum $\sum_{k>A\ell^{3/2}}\cdots$ as a negligible contribution, indeed the probability $\mathbb{P}(\sup_{m\leq k}\overline{S}_m-S_m\leq\sqrt{\ell})$ is smaller, thanks to Proposition 3.1 in [FHS11], to $e^{-\pi^2\sigma^2 j/4\ell}$ this implies that $\sum_{k>A\ell^{3/2}}\mathbb{P}(\sup_{m\leq k}\overline{S}_m-S_m\leq\sqrt{\ell})\leq e^{-\pi^2\sigma^2A\ell^{1/2}/2}$. Now if we apply (75) to the first probability above as $b\sqrt{\ell}+c'\ell^d=o(t_\ell)$, this finishes the proof. □

Lemma below is a simple extension of FKG inequality.
In the following, a function $F:\mathbb{R}^k\longrightarrow\mathbb{R}$ is said to be increasing if: for all $\mathbf{s}=(s_1,\dots,s_k)\in\mathbb{R}^k$ and $\mathbf{t}=(t_1,\dots,t_k)\in\mathbb{R}^k$, $\mathbf{s}\leq_k\mathbf{t}$ implies $F(\mathbf{s})\leq F(\mathbf{t})$ where $\mathbf{s}\leq_k\mathbf{t}$ if and only if $s_j\leq t_j$ for all $j\in\{1,\dots,k\}$.

Lemma 4.4. *Let $r>0$, $k\in\mathbb{N}^*$, $f_1,f_2:\mathbb{R}^k\longrightarrow\mathbb{R}^+$. For any $i\in\{1,2\}$, introduce $\tilde{f}_i(u_1,\dots,u_k):=f_i(u_1,u_1+u_2,\dots,u_1+u_2+\ldots+u_k)$. If $\tilde{f}_1$ and $\tilde{f}_2$ are increasing then*

$$\mathbb{E}\big[f_1(S_1,S_2,\dots,S_k)f_2(S_1,S_2,\dots,S_k)\big]\geq\mathbb{E}\big[f_1(S_1,S_2,\dots,S_k)\big]\mathbb{E}\big[f_2(S_1,S_2,\dots,S_k)\big].$$

Proof. When $\mathbb{R}^k$ is a totally order set, the first inequality above is the well known regular FKG inequality. Here, we can easly extend it to the partial order $\leq_k$. Indeed, since $\tilde{f}_i$ is increasing for any $i\in\{1,2\}$, we have, by independence of increments of S

$$\prod_{i\in\{1,2\}}\mathbb{E}\big[f_i(S_1,S_2,\dots,S_k)\big]=\prod_{i\in\{1,2\}}\mathbb{E}\big[\tilde{f}_i(S_1,S_2-S_1,\dots,S_k-S_{k-1})\big]=\mathbb{E}[\mathbf{F}_1(S_1)]\mathbb{E}[\mathbf{F}_2(S_1)],$$

with $\mathbf{F}_i(u_1) := \mathbb{E}\big[\tilde{f}_i(u_1, S_2 - S_1, \dots, S_k - S_{k-1})\big]$ for any $i \in \{1,2\}$. Since $\tilde{f}_i$ is increasing, $\mathbf{F}_i$ is also increasing so thanks to the regular FKG inequality, $\mathbb{E}[\mathbf{F}_1(S_1)]\mathbb{E}[\mathbf{F}_2(S_1)] \le \mathbb{E}[\mathbf{F}_1\mathbf{F}_2(S_1)]$. Again, using independence and stationarity of increments of S and the result follows by induction. □

Lemma 4.5. *Let (t_ℓ) a sequence of positive numbers such that $t_\ell/\ell \to 0$. For all $d \in (0, 1/2]$ such that $t_\ell/\ell^d \to +\infty$ and all $\varepsilon, B > 0$, $a \ge 0$ and $0 \le d' < d \le 1/2$ for n large enough*

$$\sum_{k \le \ell^2} \mathbb{P}\big(S_k \ge t_\ell, \max_{j \le k} H_j^S \le e^{\ell^d - a\ell^{d'}}, \underline{S}_k \ge -B, \overline{S}_k = S_k\big) \ge e^{-\frac{t_\ell}{\ell^d}(1+\varepsilon)}.$$

