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FOLDED QUANTUM INTEGRABLE MODELS AND DEFORMED $\mathcal{W}$-ALGEBRAS

EDWARD FRENKEL, DAVID HERNANDEZ, AND NICOLAI RESHETIKHIN

In memory of Hugues Charnallet

Abstract. We propose a novel quantum integrable model for every non-simply laced simple Lie algebra $\mathfrak{g}$, which we call the folded integrable model. Its spectra correspond to solutions of the Bethe Ansatz equations obtained by folding the Bethe Ansatz equations of the standard integrable model associated to the quantum affine algebra $U_q(\widehat{\mathfrak{g}'})$ of the simply-laced Lie algebra $\mathfrak{g}'$ corresponding to $\mathfrak{g}$. Our construction is motivated by the analysis of the second classical limit of the deformed $\mathcal{W}$-algebra of $\mathfrak{g}$, which we interpret as a "folding" of the Grothendieck ring of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}'})$. We conjecture, and verify in a number of cases, that the spaces of states of the folded integrable model can be identified with finite-dimensional representations of $U_q({}^L\widehat{\mathfrak{g}})$, where ${}^L\widehat{\mathfrak{g}}$ is the (twisted) affine Kac–Moody algebra Langlands dual to $\widehat{\mathfrak{g}}$. We discuss the analogous structures in the Gaudin model which appears in the limit $q \to 1$. Finally, we describe a conjectural construction of the simple $\mathfrak{g}$-crystals in terms of the folded q-characters.

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1. Introduction

1.1. **Integrable models.** Exactly solvable quantum integrable models have played a prominent role in mathematical physics ever since the groundbreaking work of Hans Bethe [B] in which he described the spectrum of the Hamiltonian of the XXX spin chain in terms of the solutions of what we call today the *Bethe Ansatz equations*. The XXX spin chain naturally corresponds to the Yangian of $\mathfrak{sl}_2$, and it can be generalized to quantum spin chain models corresponding to the Yangian of an arbitrary simple Lie algebra $\mathfrak{g}$ or the corresponding

quantum affine algebra $U_q(\widehat{\mathfrak{g}})$. In this paper, we focus on the latter model, which we call *XXZ-type model* associated to $U_q(\widehat{\mathfrak{g}})$.

Using the universal R-matrix of $U_q(\widehat{\mathfrak{g}})$, one assigns a commuting family of Hamiltonians $t_V(z), z \in \mathbb{C}$, of this model, to every finite-dimensional representation V of $U_q(\widehat{\mathfrak{g}})$, which is called an *auxiliary space*. These Hamiltonians, which are called the transfer-matrices, act on any finite-dimensional representation W of $U_q(\widehat{\mathfrak{g}})$, which is called a *space of states* (or physical space) of the model. Moreover, the transfer-matrix construction extends to a compatible family of ring homomorphisms

$$h_W : \operatorname{Rep} U_q(\widehat{\mathfrak{g}}) \to \operatorname{End}(W)[[z]] \tag{1.1}$$

sending the class of V in the Grothendieck ring $\operatorname{Rep} U_q(\widehat{\mathfrak{g}})$ of $U_q(\widehat{\mathfrak{g}})$ to the corresponding transfer-matrix $t_V(z)$ acting on W (viewed as a formal power series in z with coefficients in $\operatorname{End}(W)$). We will assume throughout that $q \in \mathbb{C}^\times$ is not a root of unity.

In a generic situation, the spectra of these quantum Hamiltonians are expected to be in one-to-one correspondence with the solutions of the generalized Bethe Ansatz equations. For the XXZ-type model associated to $U_q(\widehat{\mathfrak{g}})$, where $\widehat{\mathfrak{g}}$ is an arbitrary affine Kac–Moody algebra (twisted or untwisted) they were proposed in [OW, RW, R]. A pair of authors of the present paper subsequently conjectured in [FR3] an explicit formula for the eigenvalues of the transfer-matrices $t_V(z)$ corresponding to a given solution of these generalized Bethe Ansatz equations. This formula is written in terms of the q-character of V.

Another pair of authors of the present paper then extended the above construction to a larger algebra of quantum Hamiltonians [FH3]. Namely, the homomorphism h_W extends to a homomorphism

$$h'_W : \operatorname{Rep}' \to \operatorname{End}(W)(u)[[z]], \tag{1.2}$$

where Rep' stands for the Grothendieck ring of the category $\mathcal{O}$ introduced in [HJ] or its dual category $\mathcal{O}^*$. The corresponding transfer-matrices $t_V(z,u)$ are formal power series in z with coefficients depending on an element u of the Cartan subgroup H of the simply-connected Lie group G associated to $\mathfrak{g}$. The category $\mathcal{O}^*$ is topologically generated by the prefundamental representations $R_j^\pm(z), j \in I$, where I is the set of vertices of the Dynkin diagram of $\mathfrak{g}$. Hence the eigenvalues of the quantum Hamiltonians are determined by the eigenvalues of the *Q-operators* $Q_j^\pm(z,u) = t_{R_j^\pm}(z,u)$, the transfer-matrices associated in [FH3] to the prefundamental representations $R_j^\pm(z)$.

It was proved in [FH3] that up to a universal factor depending on the representation W, all eigenvalues of $Q_j^+(z,u)$ on W (which are *a priori* formal power series in z) are in fact *polynomials* in z. Moreover, it follows from [FH3, FH4] that the roots of these polynomials are solutions of the Bethe Ansatz equations (more precisely, the u-dependent version of the Bethe Ansatz equations) under a certain non-degeneracy condition. (This condition was subsequently dropped in [FJMM].) Finally, it was shown in [FH3] that the eigenvalues of the transfer-matrix $t_V(z,u)$ corresponding to a finite-dimensional representation V can be expressed in terms of the eigenvalues of the $Q_j^+(z,u)$ and the q-character of V, proving the conjecture of [FR3]. These results provide a link between the spectra of the XXZ-type model associated to $U_q(\widehat{\mathfrak{g}})$ and the solutions of the corresponding Bethe Ansatz equations (BAE).

Explicitly, if the space of states W is the tensor product of irreducible finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ with the Drinfeld polynomials $P_{i,k}, i \in I, k = 1, \ldots, N$, then the

BAE have the form:

$$\prod_{k=1}^{N} q_i^{\deg P_{i,k}} \frac{P_{i,k}(q_i^{-1}/w_r^{(i)})}{P_{i,k}(q_i/w_r^{(i)})} = -\prod_{s\neq r} \frac{w_r^{(i)} - w_s^{(i)} q_i^{-2}}{w_r^{(i)} - w_s^{(i)} q_i^{2}} \prod_{j\neq i}\prod_{s=1}^{m_j} \frac{w_r^{(i)} - w_s^{(j)} q^{-B_{ij}}}{w_r^{(i)} - w_s^{(j)} q^{B_{ij}}}. \tag{1.3}$$

with one equation for each root $w_r^{(i)}, r = 1, \dots, m_i$, of the ith Baxter polynomial $Q_i^+(z)$, with i ranging over the set I. Here (B_{ij}) is the symmetrized Cartan matrix of $\mathfrak{g}$: $B_{ij} = d_i C_{ij}$, where (C_{ij}) is the Cartan matrix and the d_i are relatively prime integers, and we set $q_i = q^{d_i}$.

Note that a typical factor on the right hand side of (1.3) looks like this:

$$\frac{w_r^{(i)} - w_s^{(j)} q^{-B_{ij}}}{w_r^{(i)} - w_s^{(j)} q^{B_{ij}}} \tag{1.4}$$

1.2. **Miura q-opers and folded Bethe Ansatz equations.** In a recent paper [FKSZ], certain geometric objects on $\mathbb{CP}^1$ called *Miura (G, q)-opers* were introduced. It was shown in [FKSZ] (see also the earlier work [KSZ] in the case $G = SL_n$) that there is a one-to-one correspondence between the set of Miura (G, q)-oper satisfying a non-degeneracy condition and the set of solutions of a system of equations which look very similar to the BAE associated to $U_q(\widehat{\mathfrak{g}})$.[1]

In these equations a typical factor with $i \neq j$ reads

$$\frac{(w_r^{(i)} - w_s^{(j)} q)^{-C_{ji}}}{(w_r^{(i)} - w_s^{(j)} q^{-1})^{-C_{ji}}}, \qquad \text{if} \quad i \neq j, \tag{1.5}$$

where (C_{ij}) is the Cartan matrix of $\mathfrak{g}$. If $\mathfrak{g}$ is simply-laced, formulas (1.4) and (1.5) coincide and so the equations are just the standard BAE associated to $U_q(\widehat{\mathfrak{g}})$. Therefore, in this case one obtains a "dual" description of the spectrum in terms of Miura (G, q)-opers, giving rise to what in [FKSZ] was called the *qDE/IM correspondence*.

However, if $\mathfrak{g}$ is not simply-laced, we obtain this way a different system of equations. A Yangian version of these equations first appeared in the work of Mukhin and Varchenko [MV1, MV2]. We note that for $\mathfrak{g} = B_\ell$ similar equations were also obtained in [DHKM] in the context of 3d quiver gauge theories.

In this paper, we will call these equations the *folded Bethe Ansatz equations* because they can be obtained by "folding" the BAE for the simply-laced simple Lie algebra $\mathfrak{g}'$ that gives rise to $\mathfrak{g}$ (i.e. $\mathfrak{g}'$ is equipped with an automorphism of order 2 or 3 whose invariant Lie subalgebra is $\mathfrak{g}$).

To explain this folding procedure and to illustrate the difference between the two types of BAE, consider the case of $\mathfrak{g} = C_\ell$. Then $\mathfrak{g}' = A_{2\ell-1}$. In this case $d_i = 1$ for $i = 1, \dots, \ell - 1$ and $d_\ell = 2$. Therefore, the only factors (1.4) for C_ℓ with $i \neq j$ and the powers of q different from ± 1 (which are the only powers of q appearing in the factors with $i \neq j$ in the simply-laced cases) occur for $i = \ell, j = \ell - 1$ or the other way around. The first of them is

$$\frac{w_r^{(\ell)} - w_s^{(\ell-1)} q^2}{w_r^{(\ell)} - w_s^{(\ell-1)} q^{-2}} = \frac{w_r^{(\ell)} - w_s^{(\ell-1)} q^2}{w_r^{(\ell)} - w_s^{(\ell-1)}} \frac{w_r^{(\ell)} - w_s^{(\ell-1)}}{w_r^{(\ell)} - w_s^{(\ell-1)} q^{-2}}. \tag{1.6}$$

[1] More precisely, this was proved in [FKSZ] for Miura-Plücker (G, q)-opers, but in the subsequent work [KZ] it was shown that this notion is equivalent to the notion of Miura (G, q)-oper.

On the other hand, since the entry $C_{\ell-1,\ell}$ of the Cartan matrix of $\mathfrak{g} = C_\ell$ is equal to -2, the corresponding factor (1.5) reads

$$\frac{(w_r^{(\ell)} - w_s^{(\ell-1)} q)^2}{(w_r^{(\ell)} - w_s^{(\ell-1)} q^{-1})^2}. \tag{1.7}$$

It can be obtained by folding the expression of the form

$$\frac{w_r^{(\ell)} - w_s^{(\ell-1)} q}{w_r^{(\ell)} - w_s^{(\ell-1)} q^{-1}} \frac{w_r^{(\ell)} - w_s^{(\ell+1)} q}{w_r^{(\ell)} - w_s^{(\ell+1)} q^{-1}} \tag{1.8}$$

appearing in the BAE of the simply-laced Lie algebra $\mathfrak{g}' = A_{2\ell-1}$. Indeed, the automorphism of the Dynkin diagram of $A_{2\ell-1}$ preserves the ℓth vertex and exchanges the $(\ell-1)$st and the $(\ell+1)$st vertices. If we accordingly identify the variables $w_s^{(\ell-1)}$ and $w_s^{(\ell+1)}$ in the expression (1.8), we obtain the factor (1.7). This is what we mean by folding the BAE of a simply-laced Lie algebra $\mathfrak{g}'$.

Note that the difference between (1.7) and (1.6) (as its RHS shows) is the difference between $f(w)^2$ and $f(wq)f(wq^{-1})$.

Remark 1.1. After the first version of this paper was posted on arXiv, Heng-Yu Chen and Taro Kimura informed us about their paper [CK], in which they considered two classical limits of the deformed $\mathcal{W}$-algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ in the context of the corresponding 5D fractional quiver gauge theory introduced in [KP2] (where these classical limits are interpreted as the two Nekrasov–Shatashvili limits). They obtained a version of the folded Bethe Ansatz equations of the present paper from the analysis of the partition function of this theory in one of these limits. They did not consider the folded integrable model, which is the main focus of the present paper, where these equations naturally appear from certain subspaces of finite-dimensional representations of quantum affine algebras (see Section 5). □

1.3. **Folded integrable model.** According to [FKSZ], non-degenerate Miura (G, q)-opers encode solutions of the folded BAE. But which integrable model do these equations correspond to?

In this paper (Section 5) we propose a conjectural answer to this question. Namely, we conjecture the existence of what we will call the *folded integrable model* for every non-simply laced simple Lie algebra $\mathfrak{g}$, whose spectra give rise to solutions of the folded BAE (under a genericity condition). This folded integrable model combines in a non-trivial way representations of the quantum affine algebra $U_q(\widehat{\mathfrak{g}'})$, where $\mathfrak{g}'$ is the corresponding simply-laced Lie algebra (these appear as the *auxiliary spaces* of the folded model) and representations of the quantum affine algebra $U_q({}^L\widehat{\mathfrak{g}})$, where ${}^L\widehat{\mathfrak{g}}$ is the twisted affine Kac–Moody algebra which is *Langlands dual* to $\widehat{\mathfrak{g}}$ (these appear as the *spaces of states* of the folded model).

Remark 1.2. Note that to the twisted quantum affine algebra $U_q({}^L\widehat{\mathfrak{g}})$ one can also associate an XXZ-type quantum integrable model. It is constructed in the same way as for the untwisted quantum affine algebras, and its spectra correspond to the solutions of the BAE that were proposed in [RW, R]. But this model is *different* from the folded model. Namely, the typical factors of the BAE of this model read

$$\frac{(w_r^{(i)})^{-C_{ji}} - (w_s^{(j)} q)^{-C_{ji}}}{(w_r^{(i)})^{-C_{ji}} - (w_s^{(j)} q^{-1})^{-C_{ji}}}, \qquad \text{if} \quad i \neq j, \tag{1.9}$$

so they differ from the factors (1.5).

For example, in the case of $\mathfrak{g} = C_\ell$, instead of the factor (1.7) we have

$$\frac{(w_r^{(\ell)})^2 - (w_s^{(\ell-1)}q)^2}{(w_r^{(\ell)})^2 - w_s^{(\ell-1)}q^{-1})^2} = \frac{w_r^{(\ell)} - w_s^{(\ell-1)}q}{w_r^{(\ell)} - w_s^{(\ell-1)}q^{-1}} \frac{w_r^{(\ell)} + w_s^{(\ell-1)}q}{w_r^{(\ell)} + w_s^{(\ell-1)}q^{-1}}. \tag{1.10}$$

Thus, it can be obtained by folding the expression (1.8) if we also multiply the spectral parameter by -1, i.e. identify $w_s^{(\ell-1)}$ and $-w_s^{(\ell+1)}$ in (1.8) (rather than $w_s^{(\ell-1)}$ and $w_s^{(\ell+1)}$). In other words, the difference between (1.7) and (1.10) is the difference between $f(w)^2$ and $f(w)f(-w)$. And similarly for other Lie algebras corresponding to an automorphism of order 2. In the case of $\mathfrak{g} = G_2$, when the automorphism has order 3, it's the difference between $f(w)^3$ and $f(w)f(w\epsilon)f(w\epsilon^{-1})$, where $\epsilon = e^{2\pi i/3}$. □

1.4. QQ**-system.** There is an important intermediate object between the spectra of the XXZ-type model associated to $U_q(\widehat{\mathfrak{g}})$ and the corresponding BAE called the QQ-*system*. It was introduced in [MRV1, MRV2] in the context of affine opers. In [FH4], it was shown that this QQ-system naturally arises in the context of the homomorphisms h'_W (see the above formula (1.2)). Namely, in addition to the set of prefundamental representations $R_j^+(z), j \in I$, discussed above there is another set of representations, denoted by $X_j(z), j \in I$, in the category $\mathcal{O}^*$, such that properly rescaled classes of these two sets satisfy the QQ-system. In other words, if we assign to $R_j^+(z), j \in I$, the above Q-operators $Q_j(z,u)$ and to $X_j(z), j \in I$, the transfer-matrix $\widetilde{Q}_j(z,u)$ (the image of $X_j(z)$ under the homomorphism h'_W, where W is a finite-dimensional representation of $U_q(\widehat{\mathfrak{g}})$), then these operators, properly rescaled, will satisfy the QQ-system.[2] Note that for $\mathfrak{g} = \mathfrak{sl}_2$, this is the quantum Wronskian relation introduced in [BLZ], which provided the initial motivation for this line of research.

Thus, the QQ-system encodes universal relations between the classes of the representations $R_j^+(z)$ and $X_j(z), j \in I$ in Rep$'$ which translate under the homomorphism (1.2) into relations between the corresponding transfer-matrices, and hence their eigenvalues, on any representation W.[3]

As explained in [MRV1, MRV2, FH4], the Bethe Ansatz equations (1.3) follow directly from the QQ-system under a certain non-degeneracy condition. In fact, from the point of view of the preceding paragraph, the QQ-system is more fundamental to the question of describing the spectra of the XXZ-type models than the Bethe Ansatz equations.

Likewise, for the folded BAE: As shown in [FKSZ], these equations are equivalent to the QQ-system proposed in [FKSZ] (under a non-degeneracy condition). We will call the latter system the *folded QQ-system* because it can be obtained by folding the QQ-system associated to the simply-laced Lie algebra $\mathfrak{g}'$. Thus, this system appears as an intermediate object between Miura (G,q)-opers and the folded BAE.

In this paper we show that this folded QQ-system also appears naturally as a relation satisfied by the transfer-matrices of our (conjectural) folded integrable model associated to a non-simply laced Lie algebra $\mathfrak{g}$. The folded BAE equations follow from the folded QQ-system under a non-degeneracy condition.

[2]It was called $Q\widetilde{Q}$-system in [FH4] but here, for the sake of brevity, we follow the terminology of [FKSZ] and call it the QQ-system.

[3]The fact that the same QQ-system arises both from the affine opers and the eigenvalues of the transfer-matrices is a manifestation of the affine Langlands duality proposed in [FF] and further elucidated in [FH4]. However, we will not discuss this duality in the present paper.

1.5. **Deformed $\mathcal{W}$-algebras.** Valuable insights about the folded quantum integrable model can be learned from the deformed $\mathcal{W}$-algebras introduced by two of the authors of the present paper in [FR2]. Recall that this is a two-parameter algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ associated to a simple Lie algebra $\mathfrak{g}$. Recently, the algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ found interesting applications in the study of four-dimensional supersymmetric gauge theories, see [Ne, KP1, AFO, KP2, EP].

The deformed $\mathcal{W}$-algebra has two classical limits, in which the algebra becomes commutative, and equipped with a Poisson structure: the first occurs when $t \to 1$ and the second when $q \to 1$.

The first limit, $\mathcal{W}_{q,1}(\mathfrak{g})$, is relatively well understood. It is isomorphic to the center $Z_q(\widehat{\mathfrak{g}})$ of $U_q(\widehat{\mathfrak{g}})$ at the critical level. The corresponding commutative algebra of generating fields can be identified, via a version of the transfer-matrix construction (see [RS, FR1]), with the representation ring $\operatorname{Rep} U_q(\widehat{\mathfrak{g}})$.[4] Moreover, under this identification the free field realization of $\mathcal{W}_{q,1}(\mathfrak{g})$ becomes the q-character homomorphism (this was the motivation behind the definition of the q-characters in [FR3]).

If $\mathfrak{g}$ is simply-laced, then the second classical limit, $\mathcal{W}_{1,t}(\mathfrak{g})$, coincides with the first one upon replacing t with q. But if $\mathfrak{g}$ is not simply-laced, the second limit is substantially different from the first one. In [FR2], $\mathcal{W}_{1,t}(\mathfrak{g})$ was linked to the t-deformed Drinfeld–Sokolov reduction of the loop group associated to G introduced in [FRS, SS] and some observations were made connecting elements of $\mathcal{W}_{1,t}(\mathfrak{g})$ to the q-characters of $U_q(\widehat{\mathfrak{g}}^\vee)$, where $\widehat{\mathfrak{g}}^\vee$ is the twisted affine algebra associated to $\mathfrak{g}'$ and σ. But that's pretty much all that has been known about the limit $q \to 1$ until now.

In the present paper, we argue that it is this limit that is relevant to the "folded structures" that we discuss here, including the folded Bethe Ansatz equations and the folded integrable models. Thus, we can learn a lot about these models by studying this limit. Its hybrid nature, i.e. the fact that it mixes in a non-trivial way quantum affine algebras $\widehat{\mathfrak{g}'}$ and ${}^L\widehat{\mathfrak{g}}$, shows that $\mathcal{W}_{1,t}(\mathfrak{g})$ is a fascinating Poisson algebra that deserves further investigation.

The deformed $\mathcal{W}$-algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ creates a bridge between the two classical limits, and hence between the XXZ-type quantum integrable model associated to $U_q(\widehat{\mathfrak{g}})$ and the corresponding folded quantum integrable model. However, the non-commutative nature of $\mathcal{W}_{q,t}(\mathfrak{g})$ makes deriving practical consequences of this bridge a daunting task. For this reason, in this paper we replace $\mathcal{W}_{q,t}(\mathfrak{g})$ with its simplified commutative version introduced by two of the authors in [FH2] under the name *interpolating* (q,t)*-characters*. Using a slight refinement of these objects, we make our conjectures concerning the folded quantum integrable models more precise. This also enables us to explicitly verify our conjectures in a number of non-trivial cases (see Section 7).

1.6. **Connection to qKZ equations and quantum q-Langlands correspondence.** It is known that the critical level limit of the solutions of the qKZ equations corresponding to $U_q(\widehat{\mathfrak{g}})$ give rise to eigenvectors of the XXZ-type model associated to $U_q(\widehat{\mathfrak{g}})$. Thus, the qKZ system provides a deformation of the latter model.[5] This can also be seen from the fact that for a large class of representations of $U_q(\widehat{\mathfrak{g}})$, the difference operators of the qKZ

[4]More precisely, there is a homomorphism $\operatorname{Rep} U_q(\widehat{\mathfrak{g}}) \to Z_q(\widehat{\mathfrak{g}})[[z^{\pm 1}]]$, so that every $V \in \operatorname{Rep} U_q(\widehat{\mathfrak{g}})$ gives rise to a formal power series $T_V(z)$, and the Fourier coefficients of these series topologically generate $Z_q(\widehat{\mathfrak{g}})$, see [FR3, Section 8.1].

[5]The XXZ-type model is already quantum, but here by a deformation we mean a non-commutative deformation of the commutative algebra of quantum Hamiltonians of the XXZ-type model. Therefore, it is a kind of "second quantization."

system become in the critical level limit the transfer-matrices of the XXZ-type model (see Proposition 10.1).

In [AFO], a *quantum q-Langlands correspondence* was proposed. For a simply-laced simple Lie algebra $\mathfrak{g}$, it sets up a correspondence between solutions of the qKZ system associated to $U_t(\widehat{\mathfrak{g}})$ and the deformed conformal blocks associated to $\mathcal{W}_{q,t}(\widehat{\mathfrak{g}})$. Here q depends on the level of $U_t(\widehat{\mathfrak{g}})$ in such a way that the limit $q \to 1$ corresponds to the critical level limit. In this limit the quantum q-Langlands correspondence essentially becomes the statement that the Hamiltonians of the XXZ-type model associated to $U_t(\widehat{\mathfrak{g}})$ correspond to elements of $\mathcal{W}_{1,t}(\widehat{\mathfrak{g}})$. If $\mathfrak{g}$ is simply-laced, then $\mathcal{W}_{1,t}(\widehat{\mathfrak{g}})$ indeed coincides with $\operatorname{Rep} U_t(\widehat{\mathfrak{g}})$, so this statement comes down to the existence of the homomorphisms h_W given by formula (1.1).

However, if $\mathfrak{g}$ is not simply-laced, the algebra $\mathcal{W}_{1,t}(\widehat{\mathfrak{g}})$ is no longer isomorphic to $\operatorname{Rep} U_t(\widehat{\mathfrak{g}})$ and hence does *not* give rise to the Hamiltonians of the standard XXZ-type model associated to $U_t(\widehat{\mathfrak{g}})$, which are the transfer-matrices associated to finite-dimensional representations of $U_t(\widehat{\mathfrak{g}})$. Rather, as we argue in this paper, it gives rise to the Hamiltonians of the *folded* quantum integrable model associated to $\mathfrak{g}$. These Hamiltonians correspond to the transfer-matrices associated to finite-dimensional representations of $U_t(\widehat{\mathfrak{g}'})$ rather than $U_t(\widehat{\mathfrak{g}})$.

This suggests that for non-simply laced $\mathfrak{g}$ the quantum q-Langlands correspondence might be more subtle. Namely, it follows from the preceding paragraph that the system of q-difference equations appearing on one side of this correspondence is *not* the usual qKZ system associated to $U_t(\widehat{\mathfrak{g}})$. If this were the usual qKZ system associated to $U_t(\widehat{\mathfrak{g}})$, then in the critical limit we would recover the eigenvectors of the XXZ-type model associated to $U_t(\widehat{\mathfrak{g}})$, but this would be inconsistent with the limit on the other side of the correspondence which yields $\mathcal{W}_{1,t}(\mathfrak{g})$. As we discussed above, the latter is not the algebra of Hamiltonians of the XXZ-type model associated to $U_t(\widehat{\mathfrak{g}})$, but rather the algebra of Hamiltonians of the *folded* quantum integrable model introduced in the present paper.

What should replace the qKZ system associated to $U_t(\widehat{\mathfrak{g}})$ in the quantum q-Langlands correspondence for non-simply laced $\mathfrak{g}$? The above discussion shows that this modified qKZ system should have the property that the leading terms of its solutions in the critical level limit are eigenvectors of the folded integrable system associated to $\mathfrak{g}$. As far as we know, the existence of such modified qKZ system is an open question at the moment (naive ways to "fold" the qKZ system associated to $\mathfrak{g}'$ don't seem to work, see the Appendix of this paper). But we expect that this question can be answered using the geometric and K-theoretic methods of [AFO]. Perhaps, these equations can also be constructed purely algebraically. We hope to return to this question elsewhere.

1.7. **The Gaudin limit.** To gain further insights, it is instructive to consider the limit in which the second parameter, denoted by t in the previous subsection, also goes to 1. In this limit, the XXZ-type model associated to $U_t(\widehat{\mathfrak{g}})$ becomes the Gaudin model associated to $\mathfrak{g}$ [FFR, F]; more precisely, its modification with a twist parameter χ, an element of the Cartan subalgebra of $\mathfrak{g}$ [Ry, FFT, FFRy]. It turns out that in the limit $t \to 1$ the folded integrable model associated to $\mathfrak{g}$ that we discuss in this paper becomes the Gaudin model associated to ${}^L\mathfrak{g}$. Thus, in the Gaudin limit we do not find any new quantum integrable models. In part, this is because in this limit the irreducible finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ decompose into a direct sum of irreducible representations of the finite-dimensional Lie algebra $\mathfrak{g}$, so the affine Langlands duality $\widehat{\mathfrak{g}} \to {}^L\widehat{\mathfrak{g}}$ reduces to the finite-dimensional Langlands duality $\mathfrak{g} \to {}^L\mathfrak{g}$, which was discovered in [FFR].

However, even in this limit, as we will show in Section 9, one can observe some intriguing effects related to folding. In particular, using the results of [FFRy], we will construct embeddings of tensor products of irreducible representations of ${}^L\mathfrak{g}$ into tensor products of the corresponding irreducible representations of $\mathfrak{g}'$ (see Theorem 9.5). In fact, it's a family of embeddings depending on χ (which is assumed to be regular and generic). It maps eigenvectors of the ${}^L\mathfrak{g}$-Gaudin model with the twist χ to eigenvectors of the corresponding $\mathfrak{g}'$-Gaudin model. Under a certain assumption (see Conjecture 9.10 in the case of a single irreducible representation) this embedding can be constructed explicitly.

1.8. **Plan of the paper.** In Section 2 we fix our notation for the Lie algebras and two-parameter Cartan matrices. In Section 3 we recall the definition of the deformed $\mathcal{W}$–algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ from [FR2]. We then consider its two classical limits. The first limit, $t \to 1$, is relatively well understood; it can be identified with the Grothendieck ring of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ as we recall in Section 3.3. In Section 3.4 we obtain a description of the second limit, $q \to 1$ (which was much less understood), analogous to the description of the $t \to 1$ limit (see Propositions 3.2 and 3.3). In Section 4 we recall the relation between the $t \to 1$ limit of the deformed $\mathcal{W}$-algebra and the ring of q-characters of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$. We then relate the $q \to 1$ limit to what we call the ring of *folded t-characters* of finite-dimensional representations of $U_t(\widehat{\mathfrak{g}'})$, where $\mathfrak{g}'$ is the simply-laced Lie algebra from which $\mathfrak{g}$ is obtained as the Lie subalgebra fixed by an automorphism (see Theorem 4.3). We also discuss the link between this limit and the difference Drinfeld–Sokolov reduction. Finally, in Section 4.6 we introduce the *folded Bethe Ansatz equations*.

In Section 5 we describe the *folded quantum integrable model* in which the spectra of the Hamiltonians conjecturally correspond to solutions of the folded Bethe Ansatz equations (see Conjectures 5.15 and 5.16). In Section 6 we recall the interpolating (q,t)-characters from [FH2], which may be viewed as commutative algebra analogues of elements of the non-commutative $\mathcal{W}$-algebra $\mathcal{W}_{q,t}(\mathfrak{g})$. We then construct a refined version of the interpolating (q,t)-characters. They are elements of a ring depending on the parameters q and t, which is equipped with 5 interesting specialization homomorphisms to the rings of q- and t-characters of various affine Kac–Moody algebras related to $\mathfrak{g}$ (see Theorem 6.6). In Section 6.5 we partially prove our conjectures in the important case of σ-fundamental representations (these are the irreducible finite-dimensional representations of $U_q(\widehat{\mathfrak{g}'})$ with the σ-invariant highest monomials of smallest possible degrees). In Section 7 we present a number of explicit examples confirming our Conjectures 5.15 and 5.16. In Section 8 we formulate a conjecture linking the folded t-characters to Kashiwara's extension of Nakajima's monomial model of crystals to non-simply laced Lie algebras. In Section 9, we consider the Gaudin limit of the the folded quantum integrable models. In the Appendix we discuss a possible construction of a folded version of the qKZ equations for non-simply laced Lie algebras.

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2. Notation and setup

2.1. **Lie algebra.** Let $\mathfrak{g}$ be a simple Lie algebra of rank ℓ and $I = \{1, \dots, \ell\}$ the set of vertices of the Dynkin diagram of $\mathfrak{g}$. Let $(\cdot, \cdot)$ be the invariant inner product on $\mathfrak{g}$, normalized so that the square of the maximal root equals 2. Let $\{\alpha_1, \dots, \alpha_\ell\}$ and $\{\omega_1, \dots, \omega_\ell\}$ be the sets of simple roots and of fundamental weights of $\mathfrak{g}$, respectively. We have:

$$(\alpha_i, \omega_j) = \frac{(\alpha_i, \alpha_i)}{2} \delta_{i,j}.$$

Let d be the maximal number of edges connecting two vertices of the Dynkin diagram of $\mathfrak{g}$. Thus, $d = 1$ for simply-laced $\mathfrak{g}$, $d = 2$ for B_ℓ, C_ℓ, F_4, and $d = 3$ for G_2. We set $\epsilon = e^{i\pi/d}$.

Set

$$D = \mathrm{diag}(d_1, \dots, d_\ell),$$

where

$$d_i = d\frac{(\alpha_i, \alpha_i)}{2}. \tag{2.1}$$

All d_i's are integers, which are relatively prime with each other. For simply-laced $\mathfrak{g}$, D is the identity matrix.

Now let $C = (C_{ij})_{1 \le i,j \le \ell}$ be the *Cartan matrix* of $\mathfrak{g}$. We have:

$$C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}.$$

Denote by $(I_{ij})_{1 \le i,j \le \ell}$ the *incidence matrix*,

$$I_{ij} = 2\delta_{i,j} - C_{ij}.$$

Let $B = (B_{ij})_{1 \le i,j \le \ell}$ be the following matrix:

$$B = DC,$$

i.e.,

$$B_{ij} = d(\alpha_i, \alpha_j).$$

The weights ρ and $\rho^\vee$ are defined by $(\rho^\vee, \alpha_i) = 1$, $d(\rho, \alpha_i) = d_i$ for any $1 \le i \le \ell$.

2.2. **Lie algebras involved.** We list here all Lie algebras involved in our study:

- $\mathfrak{g}$ is a simple finite-dimensional Lie algebra.
- ${}^L\mathfrak{g}$ is its Langlands dual Lie algebra. For example, if $\mathfrak{g} = B_\ell$, then ${}^L\mathfrak{g} = C_\ell$.
- $\widehat{\mathfrak{g}}$ is the untwisted affine Kac–Moody algebra, which is the central extension of $\mathfrak{g}[t, t^{-1}]$. For example, if $\mathfrak{g} = B_\ell$, then $\widehat{\mathfrak{g}} = B_\ell^{(1)}$.
- $\widehat{{}^L\mathfrak{g}}$ is the untwisted affine Kac–Moody algebra, which is the central extension of ${}^L\mathfrak{g}[t, t^{-1}]$. For example, if $\mathfrak{g} = B_\ell$, then $\widehat{{}^L\mathfrak{g}} = C_\ell^{(1)}$.
- ${}^L\widehat{\mathfrak{g}}$ is the affine Kac–Moody algebra that is *affine* Langlands dual to $\widehat{\mathfrak{g}}$. If $\mathfrak{g}$ is simply-laced, then ${}^L\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}$. But if $\mathfrak{g}$ is non-simply laced, then ${}^L\widehat{\mathfrak{g}}$ is a *twisted* affine Kac–Moody algebra. Note that ${}^L\widehat{\mathfrak{g}}$ contains ${}^L\mathfrak{g}$ as the constant Lie subalgebra. For example, if $\mathfrak{g} = B_\ell$, then ${}^L\widehat{\mathfrak{g}} = A_{2\ell-1}^{(2)}$ (whose constant subalgebra is ${}^L\mathfrak{g} = C_\ell$); and if $\mathfrak{g} = C_\ell$, then ${}^L\widehat{\mathfrak{g}} = D_{\ell+1}^{(2)}$ (whose constant subalgebra is ${}^L\mathfrak{g} = B_\ell$).
- $\mathfrak{g}'$ is the unique *simply-laced* Lie algebra equipped with an automorphism σ of order d such that the Lie subalgebra of σ-invariants in $\mathfrak{g}'$ is $\mathfrak{g}$ (i.e. $\mathfrak{g} = (\mathfrak{g}')^\sigma$). For example, if $\mathfrak{g} = B_\ell$, then $\mathfrak{g}' = D_{\ell+1}$; and if $\mathfrak{g} = C_\ell$, then $\mathfrak{g}' = A_{2\ell-1}$.

- $\widehat{\mathfrak{g}}^\vee$ is $\widehat{\mathfrak{g}}$, if $\mathfrak{g}$ is simply-laced. If $\mathfrak{g}$ is non-simply laced, then $\widehat{\mathfrak{g}}^\vee$ is the twisted affine Kac–Moody algebra corresponding to $\mathfrak{g}'$ and σ. Note that its constant Lie subalgebra is $\mathfrak{g}$ itself. For example, $(B_\ell^{(1)})^\vee = D_{\ell+1}^{(2)}$ (its constant subalgebra is B_ℓ), and $(C_\ell^{(1)})^\vee = A_{2\ell-1}^{(2)}$ (its constant subalgebra is C_ℓ). Note also that we have $\widehat{\mathfrak{g}}^\vee = {}^L(\widehat{{}^L\mathfrak{g}})$.

It might be better to denote $\widehat{\mathfrak{g}}^\vee$ by $\mathfrak{g}'^{(d)}$, but we will use below the notation $\widehat{\mathfrak{g}}^\vee$ because it was used in [FR2].

Let us denote by I' the set of vertices of the Dynkin diagram of $\mathfrak{g}'$. Then σ acts on I' and the quotient is in bijection with the set I of vertices of the Dynkin diagram of I. Note that the automorphism σ of the Dynkin diagram of $\mathfrak{g}'$ acts on the objects labeled by the nodes of this diagram (such as simple roots, fundamental weights, etc.).

2.3. **Two-parameter Cartan matrices.** We follow the notation of [FR2], Sect. 2, except that we replace q by q^{-1} (however, for $t = 1$ this notation is consistent with the notation of [FR3]).

Now let q, t be non-zero complex numbers which are not roots of unity. We will use the standard notation for $n \in \mathbb{Z}$

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Let

$$q_i = q^{d_i}.$$

We define $\ell \times \ell$ matrices $C(q,t)$, $D(q,t)$, and $B(q,t)$ by the formulas

$$C_{ij}(q,t) = (q_i t + q_i^{-1} t^{-1})\delta_{i,j} - [I_{ij}]_q, \tag{2.2}$$

$$D(q,t) = \mathrm{diag}([d_1]_q, \dots, [d_\ell]_q), \tag{2.3}$$

$$B(q,t) = D(q,t)C(q,t).$$

Thus,

$$B_{ij}(q,t) = [d_i]_q \left((q^{d_i} t + q^{-d_i} t^{-1})\delta_{i,j} - [I_{ij}]_q \right). \tag{2.4}$$

It is easy to see that the matrix $B(q,t)$ is symmetric. For simply-laced $\mathfrak{g}$,

$$C_{ij}(q,t) = B_{ij}(q,t) = (qt + q^{-1}t^{-1})\delta_{i,j} - I_{ij}.$$

We note that the determinants of these matrices are non-zero polynomials in q and t. Hence they are invertible over the field of rational functions in q and t.

Clearly, the limits of $C(q,t)$, $D(q,t)$, and $B(q,t)$ as both $q \to 1$ and $t \to 1$ coincide with C, D, and B, respectively. We also have

$$B_{ij}(q,1) = [B_{ij}]_q, \qquad C_{ij}(q,1) = (q_i + q_i^{-1})\delta_{i,j} + [C_{ij}]_q \delta_{i \neq j},$$

and

$$B_{ij}(1,t) = d_i((t + t^{-1})\delta_{ij} - I_{ij}).$$

Let $C(q) = C(q,1)$. It is invertible over the field of rational functions in q. We denote its inverse by $\widetilde{C}(q)$.

