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The bang-bang property in some parabolic bilinear optimal control problems *via* two-scale asymptotic expansions

Idriss Mazari

November 8, 2021

Abstract

We investigate the bang-bang property for fairly general classes of $L^\infty - L^1$ constrained bilinear optimal control problems in two cases: that of the one-dimensional torus, in which case we consider parabolic equations, and that of general d dimensional domains for time-discrete parabolic models. Such a study is motivated by several applications in applied mathematics, most importantly in the study of reaction-diffusion models. The main equation in the one-dimensional case writes $\partial_t u_m - \Delta u_m = m u_m + f(t, x, u_m)$, where $m = m(x)$ is the control, which must satisfy some L^∞ bounds ($0 \leq m \leq 1$ a.e.) and an L^1 constraint ($\int m = m_0$ is fixed), and where f is a non-linearity that must only satisfy that any solution of this equation is positive at any given time. The time-discrete models are simply time-discretisations of such equations. The functionals we seek to optimise are rather general; in the case of the torus, they write $\mathcal{J}(m) = \iint_{(0,T) \times \mathbb{T}} j_1(t, x, u_m) + \int_{\mathbb{T}} j_2(x, u_m(T, \cdot))$. Roughly speaking we prove in this article that, if j_1 and j_2 are increasing, then any maximiser m^* of \mathcal{J} is bang-bang in the sense that it writes $m^* = \mathbb{1}_E$ for some subset E of the torus. It should be noted that such a result rewrites as an existence property for a shape optimisation problem. We prove an analogous result for time-discrete systems in any dimension. Our proofs rely on second order optimality conditions, combined with a fine study of two-scale asymptotic expansions. In the conclusion of this article, we offer several possible generalisations of our results to more involved situations (for instance for controls of the form $m\varphi(u_m)$), and we discuss the limits of our methods by explaining which difficulties may arise in other contexts.

Keywords: Bilinear optimal control, reaction-diffusion equations, shape optimisation, qualitative properties of optimisation problems, bang-bang property, shape optimisation.

AMS classification: 35K55, 35K57, 49J30, 49K20, 49N99, 49Q10.

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1 Introduction

1.1 Scope of the paper, informal presentation of our results

In this paper, we offer a theoretical analysis of an ubiquitous constrained optimal control problem, in which one aims at optimising a criteria by acting in a bilinear way on the state of the PDE. Prototypically, the model under consideration reads as follows: for a given non-linearity $f = f(t, x, u)$ and a control $m = m(t, x)$, we let u_m be the solution of

$$\partial_t u_m - \Delta u_m = m u_m + f(t, x, u_m)$$

with variables $x \in \Omega$ and $t \in [0, T]$ under certain boundary conditions. For a certain time horizon $T > 0$, we aim at optimising criteria of the form

$$\mathcal{J}(m) = \iint_{(0,T) \times \Omega} j_1(t, x, u_m) + \int_{\Omega} j_2(x, u_m(T, \cdot))$$

under some constraints on the control m . Throughout the paper, the constraints on m will be of L^1 and L^∞ type; in other words, one constraint takes the form

$$\forall t \in [0, T], \int_{\Omega} m(t, x) dx = V_0 \text{ fixed or } \int_{\Omega} m(x) dx = V_0 \text{ (if } m \text{ does not depend on time)}$$

while the other is of the type

$$\alpha \leq m \leq \beta \text{ a.e.}$$

In this type of setting, one of the salient qualitative features of optimisers is the *bang-bang property*. In other words, is it true that any maximiser writes $m^* = \alpha + (\beta - \alpha)\mathbf{1}_E$ for some measurable subset E of Ω ? This property is linked to (non-)existence results for shape optimisation problems. There were, in recent years, several fine qualitative studies of this property in the elliptic case or in the space-discretised case; we refer to Section 1.6. However, in the context of parabolic models and despite the current activity in the study of parabolic bilinear optimal control problems, this property does not seem to be reachable by the available techniques; we refer to section 1.4 and section 1.6.

In the first part of this paper, we prove that, under reasonable assumptions on the non-linearity f that ensure the well-posedness of the parabolic system, and on the cost functions j_1, j_2 (roughly speaking, they must both be non-decreasing, and one has to be increasing), the bang-bang property holds *if we assume that admissible controls are constant in time* and that the domain is one-dimensional. This is the main contribution of this article. It hinges on the methods of [39], coupled with two-scale asymptotic techniques previously used in [37] in the context of the optimal control of initial conditions in reaction-diffusion equations. The reason why we tackle the one-dimensional periodic case will be explained later on. It should be noted that we explain in the conclusion how we may cover, with the same type of arguments, higher-dimensional orthotopes. The main explanation behind having to work with time constant controls is a technical one; this allows to gain further regularity on the solutions of the parabolic PDE under consideration. For this reason, section 4.1.2 of the conclusion contains a discussion of possible generalisations and obstructions to generalisations; we explain, for instance, how to deal with the case of controls m writing $\sum_{i=1}^N \phi_i(t) m_i(x)$. As a first side comment, it should also be noted that our analysis cover the case of some *tracking-type functionals*. This is not the main topic, and we refer to remark 5. As a second side comment, our analysis can encompass more intricate interactions between the control and the state. For instance, we provide, in section 4.1.3 of the conclusion, a generalisation of our results to the case where the control and the state are coupled *via* a term of the form $m\varphi(u_m)$ for a large class of φ .

Our second contribution deals with a semi-discretised (in time) parabolic model, where the main system of equation is given by

$$\frac{w_{\mathbf{m},k+1} - w_{\mathbf{m},k}}{\delta t} - \Delta w_{\mathbf{m},k+1} = m_{k+1} w_{\mathbf{m},k+1} + f_{k+1}(x, w_{\mathbf{m},k+1}), k = 0, \dots, N-1$$

for some time step δt , where $\mathbf{m} = (m_1, \dots, m_N)$ and each m_i satisfies L^∞ and L^1 constraints. The optimisation problem is rather, in this case

$$J(\mathbf{m}) = \sum_{i=1}^N j_i(x, u_{\mathbf{m},i}).$$

For this semi-discrete parabolic model, we prove that, provided the functions j_i are increasing, any optimiser \mathbf{m} is of bang-bang type. Here, our analysis holds in any dimension, in any smooth bounded domain for Neumann or Robin boundary conditions. The reason we deal with this semi-discretised version is twofold: first, there has recently been some interest in the discretisation of bilinear optimal control problems [43]. Second, this allows us to give more perspective on the proof of the bang-bang property for the parabolic model in general domains.

Our introduction is divided accordingly: section 1.2 is devoted to the study of parabolic problems, while section 1.3 tackles time-discrete parabolic models.

1.2 Main model and result for parabolic problems

1.2.1 The parabolic equation

Admissible controls in parabolic models In the case of parabolic models, we are working in the (one-dimensional) torus \mathbb{T} . In section 4.2.1, we explain how our methods may extend to the case of higher dimensional tori.

Regarding the time regularity of admissible resources distribution, we shall make a strong assumption: the admissible controls are constant in time. The reason is that the method we introduce and develop hinges on fine regularity properties of solutions of the associated evolution equation that can not be obtained in the case where the control m also depends on time. We also refer to remark 8 for further comments.

In this setting, denoting by u_m the state of the equation and by m the control, the only type of interaction we are interested in is bilinear; in other words, the control appears in the model *via* the term mu_m (see Remark 7 and section 4.1.3 for considerations on the case of interactions of the form $m\varphi(u_m)$). In terms of constraints, we impose two on the controls, an L^∞ and an L^1 one. Each of these constraints has a natural interpretation in different fields of applications. In spatial ecology for instance, one may think of m as a resources distributions, in which case the L^∞ constraint simply models the fact that, at any given point, there can only be a maximum amount of resources available, while the L^1 constraint accounts for the limitation of the global quantity of resources involved. For the L^∞ constraints, without loss of generality (we also refer to remark 6), we shall consider controls satisfying

$$0 \leq m \leq 1 \text{ a.e. in } \mathbb{T}.$$

For the L^1 constraint, we fix a volume constraint $V_0 \in (0, \text{Vol}(\mathbb{T}))$, and we shall consider controls satisfying

$$\int_{\mathbb{T}} m = V_0.$$

This leads to considering the admissible class

$$\mathcal{M}(\mathbb{T}) := \left\{ m \in L^\infty(\mathbb{T}) : 0 \leq m \leq 1 \text{ a.e. in } \mathbb{T}, \int_{\mathbb{T}} m = V_0 \right\}. \quad (\text{Adm})$$

Of notable interest in $\mathcal{M}(\mathbb{T})$ are *bang-bang* functions; as they are the central theme of this paper we isolate their definition here.

Definition 1. *A function $m \in \mathcal{M}(\mathbb{T})$ is called bang-bang if there exists $E \subset \mathbb{T}$ such that $m = \mathbb{1}_E$.*

Nonlinearities under consideration Our choice of nonlinearity in the parabolic model also derives from considerations in mathematical biology or chemistry. Namely, we want the solutions not only to exist but to be uniformly bounded (in time) in the L^∞ norm, as well as to enjoy a strong maximum property (in the sense that, starting from a non-zero initial condition, the solution is positive at any arbitrary positive time). The latter is not only important from a modelling point of view but also in the course of the proof, as it influences the monotonicity of the functional under consideration.

As the right hand side of the reaction-diffusion equation writes $mu + f(t, x, u)$ we shall make the following assumptions that guarantee the well-posedness of the ensuing system:

f is \mathcal{C}^1 in time, L^∞ in x , and \mathcal{C}^2 in u ,

$$\text{and, for any } K \in \mathbb{R}, \sup_{x \in \mathbb{T}, u \in [0, K], t \in [0, K]} \left(\left| \frac{\partial f}{\partial t} \right| + \left| \frac{\partial f}{\partial u} \right| + \left| \frac{\partial^2 f}{\partial u^2} \right| \right) < \infty. \quad (\mathbf{H}_1)$$

Assumption (\mathbf{H}_1) serves to derive the proper regularity of the solutions of the equation. The next assumption is used to obtain upper and lower bounds on the solution:

$$f(\cdot, \cdot, 0) \geq 0, f(\cdot, \cdot, 0) \in L^\infty((0, T) \times \mathbb{T}) \text{ and}$$

there exists $\kappa > 0$ such that for any $u \geq \kappa$, for any $t \in \mathbb{R}_+$, for a.e. $x \in \mathbb{T}$, $f(t, x, u) \leq -u$. (\mathbf{H}_2)

In the first condition, if we had $f(\cdot, \cdot, 0) = 0$ this would simply model that when no individuals are present no reaction is happening. Assuming the general inequality allows to consider non-negative source terms (*i.e.* one may take $f(t, x, u) = ug(t, x, u) + y(t, x)$ for a certain g and a non-negative source term y). It should be noted that, had we taken $-\kappa_0 \leq m \leq \kappa_1$ as L^∞ constraints, the final inequality in (\mathbf{H}_2) would rewrite $f(t, x, u) \leq -\sup\{|\kappa_0|, |\kappa_1|\}u$.

Finally the last assumption is seemingly the most restrictive one, but we explain, in Remark 2, why it is not problematic for the type of problems we have in mind.

f is, uniformly in $(t, x) \in (0, T) \times \mathbb{T}$, uniformly Lipschitz in $u \in \mathbb{R}$: there exists A such that,

$$\text{for any } (t, x) \in (0, T) \times \mathbb{T}, \text{ for any } u, u' \in \mathbb{R}, |f(t, x, u) - f(t, x, u')| \leq A|u - u'|. \quad (\mathbf{H}_3)$$

Remark 2 (Comment on (\mathbf{H}_3)). (\mathbf{H}_3) may seem restrictive, as the typical monostable logistic diffusive equation would involve the non-linearity $f(u) = -u^2$, which grossly violates the Lipschitz condition of (\mathbf{H}_3) . However, assumption (\mathbf{H}_2) ensures that, if we start from a positive bounded initial condition u^0 , then the solution remains positive and bounded uniformly in time by $\max(\|u^0\|_{L^\infty}, \kappa)$ (see lemma 13) so that it suffices to extend $f(t, x, \cdot)$ outside $(0, \max(\|u^0\|_{L^\infty}, \kappa))$ to a globally uniformly Lipschitz function on \mathbb{R} .

Initial condition We simply take an initial condition independent of m , say u^0 , satisfying

$$\inf_{\mathbb{T}} u^0 > 0, u^0 \in \mathcal{C}^2(\mathbb{T}). \quad (1.1)$$

Parabolic model We define, for any $m \in \mathcal{M}(\mathbb{T})$, u_m as the unique solution of

$$\begin{cases} \frac{\partial u_m}{\partial t} - \Delta u_m = m u_m + f(t, x, u_m) & \text{in } (0, T) \times \mathbb{T}, \\ u_m(0, \cdot) = u^0 & \text{in } \mathbb{T}. \end{cases} \quad (1.2)$$

By [45, Theorem 5.2, Chapter 1] there exists a unique solution u_m of (1.2) (we also refer to lemma 16 for further regularity information about u_m).

Optimisation problem in the parabolic context: time-constant controls We consider fairly general functionals that we seek to optimise. To define this functional, we consider two functions j_1, j_2 , a time horizon $T > 0$ and we define

$$\mathcal{J} : \mathcal{M}(\mathbb{T}) \ni m \mapsto \iint_{(0, T) \times \mathbb{T}} j_1(t, x, u_m) + \int_{\mathbb{T}} j_2(x, u_m(T, \cdot)). \quad (1.3)$$

We mentioned earlier the crucial role of the monotonicity of the functional \mathcal{J} , which hinges on that of j_1 and j_2 ; we refer to section 1.6 for further comments. We thus assume that j_1, j_2 satisfy

j_1 and j_2 are non-decreasing in the second variable on \mathbb{R}_+ ,

j_1 is \mathcal{C}^1 in its two first variable and \mathcal{C}^2 in its third variable,

j_2 is \mathcal{C}^2 in its two variables,

$\forall (t, x) \in (0, T) \times \mathbb{T}, \forall K \in \mathbb{R}_+^*$,

$$\sup_{(t, x) \in (0, T) \times \mathbb{T}, u \in [0, K]} \sup_{\alpha=0,1,2} \left| \frac{\partial j_1}{\partial t}(t, x, u) \right| + \left| \frac{\partial^\alpha j_1}{\partial u^\alpha}(t, x, u) \right| + \left| \frac{\partial^\alpha j_2}{\partial u^\alpha}(x, u) \right| < \infty,$$

and either for any $(t, x) \in (0, T) \times \mathbb{T}$, $\partial_u j_1(t, x, \cdot) > 0$ in $(0; +\infty)$ or

for any $(t, x) \in (0, T) \times \mathbb{T}$, $\partial_u j_2(t, x, \cdot) > 0$ in $(0; +\infty)$, **(H_J)**

and we say (with a slight abuse of notation, identifying \mathcal{J} with (j_1, j_2)) that \mathcal{J} satisfies **(H_J)**.

In particular, we may choose

$$j_1(t, x, u) = u^\alpha \text{ and } j_2(x, u) = u^\beta$$

for $\alpha, \beta > 0$, or $j_1 = \varphi_1(u), j_2 = \varphi_2(u)$ with φ_1, φ_2 smooth and non-decreasing and at least one with positive derivative on \mathbb{R}_+^* , or $j_1(u) = \psi(x)u, j_2(u) = \psi_2(x)u$ with $\psi_1, \psi_2 \geq 0$ and $\sup(\inf |\psi_1|, \inf |\psi_2|) > 0$. The parabolic optimisation problem reads

$$\boxed{\max_{m \in \mathcal{M}(\mathbb{T})} \mathcal{J}(m)} \quad (\mathbf{P}_{\text{parab}})$$

Remark 3 (Existence of maximisers). *The existence of a solution of the variational problems **(P_{parab})** is a standard consequence of the direct method in the calculus of variations.*

1.2.2 Main result for the parabolic problem

We state our main result:

Theorem I. *Assume f satisfies **(H₁)**-**(H₂)**-**(H₃)**. Assume \mathcal{J} satisfies **(H_J)**. Any solution m^* of **(P_{parab})** is bang-bang: there exists $E \subset \mathbb{T}$ such that*

$$m^* = \mathbf{1}_E.$$

The proof of this theorem is the core of this paper, and takes up the entire section 2.

An application to a spatial ecology problem We can apply theorem I to the following spatial ecology problem. Consider, for any $m \in \mathcal{M}(\mathbb{T})$, the logistic-diffusive equation

$$\begin{cases} \frac{\partial \theta_m}{\partial t} - \Delta \theta_m = \theta_m (m - \theta_m) & \text{in } (0, T) \times \mathbb{T}, \\ \theta_m(0, \cdot) = \theta^0 \in \mathcal{C}^2(\mathbb{T}) & \text{in } \mathbb{T}, \\ \inf_{\mathbb{T}} \theta^0 > 0. \end{cases} \quad (1.4)$$

In this context, the control m accounts for a resources distribution available to a population, the density of which is the function θ_m . A relevant query is to solve the optimisation problem

$$\sup_{m \in \mathcal{M}(\mathbb{T})} \iint_{(0, T) \times \mathbb{T}} \theta_m(t, \cdot) + \int_{\mathbb{T}} \theta_m(T, \cdot), \quad (1.5)$$

for some time horizon $T > 0$. This problem is the parabolic counterpart of a related elliptic optimisation problem that was intensively studied in the past few years, see section 1.6 and [20, 23, 24, 28, 30, 31, 35, 39, 40, 43, 44]. In the elliptic case, the bang-bang property for optimisers was, in particular, a question that drew a lot of attention [35, 40, 44] and was only recently settled in [39]. In the parabolic setting, *i.e.* for problem (1.5), we refer, for instance, to the recent [10] for the derivation of optimality conditions, as well as for some numerical simulations for a close variant of (1.5). Then, as corollary of theorem I we obtain

Corollary 4. *Any solution m^* of (1.5) is of bang-bang type.*

The same conclusion holds for the two related problems

$$\sup_{m \in \mathcal{M}(\mathbb{T})} \int_{\mathbb{T}} \theta_m(T, \cdot), \quad \sup_{m \in \mathcal{M}(\mathbb{T})} \iint_{(0, T) \times \mathbb{T}} \theta_m.$$

It should be noted that, in [10], the case $m = m(t, x)$ is also considered. Our results do not hold in this case, as already underlined. We nonetheless refer to section 4.1.2 for generalisation of theorem I to some classes of time-dependent controls.

We conclude this section on parabolic models with some remarks .

Some remarks on theorem I

Remark 5 (Tracking type functionals). *Another class of functionals is of particular interest, that of tracking-type functionals. The goal is, in this case, to solve an optimisation problem of the form*

$$\inf_{m \in \mathcal{M}(\mathbb{T})} \int_{\Omega} |u_m(T, \cdot) - y_{\text{ref}}|^2, \quad (1.6)$$

where y_{ref} is a target state. We would like to underline that such problems fall in our framework provided y_{ref} is large enough to ensure that, for any $m \in \mathcal{M}(\mathbb{T})$ and any $T > 0$, $y_{\text{ref}} > u_m(T, \cdot)$. Indeed, as will be clear throughout the proof, the assumption that $\partial_u j_1(t, x, u) > 0$ on \mathbb{R}_+^* can be restricted to

$$\forall (t, x) \in (0, T) \times \mathbb{T}, \partial_u j_1(t, x, u) > 0 \text{ in } \left(0, \sup_{m \in \mathcal{M}(\mathbb{T}), T \in \mathbb{R}_+^*} \|u_m(T, \cdot)\|_{L^\infty} \right).$$

It is the minimum requirement we can make, as we know that the solutions of some tracking-type problems are not bang-bang.

Let us now consider the case of (1.4) with $\theta^0 < 1$ and consider the optimisation problem

$$\min_{m \in \mathcal{M}(\mathbb{T})} \iint_{(0,T) \times \mathbb{T}} |\theta_m - 1|^2.$$

This optimisation problem is equivalent to

$$\max_{m \in \mathcal{M}(\mathbb{T})} \left(- \iint_{(0,T) \times \mathbb{T}} |\theta_m - 1|^2 \right).$$

As, from the maximum principle, $\theta_m < 1$, the map $j_1(x, u) = -|1 - u|^2$ is increasing and has a positive derivative in $(0; 1)$, whence we can apply theorem I to conclude that any minimiser of the initial problem is of bang-bang type. This example immediately generalises to the case where the target $y_{\text{ref}} = 1$ is replaced with any target $y_{\text{ref}} \geq 1$ a.e.

Remark 6 (Regarding the L^∞ constraints on m). It should be noted that we may consider constraints of the form

$$\kappa_0 \leq m \leq \kappa_1$$

for two parameters $\kappa_0, \kappa_1 \in \mathbb{R}$ (possibly non-positive), without changing the conclusions of the theorems. The only difference would be that a bang-bang m would in that case be of the form $m = \kappa_0 + (\kappa_1 - \kappa_0)\mathbb{1}_E$. Indeed, the proof relies on a second-order derivative argument that still holds in this case, as the key point is that $u_m(t, \cdot)$ is positive whenever $t > 0$. For an interaction of the type mu_m with a sign-changing m , this is still the case.

Remark 7 (Regarding the bilinearity of the control). It is also worth noting that our method is flexible enough to cover the case of other interactions, of the form $m\varphi(u_m)$, for suitable nonlinearities φ . In the conclusion, see theorem IV, section 4.1.3, we give a sketch of proof for a version of theorem I for this type of interactions. The main condition on φ to ensure that the bang-bang property holds is that $\varphi'/\varphi > 0$ in \mathbb{R}_+^* , which is still sufficient to cover the case of the optimisation of the carrying capacity, where one works with the equation $\partial_t y_m - \Delta y_m = y_m(1 - my_m)$. This last example is motivated by [20].

Remark 8 (Regarding the time dependency of the control). Our method also allows to cover a form of time-dependency of the control. If we consider, rather than $m(x)u_m(t, x)$, an interaction of the form $u_m(t, x) \sum_{i=1}^N \phi_i(t)m_i(x)$, where the ϕ_i are bounded, smooth functions of time, then the bang-bang property holds. We refer to theorem III, section 4.1.2. In the conclusion, see section 4.3, we explain the difficulty in obtaining the same results for general time-dependent controls.

Remark 9 (Regarding the one-dimensional setting). The reason we work in the one-dimensional torus is a technical one; while the dimension of the torus is not problematic (see section 4.2.1 of the conclusion), the space periodicity enables us to carry out rigorous two-scale expansions in the following setting: a key part of the proof is the study of the solution \dot{u} of an equation of the form

$$\partial_t \dot{u} - \Delta \dot{u} = V(t, x) \sum_{k \gg 1} \alpha_k \cos(kx),$$

and we study \dot{u} by providing an explicit expansion as all the Fourier indexes $k \gg 1$ are very large. While, in more general domains, we may replace the right-hand side in the equation above with $\sum_{k \gg 1} \alpha_k \psi_k(x)$ for some eigenfunctions ψ_k of the laplacian operator, it is not yet clear how we may reach the necessary conclusion.

