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Abstract

We consider the semilinear wave equation

$$\partial_t^2 u - \Delta u = f(u), \quad (x,t) \in \mathbb{R}^N \times [0,T), \qquad (1)$$

with $f(u) = |u|^{p-1}u \log^a(2+u^2)$, where $p > 1$ and $a \in \mathbb{R}$, with subconformal power nonlinearity. We will show that the blow-up rate of any singular solution of (1) is given by the ODE solution associated with (1), The result in one space dimension, has been proved in [28]. Our goal here is to extend this result to higher dimensions.

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Keywords: Semilinear wave equation, blow-up, non scaling invariance, higher dimensions.

1 Introduction

1.1 Motivation of the problem

This paper is devoted to the study of blow-up solutions for the following semilinear wave equation:

$$\begin{cases} \partial_t^2 u = \Delta u + f(u), \qquad (x,t) \in \mathbb{R}^N \times [0,T), \\ \\ u(x,0) = u_0(x) \in H^1_{loc,u}(\mathbb{R}^N), \qquad \partial_t u(x,0) = u_1(x) \in L^2_{loc,u}(\mathbb{R}^N), \end{cases} \qquad (1.1)$$

where $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$ with focusing nonlinearity f defined by:

$$f(u) = |u|^{p-1}u \log^a(2+u^2), \quad p > 1, \quad a \in \mathbb{R}. \qquad (1.2)$$

The spaces $L^2_{loc,u}(\mathbb{R}^N)$ and $H^1_{loc,u}(\mathbb{R}^N)$ are defined by

$$L^2_{loc,u}(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{R} / \sup_{d\in\mathbb{R}^N} \Big(\int_{|x-d|\le 1} |u(x)|^2 dx \Big) < +\infty\},$$

and

$$H^1_{loc,u}(\mathbb{R}^N) = \{u \in L^2_{loc,u}(\mathbb{R}^N), |\nabla u| \in L^2_{loc,u}(\mathbb{R}^N)\}.$$

We assume in addition that $p > 1$ and if $N \ge 2$, we further assume that

$$p < p_c \equiv 1 + \frac{4}{N-1}. \tag{1.3}$$

When $a \neq 0$, the nonlinearity in (1.2) is not homogeneous, which means that equation (1.1) is not scale invariant. This is precisely our challenge in this paper, particularly in higher dimensions, since we handled the one dimensional case in [28].

Semilinear wave equations with a nonlinearity showing a logarithmic factor have been introduced in various nonlinear physical models, for instance in the context of nuclear physics, wave mechanics, optics, geophysics etc ... see e.g. [5, 6].

The defocusing case has been studied in the mathematical literature and the first results are due to Tao [49] where the author proved a global well-posedness and scattering result for the three dimensional nonlinear wave equation $\partial_t^2 u = \Delta u - |u|^4 u \log(2+u^2)$, in the radial case. See also the work of Shih [48], where the method is refined to treat $\partial_t^2 u = \Delta u - |u|^4 u \log^c(2+u^2)$, for any $c \in (0, \frac{4}{3})$. Later, Roy extends in [44] the results (global well-posedness and scattering) to solutions of the log-log-supercritical equation $\partial_t^2 u = \Delta u - |u|^4 u \log^c\big(\log(10+u^2)\big)$, for c small, without any radial assumption. Hoping to extend the validity of our argument to the conformal case ($p = p_c$) in some forthcoming work, we may see the case $a > 0$ of (1.2) as a further step in the understanding of blow-up dynamics in the superconformal case related to equation (1.8) below.

Let us mention that the blow-up question for the semilinear heat equation $\partial_t u = \Delta u + |u|^{p-1} u \log^a(2+u^2)$ is studied by Duong-Nguyen-Zaag in [20]. More precisely, they construct for this equation a solution which blows up in finite time T, only at one blow-up point a, according to the following asymptotic dynamics:

$$u(x,t) \sim \phi(t)\Big(1 + \frac{(p-1)|x-a|^2}{4p(T-t)|\log(T-t)|}\Big)^{-\frac{1}{p-1}}, \qquad as\ t \to T, \tag{1.4}$$

where $\phi(t)$ is is the unique positive solution of the ODE

$$\phi' = |\phi|^{p-1}\phi \log^a(2+\phi^2), \qquad \lim_{t\to T} \phi(t) = +\infty. \tag{1.5}$$

Given that we have the same expression in the pure power nonlinearity case ($g(u) = |u|^{p-1}u$) with $\phi(t)$ replaced by $\kappa(T-t)^{-\frac{1}{p-1}}$ (see [11]), we see that the effect of the nonlinearity is all encapsulated in the ODE (1.5).

Equation (1.1) is well-posed in $H^1_{loc,u} \times L^2_{loc,u}$. This follows from the finite speed of propagation and the well-posedness in $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. The existence of blow-up

solutions $u(t)$ of (1.1) follows from ODE techniques or the energy-based blow-up criterion by Levine [32] (see also [33, 47, 50]). More blow-up results can be found in Caffarelli and Friedman [12, 13], Kichenassamy and Littman [29, 30]. Numerical simulations of blow-up are given by Bizoń *et al.* (see [7, 8, 9, 10]).

If u is an arbitrary blow-up solution of (1.1), we define a 1-Lipschitz curve $\Gamma = \{(x, T(x))\}$ such that the domain of definition of u is written as

$$D = \{(x,t) \mid t < T(x)\}. \tag{1.6}$$

The justification of this fact can be found in Alinhac [1], where such a set is referred to as the *maximal influence domain*. Let us motivate that particular form, and refer the interested reader to [1] for details.
Recall first that equation (1.1) is well-posed in $H^1 \times L^2(\mathbb{R}^N)$, i.e. *globally* in space (see for example Shatah and Struwe [46]). Using the finite speed of propagation and a cut-off technique, we then solve the solution *locally* in space, as Azaiez, Masmoudi and Zaag do in [3] in a similar setting. More precisely, for any $x_0 \in \mathbb{R}^N$, we derive a solution which is defined in K_{x_0,t_0}, the backward light cone defined by

$$K_{x_0,t_0} = \{(\xi,\tau) \in \mathbb{R}^N \times [0,t_0) \mid |\xi - x_0| < t_0 - \tau\}.$$

Denoting by $T(x_0)$ the supremum of such a t_0, we fall into one of these two situations:
- either for some $x_1 \in \mathbb{R}^N$, $T(x_1) = +\infty$, and in that case, the solution if defined for all $x \in \mathbb{R}^N$ and $t \geq 0$. In other words, $T(x_0) = +\infty$, for any $x_0 \in \mathbb{R}^N$;
- or, for all $x_1 \in \mathbb{R}^N$, $T(x_1) < +\infty$, and this is the case we are interested in. In particular, the domain of definition of u is the union of backward light cones of the type K_{x_0,t_0}, and such a union is clearly of the form (1.6) with $x \mapsto T(x)$ is $1-$Lipschitz. See also Sasaki [45] for a related definition of the domain of definition in some similar setting.

The time $\bar{T} = \inf_{x\in\mathbb{R}^N} T(x)$ and Γ are called the blow-up time and the blow-up graph of u. A point x_0 is non characteristic if there are

$$\delta_0 \in (0,1) \text{ and } t_0 < T(x_0) \text{ such that } u \text{ is defined on } \mathscr{C}_{x_0,T(x_0),\delta_0} \cap \{t \geq t_0\} \tag{1.7}$$

where $\mathscr{C}_{\bar{x},\bar{t},\bar{\delta}} = \{(x,t) \mid t < \bar{t} - \bar{\delta}|x - \bar{x}|\}$. If not, x_0 is said to be characteristic.

In this paper, we study the blow-up rate of any singular solution of (1.1). Before going on, it is necessary to mention that the blow-up rate in the case with pure power nonlinearity

$$\partial_t^2 u = \Delta u + |u|^{p-1}u, \ \ (x,t) \in \mathbb{R}^N \times [0,T), \tag{1.8}$$

was studied by Merle and Zaag in [34, 35, 36]. More precisely, they proved that if u is a solution of (1.8) with blow-up graph $\Gamma : \{x \mapsto T(x)\}$ and x_0 is a non-characteristic point, then, for all $t \in [\frac{3T(x_0)}{4}, T(x_0)]$,

$$0 < \varepsilon_0(p) \leq (T(x_0) - t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0) - t)^{\frac{N}{2}}} \tag{1.9}$$

$$+(T(x_0)-t)^{\frac{2}{p-1}+1}\Big(\frac{\|\partial_t u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{\frac{N}{2}}}+\frac{\|\partial_x u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{\frac{N}{2}}}\Big)\leq K_0,$$

where the constant K_0 depends only on p and on an upper bound on $T(x_0)$, $1/T(x_0)$, $\delta_0(x_0)$, together with the norm of initial data in $H^1_{loc,u}(\mathbb{R}^N)\times L^2_{loc,u}(\mathbb{R}^N)$. Namely, the blow-up rate of any singular solution of (1.8) is given by the solution of the associated ODE $u''=|u|^{p-1}u$. Note that this result about the blow-up rate is valid in the subconformal and conformal case ($1<p\leq p_c$).