3. Deformed $\mathcal{W}$–algebras and screening operators

In this section we recall the definition of the deformed $\mathcal{W}$–algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ and related objects from [FR2]. We will then look at the two classical limits $t \to 1$ and $q \to 1$, which are defined as the intersections of the kernels of two sets of classical screening operators. The $t \to 1$ limit was described in [FR2, FR3, FM] and is closely related to the Grothendieck ring $\operatorname{Rep} U_q(\widehat{\mathfrak{g}})$ and the corresponding q-characters (as we recall in the next section). A new result of this section is the analogous description of the $q \to 1$ limit (see Propositions 3.2 and 3.3).

3.1. **Heisenberg algebra $\mathcal{H}_{q,t}(\mathfrak{g})$.** Let $\mathcal{H}_{q,t}(\mathfrak{g})$ be the Heisenberg algebra with generators $a_i[n], i = 1, \dots, \ell; n \in \mathbb{Z}$, and relations

$$[a_i[n], a_j[m]] = \frac{1}{n}(q^n - q^{-n})(t^n - t^{-n})B_{ij}(q^n, t^n)\delta_{n,-m} \tag{3.1}$$

where $1 \le i, j \le \ell; n, m \in \mathbb{Z} \setminus \{0\}$.

Here and in what follows, it is understood that the 0th generator commutes with all other generators: $[a_i[0], a_j[m]] = 0$, for all $m \in \mathbb{Z}$.

The algebra $\mathcal{H}_{q,t}(\mathfrak{g})$ becomes commutative in the limit $q \to 1$ and in the limit $t \to 1$.

The generators $a_i[n]$ are "root" type generators of $\mathcal{H}_{q,t}(\mathfrak{g})$. There is a unique set of "fundamental weight" type generators, $y_i[n], i = 1, \dots, \ell; n \in \mathbb{Z}$, that satisfy:

$$[a_i[n], y_j[m]] = \frac{1}{n}(q_i^n - q_i^{-n})(t^n - t^{-n})\delta_{i,j}\delta_{n,-m}. \tag{3.2}$$

They have the following commutation relations:

$$[y_i[n], y_j[m]] = \frac{1}{n}(q^n - q^{-n})(t^n - t^{-n})M_{ij}(q^n, t^n)\delta_{n,-m}, \tag{3.3}$$

where $(M_{ij}(q,t))_{1 \le i,j \le \ell}$ is the following matrix

$$\begin{aligned} M(q,t) &= D(q,t)C(q,t)^{-1} \\ &= D(q,t)B(q,t)^{-1}D(q,t). \end{aligned} \tag{3.4}$$

We have

$$a_i[n] = \sum_{j=1}^{\ell} C_{ji}(q^n, t^n) y_j[n]. \tag{3.5}$$

We will use the colon notation for the standard normally ordered product of elements of this algebra. Introduce the generating series

$$A_i(z) = t^{2(\rho^\vee, \alpha_i)} q^{2d(\rho, \alpha_i) + 2a_i[0]} : \exp\left(\sum_{m \ne 0} a_i[m] z^{-m}\right) :, \tag{3.6}$$

$$Y_i(z) = t^{2(\rho^\vee, \omega_i)} q^{2d(\rho, \omega_i) + 2y_i[0]} : \exp\left(\sum_{m \ne 0} y_i[m] z^{-m}\right) : . \tag{3.7}$$

Recall that $(\rho^\vee, \alpha_i) = 1, d(\rho, \alpha_i) = d_i$.

Formula (3.5) implies that

$$A_i(z) =: Y_i(zq_it)Y_i(zq_i^{-1}t^{-1}) \times \prod_{j:I_{ji}=1} Y_j(z)^{-1} \prod_{j:I_{ji}=2} Y_j(zq)^{-1}Y_j(zq^{-1})^{-1} \prod_{j:I_{ji}=3} Y_j(zq^2)^{-1}Y_j(z)^{-1}Y_j(zq^{-2})^{-1} : \tag{3.8}$$

Thus, for $\mathfrak{g}$ of non-simply laced type, the two classical limits of $A_i(z)$ are quite different: when $t \to 1$, we have

$$A_i(z) = Y_i(zq_i)Y_i(zq_i^{-1}) \times \prod_{j:I_{ji}=1} Y_j(z)^{-1} \prod_{j:I_{ji}=2} Y_j(zq)^{-1}Y_j(zq^{-1})^{-1} \prod_{j:I_{ji}=3} Y_j(zq^2)^{-1}Y_j(z)^{-1}Y_j(zq^{-2})^{-1} \tag{3.9}$$

but when $q \to 1$, we have a much simpler expression

$$A_i(z) = Y_i(zt)Y_i(zt^{-1}) \prod_{j \neq i} Y_j(z)^{-I_{ji}}. \tag{3.10}$$

Remark 3.1. (1) The first limit (3.9) coincides with the monomial $A_{i,z}$ which appears in the theory of q-characters of finite-dimensional representations of quantum affine algebras introduced in [FR3]. This is not surprising because, as explained in [FR3] and in Section 4.1 below, the q-characters may be viewed as limits of the fields from $\mathbf{W}_{q,t}$ as $t \to 1$.

(2) There is a surprising connection between the second limit (3.10) and Kashiwara's extension to non-simply laced $\mathfrak{g}$ [K3] of Nakajima's monomial model for crystals of $U_t(\mathfrak{g})$ [N2]. Let us recall that Nakajima's monomial realization was originally motivated by its relation with the q-characters in the symmetric cases. It turned out that this crystal realization was a consequence of the embedding theorem [K1], which makes sense in the symmetrizable case [K3]. But the relation between the monomial model and the q-characters was lost for non-simply laced types. Here we suggest an analogous relation, in which the role of the $t \to 1$ classical limit of $\mathbf{W}_{q,t}(\mathfrak{g})$ (whose free field realization is essentially the same as the q-character homomorphism, see Remark 4.1) is played by the $q \to 1$ classical limit.

Recall that in the monomial model the vertices of the crystal are represented by certain monomials in the variables $Y_i(t^k)^{\pm 1}$, and the crystal operators are obtained by multiplying them with some special monomials corresponding to the simple roots. In the simply-laced case, these are the monomials $A_{i,a}^{\pm 1}$ occurring in the q-character theory (as in formula (3.9)). But in the non-simply laced case the monomials $A_{i,a}^{\pm 1}$ in formula (3.9) do not work. Instead, as explained in [K3], we have to replace these monomials $A_{i,a}^{\pm 1}$ with other monomials. A direct comparison shows that Kashiwara's monomials coincide with the monomials in the above formula (3.10). We formulate a precise conjecture about this in Section 8 below. □

3.2. **Screening operators and definition of $\mathcal{W}_{q,t}(\mathfrak{g})$.** Recall that we have two sets of screening operators introduced in [FR2]: $S_i^+(z)$ and $S_i^-(z)$, $i = 1, \ldots, \ell$. They satisfy the difference equations:

$$S_i^+(zq_i) =: A_i(z)S_i^+(zq_i^{-1}) :, \tag{3.11}$$

and

$$S_i^-(zt) =: A_i(z)S_i^-(zt^{-1}) : . \tag{3.12}$$

The deformed $\mathcal{W}$-algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ was defined in [FR2] as the intersection of kernels of the residues S_i^+ of $S_i^+(z), i \in I$, or of the residues S_i^- of $S_i^-(z), i \in I$.

More precisely, $\mathcal{W}_{q,t}(\mathfrak{g})$ was defined in [FR2] as the associative topological algebra depending on two parameters q and t, which is topologically generated by the Fourier coefficients of certain fields from a deformed chiral algebra $\mathbf{W}_{q,t}(\mathfrak{g})$. The latter was defined in [FR2] as the maximal subalgebra commuting with the screening operators $S_i^{\pm}, i \in I$, in a deformed chiral algebra $\mathbf{H}_{q,t}(\mathfrak{g})$ constructed from the Heisenberg algebra $\mathcal{H}_{q,t}(\mathfrak{g})$.

In the classical limits $t \to 1$ (resp. $q \to 1$), $\mathbf{H}_{q,t}(\mathfrak{g})$ becomes commutative:

$$\mathbf{H}_{q,1}(\mathfrak{g}) = \mathbb{C}[Y_j(zq^{n_j})^{\pm 1}]_{j\in I, n_j\in\mathbb{Z}}, \qquad \mathbf{H}_{1,t}(\mathfrak{g}) = \mathbb{C}[Y_j(zt^{n_j})^{\pm 1}]_{j\in I, n_j\in\mathbb{Z}}. \tag{3.13}$$

The corresponding classical limit of $\mathbf{W}_{q,t}(\mathfrak{g})$ is a commutative subalgebra of this polynomial algebra, which is equal to the intersection of the kernels of the classical screening operators S_i^+ (resp. S_i^-), $i \in I$.

We will call these limits as the *classical $\mathcal{W}$-algebras* and denote them by $\mathcal{K}_q^+(\mathfrak{g})$ and $\mathcal{K}_t^-(\mathfrak{g})$, respectively. Below we describe both of these algebras. We will see that they are quite different if $\mathfrak{g}$ is non-simply laced.

3.3. **The $t \to 1$ limit.** In the limit $t \to 1$, the family $S_i^+, i \in I$, survives and gives rise to the following derivations

$$S_i^+ : \mathbb{C}[Y_j(zq^{n_j})^{\pm 1}]_{n_j\in\mathbb{Z}} \to \left(\bigoplus_{m\in\mathbb{Z}} \mathbb{C}[Y_j(zq^{n_j})^{\pm 1}]_{n_j\in\mathbb{Z}} \otimes S_i^+(zq^m)\right) / (S_i^+(zq_i^2) - A_i(zq_i)S_i^+(z))$$

acting by the formula

$$S_i^+ \cdot Y_j(zq^{n_j})^{\pm 1} = \pm\delta_{ij} Y_j(zq^{n_j})^{\pm 1} \otimes S_i^+(zq^{n_j}). \tag{3.14}$$

The following isomorphism was proved in [FR3, Proposition 6] and [FM, Proposition 5.2]:

$$\operatorname{Ker} S_i^+ = \mathbb{C}[Y_j(zq^{n_j})^{\pm 1}]_{j\neq i; n_j\in\mathbb{Z}} \otimes \mathbb{C}[Y_i(zq^{n_i})(1 + A_i(zq^{n_i}q_i)^{-1})]_{n_i\in\mathbb{Z}}, \tag{3.15}$$

where $A_i(z)$ is given by formula (3.9).

Now set

$$\mathcal{K}_q^+(\mathfrak{g}) := \bigcap_{i\in I} \operatorname{Ker} S_i^+. \tag{3.16}$$

Thus, $\mathcal{K}_q^+(\mathfrak{g})$ is the $t \to 1$ limit of $\mathbf{W}_{q,t}(\mathfrak{g})$. The following theorem was proved in [FM, Theorem 5.1].

Theorem 3.1. *(1) The commutative algebra $\mathcal{K}_q^+(\mathfrak{g})$ is isomorphic to $\operatorname{Rep}_z U_q(\widehat{\mathfrak{g}})$, the Grothendieck ring of the tensor subcategory $\mathcal{C}_{\mathbb{Z}}$ of the category of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ whose objects are representations with the Jordan-Hölder constituents having Drinfeld polynomials with roots in $q^{\mathbb{Z}}$.*

(2) Under this isomorphism, the embedding $\mathcal{K}_q^+(\mathfrak{g}) \to \mathbf{H}_{q,1}(\mathfrak{g})$ becomes the q-character homomorphism.

From Theorem 3.1 we obtain a natural basis of $\mathcal{K}_q^+(\mathfrak{g})$ consisting of the q-characters of simple modules from $\mathcal{C}_{\mathbb{Z}}$. It is known that these are parametrized by dominant monomials, i.e. monomials in the variables $Y_j(zq^{n_j})$, $j \in I$, $n_j \in \mathbb{Z}$, with only non-negative powers. In particular, the elements of this basis corresponding to the degree one dominant monomials $Y_j(zn^j)$ coincide with the q-characters of the corresponding fundamental representations (those are known to contain a unique dominant monomial, see [FR3, Corollary 4,(1)]).

3.4. **The $q \to 1$ limit.** Now consider the limit $q \to 1$. Then it is the family $S_i^-, i \in I$, that survives and gives rise to the derivations

$$S_i^- : \mathbb{C}[Y_j(zt^{n_j})^{\pm 1}]_{n_j \in \mathbb{Z}} \to \left(\bigoplus_{m \in \mathbb{Z}} \mathbb{C}[Y_j(zt^{n_j})^{\pm 1}]_{n_j \in \mathbb{Z}} \otimes S_i^+(zt^m) \right) / (S_i^-(zt^2) - A_i(zt) S_i^-(z))$$

acting by the formula

$$S_i^- \cdot Y_j(zt^{n_j})^{\pm 1} = \pm \delta_{ij} Y_j(zt^{n_j})^{\pm 1} \otimes S_i^-(zt^{n_j}), \tag{3.17}$$

where $A_i(z) = Y_i(zt) Y_i(zt^{-1}) Y_j(z)^{-I_{ji}}$ (formula (3.10)).

We have the following analogue of the isomorphism (3.15) (it is equivalent to (3.15) if $\mathfrak{g}$ is simply-laced, but for a non-simply laced Lie algebra $\mathfrak{g}$ this statement is new, as far as we know).

Proposition 3.2. *We have*

$$\operatorname{Ker} S_i^- = \bigcap_{i \in I} \mathbb{C}[Y_j(zt^{n_j})^{\pm 1}]_{j \neq i; n_j \in \mathbb{Z}} \otimes \mathbb{C}[Y_i(zt^{n_i})(1 + A_i(zt^{n_i+1})^{-1})]_{n_i \in \mathbb{Z}} \tag{3.18}$$

where $A_i(z) = Y_i(zt) Y_i(zt^{-1}) \prod_{j \neq i} Y_j(z)^{-I_{ji}}$ *(formula* (3.10)*).*

The proof is obtained by applying the argument used in the proof of (3.15) in [FM, Proposition 5.2].

Now set

$$\mathcal{K}_t^-(\mathfrak{g}) = \bigcap_{i \in I} \operatorname{Ker} S_i^-. \tag{3.19}$$

Thus, $\mathcal{K}_q^-(\mathfrak{g})$ is the $q \to 1$ limit of $\mathbf{W}_{q,t}(\mathfrak{g})$.

Unlike the limit $t \to 1$ (see Theorem 3.1), for non-simply laced $\mathfrak{g}$ we do not have an identification of $\mathcal{K}_t^-(\mathfrak{g})$ with the Grothendieck ring of a category of representation. Nonetheless, we do have a basis in $\mathcal{K}_t^-(\mathfrak{g})$ analogous to the basis of the q-characters of simple modules in $\mathcal{K}_q^+(\mathfrak{g})$.

Proposition 3.3. *(1) Every element of $\mathcal{K}_t^-(\mathfrak{g})$ is characterized by the multiplicities of the dominant monomials contained in it (i.e. monomials in the $Y_{j,t^{n_j}}$, $j \in I$, $n_j \in \mathbb{Z}$, with only non-negative powers).*

(2) For every dominant monomial m, there is a unique element $F(m)$ of $\mathcal{K}_t^-(\mathfrak{g})$ such that m is the unique dominant monomial of $F(m)$. Therefore we obtain a basis $\{F(m)\}$ of $\mathcal{K}^-(\mathfrak{g})$ parametrized by dominant monomials m.

Proof. The proof is the same as the proof in [H1, Theorem 5.13]. All that remains is to check the existence of the $F(Y_i(zq^n))$ for rank 2 Lie algebras. For simply-laced types, this is true because we can use the ordinary q-characters of the fundamental representations. For type B_2, we find the following elements:

$$F(Y_1(z)) = Y_1(z) + Y_1(zq^2)^{-1} Y_2(zq)^2 + 2Y_2(zq) Y_2(zq^3)^{-1} + Y_2(zq^3)^{-2} Y_1(zq^2) + Y_1(zq^4)^{-1},$$
$$F(Y_2(z)) = Y_2(z) + Y_2(zq^2)^{-1} Y_1(zq) + Y_1(zq^3)^{-1} Y_2(zq^2) + Y_2(zq^4)^{-1},$$

and for type G_2 we find

$$F(Y_1(z)) = Y_1(z) + Y_1(zq^2)^{-1} Y_2(zq)^3 + 3Y_2(zq)^2 Y_2(zq^3)^{-1} + 3Y_2(zq) Y_2(zq^3)^{-2} Y_1(zq^2)$$
$$+ Y_2(zq^3)^{-3} Y_1(zq^2)^2 + 3Y_2(zq) Y_2(zq^3) Y_1(zq^4)^{-1} + 2Y_1(zq^2) Y_1(zq^4)^{-1} + 3Y_2(zq) Y_2(zq^5)^{-1}$$
$$+ Y_1(zq^4)^{-2} Y_2(zq^3)^3 + 3Y_2(zq^3)^{-1} Y_2(zq^5)^{-1} Y_1(zq^2) + 3Y_1(zq^4)^{-1} Y_2(zq^3)^2 Y_2(zq^5)^{-1}$$

$$+3Y_2(zq^3)Y_2(zq^5)^{-2} + Y_2(zq^5)^{-3}Y_1(zq^4) + Y_1(zq^6)^{-1},$$

$$F(Y_2(z)) = Y_2(z) + Y_2(zq^2)^{-1}Y_1(zq) + Y_1(zq^3)^{-1}Y_2(zq^2)^2 + 2Y_2(zq^2)Y_2(zq^4)^{-1}$$
$$+Y_2(zq^4)^{-2}Y_1(zq^3) + Y_2(zq^4)Y_1(zq^5)^{-1} + Y_2(zq^6)^{-1}.$$

This completes the proof, up to the fact that the algorithm may produce elements $F(Y_i(zq^n))$ with an infinite number of terms. But it follows from Theorem 4.3 below that the elements $F(Y_i(zq^n))$ can also be obtained as folded q-characters of fundamental representations of $U_q(\widehat{\mathfrak{g}'})$, which do have finite numbers of terms. □

It follows from Proposition 3.3 that we have natural analogues $F(Y_i(zt^n))$ of the q-characters of the fundamental representations. We also have natural analogues of q-characters of the Kirillov–Reshetikhin modules (see Section 4.1 below).

Remark 3.2. This discussion motivates the following natural question: Is there a Hopf algebra $\mathcal{A}_t(\mathfrak{g})$ (an analogue of $U_t(\widehat{\mathfrak{g}})$) together with an injective (t-character) homomorphism

$$\mathrm{Rep}_z\, \mathcal{A}_t(\mathfrak{g}) \to \mathbb{C}[Y_j(zt^{n_j})^{\pm 1}]_{j\in I; n_j\in\mathbb{Z}},$$

where $\mathrm{Rep}_z\, \mathcal{A}_t(\mathfrak{g})$ is a subring of the Grothendieck ring of the category of finite-dimensional representations of $\mathcal{A}_t(\mathfrak{g})$, whose image is $\mathcal{K}_t^-(\mathfrak{g})$?

It is tempting to try to answer this question using an automorphism σ of the quantum affine algebra $U_t(\widehat{\mathfrak{g}'})$ defined by formula (5.2) below. The subalgebra $(U_t(\widehat{\mathfrak{g}'}))^\sigma$ of σ-invariants acts on every finite-dimensional representation V of $U_t(\widehat{\mathfrak{g}'})$. However, it is not clear how to define a comultiplication on the algebra $(U_t(\widehat{\mathfrak{g}'}))^\sigma$. Hence it is not clear how one could define a quantum integrable model this way. □

3.5. **The deformed $\mathcal{W}$-algebra for general q and t.** The structure of the deformed chiral algebra $\mathbf{W}_{q,t}(\mathfrak{g})$ for general values of q and t is much more complicated than that of its classical limits discussed above. Conjecture 1 of [FR2] implies that every basis element of $\mathbf{W}_{q,1}(\mathfrak{g}) = \mathcal{K}_q^+(\mathfrak{g})$ given by the q-character of a simple module over $U_q(\widehat{\mathfrak{g}})$ can be deformed to a basis element of $\mathbf{W}_{q,t}(\mathfrak{g})$ (i.e. an element of $\mathbf{H}_{q,t}(\mathfrak{g})$ which lies in the kernel of the screening operators).

However, apart from a few explicit examples presented in [FR2], there is no proof of existence of these elements in general. On the other hand, in [FH2] a simplified, commutative version of $\mathcal{W}_{q,t}(\mathfrak{g})$ was introduced, called the space of *interpolating* (q,t)*-characters*. These are defined from certain subrings which are modeled on what we expect the kernels of the screening operators to be (based on the description of the kernels of screening operators associated to the ordinary q-characters). We will recall this construction, and add further details to it, in Section 6.

At the moment, the relation between the interpolating (q,t)-characters and the deformed $\mathcal{W}$-algebra $\mathbf{W}_{q,t}(\mathfrak{g})$ is conjectural. However, as we will see below, for our purposes the (q,t)-characters provide a good substitute for elements of $\mathbf{W}_{q,t}(\mathfrak{g})$.

4. Classical limits of the deformed $\mathcal{W}$-algebra and q-characters

In this section we first recall some details on the relation between the $t \to 1$ limit of the deformed $\mathcal{W}$-algebra and the q-characters of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$. We then relate the $q \to 1$ limit to what we call *folded* t-characters of finite-dimensional representations of $U_t(\widehat{\mathfrak{g}'})$ (see Theorem 4.3).

4.1. **Reminder on the q-characters of representations of quantum affine algebras.** First, consider the untwisted quantum affine algebra $U_q(\widehat{\mathfrak{g}})$. Let $\operatorname{Rep} U_q(\widehat{\mathfrak{g}})$ be the Grothendieck ring of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$. The q-character homomorphism [FR3] is an injective ring homomorphism

$$\chi_q : \operatorname{Rep} U_q(\widehat{\mathfrak{g}}) \to \mathcal{Y}_q = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^\times}.$$

If we replace each $Y_{i,a}$ by y_i, we recover the usual character homomorphism for the $U_q(\mathfrak{g})$-module obtained by restriction of $U_q(\widehat{\mathfrak{g}})$-module, which encodes its grading by the lattice of integral weights of a Cartan subalgebra of the Lie algebra $\mathfrak{g}$. In what follows, we will refer to these integral weights as $\mathfrak{g}$-*weights*. In particular, each monomial in $\mathcal{Y}_q$ has a $\mathfrak{g}$-weight.

It is proved in [FR3, FM] (see also Theorem 3.1 above) that

$$\operatorname{Im}(\chi_q) = \bigcap_{i\in I} \mathfrak{K}_{i,q}, \tag{4.1}$$

where

$$\mathfrak{K}_{i,q} = \mathbb{Z}[Y_{j,a}^{\pm 1}, Y_{i,a}(1 + A_{i,aq_i}^{-1})]_{j\neq i, a\in\mathbb{C}^\times} \tag{4.2}$$

and $A_{i,a}$ is defined by formula (3.9), where we replace $Y_j(za)$ with $Y_{j,a}$.

Remark 4.1. Note that in the context of deformed $\mathcal{W}$-algebras, it is convenient to restrict ourselves to the variables $Y_j(zq^{n_j}), n_j \in \mathbb{Z}$ (i.e. restrict ourselves to the multiplicative lattice of spectral parameters $a = zq^n, n \in \mathbb{Z}$). But in the context of q-characters, we usually consider all spectral parameters $a \in \mathbb{C}^\times$ and denote the corresponding variables by $Y_{j,a}$ (see [FR3, Sect. 7] for more detail).

In particular, comparing formulas (3.16) and (3.15) with formulas (4.1) and (4.2), respectively, we find that if we replace the variables $Y_j(zq^{n_j}), n_j \in \mathbb{Z}$, by the variables $Y_{j,a}, a \in \mathbb{C}^\times$, then $\mathcal{K}_q^+$ becomes $\operatorname{Im}(\chi_q)$. That's what we mean by the statement that the $t \to 1$ limit of the free field realization of $\mathbf{W}_{q,t}(\mathfrak{g})$ corresponds to the q-character homomorphism of $U_q(\widehat{\mathfrak{g}})$. □

A monomial in $\mathcal{Y}_q$ is called dominant if it is a product of positive powers of the $Y_{i,a}, i \in I, a \in \mathbb{C}^\times$. A simple $U_q(\widehat{\mathfrak{g}})$-module is uniquely characterized by the highest monomial (in the sense of its $\mathfrak{g}$-weight) in its q-character (this monomial encodes the data of the Drinfeld polynomials; for the definition of the latter, see Theorem 12.2.6 of [CP]). This monomial is dominant. An element of $\operatorname{Im}(\chi_q)$ is characterized by the multiplicities of its dominant monomials. A $U_q(\widehat{\mathfrak{g}})$-module is said to be affine-minuscule if its q-character has a unique dominant monomial.

If a dominant monomial is in $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in q^{\mathbb{Z}}}$, then the q-character of the corresponding simple module also belongs to this subring.

A Kirillov–Reshetikhin (KR) module of $U_q(\widehat{\mathfrak{g}})$ is a simple module with the highest monomial of the form $Y_{i,a}Y_{i,aq_i^2}\cdots Y_{i,aq_i^{2(k-1)}}$ with $a \in \mathbb{C}^\times$, $i \in I$ and $k \geq 0$.

It was proved in [N3, H2] that the KR modules of $U_q(\widehat{\mathfrak{g}})$ are affine-minuscule. For $k = 1$, that is for fundamental representations, this was proved in [FM].

4.2. **Twisted affine algebras.** Next, consider the twisted quantum affine algebra $U_t({}^L\widehat{\mathfrak{g}})$, and let $\operatorname{Rep} U_t({}^L\widehat{\mathfrak{g}})$ be the Grothendieck ring of the category of its finite-dimensional representations.

The twisted t-character homomorphism [H3] is an injective ring homomorphism

$$\chi_t : \operatorname{Rep} U_t({}^L\widehat{\mathfrak{g}}) \to \mathbb{Z}[Z^{\pm 1}_{i,a^{d_i^\vee}}]_{a\in\mathbb{C}^\times, i\in I},$$

where we have set

$$d_i^\vee = d + 1 - d_i. \tag{4.3}$$

These are the analogues of the d_i for the Langlands dual Lie algebra ${}^L\mathfrak{g}$.

As in the untwisted case, we have the notions of dominant monomials, affine-minuscule modules and KR modules. An element of $\operatorname{Im}(\chi_t)$ is again characterized by its dominant monomial and the KR modules of $U_t({}^L\widehat{\mathfrak{g}})$ are affine-minuscule, as proved in [H3].

If a dominant monomial is in $\mathbb{Z}[Z^{\pm 1}_{i,a^{d_i^\vee}}]_{a\in\epsilon^{\mathbb{Z}}t^{\mathbb{Z}}, i\in I}$, then the twisted t-character of the corresponding simple module also belongs to this subring.

The image of χ_t is equal to

$$\bigcap_{i\in I} \mathbb{Z}[Z^{\pm 1}_{j,a^{d_j^\vee}}, Z_{i,a^{d_i^\vee}}(1 + B^{-1}_{i,(at)^{d_i^\vee}})]_{j\neq i, a\in\mathbb{C}^\times},$$

where

$$B_{i,a} = Z_{i,at^{d_i^\vee}} Z_{i,at^{-d_i^\vee}} \times \prod_{j\sim i | d_j^\vee = d} Z^{-1}_{j,a^{d_i}} \times \prod_{j\sim i, a' | d_j^\vee = 1, (a')^{d_i^\vee} = a} Z^{-1}_{j,a'},$$

where we write $i \sim j$ if $I_{ij} \neq 0$ (recall that (I_{ij}) denotes the incidence matrix).

Note that a special definition should be used for the monomials $B_{i,a}$ in the case of type $A^{(2)}_{2n}$, but we are not considering this case here because this affine Kac–Moody algebra is not dual to an untwisted affine algebra (note that $A^{(2)}_{2n}$ does not appear in Section 2.2).

According to Theorem 3.1, the $t \to 1$ limit $\mathcal{K}_q^+(\mathfrak{g})$ of $\mathbf{W}_{q,t}(\mathfrak{g})$ is isomorphic to $\operatorname{Rep} U_q(\widehat{\mathfrak{g}})$ so that the embedding of $\mathbf{W}_{q,1}(\mathfrak{g})$ into $\mathbf{H}_{q,1}(\mathfrak{g})$ becomes the q-character homomorphism.

Our task is to relate the $q \to 1$ limit $\mathcal{K}_t^-(\mathfrak{g})$ of $\mathbf{W}_{q,t}(\mathfrak{g})$ to t-characters of representations of quantum affine algebras. We start with two examples and then derive a general result. The upshot is that $\mathcal{K}_t^-(\mathfrak{g})$ is spanned by what we will call *folded t-characters of* $U_t(\widehat{\mathfrak{g}'})$, where $\mathfrak{g}'$ is the simply-laced Lie algebra equipped with an automorphism whose invariant Lie subalgebra is $\mathfrak{g}$ (the Dynkin diagram of $\mathfrak{g}$ can be obtained by folding the Dynkin diagram of $\mathfrak{g}'$). These are the t-characters of the finite-representations of $U_t(\widehat{\mathfrak{g}'})$ in which we identify the variables $Y_i(z)$ and $Y_{\sigma(i)}(z)$ for all $i \in I'$.

4.3. **Examples.** Consider the case $\mathfrak{g} = B_\ell$. Then we have the following formula for the element $T_1(z)$ of $\mathbf{W}_{q,t}(B_\ell)$ corresponding to the first fundamental representation of $U_q(B_\ell^{(1)})$ (see [FR2], Sect. 5.1.2). Set

$$J = \{1, \dots, \ell, 0, \overline{\ell}, \dots, \overline{1}\}$$

$$\begin{aligned}
\Lambda_i(z) &=: Y_i(zq^{2i-2}t^{i-1})Y_{i-1}(zq^{2i}t^i)^{-1} :, \qquad i = 1, \dots, \ell-1,\\
\Lambda_\ell(z) &=: Y_\ell(zq^{2\ell-3}t^{\ell-1})Y_\ell(zq^{2\ell-1}t^{\ell-1})Y_{\ell-1}(zq^{2\ell}t^\ell)^{-1} :,\\
\Lambda_0(z) &= \frac{(q+q^{-1})(qt-q^{-1}t^{-1})}{q^2t-q^{-2}t^{-1}} : Y_\ell(zq^{2\ell-3}t^{\ell-1})Y_\ell(zq^{2\ell+1}t^{\ell+1})^{-1} :,\\
\Lambda_{\overline{\ell}}(z) &=: Y_{\ell-1}(zq^{2\ell-2}t^\ell)Y_\ell(zq^{2\ell-1}t^{\ell+1})^{-1}Y_\ell(zq^{2\ell+1}t^{\ell+1})^{-1} :,\\
\Lambda_{\overline{i}}(z) &=: Y_{i-1}(zq^{4\ell-2i-2}t^{2\ell-i})Y_i(zq^{4\ell-2i}t^{2\ell-i+1})^{-1} :, \qquad i = 1, \dots, \ell-1.
\end{aligned}$$

Here and below we set $Y_0(z) = 1$. According to [FR2],

$$T_1(z) = \sum_{i \in J} \Lambda_i(z) \tag{4.4}$$

commutes with the screening operators $S_i^{\pm}, i = 1, \ldots, \ell$ and hence belongs to $\mathbf{W}_{q,t}(B_\ell)$.

The $t \to 1$ limit of the rational function

$$f_\ell(q,t) = \frac{(q+q^{-1})(qt^{-1} - q^{-1}t)}{q^2t^{-1} - q^{-2}t} \tag{4.5}$$

is equal to 1. Hence the limit of the above formula for $T_1(z)$ as $t \to 1$ has $2\ell + 1$ terms, and one can check that it coincides with the q-character of the first fundamental representation of $U_q(B_\ell^{(1)})$.

Now consider the $q \to 1$ limit of (4.4). Note that $f_\ell(1,t) = 2$, so in this limit the term $\Lambda_0(z)$ appears with coefficient 2. As observed in [FR2, Sect. 6.3], the $q \to 1$ limit of $T_1(z)$ looks like the t-character of $U_t(D_{\ell+1}^{(2)})$ in which we remove all ϵ factors. For this reason, it was conjectured in [FR2, Sect. 6.3] that the $q \to 1$ limit of the elements of $\mathbf{W}_{q,t}(\mathfrak{g})$ in general (which are, by definition, elements of $\mathcal{K}_t^-(\mathfrak{g})$) should be given by the t-characters of finite-dimensional representations of $U_t(\widehat{\mathfrak{g}}^\vee)$ in which we remove the ϵ factors (note that $D_{\ell+1}^{(2)} = (B_\ell^{(1)})^\vee$).

This conjecture is likely to be true, but the problem is that we don't know how to interpret this removal of ϵ factors in terms of representation theory, and therefore this does not help us with constructing the corresponding integrable models. Hence in this paper we give a different interpretation of this limit.

To explain it, let's look more closely at the $q \to 1$ limit of the formula (4.4) for $T_1(z)$. This is an element of $\mathcal{K}_t^-(B_\ell)$ given by the same formula (4.4), where now

$$\begin{aligned}
\Lambda_i(z) &= Y_i(zt^{i-1})Y_{i-1}(zt^i)^{-1}, \qquad i = 1, \ldots, \ell-1,\\
\Lambda_\ell(z) &= Y_\ell(zt^{\ell-1})^2 Y_{\ell-1}(zt^\ell)^{-1},\\
\Lambda_0(z) &= 2Y_\ell(zt^{\ell-1})Y_\ell(zt^{\ell+1})^{-1},\\
\Lambda_{\overline{\ell}}(z) &= Y_{\ell-1}(zt^\ell)Y_\ell(zt^{\ell+1})^{-2},\\
\Lambda_{\overline{i}}(z) &= Y_{i-1}(zt^{2\ell-i})Y_i(zt^{2\ell-i+1})^{-1}, \qquad i = 1, \ldots, \ell-1.
\end{aligned}$$

Let's compare this formula with the $q \to 1$ limit of the formula for $T_1(z)$ from $\mathbf{W}_{q,t}(D_{\ell+1})$ (see [FR2], Sect. 5.1.4). It is an element $\mathcal{K}_t^-(D_{\ell+1})$ given by formula (4.4) but now with

$$J = \{1, \ldots, \ell+1, \overline{\ell+1}, \ldots, \overline{1}\}$$

$$\begin{aligned}
\Lambda_i(z) &= Y_i(zt^{i-1})Y_{i-1}(t^i)^{-1}, \qquad i = 1, \ldots, \ell-1,\\
\Lambda_\ell(z) &= Y_{\ell+1}(zt^{\ell-1})Y_\ell(zt^{\ell-1})Y_{\ell-1}(zt^\ell)^{-1},\\
\Lambda_{\ell+1}(z) &= Y_{\ell+1}(zt^{\ell-1})Y_\ell(zt^{\ell+1})^{-1},\\
\Lambda_{\overline{\ell+1}}(z) &= Y_\ell(zt^{\ell-1})Y_{\ell+1}(zt^{\ell+1})^{-1},\\
\Lambda_{\overline{\ell}}(z) &= Y_{\ell-1}(zt^\ell)Y_\ell(zt^{\ell+1})^{-1}Y_{\ell+1}(zt^{\ell+1})^{-1},\\
\Lambda_{\overline{i}}(z) &= Y_{i-1}(zt^{2\ell-i})Y_i(zt^{2\ell-i+1})^{-1}, \qquad i = 1, \ldots, \ell-1.
\end{aligned}$$

By inspecting these formulas, we obtain the following result (recall that $\mathcal{K}_t^-(\mathfrak{g})$ is defined in formula (3.19)).

Lemma 4.1. *If we identify the generators $Y_{\ell+1}(z)$ with $Y_\ell(z)$ for $D_{\ell+1}$, then the formula for $T_1(z)$ in $\mathcal{K}_t^-(D_{\ell+1})$ becomes the formula for $T_1(z)$ in $\mathcal{K}_t^-(B_\ell)$.*

Now observe that in this case $\mathfrak{g} = B_\ell$ and $\mathfrak{g}' = D_{\ell+1}$, with the corresponding automorphism σ exchanging the ℓth and the $(\ell+1)$st nodes of the Dynkin diagram. Hence the identification of Lemma 4.1 corresponds precisely to the *folding* of the Dynkin diagram of D_ℓ, which gives the Dynkin diagram of B_ℓ.

Let us apply the same procedure in the case $\mathfrak{g} = C_\ell$. Then $\mathfrak{g}' = A_{2\ell-1}$ and σ exchanges the ith and the $2\ell - i$th nodes of the Dynkin diagram. Comparing formulas in Sects. 5.1.1 and 5.1.3 of [FR2], we obtain

Lemma 4.2. *Let us identify the generators $Y_{2\ell-i}(z), i = 1, \dots, \ell-1$, with $Y_i(z)$ for $A_{2\ell-1}$. Then the formula for $T_1(z)$ in $\mathcal{K}_t^-(A_{2\ell-1})$ becomes the formula for $T_1(z)$ in $\mathcal{K}_t^-(C_\ell)$.*

4.4. **General case.** Formula (3.10) shows that if we impose the relations $Y_i(z) = Y_{\sigma(i)}(z)$ for all $i \in I'$, then the generators $A_i(z), i \in I'$ of $\mathcal{K}_t^-(\mathfrak{g}')$ go to the corresponding generators $A_i(z), i \in I$, of $\mathcal{K}_t^-(\mathfrak{g})$. Here and below, abusing notation, we identify $i \in I'$ with its image in $I = I'/\langle\sigma\rangle$.

Thus, we have natural commutative diagram

$$\begin{array}{ccc}
\bigcap_{i\in I'} \mathbb{C}[Y_j(zt^{n_j})^{\pm 1}]_{j\neq i; n_j\in\mathbb{Z}} \otimes \mathbb{C}[Y_i(zt^{n_i})(1 + A_i(zt^{n_i+1})^{-1})]_{n_i\in\mathbb{Z}} & \longrightarrow & \mathbb{C}[Y_j(zt^{n_j})^{\pm 1}]_{j\in I'; n_j\in\mathbb{Z}} \\
\downarrow & & \downarrow \\
\bigcap_{i\in I} \mathbb{C}[Y_j(zt^{n_j})^{\pm 1}]_{j\neq i; n_j\in\mathbb{Z}} \otimes \mathbb{C}[Y_i(zt^{n_i})(1 + A_i(zt^{n_i+1})^{-1})]_{n_i\in\mathbb{Z}} & \longrightarrow & \mathbb{C}[Y_j(zt^{n_j})^{\pm 1}]_{j\in I; n_j\in\mathbb{Z}}
\end{array}$$

with the vertical maps being surjective and the horizontal maps being injective. This proves the following theorem stating that $\mathcal{K}_t^-(\mathfrak{g}) = \mathbf{W}_{1,t}(\mathfrak{g})$ is spanned by the t-characters of $U_t(\widehat{\mathfrak{g}'})$ in which we identify $Y_i(z)$ with $Y_{\sigma(i)}(z)$.