1.3 Main model and result for time-discrete problems

1.3.1 The time-discrete model

In this section, we work in any dimension $d \geq 1$. We consider a bounded, connected \mathcal{C}^2 domain $\Omega \subset \mathbb{R}^d$. Rather than periodicity condition, we shall impose Neumann boundary conditions (Robin boundary conditions would also be suitable for our analysis, see Remark 12). We fix an integer $N \in \mathbb{N} \setminus \{0\}$ as well as a time step $\delta t > 0$.

Constraints and admissible controls For the same reasons as in the parabolic model, we fix a volume constraint $V_0 \in (0, \text{Vol}(\Omega))$ and we define the set

$$\mathcal{M}(\Omega) := \left\{ m \in L^\infty(\Omega) : 0 \leq m \leq 1 \text{ a.e. in } \Omega, \int_\Omega m = V_0 \right\}.$$

The set of admissible controls for the time-discrete system is the set of N -tuples (m_1, \dots, m_N) where, for any $i \in \{1, \dots, N\}$, $m_i \in \mathcal{M}(\Omega)$.

Remark 10. *The fact that we assume all m_i ($i = 1, \dots, N$) belong to the same set $\mathcal{M}(\Omega)$ is merely for notational convenience. We may introduce, for any $i \in \{1, \dots, N\}$, the set $\mathcal{M}_i := \{m \in L^\infty(\Omega), \alpha_i \leq m \leq \beta_i \text{ a.e.}, \int_\Omega m = V_i\}$ and replace the conditions " $\forall i \in \{1, \dots, N\}, m_i \in \mathcal{M}(\mathbb{T})$ " with " $\forall i \in \{1, \dots, N\}, m_i \in \mathcal{M}_i(\mathbb{T})$ ". This would not change the conclusion.*

We define

$$\mathfrak{M}(\Omega) := \mathcal{M}(\Omega)^N.$$

Generically, the notation \mathbf{m} stands for such an N -tuple and, $\mathbf{m} \in \mathfrak{M}(\Omega)$ being given, m_k denotes the k -th component of \mathbf{m} .

We shall say that $\mathbf{m} \in \mathfrak{M}(\Omega)$ is *bang-bang* if, for any $i \in \{1, \dots, N\}$, m_i is a bang-bang function.

Non-linearities under consideration The non-linearities f_k ($k = 1, \dots, N$) are assumed to satisfy the same assumptions **(H₁)**-**(H₂)**-**(H₃)** as in the parabolic case (and with uniform constants in the sense that the κ of **(H₂)** and the A of **(H₃)** do not depend on the index k), with the convention that $f_k(t, x, u) := f_k(x, u)$. We say that $\{f_k\}_{k=1, \dots, N}$ satisfies **(H_{dis})**.

Initial condition We assume that we are given $w^0 \in L^\infty(\Omega)$ with $0 \leq w^0, w^0 \neq 0$. For the sake of simplicity (since this allows us to simply use κ instead of $\max\{\kappa, \|w^0\|_{L^\infty}\}$) we further assume that

$$w^0 \leq \kappa \text{ where } \kappa \text{ is given by } \mathbf{(H}_2\mathbf{)}.$$

Time-discrete model We define, for any $\mathbf{m} \in \mathfrak{M}(\Omega)$, $w_{\mathbf{m}} = \{w_{\mathbf{m},0} \dots, w_{\mathbf{m},N}\}$ as the solution of

$$\begin{aligned} w_{\mathbf{m},0} &= w^0 \text{ and for any } k \in \{0, \dots, N-1\}, \\ \begin{cases} \frac{w_{\mathbf{m},k+1} - w_{\mathbf{m},k}}{\delta t} - \Delta w_{\mathbf{m},k+1} = m_{k+1} w_{\mathbf{m},k+1} + f_{k+1}(x, w_{\mathbf{m},k+1}) & \text{in } \Omega, \\ \frac{\partial w_{\mathbf{m},k+1}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \end{aligned} \quad (1.7)$$

In lemma 22 we recall (with proofs in appendices) that, provided $\delta t > 0$ is small enough, this system is uniquely solvable. Lemma 22 also contains the relevant regularity properties.

Optimisation problem in the time-discrete context We consider N non-linearities $j_i = j_i(x, w)$ ($i = 1, \dots, N$) and we define the functional

$$J : \mathfrak{M} \ni \mathbf{m} \mapsto \sum_{i=1}^N j_i(x, w_{\mathbf{m}, i}).$$

We assume that

For any $i \in \{1, \dots, N\}$, j_i is L^∞ in its first variable and \mathcal{C}^2 in its second variable,

$$\forall x \in \mathbb{T}, \forall K \in \mathbb{R}_+^*, \sup_{i=1, \dots, N} \sup_{(t, x) \in (0, T) \times \mathbb{T}, w \in [0, K]} |\partial_w j_i(x, w)| < \infty, \text{ and} \\ \forall i \in \{1, \dots, N\}, \forall x \in \Omega, \partial_w j_i(x, \cdot) > 0 \text{ in } (0; +\infty). \quad (\mathbf{H}_J)$$

We say that J satisfies (\mathbf{H}_J) .

The optimisation problem reads

$$\boxed{\max_{\mathbf{m} \in \mathfrak{M}(\Omega)} J(\mathbf{m})} \quad (P_{\text{dis}}^N)$$

Remark 11 (Existence of maximisers). *The existence of a solution of the optimisation problem (P_{dis}^N) is an easy consequence of the direct method in the calculus of variations.*

1.3.2 Main result for time-discrete models

The main theorem of this section is the following:

Theorem II. *Assume $\{f_k\}_{k=1, \dots, N}$ satisfies $(\mathbf{H}_{\text{dis}})$. Assume \mathcal{J} satisfies (\mathbf{H}_J) . Any solution $\mathbf{m}^* = (m_1^*, \dots, m_N^*)$ of (P_{dis}^N) is bang-bang: there exist $E_1, \dots, E_N \subset \mathbb{T}$ such that*

$$\forall i \in \{1, \dots, N\}, m_i^* = \mathbf{1}_{E_i}.$$

Remark 12 (Regarding the boundary conditions). *The main reason we consider Neumann boundary conditions is to ensure the strict positivity of the adjoint function; in other words, the only time these boundary conditions come into play is when using a strong maximum principle to obtain a positive lower-bound on the state $w_{\mathbf{m}}$. Of course, since we also have such information for Robin boundary conditions, we claim that theorem II holds when the Neumann boundary condition is replaced by the Robin boundary condition*

$$\frac{\partial w_{\mathbf{m}, k+1}}{\partial \nu} + \beta_{k+1} w_{\mathbf{m}, k+1} = 0$$

for some $\beta_k \geq 0, \beta_k < \infty$ ($k = 1, \dots, N$).

1.4 Comments on the proofs of Theorems I and II

As mentioned, the starting point of our method is an idea we introduced in [39] in the elliptic context. Roughly speaking, consider an elliptic bilinear optimal control problem of the form $\sup_{m \in \mathcal{M}(\mathbb{T})} \int_{\mathbb{T}} j(z_m) (= J_{\text{ell}}(m))$ where $\mathcal{M}(\mathbb{T})$ is defined in (Adm), subject to $-\Delta z_m = m z_m + g(x, z_m)$. We proved that, under suitable assumptions on the non-linearity g and provided j was increasing, any optimiser m_{ell}^* was a bang-bang function. The crucial point was the following lower estimate of the second order derivative $\check{J}_{\text{ell}}(m)[h, h]$ of J_{ell} at an admissible control m in the

direction of a perturbation h : if j is increasing, if $z_m \geq 0$ is a stable steady-state for any m , then for any m there exist two positive constant $\alpha, \beta > 0$ such that for any admissible h at m ,

$$\ddot{J}_{\text{ell}}(m)[h, h] \geq \alpha \int_{\mathbb{T}} |\nabla \dot{z}_m|^2 - \beta \int_{\mathbb{T}} \dot{z}_m^2. \quad (1.8)$$

In the expression above, \dot{z}_m is the derivative of $m \mapsto z_m$ at m , in the direction h . It appears that \dot{z}_m solves a linear equation of the form $\mathcal{L}_m \dot{z}_m (:= -\Delta \dot{z}_m - V_m \dot{z}_m) = h \dot{z}_m$ for some bounded potential V_m . To obtain that any maximiser m_{ell}^* is bang-bang, we argued by contradiction: if m_{ell}^* is not bang-bang, then it suffices to build a perturbation h at m_{ell}^* supported in $\{0 < m_{\text{ell}}^* < 1\}$ such that $\ddot{J}_{\text{ell}}(m_{\text{ell}}^*)[h, h] > 0$ and, given the bound (1.8), it suffices to have $\int_{\mathbb{T}} |\nabla \dot{z}_{m_{\text{ell}}^*}|^2 \gg \int_{\mathbb{T}} \dot{z}_{m_{\text{ell}}^*}^2$. We then used the spectral eigenvalues and eigenfunctions $\{\lambda_k, \psi_k\}_{k \in \mathbb{N}}$ of the operator $\mathcal{L}_{m_{\text{ell}}^*}$ to build an h such that, in the eigenfunction basis, the only non-zero Fourier modes of $\dot{z}_{m_{\text{ell}}^*}$ are high order Fourier modes:

$$\dot{z}_m = \sum_{k \geq K} \alpha_k \psi_k \text{ for some large integer } K.$$

In the case of time-discrete models, this strategy proves efficient as well; however, in the case of parabolic equations, even obtaining such a lower order estimate on the second order derivative of the functional requires some time regularity on the controls m (hence the assumption that m does not depend on time in theorem I). However, even with such an estimate at hand, the problem is still not solved. Indeed, in the parabolic case, the structure of the equation on \dot{u}_m (the Gateaux-derivative of the map $m \mapsto u_m$ at m in an admissible direction h) is rather of the form $\partial_t \dot{u}_m - \Delta \dot{u}_m = V_m(t, x) \dot{u}_m + h u_m(t, x)$, with a time-varying potential V_m , and the lower estimate assumes the form

$$\ddot{J}(m)[h, h] \geq \alpha \iint_{(0, T) \times \mathbb{T}} |\nabla \dot{u}_m|^2 - \beta \iint_{(0, T) \times \mathbb{T}} \dot{u}_m^2 - \gamma \int_T \dot{u}_m(T, \cdot)^2.$$

We refer to estimate (2.13), proposition 15, for a precise statement. But even in the one-dimensional case, finding a perturbation h such that, for a fixed (and large) integer $K \in \mathbb{N}$, we have $\dot{u}_m = \sum_{k \geq K} \alpha_k \phi_k(t) \cos(kx)$ proves impossible because the potential V_m varies in time. Thus we have to resort to some two-scale asymptotic expansions in order to attain an approximation $\dot{u}_m \approx \sum_{k \geq K} \alpha_k \phi_k(t) \cos(kx)$ that is strong enough, see proposition 19. This part, which takes up most of the proof, is in part inspired by [37] and by seminal works on two-scale expansions [2], but requires some fine improvements to be better suited to our needs. The need for these improvements essentially comes from the fact that the potential V_m is merely $W^{2,p}$ in space, and not \mathcal{C}^2 , as is customary in such queries.

1.5 Relationship with some shape optimisation problems

There is another possible outlook on theorems I and II that relates their conclusions to (non-)existence results for PDE constrained shape optimisation problems. We only detail this for theorem I, as the case of the other results is exactly similar. For any subset $E \subset \mathbb{T}$, we may define the shape functional

$$\mathcal{F}(E) := \mathcal{J}(\mathbf{1}_E),$$

and investigate the shape optimisation problem

$$\sup_{E \subset \mathbb{T}, \text{Vol}(E) = V_0} \mathcal{F}(E). \quad (1.9)$$

For this type of optimisation problems, it is usually very difficult to obtain an existence property. The most general result is that of Buttazzo and DalMaso, in the seminal [11], which states

that, if the functional \mathcal{F} is increasing with respect to the set inclusion, and is moreover upper semi-continuous for the γ -convergence of sets, then an optimal set E^* exists. In theorem I, the monotonicity (which in turn hinges on that of (j_1, j_2)) plays a crucial role, and we prove that under assumption $(\mathbf{H}_{\mathcal{J}})$, \mathcal{J} is indeed increasing. However, \mathcal{F} is not continuous for the γ convergence of sets, which thus prevents using the result of [11]. This is well-known, and usually leads to considering the relaxation of the class of admissible sets $\mathcal{A} := \{\mathbb{1}_E, E \subset \mathbb{T}, \text{Vol}(E) = V_0\}$ in the weak $L^\infty - *$ topology; this relaxation exactly corresponds to (\mathbf{Adm}) , and the relaxed version of (1.9) is $(\mathbf{P}_{\text{parab}})$. In this way, theorem I states that every solution of $(\mathbf{P}_{\text{parab}})$ belongs to \mathcal{A} , and, consequently, that (1.9) has a solution.

This remark was also one of the motivation for the present work, which continues a series of papers devoted to establishing existence results for some shape optimisation problems, see [38, 39].

1.6 Bibliographical references and comments

Bilinear optimal control problems and the optimal control of semilinear parabolic models are present in a very large number of fields of applied mathematics. It is impossible to give an exhaustive list of contributions, but we single out a few that we think are closely related to our queries.

1.6.1 Elliptic bilinear optimal control problem

Spectral optimisation problems Let us begin with a spectral optimisation problem. In this setting, one aims at minimising the first eigenvalue $\lambda(m)$ of the operator $-\Delta - m$ in a smooth bounded domain $\Omega \subset \mathbb{R}^d$, endowed with certain boundary conditions, with respect to admissible controls m that satisfy L^∞ and L^1 constraints. The reason this problem is bilinear is that the state equation assumes the form

$$-\Delta z_m = m z_m + \lambda(m) z_m.$$

The study of the minimisation problem of $\lambda(m)$ with respect to m originates in spatial ecology consideration see [13, 14, 15, 16, 17, 18, 19, 25, 26, 27, 32, 36, 46] and the references therein. For such problems, the bang-bang property is usually immediate [26] and can be deduced from the concavity of the functional at hand or from classical tools such as the bathtub principle. Similarly, following [9], the geometric properties of optimisers have been thoroughly analysed, and are by now well understood; the main tool for this query is that of rearrangements, and a key point is that the functional is energetic. We refer to [27] for up to date results in this direction, as well as to the survey [34].

A non-energetic elliptic bilinear optimal control problem A problem which displays the rich behaviour of elliptic bilinear optimal control problems is that of the total population size in logistic-diffusive equations. In this setting, the PDE writes

$$-\mu \Delta \theta_m = \theta_m (m - \theta_m), \theta_m \geq 0, \theta_m \neq 0$$

with Neumann or Robin boundary conditions. The control m is assumed to satisfy L^∞ and L^1 constraints. The functional to optimise is $J : m \mapsto \int_\Omega \theta_m$. For modelling issues, we refer to [30] and the references therein. Obtaining the bang-bang property for its maximisers is surprisingly difficult. In [35, 44] this bang-bang property is proved under several restrictive assumptions. In [39], we introduced a new method to prove this property without these assumption; we refer to section 1.4 above to see why the method of [39] does not apply in the parabolic context. Regarding the geometric features of optimisers, it was proved in [23, 40] that the BV -norm of optimisers blows

up as $\mu \rightarrow 0^+$; in [39], this blow-up rate is quantified. It would be very interesting, in the context of parabolic models, to obtain such qualitative information about the geometry of maximisers.

A discretised bilinear optimal control problem We refer to [43] for the study of a bilinear optimal control problem for a stationary, space-discrete logistic-diffusive model. What is most surprising in [43] is that the authors achieve a complete description of maximisers in the low dispersal rate. It should be noted that [43] is also a motivation for us to undertake the study of time-discrete problems.

1.6.2 Optimal bilinear control of parabolic equations

Since we adopt, in the present paper, the point of view of optimal control, we merely indicate that there is a branch of research devoted to the question of *bilinear controllability* (*i.e.* is it possible to reach an exact state using a bilinear control ?); we refer the interested reader to [1, 7, 12] and the references therein. In the field of bilinear optimal control problems, let us first point to [21, 22] for the study of bilinear control problems in connection with chemotaxis or chemorepulsion; another very interesting example of such a problem is studied in [48]. In it, the authors study an optimal control problem for brain tumor growth. Although their bilinear control only depends on time (*i.e.* their function m satisfies $m = m(t)$, which is exactly the type of case not covered in the present contribution), some emphasis is put, through numerical simulations, on the bang-bang property.

A very relevant reference for the type of problems we are studying is the recent [10], in which the exact problem of optimisation of the total population size for parabolic logistic-diffusive equations is studied under the same type of constraints we have here. The optimality conditions are derived, and several numerical simulations are carried out.

2 Proofs of theorems I

We break this section down in several parts: first, we give a basic positivity estimate on u_m . Then we compute the first and second order Gateaux derivative of the criterion by the use of an adjoint state. Moreover, we give all the regularity information that are needed, and we use them to obtain a lower estimate on the second-order derivative. Finally, we provide a fine analysis of this second order derivative using two scale asymptotic expansions.

We first have the following basic estimate on u_m :

Lemma 13. *There holds*

$$\forall m \in \mathcal{M}(\mathbb{T}), 0 < \inf_{(0,T) \times \mathbb{T}} u_m \leq \|u_m\|_{L^\infty((0,T) \times \mathbb{T})} \leq \max\{\|u_0\|_{L^\infty}, \kappa\}.$$

As this lemma is a straightforward consequence of the maximum principle, we prove it in appendix A.1.

2.1 Computation of first and second order Gateaux derivatives *via* an adjoint state

In this section we analyse the Gateaux derivatives of the criterion and comment on its monotonicity. It is standard to see that the map

$$m \mapsto u_m$$

is twice Gateaux differentiable. For a given $m \in \mathcal{M}(\mathbb{T})$ and an admissible¹ perturbation h at m , we call $\dot{u}_m[h]$ (resp. $\ddot{u}_m[h]$) the first (resp. the second) order Gateaux derivative in the direction h . When no ambiguity is possible, we use the notation \dot{u}_m (resp. \ddot{u}_m) for the first (resp. second) order Gateaux derivative at m in the direction h . It is straightforward to see that \dot{u}_m solves

$$\begin{cases} \frac{\partial \dot{u}_m}{\partial t} - \Delta \dot{u}_m - \dot{u}_m \left(m + \frac{\partial f}{\partial u} \Big|_{u=u_m} \right) = hu_m & \text{in } (0, T) \times \mathbb{T}, \\ \dot{u}_m(0, \cdot) \equiv 0 & \text{in } \mathbb{T}. \end{cases} \quad (2.1)$$

Similarly we obtain, for the first-order Gateaux derivative of \mathcal{J} at m in the direction h the expression

$$\dot{\mathcal{J}}(m)[h] = \iint_{(0, T) \times \mathbb{T}} \dot{u}_m \frac{\partial j_1}{\partial u} \Big|_{u=u_m} + \int_{\mathbb{T}} \dot{u}_m \frac{\partial j_2}{\partial u} \Big|_{u=u_m(T, \cdot)}. \quad (2.2)$$

We define

$$V_m := m + \frac{\partial f}{\partial u} \Big|_{u=u_m}. \quad (2.3)$$

and introduce the function p_m as the solution of the backward parabolic equation

$$\begin{cases} \frac{\partial p_m}{\partial t} + \Delta p_m + V_m p_m = - \frac{\partial j_1}{\partial u} \Big|_{u=u_m} & \text{in } (0, T) \times \mathbb{T}, \\ p_m(T, \cdot) = \frac{\partial j_2}{\partial u} \Big|_{u=u_m} & \text{in } \mathbb{T}. \end{cases} \quad (2.4)$$

Multiplying (2.4) by \dot{u}_m and integrating by parts, we obtain

$$\dot{\mathcal{J}}(m)[h] = \int_{\mathbb{T}} \frac{\partial j_2}{\partial u} \Big|_{u=u_m(T, \cdot)} \dot{u}_m + \iint_{(0, T) \times \mathbb{T}} \frac{\partial j_1}{\partial u} \Big|_{u=u_m} \dot{u}_m = \iint_{(0, T) \times \mathbb{T}} hu_m p_m. \quad (2.5)$$

Let us now comment on the monotonicity of the functional, which shall play a crucial role in the forthcoming analysis. Of course, none of the computations above require that h be admissible, and we may take, for h , a non-negative function, as the constraints (and hence the admissibility of h) only play a role in the derivation of optimality conditions. By monotonicity we mean the following property:

$$\forall m \in \mathcal{M}(\mathbb{T}), \forall h \in L^\infty(\Omega), h \geq 0 \Rightarrow \dot{\mathcal{J}}(m)[h] \geq 0.$$

Given (2.5) and lemma 13, this monotonicity property actually holds if p_m itself is positive. This is where assumption **(H_J)** comes into play:

Lemma 14. *If \mathcal{J} satisfies **(H_J)** then $p_m \in W^{2,2}((0, T) \times \mathbb{T})$ and, for any $\varepsilon > 0$,*

$$\inf_{[0, T-\varepsilon] \times \mathbb{T}} p_m > 0.$$

Proof of lemma 14. If we set $q_m(t, \cdot) := p_m(T - t, \cdot)$ it appears that q_m solves

$$\begin{cases} \frac{\partial q_m}{\partial t} - \Delta q_m - V_m q_m = \frac{\partial j_1}{\partial u} \Big|_{u=u_m} & \text{in } (0, T) \times \mathbb{T}, \\ q_m(0, \cdot) = \frac{\partial j_2}{\partial u} \Big|_{u=u_m} & \text{in } \mathbb{T}. \end{cases}$$

If $\partial_u j_2 > 0$ then, as $\partial_u j_1 \geq 0$ the conclusion follows from the strong maximum principle. Likewise, if on the other hand we merely have $\partial_u j_2 \geq 0$ then, as $\partial_u j_1 > 0$ in this case, we obtain the conclusion by the maximum principle. \square

¹The wording ‘‘admissible perturbation’’ means that h belongs to the tangent cone to the set $\mathcal{B}(\partial\Omega)$ at β . It corresponds to the set of functions $h \in L^\infty(\Omega)$ such that, for any sequence of positive real numbers ε_n decreasing to 0, there exists a sequence of functions $h_n \in L^\infty(\Omega)$ converging in $L^2(\Omega)$ to h as $n \rightarrow +\infty$, and $\beta + \varepsilon_n h_n \in \mathcal{B}(\partial\Omega)$ for every $n \in \mathbb{N}$.

