Topology of chaos and climate dynamics

Denisse Sciamarella (IFAECI-CNRS)

UMI - 3351 IFAECI

Joint work with Gisela Charó (CIMA-CONICET),
Mickaël Chekroun (UCLA & Weizmann), Michael Ghil (ENS & UCLA)

Perspectives on Climate Sciences
Nonlinear Processes Division

European Geosciences Union

July 7th
2021

Topology of chaos and climate dynamics

Introduction

- What is chaos topology and why is it important in the theory of dynamical systems and in nonlinear time series analysis?

Methods

- When and how can these concepts be applied?

Applications

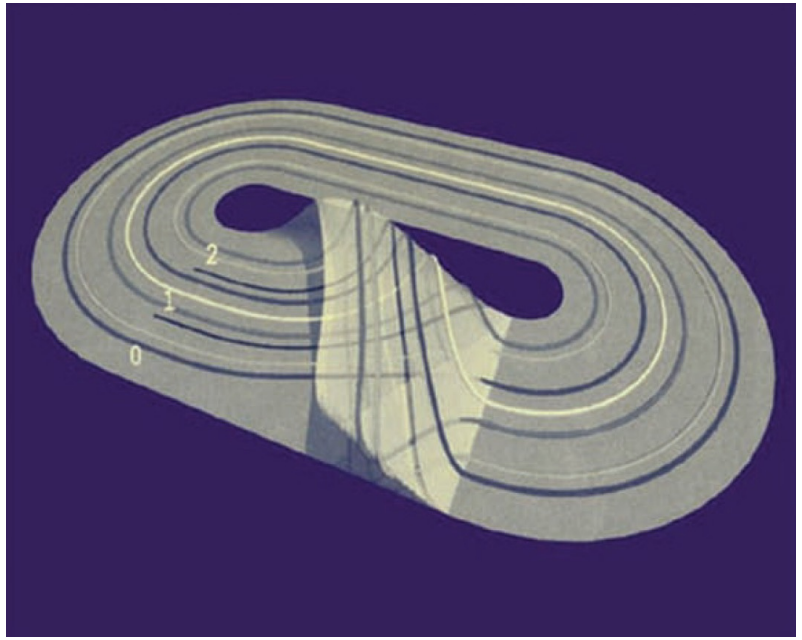
- Lagrangian analysis

Challenges

- Noise-driven chaotic dynamics

Perspectives

- Atmospheric blocking events and Tipping Points (TPs)



Introduction

- What is chaos topology and why is it important?

The first methods of time series analysis trying to associate geometric structures with experimental time series appear in 1980.

VOLUME 45, NUMBER 9 PHYSICAL REVIEW LETTERS 1 SEPTEMBER 1980

Geometry from a Time Series

N. H. Packard, J. P. Crutchfield, J. D. Farmer, and R. S. Shaw
Dynamical Systems Collective, Physics Department, University of California, Santa Cruz, California 95064
(Received 13 November 1979)

It is shown how the existence of low-dimensional chaotic dynamical systems describing turbulent fluid flow might be determined experimentally. Techniques are outlined for reconstructing phase-space pictures from the observation of a single coordinate of any dissipative dynamical system, and for determining the dimensionality of the system's attractor. These techniques are applied to a well-known simple three-dimensional chaotic dynamical system.

PACS numbers: 47.25.-c

Introduction

- What is chaos topology and why is it important?

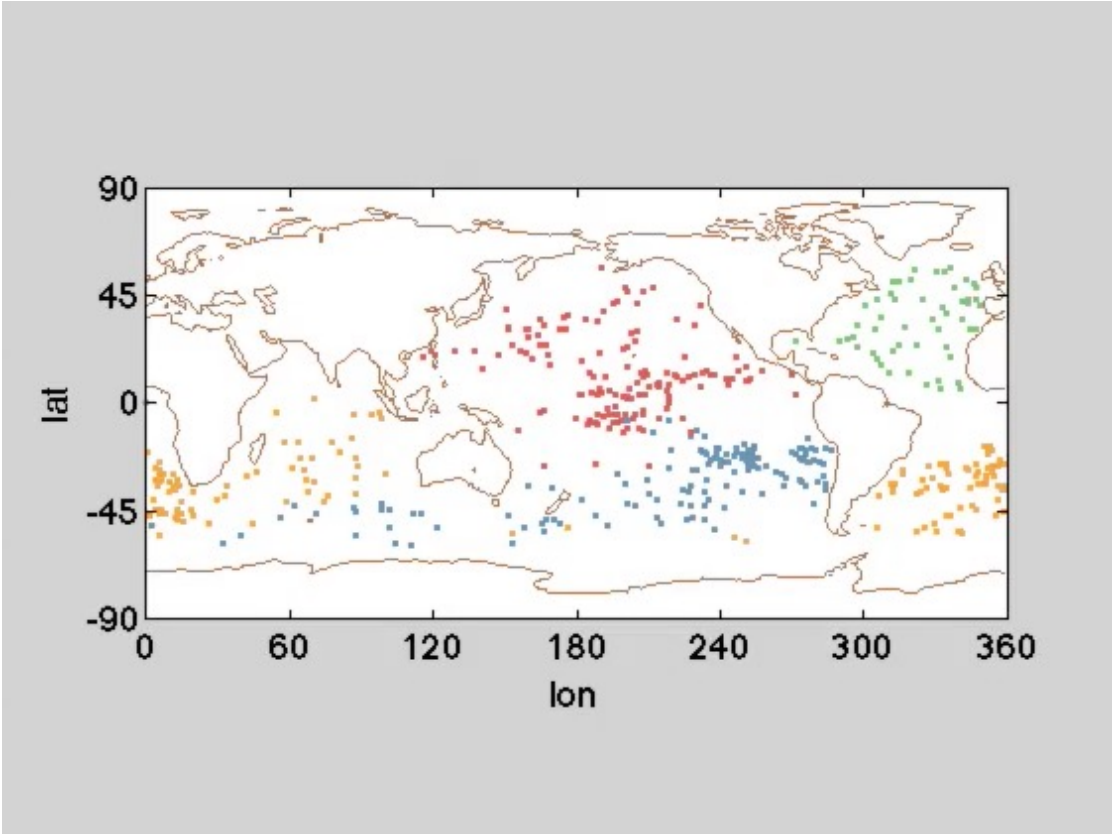
Geometric methods continue to be used, e.g., to understand datasets of Lagrangian trajectories.

Understanding the geometry of transport: Diffusion maps for Lagrangian trajectory data unravel coherent sets

Cite as: Chaos **27**, 035804 (2017); https://doi.org/10.1063/1.4971788
Submitted: 20 March 2016 . Accepted: 18 July 2016 . Published Online: 22 February 2017

Ralf Banisch, and Péter Koltai

But, is geometry the *best* lens we can use to classify data according to underlying differences in dynamics?



Introduction

- What is chaos topology and why is it important?

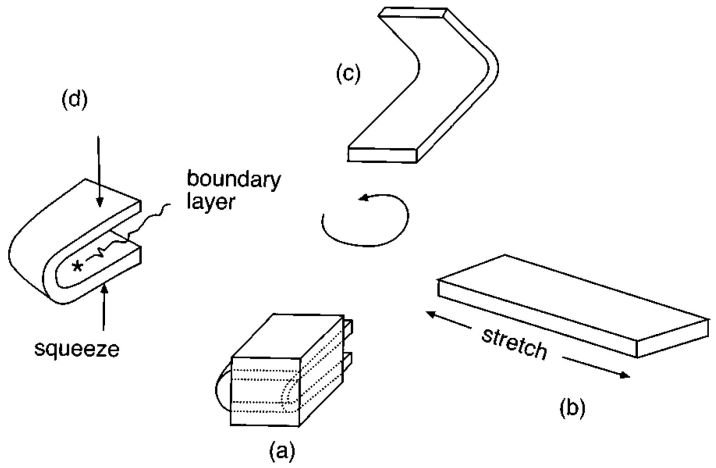
Invariants in phase space can be of different types:

a) Metric: dimensions of various types, e.g., correlation dimension (Grassberger & Procaccia, 1983), multifractal scaling functions (Halsey et al., 1986).

b) Dynamic: Lyapunov exponents (Oseledec, 1968; Wolf et al., 1985), as discussed by Eckmann & Ruelle (1985) and by Abarbanel et al. (1993).

c) Topological: linking numbers, relative rotation rates, Conway polynomials, Branched Manifolds (Birman & Williams, 1983).

Invariants (a) and (b) do not provide information on how to model the system's dynamics, while (c) actually does!

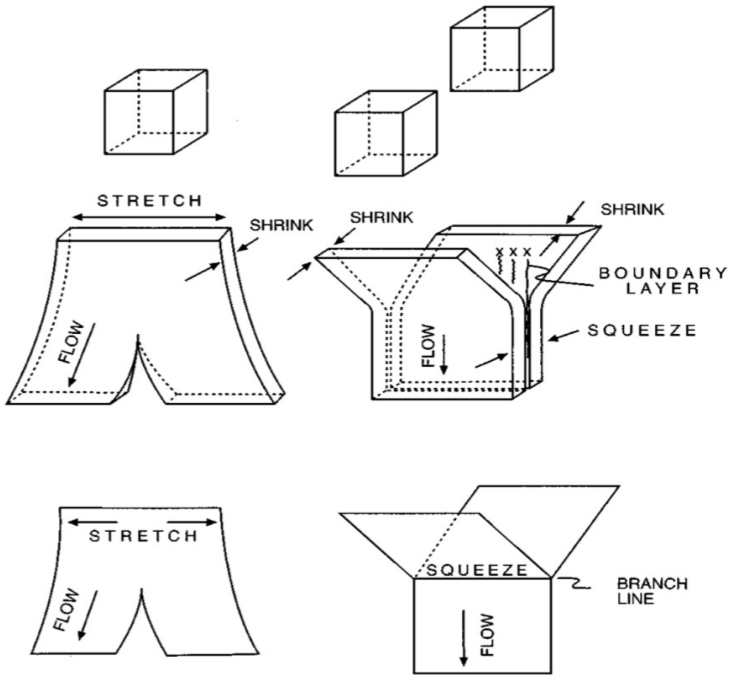


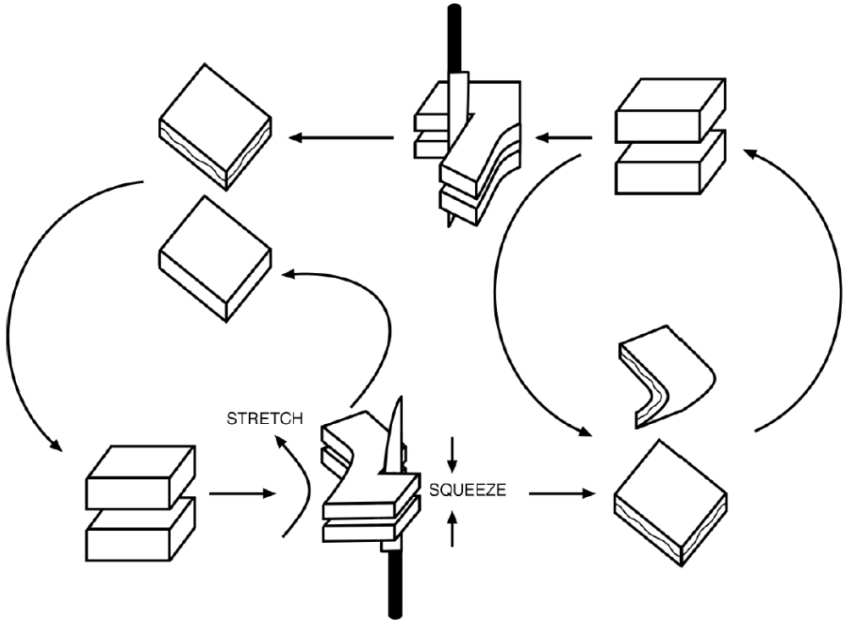
R. Gilmore, *Reviews of Modern Physics*, Vol. 70(4), October 1998

Introduction

- Why is this so?

Topology is concerned with the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling, and bending; that is, without closing holes, opening holes, gluing, or passing through itself. The animated image shows a continuous deformation of a mug into a doughnut: both objects are topologically equivalent.





Gilmore & Lefranc. *The Topology of Chaos: Alice in Stretch and Squeezeland*. Wiley-Interscience, 2002.

The “recipe” to “knead” the Lorenz(1963) attractor is a sequence of steps that are topological in nature.

Introduction

- What is chaos topology and why is it important?

> The advantage of using topology instead of geometry or fractality to describe chaos lies in the fact that topology provides information about the stretching, folding, tearing or squeezing mechanisms that act in phase space to shape the flow.

Geometry may differ, but if the underlying dynamics is equivalent, the topology should be the same.

Unveiling the topology <=> Unveiling the dynamics

Introduction

Henri Poincaré first described they way in which a dynamical system's properties depend upon its topology.

The concept of branched manifold, introduced by Robert F. Williams in 1974, was anticipated in Edward Lorenz's famous 1963 paper: on page 20, he remarks that the trajectory 'lives' on a surface and describes the architecture of the attractor in terms of "isopleths."

EXPANDING ATTRACTORS

by R. F. WILLIAMS

TABLE OF CONTENTS

Publ. Math. IHES, v. 43 (1974), p. 169–203

Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

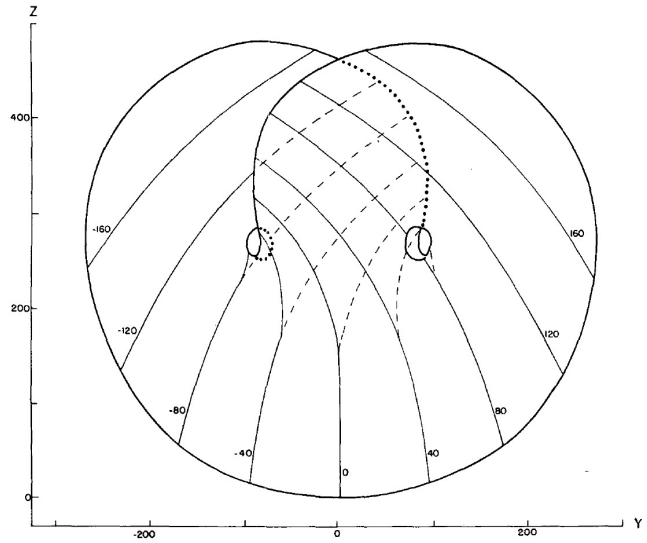


FIG. 3. Isopleths of X as a function of Y and Z (thin solid curves), and isopleths of the lower of two values of X, where two values occur (dashed curves), for approximate surfaces formed by all points on limiting trajectories. Heavy solid curve, and extensions as dotted curves, indicate natural boundaries of surfaces.

Introduction

With Joan Birman, Robert F. Williams used branched manifolds to classify chaotic attractors in terms of the way periodic orbits are "knotted" in dynamical systems.

Topology Vol. 22, No. 1, pp. 47–82, 1983
Printed in Great Britain.

0040–9383/83/010047–36$03.00/0
Pergamon Press Ltd.

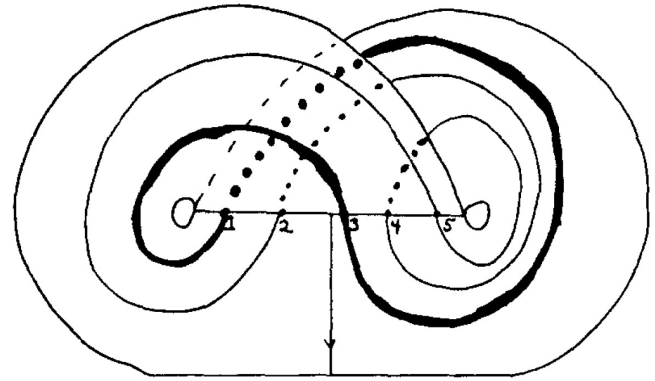
KNOTTED PERIODIC ORBITS IN DYNAMICAL SYSTEMS—I: LORENZ'S EQUATIONS

JOAN S. BIRMAN† and R. F. WILLIAMS‡

(*Received* 31 *March* 1980)

§1. INTRODUCTION

THIS PAPER is the first in a series which will study the following problem. We investigate a system of ordinary differential equations which determines a flow on the 3-sphere S^3 (or $\mathbb{R}^3$ or ultimately on other 3-manifolds), and which has one or perhaps many periodic orbits. We ask: can these orbits be knotted? What types of knots can occur? What are the implications?

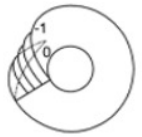
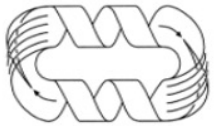
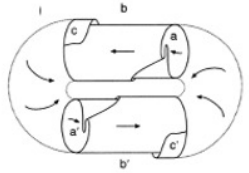
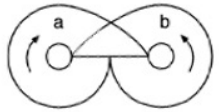


The set of unstable periodic orbits (UPOs) of the Lorenz attractor lie on a "Branched Manifold", i.e. on a 2-manifold or surface with the property that each point has a neighbourhood that is homoeomorphic to either a full 2-ball or a half 2-ball.

Birman & Williams discovered that systems whose branched manifolds have the same topology are dynamically equivalent.

Introduction

Is there a "table of elements" for the different dynamics?

