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► **To cite this version:**

Kévin Polisano, Marianne Clausel, Valérie Perrier, Laurent Condat. A convex approach to super-resolution and regularization of lines in images. *Curves & Surfaces*, Jun 2018, Arcachon, France. hal-03420326

**HAL Id: hal-03420326**

**<https://hal.science/hal-03420326>**

Submitted on 9 Nov 2021

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# A CONVEX APPROACH TO SUPER-RESOLUTION AND REGULARIZATION OF LINES IN IMAGES

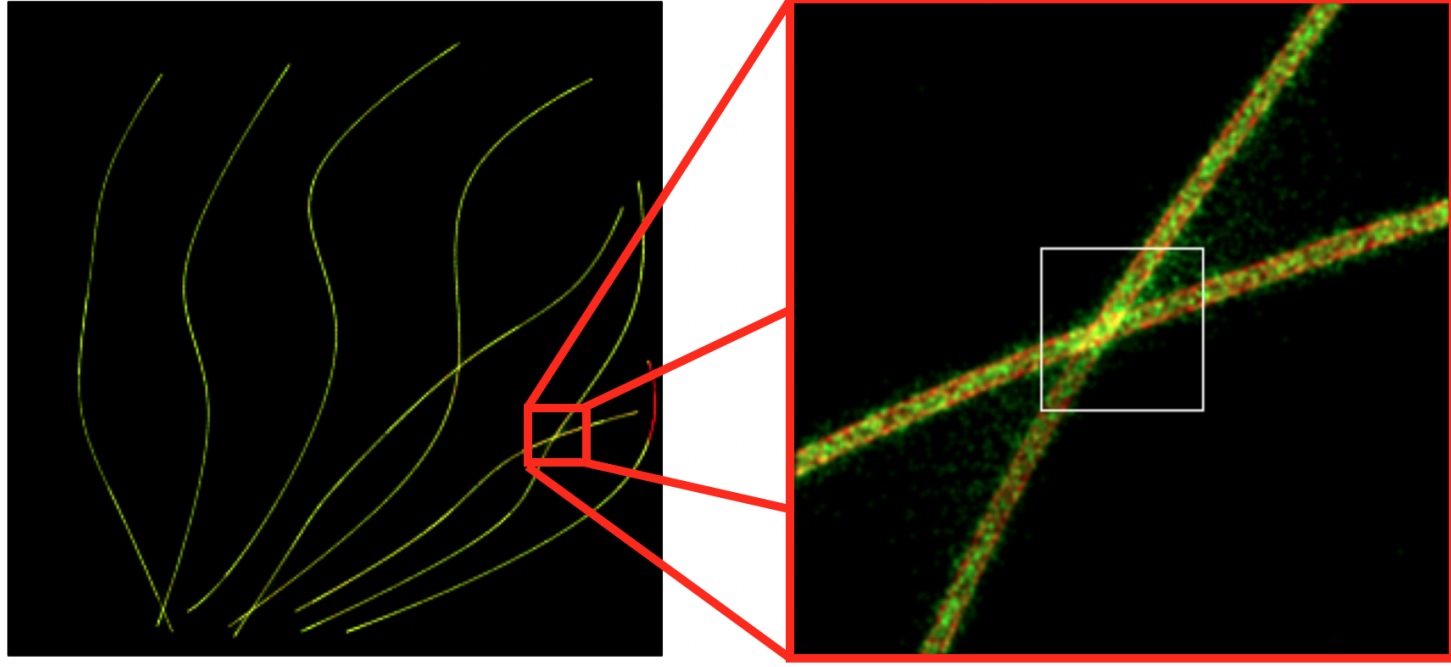
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## OBJECTIVES

We present a new convex formulation for the problem of recovering lines in degraded images. Following the recent paradigm of super-resolution, we formulate a dedicated atomic norm penalty and solve this optimization problem by a primal-dual algorithm. Then, a spectral estimation method recovers the line parameters, with subpixel accuracy.