We now move on to the computation of the second order Gateaux derivative of the functional at hand. The second order derivative of $m \mapsto u_m$ in the direction h solves

$$\begin{cases} \frac{\partial \dot{u}_m}{\partial t} - \Delta \dot{u}_m - V_m \dot{u}_m = 2h\dot{u}_m + \frac{\partial^2 f}{\partial u^2} \Big|_{u=u_m} (\dot{u}_m)^2 & \text{in } (0, T) \times \mathbb{T}, \\ \dot{u}_m(0, \cdot) \equiv 0 & \text{in } \mathbb{T} \end{cases} \quad (2.6)$$

We also have, for the second order Gateaux derivative of J at m in the direction h , the following expression:

$$\begin{aligned} \ddot{J}(m)[h, h] &= \int_{\mathbb{T}} \dot{u}_m^2(T, \cdot) \frac{\partial^2 j_2}{\partial u^2} \Big|_{u=u_m(T, \cdot)} + \int_{\mathbb{T}} \dot{u}_m(T, \cdot) \frac{\partial j_2}{\partial u} \Big|_{u=u_m(T, \cdot)} \\ &\quad + \iint_{(0, T) \times \mathbb{T}} \dot{u}_m^2 \frac{\partial^2 j_1}{\partial u^2} \Big|_{u=u_m} + \iint_{(0, T) \times \mathbb{T}} \dot{u}_m \frac{\partial j_1}{\partial u} \Big|_{u=u_m(T, \cdot)}. \end{aligned} \quad (2.7)$$

We use the adjoint state p_m again: multiplying (2.4) by \dot{u}_m and integrating by parts, we obtain

$$\begin{aligned} 2 \iint_{(0, T) \times \mathbb{T}} h \dot{u}_m p_m + \iint_{(0, T) \times \mathbb{T}} \frac{\partial^2 f}{\partial u^2} \Big|_{u=u_m(T, \cdot)} (\dot{u}_m)^2 p_m \\ = \int_{\Omega} \ddot{u}_m \frac{\partial j_2}{\partial u} \Big|_{u=u_m(T, \cdot)} + \iint_{(0, T) \times \mathbb{T}} \ddot{u}_m \frac{\partial j_1}{\partial u} \Big|_{u=u_m} \end{aligned} \quad (2.8)$$

so that

$$\begin{aligned} \ddot{J}(m)[h, h] &= \int_{\mathbb{T}} \dot{u}_m^2(T, \cdot) \frac{\partial^2 j_2}{\partial u^2} \Big|_{u=u_m(T, \cdot)} \\ &\quad + \iint_{(0, T) \times \mathbb{T}} \dot{u}_m^2 \frac{\partial^2 j_1}{\partial u^2} \Big|_{u=u_m} + 2 \iint_{(0, T) \times \mathbb{T}} h \dot{u}_m p_m + \iint_{(0, T) \times \mathbb{T}} p_m \dot{u}_m^2 \frac{\partial^2 f}{\partial u^2} \Big|_{u=u_m}. \end{aligned} \quad (2.9)$$

Rearranging the terms, we get

$$\begin{aligned} \ddot{J}(m)[h, h] &= 2 \iint_{(0, T) \times \mathbb{T}} h \dot{u}_m p_m + \int_{\mathbb{T}} \dot{u}_m^2(T, \cdot) \frac{\partial^2 j_2}{\partial u^2} \Big|_{u=u_m(T, \cdot)} \\ &\quad + \iint_{(0, T) \times \mathbb{T}} \dot{u}_m^2 \left(\frac{\partial^2 j_1}{\partial u^2} \Big|_{u=u_m} + \frac{\partial^2 f}{\partial u^2} \Big|_{u=u_m} p_m \right). \end{aligned} \quad (2.10)$$

Let us focus on the term

$$\iint_{(0, T) \times \mathbb{T}} h \dot{u}_m p_m. \quad (2.11)$$

From (2.1) we rewrite

$$h = \frac{\frac{\partial \dot{u}_m}{\partial t} - \Delta \dot{u}_m - \dot{u}_m V_m}{u_m}.$$

Let us define

$$\Psi_m := \frac{p_m}{u_m}.$$

Plugging this expression in (2.11) we obtain

$$\iint_{(0, T) \times \mathbb{T}} h \dot{u}_m p_m = \iint_{(0, T) \times \mathbb{T}} \Psi_m \dot{u}_m \left(\frac{\partial \dot{u}_m}{\partial t} - \Delta \dot{u}_m - \dot{u}_m V_m \right)$$

$$\begin{aligned}
&= \frac{1}{2} \iint_{(0,T) \times \mathbb{T}} \Psi_m \frac{\partial \dot{u}_m^2}{\partial t} \\
&+ \iint_{(0,T) \times \mathbb{T}} \Psi_m |\nabla \dot{u}_m|^2 \\
&+ \iint_{(0,T) \times \mathbb{T}} \dot{u}_m \langle \nabla \Psi_m, \nabla \dot{u}_m \rangle \\
&- \iint_{(0,T) \times \mathbb{T}} \Psi_m V_m \dot{u}_m^2 \\
&= -\frac{1}{2} \iint_{(0,T) \times \mathbb{T}} \frac{\partial \Psi_m}{\partial t} \dot{u}_m^2 + \frac{1}{2} \int_{\mathbb{T}} \Psi_m(T, \cdot) \dot{u}_m^2 \\
&+ \iint_{(0,T) \times \mathbb{T}} \Psi_m |\nabla \dot{u}_m|^2 - \frac{1}{2} \iint_{(0,T) \times \mathbb{T}} \dot{u}_m^2 \Delta \Psi_m - \iint_{(0,T) \times \mathbb{T}} \Psi_m V_m \dot{u}_m^2 \\
&= \frac{1}{2} \int_{\mathbb{T}} \Psi_m(T, \cdot) \dot{u}_m^2 + \iint_{(0,T) \times \mathbb{T}} \Psi_m |\nabla \dot{u}_m|^2 \\
&- \frac{1}{2} \iint_{(0,T) \times \mathbb{T}} \frac{\partial \Psi_m}{\partial t} \dot{u}_m^2 - \frac{1}{2} \iint_{(0,T) \times \mathbb{T}} \dot{u}_m^2 \Delta \Psi_m - \iint_{(0,T) \times \mathbb{T}} \Psi_m V_m \dot{u}_m^2 \\
&= \frac{1}{2} \int_{\mathbb{T}} \Psi_m(T, \cdot) \dot{u}_m^2 + \iint_{(0,T) \times \mathbb{T}} \Psi_m |\nabla \dot{u}_m|^2 \\
&+ \iint_{(0,T) \times \mathbb{T}} \dot{u}_m^2 \left(-\frac{1}{2} \frac{\partial \Psi_m}{\partial t} - \frac{1}{2} \Delta \Psi_m - \Psi_m V_m \right).
\end{aligned}$$

With

$$\mathcal{L}_m := -\frac{1}{2} \frac{\partial \Psi_m}{\partial t} - \frac{1}{2} \Delta \Psi_m - \Psi_m V_m + \frac{\partial^2 j_1}{\partial u^2} \Big|_{u=u_m} + \frac{\partial^2 f}{\partial u^2} \Big|_{u=u_m} p_m,$$

the second order derivative writes

$$\begin{aligned}
\ddot{\mathcal{J}}(m)[h, h] &= \int_{\mathbb{T}} \dot{u}_m^2(T, \cdot) \frac{\partial^2 j_2}{\partial u^2} \Big|_{u=u_m(T, \cdot)} + \frac{1}{2} \int_{\mathbb{T}} \Psi_m(T, \cdot) \dot{u}_m^2 \\
&+ \iint_{(0,T) \times \mathbb{T}} \Psi_m |\nabla \dot{u}_m|^2 + \iint_{(0,T) \times \mathbb{T}} \mathcal{L}_m \dot{u}_m^2. \quad (2.12)
\end{aligned}$$

We analyse this expression further in the next section.

2.2 Lower estimate on the second order Gateaux derivative of \mathcal{J}

We now prove the following lower estimate on this second order Gateaux derivative:

Proposition 15. *Let $\varepsilon > 0$ be arbitrarily small. There exist three positive constants $\alpha = \alpha(\varepsilon), \beta, \gamma > 0$ such that, for any admissible perturbation h at m , there holds*

$$\ddot{\mathcal{J}}(m)[h, h] \geq \alpha \iint_{(0, T-\varepsilon) \times \mathbb{T}} |\nabla \dot{u}_m|^2 - \beta \iint_{(0, T) \times \mathbb{T}} \dot{u}_m^2 - \gamma \int_{\mathbb{T}} \dot{u}_m^2(T, \cdot). \quad (2.13)$$

Proving this proposition requires some additional regularity on u_m, p_m . This is where the regularity of m in time (here, m is constant in time) is crucial. We gather these regularity properties in the following proposition:

Proposition 16. For any $m \in \mathcal{M}(\mathbb{T})$, for any $p \in [1; +\infty)$, there exists a constant $\mathfrak{M}_{m,p}$ such that

$$\begin{aligned} \sup_{t \in [0, T]} \|\partial_t u_m(t, \cdot)\|_{L^p(\mathbb{T})} + \sup_{t \in [0, T]} \|u_m(t, \cdot)\|_{W^{2,p}(\mathbb{T})} \\ + \sup_{t \in [0, T]} \|\partial_t p_m(t, \cdot)\|_{L^p(\mathbb{T})} + \sup_{t \in [0, T]} \|p_m(t, \cdot)\|_{W^{2,p}(\mathbb{T})} \leq \mathfrak{M}_{m,p}. \end{aligned} \quad (2.14)$$

In particular, by Sobolev embeddings, there exists a constant \mathfrak{N} such that

$$\sup_{t \in (0, T)} \|u_m(t, \cdot)\|_{\mathcal{C}^1(\mathbb{T})}, \sup_{t \in (0, T)} \|p_m(t, \cdot)\|_{\mathcal{C}^1(\mathbb{T})} \leq \mathfrak{N}.$$

The proof of this proposition is standard in the regularity theory of parabolic equations; it can be derived from classical L^p estimates (see for instance [29, Theorem 7.32, pp. 182-183]) but the setting we are working in allows for a quicker proof, that we give in appendix A.2.

With these regularity estimates we can prove proposition 15

Proof of proposition 15. First off, from lemmas 13 and 14 we have that

$$\forall \varepsilon > 0, \quad \inf_{(0, T-\varepsilon] \times \mathbb{T}} \Psi_m > 0.$$

Hence there exists a constant $\alpha_0 = \alpha_0(\varepsilon) > 0$ such that

$$\iint_{(0, T) \times \mathbb{T}} \Psi_m |\nabla \dot{u}_m|^2 \geq \alpha_0 \iint_{(0, T-\varepsilon) \times \mathbb{T}} |\nabla \dot{u}_m|^2.$$

Second, since j_2 is \mathcal{C}^2 in u and u_m is bounded, there exists a constant $\gamma > 0$ such that

$$\int_{\mathbb{T}} \dot{u}_m^2(T, \cdot) \frac{\partial^2 j_2}{\partial u^2} \Big|_{u=u_m(T, \cdot)} \geq -\gamma \int_{\mathbb{T}} \dot{u}_m^2(T, \cdot).$$

Estimates on \mathcal{L}_m As for \mathcal{L}_m , we rewrite it

$$\mathcal{L}_m = -\frac{1}{2} \frac{\partial \Psi_m}{\partial t} - \frac{1}{2} \Delta \Psi_m + \mathcal{Y}_m \quad \text{with} \quad \mathcal{Y}_m := -\Psi_m V_m + \frac{\partial^2 j_1}{\partial u^2} \Big|_{u=u_m} + \frac{\partial^2 f}{\partial u^2} \Big|_{u=u_m} p_m.$$

Of course,

$$\mathcal{Y}_m \in L^\infty.$$

Let us focus on the term

$$-\frac{\partial \Psi_m}{\partial t} - \Delta \Psi_m.$$

We compute

$$\frac{\partial \Psi_m}{\partial t} = \frac{1}{u_m} \frac{\partial p_m}{\partial t} - \frac{p_m}{u_m^2} \frac{\partial u_m}{\partial t}, \quad (2.15)$$

and (since we are working in one-dimension, $\Delta = \partial_{xx}^2$),

$$\frac{\partial^2 \Psi_m}{\partial x^2} = -2 \frac{1}{u_m^2} \frac{\partial p_m}{\partial x} \frac{\partial u_m}{\partial x} + \frac{1}{u_m} \frac{\partial^2 p_m}{\partial x^2} - \frac{p_m}{u_m^2} \frac{\partial^2 u_m}{\partial x^2} + 2 \frac{p_m}{u_m^3} \left(\frac{\partial u_m}{\partial x} \right)^2. \quad (2.16)$$

Setting

$$\mathcal{X}_m := -2 \frac{1}{u_m^2} \frac{\partial p_m}{\partial x} \frac{\partial u_m}{\partial x} + 2 \frac{p_m}{u_m^3} \left(\frac{\partial u_m}{\partial x} \right)^2$$

we have

$$\frac{\partial^2 \Psi_m}{\partial x^2} = \frac{1}{u_m} \frac{\partial^2 p_m}{\partial x^2} - \frac{p_m}{u_m^2} \frac{\partial^2 u_m}{\partial x^2} + \mathcal{X}_m.$$

From Proposition 16 we know that

$$\mathcal{X}_m \in L^\infty((0, T) \times \mathbb{T}).$$

Hence,

$$\begin{aligned} \frac{\partial \Psi_m}{\partial t} + \frac{\partial^2 \Psi_m}{\partial x^2} &= \frac{1}{u_m} \frac{\partial p_m}{\partial t} - \frac{p_m}{u_m^2} \frac{\partial u_m}{\partial t} \\ &+ \frac{1}{u_m} \frac{\partial^2 p_m}{\partial x^2} - \frac{p_m}{u_m^2} \frac{\partial^2 u_m}{\partial x^2} + \mathcal{X}_m \\ &= \mathcal{X}_m \\ &+ \frac{1}{u_m} \left(\frac{\partial p_m}{\partial t} + \frac{\partial^2 p_m}{\partial x^2} \right) \\ &- \frac{p_m}{u_m^2} \left(\frac{\partial u_m}{\partial t} + \frac{\partial^2 u_m}{\partial x^2} \right) \\ &= \mathcal{X}_m + \frac{p_m}{u_m} \left(- \frac{\partial j_1}{\partial u} \Big|_{u=u_m} \right) \\ &- \frac{p_m}{u_m^2} \left(2 \frac{\partial^2 u_m}{\partial x^2} + m u_m + f(t, x, u_m) \right), \end{aligned}$$

whence, using the fact that $j_1 \in \mathcal{C}^1$ and proposition 16, the function $\tilde{\mathcal{Z}}_m$ satisfies

$$\forall p \in [1; +\infty), \sup_{t \in [0, T]} \|\tilde{\mathcal{Z}}_m(t, \cdot)\|_{L^p(\mathbb{T})} =: \mathfrak{M}(p) < \infty. \quad (2.17)$$

This allows to estimate the last term in the second-order Gateaux derivative: from the Sobolev embedding $W^{1,2}(\mathbb{T}) \hookrightarrow \mathcal{C}^0(\mathbb{T})$ with constant C_{sob} and the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \iint_{(0, T) \times \mathbb{T}} \tilde{\mathcal{Z}}_m \dot{u}_m^2 &\geq - \int_0^T \|\dot{u}_m(t, \cdot)\|_{W^{1,2}(\mathbb{T})} \cdot \|\tilde{\mathcal{Z}}_m(t, \cdot)\|_{L^2(\mathbb{T})} \cdot \|\dot{u}_m(t, \cdot)\|_{L^2(\mathbb{T})} dt \\ &= -\mathfrak{M}(2) C_{\text{sob}} \int_0^T \|\dot{u}_m(t, \cdot)\|_{W^{1,2}(\mathbb{T})} \cdot \|\dot{u}_m(t, \cdot)\|_{L^2(\mathbb{T})} dt \\ &= -\mathfrak{M}(2) C_{\text{sob}} \int_0^T \|\nabla \dot{u}_m(t, \cdot)\|_{L^2(\mathbb{T})} \cdot \|\dot{u}_m(t, \cdot)\|_{L^2(\mathbb{T})} dt \\ &- \mathfrak{M}(2) C_{\text{sob}} \iint_{(0, T) \times \mathbb{T}} \dot{u}_m^2. \end{aligned}$$

Remark 17. *It may be argued that here we already use the fact that we are working in the one-dimensional setting, when using the Sobolev embedding $W^{1,2} \hookrightarrow \mathcal{C}^0$. However, this can be very well extended to the higher dimensional setting, in which case, we would simply have an estimate of the form $\iint \tilde{\mathcal{Z}}_m \dot{u}_m^2 \geq -C_{\text{sob}} \int_0^T \|\dot{u}_m(t, \cdot)\|_{W^{1,2}} \cdot \|\tilde{\mathcal{Z}}_m(t, \cdot)\|_{L^p} \cdot \|\dot{u}_m(t, \cdot)\|_{L^2} dt$, where we would have used the three exponents' Hölder inequality with $1/p + 1/q + 1/2 = 1$, with C_{sob} the constant of the embedding $W^{1,2} \hookrightarrow L^q$.*

We obtain

$$\begin{aligned} \ddot{\mathcal{J}}(m)[h, h] \geq & \alpha \iint_{(0, T-\varepsilon) \times \mathbb{T}} |\nabla \dot{u}_m|^2 - \mathfrak{M}(2) C_{\text{sob}} \iint_{(0, T) \times \mathbb{T}} \dot{u}^2 \\ & - \mathfrak{M}(2) C_{\text{sob}} \int_0^T \|\nabla \dot{u}_m(t, \cdot)\|_{L^2(\mathbb{T})} \|\dot{u}_m(t, \cdot)\|_{L^2(\mathbb{T})} dt - \gamma \int_{\mathbb{T}} \dot{u}^2(T, \cdot). \end{aligned} \quad (2.18)$$

We perform one last step: from the arithmetic geometric inequality, for any $\delta > 0$,

$$\|\nabla \dot{u}_m(t, \cdot)\|_{L^2(\mathbb{T})} \cdot \|u_m(t, \cdot)\|_{L^2(\mathbb{T})} \leq \delta \|\nabla \dot{u}_m(t, \cdot)\|_{L^2(\mathbb{T})}^2 + \frac{1}{\delta} \|\dot{u}_m(t, \cdot)\|_{L^2(\mathbb{T})}^2. \quad (2.19)$$

Choosing $\delta := \frac{\alpha_0}{2\mathfrak{M}(2)C_{\text{sob}}}$, setting $\alpha := \frac{\alpha_0}{2}$ and defining $\beta := (1 + \frac{1}{\delta}) \mathfrak{M}(2) C_{\text{sob}}$ we have the estimate

$$\ddot{\mathcal{J}}(m)[h, h] \geq \alpha \iint_{(0, T-\varepsilon) \times \mathbb{T}} |\nabla \dot{u}_m|^2 - \beta \iint_{(0, T) \times \mathbb{T}} \dot{u}^2 - \gamma \int_{\mathbb{T}} \dot{u}^2(T, \cdot). \quad (2.20)$$

The proof is now complete. □

This proposition indicates that a possibility to derive a proof of theorem I is as follows: first, picking a maximiser m^* of $(\mathbf{P}_{\text{parab}})$, we argue by contradiction and assume m^* is not bang-bang, so that the set $\omega = \{0 < m^* < 1\}$ has positive measure. Thus, for any admissible perturbation h supported in ω , $\dot{\mathcal{J}}(m^*)[h] = 0$. If we can pick an admissible perturbation such that $\iint_{(0, T-\varepsilon) \times \mathbb{T}} |\nabla \dot{u}_m|^2 \gg \iint_{(0, T) \times \mathbb{T}} \dot{u}_m^2 + \int_{\mathbb{T}} \dot{u}_m^2(T, \cdot)$, the second order Gateau derivative is positive, in contradiction with the optimality of m^* . To build such an h , we need to choose it highly oscillating; in other words, its Fourier series only has high order modes. Thus, the next sections are, respectively, devoted to the construction of an admissible h that only has high Fourier modes, and to the study of the ensuing \dot{u}_m via two-scale asymptotic expansions.

Throughout, we thus consider a non bang-bang maximiser m^ and define $\omega := \{0 < m^* < 1\}$.*

2.3 Construction of an admissible perturbation

The relevant function to study is \dot{u}_m , which solves the parabolic equation

$$\frac{\partial \dot{u}_m}{\partial t} - \Delta \dot{u}_m - V_m \dot{u}_m = h u_m.$$

We want to build h such that, a large integer K being fixed, h is supported in ω and that has the Fourier decomposition

$$h(x) = \sum_{k=K}^{\infty} a_k \cos(kx) + b_k \sin(kx).$$

Let us prove that such an admissible perturbation exists: let $\omega := \{0 < m^* < 1\}$. As m^* is not bang-bang, $\text{Vol}(\omega) > 0$. Consequently the space $L^2(\omega)$ is infinite dimensional. We identify each $H \in L^2(\omega)$ with $h := H \mathbb{1}_{\omega} \in L^2(\mathbb{T})$. We fix an integer $K \in \mathbb{N} \setminus \{0\}$ and we define, for any $0 \leq k \leq K-1$ the linear functionals

$$T_k^1 : L^2(\omega) \ni H \mapsto \int_{\omega} H \cos(kx) dx, T_k^2 : L^2(\omega) \ni H \mapsto \int_{\omega} H \sin(kx) dx.$$

Finally, we define

$$E_K := \bigcap_{k=0}^{K-1} (\ker(T_k^1) \cap \ker(T_k^2))$$

E_K , as an intersection of closed hyperplanes, is of finite co-dimension. It is, in particular, infinite dimensional. Hence, we can pick $H_K \in E_K$ such that $\|H_K\|_{L^2(\omega)} > 0$. By definition, $h_K := H_K \mathbb{1}_\omega$ has the Fourier decomposition

$$h_K(x) = \sum_{k=K}^{\infty} a_k \cos(kx) + b_k \sin(kx) \text{ with, up to renormalisation, } \sum_{k=K}^{\infty} a_k^2 + b_k^2 = 1.$$

We now want to study how $\dot{u}_m[h_K]$ behaves, for K large. This prompts us to considering, first, the case of single cosines and sines.

2.4 Computations for single-mode perturbations

We emphasise once again that the computations of this paragraph are formal; we refer to proposition 19 for the rigorous proof of the expansions.