Theorem 4.3. *There is a surjective homomorphism $\operatorname{Rep}_z U_t(\widehat{\mathfrak{g}'}) \to \mathcal{K}_t^-(\mathfrak{g})$ that fits in the commutative diagram*

$$\begin{array}{ccc}
\operatorname{Rep}_z U_t(\widehat{\mathfrak{g}'}) & \xrightarrow{\sim} & \bigcap_{i\in I'} \mathbb{C}[Y_j(zt^{n_j})^{\pm 1}]_{j\neq i; n_j\in\mathbb{Z}} \otimes \mathbb{C}[Y_i(zt^{n_i})(1 + A_i(zt^{n_i+1})^{-1})]_{n_i\in\mathbb{Z}} \\
\downarrow & & \downarrow \\
\mathcal{K}_t^-(\mathfrak{g}) & \xrightarrow{\sim} & \bigcap_{i\in I} \mathbb{C}[Y_j(zt^{n_j})^{\pm 1}]_{j\neq i; n_j\in\mathbb{Z}} \otimes \mathbb{C}[Y_i(zt^{n_i})(1 + A_i(zt^{n_i+1})^{-1})]_{n_i\in\mathbb{Z}}
\end{array}$$

We will call the composition of the left vertical and the lower horizontal maps the *folded t-character homomorphism* and denote it by ${}^{\mathrm{f}}\chi_t$:

$$ {}^{\mathrm{f}}\chi_t : \operatorname{Rep}_z U_t(\widehat{\mathfrak{g}'}) \to \mathbb{C}[Y_i(zt^{n_i})]_{i\in I}. \tag{4.6}$$

It extends naturally to the entire $\operatorname{Rep} U_t(\widehat{\mathfrak{g}'})$. We will also call $\mathcal{K}_t^-(\mathfrak{g})$ the *folded t-character ring.*

Thus, we obtain an interpretation of the $q \to 1$ limit $\mathcal{K}_t^-(\mathfrak{g})$ of $\mathbf{W}_{q,t}(\mathfrak{g})$ as a "folding" of the Grothendieck ring of finite-dimensional representations of $U_t(\widehat{\mathfrak{g}'})$.

4.5. **Connection to the Drinfeld–Sokolov reduction and the center of quantum affine algebra.** In this subsection we briefly discuss links between the classical limits of $\mathcal{W}_{q,t}(\mathfrak{g})$ and other (Poisson) algebras.

First, Conjecture 3 of [FR2] states that the limit $q \to 1$ of $\mathcal{W}_{q,t}(\mathfrak{g})$ is isomorphic, as a Poisson algebra, to the Poisson algebra obtained by the deformed Drinfeld–Sokolov reduction of $G((z))$ with parameter $p = t^d$. This was confirmed by an explicit computations in the case of $\mathfrak{g} = C_2$ presented in Appendix B of [FR2].

Second, recall that the $t \to 1$ limit of $\mathcal{W}_{q,t}(\mathfrak{g})$ is isomorphic, as a commutative algebra, to the center $Z_q(\widehat{\mathfrak{g}})$ of $U_q(\widehat{\mathfrak{g}})$ at the critical level. Indeed, we can associate elements of $Z_q(\widehat{\mathfrak{g}})$ to finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ using a "double" of the transfer-matrix construction [RS, FR1], and hence connect $Z_q(\widehat{\mathfrak{g}})$ to $\operatorname{Rep} U_q(\widehat{\mathfrak{g}})$, which is $\mathcal{K}_q^+(\mathfrak{g}) = \mathbf{W}_{q,1}(\mathfrak{g})$. In [FR1], it was shown that in the case of $\mathfrak{g} = A_\ell$ this is in fact an isomorphism of Poisson algebra. We expect this to be true for a general $\mathfrak{g}$ as well.

If there exists an algebra $\mathcal{A}_t(\mathfrak{g})$ satisfying the properties of Remark 3.2, then it is reasonable to expect that the center of $\mathcal{A}_t(\mathfrak{g})$ at its "critical level" is isomorphic to the $q \to 1$ limit of $\mathcal{W}_{q,t}(\mathfrak{g})$.

Remark 4.2. Note that the limit of $Z_q(\widehat{\mathfrak{g}})$ as $q \to 1$ is the center of $U(\widehat{\mathfrak{g}})$ at the critical level. According to the Feigin–Frenkel isomorphism, the latter is isomorphic to the classical $\mathcal{W}$-algebra of the Langlands dual Lie algebra ${}^L\mathfrak{g}$ (and not of $\mathfrak{g}$). This is consistent with the fact that in the $q \to 1$ limit the screening operators S_i^+, whose joint kernel is the Poisson algebra $\mathcal{W}_{q,1}(\mathfrak{g})$, become the screening operators of the classical $\mathcal{W}$-algebra of ${}^L\mathfrak{g}$. □

4.6. **Bethe Ansatz equations from deformed $\mathcal{W}$-algebras.** First, consider the limit $t \to 1$, in which we obtain the algebra of q-characters of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$. In [FR3, Sect. 6.3], it was shown how to derive the corresponding Bethe Ansatz equation (BAE) from these q-characters. Namely, according to the conjecture in [FR3, Sect. 6.1], which was proved in [FH3], the eigenvalues of the transfer-matrices on the representation $W = \bigotimes_{j=1}^N V(\mathbf{P}_j)$ can be expressed (up to an overall factor) as the q-characters in which we replace the variables $Y_{i,a}$ by the ratios $Q_i(aq_i^{-1})/Q_i(aq_i)$ of the corresponding Baxter polynomials. Then the seeming poles corresponding to the monomials of the form M and MA_{i,aq_i}^{-1} in the q-characters should cancel each other. Using formula (3.9), we obtain the following BAE:

$$\prod_{k=1}^N q_i^{\deg P_{i,k}} \frac{P_{i,k}(q_i^{-1}/w_r^{(i)})}{P_{i,k}(q_i/w_r^{(i)})} = -\prod_{s\neq r} \frac{w_r^{(i)} - w_s^{(i)} q_i^{-2}}{w_r^{(i)} - w_s^{(i)} q_i^{2}} \prod_{j\neq i}\prod_{s=1}^{m_j} \frac{w_r^{(i)} - w_s^{(j)} q^{-B_{ij}}}{w_r^{(i)} - w_s^{(j)} q^{B_{ij}}}. \tag{4.7}$$

where $\{w_k^{(i)}\}$ is the set of roots of the ith Baxter polynomial $Q_i(z)$, and $P_{j,i}(z), j = 1, \dots, N; i = 1, \dots, \ell$, are the Drinfeld polynomials of W.

Remark 4.3. This formula corrects a typo in formula (6.6) of [FR3]; namely, the entries C_{li} of the Cartan matrix there should be replaced by the entries B_{li} of the *symmetrized* Cartan matrix. □

Thus, the key point in deriving these BAE is formula (3.9) expressing variables $A_i(z)$ in terms of $Y_j(z)$.

Now consider the limit $q \to 1$ and apply the argument of [FR3, Sect. 6.3] to formula (3.10) instead of (3.9). In the same way as in [FR3] we then obtain the following equations:

$$\prod_{k=1}^N q_i^{\deg P_{i,k}} \frac{P_{i,k}(t^{-1}/w_r^{(i)})}{P_{i,k}(t/w_r^{(i)})} = -\prod_{s\neq r} \frac{w_r^{(i)} - w_s^{(i)} t^{-2}}{w_r^{(i)} - w_s^{(i)} t^{2}} \prod_{j\neq i}\prod_{s=1}^{m_j} \frac{(w_r^{(i)} - w_s^{(j)} t)^{-C_{ji}}}{(w_r^{(i)} - w_s^{(j)} t^{-1})^{-C_{ji}}}. \tag{4.8}$$

This system is equivalent to the system of Bethe Ansatz equations obtained in [FKSZ] (formula (6.16)). It can also be obtained by folding the BAE corresponding to $\mathfrak{g}'$. See Section 1.2 for the explanation of the procedure of folding. We summarize this in the following statement.

Proposition 4.4. *The system* (4.8) *corresponding to the Lie algebra* $\mathfrak{g}$ *is equivalent to the system obtained by folding the BAE* (4.7) *corresponding to the Lie algebra* $\mathfrak{g}'$*, i.e. assuming that* $m_i = m_{\sigma(i)}$ *for all* $i \in I'$*, identifying* $w_r^{(i)} \equiv w_r^{\sigma(i)}$*, and writing the equations in terms of the variables* $w_r^{(i)}, i \in I$*.*

5. Folded integrable models

In this section we describe a novel quantum integrable model in which the spectra of the Hamiltonians correspond (conjecturally) to solutions of the Bethe Ansatz equations (4.8).

5.1. **Action of an automorphism σ.** Let W be a simple finite-dimensional representation of $U_q(\widehat{\mathfrak{g}'})$ whose highest monomial is σ-invariant, i.e. it is a monomial in the elements $\widetilde{Y}_{i,a}$ defined in Section 6.1 below.[6] Denote the corresponding highest $\mathfrak{g}'$-weight by $\Lambda(W)$. Clearly, $\sigma(\Lambda(W)) = \Lambda(W)$.

Denote by P the set of all $\mathfrak{g}'$-weights and by P^σ its subset of σ-invariant $\mathfrak{g}'$-weights. For $\gamma \in P$, denote by $W_\gamma \subset W$ the weight γ subspace of W. Let $\widehat{W}$ be the direct sum of the subspaces of W corresponding to σ-invariant $\mathfrak{g}'$-weights γ:

$$\widehat{W} = \bigoplus_{\gamma \in P^\sigma} W_\gamma,$$

In particular, $\widehat{W}$ contains the one-dimensional highest weight subspace $W_{\Lambda(W)}$ of W.

Next, let $\widetilde{W} \subset W$ be the direct sum of ℓ-weight subspaces of W corresponding to the σ-invariant ℓ-weights (equivalently, σ-invariant monomials in $\chi_q(W)$). Associating to each ℓ-weight the corresponding $\mathfrak{g}'$-weight, we obtain a grading on $\widetilde{W}$ by σ-invariant $\mathfrak{g}'$-weights:

$$\widetilde{W} = \bigoplus_{\gamma \in P^\sigma} \widetilde{W}_\gamma.$$

We have an inclusion $\widetilde{W} \subset \widehat{W}$ respecting the grading by (σ-invariant) $\mathfrak{g}'$-weights:

$$\widetilde{W}_\gamma \subset W_\gamma, \qquad \gamma \in P^\sigma. \tag{5.1}$$

The restriction of this inclusion to $\widetilde{W}_{\Lambda(W)}$ is an isomorphism. Next, we define an automorphism of $U_q(\widehat{\mathfrak{g}'})$ corresponding to σ (abusing notation, we denote it in the same way).

Lemma 5.1. *There is a unique algebra automorphism* σ *of* $U_q(\widehat{\mathfrak{g}'})$ *defined on the Drinfeld generators by the formulas*

$$\sigma(h_{i,r}) = h_{\sigma(i),r}, \qquad \sigma(x_{i,m}^\pm) = x_{\sigma(i),m}^\pm, \qquad \sigma(k_i^{\pm 1}) = k_{\sigma(i)}^{\pm 1}. \tag{5.2}$$

Proof. One checks directly that the map σ preserves the relations between the Drinfeld generators. □

[6] More generally, we could consider the case when W is a tensor product of simple representations, with the same invariance property for its highest monomial (but not necessarily for each simple factor). However, we will not do so in this paper.

We associate to W its twist by the automorphism σ. Namely, if $\rho : U_q(\widehat{\mathfrak{g}'}) \to \mathrm{End}(W)$ is the action of $U_q(\widehat{\mathfrak{g}'})$ on W, we define the σ-twisted action of $U_q(\widehat{\mathfrak{g}'})$ on the same vector space by the formula

$$\rho_\sigma(g) = \rho(\sigma(g)), \qquad g \in U_q(\widehat{\mathfrak{g}'}).$$

Lemma 5.2. *(1) The representations (W, ρ) and (W, ρ_σ) are isomorphic, and there is a unique linear automorphism $\widehat{\sigma} : W \to W$, such that*

$$\rho(\sigma(g)) = \widehat{\sigma}\rho(g)\widehat{\sigma}^{-1}, \qquad \forall g \in U_q(\widehat{\mathfrak{g}'}) \tag{5.3}$$

and the restriction of $\widehat{\sigma}$ to $W_{\Lambda(W)}$ is the identity.

(2) The operator $\widehat{\sigma}$ maps every $\mathfrak{g}'$-weight (resp. ℓ-weight) subspace of W corresponding to the $\mathfrak{g}'$-weight γ (resp. monomial M in $Y_{i,a}$) to a $\mathfrak{g}'$-weight (resp. ℓ-weight) subspace corresponding to $\sigma(\gamma)$ (resp. $\sigma(M)$). In particular, it preserves the subspaces $\widehat{W}$ and $\widetilde{W}$ and their graded components corresponding to σ-invariant $\mathfrak{g}'$-weights and ℓ-weights, respectively.

Proof. (1) Drinfeld's classification of irreducible finite-dimensional representations of $U_q(\widehat{\mathfrak{g}'})$ (see [CP]) shows that such a representation W is generated by its one-dimensional highest weight subspace W_Λ and is determined by the eigenvalues of the Cartan–Drinfeld generators $h_{i,r}$ on it, which are recorded by the highest monomial of its q-character (see Section 4.1). Formulas (5.2) imply that W_Λ is still the highest weight subspace of W under the twisted representation ρ_σ. The σ-invariance of the highest monomial of the representation (W, ρ) implies that the highest monomial of (W, ρ_σ) is the same. Therefore there is an isomorphism $\widehat{\sigma} : W \to W$ intertwining the representations (W, ρ) and (W, ρ_σ) of $U_q(\widehat{\mathfrak{g}'})$ and preserving the one-dimensional subspace W_Λ. It follows that $\widehat{\sigma}$ satisfies formula (5.3). By Schur's lemma, there is a unique such $\widehat{\sigma}$ that is equal to the identity on W_Λ.

(2) Let $\{X_i, i \in I'\}$ be either $\{k_i, i \in I'\}$ or $\{h^{\mathrm{ss}}_{i,r}, i \in I'\}$ with a fixed $r \neq 0$, where $h^{\mathrm{ss}}_{i,r}$ is the semi-simplification of the action of $h_{i,r}$ on W. Suppose that v is a joint eigenvector of $\{X_i, i \in I'\}$, in W with the eigenvalues $\lambda_i, i \in I'$. Then

$$X_i \cdot \widehat{\sigma}(v) = \widehat{\sigma}(\widehat{\sigma}^{-1} X_i \widehat{\sigma}) \cdot v = \widehat{\sigma} X_{\sigma^{-1}(i)} \cdot v = \lambda_{\sigma^{-1}(i)} \widehat{\sigma}(v). \tag{5.4}$$

This completes the proof. □

Remark 5.1. Let $W^{\widehat{\sigma}} \subset W$ be the subspace of vectors fixed by $\widehat{\sigma}$. It contains $W_{\Lambda(W)}$ and is stable under the action of the subalgebra $U_q(\widehat{\mathfrak{g}'})^\sigma$ of σ-invariant elements of $U_q(\widehat{\mathfrak{g}'})$. However, in general $W^{\widehat{\sigma}}$ is not stable under the action of the Cartan generators $k_i^{\pm 1}, i \in I'$ or the Cartan–Drinfeld generators, and hence does not have a well-defined character or q-character. Besides, the invariant subalgebra $U_q(\widehat{\mathfrak{g}'})^\sigma$ does not have a natural structure of Hopf algebra and has other deficiencies, as can be shown by an argument similar to that of [H3, Section 2.7].

This is why we consider instead the subspaces $\widehat{W}$ and $\widetilde{W}$. Their graded subspaces are preserved by $\widehat{\sigma}$, and hence $\widehat{W}$ and $\widetilde{W}$ have well-defined character and q-character, respectively. □

5.2. **XXZ-type model associated to $U_q(\widehat{\mathfrak{g}'})$.** Consider the Borel subalgebra $U_q(\widehat{\mathfrak{b}'})$ of $U_q(\widehat{\mathfrak{g}'})$ and its category $\mathcal{O}^*$ which contains the prefundamental representations $R_j^+(a)$, $R_j^-(a)$ ($j \in I'$, $a \in \mathbb{C}^\times$) constructed in [HJ].

For every $j \in I'$, we have the *Q-operator* $Q_j^\pm(z, u) = t_{R_j^\pm}(z, u)$ associated in [FH3] to the prefundamental representation $R_j^\pm(z)$. This operator is a formal power series in z, and it

also depends on an element u of the Cartan subgroup H' of the simply-connected Lie group G' associated to $\mathfrak{g}'$.

More generally, for any V in the Grothendieck ring $K_0(\mathcal{O}^*)$, we have the corresponding transfer-matrix $t_V(z,u)$.

The transfer-matrices $t_V(z,u), V \in K_0(\mathcal{O}^*)$ (in particular, operators $Q_j^{\pm}(z,u), j \in I'$) commute with each other. These are the Hamiltonians of the XXZ-type model associated to $U_q(\widehat{\mathfrak{g}'})$. Every finite-dimensional representation W of $U_q(\widehat{\mathfrak{g}'})$ decomposes into a direct sum of the generalized eigenspaces of these Hamiltonians. In this paper we will focus on the eigenvalues of these Hamiltonians and ignore the structure of the corresponding Jordan blocks. Hence we introduce the *semi-simplification* $Q_i^{\pm,\mathrm{ss}}(z,u)$ of $Q_i^{\pm}(z,u)$, i.e. the unique diagonalizable operator on W whose eigenspaces and eigenvalues are the generalized eigenspaces and eigenvalues of $Q_i^{\pm}(z,u)$. It follows that each $Q_i^{\mathrm{ss},\pm}(z,u)$ can be expressed as a polynomial in the original operator $Q_i^{\pm}(z,u)$, and therefore $Q_i^{\pm,\mathrm{ss}}(z,u)$ commutes with all $Q_j^{\pm}(z,u)$ and with all $Q_j^{\pm,\mathrm{ss}}(z,u), j \in I'$.

Thus, for every $u \in H'$, we have a direct sum decomposition of W into joint eigenspaces of the operators $Q_j^{\pm,\mathrm{ss}}(z,u), j \in I'$. The following is proved in [FH3, Theorem 5.9] and [FH4].

Theorem 5.3. *Let W be a simple module over $U_q(\widehat{\mathfrak{g}'})$. Then*

(1) The eigenvalues of $Q_j^{+,\mathrm{ss}}(z,u)$ on W are polynomials in z, up to an overall function in z depending only on W. Generically, the roots of these polynomials yield a solution of the BAE (4.7) *corresponding to $\mathfrak{g}'$.*

(2) Every joint eigenspace of $Q_j^{\pm,\mathrm{ss}}(z,u), j \in I'$, on W is a subspace of a $\mathfrak{g}'$-weight subspace of W corresponding to the $\mathfrak{g}'$-weight $\Lambda(W) - \sum_{j\in I'} n_j\alpha_j$, where n_j is the degree of the Baxter polynomial encoding the eigenvalue of $Q_j^{+,\mathrm{ss}}(z,u)$.

We call the polynomials in part (1) of this theorem the *Baxter polynomials*.

Part (1) of the theorem implies that the decomposition of W into a direct sum of joint eigenspaces of $Q_j^{\pm,\mathrm{ss}}(z,u), j \in I'$, is a refinement of the decomposition of W into a direct sum of its $\mathfrak{g}'$-weight subspaces. In other words, for every $u \in H'$ and $\gamma \in P$ (the set of $\mathfrak{g}'$-weights), the corresponding component W_γ of W is a direct sum

$$W_\gamma = \bigoplus_{\lambda \in \mathcal{E}_\gamma(u)} W_{\gamma,\lambda}(u) \tag{5.5}$$

where $\mathcal{E}_\gamma(u)$ is the set of distinct joint eigenvalues of $Q_j^{\pm,\mathrm{ss}}(z,u), j \in I'$, on W_γ, and $W_{\gamma,\lambda}(u)$ is the eigenspace corresponding to $\lambda \in \mathcal{E}_\gamma(u)$. Here λ denotes a collection $\{\lambda_j^{\pm}(z,u), j \in I'\}$ of joint eigenvalues of $Q_j^{\pm,\mathrm{ss}}(z,u), j \in I'$

As shown in [FH3, Proposition 5.5], $Q_j^{+}(z,u)$ has a well-defined $u \to 0$ limit, which is equal to

$$T_j(z) = \exp\left(\sum_{m>0} z^m \frac{\widetilde{h}_{j,-m}}{[d_j]_q [m]_{q_j}}\right).$$

This is a generating series of the Cartan–Drinfeld generators

$$\widetilde{h}_{j,-m} = \sum_{k\in I'} \widetilde{C}_{k,j}(q^m) h_{k,-m}. \tag{5.6}$$

According to this formula, we can recover all Cartan–Drinfeld generators $h_{k,-m}, k \in I', m > 0$ from the $T_j(z), j \in I'$.

It follows that the set $\mathcal{E}_\gamma(u)$ has a well-defined $u \to 0$ limit, which we denote by $\mathcal{E}_\gamma(0)$. Moreover, $\mathcal{E}_\gamma(0)$ can be identified with the set of ℓ-weights with the underlying $\mathfrak{g}'$-weight γ. For each ℓ-weight $\lambda \in \mathcal{E}_\gamma(0)$, the eigenspace $W_{\gamma,\lambda}(0)$ is the corresponding ℓ-weight subspace of W_γ.

Remark 5.2. The $u \to 0$ limit of the Q-operator $Q_j^-(z,u)$ is equal to $(T_j(z))^{-1}$. This follows from a direct computation or from the decomposition of $R_{i,a}^+ \otimes R_{i,a}^-$ in $K_0(\mathcal{O}^*)$. Indeed, it is equal to the class [1] of the trivial one-dimensional representation plus the classes of simple representations whose highest weights are equal to positive linear combinations of the simple roots, which implies that they do not contribute to the $u \to 0$ limit. □

5.3. **The invariant subspace.** Let us apply the automorphism σ of the algebra $U_q(\widehat{\mathfrak{g}'})$ given by formula (5.2) to the transfer-matrices $t_V(z,u)$. Recall that they are constructed by taking the trace of u times the universal R-matrix $\mathcal{R}$ of $U_q(\widehat{\mathfrak{g}'})$ over V (see, e.g. [FH3]).

Lemma 5.4. *The automorphism σ is an automorphism of the Hopf algebra structure on $U_q(\widehat{\mathfrak{g}'})$. Moreover, it preserves the universal R-matrix $\mathcal{R}$ of $U_q(\widehat{\mathfrak{g}'})$: $(\sigma \otimes \sigma)(\mathcal{R}) = \mathcal{R}$.*

Proof. One checks directly that the comultiplication defined on the Drinfeld–Jimbo generators is invariant under σ. For the ith Drinfeld–Jimbo generators, where $i \in I'$, this is obvious. For the the 0th Drinfeld–Jimbo generators, we use a formula expressing them in terms of the Drinfeld generators from [CP, p. 393].

The universal R-matrix of $U_q(\widehat{\mathfrak{g}'})$ corresponds to the canonical element in the double of $U_q(\mathfrak{b}_+)$ under the identification of this double (modulo the Cartan subalgebra with $U_q(\widehat{\mathfrak{g}'})$), see [D]. It follows from its definition that the automorphism σ is compatible with the double structure and therefore it sends the canonical element to itself. Hence it preserves the universal R-matrix of $U_q(\widehat{\mathfrak{g}'})$. □

For any representation V in the category $\mathcal{O}^*$, denote by $\sigma^*(V)$ the twist of V by the automorphism σ defined as in Section 5.1. It is clear that σ^* induces an automorphism of the Grothendieck ring $K_0(\mathcal{O}^*)$. By construction, $\sigma^*(R_{j,a}^\pm) \simeq R_{\sigma(j),a}^\pm$. Note also that σ defines an automorphism of the Cartan subgroup H' of G'.

Lemma 5.5. *We have*

$$\sigma^{-1}(t_V(z,u)) = t_{\sigma^*(V)}(z,\sigma(u)), \qquad \sigma^{-1}(Q_j^\pm(z,u)) = Q_{\sigma(j)}^\pm(z,\sigma(u)). \tag{5.7}$$

From now on, *we will assume that u is σ-invariant*, and hence defines an element of $H = (H')^\sigma$, which is a Cartan subgroup of the group $G = (G')^\sigma$. With this assumption, formulas (5.7) become

$$\sigma^{-1}(t_V(z,u)) = t_{\sigma^*(V)}(z,u), \qquad \sigma^{-1}(Q_j^\pm(z,u)) = Q_{\sigma(j)}^\pm(z,u). \tag{5.8}$$

Now we come to a key definition.

Definition 5.6. We define the *invariant subspace* of W as

$$W(u) := \bigoplus_{\gamma \in P^\sigma} W_\gamma(u),$$

where P^σ is the set of σ-invariant $\mathfrak{g}'$-weights and

$$W_\gamma(u) := \{w \in W_\gamma \mid Q_j^{\pm,\mathrm{ss}}(z,u).w = Q_{\sigma(j)}^{\pm,\mathrm{ss}}(z,u).w, \forall j \in I'\} \subset \widehat{W}_\gamma. \tag{5.9}$$

Since the prefundamental representations (topologically) generate the entire Grothendieck ring $K_0(\mathcal{O}^*)$, we have

$$W_\gamma(u) = \{w \in W_\gamma \mid t_V^{\mathrm{ss}}(z,u).w = t_{\sigma^*(V)}^{\mathrm{ss}}(z,u).w, \forall [V] \in K_0(\mathcal{O}^*)\}.$$

Using formula (5.8) in the same way as in the proof of Lemma 5.2, we obtain the following result.

Lemma 5.7. *$W_\gamma(u)$ is preserved by $\widehat{\sigma}$.*

We can also describe $W_\gamma(u)$ as a span of joint eigenvectors of $Q_j^{\pm,\mathrm{ss}}(z,u), j \in I'$.

Lemma 5.8. *$W_\gamma(u)$ is equal to the span of the joint eigenvectors of $Q_j^{\pm,\mathrm{ss}}(z,u), j \in I'$, in W_γ with eigenvalues $\lambda_j^\pm(z,u), j \in I'$, such that $\lambda_j^\pm(z,u) = \lambda_{\sigma(j)}^\pm(z,u)$ for all $j \in I'$. In other words, in the notation of Section 5.2,*

$$W_\gamma(u) = \bigoplus_{\lambda:\sigma(\lambda)=\lambda} W_{\gamma,\lambda}(u). \tag{5.10}$$

Proof. Every vector $v \in W_\gamma$ can be written as a linear combination of joint eigenvectors of $Q_j^{\pm,\mathrm{ss}}(z,u), j \in I'$:

$$v = \sum a_\lambda v_\lambda, \tag{5.11}$$

where v_λ denotes an eigenvector with the collection of joint eigenvalues $\lambda = \{\lambda_j^\pm(z,u), j \in I'\}$. Now, if $v \in W_\gamma(u)$, then

$$\sum a_\lambda \lambda_j^\pm(z,u) v_\lambda = \sum a_\lambda \lambda_{\sigma(j)}^\pm(z,u) v_\lambda,$$

which implies that every collection $\lambda = \{\lambda_j^\pm(z,u), j \in I'\}$ appearing in the decomposition (5.11) with a non-zero coefficient a_λ must be σ-invariant, i.e. $\lambda_j^\pm(z,u) = \lambda_{\sigma(j)}^\pm(z,u)$ for all $j \in I'$. □

Now, the discussion after Theorem 5.3 implies

Lemma 5.9. *We have $W_\gamma(0) = \widetilde{W}_\gamma$, i.e. $W_\gamma(0)$ is spanned by the ℓ-weight vectors corresponding to the σ-invariant ℓ-weights whose underlying $\mathfrak{g}'$-weight is $\gamma \in P^\sigma$.*

Remark 5.3 below shows, however, that we do not expect the statement analogous to Lemma 5.9 to hold for $W_\gamma(u)$ with $u \neq 0$.

Lemma 5.7 implies that $W(u)$ is preserved by $\widehat{\sigma}$. Lemma 5.9 implies that $W(0) = \widetilde{W}$. We also record the following useful result.

Lemma 5.10. *Suppose that $v \in W$ is an eigenvector of $Q_j^{\pm,\mathrm{ss}}(z,u), j \in I'$ (resp. an ℓ-weight vector) such that $\widehat{\sigma}(v) = \mu v$, where μ is a scalar (thus, μ is a d-th root of unity). Then the set of eigenvalues of $Q_j^{\pm,\mathrm{ss}}(z,u), j \in I'$ on v (resp. the ℓ-weight of v) is σ-invariant.*

Proof. Let $\lambda_j^\pm(z,u), j \in I'$, be the eigenvalues of $Q_j^{\pm,\mathrm{ss}}(z,u), j \in I'$ on v. Using formulas (5.8) and (5.3), we obtain that

$$\lambda_{\sigma(j)}^\pm(z,u) v = Q_{\sigma(j)}^{\pm,\mathrm{ss}}(z,u) \cdot v = \widehat{\sigma} Q_j^{\pm,\mathrm{ss}}(z,u) \widehat{\sigma}^{-1}(v)$$

$$= \mu^{-1}\widehat{\sigma}Q_j^{\pm,\mathrm{ss}}(z,u)\cdot v = \mu^{-1}\lambda_j^{\pm}(z,u)\widehat{\sigma}(v) = \lambda_j^{\pm}(z,u)v,$$

for all $j \in I'$. The proof for ℓ-weight vectors is similar. □

Remark 5.3. In general, joint eigenspaces of the operators $Q_j^{\pm,\mathrm{ss}}(z,u), j \in I'$, in W_γ are *not* spanned by ℓ-weight vectors. In other words, these eigenspaces differ from their limits as $u \to 0$. For example, let us take as W the tensor product $V_1 \otimes V_a$ of two fundamental representations of $U_q(\widehat{sl}_2)$ with the highest weight monomial Y_1Y_a. The ℓ-weight vectors in it are computed for example in [H4, Example 3.3]. Let $\{w_+, w_-\}$ and $\{v_+, v_-\}$ be bases of weight vectors in V_1 and V_a, respectively, with w_+ and v_+ being the highest weight vectors. Then $w_- \otimes v_+$ is an ℓ-weight vector whose ℓ-weight is $Y_{q^2}^{-1}Y_a$.

On the other hand, the action of $Q^+(z,u)$ on the 0-weight subspace of $V_1 \otimes V_a$, which is spanned by $w_- \otimes v_+$ and $w_+ \otimes v_-$, can be computed following [FH3, Example 7.8]. The result is

$$\frac{T(z)}{1-u^2} + \frac{zu^2(q-q^{-1})}{(1-u^2)(1-u^2q^{-2})} f_1T(z)f_0,$$

where $T(z)$ is the $u \to 0$ limit of $Q^+(z,u)$, which can be expressed (see formula (5.6) above) as a generating function of the Cartan–Drinfeld elements, and f_1, f_0 are the Drinfeld–Jimbo generators.

In particular, if we denote by $g(z)$ the eigenvalue of $T(z)$ on $w_+ \otimes v_+$, then we obtain

$$Q^+(z,u).(1-u^2)(1-u^2q^{-2})(g(z))^{-1}(w_- \otimes v_+)$$
$$= (1 - zq^{-1} - u^2q^{-2} + q^{-1}zu^2)w_- \otimes v_+ + zu^2(1-q^{-2})w_+ \otimes v_-.$$

Hence $w_- \otimes v_+$ is not an eigenvector of $Q^+(z,u)$ if $u \neq 0$. □

5.4. **Eigenvalues of transfer-matrices on the invariant subspace.** Since the operators $Q_j^{\pm,\mathrm{ss}}(z,u), j \in I'$, commute with each other, they have a well-defined joint spectrum on the invariant subspace $W(u)$. Using Proposition 4.4 (the fact that the system of folded BAE (4.8) for $\mathfrak{g}$ is obtained by folding of the system of BAE (4.7) for $\mathfrak{g}'$) and Theorem 5.3, we can now link the eigenvalues of $Q_j^{\pm,\mathrm{ss}}(z,u), j \in I'$, on $W(u)$ and solutions of the folded BAE (4.8) for $\mathfrak{g}$.

Theorem 5.11. *Suppose that the Baxter polynomials encoding the joint eigenvalues of $Q_j^{+,\mathrm{ss}}(z,u)$, $j \in I'$, on a vector in $W(u)$ are generic, so that their roots satisfy the BAE (4.7) for $\mathfrak{g}'$. Then, after the identification of the roots of these polynomials corresponding to $Q_j^{+,\mathrm{ss}}(z,u)$ and $Q_{\sigma(j)}^{+,\mathrm{ss}}(z,u)$, we obtain a solution of the folded BAE (4.8).*

Next, the following result is proved in [FH3, Theorem 5.11]:

Theorem 5.12. *The eigenvalues of the transfer-matrix $t_V(z,u)$, where V is in $\mathrm{Rep}\, U_q(\widehat{\mathfrak{g}'})$, on a simple module W can be expressed (up to an overall factor) as the q-character of V, in which we replace each $Y_{i,a}, i \in I$, by a ratio of the corresponding Baxter polynomials.*

We now use this result to describe the eigenvalues of $t_V(z,u)$ on $W(u) \subset W$.

Theorem 5.13. *Let V be a finite-dimensional representation of $U_q(\widehat{\mathfrak{g}'})$. Then the transfer-matrix $t_V(z,u)$ preserves the subspace $W(u)$ and its generalized eigenvalues are given (up to an overall factor) by the folded t-character ${}^{\mathrm{f}}\chi_t(V)$ (obtained by identifying $Y_{i,a}$ with $Y_{\sigma(i),a}$ in $\chi_t(V)$) in which we further replace each $Y_{i,a}$ by a ratio of the corresponding Baxter polynomials.*

This theorem gives us the sought-after link between the spectra of commuting quantum Hamiltonians and solutions of the folded Bethe Ansatz equations. Namely, the quantum Hamiltonians are the transfer-matrices $t_V(z,u)$ of the XXZ-type model associated to $U_q(\widehat{\mathfrak{g}'})$ (so the auxiliary spaces are representations of $U_q(\widehat{\mathfrak{g}'})$, or more general objects of the corresponding category $\mathcal{O}^*$) but we restrict them to the invariant subspaces $W(u)$ of the irreducible finite-dimensional representations W of $U_q(\widehat{\mathfrak{g}'})$ and consider the corresponding spectra (so the spaces of states are the subspaces $W(u) \subset W$).

However, for non-simply laced $\mathfrak{g}$ this answer does not quite define a quantum integrable model since we have not yet described the invariant subspaces $W(u)$ as representations of a Hopf algebra. In the next subsection, we conjecture such a description. More precisely, we conjecture that there is a certain distinguished subspace $\overline{W}(u) \subset W(u)$ which is isomorphic to a representation of the twisted quantum affine algebra $U_q({}^L\widehat{\mathfrak{g}})$. If this is true, then we can indeed describe the spaces of states as representations of a Hopf algebra, and then we indeed obtain a quantum integrable model.

5.5. **Defining the folded integrable model.** As before, let W be a simple representation of $U_q(\widehat{\mathfrak{g}'})$ whose highest monomial is σ-invariant, i.e. it is a monomial in the elements $\widetilde{Y}_{i,a}$ defined in Section 6.1.

Recall that for a Lie algebra $\mathfrak{g}$, we refer to the integral weights of the Cartan subalgebra of $\mathfrak{g}$ as $\mathfrak{g}$-*weights*. We will use the following simple fact demonstrated below in Section 9.1:

Lemma 5.14. *There is a natural isomorphism between the lattice P^σ of σ-invariant $\mathfrak{g}'$-weights and the lattice ${}^LP$ of ${}^L\mathfrak{g}$-weights.*

In particular, since $W(u)$ and $\widetilde{W} = W(0)$ are, by definition, vector spaces graded by P^σ, we can view them as vector spaces graded by the lattice ${}^LP$ of ${}^L\mathfrak{g}$-weights.

Conjecture 5.15.

(i) *For generic u, there is an isomorphism $W(u) \simeq \widetilde{W}$ of vector spaces graded by ${}^L\mathfrak{g}$-weights.*

(ii) *For generic u, there is a subspace $\overline{W}(u)$ of $W(u)$ which is stable under the operators $Q_j^{\pm,\mathrm{ss}}(z,u)$ and isomorphic, as a vector space graded by ${}^L\mathfrak{g}$-weights, to the vector space underlying a $U_q({}^L\widehat{\mathfrak{g}})$-module $M(W)$ (which does not depend on u).*

In Section 7 we explicitly verify this conjecture, as well as Conjecture 5.16 below, in a number of examples. In Section 6.5 we identify the representation $M(W)$, in terms of the closely related Conjecture 5.16 below, for the simplest $U_q(\widehat{\mathfrak{g}'})$-modules with σ-invariant highest monomials (modulo Conjecture 6.11).

Remark 5.4. A possible candidate for $\overline{W}(u)$ is the subspace $\mathbf{W}(u) := \bigoplus_{\gamma\in P^\sigma} \mathbf{W}_\gamma(u)$, where

$$\mathbf{W}_\gamma(u) := \{w \in W_\gamma \mid Q_j^{\pm}(z,u).w = Q_{\sigma(j)}^{\pm}(z,u).w, \forall j \in I'\} \subset \widehat{W}_\gamma. \tag{5.12}$$

The difference with formula (5.9) is that we now consider the operators $Q_j^{\pm}(z,u)$, rather than their semi-simplifications. We expect that for generic u, $\mathbf{W}(u)$ is isomorphic to $\mathbf{W}(0) \subset W(0) = \widetilde{W}$, which is defined similarly in terms of $T_j(z)$ (not their semi-simplifications).

Another candidate for $\overline{W}(u)$ is the subspace $\mathbf{W}^\sigma(u) \subset \mathbf{W}(u)$ spanned by the σ-invariant generalized eigenvectors. □

Conjecture 5.15 means that for any non-simply laced simple Lie algebra $\mathfrak{g}$, there exists a *folded quantum integrable system* with the quantum Hamiltonians being the transfer-matrices $t_V(z,u)$, corresponding to finite-dimensional representations V of $\operatorname{Rep} U_t(\widehat{\mathfrak{g}'})$, or more general objects of the corresponding category $\mathcal{O}^*$ (these are the auxiliary spaces of this integrable model). These Hamiltonians act on vector spaces that underlie finite-dimensional representations of $U_t({}^L\widehat{\mathfrak{g}})$ (these are the spaces of states of this integrable model) and, according to Theorem 5.13, the spectra of $t_V(z,u)$ can be expressed in terms of the folded t-character of V and the corresponding Baxter polynomials. Moreover, by Theorem 5.11, the roots of these Baxter polynomials correspond to solutions of the folded BAE (4.8) associated to $\mathfrak{g}$.