Dynamical system	ODEs	Parameters	Branched manifold
Rossler	$\dot{x} = -y - z$ $\dot{y} = x + ay$ $\dot{z} = b + z(x - c)$	$(a, b, c) = (2.0, 4.0, 0.398)$	
Duffing	$\dot{x} = y$ $\dot{y} = -\delta y - x^3 + x + A\sin(\omega t)$	$(\delta, A, \omega) = (0.4, 0.4, 1.0)$	
van der Pol	$\dot{x} = by + (c - dy^2)x$ $\dot{y} = -x + A\sin(\omega t)$	(b, c, d, A, ω) $=$ $(0.7, 1.0, 10.0, 0.25, \pi/2)$	
Lorenz	$\dot{x} = -\sigma x + \sigma y$ $\dot{y} = Rx - y - xz$ $\dot{z} = -bz + xy$	$(R, \sigma, b) = (26.0, 10.0, 8/3)$	

Gilmore & Lefranc. *The Topology of Chaos: Alice in Stretch and Squeezeland*. Wiley-Interscience, 2002.

Introduction

In the late 90s, it was possible to determine whether or not two three-dimensional (3-D) dissipative dynamical systems are equivalent by using knot theory.

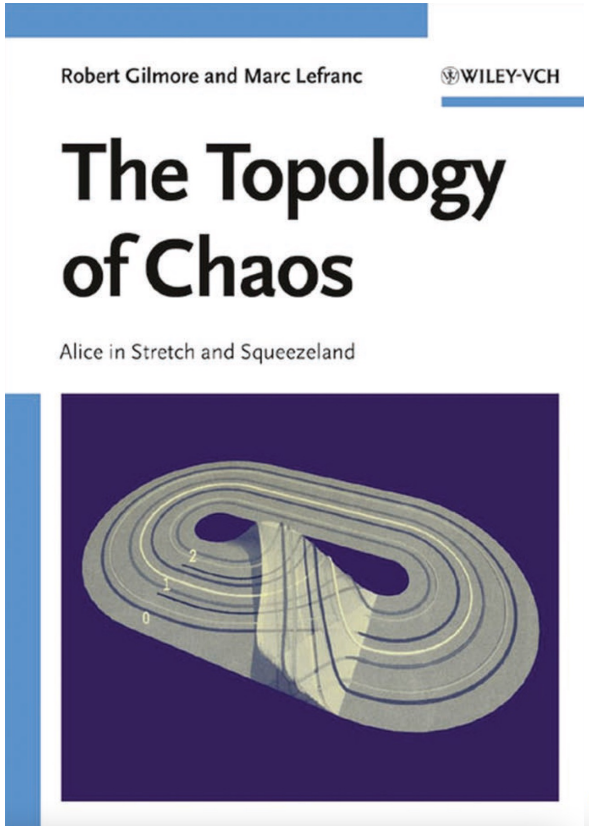
Topological analysis of chaotic dynamical systems

Robert Gilmore

Department of Physics & Atmospheric Science, Drexel University, Philadelphia, Pennsylvania 19104

Topological methods have recently been developed for the analysis of dissipative dynamical systems that operate in the chaotic regime. They were originally developed for three-dimensional dissipative dynamical systems, but they are applicable to all "low-dimensional" dynamical systems. These are systems for which the flow rapidly relaxes to a three-dimensional subspace of phase space. Equivalently, the associated attractor has Lyapunov dimension $d_L<3$. Topological methods supplement methods previously developed to determine the values of metric and dynamical invariants. However, topological methods possess three additional features: they describe how to model the dynamics; they allow validation of the models so developed; and the topological invariants are robust under changes in control-parameter values. The topological-analysis procedure depends on identifying the stretching and squeezing mechanisms that act to create a strange attractor and organize all the unstable periodic orbits in this attractor in a unique way. The stretching and squeezing mechanisms are represented by a caricature, a branched manifold, which is also called a template or a knot holder. This turns out to be a version of the dynamical system in the limit of infinite dissipation. This topological structure is identified by a set of integer invariants. One of the truly remarkable results of the topological-analysis procedure is that these integer invariants can be extracted from a chaotic time series. Furthermore, self-consistency checks can be used to confirm the integer values. These integers can be used to determine whether or not two dynamical systems are equivalent; in particular, they can determine whether a model developed from time-series data is an accurate representation of a physical system. Conversely, these integers can be used to provide a model for the dynamical mechanisms that generate chaotic data. In fact, the author has constructed a doubly discrete classification of strange attractors. The underlying branched manifold provides one discrete classification. Each branched manifold has an "unfolding" or perturbation in which some subset of orbits is removed. The remaining orbits are determined by a basis set of orbits that forces the presence of all remaining orbits. Branched manifolds and basis sets of orbits provide this doubly discrete classification of strange attractors. In this review the author describes the steps that have been developed to implement the topological-analysis procedure. In addition, the author illustrates how to apply this procedure by carrying out the analysis of several experimental data sets. The results obtained for several other experimental time series that exhibit chaotic behavior are also described. [S0034-6861(98)00304-3]

Reviews of Modern Physics, Vol. 70, No. 4, October 1998



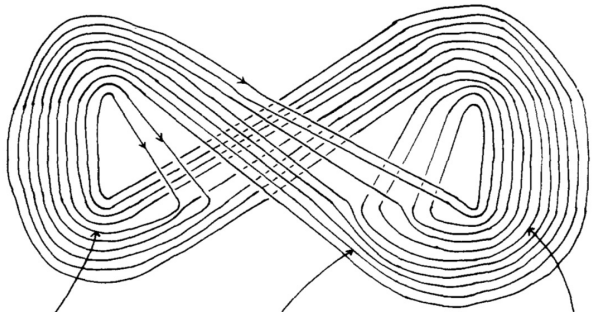
Methods

- When and how can these concepts be applied?

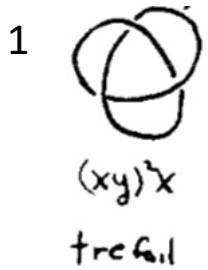
1) Approximate trajectories by closed curves.

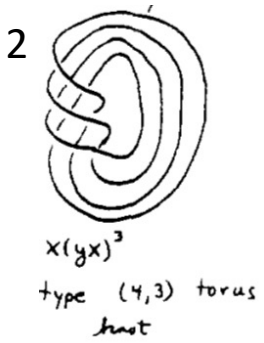
2) Find a topological representation for the orbit structure.

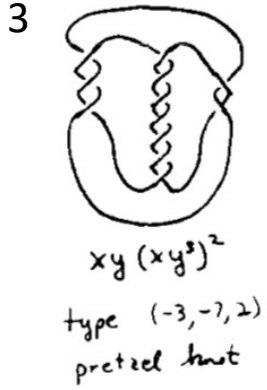
3) Obtain an algebraic description for the topological structure.



1 2 3







1) Close-return method – time series, though, must be long and noise free.

2) Knot theory – knot ≡ orbit in three dimensions.

3) Knot invariants – *e.g.*, linking numbers, Conway polynomials.

Computing topological invariants using **knots**…

3D trajectory set

Knot invariants

Knotted periodic orbits in dynamical systems. Topology: Vol.22. No. I,pp.47~81. 1983

Methods

- When and how can these concepts be applied?

When we connect the ends of a braid, we end up with a knot.

Computing topological invariants using **braids**…

3D trajectory set

Braid invariants

2.4.1 *Braid Words*

We can label the braid crossings in a similar way as in knots, this time using the additional strand information. We call σ_i the crossing where strand i goes *over* strand $i+1$, and σ_i^{-1} the opposite crossing where i goes *under* $i+1$. For a braid of $n > 1$ strands i runs from 1 to $n-1$. Hence, a braid can be described by listing the crossings of its knot projection in the parametric order. A braid can then be represented by a *braid word*. The braid in Figure 2.5 has the word $W = \sigma_1 {\sigma_2}^{-1}$. We say that a *positive braid* has no negative exponents among the σ's in its braid word.

2.4.2 *The braid group I*

Braids are also introduced as the free group of n generators with the following two restrictions: $\sigma_i\sigma_j = \sigma_j\sigma_i$, $|i-j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. The latter restriction resembles Reidemeister move III [2] in the sense that it states that a crossing between two strands can be moved to the "other side" of a third strand simply by sliding it down, as if the strands were

[2]Move I is unnecessary since defining braids as functions rules out the possibility of self-loops and move II is immediate since the group property assures that $\sigma_i\sigma_i^{-1} = Identity$.

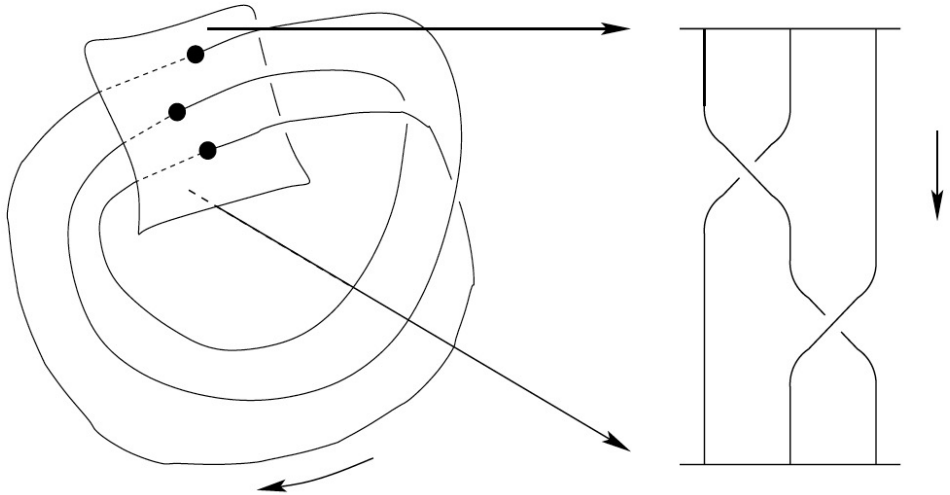
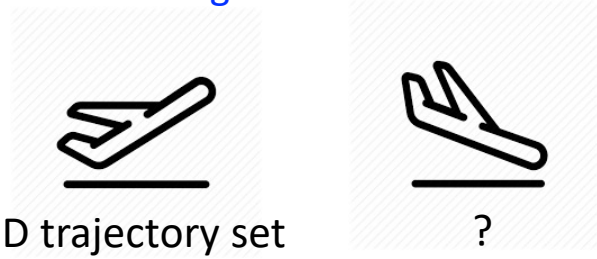


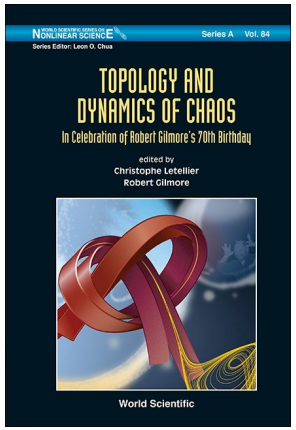
Fig. 2.5 A period-3 orbit of a dynamical system in $\mathbb{R}^2 \times \mathbb{S}^1$ and its knot projection as a braid. The arrows along the flow and braid indicate the time evolution from θ_0 to $\theta_0 + 2\pi$. These two angles are split in the braid graph (right, bottom and top) but identified in the time evolution (left, control section).

Methods

- When and how can these concepts be applied?

Computing topological invariants **without** using **knots or braids**





Chapter 7

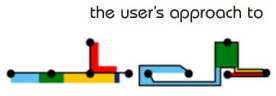
A braided view of a knotty story

Mario Natiello

Matematikcentrum-LTH, Lunds Universitet
Box 118, 221 00 Lund, Sverige

Hernán Solari

Departmento de Física, Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires, Argentina



Periodic orbits of 3-d dynamical systems admitting a Poincaré section can be described as braids. This characterisation can be transported to the Poincaré section and Poincaré map, resulting in the braid type. Information from braid types allows to estimate bounds for the topological entropy of the map while revealing detailed orbit information from the original system, such as the orbits that are necessarily present along with the given one(s) and their organisation. We review this characterisation with some examples –from a user-friendly perspective–, focusing on systems whose Poincaré section is homotopic to a disc.

6.6.3 *Homology groups*

Still, we may want to understand the topological properties of the set of periodic orbits hidden in our data. We need some "braidless" method (in the sense that knots "dissolve" into trivial objects in higher dimensions) and one method that appears to jump at hand is to consider the homology groups associated to our data [Muldoon et al. 1993, Sciamarella and Mindlin 1999; 2001].

Methods

- When and how can these concepts be applied?

Computing topology using knot theory

3-D trajectory set

Knot invariants

RESTRICTIONS

- Precision and length of time series must be good enough for orbits in phase space to be reconstructed accurately …
- Phase space dimension cannot be higher than three, since knots or braids unknot …

Computing topology using homologies

n-D point-cloud

Homologies

HOMOLOGY GROUPS

- Time series can be shorter and noisy since the method is independent of the reconstruction of trajectories in phase space (no periodic orbit).
- Applicable in *n* dimensions: the method is knotless and braidless.

Methods

- When and how can these concepts be applied?

Computing topological invariants using **homologies**

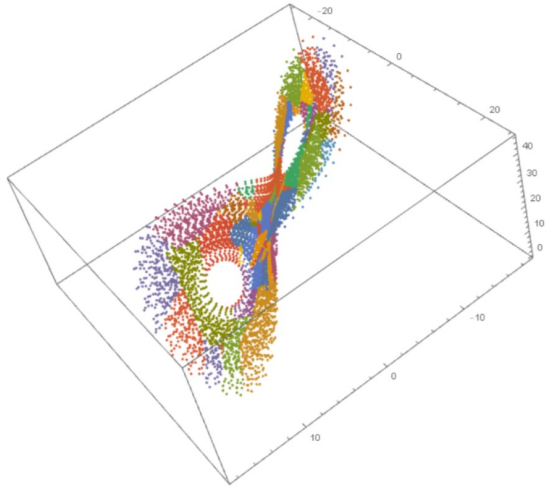
n-D point-cloud

Homologies

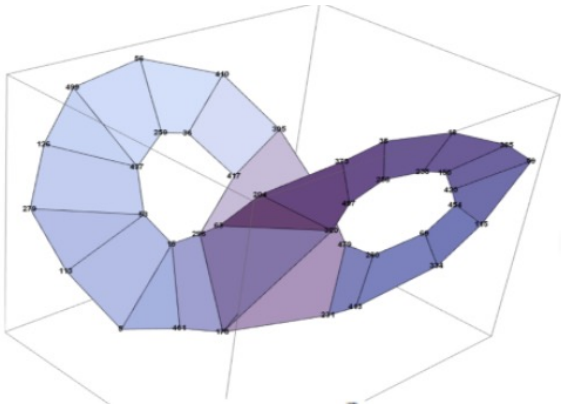
1) Approximate points as lying on a branched manifold.

2) Find a topological representation for the branched manifold.

3) Obtain an algebraic description of the topological structure.



CELL COMPLEX construction



HOMOLOGY GROUP computation

$$H_0 \approx Z^2;$$
$$H_1 \approx Z^3;$$
$$H_2 \approx 0$$

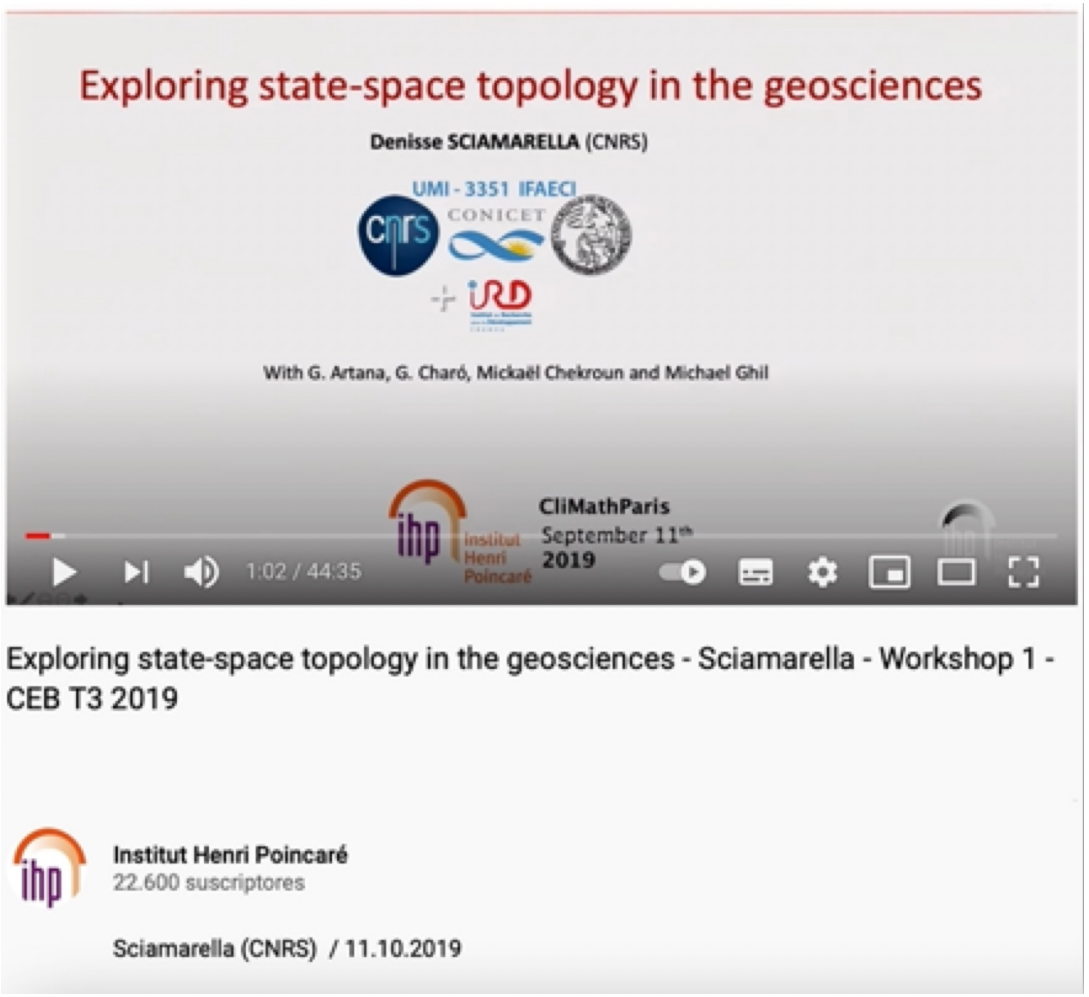
1) Local approximation by *d*-disks => short and noisy time series *can* be handled.