Case of single cosines Let, for any $k \in \mathbb{N} \setminus \{0\}$, η_k be the solution of

$$\begin{cases} \partial_t \eta_k - \Delta \eta_k - V_m \eta_k = u_m(t, x) \cos(kx) & \text{in } (0, T) \times \mathbb{T}, \\ \eta_k(0, \cdot) = 0 & \text{in } \mathbb{T}. \end{cases} \quad (2.21)$$

A natural expansion to look for is of the form

$$\eta_k(t, x) \approx \frac{1}{k^2} R_1(x, kx, t, k^2 t) + \frac{1}{k^3} R_2(x, kx, t, k^2 t) + \dots$$

By convention, we call y and s the second and fourth variables of R_1 and R_2 . Plugging this formal expansion in (2.21) we obtain the following equations:

$$\begin{cases} \partial_s R_1 - \partial_{yy}^2 R_1 = \cos(y) u_m(t, x), \\ R_1(x, y, 0, 0) = 0, \end{cases} \quad (2.22)$$

and

$$\begin{cases} \partial_s R_2 - \partial_{yy}^2 R_2 = 2 \partial_{xy}^2 R_1(x, y, t, s), \\ R_2(x, y, 0, 0) = 0. \end{cases} \quad (2.23)$$

(2.22) can be solved explicitly and we obtain

$$R_1(x, y, t, s) = u_m(t, x) (1 - e^{-s}) \cos(y). \quad (2.24)$$

This allows to derive the explicit form of (2.23). Namely, R_2 satisfies

$$\partial_s R_2 - \partial_{yy}^2 R_2 = -2 \frac{\partial u_m}{\partial x}(t, x) (1 - e^{-s}) \sin(y). \quad (2.25)$$

As a consequence, we look for R_2 under the form

$$R_2(x, y, t, s) = -2 \frac{\partial u_m}{\partial x}(t, x) \sin(y) \varphi(s).$$

The function φ satisfies

$$\varphi' + \varphi = 1 - e^{-s}$$

which can be integrated explicitly as

$$\varphi(s) = 1 - se^{-s} - e^{-s}.$$

Finally,

$$R_2(x, y, t, s) = -2 \frac{\partial u_m}{\partial x}(t, x) \sin(y) (1 - se^{-s} - e^{-s}). \quad (2.26)$$

Case of single sines We then consider the case of single sines. Let, for any $k \in \mathbb{N} \setminus \{0\}$, ζ_k be the solution of

$$\begin{cases} \partial_t \zeta_k - \Delta \zeta_k - V \zeta_k = u_m(t, x) \sin(kx) & \text{in } (0, T) \times \mathbb{T}, \\ \zeta_k(0, \cdot) = 0 & \text{in } \mathbb{T}. \end{cases} \quad (2.27)$$

Similarly, we look for an expansion in the form

$$\zeta_k(t, x) \approx \frac{1}{k^2} S_1(x, kx, t, k^2 t) + \frac{1}{k^3} S_2(x, kx, t, k^2 t) + \dots$$

By convention, we call y and s the second and fourth variables of S_1 and S_2 . Plugging this formal expansion in (2.27) we obtain the following equations:

$$\begin{cases} \partial_s S_1 - \partial_{yy}^2 S_1 = \sin(y) u_m(t, x), \\ S_1(x, y, 0, 0) = 0, \end{cases} \quad (2.28)$$

and

$$\begin{cases} \partial_s S_2 - \partial_{yy}^2 S_2 = 2 \partial_{xy}^2 S_1(x, y, t, s), \\ S_2(x, y, 0, 0) = 0. \end{cases} \quad (2.29)$$

(2.28) can be solved explicitly and we have

$$S_1(x, y, t, s) = u_m(t, x) (1 - e^{-s}) \sin(y). \quad (2.30)$$

This allows to derive the explicit form of (2.29). Namely, S_2 satisfies

$$\partial_s S_2 - \partial_{yy}^2 S_2 = 2 \frac{\partial u_m}{\partial x}(t, x) (1 - e^{-s}) \cos(y). \quad (2.31)$$

Proceeding as in the computations of R_2 we derive

$$S_2(x, y, t, s) = 2 \frac{\partial u_m}{\partial x}(t, x) \cos(y) (1 - se^{-s} - e^{-s}). \quad (2.32)$$

Of course we wish to write an approximation of the type

$$\begin{aligned} \dot{u}_m \approx \sum_{k=K}^{\infty} a_k \left(\frac{1}{k^2} R_1(x, kx, t, k^2 t) + \frac{1}{k^3} R_2(x, kx, t, k^2 t) \right) \\ + \sum_{k=K}^{\infty} b_k \left(\frac{1}{k^2} S_1(x, kx, t, k^2 t) + \frac{1}{k^3} S_2(x, kx, t, k^2 t) \right). \end{aligned} \quad (2.33)$$

In order to determine how strong this approximation should be to yield an exploitable result on \mathcal{J} , we next study the leading term of (2.13), should the expansion (2.33) hold.

2.5 Formal estimate of the leading order term

We work under the assumption that

$$\begin{aligned} \dot{u}_m &\approx \sum_{k=K}^{\infty} a_k \eta_k + \sum_{k=K}^{\infty} b_k \zeta_k \\ &\approx \sum_{k=K}^{\infty} a_k \left(u_m(t, x) \frac{1}{k^2} \cos(kx) (1 - e^{-k^2 t}) - \frac{\partial u_m}{\partial x} \frac{2}{k^3} \sin(kx) (1 - k^2 t e^{-k^2 t} - e^{-k^2 t}) \right) \end{aligned}$$

$$+ \sum_{k=K}^{\infty} b_k \left(u_m(t, x) \frac{1}{k^2} \sin(kx)(1 - e^{-k^2 t}) + \frac{\partial u_m}{\partial x} \frac{2}{k^3} \cos(kx)(1 - k^2 t e^{-k^2 t} - e^{-k^2 t}) \right)$$

In particular, we have (this is still formal, at this point)

$$\begin{aligned} \partial_x \dot{u}_m &\approx \sum_{k=K}^{\infty} a_k \left(\frac{\partial u_m}{\partial x} \frac{1}{k^2} \cos(kx)(1 - e^{-k^2 t}) - \frac{\partial^2 u_m}{\partial x^2} \frac{2}{k^3} \sin(kx)(1 - k^2 t e^{-k^2 t} - e^{-k^2 t}) \right) \\ &+ \sum_{k=K}^{\infty} a_k \left(-u_m \frac{1}{k} \sin(kx)(1 - e^{-k^2 t}) - \frac{\partial u_m}{\partial x} \frac{2}{k^2} \cos(kx)(1 - k^2 t e^{-k^2 t} - e^{-k^2 t}) \right) \\ &+ \sum_{k=K}^{\infty} b_k \left(\frac{\partial u_m}{\partial x} \frac{1}{k^2} \sin(kx)(1 - e^{-k^2 t}) + \frac{\partial^2 u_m}{\partial x^2} \frac{2}{k^3} \cos(kx)(1 - k^2 t e^{-k^2 t} - e^{-k^2 t}) \right) \\ &+ \sum_{k=K}^{\infty} b_k \left(u_m \frac{1}{k} \cos(kx)(1 - e^{-k^2 t}) - \frac{\partial u_m}{\partial x} \frac{2}{k^2} \sin(kx)(1 - k^2 t e^{-k^2 t} - e^{-k^2 t}) \right) \\ &= u_m \left\{ \sum_{k=K}^{\infty} \frac{-a_k \sin(kx) + b_k \cos(kx)}{k} (1 - e^{-k^2 t}) \right\} \quad (=: L_K) \\ &+ \frac{\partial u_m}{\partial x} \left\{ \sum_{k=K}^{\infty} \frac{a_k \cos(kx) - b_k \sin(kx)}{k^2} (-1 + e^{-k^2 t} + 2k^2 t e^{-k^2 t}) \right\} \quad (=: I_K) \\ &- 2 \frac{\partial^2 u_m}{\partial x^2} \left\{ \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k^3} (1 - k^2 t e^{-k^2 t} - e^{-k^2 t}) \right\} \quad (=: J_K). \end{aligned}$$

Thus, we should have

$$\iint_{(0, T-\varepsilon) \times \mathbb{T}} |\nabla \dot{u}_m|^2 \approx \iint_{(0, T-\varepsilon) \times \mathbb{T}} (I_K + J_K + L_K)^2. \quad (2.34)$$

Given the expressions for I_K, J_K, L_K , we expect $\iint_{(0, T-\varepsilon) \times \mathbb{T}} L_K^2$ to be leading in (2.34). For this reason we first bound the right-hand side of (2.34) from below; we shall use the inequality

$$|xy| \leq \varepsilon x^2 + \frac{1}{\varepsilon} y^2 \text{ for any } x, y \in \mathbb{R}, \varepsilon > 0 \quad (2.35)$$

as well as

$$(x + y)^2 \leq 2(x^2 + y^2) \text{ for any } x, y \in \mathbb{R}. \quad (2.36)$$

We obtain

$$\iint_{(0, T-\varepsilon) \times \mathbb{T}} (I_K + J_K + L_K)^2 = \iint_{(0, T-\varepsilon) \times \mathbb{T}} L_K^2 + 2 \iint_{(0, T-\varepsilon) \times \mathbb{T}} L_K (I_K + J_K) \quad (2.37)$$

$$+ \iint_{(0, T-\varepsilon) \times \mathbb{T}} (I_K + J_K)^2 \quad (2.38)$$

$$\geq \iint_{(0, T-\varepsilon) \times \mathbb{T}} L_K^2 - \frac{1}{2} \iint_{(0, T-\varepsilon) \times \mathbb{T}} L_K^2 \quad (2.39)$$

$$- 4 \iint_{(0, T-\varepsilon) \times \mathbb{T}} (I_K + J_K)^2 + \iint_{(0, T-\varepsilon) \times \mathbb{T}} (I_K + J_K)^2 \quad (2.40)$$

$$\text{from (2.35) with } \varepsilon = \frac{1}{2} \quad (2.41)$$

$$= \frac{1}{2} \iint_{(0, T-\varepsilon) \times \mathbb{T}} L_K^2 - 6 \iint_{(0, T-\varepsilon) \times \mathbb{T}} I_K^2 - 6 \iint_{(0, T-\varepsilon) \times \mathbb{T}} J_K^2. \quad (2.42)$$

We recall that from lemma 13 we have

$$\underline{d} := \inf_{(0, T) \times \mathbb{T}} u_m > 0.$$

This will prove crucial.

We shall estimate the three terms (*i.e.* $\iint L_K^2, \iint I_K^2, \iint J_K^2$) separately.

Estimate of L_K We have

$$\begin{aligned} \iint_{(0, T-\varepsilon) \times \mathbb{T}} L_K^2 &= \iint_{(0, T-\varepsilon) \times \mathbb{T}} u_m^2(t, x) \left\{ \sum_{k=K}^{\infty} \frac{-a_k \sin(kx) + b_k \cos(kx)}{k} (1 - e^{-k^2 t}) \right\}^2 dx dt \\ &\geq \underline{d}^2 \iint_{(0, T-\varepsilon) \times \mathbb{T}} \left\{ \sum_{k=K}^{\infty} \frac{-a_k \sin(kx) + b_k \cos(kx)}{k} (1 - e^{-k^2 t}) \right\}^2 dx dt \\ &= \underline{d}^2 \pi \sum_{k=K}^{\infty} (a_k^2 + b_k^2) \int_0^{T-\varepsilon} \left\{ \frac{1}{k^2} (1 - e^{-k^2 t})^2 \right\} dt \\ &\geq 2C_0 \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2} \end{aligned}$$

for a positive constant $C_0 > 0$. We have hence obtained

$$\iint_{(0, T-\varepsilon) \times \mathbb{T}} L_K^2 \geq 2C_0 \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2} \text{ for a constant } C_0 > 0. \quad (2.43)$$

Remark 18. We choose $2C_0$ to obtain a cleaner estimate on the second order derivative.

Estimate of I_K We first recall that from proposition 16

$$\bar{d} := \sup_{t \in [0, T]} \|u_m\|_{\mathcal{C}^1(\mathbb{T})} < \infty.$$

We then bound and compute

$$\begin{aligned} \iint_{(0, T-\varepsilon) \times \mathbb{T}} I_K^2 &= 4 \iint_{(0, T-\varepsilon) \times \mathbb{T}} \left(\frac{\partial u_m}{\partial x} \right)^2 \left\{ \sum_{k=K}^{\infty} \frac{a_k \cos(kx) - b_k \sin(kx)}{k^2} (-1 + e^{-k^2 t} + 2k^2 t e^{-k^2 t}) \right\}^2 dt dx \\ &\leq 4\bar{d}^2 \iint_{(0, T-\varepsilon) \times \mathbb{T}} \left\{ \sum_{k=K}^{\infty} \frac{a_k \cos(kx) - b_k \sin(kx)}{k^2} (-1 + e^{-k^2 t} + 2k^2 t e^{-k^2 t}) \right\}^2 dx dt \\ &= 4\pi \bar{d}^2 \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4} \int_0^{T-\varepsilon} (-1 + e^{-k^2 t} + 2k^2 t e^{-k^2 t})^2 dt \\ &= 4\pi \bar{d}^2 \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4} \int_0^{T-\varepsilon} (1 + 4e^{-2k^2 t} k^4 t^2 - 2e^{-k^2 t} - 4k^2 t e^{-k^2 t} + 4k^2 t e^{-2k^2 t} + e^{-2k^2 t}) dt. \end{aligned}$$

However, each of the integrals can be computed explicitly:

$$\begin{aligned} \int_0^{T-\varepsilon} t^2 e^{-2k^2 t} dt &= -\frac{(T-\varepsilon)^2 e^{-2k^2(T-\varepsilon)}}{2k^2} - \frac{(T-\varepsilon)e^{-2k^2(T-\varepsilon)}}{2k^4} + \frac{1}{4k^6} \left(1 - e^{-2k^2(T-\varepsilon)}\right) \\ &\leq \frac{N_{0,I}}{k^6} \text{ for some constant } N_{0,I}, \\ \int_0^{T-\varepsilon} e^{-2k^2 t} dt &\leq \frac{N_{1,I}}{k^2} \text{ for some constant } N_{1,I}, \\ \int_0^{T-\varepsilon} t e^{-k^2 t} dt &\leq \frac{N_{2,I}}{k^4} \text{ for some constant } N_{2,I}. \end{aligned}$$

Hence, there exists $K \in \mathbb{N}$ and a constant $N_{3,I}$ such that for any $k \geq K$,

$$\int_0^{T-\varepsilon} (1 + 4e^{-2k^2 t} k^4 t^2 - 2e^{-2k^2 t} - 4k^2 t e^{-k^2 t} + 4k^2 t e^{-2k^2 t} + e^{-2k^2 t}) dt \leq N_{3,I} \left(1 + \frac{1}{k^2}\right). \quad (2.44)$$

Consequently, there exists a positive constant C_1 such that

$$\iint_{(0, T-\varepsilon) \times \mathbb{T}} J_K^2 \leq \frac{C_1}{6} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4}. \quad (2.45)$$

Estimate on J_K This last term is the trickiest one. Indeed, we do not have $\frac{\partial^2 u_m}{\partial x^2} \in L^\infty$, but simply, from proposition 16,

$$\forall p \in [1; +\infty), \sup_{t \in [0, T]} \left\| \frac{\partial^2 u_m}{\partial x^2}(t, \cdot) \right\|_{L^p(\mathbb{T})} =: \mathfrak{M}(p) < \infty.$$

However, we can use the same trick as in bounding the second order derivative (see the proof of proposition 15). We indeed obtain

$$\begin{aligned} \iint_{(0, T-\varepsilon) \times \mathbb{T}} J_K^2 &= \iint_{(0, T-\varepsilon) \times \mathbb{T}} 4 \left(\frac{\partial^2 u_m}{\partial x^2} \right)^2 \underbrace{\left\{ \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k^3} (1 - k^2 t e^{-k^2 t} - e^{-k^2 t}) \right\}^2}_{=: W_K} \\ &\leq \underbrace{16 \mathfrak{M}(4)^2 C_{sob}}_{=: D_{0,J}} \int_0^{T-\varepsilon} \|W_K(t, \cdot)\|_{L^2(\mathbb{T})} \|W_K(t, \cdot)\|_{W^{1,2}(\mathbb{T})} dt \\ &\leq D_{0,J} \iint_{(0, T-\varepsilon) \times \mathbb{T}} W_K^2 + D_{0,J} \int_0^{T-\varepsilon} \|W_K(t, \cdot)\|_{L^2(\mathbb{T})} \|\nabla W_K(t, \cdot)\|_{L^2(\mathbb{T})} dt, \end{aligned}$$

where C_{sob} is the constant of the (one-dimensional) embedding $W^{1,2}(\mathbb{T}) \hookrightarrow \mathcal{C}^0(\mathbb{T})$.

We compute, for every $t \in [0, T]$, both $\|W_K(t, \cdot)\|_{L^2(\mathbb{T})}$ and $\|\nabla W_K(t, \cdot)\|_{L^2(\mathbb{T})}$. First,

$$\begin{aligned} \int_{\mathbb{T}} W_K^2(t, \cdot) &= \int_{\mathbb{T}} \left\{ \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k^3} (1 - k^2 t e^{-k^2 t} - e^{-k^2 t}) \right\}^2 dx \\ &= \pi \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^6} (1 - k^2 t e^{-k^2 t} - e^{-k^2 t})^2. \end{aligned}$$

Second,

$$\begin{aligned} \int_{\mathbb{T}} |\nabla W_K|^2(t, \cdot) &= \int_{\mathbb{T}} \left\{ \sum_{k=K}^{\infty} \frac{-a_k \sin(kx) - b_k \cos(kx)}{k^2} (1 - k^2 t e^{-k^2 t} - e^{-k^2 t}) \right\}^2 dx \\ &= \pi \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4} (1 - k^2 t e^{-k^2 t} - e^{-k^2 t})^2. \end{aligned}$$

We notice that

$$\int_{\mathbb{T}} |\nabla W_K|^2(t, \cdot) \geq K^2 \int_{\mathbb{T}} W_K^2(t, \cdot) \quad (2.46)$$

or, in other terms, that, for any $t \in [0, T]$,

$$\|W_K(t, \cdot)\|_{L^2(\mathbb{T})} \leq \frac{1}{K} \|\nabla W_K(t, \cdot)\|_{L^2(\mathbb{T})}. \quad (2.47)$$

Consequently,

$$\begin{aligned} &\iint_{(0, T-\varepsilon) \times \mathbb{T}} W_K^2 + \int_0^{T-\varepsilon} \|W_K(t, \cdot)\|_{L^2(\mathbb{T})} \|\nabla W_K(t, \cdot)\|_{L^2(\mathbb{T})} dt \\ &\leq \iint_{(0, T-\varepsilon) \times \mathbb{T}} W_K^2 + \int_0^{T-\varepsilon} \|W_K(t, \cdot)\|_{L^2(\mathbb{T})} \|\nabla W_K(t, \cdot)\|_{L^2(\mathbb{T})} dt \\ &\leq \left(\frac{1}{K^2} + \frac{1}{K} \right) \int_0^{T-\varepsilon} \|\nabla W_K\|_{L^2(\mathbb{T})}^2(t, \cdot) \\ &\leq \frac{2\pi}{K} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4} \int_0^T (1 - e^{-k^2 t} (k^2 t + 1))^2 dt. \end{aligned}$$

From the same computations that established (2.44), there exists a constant $D_{1,J}$ such that, whenever K is large enough, for any $k \geq K$, there holds

$$\int_0^T (1 - e^{-k^2 t} (k^2 t + 1))^2 dt \leq D_{1,J}, \quad (2.48)$$

and so, finally, for a constant C_2 ,

$$\iint_{(0, T-\varepsilon) \times \mathbb{T}} J_K^2 \leq \frac{C_2}{6} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4}. \quad (2.49)$$

Combining (2.42)-(2.43)-(2.45)-(2.49) we finally derive the following lower-bound on the leading term: there exists $C_{\text{lead}} > 0$ such that

$$\iint_{(0, T-\varepsilon) \times \mathbb{T}} (I_K + J_K + L_K)^2 \geq C_{\text{lead}} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2}. \quad (2.50)$$

2.6 Formal estimate of the lower order term

If we assume that

$$\dot{u}_m \approx \sum_{k=K}^{\infty} a_k \eta_k + \sum_{k=K}^{\infty} b_k \zeta_k$$

$$\begin{aligned}
&= \sum_{k=K}^{\infty} a_k \left(u(t, x) \frac{1}{k^2} \cos(kx)(1 - e^{-k^2 t}) - \frac{\partial u_m}{\partial x} \frac{2}{k^3} \sin(kx)(1 - k^2 t e^{-k^2 t} - e^{-k^2 t}) \right) \\
&+ \sum_{k=K}^{\infty} b_k \left(u(t, x) \frac{1}{k^2} \sin(kx)(1 - e^{-k^2 t}) + \frac{\partial u_m}{\partial x} \frac{2}{k^3} \cos(kx)(1 - k^2 t e^{-k^2 t} - e^{-k^2 t}) \right),
\end{aligned}$$

then, in the very same way, we obtain the existence of a constant C_{low} such that

$$\iint_{(0, T) \times \mathbb{T}} \left\{ \sum_{k=K}^{\infty} a_k \eta_k + \beta_k \zeta_k \right\}^2 \leq C_{\text{low}} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4}. \quad (2.51)$$

2.7 Strategy and comment for the proof of the asymptotic expansion

We shall now establish rigorously a strong enough approximation result. Let us define

$$\begin{aligned}
Z_K := \sum_{k=K}^{\infty} a_k \left(\frac{u_m}{k^2} \cos(kx)(1 - e^{-k^2 t}) - \frac{2\partial_x u_m}{k^3} \sin(kx)(1 - k^2 t e^{-k^2 t} - e^{-k^2 t}) \right) \\
+ \sum_{k=K}^{\infty} b_k \left(\frac{u_m}{k^2} \sin(kx)(1 - e^{-k^2 t}) + \frac{2\partial_x u_m}{k^3} \cos(kx)(1 - k^2 t e^{-k^2 t} - e^{-k^2 t}) \right).
\end{aligned}$$

From (2.51)-(2.50), we need the following proposition to prove the theorem (see also lemma 20 below, which proves that this proposition is enough):

Proposition 19. *There exists a constant C_{cont} such that, for any $\Upsilon > 0$,*

$$\begin{aligned}
\int_0^T \|\dot{u}_m - Z_K(t, \cdot)\|_{W^{1,2}(\mathbb{T})}^2 dt + \|\dot{u}_m(T, \cdot) - Z_K(T, \cdot)\|_{L^2(\mathbb{T})}^2 \leq \frac{C_{\text{cont}}}{\Upsilon} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4} \\
+ \Upsilon C_{\text{cont}} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2}. \quad (2.52)
\end{aligned}$$

The object of the next lemma is to prove that proposition 19 suffices to obtain theorem I.