Proof. Recall that $\tau_r = \inf\{i \ge 1;\ S_i \ge r\}$. First, observe that for all $j \le k \le \ell^2$, $H_j^S \le \ell^2 e^{\overline{S}_j - S_j}$ so

$$\begin{aligned}
&\sum_{k \le \ell^2} \mathbb{P}\big(S_k \ge t_\ell, \max_{j \le k} H_j^S \le e^{\ell^d - a\ell^{d'}}, \underline{S}_k \ge -B, \overline{S}_k = S_k\big) \\
&\ge \sum_{k \le \ell^2} \mathbb{P}\big(k = \tau_{t_\ell}, \max_{j \le k} \overline{S}_j - S_j \le \ell^d - a\ell^{d'} - 2\log \ell, \underline{S}_k \ge -B\big),
\end{aligned}$$

which is equal to $\mathbb{P}\big(\overline{S}_{\ell^2} \ge t_\ell, \forall j \le \tau_{t_\ell} : \overline{S}_j - S_j \le \ell^d - a\ell^{d'} - 2\log \ell, S_j \ge -B\big)$.
Now let $k_\ell = \lfloor (e^\ell t_\ell)^2 \rfloor + \ell^2$. First note that, since $\ell^2 \le k_\ell$, we have, on $\{\overline{S}_{\ell^2} \ge t_\ell\}$, $\tau_{t_\ell} = \tau_{t_\ell}^{k_\ell}$ with $\tau_{t_\ell}^{k_\ell} := k_\ell \wedge \inf\{i \le k_\ell;\ S_i \ge t_\ell\}$ so

$$\begin{aligned}
&\mathbb{P}\big(\overline{S}_{\ell^2} \ge t_\ell, \forall j \le \tau_{t_\ell} : \overline{S}_j - S_j \le \ell^d - a\ell^{d'} - 2\log \ell, S_j \ge -B\big) \\
&= \mathbb{P}\big(\overline{S}_{\ell^2} \ge t_\ell, \forall j \le \tau_{t_\ell}^{k_\ell} : \overline{S}_j - S_j \le \ell^d - a\ell^{d'} - 2\log \ell, S_j \ge -B\big).
\end{aligned}$$

For any $k \in \mathbb{N}^*$ and $r > 0$, let $\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k$ and define the $\mathbf{t}$-version $\tau_r^{k,\mathbf{t}}$ of τ_r^k that is

$$\tau_r^{k,\mathbf{t}} := k \wedge \inf\big\{i \le k;\ t_i \ge r\big\},$$

with the usual convention $\inf \varnothing = +\infty$. Then

$$\mathbb{P}\big(\overline{S}_{\ell^2} \ge t_n, \forall j \le \tau_{t_\ell}^{k_\ell} : \overline{S}_j - S_j \le \ell^d - a\ell^{d'} - 2\log \ell, S_j \ge -B\big) = \mathbb{E}\big[f_1 f_2(S_1, S_2, \dots, S_{k_\ell})\big],$$

with for all $i \in \{1,2\}$, $f_i := \mathbb{1}_{A_i^\ell}$, $f_1 f_2(\mathbf{u}) = f_1(\mathbf{u}) f_2(\mathbf{u})$ and

$$A_1^\ell := \big\{\mathbf{u} = (u_1, \dots, u_{k_\ell}) \in \mathbb{R}^{k_\ell}; \exists\, j \le \ell^2 : u_j \ge t_\ell\big\},$$

and

$$A_2^\ell := \big\{\mathbf{u} = (u_1, \dots, u_{k_\ell}) \in \mathbb{R}^{k_\ell}; \forall\, j \le \tau_{t_\ell}^{k_\ell,\mathbf{t}}, \forall i < j : u_j - u_i \ge -\ell^d + a\ell^{d'} + 2\log \ell,\ u_j \ge -B\big\}.$$