In a series of papers, Merle and Zaag [37, 38, 40, 41] (see also Côte and Zaag [14]) give a full picture of blow-up for solutions of equation (1.8) in one space dimension. Among other results, Merle and Zaag proved that characteristic points are isolated and that the blow-up set $\{(x,T(x))\}$ is $\mathscr{C}^1$ near non-characteristic points and corner-shaped near characteristic points. In higher dimensions, the method used in the one-dimensional case no longer holds because there is no classification of selfsimilar solutions of equation (1.8) in the energy space. However, in the radial case outside the origin, Merle and Zaag reduce the blow up description to the one-dimensional case with perturbation and obtain the same results as for $N=1$ (see [39] and also the extension by Hamza and Zaag in [27] to the Klein-Gordon equation and other damped lower-order perturbations of equation (1.8)). Later, Merle and Zaag could address the higher dimensional case in the subconformal case and prove the stability of the explicit selfsimilar solution with respect to the blow-up point and initial data (see [42, 43]). Considering the behavior of radial solutions at the origin, Donninger and Schörkhuber were able to prove the stability of the ODE solution $u(t)=\kappa_0(p)(T-t)^{-\frac{2}{p-1}}$ in the light cone with respect to small perturbations in initial data, in a stronger topology (see [16, 17, 18, 19]). Their approach is based in particular on a good understanding of the spectral properties of the linearized operator in selfsimilar variables, operator which is not self-adjoint. Later, thanks to suitable Strichartz estimates for the critical wave equation in similarity variables, Donninger proved in [15] the stability of the solution of the ODE with respect to small perturbations in initial data, in the energy space. Let us also mention that Killip, Stoval and Vişan proved in [31] that in superconformal and Sobolev subcritical range, an upper bound on the blow-up rate is available. This was further refined by Hamza and Zaag in [26].

In [24, 25], using a highly non-trivial perturbative method, we could obtain the blow-up rate for the Klein-Gordon equation and more generally, for equation

$$\partial_t^2 u=\Delta u+|u|^{p-1}u+f(u)+g(\partial_t u),\ \ (x,t)\in\mathbb{R}^N\times[0,T), \tag{1.10}$$

under the assumptions $|f(u)|\leq M(1+|u|^q)$ and $|g(v)|\leq M(1+|v|)$, for some $M>0$ and $q<p\leq\frac{N+3}{N-1}$. In fact, we proved a similar result to (1.9), valid in the subconformal and conformal case. Let us also mention that in [21, 22, 23], the results obtained in [24, 25] were extended by Hamza and Saidi to the strongly perturbed equation (1.10) with $|f(u)|\leq M(1+|u|^p\log^{-a}(2+u^2))$, for some $a>1$, though keeping the same condition in g. Very recently, Azaiez and Zaag derived in [4] the blow-up rate for equations of the type

$$\partial_t^2 u=a(x)(\partial_x^2 u+\frac{N-1}{x}\partial_x u)+b(x)|u|^{p-1}u+f(u)+g(x,t,\partial_x u,\partial_t u)$$

where $|f(u)| \le M(1+|u|^q)$ with $q<p$, $|g(x,t,v,z)| \le M(1+|v|\sqrt{a(x)}+|z|)$, for some $M>0$, and $a(x)$ is typically $|x|^\alpha$ with α enjoying a infinite number of values converging to 2.

In the previous works [21, 22, 23, 24, 25], we consider a class of perturbed equations where the nonlinear term is equivalent to the pure power $|u|^{p-1}u$ and we obtain the estimate (1.9). This is due to the fact that the dynamics are governed by the ODE equation: $u'' = |u|^{p-1}u$ and not influenced by perturbative terms. Furthermore, our proof remains (non trivially) perturbative with respect to the homogeneous PDE (1.8), which is scale invariant.

This leaves unanswered an interesting question: Is the scale invariance property crucial in deriving the blow-up rate?

In fact we *had the impression* that the answer was "yes", since the scaling invariance induces in similarity variables a PDE which is autonomous in the unperturbed case (1.8), and asymptotically autonomous in the perturbed case (1.10).

In this paper we *prove* that the answer is "no" from the example of the PDE (1.1) with the non homogeneous nonlinearity (1.2). In fact, our situation is different from (1.8) and (1.10), in the sense that the term $|u|^{p-1}u\log^a(2+u^2)$ is playing a fundamental role in the dynamics of the blow-up solution of (1.1). More precisely, we obtain an analogous result to (1.9) but with a logarithmic correction as shown in (1.27) below. In fact, the blow-up rate is given by the solution of the following ordinary differential equation:

$$v_T''(t) = |v_T(t)|^{p-1}v_T(t)\log^a\big(v_T^2(t)+2\big), \quad v(T)=\infty, \tag{1.11}$$

which satisfies

$$v_T(t) \sim \kappa_a \psi_T(t), \text{ as } t \to T, \quad \text{where} \quad \kappa_a = \left(\frac{2^{1-2a}(p+1)}{(p-1)^{2-a}}\right)^{\frac{1}{p-1}}, \tag{1.12}$$

and

$$\psi_T(t) = (T-t)^{-\frac{2}{p-1}}(-\log(T-t))^{-\frac{a}{p-1}} \tag{1.13}$$

(see Lemma A.2 in [28]).

Finally, let us remark that, following the analysis of Hamza in Saidi in [22], our result holds for this more general equation

$$\partial_t^2 u = \Delta u + \tilde{f}(u) + g(x,t,u,\partial_t u, \nabla u), \ \ (x,t) \in \mathbb{R}^N \times [0,T), \tag{1.14}$$

where $\tilde{f} : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^{2N+3} \to \mathbb{R}$ are $\mathscr{C}^1$ functions which satisfy the following conditions:

$$|g(x,t,u,v,w)| \le M_0(1+|u|^{\frac{p+1}{2}}+|v|+|w|), \tag{1.15}$$

and

$$\Big|\int_0^u \tilde{f}(y)\mathrm{d}y - \frac{u\tilde{f}ux)}{p+1} + \frac{2au\tilde{f}(u)}{(p+1)^2\log(2+u^2)}\Big| \le M_0\Big(1+|u|^{p+1}\log^{b-1}(2+u^2)\Big), \tag{1.16}$$

for some $M_0 > 0$, $p > 1$, $a \in \mathbb{R}$ and $b < a$. In fact, under these assumptions, we can construct a suitable Lyapunov functional for the equation (1.14) under hypothesis (1.15) and (1.16). Then, we can prove a similar result to (1.27) with the necessary modifications. However, In order to keep our analysis clear, we restrict our analysis to the problem related to equation (1.1).

1.2 Strategy of the proof

Going back to the equation under study in this paper (see (1.1) and (1.2)), we introduce the following similarity variables, defined for all $x_0 \in \mathbb{R}^N$, T_0 such that $0 < T_0 \leq T(x_0)$ by:

$$y = \frac{x - x_0}{T_0 - t}, \quad s = -\log(T_0 - t), \quad u(x,t) = \psi_{T_0}(t) w_{x_0,T_0}(y,s). \tag{1.17}$$

One may think that it would be more natural to replace $\psi_{T_0}(t)$ by $v_{T_0}(t)$ in this definition, since the latter is an exact solution of the ODE (1.11), unlike the former. That might be a good idea, however, as $v_{T_0}(t)$ has no explicit expression, the calculations will immediately become too complicated. For that reason, we preferred to replace the non-explicit $v_{T_0}(t)$ by its explicit equivalent $\psi_{T_0}(t)$ in (1.13). The fact that the latter is only an approximate solution and not an exact solution of (1.11) will have no effect on our analysis.

From (1.1), the function w_{x_0,T_0} (we write w for simplicity) satisfies the following equation for all $y \in B$ and $s \geq \max(-\log T_0, 1)$, where $B \equiv B(0,1)$ stands for the unit ball of $\mathbb{R}^N$ and throughout the paper:

$$\begin{aligned} \partial_s^2 w = {} & \frac{1}{\rho} \operatorname{div}\left(\rho \nabla w - \rho (y.\nabla w) y\right) + \frac{2a}{(p-1)s} y.\nabla w - \frac{2p+2}{(p-1)^2} w + \gamma(s) w \\ & - \Big(\frac{p+3}{p-1} - \frac{2a}{(p-1)s}\Big)\partial_s w - 2y.\nabla \partial_s w + e^{-\frac{2ps}{p-1}} s^{\frac{a}{p-1}} f(\phi(s) w), \end{aligned} \tag{1.18}$$

with $\rho(y) = (1 - |y|^2)^\alpha$,

$$\alpha = \frac{2}{p-1} - \frac{N-1}{2} > 0, \tag{1.19}$$

$$\gamma(s) = \frac{a(p+5)}{(p-1)^2 s} - \frac{a(p+a-1)}{(p-1)^2 s^2}, \tag{1.20}$$

and

$$\phi(s) = e^{\frac{2s}{p-1}} s^{-\frac{a}{p-1}}. \tag{1.21}$$

In the new set of variables (y,s), the behavior of u as $t \to T_0$ is equivalent to the behavior of w as $s \to +\infty$. Moreover, if $T_0 = T(x_0)$, then we simply write w_{x_0} instead of $w_{x_0,T(x_0)}$.

The equation (1.18) will be studied in the Hilbert space $\mathscr{H}$

$$\mathscr{H} = \Big\{(w_1, w_2), | \int_B \Big(\big(w_1^2 + |\nabla w_1|^2 - |y.\nabla w_1|^2\big) + w_2^2\Big) \mathrm{d}y < +\infty\Big\}.$$

As in the pure power case (1.8) and the perturbed case (1.10), the construction of a Lyapunov functional in similarity variables was the starting point of our strategy. In

the present case (1.1), we adopt the same strategy. We were successful in implementing that in [28], however, only in one space dimension. Indeed, our method in [28] breaks down in higher dimensions. Let us briefly explain in the following the method used in [28] and how it breaks down in higher dimensions, giving sense to the present work.