2) Build a cell complex keeping track of the gluing prescriptions => *3*-D+ *can* be handled.

3) Compute homologies and orientability properties of the cell complex => the structure can be identified.

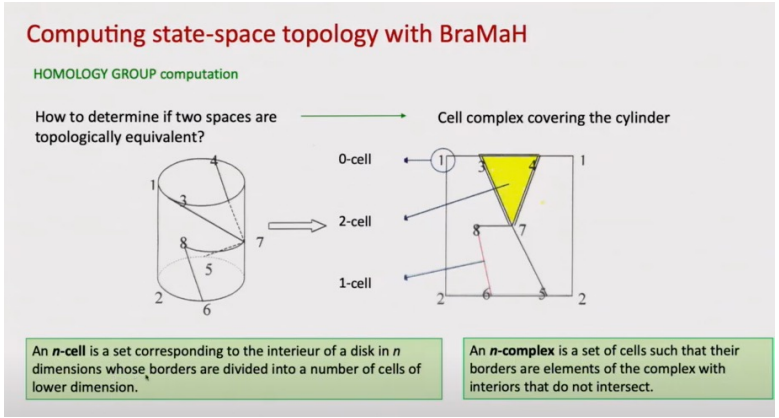
Methods

- When and how can these concepts be applied?



https://youtu.be/RH2zzE8OkgE

HOMOLOGY GROUPS



Branched **Ma**nifold Analysis through **H**omologies (**BraMAH**)

Methods

Historical steps

Computing topological invariants using **homologies**

n-D point-cloud

Homologies

Physica D 65 (1993) 1–16
North-Holland

SDI: 0167-2789(92)00026-1

1993

PHYSICA D

Topology from time series

M.R. Muldoon[a], R.S. MacKay[a], J.P. Huke[b] and D.S. Broomhead[b]
[a]*Nonlinear Systems Laboratory, Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom*
[b]*DRA at RSRE, Malvern, St. Andrew's Road, Great Malvern, Worcestershire WR14 3PS, United Kingdom*

Received 15 August 1992
Revised manuscript received 13 November 1992
Accepted 23 November 1992
Communicated by G. Ahlers

We describe methods for the study of topological pro systems. We explain how to compute such invariants as the and suggest a number of potential applications.

1999

VOLUME 82, NUMBER 7 PHYSICAL REVIEW LETTERS 15 FEBRUARY 1999

Topological Structure of Chaotic Flows from Human Speech Data

Denisse Sciamarella and G. B. Mindlin
Departamento de Física, FCEN, Universidad de Buenos Aires, Pab I, Ciudad Universitaria, cp 1428, Buenos Aires, Argentina
(Received 7 July 1998)

We report the analysis of branched manifolds through homologies, in order to extend the range of applicability of the topological approach to the analysis of human speed data. Analytic and experimental cases are discussed. [S0031-9007(99)08424-0]

PACS numbers: 47.52.+j, 02.40.Sf, 43.72.+q

2001

PHYSICAL REVIEW E, VOLUME 64, 036209

Unveiling the topological structure of chaotic flows from data

Denisse Sciamarella and G. B. Mindlin
Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Pab I, Ciudad Universitaria, Casilla de Correo 1428, Buenos Aires, Argentina
(Received 13 December 2000; published 21 August 2001)

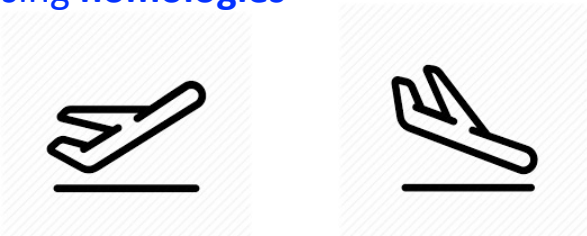
We report the analysis of branched manifolds through homologies, in order to extend the range of applicability of the topological approach to the analysis of chaotic data. Analytic and numerical cases are discussed.

DOI: 10.1103/PhysRevE.64.036209 PACS number(s): 05.45.Pq, 47.52.+j, 02.40.Sf

Methods

The topological information obtained as output by Muldoon et al. (1993) does not suffice to correctly identify the dynamics from time series.

Computing topological invariants using **homologies**



n-D point-cloud Homologies

Physica D 65 (1993) 1–16
North-Holland

SDI: 0167-2789(92)00026-1

PHYSICA D

Topology from time series

M.R. Muldoon[a], R.S. MacKay[a], J.P. Huke[b] and D.S. Broomhead[b]

[a]*Nonlinear Systems Laboratory, Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom*
[b]*DRA at RSRE, Malvern, St. Andrew's Road, Great Malvern, Worcestershire WR14 3PS, United Kingdom*

Received 15 August 1992
Revised manuscript received 13 November 1992
Accepted 23 November 1992
Communicated by G. Ahlers

We describe methods for the study of topological properties of the invariant manifolds of experimental dynamical systems. We explain how to compute such invariants as the Euler characteristic and Betti numbers using time series data, and suggest a number of potential applications.

4. Concluding remarks

We have introduced novel algorithms to extract topological information from experimental time series and have illustrated them with experimental examples. The examples involved boundaryless manifolds traversed by a dense orbit and may leave the impression that our methods are useless for systems whose attractors are not manifolds. To counter this impression, we suggest potential applications to a broader class of objects: some are not attractors and one is not a manifold.

The 1993 approach does not extract all the information that is relevant to identify a branched manifold.

Methods

- When and how can these concepts be applied?

There is much more information in a cell complex than that computed by Muldoon et al in 1993, and this information is relevant to describe a branched manifold.

Computing topological invariants using **homologies**

n-D point-cloud Homologies

In 1999, we showed that the branched manifold could be reconstructed with all its features (including torsions) from noisy datasets.

VOLUME 82, NUMBER 7 PHYSICAL REVIEW LETTERS 15 FEBRUARY 1999

Topological Structure of Chaotic Flows from Human Speech Data

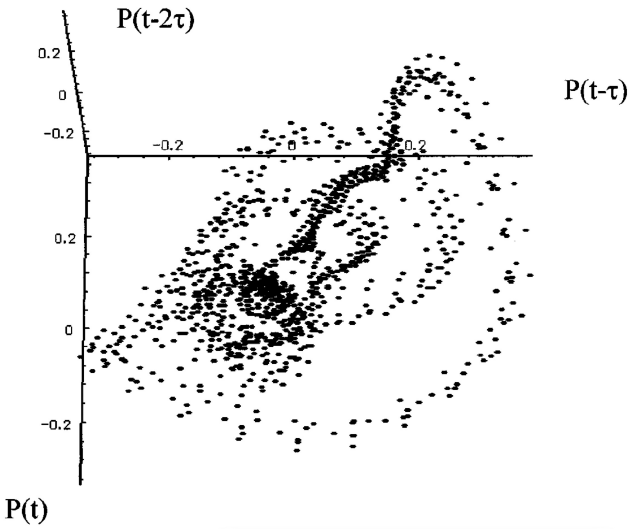
Denisse Sciamarella and G. B. Mindlin

Departamento de Física, FCEN, Universidad de Buenos Aires, Pab I, Ciudad Universitaria, cp 1428, Buenos Aires, Argentina

(Received 7 July 1998)

We report the analysis of branched manifolds through homologies, in order to extend the range of applicability of the topological approach to the analysis of human speed data. Analytic and experimental cases are discussed. [S0031-9007(99)08424-0]

PACS numbers: 47.52.+j, 02.40.Sf, 43.72.+q

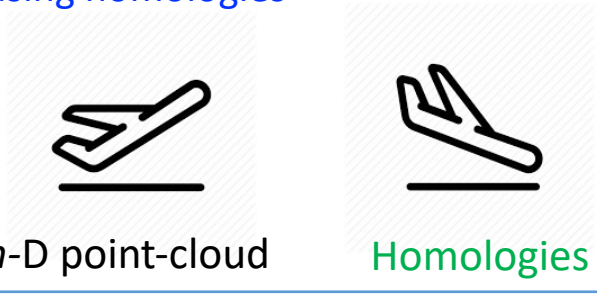


Methods

- When and how can these concepts be applied?

The approach unveils the topology of branched manifolds (including torsions) in more than three phase space dimensions.

Computing topological invariants using homologies



n-D point-cloud Homologies

PHYSICAL REVIEW E, VOLUME 64, 036209

Unveiling the topological structure of chaotic flows from data

Denisse Sciamarella and G. B. Mindlin
Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Pab I, Ciudad Universitaria, Casilla de Correo 1428, Buenos Aires, Argentina
(Received 13 December 2000; published 21 August 2001)

We report the analysis of branched manifolds through homologies, in order to extend the range of applicability of the topological approach to the analysis of chaotic data. Analytic and numerical cases are discussed.

DOI: 10.1103/PhysRevE.64.036209 PACS number(s): 05.45.Pq, 47.52.+j, 02.40.Sf

Chaotic solutions of Shilnikov type

The flow generated by a set of equations such that any three-dimensional projection of it presents self-intersections.

$$x' = -(z+2)d\{x-[a+\epsilon_3(2+w)]\}+(2-z)\{\alpha(x-2) - \beta y - \alpha(x-2)[(x-2)^2+y^2/R^2]\},$$

$$y' = -(z+2)(y-b)+(2-z)\{\beta(x-2)-\alpha y - \alpha y[(x-2)^2+y^2/R^2]\},$$

$$\epsilon_1 z' = (4-z^2)[z+2-m(x+2)]-\epsilon_1 cz,$$

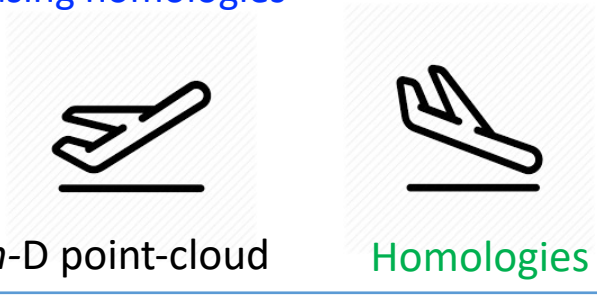
$$\epsilon_2 w' = (4-z^2)[z+2-m(x+2)]-\epsilon_2 cz$$

Methods

- When and how can these concepts be applied?

Should we encounter a Klein bottle in our data, our method would detect it.

Computing topological invariants using homologies



n-D point-cloud

Homologies

ELSEVIER

Physica D 102 (1997) 177–186

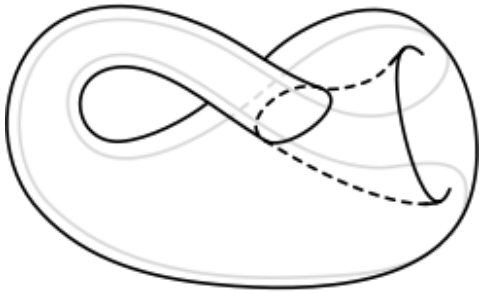
PHYSICA D

Tori and Klein bottles in four-dimensional chaotic flows

G.B. Mindlin *, H.G. Solari

Departamento de Física, FCEN, Universidad de Buenos Aires, Pab I, Ciudad Universitaria, cp 1428, Buenos Aires, Argentina

Received 11 October 1995; revised 24 June 1996; accepted 5 July 1996
Communicated by A. Albano



A Klein bottle is a a one-sided surface that is formed by passing the narrow end of a tapered tube through the side of the tube and flaring this end out to join the other end.

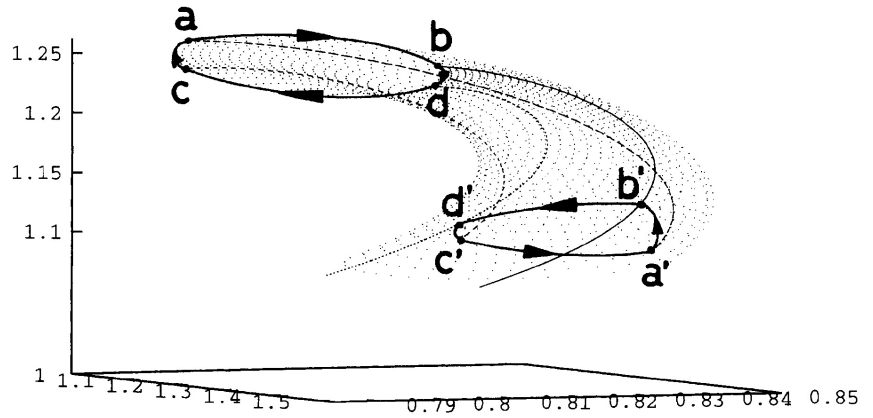
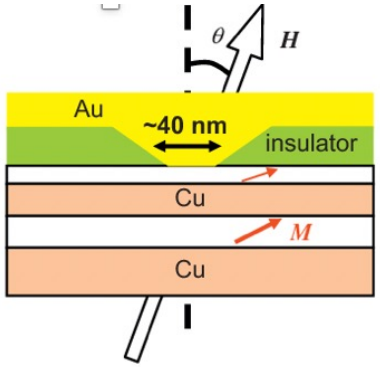


Fig. 5. A piece of the two-dimensional surface within the three-dimensional unstable manifold of the orbit displayed in Fig. 2(b). Four points are selected within the surface (A, B, C, D), and their evolutions displayed as dark lines. They reach the points A', B', C', D'. Notice that in the three-dimensional projection used (x, y, z), the surface self-intersects (the curves joining A, B, C, D and A', B', C', D' are oriented in opposite ways).

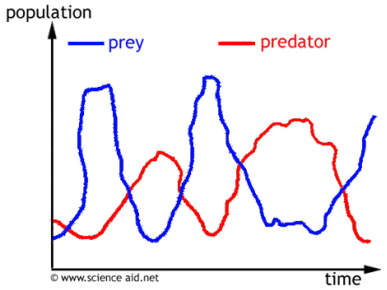
Methods

- The topological program has been applied to data in multiple fields of research.

Voice production

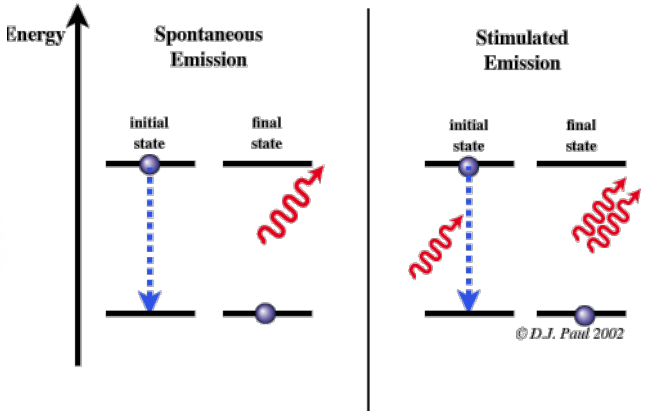


Nano-oscillators

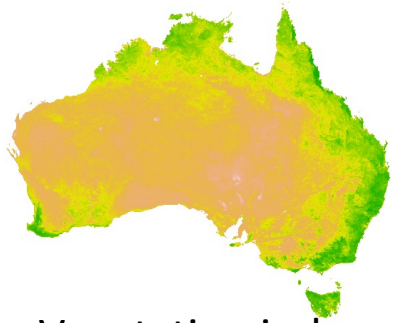


Population dynamics

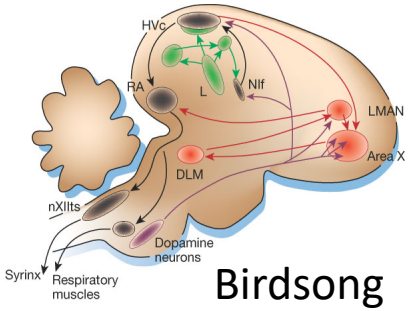
Chemical reactions



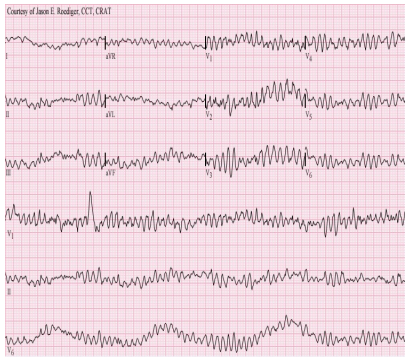
Laser optics



Vegetation index



Birdsong motor control



Cardiac arrythmia

Applications

- The topological program can be harnessed for multiple purposes.

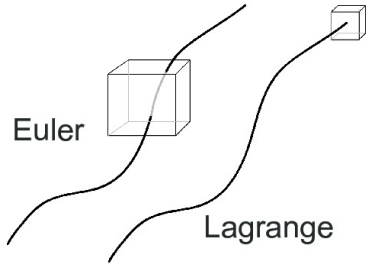
« Topological methods can be used to determine whether or not two dynamical systems are equivalent; in particular, they can determine whether a model developed from time-series data is an accurate representation of a physical system. Conversely, it can be used to provide a model for the dynamical mechanisms that generate chaotic data. »

R. Gilmore, Reviews of Modern Physics, Vol. 70, No. 4, October 1998

- ✓ Validate/refute models – simulations vs. observations.
- ✓ Comparing models – time series generated by different models.
- ✓ Comparing datasets – e.g., *in situ* versus satellite data.
- ✓ Extracting models from data – using global modeling techniques with a topological validation.
- ✓ Characterizing and labeling chaotic behaviors – towards a systematic classification.
- ✓ Classifying sets of time series according to their main dynamical traits – e.g., in Lagrangian Analysis.