Lemma 20. *Proposition 19 implies theorem I.*

Proof of Lemma 20. We use proposition 15 with the perturbation h_K constructed in Section 2.3. We study the right-hand side of (2.13). On the one-hand, we have

$$\begin{aligned}
\iint_{(0, T-\varepsilon) \times \mathbb{T}} |\nabla \dot{u}_m|^2 &\geq \frac{1}{2} \iint_{(0, T-\varepsilon) \times \mathbb{T}} |\nabla Z_K|^2 - \iint_{(0, T) \times \mathbb{T}} |\nabla \dot{u}_m - \nabla Z_K|^2 dt dx \\
&\geq \frac{C_{\text{lead}}}{2} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2} - \frac{C_{\text{cont}}}{\Upsilon} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4} - \Upsilon C_{\text{cont}} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\iint_{(0, T) \times \mathbb{T}} \dot{u}_m^2 &\leq 2 \iint_{(0, T) \times \mathbb{T}} |Z_K|^2 + 2 \iint_{(0, T) \times \mathbb{T}} |\dot{u}_m - Z_K|^2 dt dx \\
&\leq 2C_{\text{low}} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4} + \frac{2C_{\text{cont}}}{\Upsilon} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4} + 2\Upsilon C_{\text{cont}} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2}
\end{aligned}$$

for K large enough, from (2.51) and Proposition 19, and, in the same way,

$$\int_{\mathbb{T}} \dot{u}_m^2(T, \cdot) \leq \frac{2C_{\text{cont}}}{\Upsilon} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4} + 2\Upsilon C_{\text{cont}} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2}$$

Consequently,

$$\begin{aligned} \ddot{\mathcal{J}}(m^*)[h_K, h_K] &\geq \frac{\alpha C_{\text{lead}}}{2} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2} - \frac{\alpha C_{\text{cont}}}{\Upsilon} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4} - \alpha \Upsilon C_{\text{cont}} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2} \\ &\quad - 2\beta C_{\text{low}} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4} - 2\frac{\beta C_{\text{cont}}}{\Upsilon} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4} - 2\beta \Upsilon C_{\text{cont}} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2} \\ &\quad - 2\gamma C_{\text{low}} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4} - 2\gamma \frac{C_{\text{cont}}}{\Upsilon} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4} - 2\gamma \Upsilon C_{\text{cont}} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2} \\ &= \left(\sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2} \right) \left\{ \frac{\alpha C_{\text{lead}}}{2} - \alpha \Upsilon C_{\text{cont}} - 2\beta \Upsilon C_{\text{cont}} - 2\gamma \Upsilon C_{\text{cont}} \right\} \\ &\quad - \left(\sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4} \right) \left\{ \frac{\alpha C_{\text{cont}}}{\Upsilon} - 2\beta C_{\text{low}} - 2\frac{\beta C_{\text{cont}}}{\Upsilon} - 2\gamma C_{\text{low}} - 2\gamma \Upsilon C_{\text{cont}} \right\} \\ &\geq \left(\sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2} \right) \left\{ \frac{\alpha C_{\text{lead}}}{2} - \alpha \Upsilon C_{\text{cont}} - 2\beta \Upsilon C_{\text{cont}} - 2\gamma \Upsilon C_{\text{cont}} \right\} \\ &\quad - \left(\sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2} \right) \frac{\left| \frac{\alpha C_{\text{cont}}}{\Upsilon} - 2\beta C_{\text{low}} - 2\frac{\beta C_{\text{cont}}}{\Upsilon} - 2\gamma C_{\text{low}} - 2\gamma \Upsilon C_{\text{cont}} \right|}{K^2} \end{aligned}$$

We first pick $\Upsilon > 0$ small enough so that

$$\alpha'' := \frac{\alpha C_{\text{lead}}}{2} - \alpha \Upsilon C_{\text{cont}} - 2\beta \Upsilon C_{\text{cont}} - 2\gamma \Upsilon C_{\text{cont}} > 0.$$

We define

$$\beta'' := \left| \frac{\alpha C_{\text{cont}}}{\Upsilon} - 2\beta C_{\text{low}} - 2\frac{\beta C_{\text{cont}}}{\Upsilon} - 2\gamma C_{\text{low}} - 2\gamma \Upsilon C_{\text{cont}} \right|.$$

Thus we have the lower bound

$$\ddot{\mathcal{J}}(m^*)[h_K, h_K] \geq \left\{ \alpha'' - \frac{\beta''}{K^2} \right\} \left(\sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2} \right). \quad (2.53)$$

We pick K large enough to ensure that

$$\alpha'' - \frac{\beta''}{K^2} \geq \frac{\alpha''}{2}$$

and it follows that

$$\ddot{\mathcal{J}}(m^*)[h_K, h_K] > 0,$$

in contradiction with the optimality of m^* . The conclusion of the Theorem follows: every maximiser must be a bang-bang function. \square

The rest of this section is devoted to the proof of proposition 19.

2.8 Proof of proposition 19

Proof of proposition 19. We recall that, in its expanded form, Z_K writes

$$\begin{aligned} Z_K := & \sum_{k=K}^{\infty} a_k \left(\frac{u_m}{k^2} \cos(kx)(1 - e^{-k^2t}) - \frac{2\partial_x u_m}{k^3} \sin(kx)(1 - k^2te^{-k^2t} - e^{-k^2t}) \right) \\ & + \sum_{k=K}^{\infty} b_k \left(\frac{u_m}{k^2} \sin(kx)(1 - e^{-k^2t}) + \frac{2\partial_x u_m}{k^3} \cos(kx)(1 - k^2te^{-k^2t} - e^{-k^2t}) \right) \end{aligned}$$

We define the remainder term

$$R_K := \dot{u}_m - Z_K.$$

The computations needed in order to determine an explicit equation for R_K are rather lengthy. We split them up.

Define

$$T_K^1 := u_m \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} (1 - e^{-k^2t}).$$

We first have

$$\frac{\partial T_K^1}{\partial t} = \frac{\partial u_m}{\partial t} \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} (1 - e^{-k^2t}) + u_m \sum_{k=K}^{\infty} (a_k \cos(kx) + b_k \sin(kx)) e^{-k^2t}.$$

Second, we have

$$\frac{\partial T_K^1}{\partial x} = \frac{\partial u_m}{\partial x} \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} (1 - e^{-k^2t}) - u_m \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k} (1 - e^{-k^2t})$$

so that

$$\begin{aligned} \frac{\partial^2 T_K^1}{\partial x^2} = & \frac{\partial^2 u_m}{\partial x^2} \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} (1 - e^{-k^2t}) - u_m \sum_{k=K}^{\infty} (a_k \cos(kx) + b_k \sin(kx)) (1 - e^{-k^2t}) \\ & - 2 \frac{\partial u_m}{\partial x} \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k} (1 - e^{-k^2t}). \end{aligned}$$

Hence, introducing the differential operator

$$\mathcal{L}_{V_m} : \Phi \mapsto \partial_t \Phi - \Delta \Phi - V_m \Phi$$

we obtain

$$\begin{aligned} \mathcal{L}_{V_m} T_K^1 = & (\mathcal{L}_{V_m} u_m) \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} (1 - e^{-k^2t}) + u_m \sum_{k=K}^{\infty} (a_k \cos(kx) + b_k \sin(kx)) \\ & + 2 \frac{\partial u_m}{\partial x} \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k} (1 - e^{-k^2t}) \quad (2.54) \end{aligned}$$

Second, we set

$$T_K^2 := -2 \frac{\partial u_m}{\partial x} \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k^3} (1 - k^2te^{-k^2t} - e^{-k^2t}).$$

We obtain

$$\begin{aligned} \frac{\partial T_K^2}{\partial t} &= -2 \frac{\partial}{\partial x} \left(\frac{\partial u_m}{\partial t} \right) \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k^3} (1 - k^2 t e^{-k^2 t} - e^{-k^2 t}) \\ &\quad - 2 \left(\frac{\partial u_m}{\partial x} \right) \sum_{k=K}^{\infty} (a_k \sin(kx) - b_k \cos(kx)) k t e^{-k^2 t}. \end{aligned}$$

Let us define, in order to alleviate the upcoming computations,

$$\varphi(s) := 1 - s e^{-s} - e^{-s}.$$

Similarly we obtain

$$\frac{\partial T_K^2}{\partial x} = -2 \frac{\partial^2 u_m}{\partial x^2} \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k^3} \varphi(k^2 t) - 2 \frac{\partial u_m}{\partial x} \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} \varphi(k^2 t)$$

as well as (the next equation should be understood in the $W^{-1,2}(\Omega)$ sense)

$$\begin{aligned} \frac{\partial^2 T_K^2}{\partial x^2} &= -2 \frac{\partial}{\partial x} \left(\frac{\partial^2 u_m}{\partial x^2} \right) \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k^3} \varphi(k^2 t) \\ &\quad - 4 \frac{\partial^2 u_m}{\partial x^2} \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} \varphi(k^2 t) \\ &\quad + 2 \frac{\partial u_m}{\partial x} \sum_{k=K}^{\infty} \frac{a_k \sin(kx) + b_k \cos(kx)}{k} \varphi(k^2 t). \end{aligned}$$

Combining these bricks we are left with

$$\begin{aligned} \mathcal{L}_{V_m} T_K^2 &= -2 \frac{\partial}{\partial x} (\mathcal{L}_{V_m} u_m) \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k^3} \varphi(k^2 t) \\ &\quad - 2 \frac{\partial u_m}{\partial x} \sum_{k=K}^{\infty} (a_k \sin(kx) - b_k \cos(kx)) \left(k t e^{-k^2 t} + \frac{\varphi(k^2 t)}{k} \right) \\ &\quad - 4 \frac{\partial^2 u_m}{\partial x^2} \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} \varphi(k^2 t) \\ &= -2 \frac{\partial}{\partial x} (\mathcal{L}_{V_m} u_m) \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k^3} \varphi(k^2 t) \\ &\quad - 2 \frac{\partial u_m}{\partial x} \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k} \left(1 - e^{-k^2 t} \right) \\ &\quad - 4 \frac{\partial^2 u_m}{\partial x^2} \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} \varphi(k^2 t), \end{aligned}$$

where, to obtain the last equality, we simply wrote

$$k t e^{-k^2 t} + \frac{\varphi(k^2 t)}{k} = k t e^{-k^2 t} + \frac{1}{k} - k t e^{-k^2 t} - \frac{e^{-k^2 t}}{k} = \frac{1 - e^{-k^2 t}}{k}.$$

Now, it follows that R_K satisfies

$$\begin{aligned}
\mathcal{L}_{V_m} R_K &= \mathcal{L}_{V_m} \dot{u}_m - \mathcal{L}_{V_m} T_K^1 - \mathcal{L}_{V_m} T_K^2 \\
&= u_m \sum_{k=K}^{\infty} a_k \cos(kx) + b_k \sin(kx) - (\mathcal{L}_{V_m} u) \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} (1 - e^{-k^2 t}) \\
&\quad - u_m \sum_{k=K}^{\infty} (a_k \cos(kx) + b_k \sin(kx)) \\
&\quad - 2 \frac{\partial u_m}{\partial x} \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k} (1 - e^{-k^2 t}) \\
&\quad + 2 \frac{\partial}{\partial x} (\mathcal{L}_{V_m} u_m) \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k^3} \varphi(k^2 t) \\
&\quad + 2 \frac{\partial u_m}{\partial x} \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k} (1 - e^{-k^2 t}) \\
&\quad + 4 \frac{\partial^2 u_m}{\partial x^2} \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} \varphi(k^2 t) \\
&= - (\mathcal{L}_{V_m} u_m) \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} (1 - e^{-k^2 t}) \\
&\quad + 2 \frac{\partial}{\partial x} (\mathcal{L}_{V_m} u_m) \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k^3} \varphi(k^2 t) \\
&\quad + 4 \frac{\partial^2 u_m}{\partial x^2} \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} \varphi(k^2 t)
\end{aligned}$$

We need one more transformation before this is in a workable form: we observe that (still in the $W^{-1,2}(\Omega)$ sense) we have

$$\begin{aligned}
\frac{\partial}{\partial x} (\mathcal{L}_{V_m} u_m) \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k^3} \varphi(k^2 t) &= \frac{\partial}{\partial x} \left\{ \mathcal{L}_{V_m} u_m \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k^3} \varphi(k^2 t) \right\} \\
&\quad - (\mathcal{L}_{V_m} u_m) \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} \varphi(k^2 t),
\end{aligned}$$

and the equation we shall be working on is then given by

$$\begin{aligned}
\mathcal{L}_{V_m} R_K &= - (\mathcal{L}_{V_m} u_m) \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} (1 - e^{-k^2 t} + 2\varphi(k^2 t)) \\
&\quad + 2 \frac{\partial}{\partial x} \left\{ \mathcal{L}_{V_m} u_m \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k^3} \varphi(k^2 t) \right\} \\
&\quad + 4 \frac{\partial^2 u_m}{\partial x^2} \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} \varphi(k^2 t) \quad (2.55)
\end{aligned}$$

Estimating R_K We introduce the notation

$$\phi(t) := 1 - e^{-s} + 2\varphi(s).$$

As φ is bounded, so is ϕ . Let $R_{K,1}, R_{K,2}, R_{K,3}$ be the solutions of

$$\left\{ \begin{array}{l} \mathcal{L}_{V_m} R_{K,1} = -(\mathcal{L}_{V_m} u_m) \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} \phi(k^2 t), \\ \mathcal{L}_{V_m} R_{K,2} = 2 \frac{\partial}{\partial x} \left\{ \mathcal{L}_{V_m} u_m \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k^3} \varphi(k^2 t) \right\}, \\ \mathcal{L}_{V_m} R_{K,3} = -4 \frac{\partial^2 u_m}{\partial x^2} \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} \varphi(k^2 t). \end{array} \right. \quad (2.56)$$

Obviously,

$$R_K = R_{K,1} + R_{K,2} + R_{K,3},$$

and so, up to a multiplicative constant E_1 we have

$$\begin{aligned} \iint_{(0,T) \times \mathbb{T}} |\nabla R_K|^2 + \iint_{(0,T) \times \mathbb{T}} R_K^2 + \int_{\mathbb{T}} R_K^2(T, \cdot) \\ \leq E_1 \sum_{j=1}^3 \left(\iint_{(0,T) \times \mathbb{T}} |\nabla R_{K,j}|^2 + \iint_{(0,T) \times \mathbb{T}} R_{K,j}^2 + \int_{\mathbb{T}} R_{K,j}^2(T, \cdot) \right). \end{aligned} \quad (2.57)$$

We shall now estimate each of these three functions. All the upcoming estimates rely on the following, standard, parabolic regularity result (proved in appendix A.3):

Lemma 21. *Let $f \in L^2(\Omega)$ and $g \in L^2(\Omega)$. Let $q \in L^\infty((0, T) \times \mathbb{T})$. Let θ be the solution of*

$$\begin{cases} \partial_t \theta - \partial_{xx}^2 \theta - V\theta = \partial_x f + qg & \text{in } (0, T) \times \mathbb{T}, \\ \theta(0, \cdot) = 0. \end{cases} \quad (2.58)$$

Then

$$\iint_{(0,T) \times \mathbb{T}} |\partial_x \theta|^2 + \iint_{(0,T) \times \mathbb{T}} \theta^2 + \int_{\mathbb{T}} \theta^2(T, \cdot) \leq C(V, q) \iint_{(0,T) \times \mathbb{T}} (f^2 + g^2). \quad (2.59)$$

We can move back to estimating $R_{K,j}$, for $j = 1, 2, 3$. To estimate $R_{K,1}$ we apply Lemma 21 with

$$f = 0, q = -\mathcal{L}_{V_m} u_m, g = \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} \phi(k^2 t).$$

Thus we obtain

$$\begin{aligned} \iint_{(0,T) \times \mathbb{T}} |\nabla R_{K,1}|^2 + \iint_{(0,T) \times \mathbb{T}} R_{K,1}^2 + \int_{(0,T) \times \mathbb{T}} R_{K,1}(T, \cdot)^2 \\ \leq C_1 \iint_{(0,T) \times \mathbb{T}} \left\{ \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} \phi(k^2 t) \right\}^2 dt dx. \end{aligned} \quad (2.60)$$

Since ϕ is bounded by a constant, which we take equal to 1 up to changing the value of C_1 , we obtain

$$\iint_{(0,T) \times \mathbb{T}} |\nabla R_{K,1}|^2 + \iint_{(0,T) \times \mathbb{T}} R_{K,1}^2 + \int_{(0,T) \times \mathbb{T}} R_{K,1}(T, \cdot)^2 \leq C_1 \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4}. \quad (2.61)$$

For $R_{K,2}$, it suffices to apply lemma 21 with

$$f = 2\mathcal{L}_{V_m} u_m \sum_{k=K}^{\infty} \frac{a_k \sin(kx) - b_k \cos(kx)}{k^3} \varphi(k^2 t), \quad q = g = 0$$

and we obtain, since $\mathcal{L}_{V_m} u_m \in L^\infty((0, T) \times \mathbb{T})$, the existence of a constant C_2 such that

$$\iint_{(0,T) \times \mathbb{T}} |\nabla R_{K,2}|^2 + \iint_{(0,T) \times \mathbb{T}} R_{K,2}^2 + \int_{\mathbb{T}} R_{K,2}(T, \cdot)^2 \leq C_2 \sum_{k=K}^{\infty} \frac{a_k^2}{k^6}. \quad (2.62)$$

The case of $R_{K,3}$ is, on the other hand, trickier, but can be handled similarly. We first recall that, from proposition 16, for any $p \in [1; +\infty)$,

$$\mathfrak{M}(p) := \sup_{t \in [0, T]} \|u_m(t, \cdot)\|_{W^{2,p}(\mathbb{T})} < \infty.$$

From standard $W^{1,2}$ parabolic estimates, we obtain, for a constant C_3 ,

$$\begin{aligned} & \iint_{(0,T) \times \mathbb{T}} |\nabla R_{K,3}|^2 + \iint_{(0,T) \times \mathbb{T}} R_{K,3}^2 + \int_{\mathbb{T}} R_{K,3}^2(T, \cdot) \\ & \leq C_3 \iint_{(0,T) \times \Omega} \left(\frac{\partial^2 u_m}{\partial x^2} \right)^2 \left\{ \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} \varphi(k^2 t) \right\}^2. \end{aligned} \quad (2.63)$$

Thus, up to replacing C_3 with $C_3 \|\varphi\|_{L^\infty}$, we obtain

$$\begin{aligned} & \int_{\mathbb{T}} R_{K,3}^2(T, \cdot) + \iint_{(0,T) \times \mathbb{T}} R_{K,3}^2 + \iint_{(0,T) \times \mathbb{T}} |\nabla R_{K,3}|^2 \\ & \leq C_3 \iint_{(0,T) \times \Omega} \left(\frac{\partial^2 u_m}{\partial x^2} \right)^2 \left\{ \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2} \right\}^2. \end{aligned} \quad (2.64)$$

Define

$$\Psi_K := \sum_{k=K}^{\infty} \frac{a_k \cos(kx) + b_k \sin(kx)}{k^2}.$$

From Hölder's inequality, we obtain

$$\begin{aligned} \int_{\mathbb{T}} R_{K,3}^2(T, \cdot) + \iint_{(0,T) \times \mathbb{T}} R_{K,3}^2 + \iint_{(0,T) \times \mathbb{T}} |\nabla R_{K,3}|^2 & \leq C_3 \iint_{(0,T) \times \mathbb{T}} \left(\frac{\partial^2 u_m}{\partial x^2} \right)^2 \Psi_K^2 \\ & \leq \underbrace{C_3 \mathfrak{M}(4)}_{=: C'_3} \int_0^T \|\nabla \Psi_K(t, \cdot)\|_{L^2(\mathbb{T})} \|\Psi_K(t, \cdot)\|_{L^2(\mathbb{T})} dt \end{aligned}$$

$$\leq 2C'_3 \left\{ \Upsilon \iint_{(0,T) \times \mathbb{T}} |\nabla \Psi_K|^2 + \frac{1}{\Upsilon} \iint_{(0,T) \times \mathbb{T}} \Psi_K^2 \right\}.$$

However, since

$$\frac{\partial \Psi_K}{\partial x} = \sum_{k=K}^{\infty} \frac{-a_k \sin(kx) + b_k \cos(kx)}{k}$$

we obtain, on the one-hand,

$$\iint_{(0,T) \times \mathbb{T}} \Psi_K^2 = \frac{1}{2} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4},$$

and, on the other hand,

$$\iint_{(0,T) \times \mathbb{T}} |\nabla \Psi_K|^2 = \frac{1}{2} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2}.$$

Finally, we obtain the estimate

$$\int_{\mathbb{T}} R_{K,3}^2(T, \cdot) + \iint_{(0,T) \times \mathbb{T}} R_{K,3}^2 + \iint_{(0,T) \times \mathbb{T}} |\nabla R_{K,3}|^2 \leq \Upsilon C'_3 \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2} + \frac{C'_3}{\Upsilon} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4}. \quad (2.65)$$

Hence, summing (2.61)-(2.62)-(2.65) and plugging these estimates in (2.57), there exists C_{cont} such that

$$\iint_{(0,T) \times \mathbb{T}} |\nabla R_K|^2 + \iint_{(0,T) \times \mathbb{T}} R_K^2 + \int_{\mathbb{T}} R_K^2(T, \cdot) \leq \frac{C_{\text{cont}}}{\Upsilon} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^4} + \Upsilon C_{\text{cont}} \sum_{k=K}^{\infty} \frac{a_k^2 + b_k^2}{k^2},$$

thus concluding the proof. \square

Thus proposition 19 is proved. As, from lemma 20, proposition 19 implies theorem I, theorem I is established.

3 Proof of Theorem II

The first part of the proof follows the plan of the proof of theorem I: the idea is to obtain a time-discrete analog of proposition 15, see lemma 23 below. The way to use such an estimate then differs from the proof of theorem I and is simpler, as there is no need to carry out two-scale asymptotic expansions.

3.1 Preliminary analysis of the system

We recall that the system of equations we work with is

$$w_{\mathbf{m},0} = w^0 \text{ and for any } k \in \{0, \dots, N-1\},$$

$$\begin{cases} \frac{w_{\mathbf{m},k+1} - w_{\mathbf{m},k}}{\delta t} - \Delta w_{\mathbf{m},k+1} = m_{k+1} w_{\mathbf{m},k+1} + f_{k+1}(x, w_{\mathbf{m},k+1}) & \text{in } \Omega, \\ \frac{\partial w_{\mathbf{m},k+1}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.1)$$

The following Lemma contains all the information we shall need regarding the solvability of (3.4), and the regularity we will use. Since its proof relies on standard techniques in time-discretised systems, we give it in Appendix B.