The first step in [28] consists in the introduction of a functional associated to equation (1.18) which satisfies the following differential inequality:

$$\frac{d}{ds}h(s) \leq -\alpha \int_B (\partial_s w)^2 \frac{\rho(y)}{1-|y|^2}\mathrm{d}y + \frac{C}{s}h(s),$$

where α is defined in (1.19) and w is the solution of (1.18). Thanks to the above-mentioned functional, we easily derived a polynomial (in s) bound for the $H^1 \times L^2(B)$ norm of the solution of (1.18) (in **space-time** averages). More precisely, we obtain the estimates (2.1), (2.2) and (2.3) below. Let us recall that the nonlinear term $f(u)$ given by (1.2) is not a pure power. This is why the strategy used to remove the time averages in the case of pure power in Merle and Zaag [34, 35, 36], and naturally implemented in our previous papers [21, 22, 23, 24, 25] in the perturbative cases, breaks down in higher dimensions. Indeed, this method is somehow based on some interpolation results in Sobolev spaces, and some critical Gagliardo-Nirenberg estimates. However, in one dimension, the strategy used to remove the time average works since it is based on the embedding $H^1(\mathbb{R} \times [-\log T, +\infty)) \hookrightarrow L^q(\mathbb{R} \times [-\log T, +\infty))$, for any $q > 1$. Using the polynomial (in s) bound for the $H^1_{loc,u}(\mathbb{R})-$norm of the solution of (1.18) and the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, we derive a Lyapunov functional for equation (1.18) in one space dimension, which is a crucial step to obtain the optimal estimate.

Since the embedding of H^1 into L^q for any $q > 1$ is specific to dimension $1+1$ and doesn't hold in dimension $N+1$, the higher dimensional case requires new ideas, which we explain in the following.

First, we recall in (2.1), (2.2) and (2.3) the rough polynomial (in s) bound (in space-time averages) on the solution near any non characteristic point in similarity variables, was proved and stated in any dimensions in the subconformal case ($p < p_c$). Moreover, we can somehow reduce to the pure power case, up to an ε perturbation, as we write in the elementary estimates on the nonlinear term given below in (A.4), (A.7) and (A.8). In fact, by exploiting these estimates, we prove an improved version of the estimates (2.1) and (2.2), where we remove the time average. Then, using these new estimates, the embedding of $H^1(\mathbb{R}^N)$ in $L^{2^*}(\mathbb{R}^N)$ if $N \geq 3$ and in $L^q(\mathbb{R}^N)$ for any $q \geq 2$ if $N = 2$, together with the structure of the nonlinear term, we end up with the construction of a functional $g(s)$ which satisfies the following differential inequality:

$$\frac{d}{ds}g(s) \leq -\alpha \int_B (\partial_s w)^2 \frac{\rho(y)}{1-|y|^2}\mathrm{d}y + \frac{C}{s^{\frac{3}{2}}}g(s) + \frac{C}{s^{\frac{7}{4}}}, \tag{1.22}$$

where α is defined in (1.19) and w is the solution of (1.18). Naturally, by (1.22), we easily derive a Lyapunov functional for equation (1.18), valid in any dimensions in the subconformal case, and this is our main contribution in this work. With this Lyapunov functional at hand, the adaptation of the interpolation strategy from our previous papers works straightforwardly.

1.3 Statement of the results

To state our main result, we start by introducing the following functionals:

$$E(w(s),s)=\int_B\Big(\frac{1}{2}(\partial_s w)^2+\frac{1}{2}(|\nabla w|^2-(y.\nabla w)^2)+\frac{p+1}{(p-1)^2}w^2 - e^{-\frac{2(p+1)s}{p-1}}s^{\frac{2a}{p-1}}F(\phi w)\Big)\rho(y)\mathrm{d}y, \tag{1.23}$$

$$L_0(w(s),s)=E(w(s),s)-\frac{1}{s\sqrt{s}}\int_B \partial_s w w\rho(y)\mathrm{d}y, \tag{1.24}$$

where

$$F(u)=\int_0^u f(v)\mathrm{d}v=\int_0^u |v|^{p-1}v\log^a(v^2+2)\mathrm{d}v. \tag{1.25}$$

Moreover, for all $s\geq \max(1,-\log T_0)$, we define the functional

$$L(w(s),s)=\exp\Big(\frac{p+3}{\sqrt{s}}\Big)L_0(w(s),s)+\theta s^{-\frac{3}{4}}, \tag{1.26}$$

where θ is a sufficiently large constant that will be determined later. We will show that the functional $L(w(s),s)$ is a decreasing functional of time for equation (1.18), provided that s is large enough. Clearly, by (1.24) and (1.26), the functional $L(w(s),s)$ is a small perturbation of the "natural" energy $E(w(s),s)$.

Here is the statement of our main theorem in this paper.

Theorem 1. *Let $N\geq 2$, $1<p<1+\frac{4}{N-1}$ and $a\in\mathbb{R}$. Let f defined by: $f(u)=|u|^{p-1}u\log^a(2+u^2)$. Consider u a solution of (1.1) with blow-up graph $\Gamma:\{x\mapsto T(x)\}$, and x_0 a non characteristic point. Then there exists $t_1(x_0)\in[0,T(x_0))$ such that, for all $T_0\in(t_1(x_0),T(x_0)]$, for all $s\geq -\log(T_0-t_1(x_0))$, we have*

$$L(w(s+1),s+1)-L(w(s),s)\leq -\alpha\int_s^{s+1}\int_B(\partial_s w)^2\frac{\rho(y)}{1-|y|^2}\mathrm{d}y\mathrm{d}\tau,$$

where $w=w_{x_0,T_0}$ is defined in (1.17).

Remark 1.1. Since the existence of a Lyapunov functional in similarity variables is far from being trivial and represents the crucial step in this paper, we choose to state it first in our paper, and to give it the status of a "first theorem".

Remark 1.2. As we mentioned before, we can construct a suitable Lyapunov functional in similarity variables for the more general equation (1.14) under hypotheses (1.15) and (1.16), then show that Theorem 1 holds true for this functional.

Remark 1.3. Let us note that our method breaks down in the case of a characteristic point, since in the construction of the Lyapunov functional in similarity variables, we use a covering technique in our argument which is not available at a characteristic point. At this moment, we do not know whether Theorem 1 continues to hold if x_0 is a characteristic point.

As we said earlier, the existence of this Lyapunov functional $L(w(s),s)$ together with a blow-up criterion for equation (1.18) make a crucial step in the derivation of the blow-up rate for equation (1.1). Indeed, with the functional $L(w(s),s)$ and some more work, we are able to adapt the analysis performed in [34, 35, 36] for equation (1.8) and obtain the following result:

Theorem 2. (Blow-up rate for equation (1.1)**).** *Let $N \geq 2$, $1 < p < 1 + \frac{4}{N-1}$ and $a \in \mathbb{R}$. Consider u a solution of (1.1), where $f(u) = |u|^{p-1}u\log^a(2+u^2)$, with blow-up graph $\Gamma : \{x \mapsto T(x)\}$ and x_0 a non characteristic point. Then, there exists $\widehat{S}_2$ large enough such that*

i) For all $s \geq \widehat{s}_2(x_0) = \max(\widehat{S}_2, -\log\frac{T(x_0)}{4})$,

$$0 < \varepsilon_0 \leq \|w_{x_0}(s)\|_{H^1(B)} + \|\partial_s w_{x_0}(s)\|_{L^2(B)} \leq K,$$

where $w_{x_0} = w_{x_0,T(x_0)}$ is defined in (1.17), for some $\varepsilon_0 = \varepsilon_0(N,p,a)$ and $K = K(N,p,a,T(x_0),t_2(x_0),\|(u(t_2(x_0)),\partial_t u(t_2(x_0)))\|_{H^1\times L^2(B(x_0,\frac{T(x_0)-t_2(x_0)}{\delta_0(x_0)}))})$, $\psi_{T(x_0)}(t)$ is defined in (1.13) *and $\delta_0(x_0)$ is given by* (1.7)*.*

.

ii) For all $t \in [t_2(x_0), T(x_0))$, where $t_2(x_0) = T(x_0) - e^{-\widehat{s}_2(x_0)}$, we have

$$0 < \varepsilon_0 \leq \frac{1}{\psi_{T(x_0)}(t)}\frac{\|u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{\frac{N}{2}}} \tag{1.27}$$
$$+\frac{T(x_0)-t}{\psi_{T(x_0)}(t)}\Big(\frac{\|\partial_t u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{\frac{N}{2}}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0,T(x_0)-t))}}{(T(x_0)-t)^{\frac{N}{2}}}\Big) \leq K,$$

where ε_0 and K are introduced in item i).

Remark 1.4. As we said before, we are able to obtain a similar result to (1.27) for the more general equation (1.14) under hypotheses (1.15) and (1.16).

Remark 1.5. Both for the construction of the Lyapunov functional and the derivation of the bounds in this theorem, our method breaks down in the conformal case, even when $a < 0$, and we are not able to obtain the sharp estimate as in the case of a pure power nonlinearity (1.8) treated in [36].

Remark 1.6. Since we crucially need a covering technique in the argument of the construction of the Lyapunov functional, our method breaks down too in the case of a characteristic point and we are not able to obtain the sharp estimate as in the unperturbed case (1.8).

Remark 1.7. As in [35] in the pure power nonlinearity case (1.8), the proof of Theorem 2 relies on four ideas: the existence of a Lyapunov functional, interpolation in Sobolev spaces, some critical Gagliardo-Nirenberg estimates and a covering technique adapted to the geometric shape of the blow-up surface. As we said before, the first point where we construct a Lyapunov functional in similarity variables is far from being trivial and represent a crucial step. Consequently, we have chosen to present our main contribution as Theorem (1) and we write a detailed proof. However, for the other

three points, the adaption of the proof of [35] given in the pure power nonlinearity case (1.8) is straightforward except for a key argument, where we bound the nonlinear term $e^{-\frac{2(p+1)s}{p-1}} s^{\frac{2a}{p-1}} F(\phi(s)w)$. Therefore, in order to avoid unnecessary repetition, we prove this step and kindly refer to [34, 35, 36, 24, 25, 21, 22, 23] for the rest of the proof.

This paper is organized as follows: In Section 2, we obtain a polynomial (in s) bound for the $H^1 \times L^2(B)$ norm of the solution $(w, \partial_s w)$. In Section 3, thanks to this result, we prove that the functional $L(w(s), s)$ defined in (1.26) is a Lyapunov functional for equation (1.18). Thus, we get Theorem 1. Finally, applying this last theorem, we prove Theorem 2.