Applications

- Lagrangian Analysis



<u>What is Lagrangian analysis</u>?

In fluid mechanics, two viewpoints are possible.

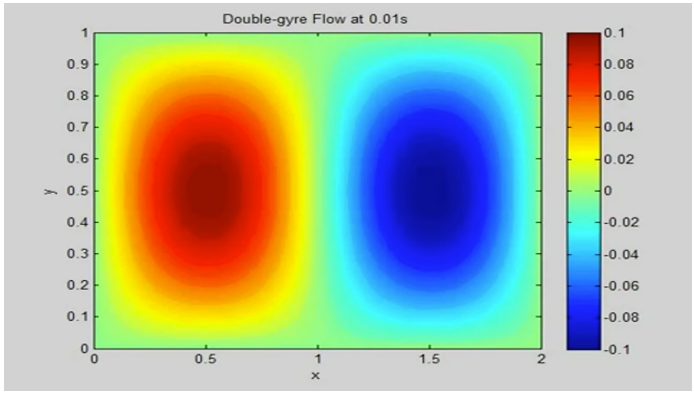
In the Eulerian viewpoint, fluid motion is observed at specific locations in space, as time passes.

In the Lagrangian viewpoint, the observer follows individual fluid particles as they move through the fluid domain.

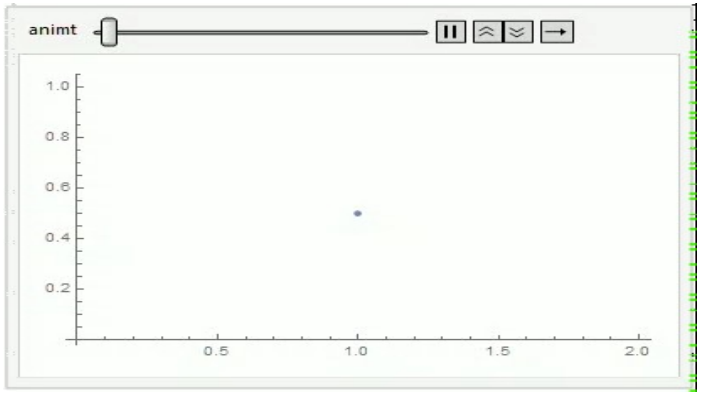
The Driven Double Gyre (DDG) system is an analytic model, often used to show how much Lagrangian patterns may differ from patterns in Eulerian fields.

It was introduced by Shadden et al. (2005) to mimic the motion of two adjacent oceanic gyres enclosed by land and, since Sulalitha Priyankara et al. (2017), it is known to present chaotic transport between the two counter-rotating laterally oscillating vortices.

From the Eulerian perspective, the DDG has a periodic and simple behaviour.



What happens, for instance, if there is an “oil spill” in the middle of the domain?



Applications

- Lagrangian Analysis

Lagrangian analysis is a powerful way to analyse fluids when tracking and understanding the fates, or origins, of fluid particles in flows.

Ocean Modelling 121 (2018) 49–75

Contents lists available at ScienceDirect

Ocean Modelling

journal homepage: www.elsevier.com/locate/ocemod

Review

Lagrangian ocean analysis: Fundamentals and practices

Erik van Sebille[*,a,b], Stephen M. Griffies[c], Ryan Abernathey[d], Thomas P. Adams[e], Pavel Berloff[f], Arne Biastoch[g], Bruno Blanke[h], Eric P. Chassignet[i], Yu Cheng[j], Colin J. Cotter[f], Eric Deleersnijder[k,l], Kristofer Döös[n], Henri F. Drake[o,p], Sybren Drijfhout[q], Stefan F. Gary[e], Arnold W. Heemink[l], Joakim Kjellsson[r,t], Inga Monika Koszalka[g,aa], Michael Lange[a,s], Camille Lique[h], Graeme A. MacGilchrist[u], Robert Marsh[q], C. Gabriela Mayorga Adame[v], Ronan McAdam[a], Francesco Nencioli[w], Claire B. Paris[j], Matthew D. Piggott[s], Jeff A. Polton[v], Siren Rühs[g], Syed H.A.M. Shah[m,l], Matthew D. Thomas[x], Jinbo Wang[y], Phillip J. Wolfram[z], Laure Zanna[t], Jan D. Zika[a,bb]

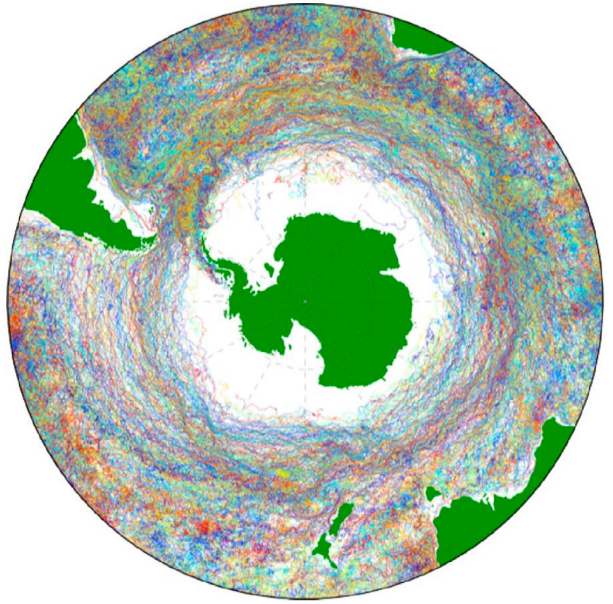


Fig. 1. Map of all the Southern Ocean observational Lagrangian surface drifters in the NOAA GDP Data Set (Lumpkin and Pazos, 2007). Each drifter is geo-located every 6 h and has a randomly assigned colour. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Lagrangian drifters in the Southern Ocean

Applications

- Lagrangian Analysis

Interesting cases correspond to dynamical systems that are nonautonomous (NDSs).

In NDSs, the phase space is not completely determined, i.e., some processes involved in the dynamics are not explicitly described.

Many authors choose to work in an "extended phase space" in which time is added as a phase space coordinate.

Chaos ARTICLE scitation.org/journal/cha

Observability of laminar bidimensional fluid flows seen as autonomous chaotic systems

Cite as: Chaos **29**, 123126 (2019); doi: 10.1063/1.5120625
Submitted: 19 July 2019 · Accepted: 2 December 2019 ·
Published Online: 18 December 2019

View Online Export Citation CrossMark

Gisela D. Charó,[1] Denisse Sciamarella,[2] Sylvain Mangiarotti,[3] Guillermo Artana,[1] and Christophe Letellier[4]

AFFILIATIONS

[1]Laboratorio de Fluidodinámica, Facultad de Ingeniería, Universidad de Buenos Aires, CONICET, C1063ACV CABA, Argentina
[2] Institut Franco-Argentin d'Études sur le Climat et ses Impacts (IFAECI), UMI 3351 (CNRS-CONICET-UBA), C1428EGA CABA, Argentina
[3]Centre d'Études Spatiales de la Biosphère, UPS-CNRS-CNES-IRD, Observatoire Midi-Pyrénées, 18 avenue Édouard Belin, 31401 Toulouse, France
[4]Normandie Université—CORIA, Campus Universitaire du Madrillet, F-76800 Saint-Etienne du Rouvray, France

But the "extended phase space" is in fact deceptive, since it assigns a double status to the time variable, which should not play the role of an independent and a dependent variable at the same time. Among other problems, this leads to an unbounded phase space where some tools from nonlinear dynamical systems theory do not necessarily apply.

Topological studies for NDSs require working in a higher-dimensional phase space that does not include time as a coordinate. This can be achieved using a knotless approach!

Applications

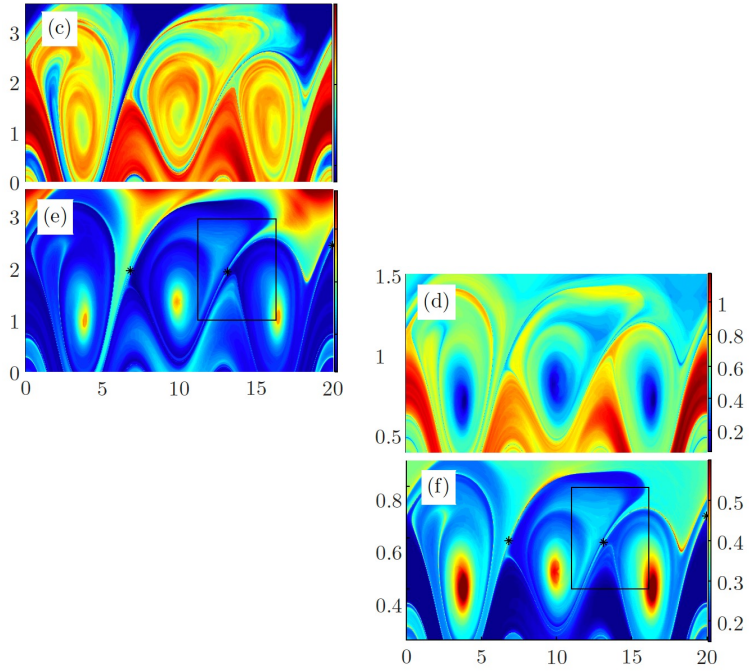
- Lagrangian Analysis

When applied to time series describing particle trajectories, BraMAH falls within a family of methods that measure "complexity of individual trajectories" to identify coherent regions, i.e., regions with qualitatively different dynamical behaviour of trajectories in fluid flows.

Rypina et al (2011), for instance, use correlation dimension as a measure of complexity.

Nonlin. Processes Geophys., 18, 977–987, 2011
www.nonlin-processes-geophys.net/18/977/2011/
doi:10.5194/npg-18-977-2011
© Author(s) 2011. CC Attribution 3.0 License.

Nonlinear Processes in Geophysics



Investigating the connection between complexity of isolated trajectories and Lagrangian coherent structures

I. I. Rypina[1], S. E. Scott[2], L. J. Pratt[1], and M. G. Brown[3]

[1]Physical Oceanography Department, Woods Hole Oceanographic Institution, Woods Hole, MA, 02543, USA
[2]Department of Mathematics, Statistics, and Computer Science, Marquette University, Milwaukee, WI, 53201, USA
[3]RSMAS, University of Miami, 4600 Rickenbacker Causeway, Miami, FL, 33149, USA

Received: 9 February 2011 – Revised: 19 September 2011 – Accepted: 2 December 2011 – Published: 15 December 2011

Applications

- Lagrangian Analysis

Introduction

- What is chaos topology and why is it important in the theory of dynamical systems and in nonlinear time series analysis?

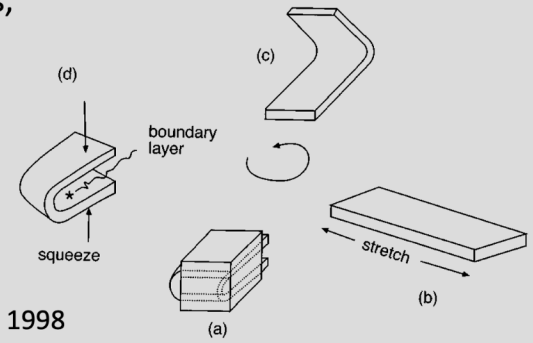
Invariants in phase space can be of different types:

a) Metric: dimensions of various types, e.g., correlation dimension (Grassberger & Procaccia, 1983), multifractal scaling functions (Halsey et al., 1986).

b) Dynamic: Lyapunov exponents (Oseledec, 1968; Wolf et al., 1985), as discussed by Eckmann & Ruelle (1985) and by Abarbanel et al. (1993).

c) Topological: linking numbers, relative rotation rates, Conway polynomials, Branched Manifolds (Birman & Williams, 1983).

Invariants (a) and (b) do not provide information on how to model the system's dynamics, while (c) actually does!



R. Gilmore, Reviews of Modern Physics, Vol. 70, Nro. 4, Octobre 1998

Let's try BRAMAH

Applications

- Lagrangian Analysis

Physica D 405 (2020) 132371

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Topology of dynamical reconstructions from Lagrangian data

Gisela D. Charó [a,b,*], Guillermo Artana [a,b], Denisse Sciamarella [c]

[a] Laboratorio de Fluidodinámica, Facultad de Ingeniería, Universidad de Buenos Aires, C1063ACV CABA, Argentina
[b] CONICET – Consejo Nacional de Investigaciones Científicas y Técnicas, C1425FQD CABA, Argentina
[c] Institut Franco-Argentin d'Études sur le Climat et ses Impacts (IFAECI), UMI 3351 (CNRS-CONICET-UBA), C1428EGA CABA, Argentina

ARTICLE INFO

Article history:
Received 16 August 2019
Received in revised form 9 December 2019
Accepted 20 January 2020
Available online 30 January 2020
Communicated by G. Froyland

Keywords:
Topology
Delay-coordinate embedding
Nonlinear time-series analysis
Homology

ABSTRACT

Branched Manifold Analysis through Homologies (BraMAH) is a technique that computes the state-space topology of a dynamical reconstruction from scalar data. This work introduces the application of this technique to Lagrangian time series. The approach unveils the topological structure underlying the behavior of a fluid particle. When applied to a set of sparse particles, the results of the analysis can be used to classify them according to the dynamics they deploy during a given time window. Topological grids can be constructed to portray the spatial organization of the topological classes. The connection between the topological grids and the transport properties of the flow is examined using streaklines. Even if demonstrated here in the context of kinematic flow models, the generality of the method allows for its potential application to experimental or observational Lagrangian data satisfying the technical requirements for the analysis.

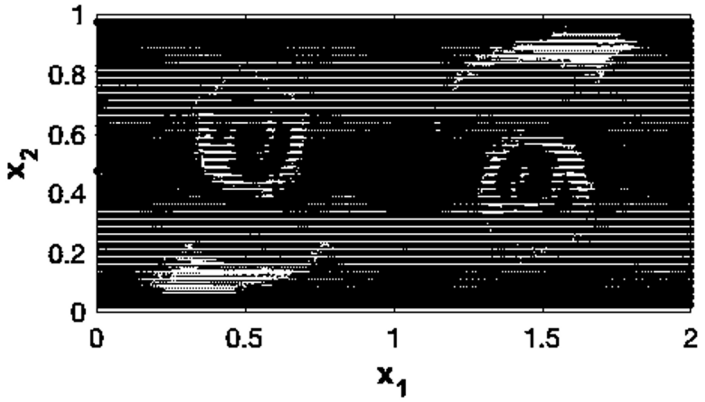
© 2020 Elsevier B.V. All rights reserved.

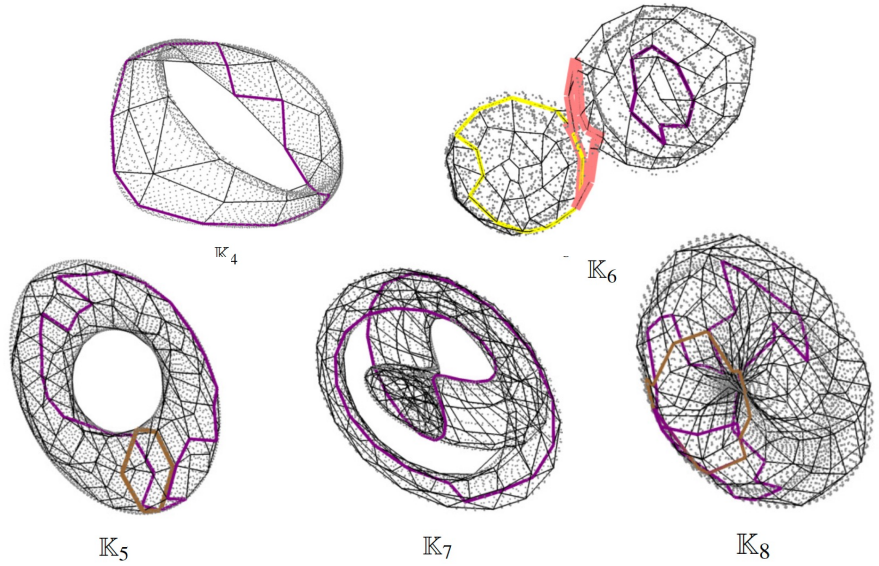
Applications

- Lagrangian Analysis

BraMAH does not look into the geometrical complexity of individual particle trajectories, but into the topology of the associated branched manifolds, which dictate the recipes that knead their dynamical behavior in phase space.

Back to the “oil spill” in the middle of the domain of the DDG system, we apply BraMAH to 8528 fluid particles in a four-dimensional reconstructed phase space. Only five different topological classes come out, and there is a Klein bottle among them, stressing the importance of working in a sufficiently high-dimensional phase space (the Klein bottle cannot be immersed in three dimensions without self-intersections).