Then, it's easy to see that for all $i \in \{1,2\}$, $\tilde{f}_i$ (see Lemma 4.4 for the definition) is increasing according to the partial order $\le_{k_\ell}$ defined above on $\mathbb{R}^{k_\ell}$ and thanks to Lemma 4.4, $\mathbb{E}[f_1 f_2(S_1, S_2, \dots, S_{k_\ell})]$ is larger than

$$\begin{aligned}
&\ge \mathbb{P}\big((S_1, S_2, \dots, S_{k_n}) \in A_1^\ell\big)\mathbb{P}\big((S_1, S_2, \dots, S_{k_\ell}) \in A_2^\ell\big) \\
&\ge \mathbb{P}(\overline{S}_{\ell^2} \ge t_\ell)\mathbb{P}\big(\forall j \le \tau_{t_\ell}^{k_\ell} : \overline{S}_j - S_j \le \ell^d - a\ell^{d'} - 2\log \ell, S_j \ge -B, \tau_{t_\ell} \le k_\ell\big)
\end{aligned}$$

Again, on $\{\tau_{t_\ell} \le k_\ell\}$, $\tau_{t_\ell}^{k_\ell} = \tau_{t_\ell}$ and thanks to [Koz76] (Theorem A), there exists $C_K > 0$ such that for ℓ large enough

$$\begin{aligned}
&\mathbb{P}\big(\forall j \le \tau_{t_\ell}^{k_\ell} : \overline{S}_j - S_j \le \ell^d - a\ell^{d'} - 2\log \ell, S_j \ge -B, \tau_{t_\ell} \le k_\ell\big) \\
&\ge \mathbb{P}\big(\forall j \le \tau_{t_\ell} : \overline{S}_j - S_j \le \ell^d - a\ell^{d'} - 2\log \ell, S_j \ge -B\big) - \mathbb{P}(\tau_{t_\ell} > k_\ell) \\
&\ge \mathbb{P}\big(\forall j \le \tau_{t_\ell} : \overline{S}_j - S_j \le \ell^d - a\ell^{d'} - 2\log \ell, S_j \ge -B\big) - C_K e^{-\ell}
\end{aligned}$$

Moreover, $t_\ell/\ell \to 0$ so $\mathbb{P}(\overline{S}_{\ell^2} \geq t_\ell) \to 1$. Finally, by (75) together with the fact that $\ell^d \sim \ell^d - a\ell^{d'} - 2\log\ell$ (as $d > d'$) for ℓ large enough $\mathbb{P}\big(\forall j \leq \tau_{t_\ell} : \overline{S}_j - S_j \leq \ell^d - a\ell^{d'} - 2\log\ell, S_j \geq -B\big) \geq 2e^{-t_\ell \ell^{-d}(1+\varepsilon)}$ and since $t_{\ell_n}/\ell^d = o(\ell)$, $C_K e^{-\ell} \leq e^{-t_\ell \ell^{-d}(1+\varepsilon)}$ so we obtain the result. □

Lemma 4.6. *Let $\alpha \in (1,2)$ and $\varepsilon_\alpha \in [0, \alpha - 1)$ and intruduce $\boldsymbol{L}_\ell := \lfloor \chi \ell^{1+\frac{\varepsilon_\alpha}{2}} \rfloor$, $\chi > 0$. For all $\varepsilon > 0$, ℓ large enough and any $k \in \{\boldsymbol{L}_\ell, \dots, \ell^2\}$*

$$\mathbb{P}\big(\max_{j \leq k} H_j^S \leq e^{\sqrt{\ell}}\big) \leq e^{-\frac{k\pi^2\sigma^2}{8\ell}(1-\varepsilon)}, \tag{78}$$

and for any $a, d, c > 0$, $b \in (0,1)$, ℓ large enough and any $k \in \{\boldsymbol{L}_\ell, \dots, \ell^2\}$

$$\mathbb{P}\big(\max_{j \leq k} H_j^S \leq e^{\sqrt{\ell} - a\ell^d}, e^{b\sqrt{\ell}} < H_k^S \leq e^{b\sqrt{\ell} + c\ell^d}, \underline{S}_k \geq 0\big) \geq e^{-\frac{k\pi^2\sigma^2}{8\ell}(1+\varepsilon)}. \tag{79}$$