Lemma 22. Assume $\{f_k\}_{k=1,\dots,n}$ satisfies (\mathbf{H}_{dis}) . Let $w^0 \in L^\infty(\Omega)$ satisfy

$$0 < \inf_{\Omega} w^0 \leq \|w^0\|_{L^\infty(\Omega)} \leq \kappa$$

where we recall that κ is given by (\mathbf{H}_2) . For any $\delta t > 0$ small enough, for any $\mathbf{m} \in \mathfrak{M}(\Omega)$ the system (3.4) is uniquely solvable. Furthermore,

$$\forall k \in \{1, \dots, N\}, 0 < \inf_{\Omega} w_{\mathbf{m},k} \leq \|w_{\mathbf{m},k}\|_{L^\infty(\Omega)} \leq \kappa$$

and

$$\forall k \in \{1, \dots, N\}, \forall p \in [1; +\infty), w_{\mathbf{m},k} \in W^{2,p}(\Omega).$$

3.2 Computation and estimate on the derivatives of the functional

The Gateaux derivative of the state solves

$$\begin{cases} \dot{w}_{\mathbf{m},0} = 0 \text{ and for any } k \in \{0, \dots, N-1\}, \\ \begin{cases} \frac{\dot{w}_{\mathbf{m},k+1}}{\delta t} - \Delta \dot{w}_{\mathbf{m},k+1} = \dot{w}_{\mathbf{m},k+1} (m_{k+1} + \partial_w f_{k+1}(x, w_{\mathbf{m},k+1})) + \frac{\dot{w}_{\mathbf{m},k}}{\delta t} + h_{k+1} w_{\mathbf{m},k+1} & \text{in } \Omega, \\ \frac{\partial \dot{w}_{\mathbf{m},k+1}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \end{cases} \quad (3.2)$$

The derivative of the criterion under consideration writes

$$j_{dis}(\mathbf{m})[\mathbf{h}] = \sum_{i=1}^N \int_{\Omega} \dot{w}_{\mathbf{m},i} \partial_w j_i(x, w_{\mathbf{m},i}). \quad (3.3)$$

Let us define, for any $k \in \{1, \dots, N\}$,

$$W_k := (m_{k+1} + \partial_w f_{k+1}(x, w_{\mathbf{m},k+1})).$$

We introduce the adjoint state, namely, the solution $q_{\mathbf{m}} = (q_{\mathbf{m},0}, \dots, q_{\mathbf{m},N-1})$ solution of

$$\begin{cases} \frac{\dot{q}_{\mathbf{m},N-1}}{\delta t} - \Delta \dot{q}_{\mathbf{m},N-1} + W_{N-1} \dot{q}_{\mathbf{m},N-1} = \partial_w j_N(x, w_{\mathbf{m},N}) \text{ and, for any } k \in \{1, \dots, N-1\}, \\ \begin{cases} \frac{\dot{q}_{\mathbf{m},k-1}}{\delta t} - \Delta \dot{q}_{\mathbf{m},k-1} = \dot{q}_{\mathbf{m},k-1} W_k + \frac{\dot{q}_{\mathbf{m},k}}{\delta t} - \partial_w j_1(x, w_{\mathbf{m},k}) & \text{in } \Omega, \\ \frac{\partial \dot{q}_{\mathbf{m},k-1}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \end{cases} \quad (3.4)$$

Multiplying (3.2) by $q_{\mathbf{m}}$ and integrating by parts in space yields

$$\begin{aligned} \frac{1}{\delta t} \sum_{k=0}^{N-1} \int_{\Omega} (\dot{w}_{\mathbf{m},k+1} - \dot{w}_{\mathbf{m},k}) q_{\mathbf{m},k} + \sum_{k=0}^{N-1} \int_{\Omega} \langle \nabla \dot{w}_{\mathbf{m},k+1}, \nabla q_{\mathbf{m},k} \rangle \\ + \sum_{k=0}^{N-1} \int_{\Omega} W_{k+1} \dot{w}_{\mathbf{m},k+1} q_{\mathbf{m},k} = \sum_{k=0}^{N-1} \int_{\Omega} h_{k+1} w_{\mathbf{m},k+1} q_{\mathbf{m},k}. \end{aligned} \quad (3.5)$$

We perform an Abel transformation:

$$\sum_{k=0}^{N-1} \int_{\Omega} (\dot{w}_{\mathbf{m},k+1} - \dot{w}_{\mathbf{m},k}) q_{\mathbf{m},k} = \sum_{k=0}^{N-1} \int_{\Omega} \dot{w}_{\mathbf{m},k+1} q_{\mathbf{m},k} - \sum_{k=0}^{N-1} \int_{\Omega} \dot{w}_{\mathbf{m},k} q_{\mathbf{m},k}$$

$$\begin{aligned}
&= \sum_{k=1}^N \int_{\Omega} \dot{w}_{\mathbf{m},k} q_{\mathbf{m},k-1} - \sum_{k=1}^{N-1} \int_{\Omega} \dot{w}_{\mathbf{m},k} q_{\mathbf{m},k} \\
&= \int_{\Omega} \dot{w}_{\mathbf{m},N} q_{\mathbf{m},N-1} + \sum_{k=1}^{N-1} \int_{\Omega} \dot{w}_{\mathbf{m},k} (q_{\mathbf{m},k-1} - q_{\mathbf{m},k}),
\end{aligned}$$

where we used repeatedly the fact that $\dot{w}_{\mathbf{m},0} = 0$.

As a consequence, we obtain

$$\begin{aligned}
\frac{1}{\delta t} \int_{\Omega} \dot{w}_{\mathbf{m},N} q_{\mathbf{m},N-1} + \frac{1}{\delta t} \sum_{k=1}^{N-1} \int_{\Omega} \dot{w}_{\mathbf{m},k} (q_{\mathbf{m},k-1} - q_{\mathbf{m},k}) + \sum_{k=0}^{N-1} \int_{\Omega} \langle \nabla \dot{w}_{\mathbf{m},k+1}, \nabla q_{\mathbf{m},k} \rangle \\
+ \sum_{k=0}^{N-1} \int_{\Omega} W_{k+1} \dot{w}_{\mathbf{m},k+1} q_{\mathbf{m},k} = \sum_{k=0}^{N-1} \int_{\Omega} h_{k+1} w_{\mathbf{m},k+1} q_{\mathbf{m},k}, \quad (3.6)
\end{aligned}$$

which rewrites

$$\begin{aligned}
\frac{1}{\delta t} \int_{\Omega} \dot{w}_{\mathbf{m},N} q_{\mathbf{m},N-1} + \frac{1}{\delta t} \sum_{k=1}^{N-1} \int_{\Omega} \dot{w}_{\mathbf{m},k} (q_{\mathbf{m},k-1} - q_{\mathbf{m},k}) + \sum_{k=1}^N \int_{\Omega} \langle \nabla \dot{w}_{\mathbf{m},k}, \nabla q_{\mathbf{m},k-1} \rangle \\
+ \sum_{k=1}^N \int_{\Omega} W_k \dot{w}_{\mathbf{m},k} q_{\mathbf{m},k-1} = \sum_{k=1}^N \int_{\Omega} h_k w_{\mathbf{m},k} q_{\mathbf{m},k-1}. \quad (3.7)
\end{aligned}$$

It follows that

$$\dot{J}(\mathbf{m})[\mathbf{h}] = \sum_{i=1}^N \int_{\Omega} \dot{w}_{\mathbf{m},i} \partial_w j_i(w, w_{\mathbf{m},i}) = \sum_{k=1}^N \int_{\Omega} h_k w_{\mathbf{m},k} q_{\mathbf{m},k-1}. \quad (3.8)$$

In the same way, we have the following system for the second-order Gateau derivative

$$\begin{cases} \ddot{w}_{\mathbf{m},0} = 0 \text{ and for any } k \in \{0, \dots, N-1\}, \\ \frac{1}{\delta t} \ddot{w}_{\mathbf{m},k+1} - \Delta \ddot{w}_{\mathbf{m},k+1} = \ddot{w}_{\mathbf{m},k+1} (m_{k+1} + \partial_w f_{k+1}(t, x, w_{\mathbf{m},k+1})) + (\dot{w}_{\mathbf{m},k+1})^2 \partial_{ww}^2 f_{k+1} \\ \quad + \frac{1}{\delta t} \ddot{w}_{\mathbf{m},k} + 2h_{k+1} \dot{w}_{\mathbf{m},k+1} \text{ in } \Omega, \\ \frac{\partial \ddot{w}_{\mathbf{m},k+1}}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases} \quad (3.9)$$

as well as the second-order derivative of the criterion

$$\begin{aligned}
\ddot{J}(\mathbf{m})[\mathbf{h}] &= \sum_{i=1}^N \ddot{w}_{\mathbf{m},i} \partial_w j_i(x, w_{\mathbf{m},i}) + \sum_{i=1}^N (\dot{w}_{\mathbf{m},i})^2 \partial_{ww}^2 j_i(x, w_{\mathbf{m},i}) \\
&= 2 \sum_{k=1}^N \int_{\Omega} h_k \dot{w}_{\mathbf{m},k} q_{\mathbf{m},k-1} + \sum_{k=1}^N \int_{\Omega} (\dot{w}_{\mathbf{m},k})^2 \partial_{ww}^2 f_{k+1} q_{\mathbf{m},k-1} + \sum_{i=1}^N (\dot{w}_{\mathbf{m},i})^2 \partial_{ww}^2 j_i(x, w_{\mathbf{m},i}). \quad (3.10)
\end{aligned}$$

The main lemma is then

Lemma 23. *Let $\mathbf{m} \in \mathfrak{M}(\Omega)$. There exist two positive constant $\alpha, \beta > 0$ such that for any admissible perturbation \mathbf{h} at \mathbf{m} there holds*

$$\ddot{J}(\mathbf{m})[\mathbf{h}, \mathbf{h}] \geq \alpha \sum_{k=0}^{N-1} \int_{\Omega} |\nabla \dot{w}_{\mathbf{m},k+1}|^2 - \beta \sum_{k=0}^{N-1} \int_{\Omega} \dot{w}_{k+1}^2.$$

Proof of Lemma 23. Let us focus on

$$\sum_{k=1}^N \int_{\Omega} h_k \dot{w}_{\mathbf{m},k} q_{\mathbf{m},k-1} = \sum_{k=0}^{N-1} \int_{\Omega} h_{k+1} \dot{w}_{\mathbf{m},k+1} q_{\mathbf{m},k}.$$

Let us observe that, for any $k \in \{0, \dots, N-1\}$, we have

$$h_{k+1} = \frac{\frac{1}{\delta t}(\dot{w}_{\mathbf{m},k+1} - \dot{w}_{\mathbf{m},k}) - \Delta \dot{w}_{\mathbf{m},k+1} - \dot{w}_{\mathbf{m},k+1} W_{k+1}}{w_{\mathbf{m},k+1}}$$

In particular, for any $k \in \{0, \dots, N-1\}$,

$$\int_{\Omega} h_{k+1} \dot{w}_{\mathbf{m},k+1} q_{\mathbf{m},k} = \int_{\Omega} q_{\mathbf{m},k} \frac{\frac{1}{\delta t}(\dot{w}_{\mathbf{m},k+1} - \dot{w}_{\mathbf{m},k}) - \Delta \dot{w}_{\mathbf{m},k+1} - \dot{w}_{\mathbf{m},k+1} W_{k+1}}{w_{\mathbf{m},k+1}} \dot{w}_{\mathbf{m},k+1}.$$

Let us define, for any $k \in \{0, \dots, N-1\}$,

$$\Psi_{\mathbf{m},k+1} := \frac{q_{\mathbf{m},k}}{w_{\mathbf{m},k+1}}.$$

The above expression rewrites

$$\begin{aligned} \int_{\Omega} h_{k+1} \dot{w}_{\mathbf{m},k+1} q_{\mathbf{m},k} &= \int_{\Omega} \left(\frac{1}{\delta t} \Psi_{\mathbf{m},k+1} - W_{k+1} \right) (\dot{w}_{\mathbf{m},k+1})^2 \\ &\quad + \frac{1}{\delta t} \int_{\Omega} \Psi_{\mathbf{m},k+1} \dot{w}_{\mathbf{m},k+1} \dot{w}_{\mathbf{m},k} \\ &\quad - \frac{1}{2} \int_{\Omega} \Delta (\dot{w}_{\mathbf{m},k+1}^2) \Psi_{\mathbf{m},k+1} + \int_{\Omega} \Psi_{\mathbf{m},k+1} |\nabla \dot{w}_{\mathbf{m},k+1}|^2 \\ &= \int_{\Omega} \left(\frac{1}{\delta t} \Psi_{\mathbf{m},k+1} - W_{k+1} - \frac{1}{2} \Delta \Psi_{\mathbf{m},k+1} \right) (\dot{w}_{\mathbf{m},k+1})^2 \\ &\quad + \int_{\Omega} \Psi_{\mathbf{m},k+1} |\nabla \dot{w}_{\mathbf{m},k+1}|^2 \\ &\quad + \frac{1}{\delta t} \int_{\Omega} \Psi_{\mathbf{m},k+1} \dot{w}_{\mathbf{m},k+1} \dot{w}_{\mathbf{m},k}. \end{aligned}$$

However, adapting the arguments of lemma 22 and using the fact that J satisfies (\mathbf{H}_J) we easily derive that

$$\forall k \in \{0, \dots, N\}, 0 < \inf_{\Omega} q_{\mathbf{m},k} \leq \|q_{\mathbf{m},k}\|_{L^\infty} < \infty.$$

With lemma 22 this implies that there exists a constant $\alpha > 0$ such that

$$\sum_{k=0}^{N-1} \int_{\Omega} \Psi_{\mathbf{m},k+1} |\nabla \dot{w}_{\mathbf{m},k+1}|^2 \geq \alpha \sum_{k=0}^{N-1} \int_{\Omega} |\nabla \dot{w}_{\mathbf{m},k+1}|^2 \quad (3.11)$$

Similarly, there exists a constant $\beta_0 \in \mathbb{R}$ such that

$$\begin{aligned} \sum_{k=0}^{N-1} \int_{\Omega} \left(\frac{1}{\delta t} \Psi_{\mathbf{m},k+1} - W_{k+1} - \Delta \Psi_{\mathbf{m},k+1} \right) (\dot{w}_{\mathbf{m},k+1})^2 + \frac{1}{\delta t} \int_{\Omega} \Psi_{\mathbf{m},k+1} \dot{w}_{\mathbf{m},k+1} \dot{w}_{\mathbf{m},k} \\ \leq \beta_0 \sum_{k=0}^{N-1} \int_{\Omega} w_{\mathbf{m},k+1}^2. \quad (3.12) \end{aligned}$$

To obtain the estimate above, in particular, we used the fact that $\dot{w}_{\mathbf{m},0} = 0$ to derive the bound

$$\sum_{k=0}^{N-1} \int_{\Omega} \Psi_{\mathbf{m},k+1} \dot{w}_{\mathbf{m},k+1} \dot{w}_{\mathbf{m},k} \leq 2 \sup_{k=0, \dots, N-1} \|\Psi_{\mathbf{m},k+1}\|_{L^\infty} \sum_{k=0}^{N-1} \dot{w}_{\mathbf{m},k+1}^2.$$

Let us also notice that from lemma 22 and explicit computations similar to [39, Estimate (2.14)-(2.15)] that

$$\forall k \in \{0, \dots, N-1\}, \Delta \Psi_{\mathbf{m},k+1} \in L^\infty(\Omega)$$

and so there exists β_1 such that

$$\sum_{k=1}^N \int_{\Omega} (\dot{w}_{\mathbf{m},k})^2 \partial_{ww}^2 f_{k+1} q_{\mathbf{m},k-1} + \sum_{i=1}^N (\dot{w}_{\mathbf{m},i})^2 \partial_{ww}^2 j_i(x, w_{\mathbf{m},i}) \leq \beta_1 \sum_{k=0}^{N-1} \int_{\Omega} \dot{w}_{\mathbf{m},k+1}^2.$$

Setting $\beta := \beta_0 + \beta_1$ we obtain the following estimate on \ddot{J} :

$$\ddot{J}(\mathbf{m})[\mathbf{h}, \mathbf{h}] \geq \alpha \sum_{k=0}^{N-1} \int_{\Omega} |\nabla \dot{w}_{\mathbf{m},k+1}|^2 - \beta \sum_{k=0}^{N-1} \int_{\Omega} \dot{w}_{k+1}^2.$$

□

With this estimate at hand, we can conclude the proof of theorem II.

Proof of Theorem II. We argue by contradiction and fix a maximiser $\mathbf{m}^* = (m_1^*, \dots, m_N^*)$ that is not bang-bang. In particular, one of the m_i^* is not bang-bang. Let i^* be such that $m_{i^*}^*$ is not bang-bang, and let

$$\omega^* := \{0 < m_{i^*}^* < 1\}.$$

By construction

$$\text{Vol}(\omega^*) > 0.$$

We now construct a perturbation \mathbf{h} at \mathbf{m}^* in order to derive the conclusion. This \mathbf{h} has the form

$$\mathbf{h} = (0, \dots, h_{i^*}, \dots, 0).$$

To yield a contradiction, we need to choose h_{i^*} such that

1.
$$\int_{\Omega} h_{i^*} = 0, h_{i^*} \text{ is supported in } \omega^* \text{ and } \|h_{i^*}\|_{L^2} = 1. \quad (3.13)$$

2. The associated $\dot{w}_{\mathbf{m}}$ must satisfy

$$\alpha \sum_{k=0}^{N-1} \int_{\Omega} |\nabla \dot{w}_{\mathbf{m},k+1}|^2 > \sum_{k=0}^{N-1} \beta \int_{\Omega} \dot{w}_{k+1}^2.$$

First observe that for such an \mathbf{h} we have $\dot{w}_{\mathbf{m},i} = 0$ if $i < i^*$. Let for any $k \in \mathbb{N}^*$ and any $i \in \{i^*, \dots, N\}$ $\phi_{k,i}$ be the i -th eigenfunction, associated with the eigenvalue $\lambda_{k,i}$, of the operator

$$\mathcal{L}_i := -\Delta - W_k + \frac{1}{\delta t},$$

endowed with Neumann boundary conditions. If we fix a large integer K , if we can choose h_{i^*} such that (3.13) holds and such that for every $i \geq i^*$ we have, in the spectral basis $\{\phi_{k,i}\}_{k \in \mathbb{N}}$,

$$\dot{w}_{\mathbf{m},i} = \sum_{k=K}^{\infty} a_{k,i} \phi_{k,i} \quad (3.14)$$

then we obtain the lower bound

$$\begin{aligned} \ddot{J}(\mathbf{m})[\mathbf{h}, \mathbf{h}] &\geq \alpha \sum_{i=i^*}^{\infty} \int_{\Omega} |\nabla \dot{w}_{\mathbf{m},i}|^2 - \beta \sum_{i=i^*}^N \int_{\Omega} \dot{w}_{\mathbf{m},i}^2 \\ &\geq \alpha \sum_{i=i^*}^{\infty} \left(\int_{\Omega} |\nabla \dot{w}_{\mathbf{m},i}|^2 - \left(W_i - \frac{1}{\delta t} \right) \dot{w}_{\mathbf{m},i}^2 \right) - \sum_{i=i^*}^N \int_{\Omega} \dot{w}_{\mathbf{m},i}^2 \left(\beta + \alpha W_i - \frac{1}{\delta t} \right) \\ &\geq \alpha \sum_{k=K}^{\infty} \sum_{i=i^*}^N \lambda_{k,i} a_{k,i}^2 - \beta' \sum_{k=K}^{\infty} \sum_{i=i^*}^N a_{k,i}^2, \end{aligned}$$

for $\beta' = \beta + \frac{1}{\delta t} + \alpha \sup_{i \in \{i^*, \dots, N\}} \|W_i\|_{L^\infty}$. For K large enough, this immediately gives $\ddot{J}(\mathbf{m})[\mathbf{h}, \mathbf{h}] > 0$.

Thus it suffices to build \mathbf{h} such that (3.13)-(3.14) hold simultaneously.

Heuristic We proceed inductively to explain our construction. First, we explain how to build $h_{i^*}^0$ such that, in the spectral basis $\{\phi_{k,i^*}\}_{k \in \mathbb{N}}$ we have

$$h_{i^*}^0 w_{\mathbf{m},i^*} = \sum_{k=K}^{\infty} a_{k,i^*}^0 \phi_{k,i^*}. \quad (3.15)$$

That such an admissible perturbation exists for any $k \in \mathbb{N}$ is guaranteed by the following fact: as $\text{Vol}(\omega^*) > 0$, the space $E := L^2(\omega^*)$ is infinite dimensional. Consider the family of functionals

$$R_0 : E \ni h \mapsto \int_{\omega^*} h \text{ and } \forall k \in \{0, \dots, K-1\}, T_{k,i^*} : E \ni h \mapsto \int_{\omega^*} h w_{\mathbf{m},i^*} \phi_{k,i^*}.$$

Each of this functional is continuous on E and thus if we define

$$F_{i^*} := \ker(R_0) \cap \bigcap_{k=0}^{K-1} \ker(T_{k,i^*})$$

then F_{i^*} has finite co-dimension. Consequently, there exists $h_{i^*}^0 \in F_{i^*} \setminus \{0\}$ supported in ω such that $\|h_{i^*}^0\|_{L^2(\omega^*)} = 1$ (which we extend by setting $H_{i^*} = h_{i^*}^0 \mathbf{1}_{\omega^*}$) and such that (3.15) holds. As $w_{\mathbf{m},i^*}$ solves

$$\mathcal{L}_{i^*} \dot{w}_{\mathbf{m},i^*} = h w_{\mathbf{m},i^*} \quad (3.16)$$

we obtain

$$\dot{w}_{\mathbf{m},i^*} = \sum_{k=K}^{\infty} \frac{a_{k,i^*}^0}{\lambda_{k,i^*}} \phi_{k,i^*}. \quad (3.17)$$

However, if we merely choose h_{i^*} like this, we may have problems when considering the spectral decomposition of $w_{\mathbf{m},i^*+1}$. Indeed, since $h_{i^*+1} = 0$ in our construction, $\dot{w}_{\mathbf{m},i^*+1}$ solves

$$\mathcal{L}_{i^*+1} \dot{w}_{\mathbf{m},i^*+1} = \frac{1}{\delta t} \dot{w}_{\mathbf{m},i^*},$$

it appears that we must choose $h_{i^*}^0$ such that in the spectral basis $\{\phi_{k,i^*+1}\}$ $\dot{w}_{\mathbf{m},i^*}$ has a decomposition of the form

$$\dot{w}_{\mathbf{m},i^*} = \sum_{k=K}^{\infty} a_{k,i^*+1}^0 \phi_{k,i^*+1}.$$

In other words, we must ensure that

$$\forall k \in \{0, \dots, K\}, T_{k,i^*+1}(h_{i^*}^0) := \int_{\Omega} \dot{w}_{\mathbf{m},i^*} \phi_{k,i^*+1} = 0.$$

Let us first note that for any $k \in \{0, \dots, K-1\}$ the map T_{k,i^*+1} is linear in $h_{i^*}^0$. Furthermore, from standard elliptic regularity, T_{k,i^*+1} is continuous on E . Thus, it would suffice to choose $h_{i^*}^0$ in the set F_{i^*+1} where F_{i^*+1} is defined as

$$F_{i^*+1} = F_{i^*} \cap \bigcap_{k=0}^{K-1} \ker(T_{k,i^*+1}).$$

This indicates how to construct the function h_{i^*} .