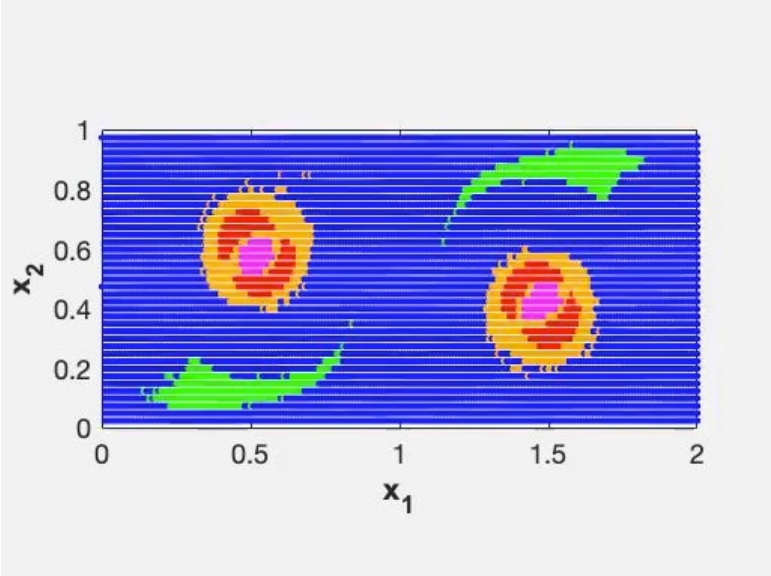




Applications

- Lagrangian Analysis

Assigning a different color to each topological class, the colors in motion define particle sets that move robustly. Let us use the term ‘separator’ to designate the frontier between differently colored regions. Such flow separators are associated with ‘Lagrangian coherent structures’, known to separate dynamically distinct regions in fluid flows (Kelley, Allshouse & Ouellette, 2013).



PHYSICAL REVIEW E **88**, 013017 (2013)

Lagrangian coherent structures separate dynamically distinct regions in fluid flows

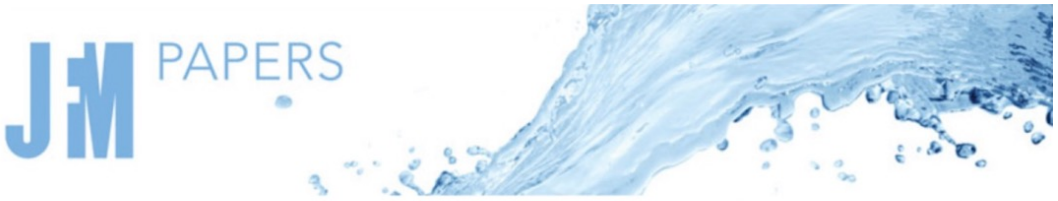
Douglas H. Kelley,[1,*] Michael R. Allshouse,[2] and Nicholas T. Ouellette[3,†]
[1]*Department of Materials Science & Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[2]*Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[3]*Department of Mechanical Engineering & Materials Science, Yale University, New Haven, Connecticut 06520, USA*
(Received 28 September 2012; published 26 July 2013)

Using filter-space techniques, we study the scale-to-scale transport of energy in a quasi-two-dimensional, weakly turbulent fluid flow averaged along the trajectories of fluid elements. We find that although the spatial mean of this Lagrangian-averaged flux is nearly unchanged from its Eulerian counterpart, the spatial structure of the scale-to-scale energy flux changes significantly. In particular, its features appear to correlate with the positions of Lagrangian coherent structures (LCS’s). We show that the LCS’s tend to lie at zeros of the scale-to-scale flux, and therefore that the LCS’s separate regions that have qualitatively different dynamics. Since LCS’s are also known to be impenetrable barriers to advection and mixing, we therefore find that the fluid on either side of an LCS is both kinematically and dynamically distinct. Our results extend the utility of LCS’s by making clear the role they play in the flow dynamics in addition to the kinematics.

DOI: 10.1103/PhysRevE.88.013017 PACS number(s): 47.10.Fg, 47.27.De, 89.75.Fb

Applications

- Lagrangian Analysis



Topological colouring of fluid particles unravels finite-time coherent sets

Gisela D. Charó[1,2,3,†], Guillermo Artana[3,4] and Denisse Sciamarella[2,5]

[1]CONICET – Universidad de Buenos Aires, Centro de Investigaciones del Mar y la Atmósfera (CIMA), Buenos Aires, Argentina

[2]CNRS – IRD – CONICET – UBA, Institut Franco-Argentin d'Études sur le Climat et ses Impacts (IRL 3351 IFAECI), Buenos Aires, Argentina

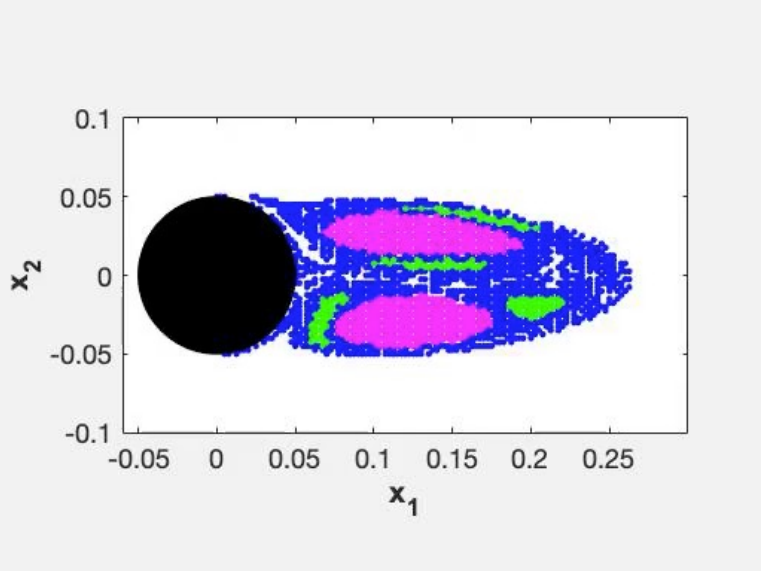
[3]CONICET – Consejo Nacional de Investigaciones Científicas y Técnicas, C1425FQD CABA, Argentina

[4]Laboratorio de Fluidodinámica, Facultad de Ingeniería, Universidad de Buenos Aires, C1063ACV CABA, Argentina

[5]CNRS – Centre National de la Recherche Scientifique, 75795 Paris, France

(Received 21 December 2020; revised 8 May 2021; accepted 18 June 2021)

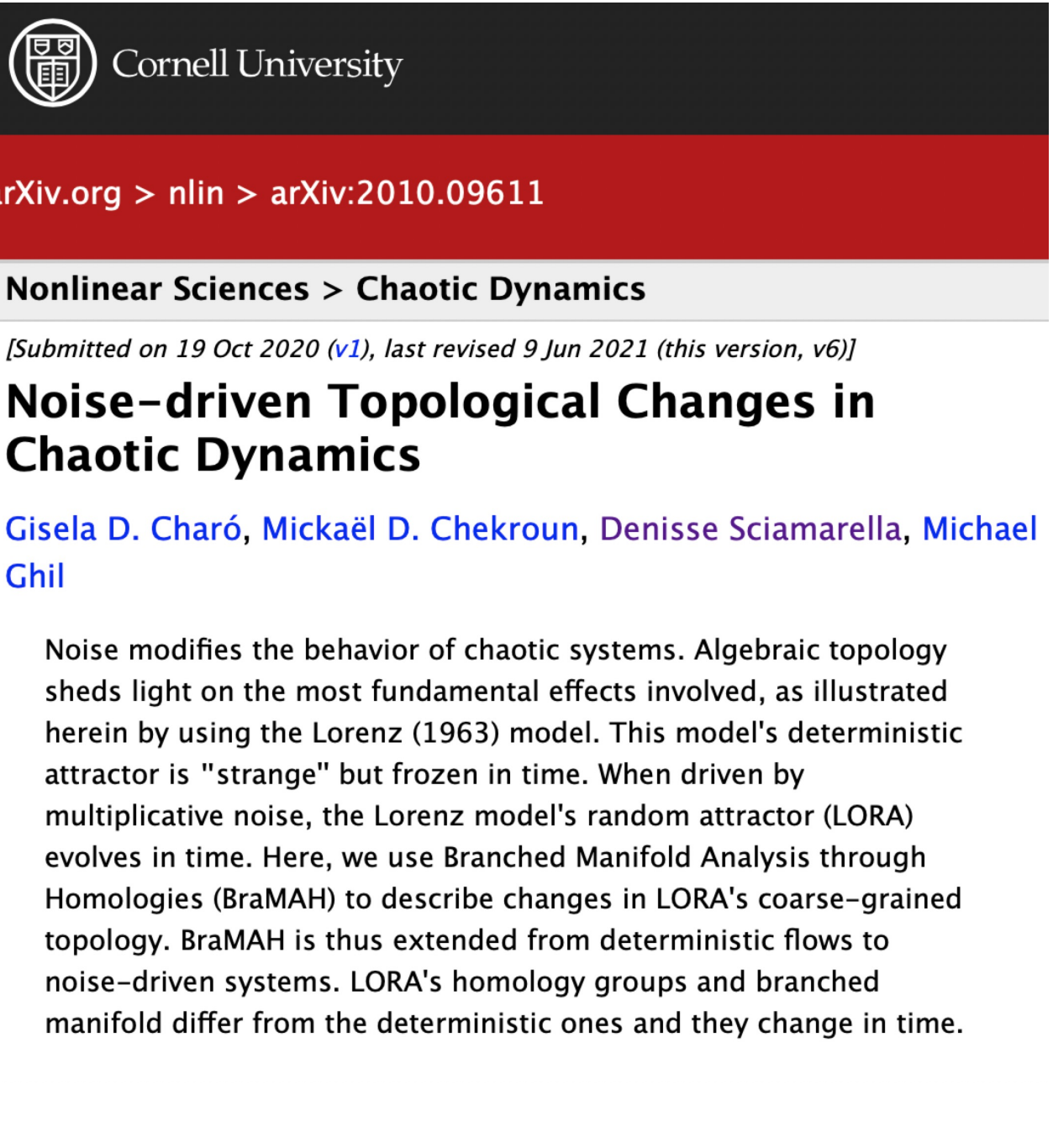
"Topological colouring of fluid particles" = using BraMAH to study the individual dynamics of a sparse particle set.



Numerically generated fluid particle behaviour in the wake past a rotary oscillating cylinder (ROC). With these applications, methodological progress is being made: BraMAH is successfully applied to non-dissipative (conservative) systems.

Challenges

- Can we take one step beyond, and extend the topological perspective to random dynamical systems (RDSs), which provide the appropriate mathematical framework to tackle ocean–atmosphere coupling and climate change?



Cornell University

arXiv.org > nlin > arXiv:2010.09611

Nonlinear Sciences > Chaotic Dynamics

[Submitted on 19 Oct 2020 (v1), last revised 9 Jun 2021 (this version, v6)]

Noise-driven Topological Changes in Chaotic Dynamics

Gisela D. Charó, Mickaël D. Chekroun, Denisse Sciamarella, Michael Ghil

Noise modifies the behavior of chaotic systems. Algebraic topology sheds light on the most fundamental effects involved, as illustrated herein by using the Lorenz (1963) model. This model's deterministic attractor is "strange" but frozen in time. When driven by multiplicative noise, the Lorenz model's random attractor (LORA) evolves in time. Here, we use Branched Manifold Analysis through Homologies (BraMAH) to describe changes in LORA's coarse-grained topology. BraMAH is thus extended from deterministic flows to noise-driven systems. LORA's homology groups and branched manifold differ from the deterministic ones and they change in time.

Challenges

In physical systems, such as those encountered in the climate sciences, time-dependent and random forcing is often present.

In a single very long integration of a stochastic system, noise has a smoothing or blurring effect.

In the so-called “pullback” approach (PBA), introduced by Chekroun, Simonnet and Ghil, one follows an ensemble of trajectories, each driven by the same noise realization.

This way, the smoothing disappears, and the fractal structure is fully captured.

Physica D 240 (2011) 1685–1700

Contents lists available at SciVerse ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Stochastic climate dynamics: Random attractors and time-dependent invariant measures

Mickaël D. Chekroun [a,b,*], Eric Simonnet [c], Michael Ghil [a,b,d]

[a] Department of Atmospheric Sciences and Institute of Geophysics and Planetary Physics, University of California, Los Angeles, CA 90095-1565, USA
[b] Environmental Research and Teaching Institute (CERES-ERTI), École Normale Supérieure, 75231 Paris Cedex 05, France
[c] Institut Non Linéaire de Nice (INLN)-UNSA, UMR 6618 CNRS, 1361, route des Lucioles 06560 Valbonne, France
[d] Geosciences Department and Laboratoire de Météorologie Dynamique (CNRS and IPSL), École Normale Supérieure, 75231 Paris Cedex 05, France

ARTICLE INFO

Article history:
Received 8 January 2010
Received in revised form
1 June 2011
Accepted 3 June 2011
Available online 14 June 2011
Communicated by H.A. Dijkstra

Keywords:
Climate dynamics
Dissipative dynamical systems
Intermittency
Pullback and random attractor
Sample invariant measure
SRB measure

ABSTRACT

This article attempts a unification of the two approaches that have dominated theoretical climate dynamics since its inception in the 1960s: the nonlinear deterministic and the linear stochastic one. This unification, via the theory of random dynamical systems (RDS), allows one to consider the detailed geometric structure of the random attractors associated with nonlinear, stochastically perturbed systems. We report on high-resolution numerical studies of two idealized models of fundamental interest for climate dynamics. The first of the two is a stochastically forced version of the classical Lorenz model. The second one is a low-dimensional, nonlinear stochastic model of the El Niño–Southern Oscillation (ENSO). These studies provide a good approximation of the two models' global random attractors, as well as of the time-dependent invariant measures supported by these attractors; the latter are shown to have an intuitive physical interpretation as random versions of Sinaï–Ruelle–Bowen (SRB) measures.

© 2011 Elsevier B.V. All rights reserved.

Challenges

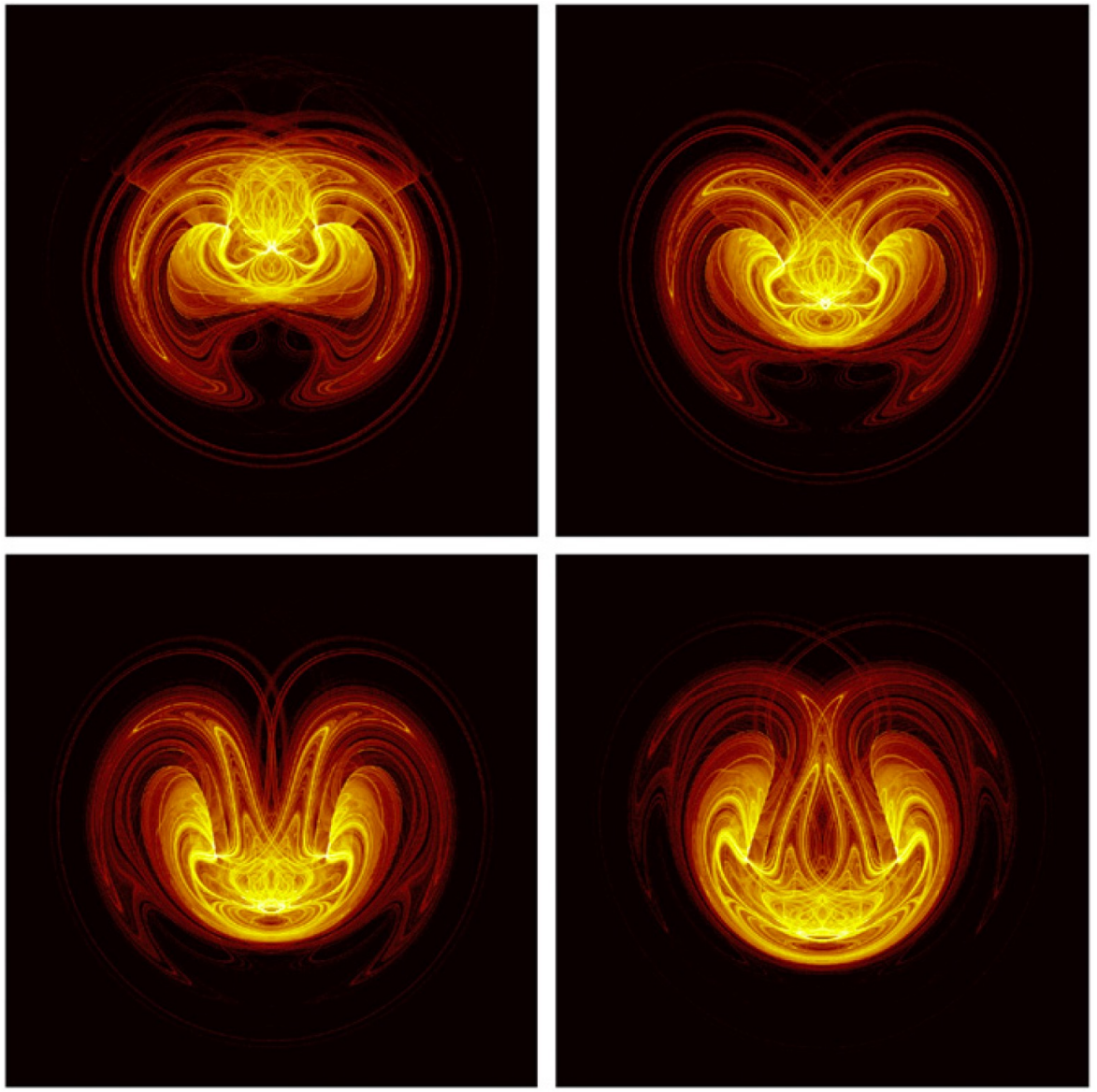
Focus: LORA the stochastic Lorenz model's pullback attractor as computed & studied by Chekroun et al. (2011).