Proof. Let us start with the upper bound. Thanks to Markov property, for any $k \in \mathbb{N}$, $k > \boldsymbol{L}_\ell$

$$\mathbb{P}\big(\max_{j \leq k} H_j^S \leq e^{\sqrt{\ell}}\big) \leq \mathbb{P}\big(\max_{j \leq k} \overline{S}_j - S_j \leq \sqrt{\ell}\big) \leq \mathbb{P}\big(\max_{j \leq \boldsymbol{L}_\ell} \overline{S}_j - S_j \leq \sqrt{\ell}\big)^{\lfloor \frac{k}{\boldsymbol{L}_\ell} \rfloor},$$

and thanks to [FHS11], for ℓ large enough $\mathbb{P}\big(\max_{j \leq \boldsymbol{L}_\ell} \overline{S}_j - S_j \leq \sqrt{\ell}\big) \leq e^{-\frac{\pi^2\sigma^2 \boldsymbol{L}_\ell}{8\ell}(1-\frac{\varepsilon}{2})}$, so for any ε, ℓ large enough and any $k > \boldsymbol{L}_\ell$

$$\mathbb{P}\big(\max_{j \leq k} \overline{S}_j - S_j \leq \sqrt{\ell}\big) \leq e^{-(1-\frac{\varepsilon}{2})\frac{\pi^2\sigma^2\boldsymbol{L}_\ell}{8\ell}\lfloor \frac{k}{\boldsymbol{L}_\ell} \rfloor} \leq e^{-(1-\varepsilon)\frac{k\pi^2\sigma^2}{8\ell}}.$$

For the lower bound, observe that for any $k \leq \ell^2$, $\mathbb{P}\big(\max_{j \leq k} H_j^S \leq e^{\sqrt{\ell} - a\ell^d}, e^{b\sqrt{\ell}} < H_k^S \leq e^{b\sqrt{\ell} + c\ell^d}, \underline{S}_k \geq 0\big)$ is larger than $\mathbb{P}\big(\max_{j \leq k} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell, b\sqrt{\ell} < \overline{S}_k - S_k \leq b\sqrt{\ell} + c\ell^d - \log\ell^2, \underline{S}_k \geq 0\big)$, where $\boldsymbol{\lambda}'_\ell := \sqrt{\ell} - a\ell^d - \log\ell^2$. As $\frac{c}{2}\ell^d \geq \log\ell^2$ ($d > 0$), the previous probability is larger than $\mathbb{P}\big(\max_{j \leq k} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell, b\sqrt{\ell} < \overline{S}_k - S_k \leq b\sqrt{\ell} + \frac{c}{2}\ell^d, \underline{S}_k \geq 0\big)$. We need independence to compute this probability so for all $k \in \mathbb{N}^*$, $\boldsymbol{L}_\ell < k \leq \ell^2$, we say that $\overline{S}_k = S_{k-\ell} \geq \boldsymbol{\lambda}'_\ell$ which gives that for all $k - \ell < j \leq k$, $\overline{S}_j \leq S_{k-\ell}$ and then, $\max_{k-\ell<j\leq k} S_{k-\ell} - S_j \leq \boldsymbol{\lambda}'_\ell$ implies that $S_j \geq S_{k-\ell} - \boldsymbol{\lambda}'_n \geq 0$ for all $k - \ell < j \leq k$. Hence