Remark 5.5. (1) In all examples we have studied so far, with W a simple module, we found a simple $U_q({}^L\widehat{\mathfrak{g}})$-module $M(W)$ that satisfies the statements of this conjecture.

(2) Suppose that a weight subspace $\widehat{W}_\gamma, \gamma \in P^\sigma$, of $\widehat{W}$ is one-dimensional. Then the automorphism $\widehat{\sigma}$ from Lemma 5.2 must act on $\widehat{W}_\gamma$ as a non-zero scalar. By Lemma 5.10, this implies that $\widehat{W}_\gamma = \widetilde{W}_\gamma$ and $\widehat{W}_\gamma = W_\gamma(u)$ for all u. Hence the statement of part (i) of the conjecture is clear for such weight subspaces.

(3) In the statement of part (ii) of the conjecture, one might be tempted to replace $U_t({}^L\widehat{\mathfrak{g}})$ with $U_t(\widehat{{}^L\mathfrak{g}})$. Indeed, the statement involves a vector space $\overline{W}(u)$ graded by ${}^L\mathfrak{g}$-weights, and therefore it could come from a representation of $U_t({}^L\mathfrak{g})$ (the quantum group of the simple finite-dimensional Lie algebra ${}^L\mathfrak{g}$), which is a subalgebra of both $U_t({}^L\widehat{\mathfrak{g}})$ and $U_t(\widehat{{}^L\mathfrak{g}})$.

In fact, in the examples we have considered so far (see Section 7), the character of $\overline{W}(u)$ is not only the character of a representation of $U_t({}^L\widehat{\mathfrak{g}})$ but also the character of a representation of $U_t(\widehat{{}^L\mathfrak{g}})$. But the representations of $U_t(\widehat{{}^L\mathfrak{g}})$ that appear here are significantly less natural than the representations of $U_t({}^L\widehat{\mathfrak{g}})$. For instance, as we illustrate below (see Sections 7.2–7.5), in most cases they are not simple, and often contain direct sums of copies of the trivial representation as direct summands (even for representations of small dimension). Even more importantly, if we consider representations of $U_t(\widehat{{}^L\mathfrak{g}})$, we can only reproduce the ordinary character of its restriction to $U_t({}^L\mathfrak{g})$, and *not* its t-character. On the other hand, in all examples we consider below, we can reproduce the t-character of a $U_t({}^L\widehat{\mathfrak{g}})$-module $M(W)$ using interpolating (q,t)-characters, see Conjecture 5.16 below.

(4) It would be interesting to write explicit formulas for the action of $U_t({}^L\widehat{\mathfrak{g}})$ on $M(W)$, at least in the cases when $M(W) \simeq \widetilde{W}$. □

We will give a more detailed description of the module $M(W)$, and hence a more precise formulation of Conjecture 5.15, using the theory of interpolating (q,t)-characters which we will recall (and refine) in Section 6 below. Namely, we have the following conjecture.

Conjecture 5.16. *The t-character $\chi_t(M(W))$ of the $U_t\left({}^L\widehat{\mathfrak{g}}\right)$-module $M(W)$ can be obtained via the specialization map Π_t from an element $X_{q,t} \in \overline{\mathcal{K}}_{q,t}(\mathfrak{g})$. In fact, $X_{q,t}$ has the following three specializations:*

$$\begin{tikzcd} & X_{q,t} \arrow[dl, "\overline{\Pi}_t"'] \arrow[d, "\Pi_t"] \arrow[dr, "\Pi'_t"] & \\ {}^{\mathrm{f}}\chi_t(W) & \chi_t(M(W)) & {}^{\mathrm{f}}\chi_t(W') \end{tikzcd}$$

where

${}^{\mathrm{f}}\chi_t(W) \in \mathcal{K}_t^-(\mathfrak{g})$ *is the folded t-character of the* $U_t\left(\widehat{\mathfrak{g}'}\right)$*-module* W*,*

${}^{\mathrm{f}}\chi_t(W') \in \mathcal{K}_t^-({}^L\mathfrak{g})$ *is the folded t-character of a* $U_t\left(\widehat{({}^L\mathfrak{g})'}\right)$*-module* W'*.*

5.6. **The subspace $\overline{W}(u)$.** Here we discuss the question of how to describe the subspace $\overline{W}(u) \subset W(u)$ in general.

Note that the fixed subspace $W^{\widehat{\sigma}} \subset W$ is not necessarily stable under the action of the operators $Q_j^{\pm,\mathrm{ss}}(z,u)$ in general. But let us introduce the subspace $W^{\sigma}(u) \subset W^{\widehat{\sigma}}$ spanned by all joint eigenvectors of $Q_j^{\pm,\mathrm{ss}}(z,u)$, $j \in I'$, which belong to $W^{\widehat{\sigma}}$.

Recall the decomposition (5.10) of $W_\gamma(u)$:

$$W_\gamma(u) = \bigoplus_{\lambda:\sigma(\lambda)=\lambda} W_{\gamma,\lambda}(u). \tag{5.13}$$

Lemma 5.17. *We have*

$$W^{\sigma}(u) = \bigoplus_{\gamma\in P^\sigma} \bigoplus_{\lambda:\sigma(\lambda)=\lambda} W^{\sigma}_{\gamma,\lambda}(u) \subset W(u) \tag{5.14}$$

where $W^{\sigma}_{\gamma,\lambda}(u)$ *is the* $\widehat{\sigma}$*-invariant subspace of* $W_{\gamma,\lambda}(u)$*.*

Proof. According to Theorem 5.3,(1), every joint eigenvector v of $Q_j^{\pm,\mathrm{ss}}(z,u)$, $j \in I'$, belongs to a weight subspace $W_\gamma \subset W, \gamma \in P$. By Lemma 5.2,(2) $\widehat{\sigma}$ maps $W_\gamma, \gamma \in P$, to $W_{\sigma(\gamma)}$. Therefore, if $\widehat{\sigma}(v) = v$, then $v \in W_\gamma, \gamma \in P^\sigma$. Next, we have the decomposition (5.5) of $W_\gamma, \gamma \in P^\sigma$, into eigenspaces of $Q_j^{\pm,\mathrm{ss}}(z,u)$. Applying formula (5.4) with $X_i = Q_i^{\pm,\mathrm{ss}}(z,u)$, $j \in I'$, we obtain that if $v \in W_{\gamma,\lambda}$, then $\widehat{\sigma}(v) \in W_{\gamma,\sigma(\lambda)}$. Hence, if $\widehat{\sigma}(v) = v$, then $\sigma(\lambda) = \lambda$. It follows that the subspace $W_{\gamma,\lambda}$ with $\gamma \in P^\sigma$ and $\sigma(\lambda) = \lambda$ is preserved by $\widehat{\sigma}$. Thus, $W_{\gamma,\lambda}$ decomposes into a direct sum of eigenspaces of $\widehat{\sigma}$, which are labeled by the d-th roots of unity. Denoting the invariant part (on which $\widehat{\sigma}$ acts the identity) by $W^{\sigma}_{\gamma,\lambda}(u)$, we obtain the decomposition (5.14). □

There is an analogue of the subspace $W^{\sigma}(u) \subset W(u)$ for $u = 0$; namely, the subspace $\widetilde{W}^{\sigma}$ spanned by all $\widehat{\sigma}$-invariant ℓ-weight vectors in W. In the same way as in the proof of Lemma 5.17 one shows that

$$\widetilde{W}^{\sigma} = \bigoplus_{\gamma\in P^\sigma} \bigoplus_{M:\sigma(M)=M} \widetilde{W}^{\sigma}_{\gamma,M} \subset \widetilde{W}, \tag{5.15}$$

where $\widetilde{W}^{\sigma}_{\gamma,M}$ is $\widehat{\sigma}$-invariant part of the ℓ-weight subspace of $\widetilde{W}_\gamma, \gamma \in P^\sigma$, whose ℓ-weight corresponds to a monomial M (note that $\widetilde{W}^{\sigma}_{\gamma,M}$ is preserved by $\widehat{\sigma}$ if and only if M is a σ-invariant monomial, see Lemma 5.2,(2)). Thus, $\widetilde{W}^{\sigma}$ is a subspace of $\widetilde{W} = W(0)$.

We expect that $\overline{W}(u)$ is a subspace of $\widetilde{W}^{\sigma}$ (although they are not equal in general, as we can see from the example in Section 7.5 below). In addition to the examples that will be presented in Section 7, some supporting evidence comes from the following result in the finite-type case.

Let W be a simple finite-dimensional representation of $U_q(\mathfrak{g}')$ with a σ-invariant highest weight. Then, as in the affine case, σ gives rise to an automorphism $\widehat{\sigma}$ of W. Let W^{σ} be the span of all weight vectors in W which belong to $W^{\widehat{\sigma}}$. In the same way as in the proof

of Lemma 5.17 one shows that

$$W^\sigma = \bigoplus_{\gamma \in P^\sigma} W^\sigma_\gamma,$$

where W^σ_γ is the subspace of $\widehat{\sigma}$-invariant vectors in $W_\gamma, \gamma \in P^\sigma$. Recall from Lemma 5.14 that we can view elements of the set P^σ of σ-invariant $\mathfrak{g}'$-weight spaces as ${}^L\mathfrak{g}$-weights. Thus, the character $\chi(W^\sigma)$ is a linear combination of ${}^L\mathfrak{g}$-weights.

Proposition 5.18. *$\chi(W^\sigma)$ is invariant under the action of the Weyl group of ${}^L\mathfrak{g}$, and hence it is the character of a virtual representation of $U_q({}^L\mathfrak{g})$.*

Proof. For $i \in I'$, denote by U_i the $U_q(\mathfrak{sl}_2)$ subalgebra of $U_q(\mathfrak{g}')$ generated by e_i, f_i, $k_i^{\pm 1}$.

Let us show that $\chi(W^\sigma)$ is invariant under the simple reflections of ${}^L\mathfrak{g}$ associated to the σ-orbits of $i \in I'$. This is clear if $\sigma(i) = i$ as W^σ is stable under the action of U_i and hence its character is invariant under the ith reflection of ${}^L\mathfrak{g}$.

Suppose $\sigma(i) \neq i$. Since $\mathfrak{g}'$ is not of type A_{2n}, the subalgebra U_i^σ of $U_q(\mathfrak{g}')$ generated by the $U_q(\mathfrak{sl}_2)$ subalgebras $U_{\sigma^k(i)}$, $1 \leq k \leq d$, is isomorphic to $U_q(\mathfrak{sl}_2)^{\otimes d}$. To simplify notation, let us assume that $d = 2$ (the proof for $d = 3$ is quite similar).

Viewed as a representation of U_i^σ, W is semi-simple:

$$W = \bigoplus_j L_j, \qquad L_j = L_j^{(1)} \otimes L_j^{(2)}, \tag{5.16}$$

where $L_j^{(1)}$ and $L_j^{(2)}$ are simple representations of the subalgebras U_i and $U_{\sigma(i)}$, respectively.

If x is a weight vector which belongs to W^σ (i.e. the weight of x is σ-invariant), then so are the vectors $\overline{e}.x$ and $\overline{f}.x$, where

$$\overline{e} = e_i e_{\sigma(i)}, \qquad \overline{f} = f_i f_{\sigma(i)}.$$

Hence W^σ is stable under $\overline{e}$ and $\overline{f}$.

Let ω be a weight such that that the weight subspace $(W^\sigma)_\omega$ is non-zero. Then $\omega(\alpha_i^\vee) = \omega(\alpha_{\sigma(i)}^\vee) = m \in \mathbb{Z}$. Suppose that $m \geq 0$. Using the decomposition (5.16), we can write any weight vector v in $(W^\sigma)_\omega$ as $v = \sum_j v_j$, where

$$v_j \in (L_j^{(1)})_m \otimes (L_j^{(2)})_m,$$

$(L_j^{(1)})_m$ and $(L_j^{(2)})_m$ being the weight subspaces corresponding to weight m in $L_j^{(1)}$ and $L_j^{(2)}$, respectively. From representation theory of $U_q(\mathfrak{sl}_2)$ we know that these weight subspaces are one-dimensional, so $v_j = v_j^{(1)} \otimes v_j^{(2)}$ is a pure tensor. Moreover, we have

$$\overline{f}^m.(v_j^{(1)} \otimes v_j^{(2)}) = (f_i^m.v_j^{(1)}) \otimes (f_{\sigma(i)}^m.v_j^{(2)}) \neq 0.$$

Hence we obtain an injective linear map :

$$\overline{f}^m : (W^\sigma)_\omega \to (W^\sigma)_{\omega - m(\alpha_i + \alpha_{\sigma(i)})}$$

and so

$$\dim(W^\sigma)_\omega \leq \dim(W^\sigma)_{\omega - m(\alpha_i + \alpha_{\sigma(i)})}$$

if $m \geq 0$.

The opposite inequality is obtained by considering the action of $\overline{e}^m$ on $(W^\sigma)_{\omega - m(\alpha_i + \alpha_{\sigma(i)})}$. This completes the proof. □

We expect that this virtual representation is an actual representation of $U_q({}^L\mathfrak{g})$ (although it is not clear to us how to construct the corresponding action of $U_q({}^L\mathfrak{g})$). In Section 9.7 we will show (assuming Conjecture 9.10) that in the limit $q \to 1$ it is possible to construct an explicit embedding of the irreducible representation of ${}^L\mathfrak{g}$ with the highest weight corresponding to that of W (which is σ-invariant by our assumption) into W^σ.

In the rest of this section, we discuss the corresponding QQ-system.

5.7. **The QQ-system.** The QQ-system (or $Q\widetilde{Q}$-system in the terminology of [FH4]) of type $\widehat{\mathfrak{g}'}$ reads

$$\left[-\frac{\alpha'_i}{2}\right] Q_{i,aq^{-1}}\widetilde{Q}_{i,aq} - \left[\frac{\alpha'_i}{2}\right] Q_{i,aq}\widetilde{Q}_{i,aq^{-1}} = \prod_{j\neq i} Q_{j,a}^{-C'_{j,i}},$$

where C' (resp. α'_i) is the Cartan matrix (resp. a simple root) of $\mathfrak{g}'$. It was written in [MRV1] in the context of affine opers, and established [FH4] as a system of relations in $K_0(\mathcal{O}^*)$, with the $Q_{i,a}$ and $\widetilde{Q}_{i,a}$ being the properly normalized classes of certain simple modules; namely, the module $R^+_{i,a}$ for $Q_{i,a}$ and another module, which we denote by $X'_{i,aq_i^{-2}}$, divided by an invertible element which does not depend on a (recall that here we consider the category $\mathcal{O}^*$, see [FH4, Remark 3.2 (iii)]). The $\left[-\frac{\alpha'_i}{2}\right]$ are classes of certain one-dimensional representations in $\mathcal{O}^*$. Then on $W(u)$, the eigenvalues of $Q_{j,a}$ are identified with those of $Q_{\sigma(j),a}$ and the eigenvalues of $\widetilde{Q}_{i,a}$ are identified with the eigenvalues of $\widetilde{Q}_{\sigma(j),a}$. Hence we obtain the following result.

Theorem 5.19. *The following QQ-system holds on the invariant subspace $W(u)$:*

$$\left[-\frac{\alpha_i}{2}\right] Q_{i,aq^{-1}}\widetilde{Q}_{i,aq} - \left[\frac{\alpha_i}{2}\right] Q_{i,aq}\widetilde{Q}_{i,aq^{-1}} = \prod_{j\neq i} Q_{j,a}^{-C_{j,i}}, \tag{5.17}$$

where C is the Cartan matrix of $\mathfrak{g}$, $Q_{i,a}$ (resp. $\widetilde{Q}_{i,a}$) is, up to an invertible constant, the Q-operator $Q^+_{i,a}$ (resp. the transfer-matrix associated to $X_{i,aq_i^{-2}}$).

Note that for a non-simply laced $\mathfrak{g}$, this is *neither* the QQ-system of type $\widehat{\mathfrak{g}}$ (as defined in [MRV2, FH4]) *nor* the QQ-system of the twisted type $\widehat{\mathfrak{g}}^\vee$ conjectured in [FH4, Section 3.3]. But this system is equivalent to the QQ-system obtained in [FKSZ] in the context of Miura (G,q)-opers (we note that a Yangian version of this system first appeared in the work of Mukhin and Varchenko [MV1, MV2]). We will call it the *folded QQ-system* associated to $\mathfrak{g}$.

According to Theorem 5.19, the spectra of the Hamiltonians of the folded integrable model introduced in this section give rise to solutions of the folded QQ-system. This is in agreement with Theorem 5.11 because, as shown in [FKSZ], under a genericity condition there is a bijection between solutions of the folded QQ-system and solutions of the folded BAE (4.8).

Remark 5.6. (1) Recall that each character $\omega \in H^*$ of the commutative group H gives a one-dimensional representation $U_q(\widehat{\mathfrak{b}})$-module $[\omega]$ which is in $\mathcal{O}$ and in $\mathcal{O}^*$. We obtain a subring $K_0(H) \subset K_0(\mathcal{O})$ of representations whose simple constituents are of this form. These are called constant elements as the associated transfer-matrices are constant (independent of the spectral parameter). The ordinary character of a representation in $\mathcal{O}$ can be viewed as an element of $K_0(H)$.

(2) According to [FH4], the precise relation between the solutions of $Q\widetilde{Q}$-system (5.17) and elements of $K_0(\mathcal{O})$ is as follows. The variables $Q_{i,z}$ and $\widetilde{Q}_{i,z}$ correspond to the classes $[L^+_{i,z}]$ and $[X_{i,zq_i^{-2}}]$, respectively, renormalized by constant invertible elements of $K_0(\mathcal{O})$.

However, Keyu Wang has pointed out that the proof of [FH4, Lemma 4.11] is incomplete. In that lemma a formula $\chi(X_{i,a}) = (1-[-\alpha_i])^{-1}\prod_{j\neq i}\chi_j^{-C_{j,i}}$ for the character of $X_{i,a}$ was given in terms of the characters $\chi_i = \chi(L^+_{i,a})$ of the prefundamental representations, and the normalization used in [FH4] (see the preceding paragraph) was based on this formula. Although we believe that this character formula is correct (and so the normalization used in [FH4] is correct), at the moment we do not have a proof of this character formula. Hence we propose to slightly modify formulas (3.1) and (3.2) in [FH4] for the normalizing factors, so we can avoid relying on this character formula. We explain this normalization in the next subsection.

(3) According to Theorem 5.19 and to Section 5.4, eigenvectors in the subspace $W(u)$ give solutions of the folded QQ-system (or, under some genericity condition, of the folded BAE (4.8)). However, we then restrict to a smaller subspace $\overline{W}(u)$ of $W(u)$, which means that we only take a subset of these solutions (we will see in the example in Section 7.5 below that $\overline{W}(u)$ can in fact be strictly smaller than $W(u)$). This is likely related to the fact that taking various limits of elements of the deformed $\mathcal{W}$-algebras, or various specializations of the corresponding interpolating (q,t)-characters (as discussed below in Section 6), may very well have different numbers of monomials. Hence it is natural to expect that there is a characterization of the solutions corresponding to the eigenvectors that belong to the subspace $\overline{W}(u)$ through a similar kind of interpolation, which would enable us to tell which solutions of the folded QQ-system (and the folded BAE) correspond to the representation $M(W)$ for the Langlands dual quantum affine and which do not. At the moment, this remains as an open question. □

5.8. Normalization of $Q_{i,z}$ and $\widetilde{Q}_{i,z}$.

Proposition 5.20. *The functions*

$$Q_{i,z} = [L^+_{i,z}]/\chi_i, \qquad \widetilde{Q}_{i,z} = [X_{i,zq_i^{-2}}]/\widetilde{\chi}_i,$$

where

$$\chi_i = \chi(L^+_{i,z}) \text{ and } \widetilde{\chi}_i = \chi(X_{i,z})\left(\left[\frac{\alpha_i}{2}\right] - \left[-\frac{\alpha_i}{2}\right]\right),$$

solve the $Q\widetilde{Q}$-system (in the notation of [FH4]*).*

Remark 5.7. Here, in contrast to [FH4], we do not specify a precise relation between the $\widetilde{\chi}_i$ and the χ_i. In addition, for symmetry, we have chosen to renormalize $[L^+_{i,z}]$ by dividing by χ_i, although in [FH4] $Q_{i,z}$ is $[L^+_{i,z}]$ without renormalization. Both normalizations are compatible, as the $Q\widetilde{Q}$-system written with the normalization in [FH4] implies that the variables in the Proposition above satisfy also the $Q\widetilde{Q}$-system. □

Proof. We give the proof of the Proposition for the category $\mathcal{O}$. The analogous result in $K_0(\mathcal{O}^*)$ is obtained in the same way.

We establish the following q-character formula

$$\chi_q(X_{i,1}) = [\widetilde{\mathbf{\Psi}}_{i,1}]\chi_{i,1}\chi(X_{i,1})(1-[\alpha_i]) \tag{5.18}$$

where $\widetilde{\mathbf{\Psi}}_{i,1}$ is the highest ℓ-weight of $X_{i,1}$ and $\chi_{i,1} = \sum_{m\geq 0}(A_{i,1}A_{i,q_i^{-2}}\cdots A_{i,q_i^{-2m}})^{-1}$. This formula implies the $Q\widetilde{Q}$-system in $K_0(\mathcal{O})$ by the arguments given in [FH4].

The proof of (5.18) is based on the results in [FH4], except that we do not use the complete proof of [FH4, Lemma 4.11], but only [FH4, Lemma 4.10]. Indeed, it implies only

$$\chi_q(X_{i,1}) = [\widetilde{\mathbf{\Psi}}_{i,1}]\sum_{m\geq 0}(A_{i,1}A_{i,q_i^{-2}}\cdots A_{i,q_i^{-2m}})^{-1}\chi(m),$$

where for $m\geq 0$, $\chi(m)\in K_0(H)$ is a constant element. It suffices to prove that $\chi(m)$ do not depend on m.

To do this, consider for $r\geq 0$ the representation $X_{i,1}^{(r)} = L(\widetilde{\mathbf{\Psi}}_{i,1}\mathbf{\Psi}_{i,q_i^{-2r}})$. Then we obtain from [H5, Theorem 8.1] that

$$\chi_q(X_{i,1}^{(r)}) = \widetilde{\mathbf{\Psi}}_{i,1}\mathbf{\Psi}_{i,q_i^{-2r}}\prod_{j\neq i}\chi_j^{-C_{i,j}}\sum_{0\leq m\leq r}(A_{i,1}A_{i,q_i^{-2}}\cdots A_{i,q_i^{-2m}})^{-1}.$$

Hence the class of $X_{i,1}\otimes L^+_{i,q_i^{-2r}}$ can be decomposed as a sum

$$[X_{i,1}\otimes L^+_{i,q_i^{-2r}}] = [X_{i,1}^{(r)}]\chi'_r + \sum_{\mathbf{\Psi}'} n_{\mathbf{\Psi}'}[L(\mathbf{\Psi}')]$$

where $\chi'_r\in K_0(H)$ is an invertible constant, the $n_{\mathbf{\Psi}'}$ are positive integers and the ℓ-weight $\mathbf{\Psi}'$ which occur in the sum are, up to a constant, of the form

$$\widetilde{\mathbf{\Psi}}_{i,1}\mathbf{\Psi}_{i,q_i^{-2r}}A_{i,1}^{-1}\cdots A_{i,q_i^{-2m}}^{-1}$$

for some $m\geq 0$. We claim that only ℓ-weights $\mathbf{\Psi}'$ with $m\geq r$ can occur, that is the class of $L(\widetilde{\mathbf{\Psi}}_{i,1}\mathbf{\Psi}_{i,q_i^{-2r}}A_{i,1}^{-1}\cdots A_{i,q_i^{-2m}}^{-1})$ cannot occur in the decomposition for $m<r$.

Indeed, for $m<r$, $A_{i,q_i^{-2m}}^{-1}$ is a factor of one of the ℓ-weight of the simple module $L(\widetilde{\mathbf{\Psi}}_{i,1}\mathbf{\Psi}_{i,q_i^{-2r}}A_{i,1}^{-1}\cdots A_{i,q_i^{-2m}}^{-1})$. This follows from an $\mathfrak{sl}_2$-reduction, as an elementary analysis shows that $A_{q^{-2m}}^{-1}$ occurs in an ℓ-weight of the representation in the $\mathfrak{sl}_2$-case

$$L(\mathbf{\Psi}_{q^2}\mathbf{\Psi}_{q^{-2m}}^{-1}\mathbf{\Psi}_{q^{-2m-2}}^{-1}\mathbf{\Psi}_{q^{-2r}}) \simeq L(\mathbf{\Psi}_{q^2}\mathbf{\Psi}_{q^{-2m}}^{-1})\otimes L(\mathbf{\Psi}_{q^{-2m-2}}^{-1}\mathbf{\Psi}_{q^{-2r}}),$$

which is the tensor product of $L(\mathbf{\Psi}_{q^2}\mathbf{\Psi}_{q^{-2m}}^{-1})$ evaluation representation of a Verma module and $L(\mathbf{\Psi}_{q^{-2m-2}}^{-1}\mathbf{\Psi}_{q^{-2r}})$ finite-dimensional representation. So, if this representation appeared in the decomposition, then all its ℓ-weights would be occur in the q-character of $X_{i,1}\otimes L^+_{i,q_i^{-2r}}$. In particular, $A_{i,q_i^{-2m}}^{-2}$ would be the factor of one of the ℓ-weights of $X_{i,1}\otimes L^+_{i,q_i^{-2r}}$, which is a contradiction.

Now, as $\chi_q(X_{i,1}\otimes L^+_{i,q_i^{-2r}}) = \chi_q(X_{i,1})[\mathbf{\Psi}^+_{i,q_i^{-2r}}]\chi_i$, by identifying for $m\leq r$ the coefficients of $(A_{i,1}A_{i,q_i^{-2}}\cdots A_{i,q_i^{-2m}})^{-1}$, we obtain :

$$\chi'_r\prod_{j\neq i}\chi_j^{-C_{i,j}} = \chi(m)\chi_i.$$

This implies that all the $\chi(m)$ are equal and we obtain the q-character formula (5.18). $\square$

6. Interpolating (q,t)-characters

Our approach to the folded quantum integrable systems associated to quantum affine algebras, as formulated in Conjectures 5.15 and 5.16, involves the interpolating (q,t)-characters introduced in [FH2] as a tool for the study of a certain Langlands duality between finite-dimensional representations of quantum affine algebras. In this section we extend and refine the definition of the interpolating (q,t)-characters from [FH2]. They are defined as elements of the refined Grothendieck ring $\overline{\mathcal{K}}_{q,t}(\mathfrak{g})$, which is defined in this section (Theorem 6.6). It turns out that they have 5 interesting specializations corresponding to various q- and t-characters. Moreover, the interpolating (q,t)-characters may be viewed as commutative analogues of elements of the deformed $\mathcal{W}$-algebra $\mathbf{W}_{q,t}(\mathfrak{g})$.

6.1. **Notation for monomials.** For $j \in I'$, a node of the Dynkin diagram of $\mathfrak{g}'$, let us set

$$\widetilde{Y}_{j,a} = \begin{cases} Y_{j,a} & \text{if } \sigma(j)=j, \\ Y_{j,a}Y_{\sigma(j),a} & \text{if } \sigma^2(j)=j \text{ and } \sigma(j)\neq j, \\ Y_{j,a}Y_{\sigma(j),a}Y_{\sigma^2(j),a} & \text{if } \sigma^3(j)=j \text{ and } \sigma(j)\neq j. \end{cases}$$

For $i \in I$, a node of the Dynkin diagram of $\mathfrak{g}$, let us set

$$\overline{Y}_{i,a} = Y_{i,a}^{1+d-d_i}, \tag{6.1}$$

$$W_{i,a} = \begin{cases} Y_{i,a} & \text{if } d_i = d, \\ Y_{i,aq^{-1}}Y_{i,aq} & \text{if } d_i = d-1, \\ Y_{i,aq^{-2}}Y_{i,a}Y_{i,aq^2} & \text{if } d_i = d-2, \end{cases} \tag{6.2}$$

$$Z_{i,a^{1+d-d_i}} = \begin{cases} Y_{i,a} & \text{if } d_i = d, \\ Y_{i,a}Y_{i,\epsilon^2 a} & \text{if } d_i = d-1, \\ Y_{i,a}Y_{i,\epsilon^2 a}Y_{i,\epsilon^4 a} & \text{if } d_i = d-2. \end{cases} \tag{6.3}$$

Note that d_i can equal $d-1$ (resp. $d-2$) only if $d=2$ (resp. $d=3$), and recall that $\epsilon = e^{i\pi/d}$.

Note that $L(W_{i,a})$ is a KR module over $U_q(\widehat{\mathfrak{g}})$. It is a fundamental representation if $d_i = d$ (in particular, for simply-laced types, these representations are always fundamental).

6.2. **Polynomial rings and specialization homomorphisms.** Now we recall the definition of the ring of interpolating (q,t)-characters from [FH2] and then define a refined version of this ring. We start with some preliminary definitions.

Consider the ring

$$\mathcal{Y}_{q,t} = \mathbb{Z}[W_{i,a}^{\pm 1}, \alpha Y_{i,a}^{\pm 1}, \alpha]_{i\in I, a\in q^{\mathbb{Z}}t^{\mathbb{Z}}} \subset \mathbb{Z}[\alpha, Y_{i,a}^{\pm 1}]_{i\in I, a\in q^{\mathbb{Z}}t^{\mathbb{Z}}}.$$

where α is an indeterminate.

Remark 6.1. For simply-laced types, $\mathcal{Y}_{q,t}$ is just $\mathbb{Z}[Y_{i,a}^{\pm 1}, \alpha]_{i\in I, a\in q^{\mathbb{Z}}t^{\mathbb{Z}}}$. □

The ring of interpolating (q,t)-characters $\mathcal{K}_{q,t}(\mathfrak{g})$ was defined in [FH2] as a subring of a quotient $\widetilde{\mathcal{Y}}_{q,t}$ of $\mathcal{Y}_{q,t}$. This quotient is defined from the specialization maps

$$\Pi_q = \Pi_{t=1,\alpha=1} : \mathcal{Y}_{q,t} \to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in q^{\mathbb{Z}}},$$

$$\Pi_t = \Pi_{q=\epsilon,\alpha=0} : \mathcal{Y}_{q,t} \to \mathbb{Z}[Z_{i,a}^{\pm 1}]_{i\in I, a\in t^{\mathbb{Z}}\epsilon^{\mathbb{Z}}},$$

by the formula

$$\widetilde{\mathcal{Y}}_{q,t} = \mathcal{Y}_{q,t}/(\mathrm{Ker}(\Pi_q) \cap \mathrm{Ker}(\Pi_t)).$$

For our purposes, we also want to use the additional specialization homomorphisms

$$\overline{\Pi}_t = \Pi_{q=1,\alpha=d} : \mathcal{Y}_{q,t} \to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in t^{\mathbb{Z}}},$$

$$\overline{\Pi}_q = \Pi_{t=1,\alpha=0} : \mathcal{Y}_{q,t} \to \mathbb{Z}[W_{i,a}^{\pm 1}]_{i\in I, a\in q^{\mathbb{Z}}}.$$

These specialization homomorphisms are well-defined on $\mathcal{Y}_{q,t}$, but $\overline{\Pi}_t$ does not descend to $\widetilde{\mathcal{Y}}_{q,t}$ if $d > 1$ (indeed, $\alpha^2 - \alpha$ projects onto 0 in $\widetilde{\mathcal{Y}}_{q,t}$, but $\overline{\Pi}_t(\alpha^2 - \alpha) = d^2 - d$). For this reason, we will work with an intermediate quotient

$$\overline{\mathcal{Y}}_{q,t} = \mathcal{Y}_{q,t}/(\mathrm{Ker}(\Pi_q) \cap \mathrm{Ker}(\Pi_t) \cap \mathrm{Ker}(\overline{\Pi}_q) \cap \mathrm{Ker}(\overline{\Pi}_t))$$

for which we have surjective ring homomorphisms

$$\mathcal{Y}_{q,t} \twoheadrightarrow \overline{\mathcal{Y}}_{q,t} \twoheadrightarrow \widetilde{\mathcal{Y}}_{q,t}.$$

Remark 6.2. (1) The interpolating (q,t)-characters are defined below as elements of a subalgebra of the commutative algebra $\overline{\mathcal{Y}}_{q,t}$. Their purpose is to imitate the properties of elements of the non-commutative deformed $\mathcal{W}$-algebra $\mathbf{W}_{q,t}(\mathfrak{g})$. In particular, the variable α is introduced in order to imitate the behavior of the rational functions like (4.5) arising in the formulas for elements of the deformed $\mathcal{W}$-algebra $\mathbf{W}_{q,t}(\mathfrak{g})$ such as (4.4). This is why in the above specializations we set α equal to $1, 0$, or d depending on the situation.

(2) There is a polynomial $P(\alpha)$ in α so that $\alpha^d P(\alpha)$ is equal to α in the quotient $\overline{\mathcal{Y}}_{q,t}$ (for example, using a Lagrange interpolating polynomial). Hence, for an arbitrary monomial m in the variables $Y_{i,a}^{\pm 1}$, as $\alpha^N m$ is in $\mathcal{Y}_{q,t}$ for N large enough, αm makes sense in $\overline{\mathcal{Y}}_{q,t}$. □

Let us recall some terminology from [FH2]. By a monomial in $\widetilde{\mathcal{Y}}_{q,t}$ we will understand an element m of the form $P(\alpha)M$, where $P(\alpha)$ is a polynomial in α and M is a monomial in the $Y_{j,a}^{\pm 1}$. Note that a monomial in $\widetilde{\mathcal{Y}}_{q,t}$ may be written in various ways as for example $\alpha Y_{i,a} = \alpha Y_{i,at}$ and $(1-\alpha)Y_{i,aq^{2d}} = (1-\alpha)Y_{i,a}$. A monomial is said to be *i-dominant* if it can be written by using only the variables α, $Y_{i,a}$ and $Y_{j,a}^{\pm 1}$, where $j \neq i$. Let B_i be the set of i-dominant monomials and for $J \subset I$, let $B_J = \cap_{j\in J} B_j$. Finally, let $B = B_I$ be the set of *dominant monomials*.

We will use the analogous definition of i-dominant (resp. dominant) monomials in $\overline{\mathcal{Y}}_{q,t}$.

Lemma 6.1. *A monomial in $\overline{\mathcal{Y}}_{q,t}$ is i-dominant if and only if its specializations under Π_q, $\overline{\Pi}_t$ and Π_t are all i-dominant.*

Proof. The direct implication is clear. For the converse, let us first consider a monomial m in $\overline{\mathcal{Y}}_{q,t}$ which is a product of various $Y_{i,a}^{\pm 1}$, $a \in q^{\mathbb{Z}} t^{\mathbb{Z}}$ (for this question, we may discard the other variables $Y_{j,a}^{\pm 1}$ with $j \neq i$). Suppose m specializes to i-dominant monomials

$$m_1 = \prod_{s\in\mathbb{Z}} Y_{i,q^s}^{u_{i,s}(m_1)}, \qquad m_2 = \prod_{s\in\mathbb{Z},\epsilon'\in\epsilon^{\mathbb{Z}}} Y_{i,\epsilon' t^s}^{u_{i,\epsilon',s}(m_2)}, \qquad m_3 = \prod_{s\in\mathbb{Z}} Y_{i,t^s}^{u_{i,s}(m_3)}$$

for the respective specializations $t = 1$, $q = \epsilon$, $q = 1$. Then the sum

$$u_i = \sum_{s\in\mathbb{Z}} u_{i,s}(m_1) = \sum_{s\in\mathbb{Z},\epsilon'\in\epsilon^{\mathbb{Z}}} u_{i,\epsilon',s}(m_2) = \sum_{s\in\mathbb{Z}} u_{i,s}(m_3) \geq 0$$

of the various powers of the various variables is the same for m_1, m_2 and m_3. We also have for any $\epsilon' = \epsilon^R$, $R \in \mathbb{Z}$, the relation

$$\sum_{s\in\mathbb{Z}} u_{i,\epsilon',s}(m_2) = \sum_{s\in R+2d\mathbb{Z}} u_{i,s}(m_1).$$

Hence we can construct an i-dominant monomial M in $\overline{\mathcal{Y}}_{q,t}$ so that its specializations at $t=1$ and $q=\epsilon$ are m_1 and m_2 respectively. Such a monomial is not unique, but, for example, the powers in the variables in M can be defined inductively by a standard combinatorial algorithm, starting from the r_0, s_0 where r_0 (resp. s_0) is minimal so that $u_{i,r_0}(m_1) \neq 0$ (resp. $u_{i,s_0}(m_3) \neq 0$). Then M specializes necessarily to m_3 at $q=1$ as for any $s \in \mathbb{Z}$

$$u_{i,s}(m_3) = \sum_{\epsilon'\in\epsilon^{\mathbb{Z}}} u_{i,s,\epsilon'}(m_2).$$

In particular m gets identified with M in $\widetilde{\mathcal{Y}}_{q,t}$ and so m is i-dominant. The case when the monomial has a factor depending of α is treated in a similar way. □

Remark 6.3. The analogous statement is not true in $\mathcal{Y}_{q,t}$ for non-simply laced types (with $r>1$). For example, $Y_{i,1}Y_{i,t}^{-1}Y_{i,tq^d}$ is not i-dominant, but specializes at $t=1$, $q=\epsilon$, $q=1$ respectively to Y_{i,q^d}, $Y_{i,1}$, $Y_{i,1}$ which are all i-dominant. But in $\overline{\mathcal{Y}}_{q,t}$ this monomial gets identified with Y_{i,q^d}, and so it is i-dominant, in accordance with Lemma 6.1. □

6.3. **Definition of interpolating (q,t)-characters.** Recall the definition of the ring $\mathcal{K}_{q,t}$ of *interpolating (q,t)-characters* from [FH2]. It is defined as the intersection of subrings $\mathcal{K}_{i,q,t} \subset \widetilde{\mathcal{Y}}_{q,t}$, $i \in I$, by analogy with the characterization of $\operatorname{Im}\chi_q$ as the intersection of the subrings $\mathfrak{K}_{i,q}$ (see formulas (4.1) and (4.2)) as well as the definition of $\mathbf{W}_{q,t}(\mathfrak{g})$ as the intersection of the kernels of the screening operators in [FR2] and Section 3.2 above.