Throughout this paper, C denotes a generic positive constant depending only on p, N and a, which may vary from line to line. In addition, we will use $K_1, K_2, K_3...$ as a positive constants depending only on $p, N, a, \delta_0(x_0)$ and initial data, which may also vary from line to line. We write $f(s) \sim g(s)$ to indicate $\lim\limits_{s\to\infty} \dfrac{f(s)}{g(s)} = 1$.

2 A polynomial bound for the $H^1 \times L^2(B)$ norm of solution of equation (1.18)

Let us first recall the rough polynomial space-time estimate of the solution u of (1.1) near any non characteristic point obtained in [28] (see Theorem 1). More precisely, we established the following results:

(Polynomial space-time estimate of solution of (1.18)). *There exists $q = q(a, p, N) > 0$ such that for any u a solution of (1.1) with blow-up graph $\Gamma : \{x \mapsto T(x)\}$ and x_0 a non characteristic point. Then, there exists $t_0(x_0) \in [0, T(x_0))$ such that, for all $T_0 \in (t_0(x_0), T(x_0)]$, for all $s \geq -\log(T_0 - t_0(x_0))$ and $x \in \mathbb{R}^N$ where $|x - x_0| \leq \frac{T_0 - t}{\delta_0(x_0)}$, we have*

$$\int_s^{s+1} \int_B \big(|\nabla w|^2 + (\partial_s w)^2\big) \mathrm{d}y \mathrm{d}\tau \leq K_1 s^q, \tag{2.1}$$

$$\frac{1}{s^a} \int_s^{s+1} \int_B |w|^{p+1} \log^a(2 + \phi^2 w^2) \mathrm{d}y \mathrm{d}\tau \leq K_1 s^q, \tag{2.2}$$

$$\int_B w^2 \mathrm{d}y \leq K_1 s^q, \tag{2.3}$$

where $w = w_{x,T^*(x)}$ is defined in (1.17), with

$$T^*(x) = T_0 - \delta_0(x_0)|x - x_0| \tag{2.4}$$

and $\delta_0(x_0)$ defined in (1.7). Note that K_1 depends on $p, a, N, \delta_0(x_0)$, $T(x_0)$, $t_0(x_0)$ and $\|(u(t_0(x_0)), \partial_t u(t_0(x_0)))\|_{H^1 \times L^2(B(x_0, \frac{T(x_0) - t_0(x_0)}{\delta_0(x_0)}))}$. Moreover, we have

$$-K_1 s^q \leq H_{m_0}(w(s), s) \leq K_1 s^q, \tag{2.5}$$

where

$$H_{m_0}(w(s),s) = E(w(s),s) - \frac{m_0}{s}\int_B w\partial_s w\rho(y)\mathrm{d}y, \tag{2.6}$$

$E(w(s),s)$ is given by (1.23) *and m_0 is a sufficiently large constant.*

This section is devoted to deriving a polynomial bound for the $H^1(B)$ norm. More precisely, this is the aim of this section.

Proposition 2.1. *There exists $q_1 = q_1(a,p,N) > 0$ such that for any solution u of (1.1) with blow-up graph $\Gamma : \{x \mapsto T(x)\}$ and x_0 a non characteristic point, there exists $t_0(x_0) \in [0, T(x_0))$ such that for all $T_0 \in (t_0(x_0), T(x_0)]$, $s \geq -\log(T_0 - t_0(x_0))$ and $x \in \mathbb{R}^N$ with $|x - x_0| \leq \frac{T_0 - t}{\delta_0(x_0)}$, we have*

$$\|w(s)\|_{H^1(B)} \leq K_2 s^{q_1}, \tag{2.7}$$

where $w = w_{x,T^*(x)}$ is defined in (1.17)*, with $T^*(x)$ given in* (2.4) *and $\delta_0(x_0)$ defined in* (1.7)*. Note that K_2 depends on $p, a, \delta_0(x_0)$, $T(x_0)$, $t_0(x_0)$ and $\|(u(t_0(x_0)), \partial_t u(t_0(x_0)))\|_{H^1\times L^2(B(x_0, \frac{T(x_0)-t_0(x_0)}{\delta_0(x_0)}))}$.*

Remark 2.1. Let us insist on the fact that the strategy of the proof works only in the subconformal case. Obviously, we can also prove that $\|\partial_s w(s)\|_{L^2(B)} \leq K_2 s^{q_1}$. However, since this estimate is not useful in the proof, we have chosen not to include it in the above proposition.

Remark 2.2. By using the Sobolev embedding and the above-mentioned proposition, we can deduce that for all $r \in [2, 2^*]$ where $2^* = \frac{2N}{N-2}$ if $N \geq 3$, and for all $r \in [2, \infty)$ if $N = 2$:

$$\|w(s)\|_{L^r(B)} \leq K_3 s^{q_1}, \quad \text{for all} \quad s \geq -\log(T^*(x) - t_0(x_0)), \tag{2.8}$$

where K_3 depends also on r.

Let us prove Proposition 2.1 in the following.

Proof of Proposition 2.1: We proceed in 2 steps:
- In Step 1, we use the covering technique and interpolation to derive a polynomial estimate related to the $L^{\frac{p+3-\varepsilon}{2}}(B)$ norm of $w(s)$, for any $\varepsilon \in (0, p-1)$.
- In Step 2, using Step 1, a Gagliardo-Nirenberg estimate and estimate (2.5) satisfied by $H_{m_0}(w(s), s)$ defined in (2.6), we easily conclude the proof of estimate (2.7).

Step 1: Control of w in $L^{\frac{p+3-\varepsilon}{2}}(B)$
We claim the following:

Lemma 2.2. *For all $\varepsilon \in (0, p-1)$ and $s \geq -\log(T^*(x) - t_0(x_0))$, we have*

$$\int_B |w(y,s)|^{\frac{p+3-\varepsilon}{2}}\mathrm{d}y \leq K_4(\varepsilon) s^q. \tag{2.9}$$

Proof: Let us recall from the expression of $\phi = \phi(s)$ defined in (1.21) that we have for all $s \geq \max(-\log T^*(x), 1)$,

$$e^{-\frac{2(p+1)s}{p-1}} s^{\frac{2a}{p-1}} \phi w f(\phi w) = \frac{1}{s^a} |w|^{p+1} \log^a(2 + \phi^2 w^2). \tag{2.10}$$

Combining (2.10), (A.4), (A.10) and (2.2), we deduce that for all $s \geq -\log(T^*(x) - t_0(x_0))$,

$$\int_s^{s+1} \int_B |w|^{p+1-\varepsilon} \mathrm{d}y\mathrm{d}\tau \leq C(\varepsilon) K_1 s^q. \tag{2.11}$$

Now, for all $s \geq -\log(T^*(x) - t_0(x_0))$, using the mean value theorem, we derive the existence of $\sigma(s) \in [s, s+1]$ such that

$$\int_B |w(y, \sigma(s))|^{p+1-\varepsilon} \mathrm{d}y = \int_s^{s+1} \int_B |w(y,\tau)|^{p+1-\varepsilon} \mathrm{d}y\mathrm{d}\tau. \tag{2.12}$$

Let us introduce the following identity for all $s \geq -\log(T^*(x) - t_0(x_0))$,

$$\int_B |w(y,s)|^{\frac{p+3-\varepsilon}{2}} \mathrm{d}y = \int_B |w(y,\sigma(s))|^{\frac{p+3-\varepsilon}{2}} \mathrm{d}y + \int_{\sigma(s)}^s \frac{d}{d\tau} \int_B |w(y,\tau)|^{\frac{p+3-\varepsilon}{2}} \mathrm{d}y\mathrm{d}\tau. \tag{2.13}$$

Combining (2.12), (2.13) and the fact that $xy \leq x^2 + y^2$, for all $x \geq 0$, $y \geq 0$, we write for all $s \geq -\log(T^*(x) - t_0(x_0))$,

$$\begin{aligned} \int_B |w(y,s)|^{\frac{p+3-\varepsilon}{2}} \mathrm{d}y \leq & \int_s^{s+1} \int_B |w(y,\tau)|^{\frac{p+3-\varepsilon}{2}} \mathrm{d}y\mathrm{d}\tau + C \int_s^{s+1} \int_B |w(y,\tau)|^{p+1-\varepsilon} \mathrm{d}y\mathrm{d}\tau \\ & + C \int_s^{s+1} \int_B (\partial_s w(y,\tau))^2 \mathrm{d}y\mathrm{d}\tau. \end{aligned} \tag{2.14}$$

Thanks to (2.14), the classical inequality $x^{\frac{p+3-\varepsilon}{2}} \leq 1 + x^{p+1-\varepsilon}$, for all $x \geq 0$, $\varepsilon \in (0, p-1)$, (2.1) and (2.11), we obtain (2.9). This concludes the Lemma 2.2. ∎

Step 2: Control of ∇w in $L^2(B)$.

As in the pure power case, we first use the Gagliardo-Nirenberg inequality in order to obtain the following:

Lemma 2.3. *There exists $\varepsilon_0 = \varepsilon_0(p, N) > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, for all $s \geq -\log(T^*(x) - t_0(x_0))$, we have*

$$\int_B |w(y,s)|^{p+1+\varepsilon} \mathrm{d}y \leq K_5 s^q \Big(\int_B |\nabla w(y,s)|^2 \mathrm{d}y \Big)^{\beta(\varepsilon)} + K_5 s^{2q}, \tag{2.15}$$

where $\beta = \beta(p, N, \varepsilon) \in (0, 1)$.