For chaotic systems subject to noise, however, this noise-induced smoothing observed in the forward approach compresses a lot of crucial information about the dynamics itself; quite to the contrary, the pullback approach brings this information into sharp focus. A quick look at Figs. 1–3 is already enlightening in this respect. All three figures refer to the invariant measure μ_ω supported by the random attractor of our stochastic Lorenz model **[SLM]**. This model obeys the following three SDEs:

$$[\mathbf{SLM}] \quad \begin{cases} dx = s(y-x)dt + \sigma x\, dW_t, \\ dy = (rx - y - xz)dt + \sigma y\, dW_t, \\ dz = (-bz + xy)dt + \sigma z\, dW_t. \end{cases} \tag{7}$$

In system (7), each of the three equations of the classical, deterministic model [3] is perturbed by linearly multiplicative noise in the Itô sense, with W_t a Wiener process and $\sigma > 0$ the noise intensity. The other parameter values are the standard ones for chaotic behavior [48], and are given in the caption of Fig. 1.

$$r = 28,\ s = 10,\ b = 8/3$$



Challenges

PBAs provide the perfect way to reveal the stretching and folding caused by the interactions between noise and chaotic nonlinearity.

The static picture of a strange attractor is replaced by a dynamic version that is even stranger, and that we refer to as noise-driven chaos.

Can we examine noise-driven chaos through the lens provided by algebraic-topology tools?

Let us redefine "branched manifold" locally as an integer-dimensional set in phase space that is a robust skeleton of the point cloud associated, at each instant, with the invariant measure supported by the random attractor.

$\sigma = 0.3$

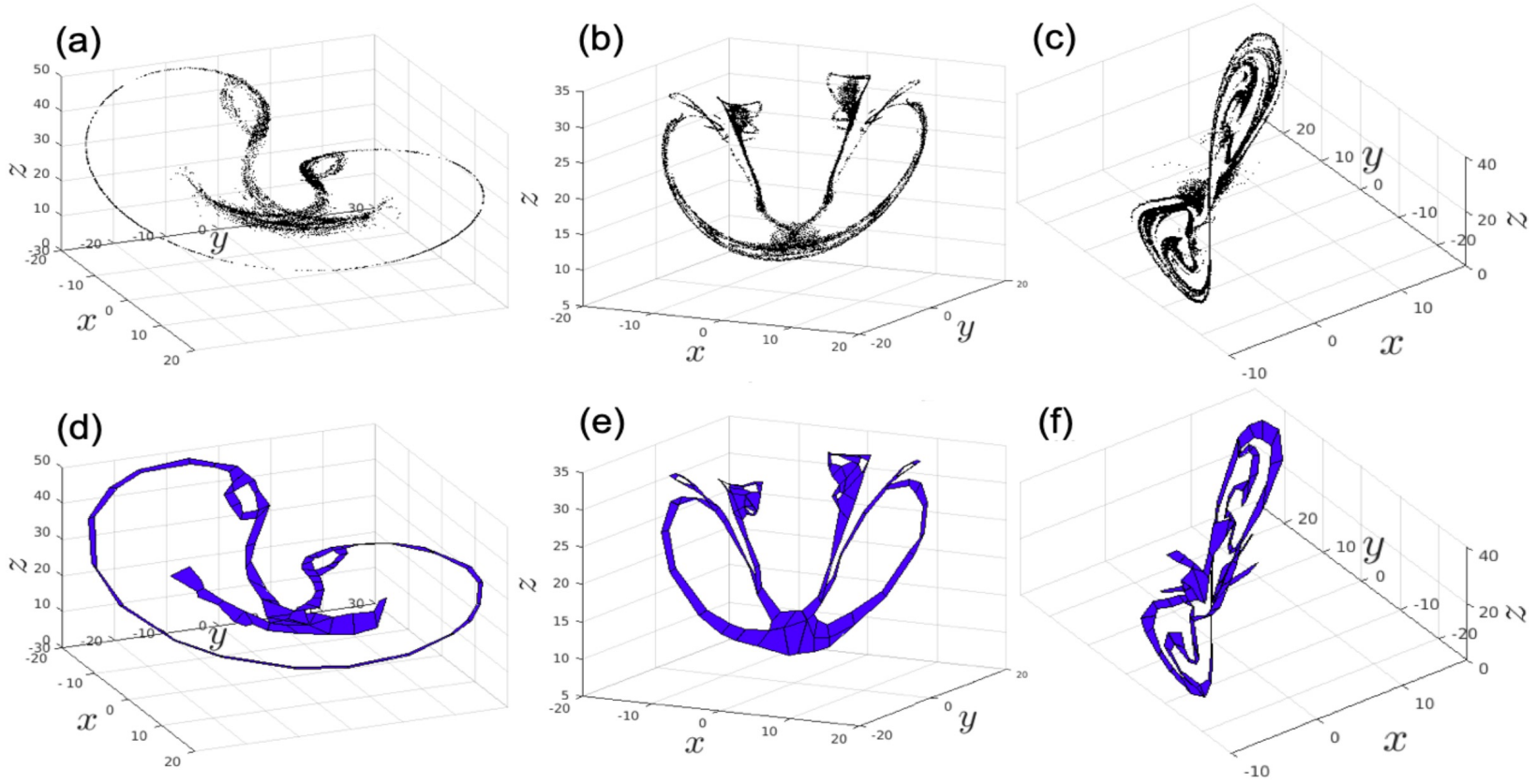
Challenges

Results!

- ❑ BraMAH captures LORA's time-evolving coarse-grained structure.
- ❑ The topologies differ from the deterministic Lorenz model's strange attractor.
- ❑ The noise-driven model's branched manifold exhibits sharp topological transitions in time.

Coarse-grained (sieved) point clouds

BraMAH cell complexes

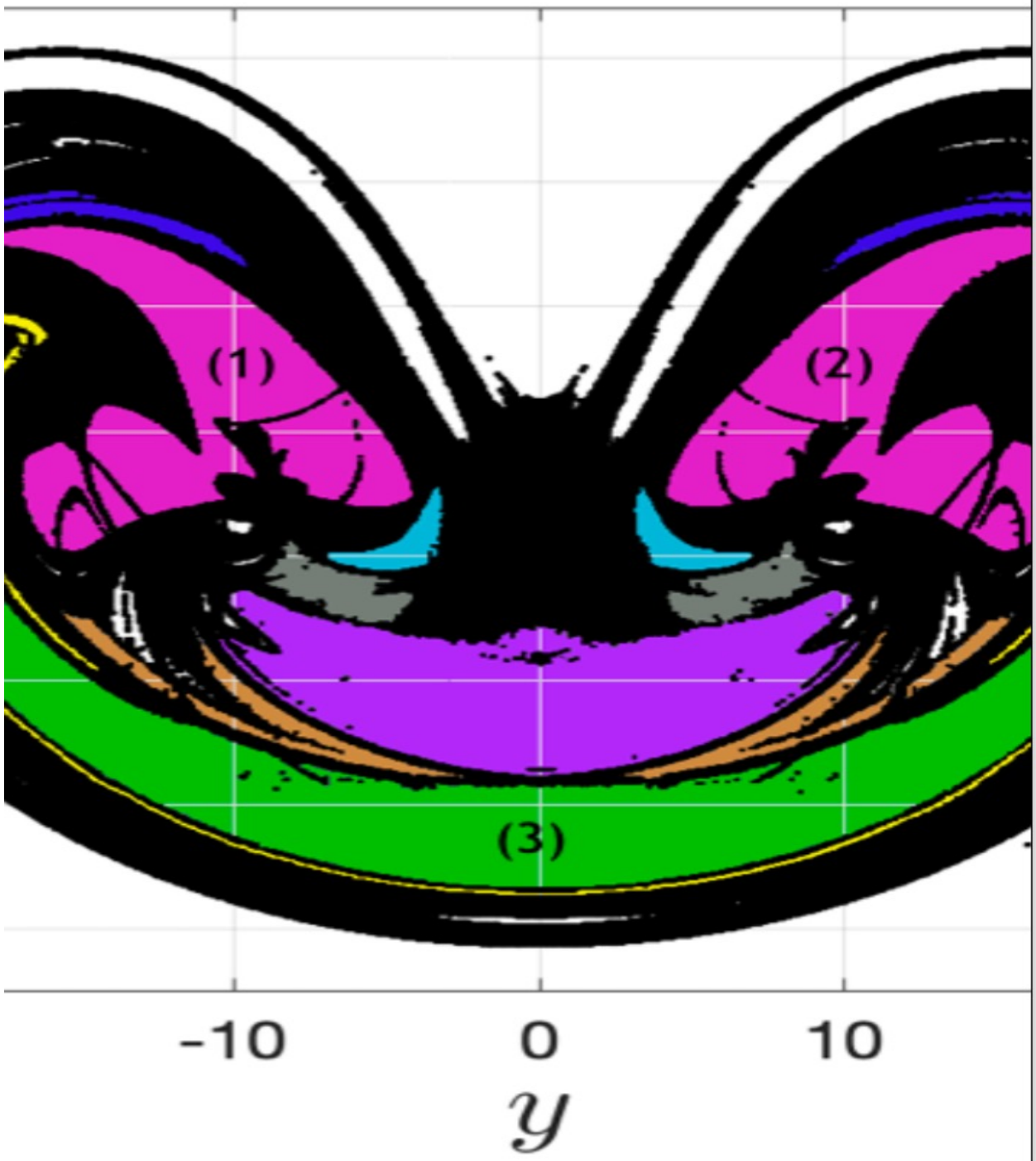


Challenges

Conclusions!

- Enlightening surprises arise if one extends the topological perspective to random dynamical systems.
- Noise modifies the behavior of a random attractor: at each instant in time, though, the random attractor's structure is still well represented by a branched manifold.
- Different "stages" in the "life" of a random attractor can be identified by monitoring the abrupt changes of the branched manifold's topology.

What about perspectives?



Perspectives

Nonlin. Processes Geophys., 27, 429–451, 2020
https://doi.org/10.5194/npg-27-429-2020
© Author(s) 2020. This work is distributed under
the Creative Commons Attribution 4.0 License.

Nonlinear Processes in Geophysics
Open Access
EGU

Review article: Hilbert problems for the climate sciences in the 21st century – 20 years later

Michael Ghil[1,2,♁]

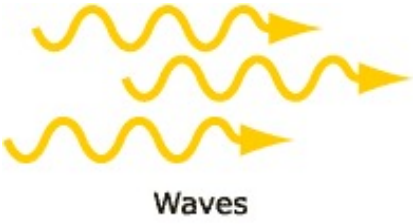
[1]Geosciences Department and Laboratoire de Météorologie Dynamique (CNRS and IPSL), Ecole normale supérieure, Paris Sciences et Lettres (PSL) University, Paris, France
[2]Atmospheric and Oceanic Sciences Department, University of California at Los Angeles, Los Angeles, California, USA
♁*Invited contribution by Michael Ghil, recipient of the EGU 2004 Lewis Fry Richardson Medal.*

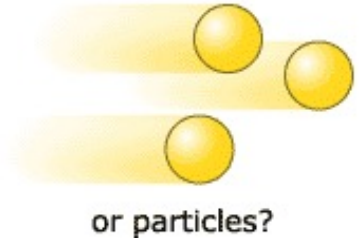
Correspondence: Michael Ghil (ghil@atmos.ucla.edu)

Received: 23 April 2020 – Discussion started: 11 May 2020
Revised: 18 July 2020 – Accepted: 30 July 2020 – Published: 17 September 2020

Perspectives

BraMAH could be used to clarify the so-called quandary of subseasonal to seasonal (S2S) prediction





"Waves" vs. "particles" in the atmosphere's phase space: A pathway to long-range forecasting?

Michael Ghil† and Andrew W. Robertson

Department of Atmospheric Sciences and Institute of Geophysics and Planetary Physics, University of California, Los Angeles, CA 90095-1567

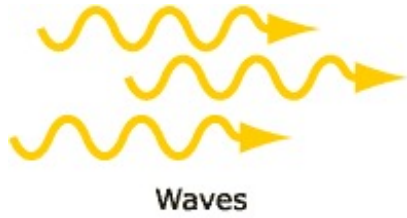
PNAS February 19, 2002, vol. 99, suppl. 1, 2493–2500

Low-frequency variability (LFV) is associated with the vast range of atmospheric processes occurring on a time scale that ranges from about a week to about a month, and no complete understanding of its nature has yet been attained.

Which type of phenomenon dominates LFV? Two apparently contradictory descriptions are possible.

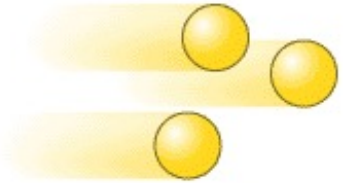
Oscillatory (wave-like) flow features or geographically fixed (particle-like) episodic flow features; e.g., blocking of the westerlies (particle-like) or intraseasonal oscillations (wave-like) with periodicities of 40–50 days.

Perspectives



How can these two apparently contradictory descriptions of LFV be reconciled?

- Blocking has been recently studied by Lucarini & Gritsun (2020) by using a new mathematical framework that extracts from the complex high-dimensional dynamics of a model its essential building blocks, given by truly nonlinear modes.
- They abandon the classic identification of weather regimes with fixed points, and directly consider the chaotic nature of the atmosphere, using the unstable periodic orbits (UPO) methodology of Gilmore.
- BraMAH could address the waves-v.-particles quandary more generally and efficiently, indicating how it might be affected by global change.



Climate Dynamics (2020) 54:575–598
https://doi.org/10.1007/s00382-019-05018-2

A new mathematical framework for atmospheric blocking events

Valerio Lucarini[1,2,3] · Andrey Gritsun[4]

Received: 21 May 2019 / Accepted: 14 October 2019 / Published online: 1 November 2019
© The Author(s) 2019

Abstract
We use a simple yet Earth-like hemispheric atmospheric model to propose a new framework for the mathematical properties of blocking events. Using finite-time Lyapunov exponents, we show that the occurrence of blockings is associated with conditions featuring anomalously high instability. Longer-lived blockings are very rare and have typically higher instability. In the case of Atlantic blockings, predictability is especially reduced at the onset and decay of the blocking event, while a relative increase of predictability is found in the mature phase. The opposite holds for Pacific blockings, for which predictability is lowest in the mature phase. Blockings are realised when the trajectory of the system is in the neighbourhood of a specific class of unstable periodic orbits (UPOs), natural modes of variability that cover the attractor the system. UPOs corresponding to blockings have, indeed, a higher degree of instability compared to UPOs associated with zonal flow. Our results provide a rigorous justification for the classical Markov chains-based analysis of transitions between weather regimes. The analysis of UPOs elucidates that the model features a very severe violation of hyperbolicity, due to the presence of a substantial variability in the number of unstable dimensions, which explains why atmospheric states can differ a lot in term of their predictability. Additionally, such a variability explains the need for performing data assimilation in a state space that includes not only the unstable and neutral subspaces, but also some stable modes. The lack of robustness associated with the violation of hyperbolicity might be a basic cause contributing to the difficulty in representing blockings in numerical models and in predicting how their statistics will change as a result of climate change. This corresponds to fundamental issues limiting our ability to construct very accurate numerical models of the atmosphere, in term of predictability of the both the first and of the second kind in the sense of Lorenz.

Perspectives

More broadly, BraMAH holds promise for the understanding of topological tipping points (TTPs).

- Tipping points (TPs) have been given a precise definition in the climate sciences as a generalization of classical bifurcations from autonomous systems to NDSs and RDSs, and they are actively being pursued.
- Topological tipping points (TTPs) can be understood as abrupt changes in the topological description of a random attractor's branched manifold.
- TTPs seem to be a further generalization of the concept that could help us apprehend sudden and drastic changes in time of model behaviour, as well as drastic changes due to mean forcing intensity.

International Journal of Bifurcation and Chaos, Vol. 21, No. 2 (2011) 399–423
© World Scientific Publishing Company
DOI: 10.1142/S0218127411028519

PREDICTING CLIMATE TIPPING AS A NOISY BIFURCATION: A REVIEW

J. MICHAEL T. THOMPSON
Department of Applied Mathematics & Theoretical Physics, Cambridge University, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WA, UK
School of Engineering (Sixth Century Professor), Aberdeen University, UK
jmtt@ucl.ac.uk

JAN SIEBER
Department of Mathematics, University of Portsmouth, Portsmouth, PO1 3HF, UK
jan.sieber@port.ac.uk

Perspectives

More broadly, BraMAH holds promise for the understanding of topological tipping points (TTPs).

- The gradual change of atmospheric concentrations of greenhouse gases and aerosols modifies global temperatures in a fairly smooth way, although it might also lead to dynamical tipping points.
- The intrinsic noise associated with cloud processes on small space and time scales affects the entire climate system through their interaction with dynamic and radiative processes on larger scales.
- This noise is considerably more complex than the one considered herein, and the climate system is infinitely more complex that we have studied.
- Still, given the huge recent increases in both computing power and data storage, it is worth exploring further whether such an interaction between large-scale dynamics and small-scale noise might not lead to striking surprises in the not-too-distant future.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A

Phil. Trans. R. Soc. A (2012) **370**, 1166–1184
doi:10.1098/rsta.2011.0306

Tipping points in open systems: bifurcation, noise-induced and rate-dependent examples in the climate system

By Peter Ashwin*, Sebastian Wieczorek, Renato Vitolo and Peter Cox

Mathematics Research Institute, University of Exeter, Exeter EX4 4QF, UK

Tipping points associated with bifurcations (B-tipping) or induced by noise (N-tipping) are recognized mechanisms that may potentially lead to sudden climate change. We focus here on a novel class of tipping points, where a sufficiently rapid change to an input or parameter of a system may cause the system to 'tip' or move away from a branch of attractors. Such rate-dependent tipping, or *R-tipping*, need not be associated with either bifurcations or noise. We present an example of all three types of tipping in a simple global energy balance model of the climate system, illustrating the possibility of dangerous rates of change even in the absence of noise and of bifurcations in the underlying quasi-static system.

Keywords: rate-dependent tipping point; bifurcation; climate system

To conclude…

Within the **deterministic** framework:

- The topology of branched manifolds provides information on the stretching, folding, tearing and squeezing mechanisms that act in phase space to shape a deterministic flow.