## ATOMIC NORM FRAMEWORK

Let  $z \in \mathbb{C}^N$  be a vector represented as a linear positive finite combination of sampled complex exponentials  $[a(f, \phi)]_i = e^{j(2\pi f_i + \phi)}$ , that is

$$z = \sum_{k=1}^K c_k a(f_k, \phi_k),$$

whose parameters  $f_k$  and  $\phi_k$  are continuously indexed in a dictionary of atoms:

$$\mathcal{A} = \{a(f, \phi) \in \mathbb{C}^N, f \in [0, 1], \phi \in [0, 2\pi)\}.$$

The atomic norm, which enforces sparsity with respect to the set  $\mathcal{A}$ , is defined as

$$\|z\|_{\mathcal{A}} = \inf_{c'_k > 0, f'_k, \phi'_k} \left\{ \sum_k c'_k : z = \sum_k c'_k a(f'_k, \phi'_k) \right\}.$$

**Theorem 1 [Caratheodory].** Let  $z = (z_n)_{n=-N+1}^{N-1}$  be a vector with Hermitian symmetry  $z_{-n} = z_n^*$ .  $z$  is a positive combination of  $K \leq N$  atoms  $a(f_k, 0)$  if and only if  $\mathbf{T}_N(z_+) \succeq 0$  and of rank  $K$ , where  $z_+ = (z_0, \dots, z_{N-1})$  and  $\mathbf{T}_N$  is the Toeplitz operator

$$\mathbf{T}_N(z_+) = \begin{pmatrix} z_0 & z_1^* & \cdots & z_{N-1}^* \\ z_1 & z_0 & \cdots & z_{N-2}^* \\ \vdots & \vdots & \ddots & \vdots \\ z_{N-1} & z_{N-2} & \cdots & z_0 \end{pmatrix}.$$

Moreover, this decomposition is unique, if  $K < N$ .

**Proposition 1.** The atomic norm  $\|z\|_{\mathcal{A}}$  can be characterized by this semidefinite program SDP( $z$ ):

$$\|z\|_{\mathcal{A}} = \min_{q \in \mathbb{C}^N} \left\{ q_0 : \mathbf{T}'_N(z, q) = \begin{pmatrix} \mathbf{T}_N(q) & z \\ z^* & q_0 \end{pmatrix} \succeq 0 \right\}.$$

## PRIMAL-DUAL ALGORITHM

Let  $\tau > 0$  and  $\sigma > 0$  such that  $\frac{1}{\tau} - \sigma \|\mathbf{L}\|^2 > \frac{\beta}{2}$ .

**Input:** The blurred and noisy data image  $y$

**Output:**  $\tilde{x}$  solution of the optimization problem (1)

- 1: Initialize all primal and dual variables to zero
- 2: **for**  $n = 1$  **to** Number of iterations **do**
- 3:  $\mathbf{X}_{n+1} = \text{prox}_{\tau G}(\mathbf{X}_n - \tau \nabla F(\mathbf{X}_n) - \tau \sum_{i=1}^Q \mathbf{L}_i^* \xi_{i,n})$
- 4: **for**  $i = 0$  **to**  $Q - 1$  **do**
- 5:  $\xi_{i,n+1} = \text{prox}_{\sigma H_i^*}(\xi_{i,n} + \sigma \mathbf{L}_i(2\mathbf{X}_{n+1} - \mathbf{X}_n))$
- 6: **end for**
- 7: **end for**

## REFERENCES

- [1] K. Polisano *et al.*, Convex super-resolution detection of lines in images, IEEE EUSIPCO, 2016.
- [2] K. Polisano *et al.*, A convex approach to super-resolution and regularization of lines in images, preprint, 2018.