Construction of h_{i^*} We define the following family of maps on $E = L^2(\omega^*)$:

1. $R_0 : E \ni h \mapsto \int_{\omega} h$
2. For any $k \leq K$ we define

$$T_{k,i^*} : E \ni h \mapsto \int_{\omega^*} h w_{\mathbf{m},i^*} \phi_{k,i^*}.$$

3. For any $i \in \{i^* + 1, \dots, N\}$, for any $k \leq K$,

$$T_{k,i} : E \ni h \mapsto \int_{\Omega} \dot{w}_{\mathbf{m},i-1} \phi_{k,i}.$$

From elliptic regularity, each of these maps is continuous on E . Consequently, the space

$$F := \ker(R_0) \cap \bigcap_{i=i^*}^N \bigcap_{k=0}^{K-1} \ker(T_{k,i})$$

has finite codimension. We pick $h_{i^*} \in F \setminus \{0\}$ and up to a rescaling assume that

$$\|h_{i^*}\|_{L^2(\omega^*)} = 1.$$

We extend h_{i^*} by 0 outside of Ω^* and, by construction, for any $i \in \{i^*, \dots, N\}$, (3.14) holds. This concludes the proof. □

4 Conclusion

4.1 Possible generalisations of theorem I

4.1.1 General comment about generalisations

Throughout these generalisations, we still assume that we are working with an initial condition $u^0 \in \mathcal{C}^2(\mathbb{T})$ with $\inf_{\mathbb{T}} u^0 > 0$. Let us draw attention to the fact that the core idea of the proof

of theorem I consists in combining two ingredients: the first one is proposition 15, which gives a lower estimate of \mathcal{J} , and the second one is a two scale asymptotic expansion. This second part is independent of the functionals j_1, j_2 , the monotonicity of which are only used in the first step. To generalise our model to other types of interactions, some assumptions will ensure that proposition 15 remain valid. A crucial part in deriving the conclusion however is estimate (2.43). To obtain it, we used the fact that in our bilinear model we have $\inf_{(0,T) \times \mathbb{T}} |u_m| > 0$. This plays a role when using the fact that \dot{u}_m solves

$$\frac{\partial \dot{u}_m}{\partial t} - \Delta \dot{u}_m - V_m \dot{u}_m = u_m h.$$

In other types of model, \dot{u}_m solves (generically) an equation of the form

$$\frac{\partial \dot{u}_m}{\partial t} - \Delta \dot{u}_m - V_m \dot{u}_m = F(u_m, m)h,$$

and other assumptions will thus ensure that $\inf_{(0,T) \times \mathbb{T}} F(u_m, m) > 0$.

4.1.2 Approximations of time dependent controls

Although we can not handle general time-dependencies, see section 4.3, we would nonetheless like to draw attention to the fact that our method covers some approximations of time-varying controls. Consider an integer N and a family of functions $\{\phi_i\}_{i=1, \dots, N}$ satisfying the following conditions:

$$\text{For any } i \in \{1, \dots, N\}, \phi_i \in \mathcal{C}^1([0, T], \mathbb{R}) \quad (\mathbf{A}_1)$$

$$\text{For any } i \in \{0, \dots, N\}, \inf_{[0, T]} |\phi_i| > 0. \quad (\mathbf{A}_2)$$

Let

$$\mathcal{M}_N(\mathbb{T}) := \left\{ m \in L^\infty((0, T) \times \mathbb{T}) \text{ that write } m = \sum_{i=1}^N \phi_i(t) m_i \right. \\ \left. \text{where for any } i \in \{1, \dots, N\}, m_i \in \mathcal{M}(\mathbb{T}) \right\}. \quad (4.1)$$

A generic $\bar{m} \in \mathcal{M}_N(\mathbb{T})$ is identified with the associated N -tuple $(m_1, \dots, m_N) \in \mathcal{M}(\mathbb{T})^N$. A function $\bar{m} \in \mathcal{M}_N(\mathbb{T})$ is called *bang-bang* if for any $i \in \{1, \dots, N\}$ m_i is a bang-bang function.

We can define $u_{\bar{m}}$ as the solution of (1.2) with m replaced with \bar{m} , and the optimisation problem is

$$\boxed{\max_{\bar{m} \in \mathcal{M}_N(\mathbb{T})} \mathcal{J}(\bar{m})}, \quad (\mathbf{P}_{\text{parab}}^N)$$

where \mathcal{J} still satisfies assumption $(\mathbf{H}_{\mathcal{J}})$ of Theorem I. We claim that, up to minor adaptations of our proof, the following result holds:

Theorem III. *Assume \mathcal{J} satisfies $(\mathbf{H}_{\mathcal{J}})$ and $\phi = \{\phi_i\}_{i=1, \dots, N}$ satisfies (\mathbf{A}_1) - (\mathbf{A}_2) . Any solution \bar{m}^* of $(\mathbf{P}_{\text{parab}}^N)$ is bang-bang: there exist $E_1, \dots, E_N \subset \mathbb{T}$ such that*

$$\bar{m}^* = \sum_{i=0}^N \phi_i(t) \mathbf{1}_{E_i}.$$

Sketch of proof of theorem III First of all, we once again have

$$\inf_{(0,T) \times \mathbb{T}} u_{\bar{m}} > 0 \text{ and for any } p \in [1; +\infty) \sup_{t \in [0, T]} \|u_{\bar{m}}(t, \cdot)\|_{W^{2,p}(\mathbb{T})} < \infty.$$

We can compute, for an admissible $\bar{m} \in \mathcal{M}_N(\mathbb{T})$ and an admissible perturbation $\bar{h} = (h_1, \dots, h_N)$ at \bar{m} , the first and second order derivatives of $\bar{m} \mapsto u_{\bar{m}}$ in the direction \bar{h} solve, respectively,

$$\begin{cases} \frac{\partial \dot{u}_{\bar{m}}}{\partial t} - \Delta \dot{u}_{\bar{m}} - V_{\bar{m}} \dot{u}_{\bar{m}} = u_{\bar{m}} \sum_{i=1}^N \phi_i(t) h_i & \text{in } (0, T) \times \mathbb{T}, \\ \dot{u}_{\bar{m}}(0, \cdot) \equiv 0 & \text{in } \mathbb{T}. \end{cases} \quad (4.2)$$

and

$$\begin{cases} \frac{\partial \ddot{u}_{\bar{m}}}{\partial t} - \Delta \ddot{u}_{\bar{m}} - V_{\bar{m}} \ddot{u}_{\bar{m}} = 2\dot{u}_{\bar{m}} \sum_{i=1}^N \phi_i(t) h_i + \frac{\partial^2 f}{\partial u^2} \Big|_{u=u_{\bar{m}}} (\dot{u}_{\bar{m}})^2 & \text{in } (0, T) \times \mathbb{T}, \\ \ddot{u}_{\bar{m}}(0, \cdot) \equiv 0 & \text{in } \mathbb{T} \end{cases} \quad (4.3)$$

with

$$V_{\bar{m}} := \left(\bar{m} + \frac{\partial f}{\partial u} \Big|_{u=u_{\bar{m}}} \right).$$

We introduce the adjoint state $p_{\bar{m}}$, solution of

$$\begin{cases} \frac{\partial p_{\bar{m}}}{\partial t} + \Delta p_{\bar{m}} + V_{\bar{m}} p_{\bar{m}} = - \frac{\partial j_1}{\partial u} \Big|_{u=u_{\bar{m}}} & \text{in } (0, T) \times \mathbb{T}, \\ p_{\bar{m}}(T, \cdot) = \frac{\partial j_2}{\partial u} \Big|_{u=u_{\bar{m}}} & \text{in } \mathbb{T}. \end{cases} \quad (4.4)$$

From the same arguments as in lemma 14 we have

$$\forall \varepsilon > 0, \inf_{(0, T-\varepsilon) \times \mathbb{T}} p_{\bar{m}} > 0 \text{ and for any } p \in [1; +\infty) \sup_{t \in [0, T]} \|p_{\bar{m}}(t, \cdot)\|_{W^{2,p}(\mathbb{T})} < \infty.$$

For any admissible perturbation \bar{h} we then have

$$\begin{aligned} \ddot{\mathcal{J}}(\bar{m})[\bar{h}, \bar{h}] &= 2 \iint_{(0, T) \times \mathbb{T}} \dot{u}_{\bar{m}} p_{\bar{m}} \left\{ \sum_{i=1}^N \phi_i(t) h_i(t) \right\} + \int_{\mathbb{T}} \dot{u}_{\bar{m}}^2(T, \cdot) \frac{\partial^2 j_2}{\partial u^2} \Big|_{u=u_{\bar{m}}(T, \cdot)} \\ &\quad + \iint_{(0, T) \times \mathbb{T}} \dot{u}_{\bar{m}}^2 \left(\frac{\partial^2 j_1}{\partial u^2} \Big|_{u=u_{\bar{m}}} + \frac{\partial^2 f}{\partial u^2} \Big|_{u=u_{\bar{m}}} p_{\bar{m}} \right). \end{aligned} \quad (4.5)$$

We then use the fact that

$$\sum_{i=1}^N \phi_i(t) h_i = \frac{\frac{\partial \dot{u}_{\bar{m}}}{\partial t} - \Delta \dot{u}_{\bar{m}} - V_{\bar{m}} \dot{u}_{\bar{m}}}{u_{\bar{m}}}.$$

From this point on, we can follow all the steps of the proof of proposition 15 to obtain the existence of three positive constants α, β, γ and of a positive $\varepsilon > 0$ such that, for any admissible perturbation \bar{h} there holds

$$\ddot{\mathcal{J}}(\bar{m})[\bar{h}, \bar{h}] \geq \alpha \iint_{(0, T-\varepsilon) \times \mathbb{T}} |\nabla \dot{u}_{\bar{m}}|^2 - \beta \iint_{(0, T) \times \mathbb{T}} \dot{u}_{\bar{m}}^2 - \gamma \int_{\mathbb{T}} \dot{u}_{\bar{m}}^2(T, \cdot).$$

We then argue by contradiction, assuming that there exists a maximiser \bar{m} and an index $j \in \{1, \dots, N\}$ such that m_j^* is not bang-bang, so that $\omega := \{0 < m_j^* < 1\}$ has positive measure.

We fix this index j and henceforth only consider perturbations \bar{h} of the form $(0, \dots, h_j, \dots, 0)$ with h_j an admissible perturbation at m_j supported in ω_j , and admitting the Fourier decomposition

$$h_j = \sum_{k=K}^{\infty} a_k \cos(kx) + b_k \sin(kx).$$

For such a perturbation, the derivative $\dot{u}_{\bar{m}}$ solves

$$\begin{cases} \frac{\partial \dot{u}_{\bar{m}}}{\partial t} - \Delta \dot{u}_{\bar{m}} - V_{\bar{m}} \dot{u}_{\bar{m}} = u_{\bar{m}} \phi_j(t) h_j & \text{in } (0, T) \times \mathbb{T}, \\ \dot{u}_{\bar{m}}(0, \cdot) \equiv 0 & \text{in } \mathbb{T}, \end{cases} \quad (4.6)$$

and given that $\inf_{(0, T) \times \mathbb{T}} |u_{\bar{m}} \phi_j| > 0$ we can conclude in exactly the same way.

4.1.3 Other types of interactions: generalisation and obstruction

One may argue that other types of interactions can be relevant. To motivate this point, let us consider another type of model from spatial ecology, where one rather aims at optimising a certain criterion for a state equation of the form

$$\begin{cases} \frac{\partial y_m}{\partial t} - \Delta y_m = f(t, x, y_m) + m \varphi(y_m) & \text{in } (0, T) \times \mathbb{T}, \\ y_m(0, \cdot) = y^0 & \text{in } \mathbb{T}, \end{cases} \quad (4.7)$$

where $y^0 \in \mathcal{C}^2(\mathbb{T})$ and $\inf_{\mathbb{T}} y^0 > 0$ is a fixed initial condition, f and φ are non-linearities that must satisfy that for any $m \in \mathcal{M}(\mathbb{T})$, y_m satisfies

$$\inf_{(0, T) \times \mathbb{T}} y_m > 0 \quad (4.8)$$

and

$$\forall T > 0, \sup_{t \in [0, T]} \|y_m(t, \cdot)\|_{L^\infty} < \infty. \quad (4.9)$$

We aim at optimising

$$\mathcal{J} : \mathcal{M}(\mathbb{T}) \ni m \mapsto \iint_{(0, T) \times \mathbb{T}} j_1(t, x, y_m) + \int_{\mathbb{T}} j_2(x, y_m(T, \cdot)) \quad (4.10)$$

and assume that \mathcal{J} satisfies $(\mathbf{H}_{\mathcal{J}})$.

We claim that, up to minor adaptation of the proof of theorem I, the following result holds:

Theorem IV. *Assume that φ is \mathcal{C}^2 on \mathbb{R}_+^* . If, for any $K \in \mathbb{R}_+^*$, for any $\varepsilon > 0$,*

$$\inf_{y \in (0, K)} \frac{\varphi'(y)}{\varphi(y)} = a_1(K) > 0, \quad \inf_{y \in (\varepsilon, K)} |\varphi(y)| = a_2(\varepsilon, K) > 0 \quad (4.11)$$

then any solution m_φ^ of the optimisation problem*

$$\sup_{m \in \mathcal{M}(\mathbb{T})} \mathcal{J}(m)$$

is bang-bang.

Let us explain why this type of setting is relevant in application: consider the case of the logistic-diffusive equation

$$\begin{cases} \frac{\partial y_m}{\partial t} - \Delta y_m = y_m(1 - my_m) & \text{in } (0, T) \times \mathbb{T}, \\ y_m(0, \cdot) = y^0 & \text{in } \mathbb{T}, \end{cases} \quad (4.12)$$

as well as the functional

$$\mathcal{J}(m) := \int_{\mathbb{T}} y_m(T, \cdot).$$

Maximising \mathcal{J} with respect to $m \in \mathcal{M}(\mathbb{T})$ amounts to optimising the total population size with respect to m , the inverse of the carrying capacity (at this stage, one may argue that it would make more sense to consider the case of m satisfying $\varepsilon \leq m \leq 1$ in the definition of $\mathcal{M}(\mathbb{T})$; given remark 6, we claim that this would not change anything to the conclusion of theorem IV). Such a problem is inspired by the considerations of [20].

Sketch of proof of theorem IV We start by noticing that

$$\inf_{(0, T) \times \mathbb{T}} y_m > 0 \text{ and for any } p \in [1; +\infty) \sup_{t \in [0, T]} \|y_m(t, \cdot)\|_{W^{2,p}(\mathbb{T})} < \infty.$$

Let $m \in \mathcal{M}(\mathbb{T})$, and consider an admissible perturbation h at m ; the first and second order derivatives of $m \mapsto u_m$ in the direction h solve,

$$\begin{cases} \frac{\partial \dot{y}_m}{\partial t} - \Delta \dot{y}_m - V_m \dot{y}_m = \varphi(y_m)h & \text{in } (0, T) \times \mathbb{T}, \\ \dot{y}_m(0, \cdot) \equiv 0 & \text{in } \mathbb{T}. \end{cases} \quad (4.13)$$

and

$$\begin{cases} \frac{\partial \ddot{y}_m}{\partial t} - \Delta \ddot{y}_m - V_m \ddot{y}_m = 2\dot{y}_m h \varphi'(y_m) + \left(m\varphi''(y_m) + \frac{\partial^2 f}{\partial u^2} \Big|_{u=y_m} \right) \dot{y}_m^2 & \text{in } (0, T) \times \mathbb{T}, \\ \ddot{y}_m(0, \cdot) \equiv 0 & \text{in } \mathbb{T} \end{cases} \quad (4.14)$$

with

$$V_m := \left(m + \frac{\partial f}{\partial u} \Big|_{u=y_m} + \varphi'(y_m) \right).$$

We introduce the adjoint state q_m , solution of

$$\begin{cases} \frac{\partial q_m}{\partial t} + \Delta q_m + V_m q_m = - \frac{\partial j_1}{\partial u} \Big|_{u=y_m} & \text{in } (0, T) \times \mathbb{T}, \\ q_m(T, \cdot) = \frac{\partial j_2}{\partial u} \Big|_{u=y_m} & \text{in } \mathbb{T}. \end{cases} \quad (4.15)$$

From the same arguments as in lemma 14 we have

$$\forall \varepsilon > 0, \inf_{(0, T-\varepsilon) \times \mathbb{T}} q_m > 0 \text{ and for any } p \in [1; +\infty) \sup_{t \in [0, T]} \|q_m(t, \cdot)\|_{W^{2,p}(\mathbb{T})} < \infty.$$

For any admissible perturbation \bar{h} we then have

$$\tilde{\mathcal{J}}(m)[\bar{h}, \bar{h}] = 2 \iint_{(0, T) \times \mathbb{T}} \dot{y}_m \varphi'(y_m) q_m h + \int_{\mathbb{T}} \dot{y}_m^2(T, \cdot) \frac{\partial^2 j_2}{\partial u^2} \Big|_{u=y_m(T, \cdot)}$$

$$+ \iint_{(0,T) \times \mathbb{T}} \dot{y}_m^2 \left(\frac{\partial^2 j_1}{\partial u^2} \Big|_{u=y_m} + \frac{\partial^2 f}{\partial u^2} \Big|_{u=y_m} q_m + m\varphi''(y_m) \right). \quad (4.16)$$

We then use the fact that

$$h = \frac{\frac{\partial \dot{y}_m}{\partial \varepsilon} - \Delta \dot{y}_m - V_m \dot{y}_m}{\varphi(y_m)}.$$

Following all the steps of the proof of proposition 15, we obtain the existence of two constants β, γ such that, for any admissible perturbation h at m , there holds

$$\ddot{\mathcal{J}}(m)[h, h] \geq \iint_{(0,T) \times \mathbb{T}} \frac{q_m \varphi'(y_m)}{\varphi(y_m)} |\nabla \dot{y}_m|^2 - \beta \iint_{(0,T) \times \mathbb{T}} \dot{y}_m^2 - \gamma \int_{\mathbb{T}} \dot{y}_m^2(T, \cdot).$$

We then use the assumption to obtain the existence of an $\varepsilon > 0$ such that we have, for any admissible perturbation h at m , the estimate

$$\begin{aligned} & \ddot{\mathcal{J}}(m)[h, h] \\ & \geq \left(\inf_{(0, T-\varepsilon) \times \mathbb{T}} q_m \right) \frac{a_1(\sup_{(0,T) \times \mathbb{T}} y_m)}{a_2(\inf_{(0,T) \times \mathbb{T}} y_m, \sup_{(0,T) \times \mathbb{T}} y_m)} \iint_{(0, T-\varepsilon) \times \mathbb{T}} |\nabla \dot{y}_m|^2 - \beta \iint_{(0,T) \times \mathbb{T}} \dot{y}_m^2 - \gamma \int_{\mathbb{T}} \dot{y}_m^2(T, \cdot) \\ & \geq \alpha \iint_{(0, T-\varepsilon) \times \mathbb{T}} |\nabla \dot{y}_m|^2 - \beta \iint_{(0,T) \times \mathbb{T}} \dot{y}_m^2 - \gamma \int_{\mathbb{T}} \dot{y}_m^2(T, \cdot) \end{aligned}$$

for a positive $\alpha > 0$. We then follow exactly the same steps.

4.1.4 Some interactions not covered by our method

However, despite their interest, our generalisations, theorems IV and III, do not cover several cases. A typical example of such an interaction between the state and the control is, typically, of the form $m\varphi(u_m)$ with φ an increasing, negative function. Indeed, the following is easily checked *via* the same computations: let φ be a smooth function such that

$$\forall K, \varepsilon > 0, \sup_{(0;K)} \frac{\varphi'}{\varphi} = -a_1 < 0, \inf_{[\varepsilon;K]} |\varphi| = a_2(\varepsilon, K) > 0,$$

and consider the solution z_m of

$$\begin{cases} \frac{\partial z_m}{\partial t} - \Delta z_m = f(t, x, z_m) + m\varphi(z_m) & \text{in } (0, T) \times \mathbb{T}, \\ z_m(0, \cdot) = z^0 & \text{in } \mathbb{T}, \end{cases} \quad (4.17)$$

where $\inf_{\mathbb{T}} z^0 > 0$ and φ and f are further chosen to satisfy

$$\forall m \in \mathcal{M}(\mathbb{T}), \inf_{(0,T) \times \mathbb{T}} z_m > 0, \sup_{t \in [0, T]} \|z_m(t, \cdot)\|_{W^{2,p}(\mathbb{T})} < \infty.$$

Then, considering a functional \mathcal{J} that satisfies (H $_{\mathcal{J}}$), there exists a positive constant $\alpha > 0$ and two constants $\beta, \gamma > 0$ such that

$$\ddot{\mathcal{J}}(m)[h, h] \leq -\alpha \iint_{(0, T-\varepsilon) \times \mathbb{T}} |\nabla \dot{z}_m|^2 + \beta \iint_{(0,T) \times \mathbb{T}} \dot{z}_m^2 + \gamma \int_{\mathbb{T}} \dot{z}_m^2(T, \cdot),$$

where \dot{z}_m is of course the derivative of $m \mapsto z_m$ at m in the direction h . In this case, our method fails to provide a bang-bang property for maximisers, but yields a bang-bang property for minimisers.

4.2 Generalisations and obstructions for Theorem I

In this section, we present several possible obstructions and generalisations of theorem I and of our methods to other contexts (*e.g* to the multi-dimensional case or to other geometries), pinpointing what the main difficulties seem to be.

4.2.1 Higher dimensional tori

We believe that our method extends, in a straightforward manner, to the case of d -dimensional tori, for any $d \in \mathbb{N} \setminus \{0\}$; once again, two steps are crucial in deriving theorem I. The first one, proposition 15, is an estimate on the second order Gateaux derivative of the functional which does not depend on the dimension, see in particular remark 17. The second one is to establish a two-scale asymptotic expansions for solutions of a linear heat equation with a highly oscillating source term. We claim that this step can be extended in a straightforward way to the d dimensional torus, provided the functions $\cos(k \cdot)$ and $\sin(k \cdot)$ are replaced with products of the form $\prod_{i=1}^d \cos(k_i x) \sin(k'_i x)$ or $\prod_{i=1}^d \cos(k_i x)$ where $(k_1, \dots, k_d, k'_1, \dots, k'_d) \in \mathbb{N}^{2d}$.

4.2.2 Possible obstructions in other domains

It would be extremely interesting and relevant, in many applications, to consider not only the case of bounded domains $\Omega \subset \mathbb{R}^d$ with, for instance, Neumann or Robin boundary conditions (Dirichlet boundary conditions may not be suitable for our needs, as we need, in a crucial manner, a uniform lower bound on solutions of the equation). In this context, we claim that the lower estimate given by proposition 15 still holds, as is clear in the proof and in remark 17. The main difficulty lies elsewhere, namely, in the possibility to attain two-scale asymptotic expansions in order to derive the bang-bang property. A possibility to do so would be to replace the $\cos(k \cdot), \sin(k \cdot)$ with ψ_k , where the $\{\psi_k\}_{k \in \mathbb{N}}$ or the Neumann (if working with Neumann boundary conditions) or Robin (if working with Robin boundary conditions) eigenfunctions of the laplacian in Ω , associated with the (increasing) sequence of eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}}$. Let us assume that we are working with Neumann boundary conditions. Then the task at hand would be, if we mimicked our approach, to find an asymptotic expansion for a solution \dot{u}_m of

$$\begin{cases} \frac{\partial \dot{u}_m}{\partial t} - \Delta \dot{u}_m - V_m \dot{u}_m = u_m(t, x) \sum_{k=K}^{\infty} a_k \psi_k(x) & \text{in } (0, T) \times \Omega, \\ \frac{\partial \dot{u}_m}{\partial \nu} = 0 & \text{on } \partial \Omega, \\ \dot{u}_m(0, \cdot) = 0 & \text{in } \Omega. \end{cases}$$

It is unclear, in this situation, which asymptotic expansion would yield a result analogous to that of proposition 19.