We will need the following analogues of the generating series $A_i(z)$ in the deformed $\mathcal{W}$-algebra $\mathbf{W}_{q,t}(\mathfrak{g})$ given by formula (3.6):

(6.4)

$$\widetilde{A}_{i,a} = Y_{i,a(q_it)^{-1}}Y_{i,aq_it} \times \prod_{j\in I, C_{j,i}=-1} Y_{j,a}^{-1} \times \prod_{j\in I, C_{j,i}=-2} Y_{j,aq^{-1}}^{-1}Y_{j,aq}^{-1} \times \prod_{j\in I, C_{j,i}=-3} Y_{j,aq^{-2}}^{-1}Y_{j,a}^{-1}Y_{j,aq^2}^{-1}.$$

Definition 6.2. Let $\mathcal{K}_{i,q,t}$ be the subring of $\widetilde{\mathcal{Y}}_{q,t}$ generated by the variables α, $W_{j,a}^{\pm 1}$, $\alpha Y_{j,a}^{\pm 1}$ ($j \neq i$, $a \in q^{\mathbb{Z}}t^{\mathbb{Z}}$), the $\alpha Y_{i,a}(1+\widetilde{A}_{i,aq_it}^{-1})$ ($a \in q^{\mathbb{Z}}t^{\mathbb{Z}}$) and

$$W_{i,a} \times \begin{cases} (1+\widetilde{A}_{i,aq_it}^{-1}) & \text{if } d_i = d, \\ (1+\alpha\widetilde{A}_{i,aq^2t}^{-1} + \widetilde{A}_{i,aq^2t}^{-1}\widetilde{A}_{i,at}^{-1}) & \text{if } d_i = d-1, \\ (1+\alpha\widetilde{A}_{i,aq^3t}^{-1} + \alpha\widetilde{A}_{i,aq^3t}^{-1}\widetilde{A}_{i,aqt}^{-1} + \widetilde{A}_{i,aq^3t}^{-1}\widetilde{A}_{i,aqt}^{-1}\widetilde{A}_{i,aq^{-1}t}^{-1}) & \text{if } d_i = d-2, \end{cases} \tag{6.5}$$

where $a \in q^{\mathbb{Z}}t^{\mathbb{Z}}$.

Following [FH2], we define the *ring $\mathcal{K}_{q,t}(\mathfrak{g})$ of interpolating (q,t)-characters associated to* $\mathfrak{g}$ as the following ring intersection:

$$\mathcal{K}_{q,t}(\mathfrak{g}) := \bigcap_{i\in I} \mathcal{K}_{i,q,t} \subset \widetilde{\mathcal{Y}}_{q,t}.$$

Remark 6.4. If $\mathfrak{g}$ is simply-laced,

$$\widetilde{A}_{i,a} = Y_{i,a(qt)^{-1}}Y_{i,aqt} \times \prod_{j\in I, I_{ji}=-1} Y_{j,a}^{-1}$$

coincides with $A_{i,qt}$. Hence $\mathcal{K}_{i,q,t}$ is the image in $\widetilde{\mathcal{Y}}_{q,t}$ of:

$$\mathbb{Z}[\alpha, Y_{j,a}^{\pm 1}, Y_{i,a}(1+A_{i,aqt}^{-1})]_{a\in q^{\mathbb{Z}}t^{\mathbb{Z}}} \subset \mathcal{Y}_{q,t}.$$

□

The following result was proved in [FH2].

Theorem 6.3.

(i) $\mathcal{K}_{q,t}(\mathfrak{g})$ *is non-zero.*

(ii) *Every element* $F \in \mathcal{K}_{q,t}(\mathfrak{g})$ *(resp.* $\in \mathcal{K}_{i,q,t}$, $i \in I$*) is uniquely determined by the multiplicities of the dominant monomials (resp.* i*-dominant monomials) occurring in* F*.*

(iii) *For each dominant monomial* m*, there is a unique* $F_{q,t}(m) \in \mathcal{K}_{q,t}(\mathfrak{g})$ *such that* m *is the unique dominant monomial occurring in* $F_{q,t}(m)$ *(moreover, there is an algorithm to construct* $F_{q,t}(m)$ *explicitly).*

In particular, the interpolating (q,t)-character $F_{q,t}(Y_{i,a})$ corresponding to the ith fundamental representation is well-defined, as are the interpolating (q,t)-characters

$$F_{q,t}(Y_{i,a}Y_{i,aq_i^2}\cdots Y_{i,q_i^{2(k-1)}})$$

corresponding to the KR modules.

Remark 6.5. (1) In general, the ring $\mathcal{K}_{q,t}(\mathfrak{g})$ cannot be lifted to a subring of $\mathcal{Y}_{q,t}$ satisfying the properties listed in Theorem 6.3. For example, suppose that $d_i = d-1$ and consider the following element:

$$\begin{aligned}\alpha^2 Y_{i,q^{-1}}(1+\widetilde{A}_{i,t}^{-1})Y_{i,q}(1+\widetilde{A}_{i,q^2t}^{-1}) - \alpha^2 W_{i,1}(1+\alpha\widetilde{A}_{i,q^2t}^{-1}+\widetilde{A}_{i,q^2t}^{-1}\widetilde{A}_{i,t}^{-1}) \\ = (\alpha^2-\alpha^3)W_{i,1}\widetilde{A}_{i,q^2t}^{-1} + \alpha^2 W_{i,1}\widetilde{A}_{i,t}^{-1}\end{aligned} \tag{6.6}$$

Viewed as an element of $\widetilde{\mathcal{Y}}_{q,t}$, it can be identified with $\alpha W_{i,1}\widetilde{A}_{i,1}^{-1}$ which belongs to $\mathcal{K}_{i,q,t}$ and is an i-dominant monomial.

However, viewed as an element of $\mathcal{Y}_{q,t}$, it does not contain any i-dominant monomials. This contradicts property (ii) of Theorem 6.3.

(2) If $\mathfrak{g}$ is simply-laced, there is no such obstruction and we can consider interpolating (q,t)-characters as elements of $\mathcal{Y}_{q,t}$. These are just the ordinary q-characters, but with the quantum parameter qt instead of q. More precisely, for $r \in \mathbb{Z}$, let $\mathcal{K}_{q,t}^{(r)}(\mathfrak{g})$ be the subring of elements of $\mathcal{K}_{q,t}(\mathfrak{g})$ involving only variables $Y_{i,q^d(tq)^s}^{\pm 1}$, $i \in I$, $s \in \mathbb{Z}$. Then $\mathcal{K}_{q,t}^{(r)}(\mathfrak{g})$ is isomorphic to $\mathcal{K}_{q,t}^{(0)}(\mathfrak{g})$ as it is obtained from $\mathcal{K}_{q,t}^{(0)}(\mathfrak{g})$ by the automorphism shifting spectral parameters by q^d. Moreover,

$$\mathcal{K}_{q,t}(\mathfrak{g}) \simeq \mathbb{Z}[\alpha] \otimes \bigotimes_{r\in\mathbb{Z}} \mathcal{K}_{q,t}^{(r)}(\mathfrak{g}).$$

By Remark 6.4, $\mathcal{K}_{q,t}^{(0)}(\mathfrak{g})$ is the image in $\widetilde{\mathcal{Y}}_{q,t}$ of

$$\mathrm{Im}(\chi_{qt}) \subset \mathcal{Y}_{q,t}$$

where

$$\chi_{qt} : K_0(\mathcal{C}_{\mathbb{Z}}) \to \mathbb{Z}[Y_{i,(qt)^r}^{\pm 1}]_{i\in I, r\in\mathbb{Z}}$$

is the ordinary q-character homomorphism, but with quantum parameter q replaced by qt and $\mathcal{C}_{\mathbb{Z}}$ is the subcategory of finite-dimensional representations of $U_{qt}(\widehat{\mathfrak{g}})$ whose simple constituents have Drinfeld polynomials have roots that are powers of qt. $\square$

Theorem 6.4. *$\mathcal{K}_{q,t}(\mathfrak{g})$ can be lifted to a subring $\overline{\mathcal{K}}_{q,t}(\mathfrak{g}) \subset \overline{\mathcal{Y}}_{q,t}$ so that we have a commutative diagram:*

$$\begin{tikzcd}
 & \mathcal{Y}_{q,t} \arrow[d, two heads] \arrow[dr, two heads, "\overline{\Pi}_t,\overline{\Pi}_q"] & \\
\overline{\mathcal{K}}_{q,t}(\mathfrak{g}) \arrow[r, hook] \arrow[d, two heads] & \overline{\mathcal{Y}}_{q,t} \arrow[r, two heads, "\overline{\Pi}_t,\overline{\Pi}_q"'] \arrow[d, two heads] & \mathbb{Z}[Y_{j,b}^{\pm 1}]_{j\in I, b\in t^{\mathbb{Z}}q^{\mathbb{Z}}} \\
\mathcal{K}_{q,t}(\mathfrak{g}) \arrow[r, hook] & \widetilde{\mathcal{Y}}_{q,t} &
\end{tikzcd}$$

This theorem follows from Theorem 6.6 below.

6.4. **The refined ring of interpolating (q,t)-characters.** Let us consider the completion $\overline{\mathcal{Y}}_{q,t}^{\infty}$ of the ring $\overline{\mathcal{Y}}_{q,t}$ which includes infinite linear combinations of monomials whose $\mathfrak{g}$-weights belong to a finite union of cones $\{\mu \preceq \omega\}$, with $\preceq$ being the standard ordering (as for the q-characters in the category $\mathcal{O}$). Next, we define the subrings $\overline{\mathcal{K}}_{i,q,t} \subset \overline{\mathcal{Y}}_{q,t}^{\infty}$ in the same way as the subrings $\mathcal{K}_{i,q,t}$ of $\widetilde{\mathcal{Y}}_{q,t}$, except that we include infinite which make sense in $\overline{\mathcal{Y}}_{q,t}^{\infty}$. Note that the elements generating $\mathcal{K}_{i,q,t}$ are well-defined in $\overline{\mathcal{Y}}_{q,t}$ and in $\overline{\mathcal{Y}}_{q,t}^{\infty}$ by (2) in Remark 6.2.

Definition 6.5. We define the *refined ring of interpolating (q,t)-characters* $\overline{\mathcal{K}}_{q,t}(\mathfrak{g})$ as the intersection

$$\bigcap_{i\in I} \overline{\mathcal{K}}_{i,q,t} \subset \overline{\mathcal{Y}}_{q,t}^{\infty}.$$

Theorem 6.6. *The refined ring of interpolating (q,t)-characters $\overline{\mathcal{K}}_{q,t}(\mathfrak{g})$ satisfies the same properties as the properties of $\mathcal{K}_{q,t}(\mathfrak{g})$ listed in Theorem 6.3. In particular, it contains a unique element*

$$\overline{F}_{q,t}(m) \in \overline{\mathcal{K}}_{q,t}(\mathfrak{g})$$

for each dominant monomial m.

Proof. All proofs in [FH2] remain valid if we replace $\widetilde{\mathcal{Y}}_{q,t}$ by $\overline{\mathcal{Y}}_{q,t}^{\infty}$. More precisely,

(i) follows from (iii).

(ii) The property for the i-dominant monomials is proved as for [FH2, Lemma 4.1]. In the simply-laced cases ($d=1$), the proof is the same as for ordinary q-character as discussed above (see Remark 6.5). If $d=2$, the crucial point is indeed that the formula (6.6) has an i-dominant monomial: its respective specializations under $\overline{\Pi}_t$, Π_q, Π_t are 0, $Y_{i,q^{-1}}Y_{i,q}A_{i,1}^{-1}$, 0, which are i-dominant. Hence expression (6.6) identifies with $\alpha(2-\alpha)Y_{i,q^{-1}}Y_{i,q}A_{i,1}^{-1}$ in $\overline{\mathcal{Y}}_{q,t}^{\infty}$ which is i-dominant. An analogous reasoning gives the result for $d=3$.

The property for dominant monomials is proved exactly as in [FH2, Lemma 4.2] from the property we just obtained for the i-dominant monomials.

(iii) This is proved as in [FH2, Section 4.2]: an algorithm is proposed which produces the $F(W_{i,a})$ for any i,a, and from which the $F(m)$ are obtained as algebraic combinations of these $F(W_{i,a})$. For the $F(W_{i,a})$, the algorithm and the proof that it does not fail are the same. This proof is obtained by induction on the rank of the Lie algebra. That is why we

have to check for $n=1$ (type A_1) and $n=2$ (types $A_1\times A_1$, A_2, C_2, G_2). The simply-laced cases are clear by Remark 6.4. For type C_2, the formulas obtained in [FH2] for $\mathcal{K}_{q,t}(\mathfrak{g})$ work as well for $\overline{\mathcal{K}}_{q,t}(\mathfrak{g})$:

$$\overline{F}_{q,t}(Y_{2,1}) = Y_{2,1} + Y_{2,q^4t^2}^{-1}Y_{1,qt}Y_{1,q^3t} + \alpha Y_{1,qt}Y_{1,q^5t^3}^{-1} + Y_{1,q^3t^3}^{-1}Y_{1,q^5t^3}^{-1}Y_{2,q^2t^2} + Y_{2,q^6t^4}^{-1}, \tag{6.7}$$

$$\begin{aligned}\overline{F}_{q,t}(W_{1,1}) &= Y_{1,q^{-1}}Y_{1,q} + \alpha Y_{1,q^{-1}}Y_{1,q^3t^2}^{-1}Y_{2,q^2t} + Y_{1,qt^2}^{-1}Y_{1,q^3t^2}^{-1}Y_{2,t}Y_{2,q^2t}\\ &+\alpha Y_{1,q^{-1}}Y_{1,q^5t^2}Y_{2,q^6t^3}^{-1} + Y_{2,q^2t}Y_{2,q^4t^3}^{-1} + Y_{1,qt^2}^{-1}Y_{1,q^5t^2}Y_{2,q^6t^3}^{-1}Y_{2,t} + \alpha Y_{1,q^{-1}}Y_{1,q^7t^4}^{-1}\\ &+Y_{2,q^4t^3}^{-1}Y_{2,q^6t^3}^{-1}Y_{1,q^3t^2}Y_{1,q^5t^2} + \alpha Y_{1,qt^2}^{-1}Y_{1,q^7t^4}^{-1}Y_{2,t} + \alpha Y_{2,q^4t^3}^{-1}Y_{1,q^3t^2}Y_{1,q^7t^4}^{-1} + Y_{1,q^5t^4}^{-1}Y_{1,q^7t^4}^{-1},\end{aligned} \tag{6.8}$$

$$\overline{F}_{q,t}(\alpha Y_{1,1}) = \alpha(Y_{1,1} + Y_{1,q^2t^2}^{-1}Y_{2,qt} + Y_{2,q^5t^3}^{-1}Y_{1,q^4t^2} + Y_{1,q^6t^4}^{-1}). \tag{6.9}$$

For the type G_2, the formulas obtained in [FH2, Section 5.2] work as well. □

Now let us consider the various specializations of $\mathcal{K}_{q,t}(\mathfrak{g})$. We have first the three specializations corresponding to the respective specializations of the deformed $\mathcal{W}$-algebras at $t=1$, $q=\epsilon$ and $q=1$.

- The specialization $\Pi_q = \Pi_{t=1,\alpha=1}$: we get the q-character of a $U_q(\widehat{\mathfrak{g}})$-module, that is an element of $\mathcal{K}_q^+(\mathfrak{g})$ (this is proved in [FH2]).
- The specialization $\Pi_t = \Pi_{q=\epsilon,\alpha=0}$: we get the t-character of a $U_t({}^L\widehat{\mathfrak{g}})$-module (this is proved in [FH2]).
- The specialization $\overline{\Pi}_t = \Pi_{q=1,\alpha=d}$: we get the t-character of an element of $\mathcal{K}_t^-(\mathfrak{g})$ (in light of Proposition 3.2, the proof is parallel to the proof for the other specializations since $\overline{\Pi}_t(\overline{\mathcal{K}}_{i,q,t}) = \mathrm{Ker}(S_i^-)$). Hence, it can also be viewed as the folded t-character of a $U_t(\widehat{\mathfrak{g}'})$-module (see Section 4.3).

There are also two additional interesting specializations:

- The specialization $\Pi'_t = \Pi_{q=1,\alpha=0}$: it is well-defined as the composition of Π_t and the homomorphism
$$\mathbb{Z}[W_{i,a}^{\pm 1}]_{i\in I, a\in t^{\mathbb{Z}}\epsilon^{\mathbb{Z}}} \to \mathbb{Z}[\overline{Y}_{i,a}^{\pm 1}]_{i\in I, a\in t^{\mathbb{Z}}}$$
(see formula (6.1) for the definition of $\overline{Y}_{i,a}$) identifying ϵ with 1. We get the t-character of an element of $\mathcal{K}_t^-({}^L\mathfrak{g})$ which is defined in terms of the variables $\overline{Y}_{i,a}$ instead of $Y_{i,a}$ (this is parallel to the proof for the other specializations as $\Pi'_t(\overline{\mathcal{K}}_{i,q,t}) = \mathrm{Ker}(S_i^-)$). Hence, it can also be viewed as the folded t-character of a $U_t\left(\widehat{({}^L\mathfrak{g})'}\right)$-module (see Section 4.3).
- The specialization $\overline{\Pi}_q = \Pi_{t=1,\alpha=0}$: we obtain elements of $\mathrm{Im}(\chi_q^L)$, i.e. q-characters of modules over the Langlands dual quantum affine algebra $U_q({}^L\widehat{\mathfrak{g}})$, as defined in [H5, Section 12] (in the context of the parametrization of simple representations of shifted quantum affine algebras).

So, in the total, we have 5 interesting specializations of the refined ring of interpolating (q,t)-characters:

$$\begin{tikzcd}
 & & \operatorname{Rep} U_t({}^L\widehat{\mathfrak{g}}) \arrow[r, "\cong"'] & \operatorname{Rep} U_t(\widehat{({}^L\mathfrak{g})'}) & \\
\operatorname{Rep} U_q(\widehat{\mathfrak{g}}) \arrow[r, "\cong"'] & \mathcal{K}_q^+(\mathfrak{g}) & \overline{\mathcal{K}}_{q,t}(\mathfrak{g}) \arrow[l, "\Pi_q"'] \arrow[u, "\Pi_t"] \arrow[r, "\overline{\Pi}_t"] \arrow[dl, "\overline{\Pi}_q"] \arrow[dr, "\Pi'_t"'] & \mathcal{K}_t^-(\mathfrak{g}) & \operatorname{Rep} U_t(\widehat{\mathfrak{g}'}) \arrow[l, two heads] \\
\operatorname{Im}(\chi_q^L) & & & \mathcal{K}_t^-({}^L\mathfrak{g}) & \operatorname{Rep} U_t(\widehat{({}^L\mathfrak{g})'}) \arrow[l, two heads]
\end{tikzcd}$$

Remark 6.6. To get the t-characters of $U_t({}^L\widehat{\mathfrak{g}})$-modules in the sense of [H3] (as recalled above), one has to change the sign in the definition of the variables $Z_{i,a}$, as described in [FH2] (formula (2) for $r = 2$ and Section 3.2 for $r = 3$). In this paper, to simplify our notation, we do not make this sign change. □

Remark 6.7. In [FH2, Lemma 4.16] we proved that the generators of $\mathcal{K}_{q,t}(\mathfrak{g})$, i.e. the fundamental elements $F_{q,t}(W_{i,a})$, are finite linear combinations of monomials. In the case of the refined ring $\overline{\mathcal{K}}_{q,t}(\mathfrak{g})$ the algorithm in the proof of Theorem 6.6 produces the fundamental elements $\overline{F}_{q,t}(W_{i,a})$ but so far we have only been able to prove that they are (possibly infinite) linear combinations of monomials with weights in the union of finitely many cones. That's why we have allowed such linear combinations in the definition of the $\overline{\mathcal{K}}_{i,q,t}$ above. However, we also know that the specializations under Π_q, Π_t, $\overline{\Pi}_t$ and Π'_t of an interpolating (q,t)-character with a finite number of dominant monomials are finite linear combinations. This follows from Proposition 6.10 below. Therefore finiteness of elements of $\overline{\mathcal{K}}_{q,t}(\mathfrak{g})$ follows from Conjecture 6.11 saying that the coefficients of all monomials in $\overline{F}_{q,t}(W_{i,a})$ are positive (see Corollary 6.12). □

Example 6.7. *Let us consider the interpolating (q,t)-character (6.7) corresponding to the second fundamental representation of $U_q(C_2^{(1)})$. In this case, we have ${}^L\widehat{\mathfrak{g}} = D_3^{(2)}, \widehat{\mathfrak{g}'} = A_3^{(1)}, \widehat{({}^L\mathfrak{g})'} = D_3^{(1)} \simeq A_3^{(1)}$ (the last isomorphism involves switching the indices $1 \leftrightarrow 2 \in \{1,2,3\} = I'$).*

The above 5 specializations (listed in the same clockwise order) are, respectively,

- *the q-character of the fundamental representation $L(Y_{2,1})$ of $U_q(C_2^{(1)})$:*
$$Y_{2,1} + Y_{2,q^4}^{-1} Y_{1,q} Y_{1,q^3} + Y_{1,q} Y_{1,q^5}^{-1} + Y_{1,q^3}^{-1} Y_{1,q^5}^{-1} Y_{2,q^2} + Y_{2,q^6}^{-1},$$
- *the twisted t-character of the fundamental representation $L(Z_{2,1})$ of $U_t(D_3^{(2)})$:*
$$Z_{2,1} + Z_{2,t^2}^{-1} Z_{1,-t^2} + Z_{1,-t^6}^{-1} Z_{2,-t^2} + Z_{2,-t^4}^{-1},$$
- *the t-character in $\mathcal{K}_t^-(C_2)$ of highest monomial $Y_{2,1}$:*
$$Y_{2,1} + Y_{2,t^2}^{-1} Y_{1,t}^2 + 2 Y_{1,t} Y_{1,t^3}^{-1} + Y_{1,t^3}^{-2} Y_{2,t^2} + Y_{2,t^4}^{-1},$$
also equal to the folded t-character of the fundamental representation $L(Y_{2,1})$ of $U_t(A_3^{(1)})$,
- *the t-character in $\mathcal{K}_t^-({}^LC_2)$ of highest monomial $\overline{Y}_{2,1}$:*
$$\overline{Y}_{2,1} + \overline{Y}_{2,t^2}^{-1} \overline{Y}_{1,t} + \overline{Y}_{1,t^3}^{-1} \overline{Y}_{2,t^2} + \overline{Y}_{2,t^4}^{-1},$$

which after switching the indices 1 *and* 2 *is equal to the folded* t*-character of the fundamental representation* $L(Y_{1,1})$ *of* $U_t(A_3^{(1)})$*, defined in terms of the variables* $\overline{Y}_{i,a}$ *instead of* $Y_{i,a}$*.*

- *the Langlands dual* q*-characters of the fundamental representation* $L(Z_{2,1})$ *of* $U_q(D_3^{(2)})$

$$W_{2,1} + W_{2,q^4}^{-1} W_{1,q^2} + W_{1,q^4}^{-1} W_{2,q^2} + W_{2,q^6}^{-1}.$$

Example 6.8. *Let us consider the interpolating* (q,t)*-character (6.8) corresponding to the representation* $L(W_{1,1})$ *of* $U_q(C_2^{(1)})$*. The* 5 *specializations are, respectively,*

- *the* q*-character of the* 11*-dimensional representation* $L(Y_{1,q^{-1}}Y_{1,q})$ *of* $U_q(C_2^{(1)})$*.*
- *the twisted* t*-character of the fundamental representation* $L(Z_{1,-1})$ *of* $U_t({}^L\widehat{\mathfrak{g}}) = U_t(D_3^{(2)})$*.*

$$Z_{1,-1} + Z_{1,-t^4}^{-1} Z_{2,t} Z_{2,-t} + Z_{2,-t} Z_{2,t^3}^{-1} + Z_{2,-t^3}^{-1} Z_{2,t} + Z_{2,t^3}^{-1} Z_{2-t^3}^{-1} Z_{1,-t^4} + Z_{1,-t^8}^{-1}.$$

- *the* t*-character in* $\mathcal{K}_t^-(C_2)$ *of highest monomial* $Y_{1,1}^2$*:*

$$(Y_{1,1} + Y_{1,t^2}^{-1} Y_{2,t} + Y_{2,t^3}^{-1} Y_{1,t^2} + Y_{1,t^4}^{-1})^2,$$

also equal to the folded t*-character of the representation* $L(Y_{1,1}Y_{3,1})$ *of* $U_t(A_3^{(1)})$

- *the* t*-character in* $\mathcal{K}_t^-({}^LC_2)$ *of highest monomial* $\overline{Y}_{1,1}$*:*

$$\overline{Y}_{1,1} + \overline{Y}_{1,t^2}^{-1} \overline{Y}_{2,t}^2 + 2\overline{Y}_{2,t} \overline{Y}_{2,t^3}^{-1} + \overline{Y}_{2,t^3}^{-2} \overline{Y}_{1,t^2} + \overline{Y}_{1,t^4}^{-1}$$

which after switching the indices 1 *and* 2 *is equal to the folded* t*-character of the fundamental representation* $L(Y_{2,1})$ *of* $U_t(A_3^{(1)})$*, defined in terms of the variables* $\overline{Y}_{i,a}$ *instead of* $Y_{i,a}$*.*

- *the Langlands dual* q*-characters of the fundamental representation* $L(Z_{2,1})$ *of* $U_q(D_3^{(2)})$

$$W_{1,1} + W_{1,q^2}^{-1} W_{2,1} W_{2,q^2} + W_{2,q^2} W_{2,q^4}^{-1} + W_{1,q^2}^{-1} W_{1,q^4} W_{2,q^6}^{-1} Y_{2,t} + W_{2,q^4}^{-1} W_{2,q^6}^{-1} W_{1,q^4} + W_{1,q^6}^{-1}.$$

Example 6.9. *Let us consider the interpolating* (q,t)*-character (6.9) corresponding to the first fundamental representation of* $U_q(C_2^{(1)})$*. The specializations under* Π_t*,* Π_t'*,* $\overline{\Pi}_q$ *are zero because of the* α *factor. The specializations under* Π_q *and* $\overline{\Pi}_t$ *are respectively:*

- *the* q*-character of the fundamental representation* $L(Y_{1,1})$ *of* $U_q(C_2^{(1)})$*:*

$$Y_{1,1} + Y_{1,q^2}^{-1} Y_{2,q} + Y_{2,q^5}^{-1} Y_{1,q^4} + Y_{1,q^6}^{-1},$$

- *the* t*-character in* $\mathcal{K}_t^-(C_2)$ *of highest monomial* $2Y_{1,1}$*:*

$$2(Y_{1,1} + Y_{1,t^2}^{-1} Y_{2,t} + Y_{2,t^3}^{-1} Y_{1,t^2} + Y_{1,t^4}^{-1})$$

also equal to (the double of) the folded t*-character of the representation* $L(Y_{1,1})$ *of* $U_t(A_3^{(1)})$*.*

6.5. **σ-fundamental interpolating (q,t)-characters.** In this subsection we identify the representation $M(W)$ of $U_q({}^L\widehat{\mathfrak{g}})$ in terms of Conjecture 5.16 for the simplest $U_q(\widehat{\mathfrak{g}'})$-modules with σ-invariant highest monomials. These monomials have the form $Y_{i,a}$ if $\sigma(i) = i$ and $\prod_{1\le k\le d} Y_{\sigma^k(i),a}$ if $\sigma(i) \neq i$. For this reason, we call these $U_q(\widehat{\mathfrak{g}'})$-modules *$\sigma$-fundamental*. Note however that the corresponding $U_t({}^L\widehat{\mathfrak{g}})$-modules (see the second part of Proposition 6.10) are in fact fundamental.

Consider a $U_q(\widehat{\mathfrak{g}'})$-module of the form $L(\widetilde{Y}_{i,a}), i \in I$ (the set of vertices of the Dynkin diagram of $\mathfrak{g} = (\mathfrak{g}')^\sigma$) and with $a \in q^{\mathbb{Z}}$.

Let

$$X_{q,t}^{(i)} := \overline{F}_{q,t}(W_{i,a}) \in \overline{\mathcal{K}}_{q,t}(\mathfrak{g}).$$

We call it the ith *σ-fundamental interpolating (q,t)-character*. The following proposition shows that $X_{q,t}^{(i)}$ satisfies the properties of the element $X_{q,t}$ whose existence is stated in Conjecture 5.16 in the case when $W = L(\widetilde{Y}_{i,a})$.

Proposition 6.10. *The specializations of $X_{q,t}^{(i)}$ under Π_q, Π_t, $\overline{\Pi}_t$, Π'_t, $\overline{\Pi}_q$ are respectively*

- *the q-character of the simple $U_q(\widehat{\mathfrak{g}})$-module of highest monomial $W_{i,a}$. It is a KR module (a fundamental representation if $d_i = d$).*
- *the t-character of the fundamental $U_t({}^L\widehat{\mathfrak{g}})$-module of highest monomial $Z_{i,(a_{q=\epsilon})^{d_i^\vee}}$ (see (4.3) for the definition of $d_i^\vee$), with $a_{q=\epsilon}$ the specialization of a at $q=\epsilon$.*
- *the t-character of the element $F(\overline{Y}_{i,a}) = F(Y_{i,a})^{d_i^\vee}$ in $\mathcal{K}_t^-(\mathfrak{g})$ (defined in terms of the variables $Y_{j,b}$) of highest monomial $\overline{Y}_{i,a}$. It is also the folded t-character of the simple $U_t(\widehat{\mathfrak{g}'})$-module of highest monomial $\widetilde{Y}_{i,a}$ (this is a tensor product of the fundamental representations).*
- *the t-character of the element $F(\overline{Y}_{i,a})$ in $\mathcal{K}_t^-({}^L\mathfrak{g})$ (which is defined in terms of the variables $\overline{Y}_{j,b}$ instead of $Y_{j,b}$) of highest monomial $\overline{Y}_{i,a}$. It is also, after the appropriate permutation of indices, equal to the folded t-character of the fundamental $U_t\left(\widehat{({}^L\mathfrak{g})'}\right)$-module of highest monomial $Y_{i,a}$.*
- *the Langlands dual q-character of the fundamental $U_q({}^L\widehat{\mathfrak{g}})$-module of highest monomial $Z_{i,a^{d_i^\vee}}$.*

Proof. By construction, following the algorithm in [FH2, Section 4.2.4], the interpolating (q,t)-character $F_{q,t}(W_{i,a})$ is the sum of $W_{i,a}$ plus other monomials of the form

$$W_{i,a}\widetilde{A}_{i,aq^dt}^{-1}\widetilde{A}_{j_1,b_1}^{-1}\cdots\widetilde{A}_{j_N,b_N}^{-1}$$

where $j_k \in I$, $b_j \in aq^{\mathbb{Z}}t^{\mathbb{Z}}$ and the $\widetilde{A}_{j,b}$ are given by formula (6.4). Though the argument in [FH2] concerns $F_{q,t}(W_{i,a})$, the algorithm constructing $X_{q,t}^{(i)} = \overline{F}_{q,t}(W_{i,a})$ in $\overline{\mathcal{Y}}_{q,t}$ is the same, hence we obtain that $X_{q,t}^{(i)}$ satisfies the same property. This implies that each of the 5 specializations considered in the statement of the proposition has a unique dominant monomial; namely, the corresponding specialization of the highest monomial $W_{i,a}$. Then, as explained in [FH2], the first two specializations are respectively the q-character of a KR module of $U_q(\widehat{\mathfrak{g}})$ and the t-character of a fundamental module of $U_t({}^L\widehat{\mathfrak{g}})$, which are known to have a unique dominant monomial by [H2] and [H3]. The third specialization is an element of $\mathcal{K}_t^-(\mathfrak{g})$ with a unique dominant monomial $\overline{Y}_{i,a}$, hence it is equal to $F(\overline{Y}_{i,a})$. Moreover, the simple $U_t(\widehat{\mathfrak{g}'})$-module of highest monomial $\widetilde{Y}_{i,a}$ is a simple tensor product of fundamental representations, and, as a consequence of [FM], its (folded) t-character belongs to $\overline{Y}_{i,a} + \overline{Y}_{i,a}A_{i,at}^{-1}\mathbb{Z}[A_{j,b}^{-1}]_{j\in I, b\in at^{\mathbb{Z}}}$, with the $A_{j,b}$ as in formula (3.10). Hence it has a unique dominant monomials and is equal to $F(\overline{Y}_{i,a})$. We use an analogous argument for the fourth specialization. For the last specialization, this is the Langlands dual q-character which is precisely defined as the specialization of the interpolating (q,t)-character $F_{q,t}(W_{i,a})$. $\square$

The following conjecture is true in all examples known to us.

Conjecture 6.11. *$X_{q,t}^{(i)}$ can be written in such a way that all coefficients of its monomials are positive.*

Corollary 6.12. *If Conjecture 6.11 holds, then $X_{q,t}^{(i)}$ is a polynomial and therefore every element of $\overline{\mathcal{K}}_{q,t}(\mathfrak{g})$ is a polynomial (a finite linear combination of monomials).*

Proof. Positivity of coefficients of $X_{q,t}^{(i)}$ implies that if $X_{q,t}^{(i)}$ were an infinite combination of monomials, then so would be $\overline{\Pi}_t(X_{q,t}^{(i)})$, which is not the case. Since the elements $X_{q,t}^{(i)}$ generate $\overline{\mathcal{K}}_{q,t}(\mathfrak{g})$, it follows that every element of $\overline{\mathcal{K}}_{q,t}(\mathfrak{g})$ is finite as well. □

Remark 6.8. The statement analogous to Conjecture 6.11 for the elements $F_{q,t}(W_{i,a})$ of the original ring $\mathcal{K}_{q,t}(\mathfrak{g})$ is also a conjecture. At present, we are not aware of a uniform proof of this statement or of Conjecture 6.11. We expect, however, that it is possible to write an explicit positive and finite expression for these elements for Lie algebras $\mathfrak{g}$ of classical types, and to check the statement with the help of a computer for $\mathfrak{g}$ of exceptional types. There is a similar question for the corresponding elements of the deformed $\mathcal{W}$-algebra. We hope to discuss this in another paper.

Assuming Conjecture 6.11, we obtain a proof of the second part of Conjecture 5.15,(ii) (modulo Conjecture 5.15,(i)) and Conjecture 5.16 for $W = L(\widetilde{Y}_{i,a})$.

Theorem 6.13. *Let $W = L(\widetilde{Y}_{i,a})$, $i \in I$, $a \in \mathbb{C}^\times$. Suppose that Conjecture 6.11 holds. Then there exists a subspace $\overline{W} \subset \widetilde{W}$ isomorphic, as a vector space graded by ${}^L\mathfrak{g}$-weights, to a $U_q({}^L\widehat{\mathfrak{g}})$-module $M(W)$, which is the simple $U_t({}^L\widehat{\mathfrak{g}})$-module of highest monomial $Z_{i,(a_{q=\epsilon})^{d_i^\vee}}$. Moreover, it satisfies Conjecture 5.16 for the interpolating (q,t)-character $X_{q,t}^{(i)}$.*

Proof. We have seen in Proposition 6.10 that $\overline{\Pi}_t(X_{q,t}^{(i)})$ is the folded t-character of W and that $\Pi_t(X_{q,t}^{(i)}) = \chi_t(M(W))$ where $M(W)$ is the fundamental $U_t({}^L\widehat{\mathfrak{g}})$-module of highest monomial $Z_{i,(a_{q=\epsilon})^{d_i^\vee}}$. The positivity of $X_{q,t}^{(i)}$ implies that the multiplicities of weights in $\Pi_t(X_{q,t}^{(i)})$ (for which $\alpha = 0$) are lower than in $\overline{\Pi}_t(X_{q,t}^{(i)})$ (for which $\alpha = d$). Hence the result. □

7. Examples

In this section we present five explicit examples confirming Conjectures 5.15 and 5.16.

7.1. **First example: the fundamental representation $L(Y_{2,1})$ of $U_q(A_3^{(1)})$.** Consider the Lie algebra $\mathfrak{g}'$ of type A_3 with the automorphism σ exchanging the nodes 1 and 3 of its Dynkin diagram. We have the Lie algebra $\mathfrak{g} = C_2$ with $d_1 = 1$, $d_2 = 2$ and its Langlands dual is ${}^L\mathfrak{g} = B_2$. We also have $\left({}^L\mathfrak{g}\right)' = A_3$, but with the nodes 1 and 3 exchanged in comparison to the original $\mathfrak{g}'$, and ${}^L\widehat{\mathfrak{g}} = D_3^{(2)}$.

Then the algebra $U_q(\widehat{\mathfrak{g}'}) = U_q(A_3^{(1)})$ acts on its 6-dimensional fundamental representation $W = L(Y_{2,1})$ whose q-character is

$$Y_{2,1} + Y_{2,q^2}^{-1}Y_{1,q}Y_{3,q} + Y_{1,q^3}^{-1}Y_{3,q} + Y_{3,q^3}^{-1}Y_{1,q} + Y_{2,q^2}Y_{1,q^3}^{-1}Y_{3,q^3}^{-1} + Y_{2,q^4}^{-1}.$$

We have the Q-operators $Q_i^{\pm}(z,u) = t_{R_i^{\pm}}(z,u)$ associated to the prefundamental representations $R_i^+(z), i = 1,2,3$, of $U_q(\widehat{\mathfrak{g}'})$. The roots of the corresponding Baxter polynomials satisfy the BAE (4.7) of type $A_3^{(1)}$. Now we consider the subspace $W(u)$ of W where the actions of $Q_1^{\pm,\mathrm{ss}}(z,u)$ and $Q_3^{\pm,\mathrm{ss}}(z,u)$ coincide. We also have the 4-dimensional subspace $\widehat{W} \subset W$ which is the direct sum of the σ-invariant weight subspaces W and the subspace $\widetilde{W}$ which is the direct sum of the ℓ-weight subspaces corresponding to the σ-invariant ℓ-weights (or monomials in the q-character).

In the present case all σ-invariant weights have multiplicity 1. Therefore, by Remark 5.5,(2), for all values of u we have

$$W(u) = \widetilde{W} = \widehat{W} = W^{\sigma}(u).$$

Thus, part (i) of Conjecture 5.15 is verified in this case.

This implies that only σ-invariant solutions of the BAE (4.7) appear in this case. Identifying the eigenvalues of the operators $Q_1^{\pm,\mathrm{ss}}(z) = Q_3^{\pm,\mathrm{ss}}(z)$, we obtain the folded BAE (4.8) corresponding to the Lie algebra $\mathfrak{g} = C_2$.

The q-character of the subspace $\widetilde{W}$ is given by the formula

$$Y_{2,1} + Y_{2,q^2}^{-1}(Y_{1,q}Y_{3,q}) + Y_{2,q^2}(Y_{1,q^3}Y_{3,q^3})^{-1} + Y_{2,q^4}^{-1}.$$

Setting $\widetilde{Y}_{1,q} = Y_{1,q}Y_{3,q}$, one gets

$$Y_{2,1} + Y_{2,q^2}^{-1}\widetilde{Y}_{1,q} + Y_{2,q^2}\widetilde{Y}_{1,q^3}^{-1} + Y_{2,q^4}^{-1}. \tag{7.1}$$

The corresponding character

$$y_2 + y_2^{-1}y_1 + y_1^{-1}y_2 + y_2^{-1}$$

is equal to the character of a fundamental representation of $U_t({}^L\widehat{\mathfrak{g}}) = U_t(D_3^{(2)})$. Thus, part (ii) of Conjecture 5.15 is verified.