## SUPER-RESOLUTION AND REGULARIZATION OF LINES

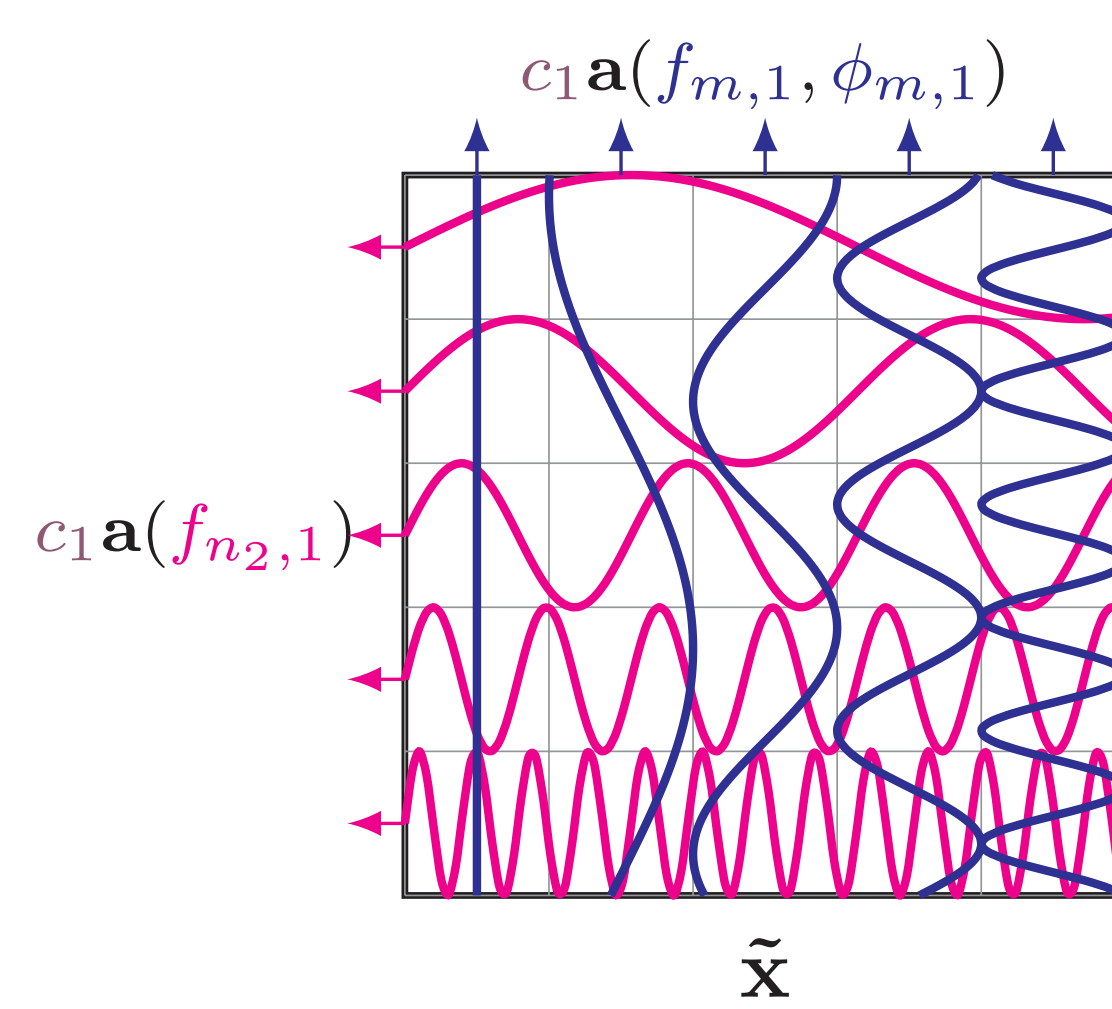
**The Model.** A sum of  $K$  perfect lines of infinite length, with angle  $\theta_k \in (-\pi/2, \pi/2]$ , amplitude  $\alpha_k > 0$  and offset  $\eta_k \in \mathbb{R}$ , is defined as the distribution

$$x^\sharp(t_1, t_2) = \sum_{k=1}^K \alpha_k \delta(\cos \theta_k (t_1 - \eta_k) + \sin \theta_k t_2).$$