Proof: We distinguish two cases:

- First case ($N = 2$):
  Let $\varepsilon > 0$ and $r = r(p, \varepsilon) > p + 1 + \varepsilon$. By interpolation, we write

$$\int_B |w(y,s)|^{p+1+\varepsilon} \mathrm{d}y \leq \Big(\int_B |w(y,s)|^{\frac{p+3-\varepsilon}{2}} \mathrm{d}y \Big)^{\eta} \Big(\int_B |w(y,s)|^r \mathrm{d}y \Big)^{1-\eta}, \tag{2.16}$$

  where

$$\eta = \frac{r - (p+1+\varepsilon)}{r - \frac{p+3-\varepsilon}{2}}.$$

  By using (2.16) and the Sobolev embedding $H^1(B) \hookrightarrow L^r(B)$ we get

$$\int_B |w(y,s)|^{p+1+\varepsilon} \mathrm{d}y \leq \Big(\int_B w^{\frac{p+3-\varepsilon}{2}}(y,s) \mathrm{d}y \Big)^{\eta} \Big(\int_B |w(y,s)|^2 \mathrm{d}y + \int_B |\nabla w(y,s)|^2 \mathrm{d}y \Big)^{\beta}. \tag{2.17}$$

  where

$$\beta(r,p,\varepsilon) = \frac{r(p-1+3\varepsilon)}{4r - 2(p+3-\varepsilon)}.$$

  The combination of $\lim_{r\to\infty} \beta(r,p,\varepsilon) = \dfrac{p-1+3\varepsilon}{4}$ and $\frac{p-1}{4} < 1$ implies that there exists $\varepsilon_0 = \varepsilon_0(p) = \frac{5-p}{10} > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$ we can choose r large enough such that $\beta = \beta(r,p,\varepsilon) \in (0,1)$. The result (2.15) follows immediately from (2.17), (2.3) and (2.9).

- Second case ($N \geq 3$):
  Let $\varepsilon \in (0, p-1)$. Let us write the following Gagliardo-Nirenberg inequality:

$$\int_B |w(y,s)|^{p+1+\varepsilon} \mathrm{d}y \leq \Big(\int_B w^{\frac{p+3-\varepsilon}{2}}(y,s) \mathrm{d}y \Big)^{\eta} \Big(\int_B |w(y,s)|^2 \mathrm{d}y + \int_B |\nabla w(y,s)|^2 \mathrm{d}y \Big)^{\beta}, \tag{2.18}$$

  where

$$\eta = \frac{1 - \frac{p+1+\varepsilon}{2^*}}{1 - \frac{p+3-\varepsilon}{2\cdot 2^*}}, \quad \beta = \beta(p,N,\varepsilon) = \frac{\frac{p-1+3\varepsilon}{4}}{1 - \frac{p+3-\varepsilon}{2\cdot 2^*}} \quad \text{and} \quad \frac{1}{2^*} = \frac{N-2}{2N}.$$

  Observe that the function $p \hookrightarrow \beta(p,N,\varepsilon)$ is an increasing function on $(1, p_c)$, hence,

$$\beta(p,N,\varepsilon) < \beta(p_c,N,\varepsilon), \qquad \forall p \in (1,p_c). \tag{2.19}$$

  Thanks to (2.19), the fact $\beta(p_c,N,0) = 1$ and by continuity, we infer that there exists $\varepsilon_0 = \varepsilon_0(p,N) > 0$ small enough such that for $\varepsilon \in (0,\varepsilon_0]$ we have $\beta = \beta(p,N,\varepsilon) \in (0,1)$. The result (2.15) follows immediately from (2.18), (2.3) and (2.9), which ends the proof of Lemma 2.3. ■

Now, we are ready to give the proof of Proposition 2.1.

Proof of Proposition 2.1: Let us first use the following covering lemma from [35]: using Proposition 3.3 in that paper with $\eta = \frac{2(p+1)}{p-1} - N$ (which is positive), $q = 2$ and $f = \nabla u$, together with the selfsimilar change of variables (1.17), we write

$$\sup_{\{x \,|\, |x-x_0| \leq \frac{T_0 - t}{\delta_0}\}} \int_B |\nabla w|^2 \, \mathrm{d}y \leq C(\delta_0) \sup_{\{x \,|\, |x-x_0| \leq \frac{T_0 - t}{\delta_0}\}} \int_{B(0,\frac{1}{2})} |\nabla w|^2 \, \mathrm{d}y, \tag{2.20}$$

where $w = w_{x,T^*(x)}(y,s)$ is defined in (1.17), with $T^*(x)$ given in (2.4) and $s = -\log(T^*(x)-t)$.

From (2.5), the definition (2.6) of $H_{m_0}(w(s),s)$, we see that for all $s \geq -\log(T^*(x)-t_0(x_0))$,

$$\int_B |\nabla w(y,s)|^2(1-|y|^2)\rho(y)\mathrm{d}y + \int_B (\partial_s w(y,s))^2 w\mathrm{d}y - \frac{2m_0}{s}\int_B w\partial_s w\rho(y)\mathrm{d}y \\ \leq 2\int_B e^{-\frac{2(p+1)s}{p-1}} s^{\frac{2a}{p-1}} F(\phi w)\rho(y)\mathrm{d}y + 2K_1 s^q. \tag{2.21}$$

By the use of the basic inequality $2ab \leq a^2+b^2$, we write

$$\frac{2m_0}{s}\int_B w\partial_s w\rho(y)\mathrm{d}y \leq \int_B (\partial_s w(y,s))^2 w\mathrm{d}y + \frac{(m_0)^2}{s^2}\int_B w^2\rho(y)\mathrm{d}y. \tag{2.22}$$

Plugging (2.22) and (2.3) into (2.21), we obtain

$$\int_B |\nabla w(y,s)|^2(1-|y|^2)\rho(y)\mathrm{d}y \leq 2\int_B e^{-\frac{2(p+1)s}{p-1}} s^{\frac{2a}{p-1}} F(\phi w)\rho(y)\mathrm{d}y + K_6 s^q. \tag{2.23}$$

Thanks to (A.9) and (2.23), we conclude for all $s \geq -\log(T^*(x)-t_0(x_0))$,

$$\int_B |\nabla w(y,s)|^2(1-|y|^2)\rho(y)\mathrm{d}y \leq K_7\int_B |w(y,s)|^{p+\varepsilon+1}\rho(y)\mathrm{d}y + K_7 s^q. \tag{2.24}$$

According to (2.24) together with Lemma 2.3, we have for all $s \geq -\log(T^*(x)-t_0(x_0))$,

$$\int_{B(0,\frac{1}{2})} |\nabla w(y,s)|^2\mathrm{d}y \leq C\int_B |\nabla w(y,s)|^2(1-|y|^2)\rho(y)\mathrm{d}y \\ \leq K_8 s^q\Big(\int |\nabla w(y,s)|^2\mathrm{d}y\Big)^\beta + K_8 s^{2q}. \tag{2.25}$$

Therefore,

$$\sup_{\{x\,|\,|x-x_0|\leq\frac{T_0-t}{\delta_0}\}}\int_{B(0,\frac{1}{2})} |\nabla w(y,s)|^2\mathrm{d}y \leq K_8 s^q\left(\sup_{|x-x_0|\leq\frac{T_0-t}{\delta_0}}\int_B |\nabla w(y,s)|^2\,\mathrm{d}y\right)^\beta + K_8 s^{2q}. \tag{2.26}$$

where $\beta \in (0,1)$. From (2.20) and (2.26), we see that

$$\sup_{\{x\,|\,|x-x_0|\leq\frac{T_0-t}{\delta_0}\}}\int_B |\nabla w(y,s)|^2\mathrm{d}y \leq K_9 s^q\left(\sup_{|x-x_0|\leq\frac{T_0-t}{\delta_0}}\int_B |\nabla w(y,s)|^2\,\mathrm{d}y\right)^\beta + K_9 s^{2q}. \tag{2.27}$$

It suffices to combine (2.27) and the fact that $\beta < 1$, to obtain that

$$\sup_{\{x\,|\,|x-x_0|\leq\frac{T_0-t}{\delta_0}\}}\int_B |\nabla w(y,s)|^2\mathrm{d}y \leq K_{10} s^{\frac{2q}{1-\beta}}. \tag{2.28}$$

Clearly, by using (2.28) and (2.3), we conclude (2.7), where $q_1 = \frac{1}{1-\beta}q$, which yields the conclusion of Proposition 2.1. ∎

3 Proof of Theorem 1 and Theorem 2

In this section, we prove Theorem 1 and Theorem 2 here thanks to Proposition 2.1. This section is divided into two parts:

- In subsection 3.1, we state a general version of Theorem 1, uniform for x near x_0 and prove it.
- In subsection 3.2, we prove Theorem 2.

3.1 A Lyapunov functional

In this subsection, our aim is to construct a Lyapunov functional for equation (1.18). Note that this functional is far from being trivial and makes our main contribution. More precisely, thanks to the rough estimate obtained in the Proposition 2.1, we derive here that the functional $L(w(s), s)$ defined in (1.26) is a decreasing functional of time for equation (1.18), provided that s is large enough. First, thanks to the additional information obtained in Section 2, we can write this important lemma which plays a key role in our analysis. More precisely, we claim the following:

Lemma 3.1. *For all $s \geq -\log(T^*(x) - t_0(x_0))$, we have*

$$\int_B |w|^{p+1} \log^a(2+\phi^2 w^2) \log(2+w^2)\rho(y)\mathrm{d}y \leq K_{11} s^{\frac{1}{4}} \int_B |w|^{p+1} \log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y + K_{11} s^{a+\frac{1}{4}}. \tag{3.1}$$

Remark 3.1. Let us mention that, in the first term on the right-hand side, the choice of the power $\frac{1}{4}$ is not optimal. In fact, with the same proof, one can show the same estimate with the power ν, for any $\nu > 0$, instead of the power $\frac{1}{4}$. Let us remark that we can construct a Lyapunov functional, when we have the estimate above for some power ν such that $\nu \in (0,1)$ instead of the power $\frac{1}{4}$.