Finally, we discuss Conjecture 5.16 in the present case. We have the interpolating (q,t)-character (6.7) of the second fundamental representation of $U_q(C_2^{(1)})$ studied in Example 6.7. We verify Conjecture 5.16:

The specialization of the interpolating (q,t)-character (6.7) under Π_t is equal to the t-character of the fundamental representation $M(W)$ of $U_t({}^L\widehat{\mathfrak{g}}) = U_t(D_3^{(2)})$.

Its specialization under $\overline{\Pi}_t$ is the folded t-character of the $U_t(\widehat{\mathfrak{g}'}) = U_t(A_3^{(1)})$-module W.

Its specialization under Π_t' is (after switching the indices 1 and 2) the folded t-character of W', the fundamental representation $L(Y_{1,1})$ of $U_t(A_3^{(1)})$, defined in terms of the variables $\overline{Y}_{i,a}$ instead of $Y_{i,a}$.

Thus, we find that both Conjectures 5.15 and 5.16 hold in this case. In other words, we obtain an example of the folded quantum integrable system associated to $\mathfrak{g} = C_2$ whose spectra correspond to the solutions of the folded BAE equation (4.8) associated to $\mathfrak{g} = C_2$, with the space of states being isomorphic to a representation of $U_q({}^L\widehat{\mathfrak{g}} = D_3^{(2)})$.

7.2. **Second example: representation $L(Y_{1,1}Y_{2n-1,1})$ of $U_q(A_{2n-1}^{(1)})$.** We work with the Lie algebras $\mathfrak{g}' = A_{2n-1}$, $\mathfrak{g} = C_n$, ${}^L\mathfrak{g} = B_n$, $\left({}^L\mathfrak{g}\right)' = D_{n+1}$ and ${}^L\widehat{\mathfrak{g}} = D_{n+1}^{(2)}$ (we assume that $n \geq 2$).

Consider the representation $W = L(Y_{1,1}Y_{2n-1,1})$ of $U_q(A^{(1)}_{2n-1})$. It is $4n^2$-dimensional and is isomorphic to the tensor product of the fundamental representations $L(Y_{1,1}) \otimes L(Y_{2n-1,1})$. Its q-character has $2n$ σ-invariant monomials, including one with multiplicity 2:

$$\chi_q(\widetilde{W}) = Y_{1,1}Y_{2n-1,1} + Y^{-1}_{1,q^2}Y^{-1}_{2n-1,q^2}Y_{2,q}Y_{2n-2,q} + \cdots + Y^{-1}_{n-1,q^n}Y^{-1}_{n,q^n}Y^2_{n,q^{n-1}}$$

$$+2Y_{n,q^{n-1}}Y^{-1}_{n,q^{n+1}} + Y^{-2}_{n,q^{n+1}}Y_{n-1,q^n}Y_{n+1,q^n} + \cdots + Y^{-1}_{1,q^{2n}}Y^{-1}_{2n-1,q^{2n}}.$$

All weight subspaces of W are one-dimensional, except for the 0-weight subspace W_0 of W which is $2n$-dimensional, containing as a proper subspace its intersection with $\widetilde{W}$, which is the 2-dimensional ℓ-weight space associated to the σ-invariant monomial $Y_{n,q^{n-1}}Y^{-1}_{n,q^{n+1}}$ (the other $n-1$ monomials of weight 0 are not σ-invariant). This ℓ-subspace is therefore precisely the intersection $\widetilde{W} \cap W_0$, and Lemma 5.2,(2) implies that $\widehat{\sigma}$ preserves this ℓ-weight subspace. We are going to show that $\widehat{\sigma}$ acts on it as the identity.

Let v be a generating vector of the weight subspace W_{α_n}, which is one-dimensional, and hence is an ℓ-weight subspace. The corresponding monomial is $Y^2_{n,q^{n-1}}Y^{-1}_{n-1,q^n}Y^{-1}_{n-1,q^n}$. Under the action of the $U_q(\widehat{\mathfrak{sl}}_2)$ subalgebra corresponding to the node n, the vector v generates a representation of dimension 4 with a 2-dimensional 0-weight space, which is spanned by the $x^-_{n,m}.v$, $m \in \mathbb{Z}$. Since W_{α_n} is preserved by $\widehat{\sigma}$, it follows that $\widehat{\sigma}(v) = \pm v$. In fact, it is easy to see that $\widehat{\sigma}(v) = v$ by restricting W to $U_q(\mathfrak{g})$ and taking the limit $q \to 1$. Then W decomposes into the direct sum of the adjoint representation of A_{2n-1} and the trivial one-dimensional representation, and the weight subspace W_{α_n} appears in the former. It is easy to see that the operator $\widehat{\sigma}$ acts on the adjoint representation as the automorphism induced by the automorphism σ of the Dynkin diagram of A_{2n-1} (whose invariant Lie subalgebra is C_n). From this we readily obtain that $\widehat{\sigma}$ acts as the identity on W_{α_n}.

Next, we have $\sigma(x^-_{n,m}) = x^-_{n,m}$ for any $m \in \mathbb{Z}$, so we obtain that $\widehat{\sigma}(x^-_{n,m}.v) = x^-_{n,m}.v$. This implies that the vectors in $\widetilde{W} \cap W_0$ are indeed $\widehat{\sigma}$-invariant. This implies that $\widetilde{W}^\sigma = \widetilde{W}$.

Now we derive from this that for generic u the intersection $W(u) \cap W_0$ is two-dimensional.

We have the $2n$-dimensional weight subspace W_0 of W. By analyzing the q-character of W, we have found that it decomposes into a direct sum of $n-1$ two-dimensional subspaces, each containing two ℓ-weight subspaces corresponding to a pair of monomials $M_1 \neq M_2$ such that $\sigma(M_1) = M_2$ (and therefore $\widehat{\sigma}$ interchanges them, according to Lemma 5.2,(2)), and the two-dimensional ℓ-weight subspace corresponding to the σ-invariant monomial $Y_{n,q^{n-1}}Y^{-1}_{n,q^{n+1}}$ on which $\widehat{\sigma}$ acts as the identity, as we have shown above. This implies that the trace of $\widehat{\sigma}$ on W_0 is equal to 2.

On the other hand, W_0 also has a basis of joint eigenvectors of $Q^{\pm,\mathrm{ss}}_i(u), i \in I'$, with eigenvalues $\lambda(u) = (\lambda^\pm_i(u))$. By applying the argument of Lemma 5.2,(2), we find that $\widehat{\sigma}$ maps such an eigenvector to another one with eigenvalues $\sigma(\lambda(u))$. In the limit $u \to 0$ the eigenspaces of $Q^{\pm,\mathrm{ss}}_i(u), i \in I'$ become ℓ-weight spaces. Hence we find from the preceding paragraph that for generic u we have at least $2(n-1)$ eigenvalues $\lambda(u)$ which are not σ-invariant, and the set of these eigenvalues breaks into $n-1$ pairs, with the eigenvalues in each pair (and the corresponding one-dimensional eigenspaces) exchanged by σ. If the remaining 2 eigenvalues were not σ-invariant, then the corresponding 2 one-dimensional eigenspaces would have to be exchanged by σ. But then the trace of $\widehat{\sigma}$ on W_0 would be equal to 0, which is a contradiction. Therefore for generic u there must be one σ-invariant eigenvalue $\lambda(u)$ with multiplicity 2, such that the corresponding two-dimensional eigenspace is $\widehat{\sigma}$-invariant (this is the necessary condition for the trace of $\widehat{\sigma}$ on this subspace, and hence

on W_0, to be equal to 2). But then this two-dimensional subspace is contained in $W(u)$, and moreover in $W^\sigma(u)$. This implies that $W(u) \cap W_0 = W^\sigma(u) \cap W_0$ is two-dimensional.

Since the other weight spaces are 1-dimensional, it follows from Remark 5.5,(2) that for generic u we have

$$W(u) \simeq \widetilde{W}$$

as vector spaces graded by ${}^L\mathfrak{g}$-weights, in agreement with Conjecture 5.15,(1). Moreover, we obtain that $W(u) = W^\sigma(u)$.

Now, setting $\widetilde{Y}_{i,a} = Y_{i,a}Y_{2n-i,a}$ for $1 \le i \le n-1$, we obtain that

$$\chi_q(\widetilde{W}) = \widetilde{Y}_{1,1} + \widetilde{Y}_{1,q^2}^{-1}\widetilde{Y}_{2,q} + \cdots + \widetilde{Y}_{n-1,q^n}^{-1}Y_{n,q^{n-1}}^2 + 2Y_{n,q^{n-1}}Y_{n,q^{n+1}}^{-1} + \widetilde{Y}_{n-1,q^n}Y_{n,q^{n+1}}^{-2} + \cdots + \widetilde{Y}_{1,q^{2n}}^{-1}.$$

The corresponding character is

$$y_1 + y_1^{-1}y_2 + \cdots + y_{n-1}^{-1}y_n^2 + 2 + y_n^{-2}y_{n-1} + \cdots + y_1^{-1},$$

which coincides with the character of the $(2n+2)$-dimensional fundamental representation $M(W)$ of $U_t({}^L\widehat{\mathfrak{g}}) = U_t(D_{n+1}^{(2)})$. Moreover, as we will see below, the specialization Π_t of corresponding interpolating (q,t)-character gives rise to the t-character of this representation of $U_t(D_{n+1}^{(2)})$.

The above character can also be interpreted as the character of the direct sum of the $(2n+1)$-dimensional fundamental representation and the trivial representation of $U_t(\widehat{{}^L\mathfrak{g}}) = U_t(B_n^{(1)})$. But we cannot obtain its t-character from the interpolating (q,t)-character below because this t-character contains the monomial 1 (compare with the discussion in Remark 5.5,(3)).

The interpolating (q,t)-character of the simple representation $L(Y_{1,q^{-1}}Y_{1,q})$ of $U_q(C_n^{(1)})$ is given by formula (6.8) in Example 6.8 for $n=2$. For general n, the statements about its specializations follow from Proposition 6.10:

Its specialization under $\overline{\Pi}_t$ it is equal to ${}^{\mathrm{f}}\chi_t(W)$.

Its specialization under Π_t is the t-character of the fundamental representation $M(W)$ of $U_t({}^L\widehat{\mathfrak{g}}) = U_t(D_{n+1}^{(2)})$.

Its specialization under Π'_t is (after switching the indices i and $n+1-i$) is the folded t-character of W', the fundamental representation $L(Y_{n,1})$ of $U_t(A_{2n-1}^{(1)})$, defined in terms of the variables $\overline{Y}_{i,a}$ instead of $Y_{i,a}$.

All of this is in agreement with Conjectures 5.15 and 5.16.

7.3. **Third example: the fundamental representation $L(Y_{3,1})$ of $U_q(A_5^{(1)})$.** Consider the Lie algebra $\mathfrak{g}'$ of type A_5 with σ being the unique automorphism of the Dynkin diagram of order 2. Then $\mathfrak{g} = C_3$ with $d_1 = d_2 = 1$, $d_3 = 2$ and its Langlands dual Lie algebra is ${}^L\mathfrak{g} = B_3$. We also have $\left({}^L\mathfrak{g}\right)' = D_4$ and ${}^L\widehat{\mathfrak{g}} = D_5^{(2)}$.

The fundamental representation $L(Y_{3,1})$ of $U_q(\widehat{\mathfrak{g}}') = U_q(A_5^{(1)})$ is 20-dimensional. It has 8 invariant monomials:

$$\chi_q(\widetilde{W}) = Y_{3,1} + Y_{3,q^2}^{-1}Y_{4,q}Y_{2,q} + Y_{1,q^2}Y_{5,q^2}Y_{2,q^3}^{-1}Y_{4,q^3}^{-1}Y_{3,q^2} + Y_{1,q^2}Y_{5,q^2}Y_{3,q^4}^{-1} + Y_{3,q^2}Y_{1,q^4}^{-1}Y_{5,q^4}^{-1}$$
$$+ Y_{1,q^4}^{-1}Y_{5,q^4}^{-1}Y_{2,q^3}Y_{4,q^3}Y_{3,q^4}^{-1} + Y_{3,q^4}Y_{2,q^5}^{-1}Y_{4,q^5}^{-1} + Y_{3,q^6}^{-1}.$$

In this case all ordinary weights have multiplicity 1. Therefore Remark 5.5,(2) implies that for any u we have

$$W(u) = \widetilde{W} = \widehat{W} = W^\sigma(u).$$

Setting $\widetilde{Y}_{1,a} = Y_{1,a}Y_{5,a}$ and $\widetilde{Y}_{2,a} = Y_{2,a}Y_{4,a}$, we obtain that $\chi_q(\widetilde{W})$ equals

$$Y_{3,1} + Y_{3,q^2}^{-1}\widetilde{Y}_{2,q} + \widetilde{Y}_{1,q^2}\widetilde{Y}_{2,q^3}^{-1}Y_{3,q^2} + \widetilde{Y}_{1,q^2}Y_{3,q^4}^{-1} + Y_{3,q^2}\widetilde{Y}_{1,q^4}^{-1} + \widetilde{Y}_{1,q^4}^{-1}\widetilde{Y}_{2,q^3}Y_{3,q^4}^{-1} + Y_{3,q^4}\widetilde{Y}_{2,q^5}^{-1} + Y_{3,q^6}^{-1}.$$

The corresponding character is

$$y_3 + y_2y_3^{-1} + y_3y_2^{-1}y_1 + y_3^{-1}y_1 + y_3y_1^{-1} + y_3^{-1}y_2y_1^{-1} + y_2^{-1}y_3 + y_3^{-1},$$

which is the character of the third fundamental representation $M(W)$ of $U_t({}^L\widehat{\mathfrak{g}}) = U_t(D_5^{(2)})$.

Actually, it is also the character of the fundamental representation $L(Y_{3,1})$ of $U_t(\widehat{{}^L\mathfrak{g}}) = U_t(B_3^{(1)})$. But we cannot obtain the t-character of this representation from the interpolating (q,t)-character, whereas we can do it for the third fundamental representation of $U_t(D_5^{(2)})$ (see the discussion in Remark 5.5,(3)).

In fact, the interpolating (q,t)-character of the simple representation $L(Y_{3,1})$ of $U_q(\widehat{\mathfrak{g}}) = U_q(C_3^{(1)})$ was computed in [FH2, Section 4.5] (note however that there was a typo there for the monomials Y_{2,q^5t^3} and Y_{2,q^7t^5}, which we correct here):

$$\begin{aligned}
&Y_{3,1} + Y_{3,q^4t^2}^{-1}Y_{2,qt}Y_{2,q^3t} + \alpha Y_{1,q^4t^2}Y_{2,qt}Y_{2,q^5t^3}^{-1} + Y_{3,q^2t^2}Y_{2,q^3t^3}^{-1}Y_{2,q^5t^3}^{-1}Y_{1,q^4t^2}Y_{1,q^2t^2} + \alpha Y_{1,q^6t^4}^{-1}Y_{2,qt}\\
&+Y_{3,q^6t^4}^{-1}Y_{1,q^4t^2}Y_{1,q^2t^2}+\alpha Y_{3,q^2t^2}Y_{2,q^3t^3}^{-1}Y_{1,q^6t^4}^{-1}Y_{1,q^2t^2}+\alpha Y_{3,q^6t^4}^{-1}Y_{1,q^6t^4}^{-1}Y_{1,q^2t^2}Y_{2,q^5t^3}+Y_{3,q^2t^2}Y_{1,q^6t^4}^{-1}Y_{1,q^4t^4}^{-1}\\
&+\alpha Y_{1,q^2t^2}Y_{2,q^7t^5}^{-1}+Y_{3,q^6t^4}^{-1}Y_{2,q^3t^3}Y_{2,q^5t^3}Y_{1,q^6t^4}^{-1}Y_{1,q^4t^4}^{-1}+\alpha Y_{2,q^3t^3}Y_{2,q^7t^5}^{-1}Y_{1,q^4t^4}^{-1}+Y_{2,q^5t^5}^{-1}Y_{2,q^7t^5}^{-1}Y_{3,q^4t^4}+Y_{3,q^8t^6}^{-1}.
\end{aligned}$$

There are 14-monomials, 8 with multiplicity 1 and 6 with multiplicity α.

Its specialization under $\overline{\Pi}_t$, has 20 terms, and it is equal to $\chi_t(W)$ after identification of the variables $Y_{1,a} \sim Y_{5,a}$, $Y_{2,a} \sim Y_{4,a}$.

Its specialization under Π_t, has 8 terms:

$$\begin{aligned}
&Z_{3,1} + Z_{3,t^2}^{-1}Z_{2,-t^2} + Z_{3,-t^2}Z_{2,-t^6}^{-1}Z_{1,t^4} + Z_{3,-t^4}^{-1}Z_{1,t^4}\\
&+Z_{3,-t^2}Z_{1,t^8}^{-1} + Z_{3,-t^4}^{-1}Z_{2,-t^6}Z_{1,t^8}^{-1} + Z_{2,-t^{10}}^{-1}Y_{3,t^4} + Z_{3,t^6}^{-1},
\end{aligned}$$

and is the t-character of the fundamental representation $M(W)$ of $U_t({}^L\widehat{\mathfrak{g}}) = U_t(D_5^{(2)})$.

Its specialization under Π'_t is the folded t-character

$$\overline{Y}_{3,1}+\overline{Y}_{3,t^2}^{-1}\overline{Y}_{2,t}+\overline{Y}_{3,t^2}\overline{Y}_{2,t^3}^{-1}\overline{Y}_{1,t^2}+\overline{Y}_{3,t^4}^{-1}\overline{Y}_{1,t^2}+\overline{Y}_{3,t^2}\overline{Y}_{1,t^4}^{-1}+\overline{Y}_{3,t^4}^{-1}\overline{Y}_{2,t^3}\overline{Y}_{1,t^4}^{-1}+\overline{Y}_{2,t^5}^{-1}\overline{Y}_{3,t^4}+\overline{Y}_{3,t^6}^{-1}$$

of the fundamental representation $L(Y_{3,1})$ of $U_t\left(\widehat{({}^L\mathfrak{g})'}\right) = U_t(D_4^{(1)})$, defined in terms of the variables $\overline{Y}_{i,a}$ instead of $Y_{i,a}$.

All of this is in agreement with Conjectures 5.15 and 5.16.

7.4. **Fourth example: the trivalent fundamental representation $L(Y_{1,1})$ of $U_q(D_4^{(1)})$.** Now we consider the Lie algebra $\mathfrak{g}'$ of type D_4 with an automorphism σ of order 3 and denote by $i = 1$ the trivalent node. Then $\mathfrak{g} = G_2$ with $d_1 = 3$, $d_2 = 1$ and its Langlands dual Lie algebra is ${}^L\mathfrak{g} = G_2$ with $d_1^\vee = 1$ and $d_2^\vee = 3$. We also have $\left({}^L\mathfrak{g}\right)' = D_4$ (but with the trivalent node now being $i = 2$) and ${}^L\widehat{\mathfrak{g}} = D_4^{(3)}$.

Let us consider the example of the fundamental representation $W = L(Y_{1,1})$ of $U_q(\widehat{\mathfrak{g}'}) = U_q(D_4^{(1)})$. Its q-character is computed in [N1, Example 5.3.2]. It has 27 monomials, all of multiplicity 1 except one $(Y_{1,q^2}Y_{1,q^4}^{-1})$ of multiplicity 2. It has 8 invariant monomials for an automorphism σ of order 3, including one of multiplicity 2:

$$\chi_q(\widetilde{W}) = Y_{1,1} + Y_{1,q^2}^{-1}Y_{2,q}Y_{3,q}Y_{4,q} + Y_{1,q^2}^2Y_{2,q^3}^{-1}Y_{3,q^3}^{-1}Y_{4,q^3}^{-1} + 2Y_{1,q^2}Y_{1,q^4}^{-1} + Y_{1,q^4}^{-2}Y_{2,q^3}Y_{3,q^3}Y_{4,q^3}$$

$$+Y_{1,q^4}Y_{2,q^5}^{-1}Y_{3,q^5}^{-1}Y_{4,q^5}^{-1}+Y_{1,q^6}^{-1}.$$

All weight spaces of W are one-dimensional, except for the 0-weight space W_0 of W which is 5-dimensional, containing as a proper subspace its intersection with $\widetilde{W}$, which is the 2-dimensional ℓ-weight space associated to $Y_{1,q^2}Y_{1,q^4}^{-1}$ (the other 3 monomials of weight 0 are $Y_{2,q}Y_{2,q^5}^{-1}$, $Y_{3,q}Y_{3,q^5}^{-1}$, $Y_{4,q}Y_{4,q^5}^{-1}$, which are not σ-invariant).

Let v be a generating vector of the weight space W_{α_1}, which is one-dimensional and whose corresponding monomial is $Y_{1,q^2}^2Y_{2,q}^{-1}Y_{3,q}^{-1}Y_{4,q}^{-1}$. Under the action of the $U_q(\widehat{\mathfrak{sl}}_2)$-subalgebra corresponding to the node 1, the vector v generates a representation of dimension 4 with a 2-dimensional 0-weight space. We obtain as in Section 7.2 above that for generic u we have $\dim W(u)\cap W_0=\dim\widetilde{W}\cap W_0=2$.

Since the other weight spaces are 1-dimensional, it follows from Remark 5.5,(2) that for generic u we have

$$W(u)\simeq\widetilde{W}$$

as vector spaces graded by ${}^L\mathfrak{g}$-weights. Moreover, it follows that $W^\sigma(u)=W(u)$ and $\widetilde{W}^\sigma=\widetilde{W}$.

Setting $\widetilde{Y}_{2,a}=Y_{2,a}Y_{3,a}Y_{4,a}$, we obtain that

$$\chi_q(\widetilde{W})=Y_{1,1}+Y_{1,q^2}^{-1}\widetilde{Y}_{2,q}+Y_{1,q^2}^2\widetilde{Y}_{2,q^3}^{-1}+2Y_{1,q^2}Y_{1,q^4}^{-1}+Y_{1,q^4}^{-2}\widetilde{Y}_{2,q^3}+Y_{1,q^4}\widetilde{Y}_{2,q^5}^{-1}+Y_{1,q^6}^{-1}.$$

The corresponding character

$$y_1+y_2y_1^{-1}+y_1^2y_2^{-1}y_1+2+y_2y_1^{-2}+y_2^{-1}y_1+y_1^{-1}$$

is the character of the first fundamental representation $M(W)$ of $U_t({}^L\widehat{\mathfrak{g}})=U_t(D_4^{(3)})$.

Note that it is also the character of the direct sum of the first fundamental representation and the trivial one-dimensional representation of $U_t(\widehat{{}^L\mathfrak{g}})=U_t(G_2^{(1)})$. Therefore we cannot obtain the t-character of this representation of $U_t(G_2^{(1)})$ from the corresponding interpolating (q,t)-character, whereas we can do it for the first fundamental representation of $U_t(D_4^{(3)})$ (compare with the discussion in Remark 5.5,(3)).

In fact, the interpolating (q,t)-character of the simple representation $L(Y_{1,1})$ of $U_q(G_2^{(1)})$ was computed in [FH2, Section 5.2]. It has 15 monomials, 8 with multiplicity 1 and 7 with multiplicity α.

Its specialization under $\overline{\Pi}_t$ has 29 terms and is equal to ${}^{\mathrm{f}}\chi_t(W)$.

Its specialization under Π_t, has 8 terms:

$$Z_{1,1}+Z_{1,t^2}^{-1}Z_{2,-t^3}+Z_{2,-t^9}^{-1}Z_{1,-\epsilon t^2}Z_{1,\epsilon^2t^2}+Z_{1,\epsilon t^2}Z_{1,\epsilon^2t^4}^{-1}+Z_{1,\epsilon^2t^2}Z_{1,-\epsilon t^4}^{-1}$$
$$+Z_{1,\epsilon^2t^4}^{-1}Z_{1,-\epsilon t^4}^{-1}Z_{-2,t^9}+Z_{-2,t^{15}}Z_{1,t^4}+Z_{1,t^6}^{-1}$$

and is the t-character of the fundamental representation $M(W)$ of $U_t({}^L\widehat{\mathfrak{g}})=U_t(D_4^{(3)})$.

Its specialization under Π_t' is the folded t-character

$$\overline{Y}_{1,1}+\overline{Y}_{1,t^2}^{-1}\overline{Y}_{2,t}+\overline{Y}_{2,t^3}^{-1}\overline{Y}_{1,t^2}^2+2\overline{Y}_{1,t^4}^{-1}\overline{Y}_{1,t^2}+\overline{Y}_{2,t^3}\overline{Y}_{1,t^4}^{-2}+\overline{Y}_{2,t^5}^{-1}\overline{Y}_{1,t^4}+\overline{Y}_{1,t^6}^{-1}$$

of the fundamental representation $L(Y_{1,1})$ of $U_t\left(\widehat{({}^L\mathfrak{g})}'\right)=U_t(D_4^{(1)})$, defined in terms of the variables $\overline{Y}_{i,a}$ instead of $Y_{i,a}$.

All of this is in agreement with Conjectures 5.15 and 5.16.

7.5. **Fifth example: a simple tensor product of 4 fundamental representations of $U_q(A_3^{(1)})$.** Here we study an example in which $\overline{W}(u)$ is not equal to the whole space $W(u)$.

We work with the same Lie algebras $\mathfrak{g}' = A_3$, $\mathfrak{g} = C_2$, ${}^L\mathfrak{g} = B_2$, $\left({}^L\mathfrak{g}\right)' = A_3$ and ${}^L\widehat{\mathfrak{g}} = D_3^{(2)}$ as in the first example.

Consider the following simple tensor product of 4 fundamental representations of $U_q(\widehat{\mathfrak{g}'}) = U_q(A_3^{(1)})$,

$$W = L(Y_{1,1}^2 Y_{3,1}^2) \simeq L(Y_{1,1})^{\otimes 2} \otimes L(Y_{3,1})^{\otimes 2}.$$

Its highest monomial $Y_{1,1}^2 Y_{3,1}^2$ is σ-invariant.

It is the tensor square of the 16-dimensional representation $W_1 = L(Y_{1,1}Y_{3,1})$ studied in Section 7.2 (with $n = 2$), whose invariant subspace $\widetilde{W}_1 = W_1(0)$ has dimension 6. Its dimension is 256 and its subspace $\widetilde{W} = W(0)$ contains a subspace of dimension $6^2 = 36$ corresponding to the square of the q-character of the invariant subspace $\widetilde{W}_1$ of $L(Y_{1,1}Y_{3,1})$:

(7.2)

$$\widetilde{Y}_{1,1}^2 + 2\widetilde{Y}_{1,1}\widetilde{Y}_{1,q^2}^{-1}\widetilde{Y}_{2,q}^2 + 4\widetilde{Y}_{1,1}\widetilde{Y}_{2,q}\widetilde{Y}_{2,q^3}^{-1} + \widetilde{Y}_{1,q^2}^{-2}\widetilde{Y}_{2,q}^4 + 4\widetilde{Y}_{1,q^2}^{-1}\widetilde{Y}_{2,q}^3\widetilde{Y}_{2,q^3}^{-1} + 2\widetilde{Y}_{1,1}\widetilde{Y}_{1,q^2}\widetilde{Y}_{2,q^3}^{-2} + 6\widetilde{Y}_{2,q}^2\widetilde{Y}_{2,q^3}^{-2}$$
$$+2\widetilde{Y}_{1,1}\widetilde{Y}_{1,q^4}^{-1} + 4\widetilde{Y}_{2,q}\widetilde{Y}_{2,q^3}^{-3}\widetilde{Y}_{1,q^2} + 2\widetilde{Y}_{1,q^2}^{-1}\widetilde{Y}_{1,q^4}^{-1}\widetilde{Y}_{2,q}^2 + \widetilde{Y}_{2,q^3}^{-4}\widetilde{Y}_{1,q^2}^2 + 4\widetilde{Y}_{2,q}\widetilde{Y}_{2,q^3}^{-1}\widetilde{Y}_{1,q^4}^{-1} + 2\widetilde{Y}_{2,q^3}^{-2}\widetilde{Y}_{1,q^2}\widetilde{Y}_{1,q^4}^{-1} + \widetilde{Y}_{1,q^4}^{-2}.$$

However, the analysis of the q-character

$$\chi_q(W) = (Y_{1,1} + Y_{1,q^2}^{-1}Y_{2,q} + Y_{2,q^3}^{-1}Y_{3,q^2} + Y_{3,q^4}^{-1})^2(Y_{3,1} + Y_{3,q^2}^{-1}Y_{2,q} + Y_{2,q^3}^{-1}Y_{1,q^2} + Y_{1,q^4}^{-1})^2$$

shows that $\widetilde{W}$ is larger: it has dimension 54 and $\chi_q(\widetilde{W})$ equals

$$\widetilde{Y}_{1,1}^2 + 4\widetilde{Y}_{1,1}\widetilde{Y}_{1,q^2}^{-1}\widetilde{Y}_{2,q}^2 + 8\widetilde{Y}_{1,1}\widetilde{Y}_{2,q}\widetilde{Y}_{2,q^3}^{-1} + \widetilde{Y}_{1,q^2}^{-2}\widetilde{Y}_{2,q}^4 + 4\widetilde{Y}_{1,q^2}^{-1}\widetilde{Y}_{2,q}^3\widetilde{Y}_{2,q^3}^{-1} + 4\widetilde{Y}_{1,1}\widetilde{Y}_{1,q^2}\widetilde{Y}_{2,q^3}^{-2} + 6\widetilde{Y}_{2,q}^2\widetilde{Y}_{2,q^3}^{-2}$$
$$+4\widetilde{Y}_{1,1}\widetilde{Y}_{1,q^4}^{-1} + 4\widetilde{Y}_{2,q}\widetilde{Y}_{2,q^3}^{-3}\widetilde{Y}_{1,q^2} + 4\widetilde{Y}_{1,q^2}^{-1}\widetilde{Y}_{1,q^4}^{-1}\widetilde{Y}_{2,q}^2 + \widetilde{Y}_{2,q^3}^{-4}\widetilde{Y}_{1,q^2}^2 + 8\widetilde{Y}_{2,q}\widetilde{Y}_{2,q^3}^{-1}\widetilde{Y}_{1,q^4}^{-1} + 4\widetilde{Y}_{2,q^3}^{-2}\widetilde{Y}_{1,q^2}\widetilde{Y}_{1,q^4}^{-1} + \widetilde{Y}_{1,q^4}^{-2}.$$

But the corresponding character is *not* equal to the character of a representation of the twisted quantum affine algebra $U_q(D_3^{(2)})$. Indeed, the dimensions of the simple $U_q(D_3^{(2)})$-modules whose highest ${}^L\mathfrak{g}$-weight is y_1^2 are 36 (simple tensor square of the first fundamental representation), 35 ($L(Z_{1,t}Z_{1,t^4})$) or 32 (KR modules). The dimension of the remaining space and the multiplicity of the highest weight y_2^2 within it are respectively 18 and $2y_2^2$, 19 and $2y_2^2$, 22 and $3y_2^2$. But the dimension of the simple modules of highest weight y_2^2 are 16 (simple tensor product of fundamental representation), 15 ($L(Z_{2,t}Z_{2,t^4})$) or 10 (KR modules). This means that this is not the character of a representation of $U_q(D_3^{(2)})$.

Hence in this case $\overline{W}(u)$ cannot be equal to the whole $W(u)$. Rather, it should be a subspace of the space isomorphic to the one whose q-character is (7.2). We expect that $M(W)$ is a simple tensor square of the first fundamental representation of $U_q(D_3^{(2)})$. The computation of the specialization Π_t of the corresponding interpolating (q,t)-interpolating character confirms it.

We also note that $M(W)$ is smaller than $\widetilde{W}^\sigma$ in this case. Indeed, the weight spaces associated to the ${}^L\mathfrak{g}$-weights $2\omega_1^\vee$, $2\omega_2^\vee$ and $\omega_1^\vee$ have respective dimensions 1, 4 and 8 in $\widetilde{W}$ of dimension 54. The dimensions of the corresponding weight spaces in $\widetilde{W}^\sigma$ are 1, 3 and 6 (indeed, denoting by v a highest weight vector of W, the generators for $2\omega_2^\vee$ are $x_{1,0}^- x_{3,0}^-.v$, $x_{1,1}^-.x_{3,1}^-.v$, $(x_{1,0}^- x_{3,1}^- + x_{1,1}^- x_{3,0}^-).v$, and for $\omega_1^\vee$ their images under the action of $x_{2,0}^-$ and $x_{2,1}^-$). So the dimension of $\widetilde{W}^\sigma$ is smaller than 51, and then by a symmetry argument (by considering the opposite weights), it is smaller than 48. We have seen that the dimensions

of the simple modules over $U_q(D_3^{(2)})$ of highest weight $\omega_1^\vee$ are 36, 35 or 32. The dimension of the remaining space in $\widetilde{W}^\sigma$ is smaller than 12, 13 and 20 respectively, and the multiplicity of the weights $2\omega_2^\vee$, $\omega_1^\vee$ within are respectively $y_2^2+2y_1$, $y_2^2+2y_1$ and $2y_2^2+3y_1$. But the dimensions of the simple modules of highest weight $2\omega_2^\vee$ are 16, 15 or 10. Only the last case is possible and it corresponds to a KR-module which contributes to $\omega_1^\vee$ with multiplicity 1. In all cases, the remaining space has dimension smaller than 3 with a space associated to $\omega_1^\vee$ of dimension 1. This means that this is not the character of a representation of $U_q(D_3^{(2)})$.

Now let us interpret this in terms of the interpolating (q,t)-character of the simple representation $L(Y_{1,q^{-1}}Y_{1,q})^{\otimes 2}$ of $U_q(C_2^{(1)})$, that is the square of the formula (6.8) studied in Example 6.8.

Its specialization under $\overline{\Pi}_t$ is equal to ${}^{\mathrm{f}}\chi_t(W)$.

Its specialization under Π_t is the t-character of the 36-dimensional simple representation $M(W)=L(Z_{1,-1})^{\otimes 2}$ of $U_t({}^L\widehat{\mathfrak{g}})=U_t(D_3^{(2)})$.

Its specialization under Π'_t is (after switching the indices 1 and 2) the folded t-character of W', the 36-dimensional simple representation $L(Y_{2,1})^{\otimes 2}$ of $U_t(A_3^{(1)})$, defined in terms of the variables $\overline{Y}_{i,a}$ instead of $Y_{i,a}$.

8. Connection to monomial crystals

In this section we formulate a conjecture linking folded t-characters to Kashiwara's extension of Nakajima's monomial model of crystals to non-simple laced Lie algebras. We have previously mentioned it in Remark 3.1,(2).

Recall that a $\mathfrak{g}$-*crystal* is a set C together with an assignment to each element $c\in C$ a $\mathfrak{g}$-weight $\mathrm{wt}(c)$; the crystal operators $\widetilde{e}_i,\widetilde{f}_i : C\to C\sqcup\{0\}, i\in I$ (crystallized versions of Chevalley operators of $U_q(\mathfrak{g})$); and maps $\epsilon_i,\phi_i : C\to\mathbb{Z}$ satisfying the axioms of the crystal theory (in general, ϵ_i,ϕ_i could have infinite values, but we do not consider this possibility here). There is a corresponding notion of morphisms and isomorphisms of $\mathfrak{g}$-crystals as well as that of $\mathfrak{g}$-subcrystals. For example, each simple finite-dimensional representation $V(\lambda)$ of $U_q(\mathfrak{g})$ of highest weight λ is known to have a crystal basis. The set of its elements has the structure of a $\mathfrak{g}$-crystal, which is denoted by $\mathcal{B}(\lambda)$. This $\mathfrak{g}$-crystal is called a simple crystal and it has various realizations. One of them is the *monomial crystal* which was introduced by Nakajima [N2] and further studied by Kashiwara [K3]. We now recall the definition.

Let $s\colon I\to\{0,1\}$ $(i\mapsto s_i)$ be a map such that $C_{i,j}\le -1$ implies $s_i+s_j=1$. Consider the set of monomials $\mathcal{M}$ generated by the $Y_{i,q^r}^{\pm 1}$ such that $r\equiv s_i \bmod [2]$.

For $m=\prod_{j\in I,l\in\mathbb{Z}} Y_{j,q^l}^{u_{j,l}}\in\mathcal{M}$, define its $\mathfrak{g}$-weight by the formula

$$\mathrm{wt}(m)=\sum_{j\in I,l\in\mathbb{Z}} u_{j,l}\omega_j.$$

Next, set for $i\in I$,

$$\phi_i(m)=\max\{\phi_{i,L}(m)\mid L\in\mathbb{Z}\} \text{ where } \phi_{i,L}(m)=\sum_{l\le L} u_{i,l}(m),$$

$$\epsilon_i(m)=\max\{\epsilon_{i,L}(m)\mid L\in\mathbb{Z}\} \text{ where } \epsilon_{i,L}(m)=-\sum_{l\ge L} u_{i,l}(m).$$

Finally, define $\widetilde{e}_i, \widetilde{f}_i \colon \mathcal{M} \to \mathcal{M} \cup \{0\}$ for $i \in I$ by the formulas

$$\widetilde{e}_i(m) = \begin{cases} 0 & \text{if } \epsilon_i(m) = 0, \\ mA_{i,q^{p_i(m)-1}} & \text{if } \epsilon_i(m) > 0, \end{cases}$$

$$\widetilde{f}_i(m) = \begin{cases} 0 & \text{if } \phi_i(m) = 0, \\ mA^{-1}_{i,q^{q_i(m)+1}} & \text{if } \phi_i(m) > 0, \end{cases}$$

where the A_{j,q^d} are given by formula (3.10) and

$$p_i(m) = \max\{L \in \mathbb{Z} \mid \epsilon_{i,L}(m) = \epsilon_i(m)\} = \max\{L \in \mathbb{Z} \mid \sum_{l<L} u_{i,l}(m) = \phi_i(m)\},$$

$$q_i(m) = \min\{L \in \mathbb{Z} \mid \phi_{i,L}(m) = \phi_i(m)\} = \min\{L \in \mathbb{Z} \mid -\sum_{l>L} u_{i,l}(m) = \epsilon_i(m)\}.$$

Theorem 8.1 ([N2, K3])**.** *(1) The collection* $(\mathcal{M}, \mathrm{wt}, \epsilon_i, \phi_i, \widetilde{e}_i, \widetilde{f}_i)$ *is a* $\mathfrak{g}$*-crystal. It is called the monomial crystal.*

(2) For a dominant $m \in \mathcal{M}$*, let* $\mathcal{M}(m) \subset \mathcal{M}$ *be the* $\mathfrak{g}$*-subcrystal generated by* m*. Then* $\mathcal{M}(m)$ *is isomorphic, as a* $\mathfrak{g}$*-crystal, to* $\mathcal{B}(\mathrm{wt}(m))$*.*

For example, for $\mathfrak{g} = A_2$ we have

$$\mathcal{M}(Y_{1,1}) = \{Y_{1,1}, Y_{2,q}Y^{-1}_{1,q^2}, Y^{-1}_{2,q^3}\} \simeq \mathcal{B}(\omega_1) \tag{8.1}$$

and for $\mathfrak{g} = C_2$ we have

$$\mathcal{M}(Y_{2,1}) = \{Y_{2,1}, Y^{-1}_{2,q^2}Y^2_{1,q}, Y_{1,q}Y^{-1}_{1,q^3}, Y^{-2}_{1,q^3}Y_{2,q^2}, Y^{-1}_{2,q^4}\} \simeq \mathcal{B}(\omega_2). \tag{8.2}$$

Observe that the first set (8.1) is the set of monomials of the q-character of a fundamental representation of $U_q(A_2^{(1)})$, but the second set (8.2) in not the set of monomials of the q-character of a representation of $U_q(C_2^{(1)})$.