$$\mathbb{P}\big(\max_{j \leq k} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell, b\sqrt{\ell} < \overline{S}_k - S_k \leq b\sqrt{\ell} + \frac{c}{2}\ell^d, \underline{S}_k \geq 0\big) \geq \mathbb{P}(A_{k,\ell} \cap B_{k,\ell}) = \mathbb{P}(A_{k,\ell})\mathbb{P}(B_{k,\ell}),$$

with

$$A_{k,\ell} := \big\{ \max_{j \leq k-\ell} \overline{S}_j - S_j \leq \boldsymbol{\lambda}'_\ell, \underline{S}_{k-\ell} \geq 0, S_{k-\ell} = \overline{S}_{k-\ell} \geq \boldsymbol{\lambda}'_\ell \big\},$$

and

$$B_{k,\ell} := \big\{ \forall\, k - \ell < j \leq k, S_{k-\ell} - S_j \leq \boldsymbol{\lambda}'_\ell, S_j \leq S_{k-\ell}, b\sqrt{\ell} < S_{k-\ell} - S_k \leq b\sqrt{\ell} + \frac{c}{2}\ell^d \big\}.$$

Let $\boldsymbol{S} := -S$. $\mathbb{P}(B_{k,\ell})$ is nothing but

$$\begin{aligned}\mathbb{P}\big(\overline{\boldsymbol{S}}_\ell \leq \boldsymbol{\lambda}'_\ell, \underline{\boldsymbol{S}}_\ell \geq 0, \boldsymbol{S}_\ell \in (b\sqrt{\ell}, b\sqrt{\ell} + \frac{c}{2}\ell^d]\big) =& \mathbb{P}(\underline{\boldsymbol{S}}_\ell > 0)\mathbb{P}\big(\boldsymbol{S}_\ell \in (b\sqrt{\ell}, b\sqrt{\ell} + \frac{c}{2}\ell^d] | \underline{\boldsymbol{S}}_\ell \geq 0\big) \\ &\times \mathbb{P}\big(\overline{\boldsymbol{S}}_\ell \leq \boldsymbol{\lambda}'_\ell | \underline{\boldsymbol{S}}_\ell > 0, \boldsymbol{S}_\ell \in (b\sqrt{\ell}, b\sqrt{\ell} + \frac{c}{2}\ell^d]\big),\end{aligned}$$

which is larger than C/ℓ for ℓ large enough (see Lemma 4.3).
We then deal with $\mathbb{P}(A_{k,\ell})$. Thanks to Lemma 4.4, this probability is larger than

$$\mathbb{P}\big(\max_{j\le k-\ell}\overline{S}_j - S_j \le \boldsymbol{\lambda}'_\ell\big)\mathbb{P}\big(\overline{S}_{k-\ell}\ge \boldsymbol{\lambda}'_\ell\big)\mathbb{P}\big(\underline{S}_{k-\ell}\ge 0\big)^2,$$

and again, using [Koz76] together with the fact that $\mathbb{P}(\overline{S}_{\boldsymbol{L}_\ell}\ge \boldsymbol{\lambda}'_\ell)\to 1$, there exists $C>0$ such that for ℓ large enough and any $k\in\{\boldsymbol{L}_\ell,\dots,\ell^2\}$,

$$\mathbb{P}\big(\overline{S}_{k-\ell}\ge\sqrt{\ell}\big)\mathbb{P}\big(\underline{S}_{k-\ell}\ge 0\big)\ge \mathbb{P}\big(\overline{S}_{\boldsymbol{L}_\ell}\ge\sqrt{\ell}\big)\mathbb{P}\big(\underline{S}_{\ell^2}\ge 0\big)^2\ge\frac{C}{\ell^2}.$$