4.2.3 Possible obstructions for other diffusion operators

A very relevant query, if we keep application to mathematical biology in mind, is the analysis of heterogeneous diffusion operators. In other words, following, for instance, [8], one may rather be interested in state equations assuming the form

$$\frac{\partial u_m}{\partial t} - \nabla \cdot (A \nabla u_m) = m u_m + f(t, x, u)$$

where $A = A(t, x)$ accounts for some heterogeneity. It is likely that our methods extend to this case, provided A is smooth enough to guarantee uniform (in time) $W^{2,p}(\mathbb{T})$ (in space) estimates on the solution u_m . Although this can be of interest, since our main goal is the analysis of shape optimisation problems in optimal control problems, this is not the most relevant analytical setup for

us. Indeed, in the context of spatial ecology, the diffusion matrix A and the resources distribution m are often linked, which will lead to very intricate situations from the regularity point of view. For an example of such an optimisation problem and of the wealth of qualitative and technical issues it can lead to we refer to the elliptic optimisation problem studied in [36].

4.2.4 Some possible open questions

Of course, the bang-bang property is one of the many qualitative aspects of bilinear optimal control problems. Even when m does not depend on time, and we know that maximisers are bang-bang, what do these optimisers look like from a geometric point of view? In other words, considering a maximiser $m^* = \mathbb{1}_{E^*}$, what are the geometric and topological features of E^* ? Is it connected, disconnected, and how may we quantify such information? Let us underline here that the functionals under consideration in this paper are non-energetic, which prohibits the use of rearrangement techniques. Such techniques, developed in the context of mathematical biology in [9] for instance, although very powerful for energetic or spectral optimisation problems in the elliptic case [27] or for the study of concentration phenomena in parabolic models [3, 4, 5, 6, 33, 41, 42, 47] can not yield satisfying results for non-energetic problems. As an example, we refer to the elliptic problem of optimising the total population size described in section 1.6 and, more specifically, to the results of [23, 39, 40]: in the elliptic context, depending on the dispersal rate of the population, optimal resources distributions are either concentrated (and when that dispersal rate is high enough we can apply symmetrisation properties) or display hectic oscillations (this corresponds to the low dispersal rate limit, and is clearly a case where it is hopeless to apply rearrangements). The study of these properties in parabolic models seems challenging, and we plan on studying it in further works.

4.3 The difficulty with general time dependent controls

Finally, let us underline the core difficulties in reaching the bang-bang property for general time-dependent controls. In other words, assume we are working with controls m satisfying

$$m \in \mathcal{M}((0, T) \times \mathbb{T}) \\ = \left\{ m \in L^\infty((0, T) \times \mathbb{T}) : 0 \leq m \leq 1 \text{ a.e. and for a.e. } t \in [0, T] \int_{\mathbb{T}} m(t, \cdot) = m_0 \right\},$$

and we are studying the maximisation of \mathcal{J} over $\mathcal{M}((0, T) \times \mathbb{T})$. Two problems rapidly arise. The first one is that, as noted several times, the regularity of u_m is crucial in deriving proposition 15; this necessarily requires some *a priori* assumptions on the regularity of the control m in time. The second difficulty is in defining, for a given maximiser m^* that would, arguing by contradiction, not be bang-bang, a highly oscillating perturbation h . One may be tempted to reason in a way analogous to that of theorem II and to choose a perturbation h supported in the right set (*i.e.* in the set $\{0 < m^* < 1\}$) by reasoning as follows: define, for a.e. $t \in [0, T]$, $\omega(t) := \{t\} \times \{0 < m^* < 1\}$ and consider a function h_t supported in $\omega(t)$ that only has high (enough) Fourier modes. One would then define the perturbation as $h(t, x) = h_t(x)$. The problem here is that there is no guarantee that such a function h is measurable in (t, x) .

On the other hand, enforcing strong time regularity constraints on the control m (such as, for instance, $m \in \mathcal{C}^1([0, T]; \mathcal{M}(\mathbb{T}))$) may allow such constructions to work. This is however, beyond the scope of our article, and we plan on investigating the influence of time-regularity constraints on the bang-bang property in the future.

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A Study of the parabolic model

A.1 Proof of lemma 13

Proof of lemma 13. Given that f satisfies (\mathbf{H}_3) , u_m satisfies the inequality

$$\begin{cases} \frac{\partial u_m}{\partial t} - \Delta u_m \geq mu_m + f(t, x, 0) - A|u_m| \geq mu_m - A|u_m| & \text{in } (0, T) \times \mathbb{T} \\ u_m(0, \cdot) = u^0 & \text{in } \mathbb{T}. \end{cases} \quad (\text{A.1})$$

Multiplying this equation by the negative part u_m^- of u_m and integrating by parts gives

$$\frac{\partial}{\partial t} \int_{\mathbb{T}} (u_m^-)^2 + \int_{\mathbb{T}} (u_m^-)^2 \leq A \int_{\Omega} (u_m^-)^2$$

and since $u_m^-(0, \cdot) \equiv 0$ we obtain

$$u_m \geq 0 \text{ in } (0, T) \times \mathbb{T}.$$

To derive that

$$\inf_{(0, T) \times \mathbb{T}} u_m > 0$$

it suffices to apply the strong maximum principle.

For the upper bound, let κ be given by (\mathbf{H}_2) . Up to replacing κ with $\max\{\|u^0\|_{L^\infty}, \kappa\}$ we may assume that

$$\kappa \geq \|u^0\|_{L^\infty}.$$

Let $z_m := u_m - \kappa$. z_m solves the partial differential equation

$$\begin{aligned} \frac{\partial z_m}{\partial t} - \Delta z_m &= mu_m + f(t, x, u_m) = mz_m + f(t, x, u_m) + \kappa m \\ &= mz_m + f(t, x, u_m) - f(t, x, \kappa) + \kappa m + f(t, x, \kappa) \\ &\leq mz_m + f(t, x, u_m) - f(t, x, \kappa) \\ &\leq (m + A)|z_m|. \end{aligned}$$

As $z_m(0, \cdot) \leq 0$, the conclusion follows: $z_m \leq 0$ in $(0, T) \times \mathbb{T}$ so that $u_m \leq \kappa$. \square

A.2 Regularity results: proof of proposition 16

To prove proposition 16 we need an auxilliary result:

Lemma 24. *For any $p \in [1; +\infty)$ there exists a constant C_p such that the following holds: there exists $q > 1$ such that, for any $\theta^0 \in \mathcal{C}^\infty(\mathbb{T})$, for any $\Theta, W \in W^{1,q}(0, T; L^q(\mathbb{T})) \cap L^\infty((0, T) \times \mathbb{T})$, if θ be the unique $L^p(0, T; W^{1,p}(\mathbb{T}))$ solution of*

$$\begin{cases} \partial_t \theta - \partial_{xx} \theta - W\theta = \Theta & \text{in } (0, T) \times \mathbb{T}, \\ \theta(0, \cdot) = \theta^0 & \text{in } \mathbb{T}. \end{cases} \quad (\text{A.2})$$

Then

$$\begin{aligned} \sup_{t \in [0, T]} \|\partial_t \theta(t, \cdot)\|_{L^p(\mathbb{T})} + \sup_{t \in [0, T]} \|\theta(t, \cdot)\|_{W^{2,p}(\mathbb{T})} \\ \leq C_p \left(\|\Theta\|_{W^{1,q}(0, T; L^q(\mathbb{T}))} + \|W\|_{W^{1,q}(0, T; L^q(\Omega))} + \|\theta^0\|_{\mathcal{C}^2(\mathbb{T})} \right). \end{aligned} \quad (\text{A.3})$$

It suffices to take $q = 4\lfloor p \rfloor$.

Proof of Lemma . Let us first prove that for any $p \in [1; +\infty)$ there exists a constant C_p^0 such that

$$\sup_{t \in [0, T]} \|\theta(t, \cdot)\|_{L^p(\mathbb{T})} \leq C_p^0 (\|\Theta\|_{L^\infty((0, T) \times \mathbb{T})} + \|\theta^0\|_{L^q}), \quad (\text{A.4})$$

for q large enough. We first recall the inclusion of Lebesgue spaces: if $p_1 > p_0$ then $L^{p_1}(\mathbb{T}) \hookrightarrow L^{p_0}(\mathbb{T})$. By this inclusion of Lebesgue spaces it suffices to prove (A.4) for $p = 2k$, $k = 1, \dots, n, \dots$. Let $k \in \mathbb{N} \setminus \{0\}$. To obtain (A.4) for $p = 2k$ we use

$$v := 2k\theta^{2k-1}$$

as a test function in the weak formulation of (A.2). We obtain, for a.e. $t \in (0, T)$,

$$\int_{\mathbb{T}} 2k(\partial_t \theta) \theta^{2k-1} + 2k(2k-1) \int_{\mathbb{T}} \theta^{2k-2} |\nabla \theta|^2 - 2k \int_{\mathbb{T}} W \theta^{2k} = 2k \int_{\mathbb{T}} \Theta \theta^{2k-1}.$$

In particular we obtain

$$\partial_t \int_{\mathbb{T}} \theta^{2k} - \|W\|_{L^\infty((0, T) \times \mathbb{T})} \int_{\mathbb{T}} \theta^{2k} \leq 2k \int_{\mathbb{T}} |\Theta| \cdot |\theta|^{2k-1}.$$

We bound the right-hand side using Hölder's inequality:

$$2k \int_{\mathbb{T}} |\Theta| \cdot |\theta|^{2k-1} \leq 2k \|\Theta(t, \cdot)\|_{L^{2k}(\mathbb{T})} \left(\int_{\mathbb{T}} \theta^{2k} \right)^{\frac{2k-1}{2k}}$$

Defining $c_0 := \|W\|_{L^\infty((0, T) \times \mathbb{T})}$, $c_1(t) := 2k \|\Theta(t, \cdot)\|_{L^{2k}(\mathbb{T})}$ and setting

$$y(t) := \int_{\mathbb{T}} \theta^{2k}(t, \cdot)$$

we are left with the differential inequality

$$y'(t) - c_0 y(t) \leq c_1(t) y(t)^{1 - \frac{1}{2k}},$$

which in turn yields

$$y' y^{\frac{1}{2k} - 1} - c_0 y^{\frac{1}{2k}} \leq c_1.$$

We set $\bar{z} = y^{\frac{1}{2k}} e^{-c_0 t}$, which leads to

$$\bar{z}'(t) \leq c_1(t) e^{-c_0 t},$$

and it suffices to integrate this inequality to obtain

$$y_k(t)^{\frac{1}{2k}} \leq e^{c_0 t} y_k(0)^{\frac{1}{2k}} + 2k e^{c_0 t} \int_0^t e^{-c_0 \tau} \|\Theta(\tau, \cdot)\|_{L^{2k}(\mathbb{T})} d\tau.$$

We bound brutally $t \leq T$ in the exponentials, and we obtain, for a constant c_3 ,

$$y_k(t)^{\frac{1}{2k}} \leq 2k c_3 \left(y_k(0)^{\frac{1}{2k}} + \int_0^t \|\Theta(\tau, \cdot)\|_{L^{2k}(\mathbb{T})} d\tau \right).$$

Finally, we use Jensen's inequality:

$$\int_0^t \|\Theta(\tau, \cdot)\|_{L^{2k}(\mathbb{T})} d\tau \leq t^{1 - \frac{2}{k}} \left(\int_0^t \int_{\mathbb{T}} \Theta^{2k} \right)^{\frac{1}{2k}} \leq T^{1 - \frac{2}{k}} \|\Theta\|_{L^{2k}((0, T) \times \mathbb{T})}.$$

Finally:

$$\|\theta(t, \cdot)\|_{L^p(T)} \leq 2kc_3 \left(\|\theta^0\|_{L^{2k}(\mathbb{T})} + T^{1-\frac{2}{k}} \|\Theta\|_{L^{2k}((0,T) \times \mathbb{T})} \right).$$

To derive (A.3) we differentiate (A.2) with respect to time. It appears that $q := \partial_t \theta$ solves

$$\begin{cases} \partial_t q - \Delta q - Wq = \partial_t \Theta + \theta \partial_t W & \text{in } (0, T) \times \mathbb{T}, \\ q(0, \cdot) = W\theta^0 + \Delta\theta^0 + \Theta(0, \cdot) & \text{in } \mathbb{T}. \end{cases} \quad (\text{A.5})$$

We reason once again using only $p = 2k$, $k \in \mathbb{N} \setminus \{0\}$. It suffices to apply (A.4) to obtain

$$\begin{aligned} & \sup_{t \in (0, T)} \|q(t, \cdot)\|_{L^{2k}(\mathbb{T})} \\ & \leq c_3 \left(\|W\theta^0\|_{L^{2k}(\mathbb{T})} + \|\Delta\theta^0\|_{L^{2k}(\mathbb{T})} + \|\Theta(0, \cdot)\|_{L^{2k}(\mathbb{T})} + T^{1-\frac{2}{k}} \|\partial_t \Theta + \theta \partial_t W\|_{L^{2k}((0, T) \times \mathbb{T})} \right) \end{aligned} \quad (\text{A.6})$$

We bound the first terms as follows:

$$\|W\theta^0\|_{L^{2k}(\mathbb{T})} + \|\Delta\theta^0\|_{L^{2k}(\mathbb{T})} + \|\Theta(0, \cdot)\|_{L^{2k}(\mathbb{T})} \leq \|W\theta^0\|_{L^\infty((0, T) \times \mathbb{T})} + \|\theta^0\|_{\mathcal{C}^2(\mathbb{T})} + \|\Theta(0, \cdot)\|_{L^\infty((0, T) \times \mathbb{T})}.$$

It remains to bound

$$\|\partial_t \Theta + \theta \partial_t W\|_{L^{2k}((0, T) \times \mathbb{T})}.$$

To control this term we use the arithmetic-geometric inequality to obtain

$$\begin{aligned} \|\theta \partial_t W\|_{L^{2k}((0, T) \times \mathbb{T})} & \leq \frac{1}{2} \|\theta^2 + (\partial_t W)^2\|_{L^{2k}((0, T) \times \mathbb{T})} \\ & \leq \frac{1}{2} \left(\|\theta\|_{L^{4k}((0, T) \times \mathbb{T})}^2 + \|\partial_t W\|_{L^{4k}((0, T) \times \mathbb{T})}^2 \right) \end{aligned}$$

Thus,

$$\sup_{t \in (0, T)} \|\partial_t \theta(t, \cdot)\|_{L^p(\mathbb{T})} < C$$

and the constant C only depends on the $W^{1,p}(0, T; L^r(\mathbb{T}))$ norms of all the functions involved. To obtain the uniform $W^{2,p}$ estimate, we simply observe that for a.e. $t \in (0, T)$ $\theta(t, \cdot)$ solves

$$-\Delta\theta(t, \cdot) = G := \Theta - \partial_t \theta + W\theta$$

and to apply standard $W^{2,p}$ elliptic regularity estimates. □

We can now give the proof of Proposition 16

Proof of Proposition 16. We observe that u_m solves

$$\partial_t u_m - \frac{\partial^2 u_m}{\partial x^2} - m u_m = f(t, x, u_m).$$

Now define

$$q := \partial_t u_m.$$

As m does not depend on time, we obtain the following equation on q :

$$\begin{cases} \frac{\partial q}{\partial t} - \frac{\partial^2 q}{\partial x^2} - \left(m + \frac{\partial f}{\partial u}(t, x, u_m) \right) q = \frac{\partial f}{\partial t}(t, x, u_m) & \text{in } (0, T) \times \mathbb{T} \\ q(0, \cdot) = \Delta u^0 + m u^0 + f(0, x, u^0). \end{cases}$$

As $u_m \leq K$, $\partial_f(t, x, u_m), \partial_t f(t, x, u_m) \in L^\infty((0, T) \times \mathbb{T})$. From Lemma A.2 we obtain, for any $p \in [1; +\infty)$,

$$\sup_{t \in [0, T]} \|\partial_t u_m\|_{L^p(\mathbb{T})} + \sup_{t \in [0, T]} \|u_m\|_{W^{2,p}(\mathbb{T})} < \infty.$$

In the same way, we derive the desired estimate on p_m . \square

A.3 Proof of lemma 21

Proof of lemma 21. Multiplying the equation by θ , integrating by parts in space and using the fact that $V \in L^\infty((0, T) \times \mathbb{T})$, there exists a constant $M > 0$ such that

$$\begin{aligned} \frac{1}{2} \partial_t \int_{\mathbb{T}} \theta^2 + \int_{\mathbb{T}} |\nabla \theta|^2 - \frac{M}{2} \int_{\mathbb{T}} \theta^2 &\leq \frac{1}{2} \partial_t \int_{\mathbb{T}} \theta^2 + \int_{\mathbb{T}} |\nabla \theta|^2 - \int_{\mathbb{T}} V \theta^2 \\ &= \int_{\mathbb{T}} \theta \partial_x f + \int_{\mathbb{T}} \theta q g \\ &= - \int_{\mathbb{T}} f \partial_x \theta + \int_{\mathbb{T}} \theta q g \\ &\leq \int_{\mathbb{T}} \frac{f^2 + (\partial_x \theta)^2}{2} + \frac{\|q\|_{L^\infty}}{2} \int_{\mathbb{T}} g^2 + \frac{\|q\|_{L^\infty}}{2} \int_{\mathbb{T}} \theta^2. \end{aligned}$$

Thus, there exists two constants M', M'' such that

$$\partial_t \int_{\mathbb{T}} \theta^2 + \int_{\mathbb{T}} |\nabla \theta|^2 - M' \int_{\mathbb{T}} \theta^2 \leq M'' \left(\int_{\mathbb{T}} f^2(t, \cdot) + \int_{\mathbb{T}} g^2(t, \cdot) \right),$$

so that, for a.e. t , we have

$$e^{-Mt} \int_{\mathbb{T}} \theta^2(t, \cdot) + \int_0^t e^{-M\tau} \int_{\mathbb{T}} |\nabla \theta|^2 \leq M'' \int_0^t e^{-M\tau} \left(\int_{\mathbb{T}} f^2(\tau, \cdot) + \int_{\mathbb{T}} g^2(\tau, \cdot) \right) d\tau.$$

The conclusion follows by the using the inequality $e^{-MT} \leq e^{-Mt} \leq 1$. \square

B Study of the time-discrete model

B.1 Proof of Lemma 22

Proof of Lemma 22. To derive existence, uniqueness, positivity and regularity, we simply prove that, for any w^0 satisfying the assumptions of Lemma 22, for any $\mathbf{m} \in \mathfrak{M}(\Omega)$, $w_{\mathbf{m},1}$ exists, is unique, positive and enjoys the proper regularity properties, provided $\delta t \in (0; \bar{\delta})$, where $\bar{\delta}$ only depends on f and on the upper bound of m_1 . It then suffices to proceed inductively.

1. Existence and uniqueness of $w_{\mathbf{m},1}$ We introduce the anti-derivative F of f as

$$F_1(x, w) := \int_0^w f_1(x, \cdot).$$

As f_1 satisfies **(H₂)**-**(H₃)** there exist two constant A_0, A_1 such that

$$\forall x \in \Omega, \forall w \geq 0, f_1(x, w) \leq A_0 - A_1 w.$$

Consequently, there exist two constants B_0, B_1 such that

$$\forall x \in \Omega, \forall w \in \mathbb{R}, F_1(x, w) \geq -B_0 w - B_1 w^2.$$

We introduce the energy functional

$$E : W^{1,2}(\Omega) \ni w \mapsto \frac{1}{2} \int_{\Omega} |\nabla w|^2 + \frac{1}{2} \int_{\Omega} \left(\frac{1}{\delta t} - m_1 \right) w^2 - \int_{\Omega} F_1(x, w) - \frac{1}{\delta t} \int_{\Omega} w^0 w.$$

If we pick $\delta t > 0$ small enough to guarantee that

$$\frac{1}{\delta t} > 1 + B_1$$

then E is a coercive functional, so that a minimiser w_1 exists. To prove that it is unique, we consider two different solutions w_1, w'_1 of

$$\begin{cases} \frac{w-w^0}{\delta t} - \Delta w = f_1(x, w) & \text{in } \Omega, \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

The function $z := w_1 - w'_1$ satisfies

$$\frac{z}{\delta t} - \Delta z = f_1(x, w_1) - f_1(x, w'_1) \leq A|w_1 - w'_1| \leq A|z|$$

where A is given by (H₃). Multiplying this equation by the positive part z^+ of z and integrating by parts we obtain

$$\frac{1}{\delta t} \int_{\Omega} (z^+)^2 + \int_{\Omega} |\nabla z^+|^2 \leq A \int_{\Omega} (z^+)^2.$$

Taking δt small enough to ensure that

$$\frac{1}{\delta t} > A$$

we get $z^+ = 0$. As the roles of w_1, w'_1 are symmetric, we finally get $z = 0$, and we thus have existence and uniqueness of $w_{m,1}$.

2. Upper and lower bounds on $w_{m,1}$ We define $\phi \equiv \kappa$. As $f_1(x, \kappa) < 0$ and $w^0 \leq \kappa$ we obtain in particular

$$\frac{\phi - w^0}{\delta t} - \Delta \phi \geq f_1(x, \kappa) = f_1(x, \phi).$$

Defining $z := \phi - w_{m,1}$ it appears that z satisfies

$$\frac{z}{\delta t} - \Delta z \geq f_1(x, \phi) - f_1(x, w_{m,1}) \geq -A|w_{m,1} - \Phi| \geq -A|z|.$$

We multiply this equation by the negative part z^- of z and integrate by parts to obtain

$$-\frac{1}{\delta t} \int_{\Omega} (z^-)^2 - \int_{\Omega} |\nabla z^-|^2 \geq -A \int_{\Omega} (z^-)^2.$$

If δt is small enough that

$$\frac{1}{\delta t} > A$$

we get $z^- = 0$ and so

$$w_{m,1} \leq \Phi \equiv \kappa.$$

To derive the non-negativity of $w_{\mathbf{m},1}$ we proceed in the same way: as $f_1(x, 0), w^0 \geq 0$, $w_{\mathbf{m},1}$ solves the differential inequality

$$\frac{w_{\mathbf{m},1}}{\delta t} - \Delta w_{\mathbf{m},1} \geq f_1(x, w_{\mathbf{m},1}) - f_1(x, 0) + f_1(x, 0) \geq f_1(x, w_{\mathbf{m},1}) - f_1(x, 0) \geq -A|w_{\mathbf{m},1}|.$$

The conclusion follows in the same way and for $\delta t > 0$ small enough $w_{\mathbf{m},1} \geq 0$.

To obtain

$$\inf_{\Omega} w_{\mathbf{m},1} > 0$$

we simply use the proof of the strong maximum principle.

3. Regularity of $w_{\mathbf{m},1}$ As $0 \leq w_{\mathbf{m},1} \leq \kappa$ the $W^{2,p}$ regularity is a standard application of the L^p elliptic regularity theory.

□