Proof: Let $\varepsilon \in (0,1)$. By using the inequality $\log(2+z^2) \leq C(\varepsilon) + |z|^{\varepsilon^2}$, for all $z \in \mathbb{R}$, we conclude that

$$\int_B |w|^{p+1} \log^a(2+\phi^2 w^2) \log(2+w^2)\rho(y)\mathrm{d}y \leq C \int_B |w|^{p+1} \log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y + \int_B |w|^{p+1+\varepsilon^2} \log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y. \tag{3.2}$$

Furthermore, we apply the interpolation in Lebesgue spaces to get

$$\int_B |w|^{p+1+\varepsilon^2} \log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y \leq \Big(\int_B |w|^{p+1} \log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y \Big)^{1-\varepsilon} \Big(\int_B |w|^{p+1+\varepsilon} \log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y \Big)^{\varepsilon}. \tag{3.3}$$

By combining(A.4), (A.9) and the inequality $|z|^{\varepsilon} \leq 1+|z|^{p+1+2\varepsilon}$, for all $z \in \mathbb{R}$, we obtain

$$\frac{1}{s^a}\int_B |w|^{p+1+\varepsilon}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y \leq C + C\int_B |w|^{p+1+2\varepsilon}\mathrm{d}y. \tag{3.4}$$

Since $p < p_c = \frac{N+3}{N-1}$, we then choose $\varepsilon_1 \leq \varepsilon_0$ small enough, such that for all $\varepsilon \in (0,\varepsilon_1]$ we have $p+1+2\varepsilon < 2^*$ where $2^* = \frac{2N}{N-2}$, if $N \geq 3$ and $2^* = \infty$, if $N = 2$. Therefore, estimate (2.8) implies that, for all $s \geq -\log(T^*(x) - t_0(x_0))$

$$\int_B |w|^{p+1+2\varepsilon}\mathrm{d}y \leq (K_3 s^{q_1})^{p+1+2\varepsilon}, \quad \forall \varepsilon \in [0,\varepsilon_1]. \tag{3.5}$$

By combining (3.3), (3.4) and (3.5), we deduce that, for all $s \geq -\log(T^*(x) - t_0(x_0))$, for all $\varepsilon \in (0,\varepsilon_1]$.

$$\int_B |w|^{p+1+\varepsilon^2}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y \leq K_{12}s^{q_1(p+1+2\varepsilon)\varepsilon}s^{\varepsilon a}\Big(\int_B |w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y\Big)^{1-\varepsilon}. \tag{3.6}$$

Thanks to the basic inequality $|X|^{\nu}|Y|^{1-\nu} \leq C|X| + C|Y|$, for all $X, Y \in \mathbb{R}$, for all $\nu \in (0,1)$, we conclude that, for all $s \geq -\log(T^*(x) - t_0(x_0))$, for all $\varepsilon \in (0,\varepsilon_1]$.

$$\int_B |w|^{p+1+\varepsilon^2}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y \leq K_{13}s^{q_1(p+1+2\varepsilon)\varepsilon}\Big(s^a + \int_B |w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y\Big). \tag{3.7}$$

We choose $\varepsilon_2 \in (0,\varepsilon_1]$, such that $q_1(p+1+2\varepsilon_2)\varepsilon_2 < \frac{1}{4}$. Then, by (3.2) and (3.7), we easily obtain (3.1). This concludes the proof of Lemma 3.1. ■

Thanks to estimate (3.1), we can improve the estimate related to the control of the time derivative of the functional $E(w(s),s)$. More precisely, we prove the following lemma:

Lemma 3.2. *There exists $S_1 > 0$ such that for all $s \geq \max(-\log(T^*(x) - t_0(x_0)), S_1)$, we have*

$$\begin{aligned}\frac{d}{ds}E(w(s),s) \leq &-\frac{3\alpha}{2}\int_B (\partial_s w)^2\frac{\rho(y)}{1-|y|^2}\mathrm{d}y + \frac{K_{14}}{s^{a+\frac{7}{4}}}\int_B |w|^{p+1}\log^a(2+\phi^2w^2)\rho(y)\mathrm{d}y\\ &+\frac{C}{s^2}\int_B |\nabla w|^2(1-|y|^2)\rho(y)\mathrm{d}y + \frac{C}{s^2}\int_B w^2\rho(y)\mathrm{d}y + \frac{K_{14}}{s^{\frac{7}{4}}}.\end{aligned} \tag{3.8}$$

Proof: Multiplying (1.18) by $\partial_s w\,\rho(y)$ and integrating over B, we obtain

$$\begin{aligned}\frac{d}{ds}E(w(s),s) = &-2\alpha\int_B (\partial_s w)^2\frac{\rho(y)}{1-|y|^2}\mathrm{d}y \\ &+\underbrace{\frac{a}{(p+1)s^{a+1}}\int_B |w|^{p+1}\log^{a-1}(2+\phi^2w^2)\Big(\log(2+\phi^2w^2) - \frac{4s}{p-1}\Big)\rho(y)\mathrm{d}y}_{\chi_1(s)}\end{aligned} \tag{3.9}$$

$$+\underbrace{\frac{2e^{-\frac{2(p+1)s}{p-1}}}{p-1}s^{\frac{2a}{p-1}}\int_B\Big((p+1)F_2(\phi w)-\frac{a}{s}F_1(\phi w)-\frac{a}{s}F_2(\phi w)\Big)\rho(y)\mathrm{d}y}_{\chi_2(s)}$$
$$+\underbrace{\gamma(s)\int_B w\partial_s w\rho(y)\mathrm{d}y+\frac{2a}{(p-1)s}\int_B(\partial_s w)^2\rho(y)\mathrm{d}y}_{\chi_3(s)}$$
$$+\underbrace{\frac{2a}{(p-1)s}\int_B y.\nabla w\partial_s w\rho(y)\mathrm{d}y}_{\chi_4(s)},$$

where F_1 and F_2 are defined by

$$F_1(x)=-\frac{2a}{(p+1)^2}|x|^{p+1}\log^{a-1}(2+x^2), \tag{3.10}$$

and

$$F_2(x)=F(x)-\frac{xf(x)}{p+1}-F_1(x). \tag{3.11}$$

Note that, in (3.9) we grouped the main terms together. In fact, it is easy to control the terms $\chi_2(s)$, $\chi_3(s)$ and $\chi_4(s)$. However, the control of the term $\chi_1(s)$ needs the use of the additional information obtained in Lemma 3.1. More precisely, for all $s\geq -\log(T^*(x)-t_0(x_0))$, we divide B into two parts

$$A_1(s)=\{y\in B\mid \phi(s)w^2(y,s)\leq 1\}\text{ and }A_2(s)=\{y\in B\mid \phi(s)w^2(y,s)\geq 1\}. \tag{3.12}$$

Accordingly, we write $\chi_1(s)=\chi_1^1(s)+\chi_1^2(s)$, where

$$\chi_1^1(s)=\frac{a}{(p+1)s^{a+1}}\int_{A_1(s)}|w|^{p+1}\log^{a-1}(2+\phi^2w^2)\Big(\log(2+\phi^2w^2)-\frac{4s}{p-1}\Big)\rho(y),$$
$$\chi_1^2(s)=\frac{a}{(p+1)s^{a+1}}\int_{A_2(s)}|w|^{p+1}\log^{a-1}(2+\phi^2w^2)\Big(\log(2+\phi^2w^2)-\frac{4s}{p-1}\Big)\rho(y)\mathrm{d}y.$$

On the one hand, by using the definition of the set $A_1(s)$ given in (3.12) and the expression of $\phi(s)$ in (1.21), we get, for all $s\geq -\log(T^*(x)-t_0(x_0))$,

$$|w|^{p+1}\log^a(2+\phi^2w^2)\leq C\phi^{-\frac{p+1}{2}}(s)\log^{|a|}(2+\phi(s))\leq Ce^{-\frac{ps}{p-1}}. \tag{3.13}$$

If we integrate (3.13) over $A_1(s)$, we obtain

$$\chi_1^1(s)\leq Ce^{-s}. \tag{3.14}$$

On the other hand, by using the definition of the $\phi(s)$ given by (1.21), we write the identity

$$\log(2+\phi^2w^2)-\frac{4s}{p-1}=\log(2\phi^{-2}+w^2)-\frac{2a\log s}{p-1}. \tag{3.15}$$

Furthermore, the exists $S_0 > 0$ such that for all $s \geq S_0$, we have $\phi(s) \geq 1$. Therefore, by exploiting (3.15), we write for all $s \geq \max(-\log(T^*(x) - t_0(x_0)), S_0)$,

$$\log(2 + \phi^2 w^2) - \frac{4s}{p-1} \leq \log(2 + w^2) + C \log s. \tag{3.16}$$

Also, by using the definition of the set $A_2(s)$ defined in (3.12), we can write for all $s \geq -\log(T^*(x) - t_0(x_0))$, if $y \in A_2(s)$, we have

$$\log(2 + \phi^2 w^2) \geq \log(\phi(s)) \geq \frac{2s}{p-1} - \frac{a \log s}{p-1}. \tag{3.17}$$

Clearly, the exists $S_1 > S_0$ such that for all $s \geq S_1$, we have $\frac{2s}{p-1} - \frac{a \log s}{p-1} \geq \frac{s}{p-1}$. Therefore, by exploiting (3.16) and (3.17) we have for all $s \geq \max(-\log(T^*(x) - t_0(x_0)), S_1)$,

$$\begin{aligned}\chi_1^2(s) \leq & \frac{C}{s^{a+2}} \int_B |w|^{p+1} \log^a(2 + \phi^2 w^2) \log(2 + w^2) \rho(y) \mathrm{d}y \\ & + \frac{C \log s}{s^{a+2}} \int_B |w|^{p+1} \log^a(2 + \phi^2 w^2) \rho(y) \mathrm{d}y. \end{aligned} \tag{3.18}$$