In fact, if $\mathfrak{g}$ is a simply-laced Lie algebra, there is a precise relation between the monomial crystals and the q-characters of modules over $U_q(\widehat{\mathfrak{g}})$ discovered by Nakajima [N2]. Namely, it is proved in [N2, Theorem 3.3] that for a dominant monomial m in $\mathcal{M}$, the set $\mathcal{V}(m)$ of monomials (without multiplicities) of the q-character of the standard module $V(m)$ associated to m (i.e. $V(m)$ is the corresponding tensor product of the fundamental representations) is a $\mathfrak{g}$-subcrystal of the monomial crystal $\mathcal{M}$ (in particular, its union with $\{0\}$ is stable by the crystal operators). Moreover, for each dominant $\mathfrak{g}$-weight λ, there is a choice of a dominant monomial $m \in \mathcal{M}$ of weight $\mathrm{wt}(m) = \lambda$ so that the $\mathfrak{g}$-crystal $\mathcal{V}(m)$ is equal to $\mathcal{M}(m)$. By Theorem 8.1,(2), $\mathcal{V}(m)$ is isomorphic, as a $\mathfrak{g}$-crystal, to the corresponding simple crystal $\mathcal{B}(\lambda)$ [N2, Proposition 3.4] (for a fundamental weight $\lambda = \omega_i$ we can choose $m = Y_{i,a}$).

We will now conjecture an analogous result for non-simply laced $\mathfrak{g}$, with the q-characters of representations of quantum affine algebras in $\mathcal{K}_q^+(\mathfrak{g})$ replaced by the t-characters in the *folded t-character ring* $\mathcal{K}_t^-(\mathfrak{g})$ introduced in Section 4.4. Note that we now denote the quantum parameter in $\mathcal{M}$ by t instead of q to fit the notation of the previous sections.

Conjecture 8.2. *(1) The set of monomials (without multiplicities) of a product of the fundamental elements* $F(Y_{i,t^r}) \in \mathcal{K}_t^-(\mathfrak{g})$*, with* $Y_{i,t^r} \in \mathcal{M}$*, is a* $\mathfrak{g}$*-subcrystal of the monomial crystal* $\mathcal{M}$*.*

(2) For each dominant weight λ, there is a product of the fundamental elements $F(Y_{i,a})$ with highest monomial $m \in \mathcal{M}$ of weight λ whose set of monomials is equal to $\mathcal{M}(m)$ and hence is a $\mathfrak{g}$-crystal isomorphic to $\mathcal{B}(\lambda)$.

In particular, for $i \in I$, we expect that the set of monomials (without multiplicities) occurring in $F(Y_{i,1}) \in \mathcal{K}_t^-(\mathfrak{g})$ is equal to $\mathcal{M}(Y_{i,1})$ and so is isomorphic, as a $\mathfrak{g}$-crystal, to $\mathcal{B}(\omega_i)$. We also expect that $\widetilde{W} = W'$ in this case (see Conjecture 5.16).

For example, we have the following element in $\mathcal{K}_t^-(C_2)$:

$$Y_{2,1} + Y_{2,t^2}^{-1} Y_{1,t}^2 + 2Y_{1,t} Y_{1,t^3}^{-1} + Y_{1,t^3}^{-2} Y_{2,t^2} + Y_{2,t^4}^{-1}.$$

The set of its monomials is the set (8.2), which is equal to $\mathcal{M}(Y_{2,1})$, and so it is isomorphic, as a $\mathfrak{g} = C_2$-crystal, to $\mathcal{B}(\omega_2)$.

One of our motivations for this Conjecture in the following. Consider the representation $L(\widetilde{Y}_{i,1})$ of $U_q(\widehat{({}^L\mathfrak{g})'})$. It is a simple tensor product of fundamental representations. By Theorem 8.1, the set of monomials of its q-character, $\mathcal{M}(L(\widetilde{Y}_{i,1}))$, has the structure of a simple $\left({}^L\mathfrak{g}\right)'$-crystal.

Now, the action of σ on $\mathcal{M}(L(\widetilde{Y}_{i,1}))$ is an automorphism of $\left({}^L\mathfrak{g}\right)'$-crystal. It then follows from the general results on crystals in [K2] that the subset of monomials $(\mathcal{M}(L(\widetilde{Y}_{i,1})))^\sigma$ fixed by σ has the structure of a simple ${}^L({}^L\mathfrak{g}) = \mathfrak{g}$-crystal isomorphic to $\mathcal{B}(\omega_i)$.

On the other hand, the identification of variables $Y_{j,b} \equiv Y_{j,\sigma(b)}$ is injective on $(\mathcal{M}(L(\widetilde{Y}_{i,a})))^\sigma$. Hence the set $(\mathcal{M}(L(\widetilde{Y}_{i,1})))^\sigma$ can be identified with a subset $(\mathcal{M}(L(\widetilde{Y}_{i,a})))^{\sigma,f}$ of the set of monomials of the folded t-character

$$\chi_t^f(L(\widetilde{Y}_{i,a})) = F(\overline{Y}_{i,a}) \in \mathcal{K}_t^-({}^L\mathfrak{g}).$$

Hence this subset $(\mathcal{M}(L(\widetilde{Y}_{i,a})))^{\sigma,f}$ inherits the structure of a simple $\mathfrak{g}$-crystal.

In Proposition 6.10 we have established that $\chi_t^L(L(\widetilde{Y}_{i,a}))$ is the specialization of the interpolating (q,t)-character $F_{q,t}(W_{i,1}) \in \overline{\mathcal{K}}_{q,t}({}^L\mathfrak{g})$ under $\overline{\Pi}_t$. But the specialization under Π'_t of the same interpolating (q,t)-character $F_{q,t}(W_{i,1})$ is $F(\overline{Y}_{i,1}) \in \mathcal{K}_t^-(\mathfrak{g})$ (defined in terms of the variables $\overline{Y}_{j,b}$ instead of $Y_{j,b}$).

We expect that the monomials of $F(\overline{Y}_{i,1})$ correspond through this interpolation to the monomials in the subset $(\mathcal{M}(L(\widetilde{Y}_{i,a})))^{\sigma,f}$, and so the set of these monomials should inherit the structure of a $\mathfrak{g}$-crystal. This leads us to the statement of Conjecture 8.2. Proving this identification requires a finer analysis. We plan to come back to this question in another paper.

For example, consider $\mathfrak{g} = C_2$ as above. Then $\left({}^L\mathfrak{g}\right)' = A_3$ and the set of monomials of $L(\widetilde{Y}_{1,1}) \simeq L(Y_{1,0}) \otimes L(Y_{3,1})$ has 15 elements

$$\mathcal{M}(\widetilde{Y}_{1,1}) = \{Y_{1,1}, Y_{1,q^2}^{-1} Y_{2,q}, Y_{2,q^3}^{-1} Y_{3,q^2}, Y_{3,q^4}^{-1}\} \times \{Y_{3,1}, Y_{3,q^2}^{-1} Y_{2,q}, Y_{2,q^3}^{-1} Y_{1,q^2}, Y_{1,q^4}^{-1}\},$$

($Y_{2,1} Y_{2,3}^{-1}$ occurs with multiplicity 2 in the q-character). The set of fixed monomials is

$$\mathcal{M}(\widetilde{Y}_{1,1})^\sigma = \{\widetilde{Y}_{1,1}, \widetilde{Y}_{1,q^2}^{-1} \widetilde{Y}_{2,q}^2, \widetilde{Y}_{2,q} \widetilde{Y}_{2,q^3}^{-1}, \widetilde{Y}_{2,q^3}^{-2} \widetilde{Y}_{1,q^2}, \widetilde{Y}_{1,q^4}^{-1}\}.$$

It has the structure of a C_2-crystal isomorphic to $\mathcal{B}(\omega_2)$ by the general results mentioned above. In this explicit example, we can check directly that the above identification gives the C_2-crystal (8.2). Indeed, by folding, this set is identified with

$$\mathcal{M}(\widetilde{Y}_{1,1})^{\sigma,f} = \{Y_{1,1}^2, Y_{1,q^2}^{-2} Y_{2,q}^2, Y_{2,q} Y_{2,q^3}^{-1}, Y_{2,q^3}^{-2} Y_{1,q^2}^2, Y_{1,q^4}^{-2}\},$$

which is a subset of the set of monomials of $\chi_t^f(L(\widetilde{Y}_{1,1}))$. The corresponding interpolating (q,t)-character $\overline{F}_{q,t}(W_{1,1})$ is given by Formula (6.8) in type B_2 (with 1 and 2 exchanged), see Example 6.8. Its specialization under $\overline{\Pi}_t$ is the folded t-character of the representation $L(\widetilde{Y}_{1,1})$ above. Its specialization under Π'_t is the t-character of an element in $\mathcal{K}_t^-(C_2)$, whose set of monomials is the set $\mathcal{M}(\widetilde{Y}_{1,1})^{\sigma,f}$ and is equal to

$$\{\overline{Y}_{1,1}, \overline{Y}_{1,q^2}^{-1}\overline{Y}_{2,q}^2, \overline{Y}_{2,q}\overline{Y}_{2,q^3}^{-1}, \overline{Y}_{2,q^3}^{-2}\overline{Y}_{1,q^2}, \overline{Y}_{1,q^4}^{-1}\}.$$

Exchanging the indices 1 and 2, we recover the set (8.2) (with the variables $\overline{Y}_{i,a}$ instead of the $Y_{i,a}$) as expected.

9. The Gaudin model

In this section, we consider the Gaudin limit of the folded integrable model discussed above. In this case, Bethe Ansatz equations simplify and we can study more directly the links between the objects associated to the Lie algebras $\mathfrak{g}, {}^L\mathfrak{g}$, and $\mathfrak{g}'$.

9.1. **The appearance of ${}^L\mathfrak{g}$.** First, some general observations about finite-dimensional representations of these three Lie algebras. Recall that $\mathfrak{g}'$ is a simply-laced Lie algebra with an automorphism σ of order 2 or 3, and $\mathfrak{g}$ is its Lie subalgebra of σ-invariant elements. Let $\mathfrak{h}'$ be a Cartan subalgebra of $\mathfrak{g}'$. Then its subspace $\mathfrak{h}$ of σ-invariant elements is a Cartan subalgebra of $\mathfrak{g}$.

Consider for simplicity the case when σ has order 2. The case of order 3 can be analyzed in a similar way.

The Cartan subalgebra $\mathfrak{h}'$ is generated by the coroots $\check{\alpha}'_i, i \in I'$. Here I' is the set of vertices of the Dynkin diagram of $\mathfrak{g}'$. Recall that σ acts on I', and the set I of vertices of the Dynkin diagram of $\mathfrak{g}$ is the quotient of I' by this action. For each $i \in I$, let J_i be the preimage of i in I'. It either consists of one element stable by σ, or two elements exchanged by σ. The coroot generators $\check{\alpha}_i, i \in I$, of $\mathfrak{h}$ are

$$\check{\alpha}_i = \sum_{j \in J_i} \check{\alpha}'_j. \tag{9.1}$$

In other words, they have the form

$$\check{\alpha}'_i, \quad \text{if } \sigma(i) = i; \qquad \check{\alpha}'_i + \check{\alpha}'_{\sigma(i)}, \quad \text{if } \sigma(i) \neq i. \tag{9.2}$$

Consider now the dual spaces $(\mathfrak{h}')^*$ and $\mathfrak{h}^*$. We have a surjective map $(\mathfrak{h}')^* \twoheadrightarrow \mathfrak{h}^*$ dual to the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{h}'$. We can try to embed $\mathfrak{h}^*$ into $(\mathfrak{h}')^*$ so that the pairing between the image of this embedding of $\mathfrak{h}^*$ and $\mathfrak{h} \subset \mathfrak{h}'$ is the pairing we have on $\mathfrak{h}$, but then an interesting thing happens: if $\sigma(i) = i$, then we can take as the image of the corresponding fundamental weight ω_i of $\mathfrak{g}$ in $(\mathfrak{h}')^*$ to be equal to ω'_i, the ith fundamental weight of $\mathfrak{g}'$. But if $\sigma(i) \neq i$, then we have to take *half* the sum of the fundamental weights:

$$\frac{1}{2}(\omega'_i + \omega'_{\sigma(i)}). \tag{9.3}$$

Indeed, we need to have $\langle \omega_i, \check{\alpha}_j \rangle = \delta_{ij}$, and then formula (9.2) shows that we must insert the factor $\frac{1}{2}$. The same factor appears in the expressions for the simple roots of $\mathfrak{g}$.

This has an important consequence: the integral weight lattice associated to $\mathfrak{g}$ does *not* embed into the integral weight lattice of $\mathfrak{g}'$. Instead, we have an embedding of the integral weight lattice associated not to $\mathfrak{g}$ but to ${}^L\mathfrak{g}$ – the Langlands dual Lie algebra! – into the integral weight lattice of $\mathfrak{g}'$.

Actually, this is clear from the fact that ${}^L\mathfrak{h} = \mathfrak{h}^*$, and so $({}^L\mathfrak{h})^* = \mathfrak{h}$, which, as we have seen above, naturally embeds into $\mathfrak{h}'$. We then identify the latter with $(\mathfrak{h}')^*$ using the unique W-invariant bilinear form normalized so that the square length of each root of $\mathfrak{g}'$ is equal to 2. Thus, we obtain an embedding $({}^L\mathfrak{h})^* \hookrightarrow (\mathfrak{h}')^*$. Under this embedding, a fundamental weight of ${}^L\mathfrak{g}$ is mapped to

$$\omega'_i + \omega'_{\sigma(i)}. \tag{9.4}$$

Unlike formula (9.3), there is *no* factor $\frac{1}{2}$ now. The same holds for the simple roots of ${}^L\mathfrak{g}$. Thus, we have proved the statement of Lemma 5.14 that there is a natural isomorphism between the lattice of integral ${}^L\mathfrak{g}$-weights and the lattice of σ-invariant $\mathfrak{g}'$-weights.

Now suppose that we have a finite-dimensional irreducible representation W of $\mathfrak{g}'$ whose highest weight λ is σ-invariant. By analogy with Lemma 5.2, we then prove that there is a unique automorphism $\widehat{\sigma}$ of W which intertwines the representation of $\mathfrak{g}'$ on W with its σ-twist and acts as identity on the highest weight subspace. As in Section 5.6, define the subspace $W^\sigma \subset W$ as the direct sum of the $\widehat{\sigma}$-invariant parts of the weight subspaces of W corresponding to σ-invariant weights. According to Proposition 5.18, its character is the character of a virtual representation of ${}^L\mathfrak{g}$. In Section 9.7, we will show that in fact it is possible (modulo Conjecture 9.10) to explicitly embed the irreducible representation of ${}^L\mathfrak{g}$ with the highest weight corresponding to λ into W^σ.

So, an interesting switch happens: we start with a simply-laced Lie algebra $\mathfrak{g}'$ with an automorphism σ whose fixed-point Lie subalgebra is $\mathfrak{g}$. Of course, every irreducible representation W of $\mathfrak{g}'$ restricts to a representation of $\mathfrak{g}$. However, because the integral weight lattice of $\mathfrak{g}$ does not embed into the integral weight lattice of $\mathfrak{g}'$, it's not natural to describe it in terms of weight spaces corresponding to the σ-invariant weights of $\mathfrak{g}'$. Instead, it turns out that we can construct inside the direct sum of σ-invariant weight subspaces of W (actually, inside its subspace W^σ) an irreducible representation of the *Langlands dual Lie algebra* ${}^L\mathfrak{g}$, which at first glance has nothing to do with $\mathfrak{g}'$. This can be done using the the results of [FFRy] on the spectra of the Hamiltonians of the Gaudin model.

Remark 9.1. In [FH1] and references therein it was shown that representations of ${}^L\mathfrak{g}$ can be extracted from irreducible finite-dimensional representations of $\mathfrak{g}$. In contrast, here we aim to extract representations of ${}^L\mathfrak{g}$ from irreducible representations of $\mathfrak{g}'$. It would be interesting to see whether there is a connection between the two approaches. □

9.2. **Gaudin model for $\mathfrak{g}'$.** Here we discuss the Gaudin model associated to $\mathfrak{g}'$, more precisely, its modification with the twist parameter $\chi' \in \mathfrak{h}'$. It appears as the $q \to 1$ limit of the XXZ-type quantum integrable model associated to $U_q(\widehat{\mathfrak{g}'})$ (χ' is the analogue of the parameter $u \in H'$ of the XXZ-type model).

We will use the results of [FFR, F, Ry, FFT, FFRy]. Let $\lambda_k, k = 1, \ldots, N$, be a collection of dominant integral weights of $\mathfrak{g}'$. Denote by V'_{λ_k} the corresponding irreducible finite-dimensional representations of $\mathfrak{g}'$. For a collection $z_k, k = 1, \ldots, N$, of distinct complex numbers, let $V'_{\lambda_k}(z_k), k = 1, \ldots, N$, be the corresponding evaluation representations of the current algebra $\mathfrak{g}'[z]$. Consider their tensor product

$$\bigotimes_{k=1}^N V'_{\lambda_k}(z_k) \tag{9.5}$$

and its subspace of weight

$$\gamma = \sum_k \lambda_k - \sum_{j=1}^m \alpha'_{i_j}. \tag{9.6}$$

These are the spaces of states of the Gaudin model associated to $\mathfrak{g}'$.

Next we discuss the Bethe Ansatz equations. Their solutions are in one-to-one correspondence with the *Miura* $\mathfrak{g}'$*-opers* on $\mathbb{CP}^1$ with trivial monodromy representation. They can be represented by $\mathfrak{g}'$-valued connections on $\mathbb{CP}^1$ of the form

$$\partial_z + \sum_{i \in I'} f'_i - \sum_{k=1}^N \frac{\check{\lambda}_k}{z - z_k} + \sum_{j=1}^m \frac{\check{\alpha}'_{i_j}}{z - w_j} + \chi'. \tag{9.7}$$

Here the $f'_i, i \in I'$, are generators of the lower nilpotent subalgebra of $\mathfrak{g}'$, and we are using the above identification between $\mathfrak{h}'$ and $(\mathfrak{h}')^*$ by means of the normalized bilinear form; namely, $\{\check{\lambda}_k\}$ denote the elements of $\mathfrak{h}' \subset \mathfrak{g}'$ corresponding to $\{\lambda_k \in (\mathfrak{h}')^*\}$, and $\{\check{\alpha}'_i, i \in I'\}$ is the set of simple coroots of $\mathfrak{g}'$, which correspond to the simple roots $\{\alpha'_i, i \in I'\}$ under the identification $(\mathfrak{h}')^* \simeq \mathfrak{h}$.

We will assume that w_j's are distinct complex numbers such that $w_j \neq z_k$ for all j and k.

Proposition 9.1 ([FFR, F, FFT])**.** *The condition that connection* (9.7) *has trivial monodromy is equivalent to the following system of Bethe Ansatz equations:*

$$\sum_{k=1}^N \frac{\langle \alpha'_{i_j}, \check{\lambda}_k \rangle}{w_j - z_k} - \sum_{s \neq j} \frac{\langle \alpha'_{i_j}, \check{\alpha}'_{i_s} \rangle}{w_j - w_s} = \langle \alpha'_{i_j}, \chi' \rangle, \qquad j = 1, \ldots, m. \tag{9.8}$$

These equations can be obtained by taking the limit $q \to 1$ of the Bethe Ansatz equations (4.7) of the XXZ-type model associated to $U_q(\widehat{\mathfrak{g}'})$. The monic polynomial with the roots w_j where $i_j = i$ (i.e. those points on $\mathbb{CP}^1 \backslash \infty$ at which the connection (9.7) has residue $\check{\alpha}'_i$),

$$Q_i(z) = \prod_{i_j = i} (z - w_j), \tag{9.9}$$

is the analogue of the Baxter polynomial $Q_i(z), i \in I'$ (see Theorem 5.3) in the Gaudin model.

Note that if we apply a permutation to the set of roots of each polynomial $Q_i(z)$ (which we recall are assumed to be distinct) we also obtain a solution of the system (9.8). In what follows, by a *solution* of Bethe Ansatz equations we will understand an equivalence class of solutions under these permutations.

In [FFR, F, FFT] a joint eigenvector of the Gaudin Hamiltonians (Bethe vector) is constructed for each solution of the BAE (9.8). It is known, however, that for general λ_k's and γ these Bethe vectors do not yield a basis of the weight subspace of (9.5) of weight γ given by formula (9.6). In fact, explicit examples have been constructed in [MV3] showing that for $\chi = 0$ this is so even for generic z_k's (and fixed λ_k's and γ). We are not aware of such counterexamples for generic χ', so it is possible that in this case Bethe vectors do yield a basis for generic z_k's. For $N = 1$, this is the statement of Conjecture 9.10 below.

As explained in [F, Sect. 5.5] (in the case $\chi' = 0$, but the picture is similar for all χ') that the true parameters of the spectra of the quantum Gaudin Hamiltonians are not the Miura $\mathfrak{g}'$-opers of the form (9.7) but rather $\mathfrak{g}'$-opers on $\mathbb{CP}^1$ with trivial monodromy, regular

singularities at $z_1, \ldots, z_N$ with respective residues $\check{\lambda}_1, \ldots, \check{\lambda}_N$ (coweights of $\mathfrak{g}'$ corresponding to the weight $\lambda_1, \ldots, \lambda_N$), and irregular singularity at ∞ of order 2 with 2-residue corresponding to χ' (see [FFT, FFRy] for more details). Denote the set of these $\mathfrak{g}'$-opers by $\mathrm{Op}^{\chi}_{(\check{\lambda}_i),(z_i)}(\mathfrak{g}')$.

Theorem 9.2 ([FFRy])**.** *For any regular $\chi' \in \mathfrak{h}'$ and any collection $z_1 \ldots, z_N$ there is a bijection between $\mathrm{Op}^{\chi}_{(\check{\lambda}_i),(z_i)}(\mathfrak{g}')$ and the spectrum of the Gaudin Hamiltonians on the space* (9.5)*. Moreover, the eigenspace corresponding to every $\mathfrak{g}'$-oper in $\mathrm{Op}^{\chi}_{(\check{\lambda}_i),(z_i)}(\mathfrak{g}')$ is always one-dimensional. In addition, for generic $z_1 \ldots, z_N$ and χ' the Gaudin Hamiltonians are diagonalizable.*

In other words, for any regular χ and general $z_1 \ldots, z_N$ there is at most one Jordan block for each joint eigenvalue of the Gaudin Hamiltonians (this is expressed in [FFRy, Corollary 5] as the statement that the commutative algebra of Gaudin Hamiltonians has a cyclic vector in (9.5)). Moreover, for generic χ and $z_1 \ldots, z_N$ all Jordan blocks have size 1 (see [FFRy, Corollary 6]).

Following the argument in [F, Sect. 5.5] in the case $\chi' = 0$, one can show that every $\mathfrak{g}'$-oper in $\mathrm{Op}^{\chi}_{(\check{\lambda}_i),(z_i)}(\mathfrak{g}')$ corresponds to a unique Miura $\mathfrak{g}'$-oper of the form

$$\partial_z + \sum_{i \in I'} f'_i - \sum_{k=1}^{N} \frac{g_k(\check{\lambda}_k + \check{\rho}') - \check{\rho}'}{z - z_k} + \sum_{j=1}^{m} \frac{\check{\rho}' - \widetilde{g}_j(\check{\rho}')}{z - w_j} + \chi', \tag{9.10}$$

where the $w_j, j = 1, \ldots, m$, are distinct complex numbers such that $w_j \neq z_k$ for all j and k; $g_k, k = 1, \ldots, N$, and $\widetilde{g}_j, i = 1, \ldots, m$, are elements of the Weyl group of $\mathfrak{g}'$; and $\check{\rho}'$ is the sum of the dominant coweights of $\mathfrak{g}'$.

A Miura $\mathfrak{g}'$-oper of the form (9.10), and the corresponding $\mathfrak{g}'$-oper, are called *non-degenerate* (see [F, Sect. 5.2]) if $g_k = 1$ for all $k = 1, \ldots, N$, and each $\widetilde{g}_j$ is a simple reflection s_{i_j} from the Weyl group of $\mathfrak{g}'$. Since $\check{\rho}' - s_{i_j}(\check{\rho}') = \check{\alpha}'_{i_j}$, a Miura $\mathfrak{g}'$-oper of the form (9.10) is non-degenerate $\mathfrak{g}'$-oper if and only if it has the form (9.7). Further, according to Proposition 9.1, the no-monodromy condition is then equivalent to BAE (9.8).

Theorem 9.2 implies that non-degenerate Miura $\mathfrak{g}'$-opers of the form (9.7) satisfying (9.8) correspond to a subset in the spectrum of the Gaudin Hamiltonians on the weight subspace of (9.5) of weight γ. But in general, this is not the entire spectrum; there may be other joint eigenvalues of the Gaudin Hamiltonians on this weight subspace which correspond to degenerate Miura $\mathfrak{g}'$-opers of the form (9.10) with at least one g_k not equal to the identity or at least one of the $\widetilde{g}_j$ not equal to a simple reflection, or both. For such Miura $\mathfrak{g}'$-opers, the equations expressing the no-monodromy condition are more complicated, as are the corresponding eigenvectors (though they can be constructed in principle by a certain algorithm). See [F, Sect. 5.5] for more details.

9.3. **Gaudin model for ${}^L\mathfrak{g}$.** Next, consider the Gaudin model for the Lie algebra ${}^L\mathfrak{g}$ with a twist by $\chi \in ({}^L\mathfrak{h})^* = \mathfrak{h}$. The space of states of the model is then a tensor product of evaluation representations of the current algebra ${}^L\mathfrak{g}[z]$ corresponding to irreducible finite-dimensional representations of ${}^L\mathfrak{g}$:

$$\bigotimes_{k=1}^{N} V_{\mu_k}(z_k) \tag{9.11}$$

where $\mu_1, \dots, \mu_N$ are dominant integral weights of ${}^L\mathfrak{g}$ and $z_1, \dots, z_N$ are distinct complex numbers. According to the results of [FFR, F, FFT, FFRy], the spectrum of the Gaudin Hamiltonians exhibits *Langlands duality*; namely, the joint eigenvalues of the ${}^L\mathfrak{g}$-Gaudin Hamiltonians are described in terms of $\mathfrak{g}$-opers rather than ${}^L\mathfrak{g}$-opers.

More precisely, let $\check{\mu}_k \in \mathfrak{h} = ({}^L\mathfrak{h})^*$ be the integral coweight of $\mathfrak{g}$ corresponding to μ_k. Let $\mathrm{Op}^{\chi}_{(\check{\mu}_i),(z_i)}(\mathfrak{g})$ be the set of $\mathfrak{g}$-opers on $\mathbb{CP}^1$ with trivial monodromy, regular singularities at $z_1, \dots, z_N$ with respective residues $\check{\mu}_1, \dots, \check{\mu}_N$, and irregular singularity at ∞ of order 2 with 2-residue corresponding to χ (see [FFT, FFRy] for details). Then we have the following analogue of Theorem 9.2.

Theorem 9.3 ([FFRy]). *For any regular $\chi \in \mathfrak{h}$ and any collection $z_1 \dots, z_N$ there is a bijection between $\mathrm{Op}^{\chi}_{(\check{\mu}_i),(z_i)}(\mathfrak{g})$ and the spectrum of the ${}^L\mathfrak{g}$-Gaudin Hamiltonians on the space of* (9.11)*. Moreover, the eigenspace corresponding to every $\mathfrak{g}$-oper in $\mathrm{Op}^{\chi}_{(\check{\mu}_i),(z_i)}(\mathfrak{g})$ is always one-dimensional. In addition, for generic $z_1 \dots, z_N$ and χ the Gaudin Hamiltonians are diagonalizable.*

9.4. **Embedding of the space of states.** This theorem has an intriguing consequence. Note that each coweight $\check{\mu}_k$ of $\mathfrak{g}$ defines a coweight $\check{\mu}'_k$ of $\mathfrak{g}'$ while the corresponding weight μ_k of ${}^L\mathfrak{g}$ defines a weight of $\mathfrak{g}'$, and the element $\chi \in \mathfrak{h}$ defines an element $\chi' \in \mathfrak{h}'$.

Next, define the nilpotent generators $f_i, i \in I$, of $\mathfrak{g}$ by the formulas

$$f_i = f'_i \quad \text{if} \quad \sigma(i) = i, \qquad f_i = f'_i + f'_{\sigma(i)} \quad \text{if} \quad \sigma(i) \neq i.$$

Then we obtain that the embedding $\mathfrak{g} \hookrightarrow \mathfrak{g}'$ maps the principal nilpotent element of $\mathfrak{g}$ to that of $\mathfrak{g}'$:

$$p_- := \sum_{i \in I} f_i \quad \mapsto \quad p'_- := \sum_{i \in I'} f'_i. \tag{9.12}$$

In the same way, we obtain that the element $\check{\rho}$ of $\mathfrak{g}$ (the sum of its dominant coweights) maps to the corresponding element $\check{\rho}'$ of $\mathfrak{g}'$. Recall that there is a unique nilpotent element $p_+ \in \mathfrak{g}$ which is a linear combination of the generators $e_i, i \in I$, such that $\{p_-, 2\check{\rho}, p_+\}$ is an $\mathfrak{sl}_2$ triple in $\mathfrak{g}$. Likewise, we have an $\mathfrak{sl}_2$ triple $\{p'_-, 2\check{\rho}', p'_+\}$ in $\mathfrak{g}'$. Uniqueness implies that p'_+ is the image of p_+ in $\mathfrak{g}'$.

Using these $\mathfrak{sl}_2$ triples, we construct the Kostant slices $K(\mathfrak{g})$ and $K(\mathfrak{g}')$ of regular conjugacy classes in $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. Namely,

$$K(\mathfrak{g}) = p_- + \mathrm{span}\{p_j\}_{j=1,\dots,\ell},$$

where $\{p_j\}_{j=1,\dots,\ell}$ is a basis of the subspace of $\mathrm{ad}\, p_+$-invariants in the upper nilpotent subalgebra $\mathfrak{n}_+$ of $\mathfrak{g}$, such that $[\check{\rho}, p_j] = d_j p_j$, with $\{d_1, \dots, d_\ell\}$ being the set of exponents of $\mathfrak{g}$. The Kostant slice $K(\mathfrak{g}')$ in $\mathfrak{g}'$ is defined similarly. The fact that the embedding $\mathfrak{g} \hookrightarrow \mathfrak{g}'$ maps the above $\mathfrak{sl}_2$ triples to each other then implies that we have a natural embedding

$$K(\mathfrak{g}) \hookrightarrow K(\mathfrak{g}'). \tag{9.13}$$

Beilinson and Drinfeld have constructed [BD2] canonical representatives of $\mathfrak{g}$-opers in terms of $K(\mathfrak{g})$ (see [FFT, Sect. 4.2]). Using the embedding (9.13), we obtain the following.

Lemma 9.4. *There is a natural embedding*

$$\mathrm{Op}^{\chi}_{(\check{\mu}_i),(z_i)}(\mathfrak{g}) \hookrightarrow \mathrm{Op}^{\chi'}_{(\check{\mu}'_i),(z_i)}(\mathfrak{g}'). \tag{9.14}$$

Now we derive from Theorem 9.3 and Lemma 9.4 the following surprising result.

Theorem 9.5. *For generic regular $\chi \in \mathfrak{h}$, such that $\chi' \in \mathfrak{h}'$ is regular, and generic $z_1, \ldots, z_N$ there exists an embedding of the tensor product* (9.11) *of simple ${}^L\mathfrak{g}$-modules with highest weights μ_k into the tensor product of the simple $\mathfrak{g}'$-modules with the corresponding highest weights μ'_k:*

$$\bigotimes_{k=1}^N V_{\mu_k}(z_k) \hookrightarrow \bigotimes_{k=1}^N V'_{\mu'_k}(z_k). \tag{9.15}$$

Proof. By Theorem 9.3, for generic χ and $z_1, \ldots, z_N$ there exists an eigenbasis $\{e_m\}$ of the Gaudin Hamiltonians in (9.11) labeled by $\mathfrak{g}$-opers $m \in \mathrm{Op}^{\chi}_{(\check{\mu}_i),(z_i)}(\mathfrak{g})$ with each $\mathfrak{g}$-oper corresponding to exactly one basis vector. Under the embedding (9.14), each of these $\mathfrak{g}$-opers gives a $\mathfrak{g}'$-oper $m' \in \mathrm{Op}^{\chi'}_{(\check{\mu}'_i),(z_i)}(\mathfrak{g}')$. Applying Theorem 9.3 in the case of the Lie algebra $\mathfrak{g}'$ and the tensor product of simple $\mathfrak{g}$-modules with the corresponding highest weights $\mu'_k, k = 1, \ldots, N$, we find that each m' obtained this way corresponds to the one-dimensional eigenspace $E_{m'}$ of the Hamiltonians of the $\mathfrak{g}'$-Gaudin model. Mapping each basis vector e_m in (9.11) to a non-zero vector in $E_{m'}$, we obtain the desired embedding. □

The embedding (9.15) is not unique since we can rescale the images of the basis elements e_m by arbitrary non-zero numbers. Another issue is that the embedding (9.14) is defined in a rather abstract way (using Kostant slices). It turns out that we can obtain a more concrete realization of (9.14) if all $\mathfrak{g}$-opers in $\mathrm{Op}^{\chi}_{(\check{\mu}_i),(z_i)}(\mathfrak{g})$ are non-degenerate, and hence correspond to solutions of the Bethe Ansatz equations. In this case we can also construct the corresponding eigenvectors (Bethe vectors) explicitly, so we obtain a more concrete realization of the embedding (9.15) as well. This may be viewed as an analogue of the statement of our main Conjecture 5.15 in the case of simple Lie algebras. We explain it in the next subsection.

Remark 9.2. Alternatively, under the conditions of Theorem 9.5 we can construct a surjective map

$$\bigotimes_{k=1}^N V'_{\mu'_k}(z_k) \twoheadrightarrow \bigotimes_{k=1}^N V_{\mu_k}(z_k) \tag{9.16}$$

instead of an embedding (9.15). Indeed, the inclusion (9.14) gives rise to a surjective homomorphism of the corresponding algebras of functions

$$\mathrm{Fun}\,\mathrm{Op}^{\chi'}_{(\check{\mu}'_i),(z_i)}(\mathfrak{g}') \twoheadrightarrow \mathrm{Fun}\,\mathrm{Op}^{\chi}_{(\check{\mu}_i),(z_i)}(\mathfrak{g}). \tag{9.17}$$

The corresponding algebras of $\mathfrak{g}$- and $\mathfrak{g}'$-Gaudin Hamiltonians were denoted in [FFRy] by $\mathcal{A}_\chi(z_1, \ldots, z_N)$ and $\mathcal{A}_{\chi'}(z_1, \ldots, z_N)$. According to Theorem 9.3, for generic regular $\chi \in \mathfrak{h}$ and generic $z_1, \ldots, z_N$, the algebra $\mathrm{Fun}\,\mathrm{Op}^{\chi}_{(\check{\mu}_i),(z_i)}(\mathfrak{g})$ is equal to the image of $\mathcal{A}_\chi(z_1, \ldots, z_N)$ in the algebra of endomorphisms of $\bigotimes_{k=1}^N V_{\mu_k}(z_k)$. Moreover, $\mathrm{Fun}\,\mathrm{Op}^{\chi}_{(\check{\mu}_i),(z_i)}(\mathfrak{g})$ then has a cyclic vector in $\bigotimes_{k=1}^N V_{\mu_k}(z_k)$. If we choose such a cyclic vector, we obtain an isomorphism of vector spaces

$$\bigotimes_{k=1}^N V_{\mu_k}(z_k) \simeq \mathrm{Fun}\,\mathrm{Op}^{\chi}_{(\check{\mu}_i),(z_i)}(\mathfrak{g}). \tag{9.18}$$

On the other hand, according to Theorem 9.2, $\operatorname{Fun}\operatorname{Op}^{\chi'}_{(\check{\mu}'_i),(z_i)}(\mathfrak{g}')$ is the quotient of the image of $\mathcal{A}_{\chi'}(z_1,\dots,z_N)$ in the algebra of endomorphisms of $\bigotimes_{k=1}^N V_{\mu_k}(z_k)$ by its radical. Choosing a cyclic vector of $\mathcal{A}_{\chi'}(z_1,\dots,z_N)$ in $\bigotimes_{k=1}^N V'_{\mu'_k}(z_k)$ and using the homomorphism (9.17), we obtain a surjective map (9.16).

This construction can be generalized to the case when χ is a regular nilpotent element of $\mathfrak{g}$ and χ is the corresponding element of $\mathfrak{g}'$. Then we can choose as the cyclic vectors the highest weight vectors in the tensor products (9.16), so that the resulting map becomes canonical up to a scalar.