We now turn to the most important part: $\mathbb{P}\big(\max_{j\le k-\ell}\overline{S}_j - S_j\le\boldsymbol{\lambda}'_\ell\big)$. We follow the same lines as the proof of (78): for any $k\in\{\boldsymbol{L}_\ell,\dots,\ell^2\}$, $k-\ell>\boldsymbol{L}_\ell-\ell$ so $\max_{j\le\boldsymbol{L}_\ell-\ell}\overline{S}_j-S_j\le\boldsymbol{\lambda}'_\ell$ together with $\overline{S}_{\boldsymbol{L}_\ell-\ell}=S_{\boldsymbol{L}_\ell-\ell}\le S_j$ and $\max_{\boldsymbol{L}_\ell-\ell<i\le j}S_i-S_j\le\boldsymbol{\lambda}'_\ell$ for all $\boldsymbol{L}_\ell-\ell<j\le k-\ell$ implies that $\max_{j\le k-\ell}\overline{S}_j-S_j\le\boldsymbol{\lambda}'_\ell$. It follows that $\mathbb{P}\big(\max_{j\le k-\ell}\overline{S}_j-S_j\le\boldsymbol{\lambda}'_\ell\big)$ is larger than

$$\begin{aligned}&\mathbb{P}\big(\max_{j\le\boldsymbol{L}_\ell-\ell}\overline{S}_j-S_j\le\boldsymbol{\lambda}'_\ell,\overline{S}_{\boldsymbol{L}_\ell-\ell}=S_{\boldsymbol{L}_\ell-\ell},\max_{\boldsymbol{L}_\ell-\ell<i\le j}S_i-S_j\le\boldsymbol{\lambda}'_\ell,S_j\ge S_{\boldsymbol{L}_\ell-\ell}\ \forall\ \boldsymbol{L}_\ell-\ell<j\le k-\ell\big)\\&=\mathbb{P}\big(\max_{j\le\boldsymbol{L}_\ell-\ell}\overline{S}_j-S_j\le\boldsymbol{\lambda}'_\ell,\overline{S}_{\boldsymbol{L}_\ell-\ell}=S_{\boldsymbol{L}_\ell-\ell}\big)\mathbb{P}\big(\max_{j\le k-\ell-(\boldsymbol{L}_\ell-\ell)}\overline{S}_j-S_j\le\boldsymbol{\lambda}'_\ell,\underline{S}_{k-\ell-(\boldsymbol{L}_\ell-\ell)}\ge 0\big).\end{aligned}$$

Moreover, by Lemma 4.4, $\mathbb{P}\big(\max_{j\le k-\ell-(\boldsymbol{L}_\ell-\ell)}\overline{S}_j-S_j\le\boldsymbol{\lambda}'_\ell,\underline{S}_{k-\ell-(\boldsymbol{L}_\ell-\ell)}\ge 0\big)$ is larger than $\mathbb{P}(\max_{j\le k-\ell-(\boldsymbol{L}_\ell-\ell)}\overline{S}_j-S_j\le\boldsymbol{\lambda}'_\ell)\mathbb{P}(\underline{S}_{k-\ell-(\boldsymbol{L}_\ell-\ell)}\ge 0)$. By induction, we get that $\mathbb{P}\big(\max_{j\le k-\ell}\overline{S}_j-S_j\le\boldsymbol{\lambda}'_\ell\big)$ is larger than

$$\mathbb{P}\big(\max_{j\le\boldsymbol{L}_\ell-\ell}\overline{S}_j-S_j\le\boldsymbol{\lambda}'_\ell,\overline{S}_{\boldsymbol{L}_\ell-\ell}=S_{\boldsymbol{L}_\ell-\ell}\big)^{\boldsymbol{L}_\ell(k)}\prod_{i\le\boldsymbol{L}_\ell(k)}\mathbb{P}\big(\underline{S}_{k-\ell-i(\boldsymbol{L}_\ell-\ell)}\ge 0\big)$$