Adding (3.1) and (3.18) we have for all $s \geq \max(-\log(T^*(x) - t_0(x_0)), S_1)$,

$$\chi_1^2(s) \leq \frac{K_{15}}{s^{a+\frac{7}{4}}} \int_B |w|^{p+1} \log^a(2 + \phi^2 w^2) \rho(y) \mathrm{d}y + \frac{K_{15}}{s^{\frac{7}{4}}}. \tag{3.19}$$

Note that, by using the fact $\chi_1(s) = \chi_1^1(s) + \chi_1^2(s)$, (3.14) and (3.19), we get

$$\chi_1(s) \leq \frac{K_{16}}{s^{a+\frac{7}{4}}} \int_B |w|^{p+1} \log^a(2 + \phi^2 w^2) \rho(y) \mathrm{d}y + \frac{K_{16}}{s^{\frac{7}{4}}}. \tag{3.20}$$

Note from (A.5) and (A.6) that

$$\frac{1}{s}|F_1(\phi w)| + |F_2(\phi w)| \leq C + C \frac{\phi w}{s^2} f(\phi w). \tag{3.21}$$

By (3.9), (3.21) and (2.10), we have, for all $s \geq -\log(T^*(x) - t_0(x_0))$,

$$\chi_2(s) \leq \frac{C}{s^{a+2}} \int_B |w|^{p+1} \log^a(2 + \phi^2 w^2) \rho(y) \mathrm{d}y + C e^{-2s}. \tag{3.22}$$

Finally, by using the following basic inequality

$$ab \leq \nu a^2 + \frac{1}{\nu} b^2, \ \forall \nu > 0, \tag{3.23}$$

and the expression of $\gamma(s)$ defined in (1.20), we write, for all $s \geq -\log T^*(x) - t_0(x_0)$

$$\chi_3(s) + \chi_4(s) \leq \frac{\alpha}{2} \int_B (\partial_s w)^2 \frac{\rho(y)}{1 - |y|^2} \mathrm{d}y + \frac{C}{s^2} \int_B \Big(|\nabla w|^2 (1 - |y|^2) + w^2 \Big) \rho(y) \mathrm{d}y. \tag{3.24}$$

The result (3.8) derives immediately from (3.9), (3.24), (3.20), (3.22), and the identity (3.9), which ends the proof of Lemma 3.2 ■

Let us now recall the following result from [28], where we write an estimate on the functional $J(w(s), s)$ defined by:

$$J(w(s), s) = -\frac{1}{s} \int_B w \partial_s w \rho(y) \mathrm{d}y. \tag{3.25}$$

Lemma 3.3. *For all $s \geq \max(-\log T^*(x), 1)$, we have*

$$\begin{aligned}\frac{d}{ds}J(w(s),s) \quad &\leq \quad \frac{p+3}{2s}E(w(s),s) - \frac{p+7}{4s}\int_B (\partial_s w)^2\rho(y)\mathrm{d}y \\ &- \frac{p-1}{4s}\int_B (|\nabla w|^2 - (y.\nabla w)^2)\rho(y)\mathrm{d}y - \frac{p+1}{2(p-1)s}\int_B w^2\rho(y)\mathrm{d}y \\ &- \frac{p-1}{2(p+1)s^{a+1}}\int_B |w|^{p+1}\log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y + \Sigma_2(s),\end{aligned} \tag{3.26}$$

where $\Sigma_2(s)$ satisfies

$$\begin{aligned}\Sigma_2(s) \leq &\frac{C}{\sqrt{s}}\int_B (\partial_s w)^2\frac{\rho(y)}{1-|y|^2}\mathrm{d}y + \frac{C}{s\sqrt{s}}\int_B |\nabla w|^2(1-|y|^2)\rho(y)\mathrm{d}y \\ &+ \frac{C}{s\sqrt{s}}\int_B w^2\rho(y)\mathrm{d}y + \frac{C}{s^{a+2}}\int_B |w|^{p+1}\log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y + Ce^{-2s}.\end{aligned} \tag{3.27}$$

Proof: See Lemma 2.2 in [28].

■

With Lemmas 3.2 and 3.3, we are in a position to state and prove Theorem 1', which is a uniform version of Theorem 1 for x near x_0.

Theorem 1' *(Existence of a Lyapunov functional for equation (1.18))*
Consider u a solution of (1.1) with blow-up graph $\Gamma : \{x \mapsto T(x)\}$ and x_0 a non characteristic point. Then there exists $t_1(x_0) \in [0, T(x_0))$ such that, for all $T_0 \in (t_1(x_0), T(x_0)]$, for all $s \geq -\log(T_0 - t_1(x_0))$ and $x \in \mathbb{R}$, where $|x - x_0| \leq \frac{T-t}{\delta_0(x_0)}$, we have

$$L(w(s+1), s+1) - L(w(s), s) \leq -\alpha\int_s^{s+1}\int_B (\partial_s w)^2\frac{\rho(y)}{1-|y|^2}\mathrm{d}y\mathrm{d}\tau, \tag{3.28}$$

where $w = w_{x,T^*(x)}$ and $T^*(x)$ is defined in (2.4).

Proof of Theorem 1': By exploiting the definition of $L_0(w(s), s)$ in (1.24), we can write easily

$$\frac{d}{ds}L_0(w(s),s) = \frac{d}{ds}E(w(s),s) + \frac{1}{\sqrt{s}}\frac{d}{ds}J(w(s),s) - \frac{1}{2s\sqrt{s}}J(w(s),s). \tag{3.29}$$

With Lemmas 3.2 and 3.3 and the following inequality

$$\frac{1}{2s^2\sqrt{s}}\int_B w\partial_s w\rho(y)\mathrm{d}y + \frac{p+3}{2s^3}\int_B w\partial_s w\rho(y)\mathrm{d}y \leq \frac{C}{s^2}\int_B (\partial_s w)^2\rho(y)\mathrm{d}y + \frac{C}{s^2}\int_B w^2\rho(y)\mathrm{d}y,$$

allows to prove that for all $s \geq \max(-\log(T^*(x) - t_0(x_0)), S_1)$, we have

$$\begin{aligned}\frac{d}{ds}L_0(w(s),s) \leq &-(\frac{3\alpha}{2} - \frac{C}{s})\int_B (\partial_s w)^2\frac{\rho(y)}{1-|y|^2}\mathrm{d}y + \frac{p+3}{2s\sqrt{s}}L_0(w(s),s) \\ &- \frac{1}{s\sqrt{s}}(\frac{p+1}{2(p-1)} - \frac{C}{\sqrt{s}})\int_B w^2\rho(y)\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{s\sqrt{s}}\Big(\frac{p+7}{4} - \frac{C}{\sqrt{s}}\Big)\int_B (\partial_s w)^2 \rho(y)\mathrm{d}y \\
&- \frac{1}{s\sqrt{s}}\Big(\frac{p-1}{4} - \frac{C}{\sqrt{s}}\Big)\int_B |\nabla w|^2(1-|y|^2)\rho(y)\mathrm{d}y \\
&- \frac{1}{s^{a+\frac{3}{2}}}\Big(\frac{p-1}{2(p+1)} - \frac{K_{14}}{s^{\frac{1}{4}}} - \frac{C}{s}\Big)\int_B |w|^{p+1}\log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y \\
&+ C\frac{e^{-2s}}{\sqrt{s}} + \frac{K_{14}}{s^{\frac{7}{4}}}.
\end{aligned}
$$

Again, choosing $S_2 > -\log(T(x_0) - t_0(x_0))$ large enough, this implies that for all $s \geq \max(-\log(T^*(x) - t_0(x_0)), S_2)$, we have

$$
\frac{d}{ds}L_0(w(s),s) \leq -\alpha\int_B (\partial_s w)^2 \frac{\rho(y)}{1-|y|^2}\mathrm{d}y + \frac{p+3}{2s\sqrt{s}}L_0(w(s),s) + \frac{K_{15}}{s^{\frac{7}{4}}}. \tag{3.30}
$$

Recalling that,

$$
L(w(s),s) = \exp\Big(\frac{p+3}{\sqrt{s}}\Big)L_0(w(s),s) + \frac{\theta}{s^{\frac{3}{4}}},
$$

we get from straightforward computations

$$
\frac{d}{ds}L(w(s),s) = -\frac{p+3}{2s\sqrt{s}}\exp\Big(\frac{p+3}{\sqrt{s}}\Big)L_0(w(s),s) + \exp\Big(\frac{p+3}{\sqrt{s}}\Big)\frac{d}{ds}L_0(w(s),s) - \frac{3\theta}{4s^{\frac{7}{4}}}. \tag{3.31}
$$

Therefore, estimates (3.30) and (3.31) lead to the following crucial estimate:

$$
\frac{d}{ds}L(w(s),s) \leq -\alpha\exp\Big(\frac{p+3}{\sqrt{s}}\Big)\int_B (\partial_s w)^2\frac{\rho(y)}{1-|y|^2}\mathrm{d}y + \Big(K_{15}\exp\Big(\frac{p+3}{\sqrt{s}}\Big) - \frac{3\theta}{4}\Big)\frac{1}{s^{\frac{7}{4}}}. \tag{3.32}
$$

Since we have $1 \leq \exp\Big(\frac{p+3}{\sqrt{s}}\Big) \leq \exp\Big(\frac{p+3}{\sqrt{S_2}}\Big)$, we then choose θ large enough, so that $K_{15}\exp\Big(\frac{p+3}{\sqrt{s}}\Big) - \frac{3\theta}{4} \leq 0$, which yields, for all $s \geq \max(-\log(T^*(x) - t_0(x_0)), S_2)$,

$$
\frac{d}{ds}L(w(s),s) \leq -\alpha\int_B (\partial_s w)^2\frac{\rho(y)}{1-|y|^2}\mathrm{d}y.
$$

A simple integration between s and $s+1$ ensures the result (3.28), where

$$
t_1(x_0) = \max(T(x_0) - e^{-S_2}, t_0(x_0)). \tag{3.33}
$$

This concludes the proof of Theorem 1'. ■

We now claim the following lemma:

Lemma 3.4. *There exists $S_3 \geq S_2$ such that, if $L(w(s_3),s_3) < 0$ for some $s_3 \geq \max(S_3, -\log(T^*(x) - t_1(x_0)))$, then w blows up in some finite time $s_4 > s_3$.*

Proof: The argument is the same as in Lemma 3.4 in [28].