The blurred image  $b^\sharp$  of size  $W \times H$  is obtained by the convolution of  $x^\sharp$  with a separated blur function  $\phi$ , following by a sampling with unit step  $\Delta$ :

$$b^\sharp[n_1, n_2] = (x^\sharp * \phi)(n_1, n_2).$$

We denote by  $\hat{x}^\sharp = \mathcal{F}_1 x^\sharp$  (resp.  $\hat{b}^\sharp$ ) the horizontal Fourier transform of  $x^\sharp$  (resp.  $b^\sharp$ ), whose expression



$$\tilde{x} \in \arg \min_{\tilde{x}, \mathbf{q} \in \mathcal{X} \times \mathcal{Q}} \frac{1}{2} \|\mathbf{H}\tilde{x} - \mathbf{y}\|_{\mathbb{F}}^2$$

$$\begin{cases} \forall n_2 = 0, \dots, H-1 \\ \forall m = 0, \dots, M \\ \tilde{x}[0, n_2] = \tilde{x}[0, 0] \leq c \\ \mathbf{q}[m, 0] \leq c \\ \mathbf{T}'_{H_S}(\tilde{x}[m, :], \mathbf{q}[m, :]) \succeq 0 \\ \mathbf{T}_{M+1}(\tilde{x}[:, n_2]) \succeq 0 \end{cases}$$

$$\bullet \mathbf{l}_{n_2}^\sharp = \hat{x}^\sharp[:, n_2] = \sum_{k=1}^K c_k a(f_{n_2, k})$$

$$c_k = \frac{\alpha_k}{\cos \theta_k}, \quad f_{n_2, k} = \frac{\tan \theta_k n_2 - \eta_k}{W}.$$

$$\bullet \mathbf{t}_m^\sharp = \hat{x}^\sharp[m, :] = \sum_{k=1}^K c_k a(f_{m, k}, \phi_{m, k})^\top$$

$$f_{m, k} = \frac{\tan \theta_k m}{W}, \quad \phi_{m, k} = -\frac{2\pi \eta_k m}{W}, \quad d_{m, k} = c_k \phi_{m, k}.$$

**Lines regularization.** Minimizing these atomic norms simultaneously enables to enforce sparsity decomposition on rows and columns of the solution:

- $\|\mathbf{l}_{n_2}^\sharp\|_{\mathcal{A}} = \sum_{k=1}^K c_k = \hat{x}^\sharp[0, n_2]$  (Theorem 1)
- $\|\mathbf{t}_m^\sharp\|_{\mathcal{A}} = \text{SDP}(\mathbf{t}_m^\sharp) \leq \sum_{k=1}^K c_k$  (Proposition 1)

after sampling is the following Hermitian matrix:

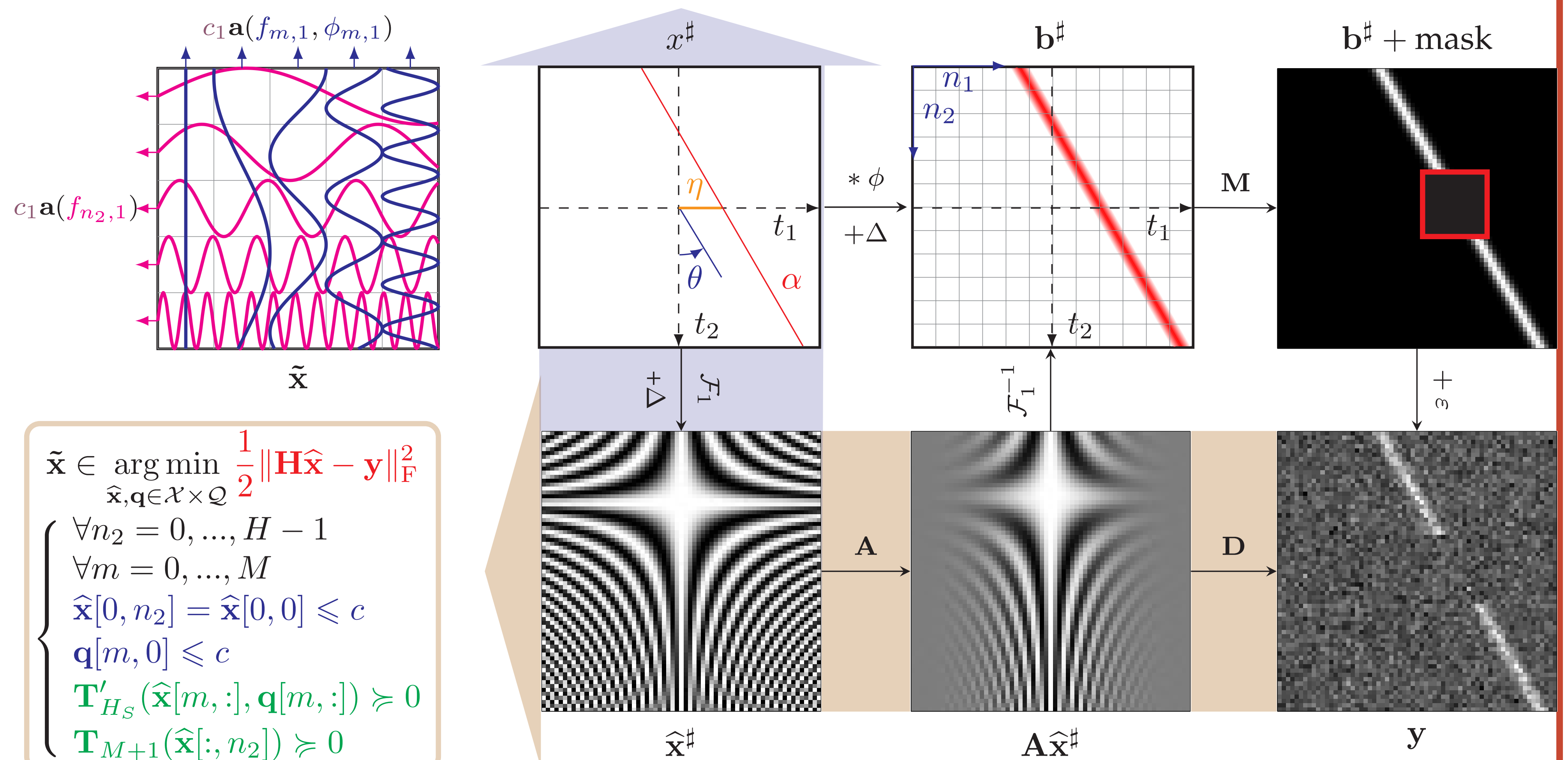
$$\hat{x}^\sharp[m, n_2] = \sum_{k=1}^K \frac{\alpha_k}{\cos \theta_k} e^{j2\pi(\tan \theta_k n_2 - \eta_k)m/W},$$

for  $m = -M, \dots, M, n_2 = 0, \dots, H, M = (W-1)/2$ . In the discrete paradigm, the blur function  $\phi$  with suitable assumptions corresponds in Fourier to a linear operator  $\mathbf{A}$ , leading to the inverse problem:

$$\mathbf{A}\hat{x}^\sharp = \hat{b}^\sharp.$$

The observed image  $y$  is possibly affected by an inpainting mask  $\mathbf{M}$  and some white noise  $\epsilon$ , that is

$$y = \mathbf{M}\mathcal{F}_1^{-1}\mathbf{A}\hat{x}^\sharp + \epsilon = \mathbf{H}\hat{x}^\sharp + \epsilon.$$