- These concepts and tools have been recently extended to address nonautonomous dynamical systems. They provide an unprecedented nonlinear perspective that can be fruitful for the climate sciences in several ways.

Within the **stochastic** framework:

- The use of branched manifolds can be extended beyond the deterministic framework by investigating the evolution in time of the topological structure of Random Dynamical Systems.

- The branched manifold is found to undergo abrupt changes for some values of the noise forcing, suggesting that the effects of noise on chaotic dynamics can be robustly addressed in this manner.

Thank you!

Perspectives on Climate Sciences
Nonlinear Processes Division

Questions?

Perspectives on Climate Sciences
Nonlinear Processes Division

Extra slides

In the same year, Otto E. Rössler came up with an attempt at constructing a hierarchy of chaos. In this hierarchy, branched manifolds are the finest description of the structure underlying complex dynamical systems.

The Chaotic Hierarchy

Otto E. Rössler

Z. Naturforsch. **38 a**, 788 – 801 (1983); received December 4, 1982

The complexity of dynamical behavior possible in nonlinear (for example, electronic) systems depends only on the number of state variables involved. Single-variable dissipative dynamical systems (like the single-transistor flip-flop) can only possess point attractors. Two-variable systems (like an LC-oscillator) can possess a one-dimensional attractor (limit cycle). Three-variable systems admit two even more complicated types of behavior: a toroidal attractor (of doughnut shape) and a chaotic attractor (which looks like an infinitely often folded sheet). The latter is easier to obtain. In four variables, we analogously have the hyper-toroidal and the hyper-chaotic attractor, respectively; and so forth. In every higher-dimensional case, all of the lower forms are also possible as well as "mixed cases" (like a combined hypertoroidal and chaotic motion, for example). Ten simple ordinary differential equations, most of them easy to implement electronically, are presented to illustrate the hierarchical tree. A second tree, in which one more dimension is needed for every type, is called the weak hierarchy because the chaotic regimes contained cannot be detected physically and numerically. The relationship between the two hierarchies is posed as an open question. It may be approached empirically – using electronic systems, for example.

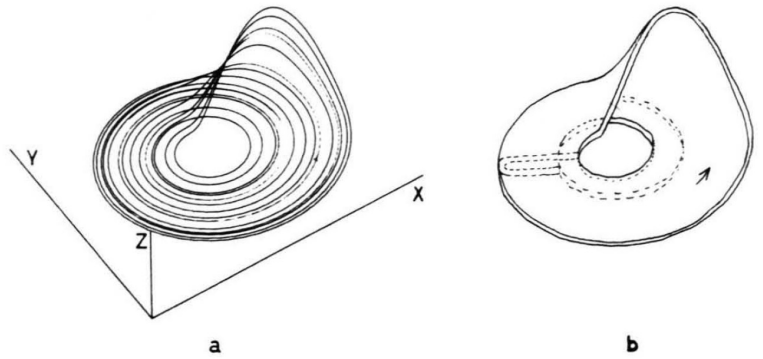
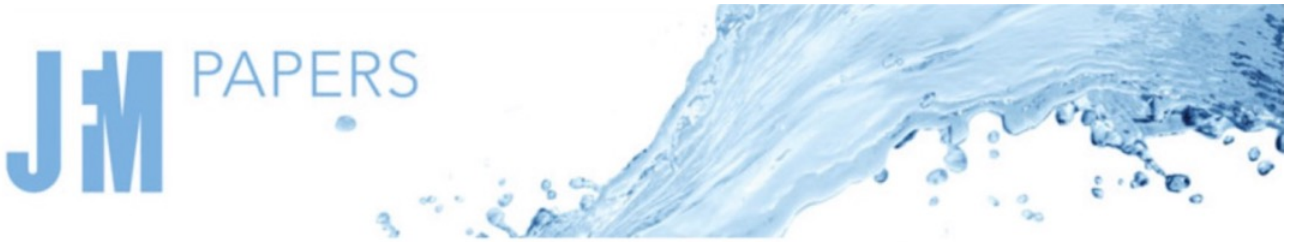


Fig. 2. The flow generated by (1) in 3-space. The same picture is shown twice (the second time schematically). Numerical simulation of (1), performed on an HP 9845B desk-top computer with peripherals using a standard Runge-Kutta-Merson integration routine. Initial conditions: $x(0) = 10$, $y(0) = 1$, $z(0) = 0$; $t_{end} = 116$. Axes: $-20 \ldots 20$ for x, $-20 \ldots 15$ for y, $0 \ldots 30$ for z.

Extra slides

- Lagrangian Analysis



Topological colouring of fluid particles unravels finite-time coherent sets

Gisela D. Charó[1,2,3,†], Guillermo Artana[3,4] and Denisse Sciamarella[2,5]

[1]CONICET – Universidad de Buenos Aires, Centro de Investigaciones del Mar y la Atmósfera (CIMA), Buenos Aires, Argentina

[2]CNRS – IRD – CONICET – UBA, Institut Franco-Argentin d'Études sur le Climat et ses Impacts (IRL 3351 IFAECI), Buenos Aires, Argentina

[3]CONICET – Consejo Nacional de Investigaciones Científicas y Técnicas, C1425FQD CABA, Argentina

[4]Laboratorio de Fluidodinámica, Facultad de Ingeniería, Universidad de Buenos Aires, C1063ACV CABA, Argentina

[5]CNRS – Centre National de la Recherche Scientifique, 75795 Paris, France

(Received 21 December 2020; revised 8 May 2021; accepted 18 June 2021)

In this work, BraMAH is applied to the numerically generated fluid particle behaviour in the wake past a rotary oscillating cylinder (ROC) at a low Reynolds number.

The results are compared with those obtained when applying BraMAH to another paradigmatic analytical model of Lagrangian motion: the Driven Double Gyre (DDG).

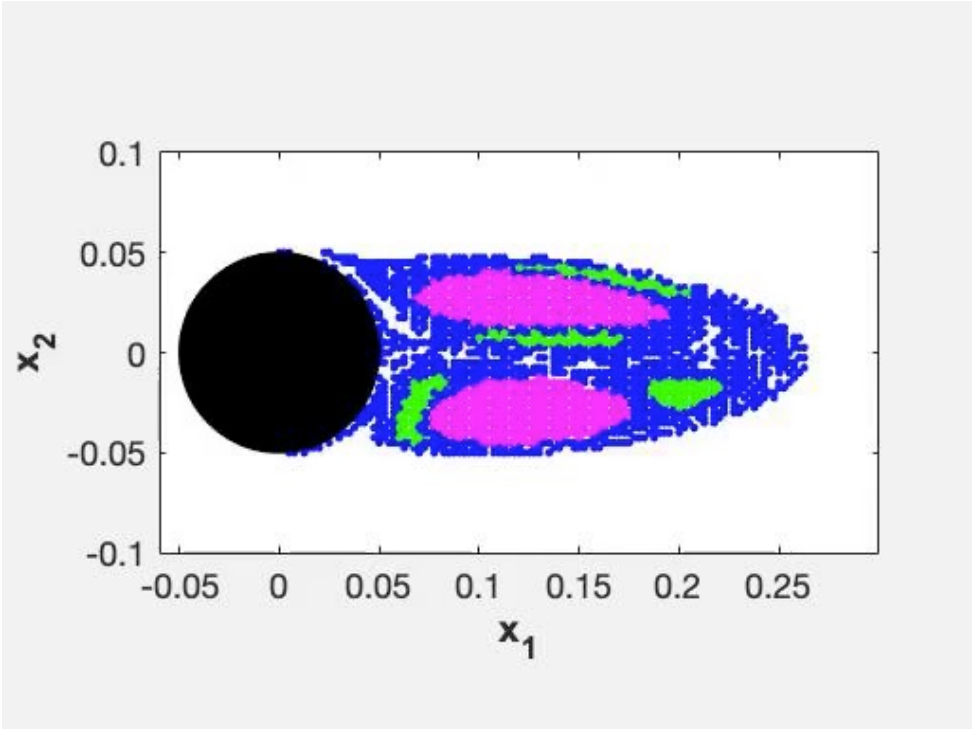
Both systems, the ROC and the DDG, present a background region with several groups of particles forming islands, which have been studied as resonance phenomena for Hamiltonian systems using a Lagrangian variational formulation.

The interesting point is that BraMAH can be used to relate particle behaviour in *a priori* unrelated flows, provided the analysis is performed in equivalently scaled time windows, and that the recirculation cells of the ROC can be seen as a 'materialisation' of the DDG.

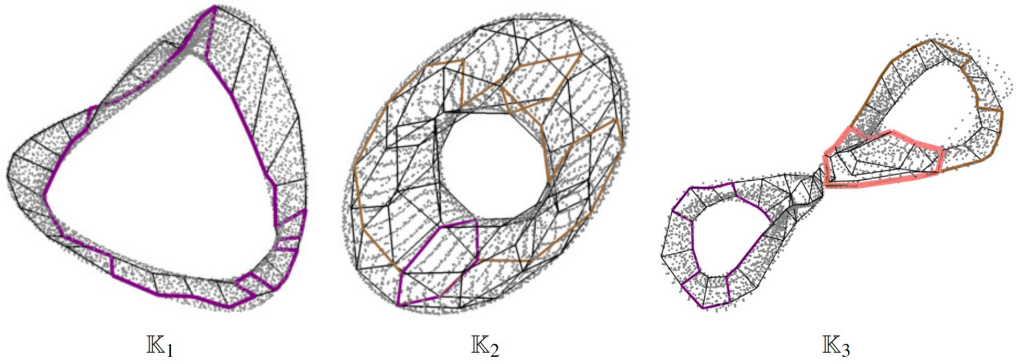
Extra slides

- Lagrangian Analysis

Topological colouring of 4 797 fluid particles using BraMAH for the ROC system.



Cell complex	Topological class	Colour
$\mathbb{K}_1$	I	Green
$\mathbb{K}_2$	II	Magenta
$\mathbb{K}_3$	III	Blue



Extra slides

- Lagrangian Analysis

The ROC system is found to share many features encountered in the DDG, with several groups of particles forming islands, which have been studied as resonance phenomena for Hamiltonian systems using a Lagrangian variational formulation.

Symplectic maps, variational principles, and transport

J. D. Meiss

Program in Applied Mathematics, University of Colorado, Boulder, Colorado 80309

Symplectic maps are the discrete-time analog of Hamiltonian motion. They arise in many applications including accelerator, chemical, condensed-matter, plasma, and fluid physics. Twist maps correspond to Hamiltonians for which the velocity is a monotonic function of the canonical momentum. Twist maps have a Lagrangian variational formulation. One-parameter families of twist maps typically exhibit the full range of possible dynamics–from simple or integrable motion to complex or chaotic motion. One class of orbits, the minimizing orbits, can be found throughout this transition; the properties of the minimizing orbits are discussed in detail. Among these orbits are the periodic and quasiperiodic orbits, which form a scaffold in the phase space and constrain the motion of the remaining orbits. The theory of transport deals with the motion of ensembles of trajectories. The variational principle provides an efficient technique for computing the flux escaping from regions bounded by partial barriers formed from minimizing orbits. Unsolved problems in the theory of transport include the explanation for algebraic tails in correlation functions, and its extension to maps of more than two dimensions.

Rev. Mod. Phys., Vol. 64, No. 3, July 1992

With these applications, theoretical progress is also being made, since BraMAH is successfully applied to non-dissipative (conservative) systems, characterizing both the quasi-periodic resonant islands and the so-called chaotic sea.

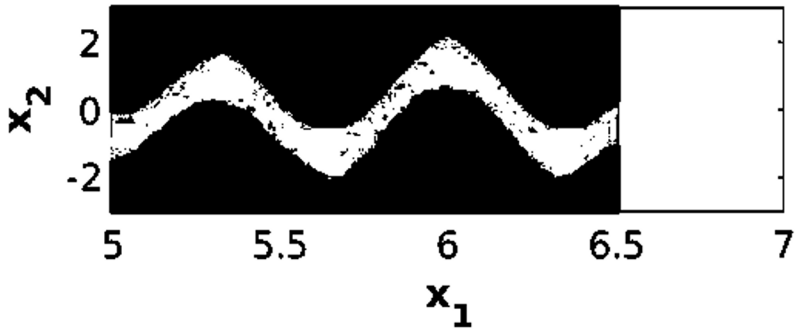
Even if the ROC system is an open flow, only particles in the recirculation cells are being studied.

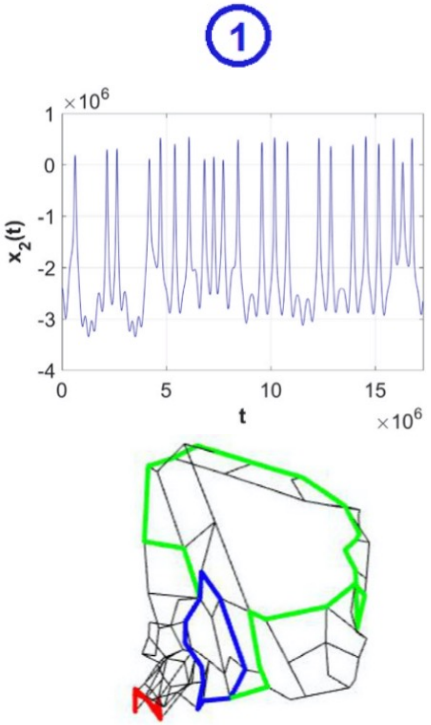
Can BraMAH be applied to an open flow?

Extra slides

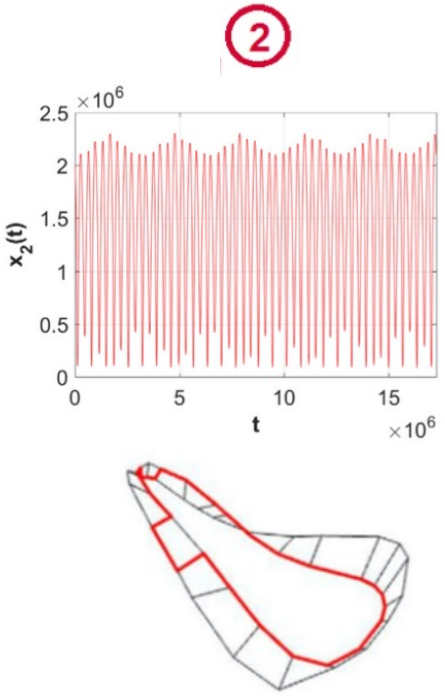
- Lagrangian Analysis

BraMAH applied to a set of 9 516 particles of the Bickley jet.

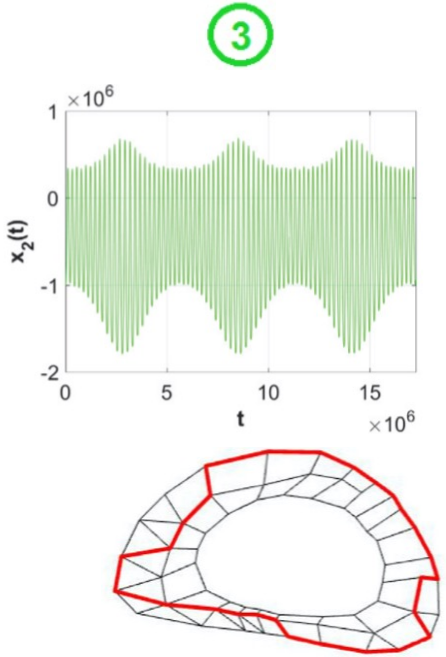




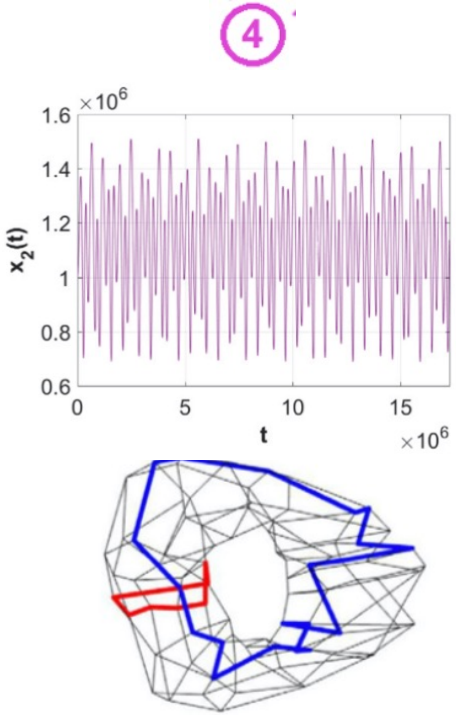
3-loop structure



Moebius strip



Standard strip

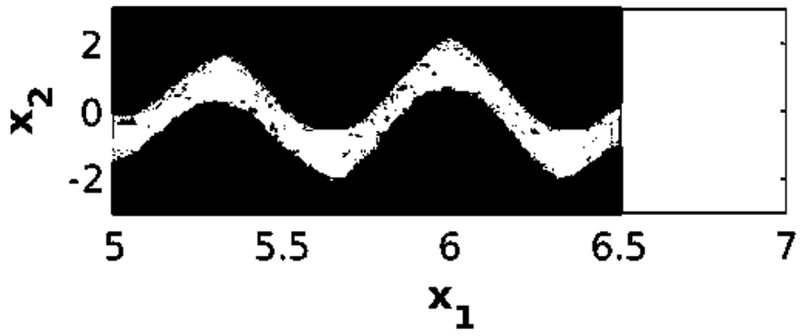


Torus

Extra slides

- Lagrangian Analysis

The Bickley jet is an analytic model of an open flow that captures the essential characteristics of the stratospheric polar vortex, in which a zonal jet acts as a transport barrier.