Constructing surjective maps (9.16) may seem more appealing than constructing the embeddings (9.15). However, in the case when all $\mathfrak{g}$-opers in $\operatorname{Op}^{\chi}_{(\check{\mu}_i),(z_i)}(\mathfrak{g})$ are non-degenerate, the embedding (9.15) can be linked to the inclusion of the sets of solutions of the corresponding Bethe Ansatz equations, as we explain in the next subsection. Therefore we can make contact to the folded integrable models. (Moreover, the embedding (9.15) can then be constructed rather explicitly using Bethe vectors, see Section 9.7.) It is for this reason that we focus on the embeddings (9.15) rather than the surjections (9.16), even though the latter also deserve to be studied. □

9.5. **Embedding in the case of non-degenerate $\mathfrak{g}$-opers.** As in the case of $\mathfrak{g}'$-opers discussed in Section 9.2, each $\mathfrak{g}$-oper in $\operatorname{Op}^{\chi}_{(\check{\mu}_i),(z_i)}(\mathfrak{g})$ can be represented in a unique way by a Miura $\mathfrak{g}$-oper given by a formula similar to (9.10). Among them are the non-degenerate Miura $\mathfrak{g}$-opers of the form

$$\partial_z + \sum_{i\in I} f_i - \sum_{k=1}^N \frac{\check{\mu}_k}{z-z_k} + \sum_{j=1}^m \frac{\check{\alpha}_{i_j}}{z-w_j} + \chi, \tag{9.19}$$

where $\check{\alpha}_i, i\in I$, are the simple coroots of $\mathfrak{g}$. The no-monodromy condition on (9.19) is equivalent to the Bethe Ansatz equations similar to (9.8) (see equation (6.11) in [FFT]):

$$\sum_{k=1}^N \frac{\langle \alpha_{i_j}, \check{\mu}_k\rangle}{w_j - z_k} - \sum_{s\neq j} \frac{\langle \alpha_{i_j}, \check{\alpha}_{i_s}\rangle}{w_j - w_s} = \langle \alpha_{i_j}, \chi\rangle, \qquad j = 1,\dots,m. \tag{9.20}$$

The image of a non-degenerate Miura $\mathfrak{g}$-oper (9.19) under the embedding (9.14) is particularly easy to describe. Namely, we interpret formula (9.19) as a Miura $\mathfrak{g}'$-oper of the form (9.7) as follows: According to formula (9.12), the first summation in (9.19) is equal to the first summation in (9.7). Next, the second summation in (9.19) is equal to the second summation in (9.7) with $\check{\lambda}_k = \check{\mu}'_k$.

Finally, every term in the third summation in (9.19) can be written as the sum of terms appearing in the third summation of (9.7). Namely, for $i\in I'$ such that $\sigma(i)=i$, we map

$$\frac{\check{\alpha}_i}{z-w_j} \mapsto \frac{\check{\alpha}'_i}{z-w_j}, \tag{9.21}$$

and for $i\in I$ such that if $\sigma(i)\neq i$, we map

$$\frac{\check{\alpha}_i}{z-w_j} \mapsto \frac{\check{\alpha}'_i}{z-w_j} + \frac{\check{\alpha}'_{\sigma(i)}}{z-w_j}. \tag{9.22}$$

In other words, each term corresponding to $i\in I$ such that $\sigma(i)=i$ in (9.19) gives us the corresponding term in (9.7), whereas in the case $\sigma(i)\neq i$ it gives us the sum of two terms in (9.7).

Observe that if the terms (9.22) are present in (9.19), then the corresponding $\mathfrak{g}'$-Miura oper (9.7) is *degenerate* because its residue at the corresponding point w_j is equal to

$$\check{\alpha}'_i + \check{\alpha}'_{\sigma(i)} = \check{\rho}' - s_i s_{\sigma(i)}(\check{\rho}') \tag{9.23}$$

(the simple reflections s_i and $s_{\sigma(i)}$ commute with each other, so it doesn't matter in which order we take their product).

Thus, the Miura $\mathfrak{g}'$-opers we obtain this way have the form (9.10), with $g_k = 1$ for all $k = 1, \ldots, N$, and $\widetilde{g}_j$ being equal to the simple reflection s_i of the Weyl group of $\mathfrak{g}'$ if the residue of this Miura $\mathfrak{g}'$-oper at w_j is equal to a σ-invariant simple coroot $\check{\alpha}'_i$ of $\mathfrak{g}'$, but $\widetilde{g}_j$ is equal to the product of two simple reflections, $s_i s_{\sigma(i)}$, if residue at w_j is equal to (9.23). We have obtained the following result.

Proposition 9.6. *Under the embedding* (9.14)*, every non-degenerate* $\mathfrak{g}$*-oper in* $\mathrm{Op}^{\chi}_{(\check{\mu}_i),(z_i)}(\mathfrak{g})$ *which is represented by a Miura* $\mathfrak{g}$*-oper of the form* (9.19) *maps to the* $\mathfrak{g}'$*-oper in* $\mathrm{Op}^{\chi'}_{(\check{\mu}'_i),(z_i)}(\mathfrak{g}')$ *represented by the Miura* $\mathfrak{g}'$*-oper*

$$\partial_z + \sum_{i \in I'} f_i - \sum_{k=1}^{N} \frac{\check{\mu}'_k}{z - z_k} + \sum_{\sigma(i_j) = i_j} \frac{\check{\alpha}'_{i_j}}{z - w_j} + \sum_{\sigma(i_j) \neq i_j} \frac{\check{\alpha}'_{i_j} + \check{\alpha}'_{\sigma(i_j)}}{z - w_j} + \chi'. \tag{9.24}$$

According to Theorem 9.2, the Miura $\mathfrak{g}'$-oper (9.24) defines a point in the spectrum of the $\mathfrak{g}'$-Gaudin model.

9.6. **Folded Bethe Ansatz equations.** From the point of view of the Bethe Ansatz equations, we interpret this as follows. In the system of BAE (9.8), it makes sense to impose the condition

$$\{w_j \,|\, i_j = i\} = \{w_j \,|\, i_j = \sigma(i)\} \tag{9.25}$$

for all $i \in I'$. That's because for $i \neq \sigma(i)$ we have $\langle \check{\alpha}'_i, \check{\alpha}'_{\sigma(i)} \rangle = 0$, so no singularities occur in equations (9.8). Equivalently, this condition may be expressed as $Q_i(z) = Q_{\sigma(i)}(z)$ for all $i \in I$, so this is the Gaudin model analogue of the condition we used to define the folding of the Bethe Ansatz equations in the XXZ-type model associated to $U_q(\widehat{\mathfrak{g}'})$ (see Proposition 4.4).

Lemma 9.7. *The system* (9.8) *of Bethe Ansatz equations of the* $\mathfrak{g}'$*-Gaudin model together with the condition* $Q_i(z) = Q_{\sigma(i)}(z)$ *for all* $i \in I'$ *is equivalent to the system* (9.20) *of Bethe Ansatz equations of the* ${}^L\mathfrak{g}$*-Gaudin model.*

Thus, the folding of the BAE of the $\mathfrak{g}'$-Gaudin model gives the BAE of the ${}^L\mathfrak{g}$-Gaudin model. In particular, we obtain that the $q \to 1$ limit of the system (4.8) of folded Bethe Ansatz equations for $U_q(\widehat{\mathfrak{g}'})$ coincides with the Bethe Ansatz equations of the ${}^L\mathfrak{g}$-Gaudin model.

Now we define an analogue of the space $W(u)$ from Section 5.3, with the role of the twist parameter u played by χ (more precisely, this is an analogue of the subspace $\mathbf{W}(u) \subset W(u)$ introduced in Remark 5.4). Recall that we have identified the lattice of integral ${}^L\mathfrak{g}$-weights (equivalently, integral $\mathfrak{g}$-coweights) with the lattice of σ-invariant integral $\mathfrak{g}'$-weights (equivalently, integral $\mathfrak{g}'$-coweights). We have a collection $\check{\mu}_k, k = 1, \ldots, N$, of $\mathfrak{g}$-coweights. Let $\mu_k, k = 1, \ldots, N$, be the corresponding ${}^L\mathfrak{g}$-weights and $\mu'_k, k = 1, \ldots, N$,

the corresponding $\mathfrak{g}'$-weights. Consider the following representation of $\mathfrak{g}'$:

$$W = \bigotimes_{k=1}^{N} V'_{\mu'_k}(z_k). \tag{9.26}$$

Definition 9.8. The subspace $W(\chi)$ of W is the span of eigenvectors of the $\mathfrak{g}'$-Gaudin Hamiltonians with the twist parameter χ whose eigenvalues correspond to solutions of the BAE (9.8) satisfying the condition $Q_i(z) = Q_{\sigma(i)}(z), i \in I'$.

Theorems 9.2 and 9.3 and Lemma 9.7 imply the following.

Theorem 9.9. *Suppose that the ${}^L\mathfrak{g}$-Gaudin Hamiltonians are diagonalizable on the tensor product* (9.11) *and all $\mathfrak{g}$-opers in $\mathrm{Op}^{\chi}_{(\check{\mu}_i),(z_i)}(\mathfrak{g})$ are non-degenerate, so they correspond to non-degenerate Miura $\mathfrak{g}$-opers of the form* (9.19) *or equivalently, solutions of the corresponding Bethe Ansatz equations. Then there is a natural embedding of the corresponding set of solutions of the BAE of this ${}^L\mathfrak{g}$-Gaudin model into the spectrum of the $\mathfrak{g}'$-Gaudin model with the spaces of states* (9.26)*. Moreover, we obtain an embedding*

$$\bigotimes_{k=1}^{N} V_{\mu_k}(z_k) \xrightarrow{\sim} W(\chi) \subset W \tag{9.27}$$

where W is the tensor product (9.26) *of representations of $\mathfrak{g}'$ and $W(\chi)$ is given in Definition 9.8.*

We can construct the embedding (9.27) explicitly using the formulas given in [FFT] for the eigenvectors (Bethe vectors) of the ${}^L\mathfrak{g}$-Gaudin model associated to solutions of the corresponding BAE. In the next subsection we will explain this in the case $N = 1$.

9.7. **The case $N = 1$.** If $N = 1$, the space (9.11) is an irreducible representation V_μ of ${}^L\mathfrak{g}$ with highest weight $\mu \in ({}^L\mathfrak{h})^*$ and (9.26) is the irreducible representation $V'_{\mu'}$ of $\mathfrak{g}'$ with the σ-invariant $\mathfrak{g}'$-highest weight μ' corresponding to μ. As before, we denote by $\check{\mu}$ and $\check{\mu}'$ the corresponding coweights of $\mathfrak{g}$ and $\mathfrak{g}'$. Let χ be a regular element of the Cartan subalgebra of $\mathfrak{g}$. As before, we will denote by χ' its image in $\mathfrak{h}'$, and we will assume that it is a regular element of $\mathfrak{h}'$ as well.

The Gaudin model for general N is invariant under simultaneous shifts of the spectral parameters z_i. Using this symmetry, we will set the only parameter z_1 in the case $N = 1$ equal to 0. The algebra of Gaudin Hamiltonians in this case is known as the *shift of argument subalgebra* $\mathcal{A}_\chi$ of the universal enveloping algebra $U(\mathfrak{g})$ (see [Ry, FFT, FFRy]).

The corresponding non-degenerate Miura $\mathfrak{g}$-opers (9.19) have the form

$$\partial_z + \sum_{i\in I} f_i - \frac{\check{\mu}}{z} + \sum_{j=1}^{m} \frac{\check{\alpha}_{i_j}}{z - w_j} + \chi. \tag{9.28}$$

This Miura $\mathfrak{g}$-oper has trivial monodromy if and only if the following BAE are satisfied (see formula (6.16) of [FFT]):

$$\frac{\langle \alpha_{i_j}, \check{\mu}\rangle}{w_j} - \sum_{s\neq j} \frac{\langle \alpha_{i_j}, \check{\alpha}_{i_s}\rangle}{w_j - w_s} = \langle \alpha_{i_j}, \chi\rangle, \quad j = 1, \dots, m. \tag{9.29}$$

Suppose that these equations are satisfied. Then, as shown in [FFT], the following Bethe vector is an eigenvector of the Gaudin Hamiltonians (provided that it is non-zero) of weight

$\gamma = \mu - \sum_{j=1}^m \alpha_{i_j}$:

$$\phi(w_1^{i_1}, \ldots, w_m^{i_m}) = \sum_{\tau \in S_m} \frac{f_{i_{\tau(1)}} f_{i_{\tau(2)}} \cdots f_{i_{\tau(m)}}}{(w_{\tau(1)} - w_{\tau(2)})(w_{\tau(2)} - w_{\tau(3)}) \ldots (w_{\tau(m-1)} - w_{\tau(m)}) w_{\tau(m)}} v_\mu, \tag{9.30}$$

where the sum is over all permutations τ on m letters (see formula (6.15) of [FFT]).

According to Theorem 9.3, for generic χ the spectrum of the algebra $\mathcal{A}_\chi$ of Gaudin Hamiltonians on V_μ is simple and in bijection with the set $\mathrm{Op}^\chi_{\check{\mu},0}(\mathfrak{g})$ of $\mathfrak{g}$-opers on $\mathbb{CP}^1$ with trivial monodromy, regular singularity at 0 with the residue $\check{\mu}$, and irregular singularity of order 2 at ∞ with the 2-residue χ.

Conjecture 9.10. *For generic regular $\chi \in \mathfrak{h}$, all $\mathfrak{g}$-opers appearing in the spectrum of this Gaudin model are non-degenerate, and so the spectrum is parametrized by Miura $\mathfrak{g}$-opers* (9.28) *with trivial monodromy, or equivalently, by collections $\{w_1, \ldots, w_m\}$ and $\{\check{\alpha}_{i_1}, \ldots, \check{\alpha}_{i_m}\}$ solving the system of BAE* (9.29)*. Moreover, the corresponding Bethe vectors* (9.30) *form an eigenbasis of $\mathcal{A}_\chi$ in the irreducible representation V_μ of $\mathfrak{g}$.*

Assuming this conjecture, we can construct explicitly an embedding of V_μ into the irreducible representation $V'_{\mu'}$ of $\mathfrak{g}'$. Namely, suppose that we have collections $\{w_1, \ldots, w_m\}$ and $\{\check{\alpha}_{i_1}, \ldots, \check{\alpha}_{i_m}\}$ satisfying the BAE (9.29) of the ${}^L\mathfrak{g}$-Gaudin model. Then we have the corresponding non-degenerate Miura $\mathfrak{g}$-oper (9.28) with trivial monodromy, to which we associate a Miura $\mathfrak{g}'$-oper with trivial monodromy via Proposition 9.6. It has the form (9.24):

$$\partial_z + \sum_{i \in I'} f_i - \frac{\check{\mu}'}{z} + \sum_{\sigma(i_j) = i_j} \frac{\check{\alpha}'_{i_j}}{z - w_j} + \sum_{\sigma(i_j) \neq i_j} \frac{\check{\alpha}'_{i_j} + \check{\alpha}'_{\sigma(i_j)}}{z - w_j} + \chi'. \tag{9.31}$$

We wish to associate to this Miura $\mathfrak{g}'$-oper an eigenvector of the $\mathfrak{g}'$-Gaudin Hamiltonians in $V'_{\mu'}$. There are two ways to do it, which give the same result. The first is to apply the construction of [FFT, Sect. 6.3] but insert at every point w_j such that $\sigma(i_j) \neq i_j$ the vector $e^R_{i_j,-1} e^R_{\sigma(i_j),-1}|0\rangle$ in the corresponding Wakimoto module, rather than $e^R_{i_j,-1}|0\rangle$ (see formula (6.4) in [FFT]). The second way (which is more direct) is to apply formula (9.30) directly to the $\mathfrak{g}'$-oper (9.31). The problem is that if i_j is such that $\sigma(i_j) \neq i_j$, then the corresponding w_j appears twice giving rise to seeming singularities in (9.30). However, since for $\sigma(i) \neq i$ we always have $[f_i, f_{\sigma(i)}] = 0$, it is easy to see that these singularities cancel out, so we do obtain a well-defined vector $\phi'(w_1^{i_1}, \ldots, w_m^{i_m})$ in $V'_{\mu'}$. It then follows that it is an eigenvector of the $\mathfrak{g}'$-Gaudin Hamiltonians with the eigenvalue corresponding to the $\mathfrak{g}'$-oper in $\mathrm{Op}^{\chi'}_{\check{\mu}',0}(\mathfrak{g}')$ represented by (9.31). Moreover, it is a weight vector of the same weight as $\phi(w_1^{i_1}, \ldots, w_m^{i_m})$; namely, $\gamma = \mu - \sum_{j=1}^m \alpha_{i_j}$ (here we identify ${}^L\mathfrak{g}$-weights with σ-invariant $\mathfrak{g}'$-weights).

Now recall that in Section 5.6 we defined the invariant subspace $(V'_{\mu'})^\sigma$ of $V'_{\mu'}$ as the direct sum of the subspaces of $\widehat{\sigma}$-invariant vectors in all weights subspaces of $V'_{\mu'}$ corresponding to σ-invariant weights. Using the explicit formula for $\phi'(w_1^{i_1}, \ldots, w_m^{i_m})$, we obtain that it belongs to $(V'_{\mu'})^\sigma$. We summarize this as follows.

Lemma 9.11. *The vector $\phi'(w_1^{i_1}, \ldots, w_m^{i_m}) \in V'_{\mu'}$ is a well-defined eigenvector of the $\mathfrak{g}'$-Gaudin Hamiltonians corresponding to the $\mathfrak{g}'$-oper in $\mathrm{Op}^{\chi'}_{\check{\mu}',0}(\mathfrak{g}')$ which is the image of the*

$\mathfrak{g}$-oper in $\mathrm{Op}^{\chi}_{\mu,0}(\mathfrak{g})$ represented by (9.28) *under the embedding* (9.14)*. Moreover, it is σ-invariant, so it belongs to $(V'_{\mu'})^{\sigma}$.*

This implies the following statement.

Theorem 9.12. *Suppose that χ is generic, so that Conjecture 9.10 holds. Assume also that the vectors $\phi'(w_1^{i_1},\ldots,w_m^{i_m})$ are non-zero in $V'_{\mu'}$. Then the map $V_\mu \to V'_{\mu'}$ sending $\phi(w_1^{i_1},\ldots,w_m^{i_m}) \in V_\mu$ to $\phi'(w_1^{i_1},\ldots,w_m^{i_m}) \in V'_{\mu'}$ defines an embedding of V_μ into $V'_{\mu'}$ whose image is the subspace $V'_{\mu'}(\chi)$ (see Definition 9.8). Moreover, $V'_{\mu'}(\chi)$ is contained in $(V'_{\mu'})^{\sigma}$.*

We view this result as an analogue of our main Conjecture 5.15.

9.8. **Example.** Let $V'_{\alpha'_{\max}}$ be the adjoint representation of $\mathfrak{g}'$. Its highest weight is the maximal root $\alpha'_{\max}$ of $\mathfrak{g}'$, which is σ-invariant. It is easy to describe the automorphism $\widehat{\sigma} : \mathfrak{g}' \to \mathfrak{g}'$ from Lemma 5.2 (adapted to simple Lie algebras) in this case.

Lemma 9.13. *The automorphism $\widehat{\sigma} : V'_{\alpha'_{\max}} \to V'_{\alpha'_{\max}}$ is equal to σ^{-1}.*

Proof. For $g \in \mathfrak{g}'$, we have $\rho(g)\cdot x = [g,x], \forall x \in W = \mathfrak{g}'$. Hence $\rho_\sigma(g)\cdot x = [\sigma(g),x]$. The operator $\widehat{\sigma}$ is uniquely defined by the equation

$$\widehat{\sigma}\rho(g)\widehat{\sigma}^{-1} = \rho(\sigma(g)), \qquad \forall g \in \mathfrak{g}' \tag{9.32}$$

and the condition that the restriction of $\widehat{\sigma}$ to the highest weight subspace of W is the identity. We claim that $\widehat{\sigma} = \sigma^{-1}$ satisfies these conditions. Indeed, it acts as the identity on the highest weight subspace of $W = \mathfrak{g}'$, and the equation (9.32) is equivalent to

$$\widehat{\sigma}\cdot[\sigma(g),\widehat{\sigma}^{-1}(x)] = [g,x], \forall x \in \mathfrak{g}'.$$

Setting $\widehat{\sigma} = \sigma^{-1}$, we obtain

$$[\sigma(g),\sigma(x)] = \sigma([g,x]),$$

which follows from the fact that σ is an automorphism of $\mathfrak{g}'$. □

Now let $\mathfrak{g}' = A_{2n-1} = \mathfrak{sl}_{2n}$ with its order two automorphism σ. Then $\mathfrak{g} = C_n = sp_{2n}$ and ${}^L\mathfrak{g} = B_n = so_{2n+1}$. The highest weight $\alpha'_{\max}$ of the adjoint representation of $\mathfrak{g}'$ is equal to $\omega'_1 + \omega'_{2n-1}$ in this case. By the above lemma, $\widehat{\sigma} = \sigma^{-1} = \sigma$. Therefore, we have following description of the invariant subspace $(V'_{\omega'_1+\omega'_{2n-1}})^{\sigma}$ (the direct sum of the subspaces of $\widehat{\sigma}$-invariant vectors in all weights subspaces of $V'_{\omega'_1+\omega'_{2n-1}}$ corresponding to σ-invariant weights).

Lemma 9.14. *$(V'_{\omega'_1+\omega'_{2n-1}})^{\sigma}$ is the direct sum of the root subspaces of $V'_{\omega'_1+\omega'_{2n-1}}$ corresponding to the roots*

$$\pm\alpha_n \qquad \text{and} \qquad \pm(\alpha_i+\alpha_{i+1}+\ldots+\alpha_{2n-i-1}+\alpha_{2n-i}), \qquad i=1,\ldots,n-1,$$

as well as the n-dimensional invariant subspace of the zero weight subspace of W, which is the Cartan subalgebra of $\mathfrak{g} = C_n$ inside the Cartan subalgebra of $\mathfrak{g}'$.

Expressing σ-invariant $\mathfrak{g}'$-weights as ${}^L\mathfrak{g}$-weights, we obtain:

Lemma 9.15. *$(V'_{\omega'_1+\omega'_{2n-1}})^{\sigma}$ is isomorphic, as a vector space graded by ${}^L\mathfrak{g}$-weights, to the direct sum of the first fundamental representation $V_{\omega_1} = \mathbb{C}^{2n+1}$ of ${}^L\mathfrak{g} = so_{2n+1}$ and $(n-1)$ copies of the trivial representation of ${}^L\mathfrak{g}$.*

Assuming Conjecture 9.10, we obtain that we can embed V_{ω_1} into $V'_{\omega'_1+\omega'_{2n-1}}$ in such a way that the image is the subspace $V'_{\omega'_1+\omega'_{2n-1}}(\chi)$ (the span of eigenvectors of the $\mathfrak{g}'$-Gaudin Hamiltonians for which $Q_i(z) = Q_{\sigma(i)}(z), i \in I'$). The latter is the analogue of the subspace $W(u)$ from Section 5.3 which we used to define the folded integrable model. Moreover, using Proposition 9.12 we can construct this embedding explicitly, using the Bethe vectors (9.30), and this shows that $V'_{\omega'_1+\omega'_{2n-1}}(\chi)$ is contained in $(V'_{\omega'_1+\omega'_{2n-1}})^\sigma$. The weight subspaces of these two vector spaces coincide for all non-zero weights, but their weight 0 subspaces are different: the former is 1-dimensional and the latter is n-dimensional.

Remark 9.3. At first glance, it may appear that this result is in contradiction with the results of Section 7.2, where we considered a similar example in the case of the quantum affine algebras associated to $\mathfrak{g}' = A_{2n-1}$ and $\mathfrak{g} = C_n$. Namely, we showed that if we take as W the irreducible representation $L(Y_{1,1}Y_{2n-1,1})$ of $U_q(A^{(1)}_{2n-1})$, then its subspace $W(u)$ defined in Section 5.6 is $(2n+2)$-dimensional for generic u. This means that the corresponding subspace in the limit $q \to 1$ should have dimension at least $2n+2$. And yet, we have found that in the case of the adjoint representation of A_{2n-1} the analogous subspace $V'_{\omega'_1+\omega'_{2n-1}}(\chi)$ (which is the image of V_{ω_1} in $V'_{\omega'_1+\omega'_{2n-1}}$) has dimension $2n+1$ for generic χ.

The explanation is that the evaluation representation of $U_q(A^{(1)}_{2n-1})$ corresponding to the adjoint representation of A_{n-1} (of dimension $4n^2-1$) is not $L(Y_{1,1}Y_{2n-1,1})$ but $L(Y_{1,1}Y_{2n-1,q^{2n}})$. The highest monomial of $L(Y_{1,1}Y_{2n-1,q^{2n}})$ is not σ-invariant and therefore we cannot define the operator $\widehat{\sigma}$ and related structures on $L(Y_{1,1}Y_{2n-1,q^{2n}})$. On the other hand, the highest monomial of $L(Y_{1,1}Y_{2n-1,1})$ is σ-invariant, but the dimension of this representation is $4n^2$, i.e. it is greater than the dimension of the adjoint representation $V'_{\omega'_1+\omega'_{2n-1}}$ of A_{n-1} by 1. Hence it is not surprising that the subspace $W(u)$ of $W = L(Y_{1,1}Y_{2n-1,1})$ has dimension greater by 1 than the dimension of $V'_{\omega'_1+\omega'_{2n-1}}(\chi)$. In fact, as shown in Section 7.2, the weight 0 subspace of $W(u)$ is two-dimensional and can be identified with the weight 0 subspace of a 4-dimensional irreducible representation of the $U_q(\widehat{\mathfrak{sl}}_2)$ subalgebra of $U_q(A^{(1)}_{2n-1})$ corresponding to node n of the Dynkin diagram of A_{2n-1}. But the corresponding irreducible representation of the $\mathfrak{sl}_2$ Lie subalgebra corresponding to node n is 3-dimensional and so its weight 0 subspace is 1-dimensional. This is the weight 0 subspace of $V'_{\omega'_1+\omega'_{2n-1}}(\chi)$.

As the result, $W(u)$ is actually isomorphic, as a vector space graded by weights of ${}^L\mathfrak{g} = B_n$, to an irreducible representation of $U_q({}^L\widehat{\mathfrak{g}})$, where ${}^L\widehat{\mathfrak{g}} = D^{(2)}_{n+1}$, whereas $V'_{\omega'_1+\omega'_{2n-1}}(\chi)$ is isomorphic to an irreducible representation of ${}^L\mathfrak{g} = B_n$. □

Remark 9.4. The above example gives us a nice illustration of how the *affine* Langlands duality of the folded integrable model becomes in the limit $q \to 1$ the finite-dimensional Langlands duality of the Gaudin model. The point is that when we take the $q \to 1$ limit, we restrict finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ to $U_q(\mathfrak{g})$ and then take the limit $q \to 1$ (the latter is an equivalence of categories if q is generic). The smallest finite-dimensional representation W of $U_q(\widehat{\mathfrak{g}})$ with σ-invariant highest monomial that has μ as the highest ${}^L\mathfrak{g}$-weight is usually isomorphic, as a $U_q(\mathfrak{g})$-module, to the direct sum of V_μ

and many other irreducible representations.[7] So it is not surprising that the intersection of an irreducible representation $M(W)$ of $U_q({}^L\widehat{\mathfrak{g}})$, which is embedded into W according to Conjecture 5.15,(2), with V_μ would be an irreducible representation of $U_q({}^L\mathfrak{g})$.

This is closely related to another seeming paradox. According to a result of [FFRy] that we used in this section, every eigenspace in the space of states of the Gaudin model is one-dimensional (if the Gaudin Hamiltonians are not diagonalizable, this means that there can be at most one Jordan block for every joint eigenvalue). In particular, for every irreducible representation V_μ of ${}^L\mathfrak{g}$, the joint eigenspace of the algebra $\mathcal{A}_\chi$ is one-dimensional for all joint eigenvalues. On the other hand, we expect that the analogous spaces in the folded integrable model for generic q can have dimensions greater than one. The explanation is that components other than V_μ in the decomposition of W as a representation of $U_q(\mathfrak{g})$ may well contain (in the limit $q \to 1$) additional eigenvectors of $\mathcal{A}_\chi$. □

9.9. **Conclusion.** It follows that, somewhat surprisingly, we can realize the ${}^L\mathfrak{g}$-Gaudin model inside the $\mathfrak{g}'$-Gaudin model by embedding the space of states (9.11) of the former into the space of states (9.26) of the latter in such a way that on the dual side this corresponds to the embedding of the Miura $\mathfrak{g}$-opers (9.19) into the Miura $\mathfrak{g}'$-opers (9.7). This is an intriguing consequence of the Langlands duality in the Gaudin models discovered in [FFR] and the results of [FFRy], which deserves further study.

The embedding of the Miura $\mathfrak{g}$-opers (9.19) into the Miura $\mathfrak{g}'$-opers (9.7) means that the Baxter polynomials $Q_i(z)$ and $Q_{\sigma(i)}(z)$ (see formula (9.9)) associated to a $\mathfrak{g}'$-oper obtained this way will be equal. Thus, the folding of the $\mathfrak{g}'$-Gaudin model is equivalent to the ${}^L\mathfrak{g}$-Gaudin model. This statement may be viewed as a $q \to 1$ limit of Conjecture 5.15, so it provides support for this conjecture.

10. Appendix: Toward constructing a folded version of the qKZ equations for non-simply laced Lie algebras

The qKZ equations [FrR] for $U_q(\widehat{\mathfrak{g}})$ (with the twist parameter u, which is an element of the Cartan subgroup H as above) can be written in the form

$$\Psi(z_1,\dots,pz_j,\dots,z_N) = R_{j,j-1}^{V_jV_{j-1}}(pz_j/z_{j-1})\dots R_{j,1}^{V_jV_1}(pz_j/z_1)\cdot u_j\cdot R_{j,N}^{V_jV_N}(z_j/z_N)\dots R_{j,j+1}^{V_jV_{j+1}}(z_j/z_{j+1})\Psi(z_1,\dots,z_j,\dots,z_N). \tag{10.1}$$

Here each V_i denotes an irreducible finite-dimensional representation of $U_q(\widehat{\mathfrak{g}})$, $V_i(z_i)$ is its shift by the spectral parameter z_i, and $R_{j,i}^{V_jV_i}(z_j/z_i)$ is the R-matrix acting on $V_j(z_j)\otimes V_i(z_i)$ normalized so that it acts as the identity on $v_j\otimes v_i$, where v_j and v_i are highest weight vectors in $V_j(z_j)$ and $V_i(z_i)$, respectively. Also, u_j denotes $u|_{V_j}$.

Let us denote the operator on the right hand side of the jth equation by $K_j(p)$.

The critical level limit corresponds to $p \to 1$. In this limit we have

$$K_j(1) = R_{j,j-1}^{V_jV_{j-1}}(z_j/z_{j-1})\dots R_{j,1}^{V_jV_1}(z_j/z_1)\cdot u_j\cdot R_{j,N}^{V_jV_N}(z_j/z_N)\dots R_{j,j+1}^{V_jV_{j+1}}(z_j/z_{j+1}). \tag{10.2}$$

Recall that for an auxiliary representation V of $U_q(\widehat{\mathfrak{g}})$ we have the transfer matrix $t_V(z,u)$. These transfer-matrices commute with each other for a fixed u and different V and z. In

[7] In the case of $\mathfrak{g} = \mathfrak{sl}_n$, every finite-dimensional irreducible representation of $U_q(\mathfrak{sl}_n)$ can be lifted to $U_q(\widehat{\mathfrak{sl}}_n)$, but the highest monomial of the resulting representation of $U_q(\widehat{\mathfrak{sl}}_n)$ is not σ-invariant, see the above example. The representations with σ-invariant monomials are generally much larger.

what follows, we will use the same notation $t_V(z,u)$ for the action of the transfer-matrix $t_V(z,u)$ on the tensor product $V_N(z_N)\otimes\ldots\otimes V_1(z_1)$. Thus,

$$t_V(z,u)=\mathrm{Tr}_a(u_a R^{VV_N}_{a,N}(z/z_N)\ldots R^{VV_1}_{a,1}(z/z_1)), \tag{10.3}$$

where the subscript a indicates the auxiliary representation $V(z)$.

Proposition 10.1. *Suppose that V_j is such that up to a scalar, $R^{V_jV_j}(1)=P$, the permutation operator on $V_j(z)\otimes V_j(z)$ sending $x\otimes y$ to $y\otimes x$. Then $K_j(1)=t_{V_j}(z_j,u)$.*

Proof. If $R^{V_jV_j}(1)=P$, then formula (10.3) with $V(z)=V_j(z_j)$ becomes

$$t_V(z,u)= \mathrm{Tr}_a(u_a R^{V_jV_N}_{a,N}(z_j/z_N)\ldots R^{V_jV_{j+1}}_{a,j+1}(z_j/z_{j+1})P_{aj}R^{V_jV_{j-1}}_{a,j-1}(z_j/z_{j-1})\ldots R^{V_jV_1}_{a,1}(z_j/z_1)). \tag{10.4}$$

Using the identify $A_aP_{aj}=P_{aj}A_j$, we rewrite the RHS of (10.4) as

$$\mathrm{Tr}_a(P_{aj}u_jR^{V_jV_N}_{j,N}(z_j/z_N)\ldots R^{V_jV_{j+1}}_{j,j+1}(z_j/z_{j+1})R^{V_jV_{j-1}}_{a,j-1}(z_j/z_{j-1})\ldots R^{V_jV_1}_{a,1}(z_j/z_1)). \tag{10.5}$$

Next, using the cyclic property of the trace, we rewrite (10.5) as

$$\mathrm{Tr}_a(R^{V_jV_{j-1}}_{a,j-1}(z_j/z_{j-1})\ldots R^{V_jV_1}_{a,1}(z_j/z_1)P_{aj}u_jR^{V_jV_N}_{j,N}(z_j/z_N)\ldots R^{V_jV_{j+1}}_{a=j,j+1}(z_j/z_{j+1})). \tag{10.6}$$

Using formula $A_aP_{aj}=P_{aj}A_j$ again, we rewrite (10.6) as

$$\mathrm{Tr}_a(P_{aj}R^{V_jV_{j-1}}_{j,j-1}(z_j/z_{j-1})\ldots R^{V_jV_1}_{j,1}(z_j/z_1)u_jR^{V_jV_N}_{j,N}(z_j/z_N)\ldots R^{V_jV_{j+1}}_{a=j,j+1}(z_j/z_{j+1})). \tag{10.7}$$

In the last formula, the only operator depending on the auxiliary space is P_{aj} and its trace over the auxiliary space is the identity operator on $V_j(z_j)$. Hence (10.7) is equal to $K_j(1)$. □

Next, we discuss under what conditions $R^{VV}(1)=P$.

Lemma 10.2. *Let V be an irreducible representation of $U_q(\widehat{\mathfrak{g}})$ such that $V\otimes V$ is also irreducible. Then $R^{VV}(1)=P$.*

Proof. It follows from the definition that $P\circ R^{VV}(1)$ is an intertwining operator $V\otimes V\to V\otimes V$. If $V\otimes V$ is irreducible, then by Schur's lemma it is a scalar operator. Under our normalization, it then has to be the identity operator, and hence $R^{VV}(1)=P$. □

Note that a representation V satisfying the condition of Lemma 10.2 is called *real* in [HL1]. Not all irreducible representations of $U_q(\widehat{\mathfrak{g}})$ are real, as shown in [L], see [HL1, Sect. 13.6]. However, there is a large class of real representations: Kirillov–Reshetikhin modules.

Proposition 10.3. *Let V be any Kirillov–Reshetikhin module over $U_q(\widehat{\mathfrak{g}})$. Then $V\otimes V$ is irreducible.*

Proof. There are several possible arguments. It follows from [C] that $V\otimes V$ is cyclic and cocyclic, and hence is irreducible. The statement also follows from the fact that the square of the q-character of a Kirillov–Reshetikhin module has a unique dominant monomial; namely, its highest monomial. The latter follows from the results of [HL2, Sect. 3.2.2]. In addition, the statement has been established by a different method in [KKOP]. □

Note however that even if $V \otimes V$ is reducible, we may still have $R^{VV}(1) = P$. It would be interesting to describe all irreducible representations V of $U_q(\widehat{\mathfrak{g}})$ satisfying the condition $R^{VV}(1) = P$.

In any case, the above results readily imply that in the case when all V_j's are Kirillov–Reshetikhin modules, the operators $K_j(1), j = 1, \ldots, N$, are commuting Hamiltonians of the XXZ-type integrable model associated to $U_q(\widehat{\mathfrak{g}})$. It is in this sense that we say that one recovers this integrable model in the critical level limit of the qKZ equations.

Similarly, under the above condition on V_j, the operator $K_j(p)$ is the transfer-matrix $t_{V_j}(z_j, u)$ acting on

$$V_N(z_N) \otimes \ldots \otimes V_j(z_j) \otimes V_{j-1}(z_{j-1}p^{-1}) \otimes V_1(z_1 p^{-1}).$$

However, because of the shifts by p^{-1} these operators do not commute with each other if $p \neq 1$.

Now suppose that $\mathfrak{g}$ is a non-simply laced simple Lie algebra. We would like to construct a "folded qKZ system" such that in the critical level limit the operators on the right hand side become the Hamiltonians of the *folded quantum integrable model* described in this paper. This means, in particular, that they should correspond to transfer-matrices of $U_q(\widehat{\mathfrak{g}'})$ rather than $U_q(\widehat{\mathfrak{g}})$. Thus, each $V_i(z_i)$ should be a representation of $U_q(\widehat{\mathfrak{g}'})$.

Unfortunately, naive attempts to construct this folded qKZ system appear to fail:

(1) For each $V_i(z_i)$, we have its subspace $(V_i(z_i))(u)$ defined as above and we can take the tensor product of these subspaces, $\otimes_{i=1}^N (V_i(z_i))(u)$. However, it is not clear why this subspace would be preserved by the operators K_j or their p-deformed versions $K_j(p)$ appearing on the RHS of (10.1).

(2) We take the subspace $V(u)$ of the entire tensor product $V = \otimes_{i=1}^N V_i(z_i)$. According to Conjecture 5.15,(ii), it contains a subspace isomorphic to a representation $M(V)$ of $U_q({}^L\widehat{\mathfrak{g}})$. Moreover, since the algebra of transfer-matrices of $U_q(\widehat{\mathfrak{g}'})$ commutes with the Baxter operators $Q_j(z), j \in I'$, it follows that all transfer-matrices of $U_q(\widehat{\mathfrak{g}'})$ preserve this subspace $V(u)$. In particular, the operators K_j given by formula (10.2), being the transfer-matrices of $U_q(\widehat{\mathfrak{g}'})$, preserve $V(u)$. But the problem is that on the right hand side of these qKZ equations we have the operators $K_j(p)$ with $p \neq 1$, which are the transfer matrices acting on the tensor product of representations in some of which (namely, the ones with $i = 1, \ldots, j-1$) there is a multiplicative shift in the spectral parameter by p^{-1}. It is not clear why these operators should preserve $V(u)$ (where V is the tensor product of the representations $V_i(z_i)$ without any shift by p^{-1}).

Hence at the moment it is unclear to us how to fold the qKZ equations for $U_q(\widehat{\mathfrak{g}'})$ in such a way that in the critical level limit we recover the commuting Hamiltonians of the folded quantum integrable model associated to $U_q(\widehat{\mathfrak{g}})$.

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