with $\boldsymbol{L}_\ell(k):=\lfloor(k-\ell)/(\boldsymbol{L}_\ell-\ell)\rfloor$. Again, by Lemma 4.4, $\mathbb{P}(\max_{j\le\boldsymbol{L}_\ell-\ell}\overline{S}_j-S_j\le\boldsymbol{\lambda}'_\ell,\overline{S}_{\boldsymbol{L}_\ell-\ell}=S_{\boldsymbol{L}_\ell-\ell}\big)\ge\mathbb{P}\big(\max_{j\le\boldsymbol{L}_\ell-\ell}\overline{S}_j-S_j\le\boldsymbol{\lambda}'_\ell\big)\mathbf{P}(\underline{S}_{\boldsymbol{L}_\ell-\ell}\ge 0)$ and as $k\le\ell^2$, $\mathbb{P}(\underline{S}_{k-\ell-i(\boldsymbol{L}_\ell-\ell)}\ge 0)\ge\mathbb{P}\big(\underline{S}_k\ge 0\big)\ge\mathbb{P}\big(\underline{S}_{\ell^2}\ge 0\big)$. Hence, by [Koz76]

$$\mathbb{P}\big(\max_{j\le k-\ell}\overline{S}_j-S_j\le\boldsymbol{\lambda}'_\ell\big)\ge\Big(\frac{C}{\ell\sqrt{\boldsymbol{L}_\ell-\ell}}\mathbb{P}\big(\max_{j\le\boldsymbol{L}_\ell-\ell}\overline{S}_j-S_j\le\boldsymbol{\lambda}'_\ell\big)\Big)^{\boldsymbol{L}_\ell(k)}$$

for some $C>0$. Then, thanks to [FHS11], for all $\varepsilon>0$ and ℓ large enough $\mathbb{P}(\max_{j\le\boldsymbol{L}_\ell-\ell}\overline{S}_j-S_j\le\boldsymbol{\lambda}'_\ell)\ge e^{-(1+\frac{\varepsilon}{4})\frac{\pi^2\sigma^2(\boldsymbol{L}_\ell-\ell)}{8}(\boldsymbol{\lambda}'_\ell)^{-2}}$ so for ℓ large enough and any $k\in\{\boldsymbol{L}_\ell,\dots,\ell^2\}$, $\mathbb{P}\big(\max_{j\le k-\ell}\overline{S}_j-S_j\le\boldsymbol{\lambda}'_\ell\big)$ is larger than

$$\Big(\frac{C}{\ell\sqrt{\boldsymbol{L}_\ell-\ell}}e^{-(1+\frac{\varepsilon}{4})\frac{\pi^2\sigma^2(\boldsymbol{L}_\ell-\ell)}{8(\boldsymbol{\lambda}'_\ell)^2}}\Big)^{\boldsymbol{L}_\ell(k)}\ge e^{-(1+\frac{\varepsilon}{3})\frac{\pi^2\sigma^2(\boldsymbol{L}_\ell-\ell)}{8(\boldsymbol{\lambda}'_\ell)^2}\boldsymbol{L}_\ell(k)}\ge e^{-(k-\ell)(1+\frac{\varepsilon}{2})\frac{\pi^2\sigma^2}{8(\boldsymbol{\lambda}'_\ell)^2}},$$

where we have used for the first inequality that $e^{-\eta\frac{\pi^2\sigma^2(\boldsymbol{L}_\ell-\ell)}{8(\boldsymbol{\lambda}'_\ell)^2}}$ is smaller than $\frac{1}{\ell^{\eta'}}$ for any $\eta,\eta'>0$. Collecting previous inequalities, we obtain

$$\mathbb{P}(A_{k,\ell})\ge\frac{C}{\ell^2}e^{-(k-\ell)(1+\frac{\varepsilon}{2})\frac{\pi^2\sigma^2}{8(\boldsymbol{\lambda}'_\ell)^2}}.$$

Finally, observe that $\boldsymbol{\lambda}'_\ell\sim\sqrt{\ell}$ and then for any $k\in\{\boldsymbol{L}_\ell,\dots,\ell^2\}$

$$\mathbb{P}(A_{k,\ell})\ge e^{-\frac{k\pi^2\sigma^2}{8\ell}(1+\varepsilon)},$$

which completes the proof. □

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