**Resolution.** The optimization problem is rewritten:

$$\tilde{\mathbf{X}} = \arg \min_{\tilde{\mathbf{X}} \in \mathcal{H}} \left\{ F(\tilde{\mathbf{X}}) + G(\tilde{\mathbf{X}}) + \sum_{i=0}^{Q-1} H_i \circ L_i(\tilde{\mathbf{X}}) \right\}, \quad (1)$$

with  $F(\mathbf{X}) = \frac{1}{2} \|\mathbf{H}\tilde{x} - \mathbf{y}\|_{\mathbb{F}}^2$ ,  $\mathbf{X} = (\tilde{x}, \mathbf{q})$ ,  $\nabla F$  a  $\beta$ -Lipschitz gradient, a proximable indicator  $G = \iota_{\mathcal{B}}$  where  $\mathcal{B}$  are the two first boundary constraints, and  $Q = M + 1 + H$  linear composite terms, where  $H_i = \iota_{\mathcal{C}}$  with  $\mathcal{C}$  the cone of semidefinite positive matrices and  $L_i \in \{\mathbf{L}_m^{(1)}, \mathbf{L}_{n_2}^{(2)}\}$ ,  $\mathbf{L}_m^{(1)}(\mathbf{X}) = \mathbf{T}'_{H_S}(\tilde{x}[m, :], \mathbf{q}[m, :])$  and  $\mathbf{L}_{n_2}^{(2)}(\mathbf{X}) = \mathbf{T}_{M+1}(\tilde{x}[:, n_2])$ .  $\mathbf{L}$  denotes the concatenation of the  $L_i$  operators in the next algorithm.

## SPECTRAL ESTIMATION BY A PRONY-LIKE METHOD

Let be  $d_k \in \mathbb{C}$ ,  $f_k \in [-1/2, 1/2)$ ,  $\zeta_k = e^{j2\pi f_k}$  and

$$z_i = \sum_{k=1}^K d_k (e^{j2\pi f_k})^i, \quad \forall i = 0, \dots, N-1.$$

The **annihilating polynomial filter** is defined by:

$$H(\zeta) = \prod_{l=1}^K (\zeta - \zeta_l) = \sum_{l=0}^K h_l \zeta^{K-l} \text{ with } h_0 = 1,$$

$$\sum_{l=0}^K h_l z_{r-l} = \sum_{k=1}^K d_k \zeta_k^{r-K} \underbrace{\left( \sum_{l=0}^K h_l \zeta_k^{K-l} \right)}_{H(\zeta_k)=0} = 0.$$

$$\mathbf{P}_K(z)\mathbf{h} = \begin{pmatrix} z_K & \cdots & z_0 \\ \vdots & \ddots & \vdots \\ z_{N-1} & \cdots & z_{N-K-1} \end{pmatrix} \begin{pmatrix} h_0 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

- 1 From  $\mathbf{P}_K(z)$ , compute  $\mathbf{h}$  by a SVD. Form  $H$  whose roots give access to the frequencies  $f_k$ .
- 2 Since  $z = \mathbf{U}\mathbf{d}$  with  $\mathbf{U} = (a(f_1), \dots, a(f_K))$ , find amplitudes by LS:  $\mathbf{d} = (\mathbf{U}^H \mathbf{U})^{-1} \mathbf{U}^H z$ .

### Procedure for retrieving the line parameters

- 1 For each column  $\tilde{x}[m, :]$  compute  $\{\tilde{f}_{m,k}\}_k$  by 1
- 2 For each column  $\tilde{x}[m, :]$  compute  $\{\tilde{d}_{m,k}\}_k$  by 2
- 3  $\{f_{m,k}\}_m = \{\frac{\tan \theta_k m}{W}\}_m$  lin. regression  $\rightarrow \{\tilde{\theta}_k\}$
- 4  $\tilde{\alpha}_{m,k} = |\tilde{d}_{m,k}| \cos(\tilde{\theta}_k) \rightarrow \{\tilde{\alpha}_k\}_k = \mathbb{E}\{\{\tilde{\alpha}_{m,k}\}_m\}$
- 5  $d_{m,k}/|d_{m,k}| = (e^{-j2\pi \frac{\eta_k}{W}})^m \rightarrow \{\tilde{\eta}_k\}_k$  by 1

This procedure enables to super-resolve the line parameters from the solution  $\tilde{x}$  of the problem (1).