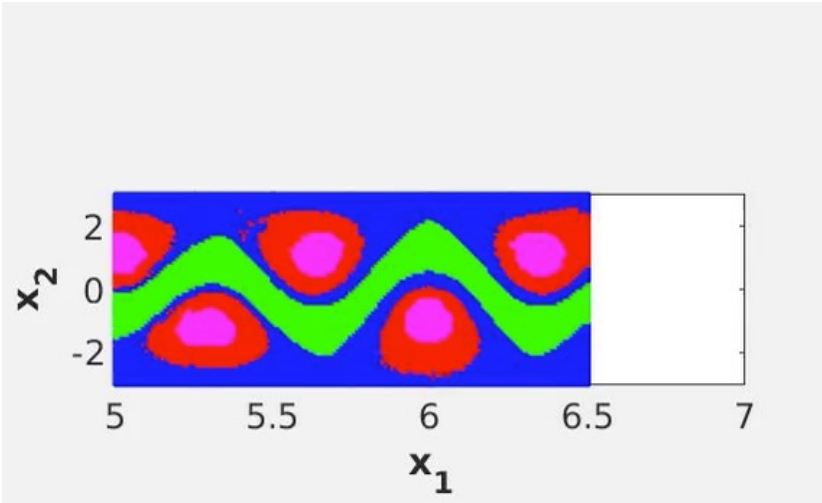


$$\psi(x_1, x_2, t) = \psi_0(x_2) + \psi_1(x_1, x_2, t),$$

$$\psi_0(x_2) = -U_0 L_0 \tanh\left(\frac{x_2}{L_0}\right)$$

$$\psi_1(x_1, x_2, t) = U_0 L_0 \operatorname{sech}^2\left(\frac{x_2}{L_0}\right) \sum_{n=1}^{3} \epsilon_n \cos\left(k_n(x_1 - c_n t)\right)$$

The stream function $\Psi(x, y, t)$ has a steady background flow $\Psi_0(y)$ and three traveling Rossby waves are superimposed.



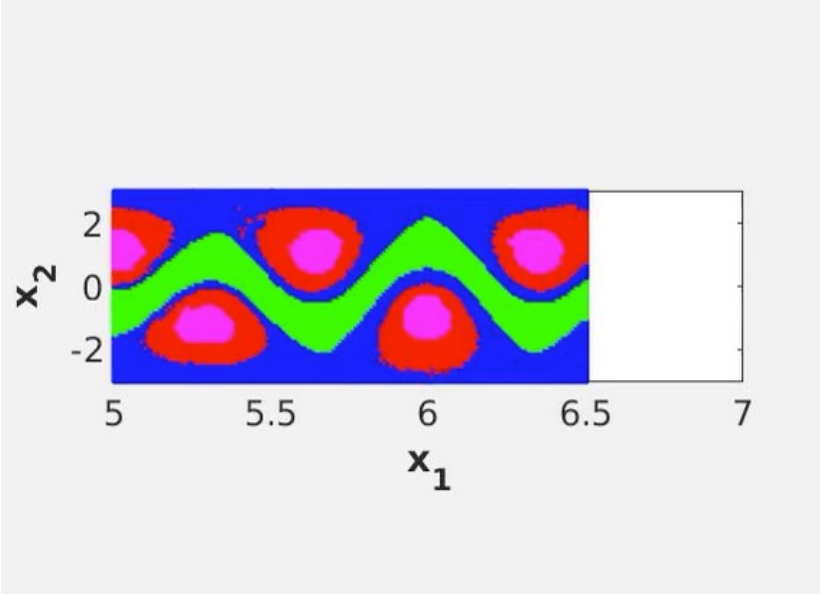
Topological colouring of a collection of 9 516 particles of the Bickley jet.

(1) 3-loop structure → Blue
(2) Moebius strip → Red
(3) Standard strip → Green
(4) Torus → Magenta

Extra slides

- Lagrangian Analysis

BraMAH applied to a collection of 9 516 particles of the Bickley jet.



(1) 3-loop structure → Blue
(2) Moebius strip → Red
(3) Standard strip → Green
(4) Torus → Magenta

At each position of the grid, the colour/label corresponding to the topological class of a particle that is passing through that position at an instant t for a time series defined for instance in $[t\ ,\ t + T_w]$ (or $[t - T_w\ ,\ t]$).

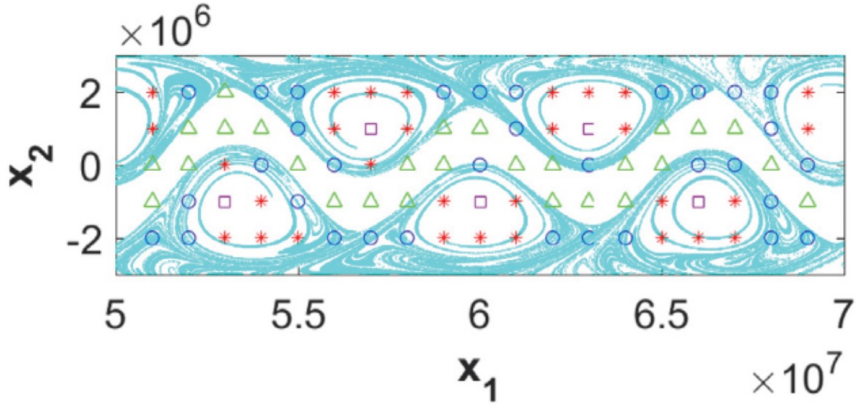


Fig. 11. Streakline associated to the Bickley Jet flow at $t_s = 40$ days with injection locations at $\mathbf{p} = (x_1, x_2)\ /\ x_1 = 1 \times 10^6 \wedge x_2 \in [-4 : 0.05 : -2] \times 10^6 \cup [2 : 0.05 : 4] \times 10^6$, juxtaposed with the topological grid where (a) $\mathbb{K}_1$ (△), (b) $\mathbb{K}_2$ (◯) , (c) $\mathbb{K}_3$ (□) and (d) $\mathbb{K}_4$ (∗). Length units are measured in meters.

Extra slides

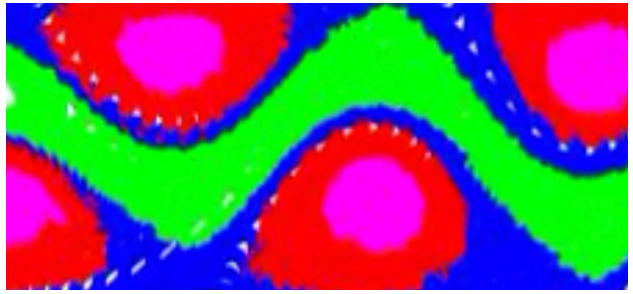
- Lagrangian Analysis

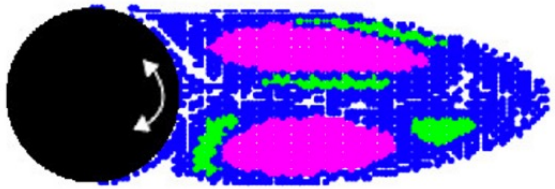
Classifying topologies (= classifying dynamics) can be used:

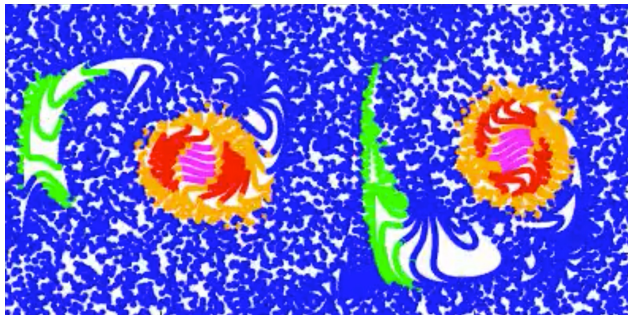
- to unravel finite-time coherent sets,
- to characterize particle dynamics within each coherent set,
- to compare non neighbouring regions of a flow,
- to compare the behaviour of particles in different flows.

Although the cases we discuss are 2-D incompressible flows, the technique is not restricted to them.

The prospects of this methodology are multiple and diverse, opening the possibility of addressing Lagrangian studies from an unprecedented perspective.







Extra slides

Several research groups carried out an important extension of the dynamical systems and model hierarchy framework of Ghil (2001) during the past 2 decades, from deterministically autonomous to nonautononomous and random dynamical systems (NDSs and RDSs: e.g., Ghil et al., 2008; Chekroun et al., 2011; Bódai & Tél, 2012; Ghil & Lucarini, 2020; Tél et al., 2020).

This framework allows one to deal, in a self-consistent way, with the increasing role of time-dependent forcing applied to the Earth system by humanity and by natural processes, such as solar variability and volcanic eruptions.

Among the 10 problems proposed in Ghil (2001), the first two were:

1. What is the coarse-grained structure of low-frequency atmospheric variability, and what is the connection between its episodic and oscillatory description?

2. What can we predict beyond 1 week, for how long, and by what methods?

Nonlinear Processes in Geophysics (2001) 8: 211–222

Nonlinear Processes in Geophysics
©European Geophysical Society 2001

Hilbert problems for the geosciences in the 21st century

M. Ghil

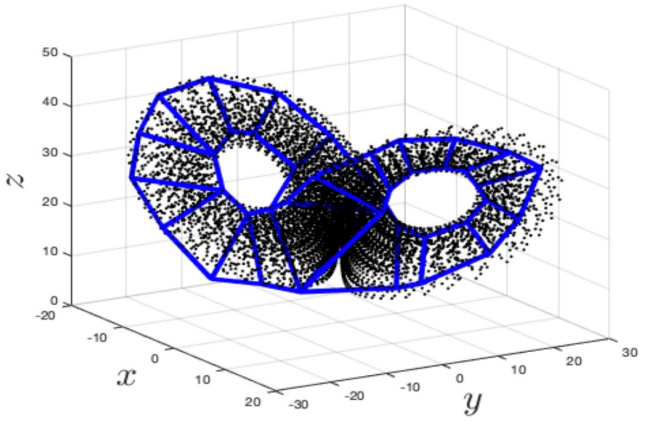
Dept. of Atmospheric Sciences and Institute of Geophysics and Planetary Physics, University of California, Los Angeles, CA 90095-1565, USA

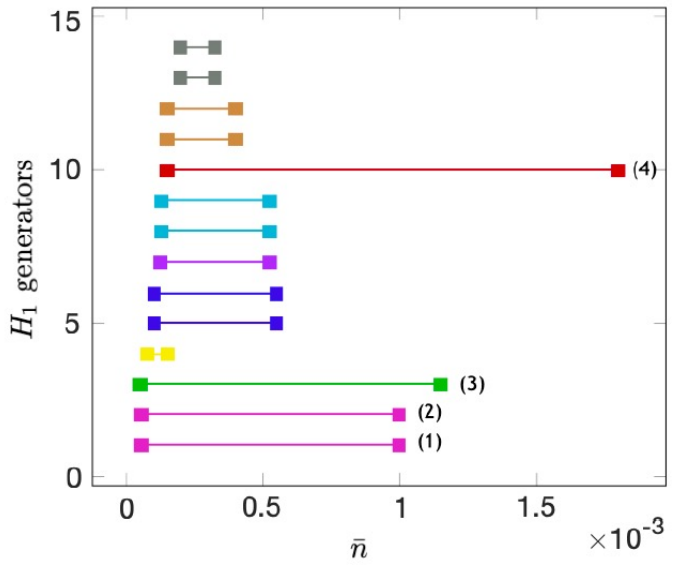
Received: 7 February 2001 – Accepted: 30 April 2001

Extra slides

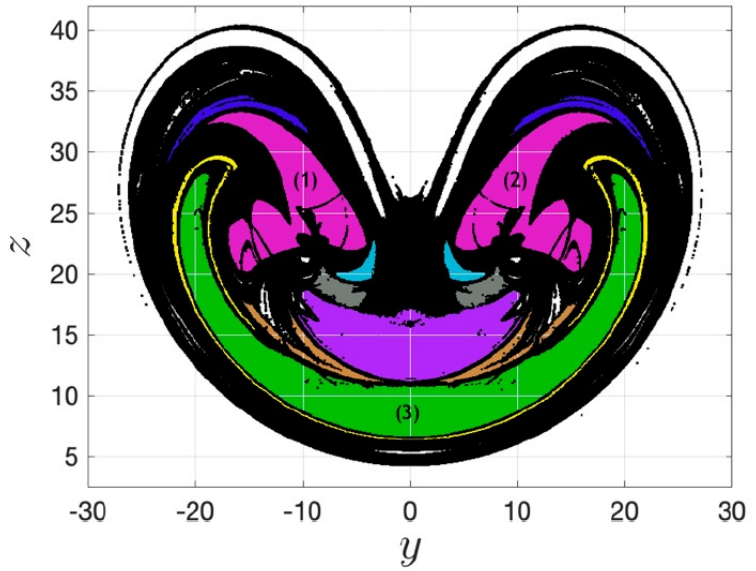
The point cloud density in the deterministic Lorenz attractor is quite homogeneous, but LORA is quite inhomogeneous in this respect.

We are interested in LORA's coarse-grained topology, so we will construct BraMAH complexes with the most populated regions of the point cloud.

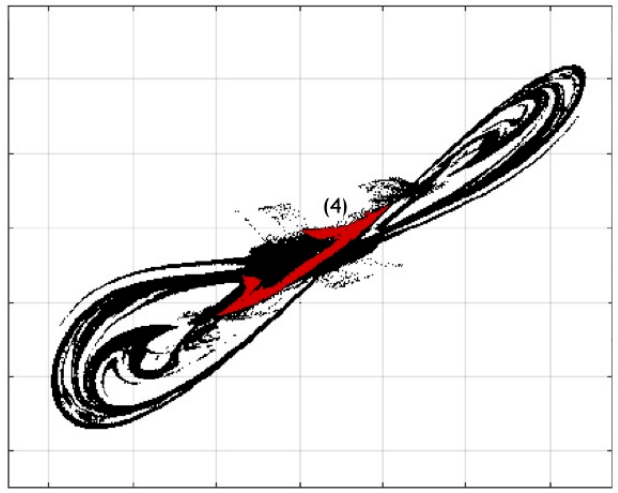




(a)

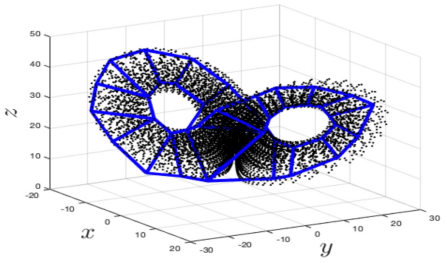


(b)



(c)

Extra slides



- At *t=40.27*, there are four *holes* that perdure as the value of the threshold $\bar{n}$ increases.
- In (b) and (c) we use a sieve of $\bar{n}$ = 2x10^{-4} for the estimated sample measure $\hat{\mu}_t$.
- We construct the BraMAH complexes with the most populated regions of the point cloud.

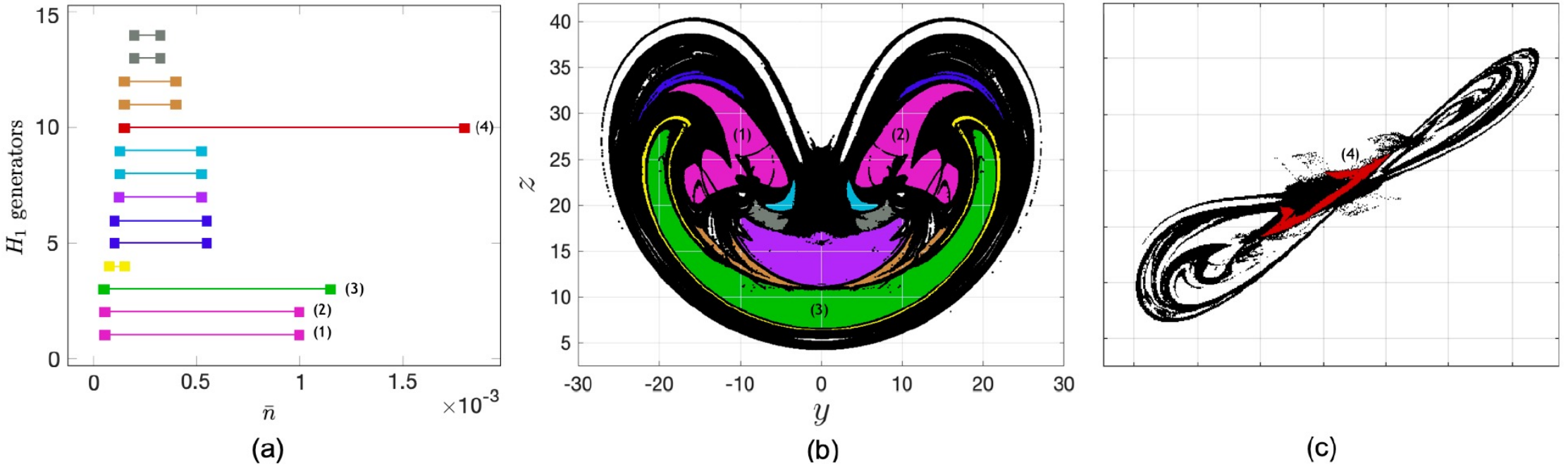


FIG. 2. LORA snapshot at $t = 40.27$, for $\sigma = 0.3$. (a) perdurance of the 1-holes as the density threshold $\bar{n}$ is increased; the most perdurant holes are labeled with numbers (1)–(4); (b) (y, z) projection of the sieved point cloud ($\bar{n} = 2 \times 10^{-4}$) with 13 colored 1-holes; (c) projection onto the plane $-2.25x - 20y + 6z = 0$ of the sieved point cloud with the 1-hole labeled (4) in red.