■

3.2 Proof of Theorem 2

In this subsection, we prove Theorem 2. Note that the lower bound follows from the finite speed of propagation and the wellposedness in $H^1 \times L^2$. For a detailed argument in the similar case of equation (1.8) in [35] (see Lemma 3.1, page 1136).
We consider u a solution of (1.1) with blow-up graph $\Gamma : \{x \mapsto T(x)\}$ and x_0 is a non-characteristic point. Let

$$t_2(x_0) = \max(T(x_0) - e^{-S_3}, t_1(x_0)). \tag{3.34}$$

Given some $T_0 \in (t_2(x_0), T(x_0)]$, for all $x \in \mathbb{R}$ is such that $|x - x_0| \leq \frac{T_0 - t_2(x_0)}{\delta_0(x_0)}$, where $\delta(x_0)$ is defined in (1.7), we aim at bounding $\|(w, \partial_s w)(s)\|_{H^1 \times L^2(B)}$ for s large.

As in [25, 22], by combining Theorem 1' and Lemma 3.4 we get the following bounds:

Corollary 3.5. *(Bound on $L_0(w(s), s)$). For all $T_0 \in (t_2(x_0), T(x_0)]$, for all $s \geq -\log(T_0 - t_2(x_0))$ and $x \in \mathbb{R}^N$ where $|x - x_0| \leq \frac{T_0 - t}{\delta_0(x_0)}$, we have*

$$-C \leq L_0(w(s), s) \leq C L_0(w(\tilde{s}_2), \tilde{s}_2) + C, \tag{3.35}$$

where $\tilde{s}_2 = -\log(T^*(x) - t_2(x_0))$.
Moreover, for all $s \geq -\log(T^*(x) - t_2(x_0))$, we have

$$\int_s^{s+1} \int_B (\partial_s w)^2 \frac{\rho(y)}{1 - |y|^2} \mathrm{d}y \mathrm{d}s \leq K_{16}, \tag{3.36}$$

where $K_{16} = K_{16}(a, p, T^*(x), \|(u(t_2), u_t(t_2))\|_{H^1 \times L^2(B(x_0, \frac{T_0 - t_2(x_0)}{\delta_0(x_0)}))})$, $C = C(a, p)$ and $\delta_0(x_0) \in (0, 1)$ is defined in (1.7).

Remark 3.2. Using the definition of (1.17) of $w_{x,T^*(x)} = w$, we write easily

$$L_0(w(\widetilde{s_2}), \widetilde{s_2}) \leq K_{17}, \tag{3.37}$$

where $K_{17} = K_{17}(T(x_0) - t_2(x_0), \|(u(t_2(x_0)), \partial_t u(t_2(x_0)))\|_{H^1 \times L^2(B(x_0, \frac{T(x_0) - t_2(x_0)}{\delta_0(x_0)}))})$.

Starting from these bounds, the proof of Theorem 2 is similar to the proof in [34] (see Corollary 2.3, page 1151) and [35] (see Proposition 2.1, page 1134) except for the treatment of the nonlinear terms and of the perturbation terms. In our opinion, handling these terms is straightforward in all the steps of the proof, except for the first step, where we bound the time averages of the nonlinear term and second step, where we remove the time averages. However, the third step where we conclude the boundedness of the $H^1_{loc,u}(\mathbb{R}^N)$ norm of solution of equation (1.18) from Proposition 3.6 is the same as in Proposition 2.1 (up to some very minor changes). For that reason, we only give the first two step and refer to the proof in [34] (see Corollary 2.3, page 1151) and [35] (see Proposition 2.1, page 1134) and the similar part in section 2 in this paper for the remaining steps in the proof of Theorem 2. This is the step we prove here.

Proposition 3.6. *For all $s \geq 1 - \log(T^*(x) - t_3(x_0))$, for some $t_3(x_0) \in [t_2(x_0), T(x_0))$,*

$$\frac{1}{s^a}\int_s^{s+1}\int_B |w|^{p+1}\log^a(2+\phi^2 w^2)\rho(y)\mathrm{d}y\mathrm{d}\tau \leq K_{18}. \tag{3.38}$$

Proof: For $s \geq 1 - \log(T^*(x) - t_2(x_0))$, let us work with time integrals between s_1 et s_2 where $s_1 \in [s-1, s]$ and $s_2 \in [s+1, s+2]$. By integrating the expression (1.24) of $L_0(w(s), s)$ in time between s_1 and s_2, where $s_2 > s_1 > -\log(T^*(x) - t_2(x_0))$, we obtain:

$$\begin{aligned}\int_{s_1}^{s_2} L_0(w(s),s)\mathrm{d}s = &\int_{s_1}^{s_2}\int_B \Big(\frac{1}{2}(\partial_s w)^2 + \frac{p+1}{(p-1)^2}w^2 - e^{-\frac{2(p+1)s}{p-1}} s^{\frac{2a}{p-1}} F(\phi w)\Big)\rho(y)\mathrm{d}y\mathrm{d}s \\ &+ \frac{1}{2}\int_{s_1}^{s_2}\int_B (|\nabla w|^2 - |y.\nabla w|^2)\rho(y)\mathrm{d}y\mathrm{d}s - \int_{s_1}^{s_2}\frac{1}{s\sqrt{s}}\int_B w\partial_s w\rho(y)\mathrm{d}y\mathrm{d}s.\end{aligned} \tag{3.39}$$

By multiplying the equation (1.18) by $w\rho(y)$ and integrating both in time and in space over $B \times [s_1, s_2]$ we obtain the following identity, after some integration by parts:

$$\begin{aligned}&\Big[\int_B \Big(w\partial_s w + (\frac{p+3}{2(p-1)} - N)w^2\Big)\rho(y)\mathrm{d}y\Big]_{s_1}^{s_2} = \int_{s_1}^{s_2}\int_B (\partial_s w)^2\rho(y)\mathrm{d}y\mathrm{d}s \\ &- \int_{s_1}^{s_2}\int_B \big(|\nabla w|^2 - (y.\nabla w)^2\big)\rho(y)\mathrm{d}y\mathrm{d}s - \frac{2p+2}{(p-1)^2}\int_{s_1}^{s_2}\int_B w^2\rho(y)\mathrm{d}y\mathrm{d}s \\ &+ \int_{s_1}^{s_2}\int_B e^{-\frac{2ps}{p-1}} s^{\frac{a}{p-1}} w f(\phi w)\rho(y)\mathrm{d}y\mathrm{d}s - 2\alpha\int_{s_1}^{s_2}\int_B w\partial_s w\frac{|y|^2\rho(y)}{1-|y|^2}\mathrm{d}y\mathrm{d}s \\ &+2\int_{s_1}^{s_2}\int_B y.\nabla w\partial_s w\rho(y)\mathrm{d}y\mathrm{d}s + \frac{2a}{p-1}\int_{s_1}^{s_2}\int_B \frac{1}{s}y.\nabla w w\rho(y)\mathrm{d}y\mathrm{d}s \\ &+\int_{s_1}^{s_2}\int_B \gamma(s)w^2\rho(y)\mathrm{d}y\mathrm{d}s + \frac{2a}{p-1}\int_{s_1}^{s_2}\int_B \frac{1}{s}\partial_s w w\rho(y)\mathrm{d}y\mathrm{d}s.\end{aligned} \tag{3.40}$$

Note that, by using the identity (3.11), we get

$$\begin{aligned}e^{-\frac{2(p+1)s}{p-1}} s^{\frac{2a}{p-1}}\Big(\frac{\phi w}{2}f(\phi w) - F(\phi w)\Big) = &\frac{p-1}{2}e^{-\frac{2(p+1)s}{p-1}} s^{\frac{2a}{p-1}} F(\phi w) \\ &- \frac{p+1}{2}e^{-\frac{2(p+1)s}{p-1}} s^{\frac{2a}{p-1}}\Big(F_1(\phi w) + F_2(\phi w)\Big).\end{aligned} \tag{3.41}$$

By combining the identities (3.39), (3.40) and exploiting (3.41), we obtain

$$\begin{aligned}&\frac{p-1}{2}\int_{s_1}^{s_2}\int_B e^{-\frac{2(p+1)s}{p-1}} s^{\frac{2a}{p-1}} F(\phi w)\rho(y)\mathrm{d}y\mathrm{d}s \\ = \;&\frac{1}{2}\Big[\int_B \Big(w\partial_s w + (\frac{p+3}{2(p-1)} - N)w^2\Big)\rho(y)\mathrm{d}y\Big]_{s_1}^{s_2} - \int_{s_1}^{s_2}\int_B (\partial_s w)^2\rho(y)\mathrm{d}y\mathrm{d}s \\ &+ \int_{s_1}^{s_2} L_0(w(s),s)\mathrm{d}s + \alpha\int_{s_1}^{s_2}\int_B w\partial_s w\frac{|y|^2\rho(y)}{1-|y|^2}\mathrm{d}y\mathrm{d}s \\ &-\int_{s_1}^{s_2}\int_B y.\nabla w\partial_s w\rho(y)\mathrm{d}y\mathrm{d}s \underbrace{-\frac{a}{p-1}\int_{s_1}^{s_2}\int_B \frac{1}{s}y.\nabla w w\rho(y)\mathrm{d}y\mathrm{d}s}_{A_1}\end{aligned}$$

$$\underbrace{-\frac{1}{2}\int_{s_1}^{s_2}\int_B \gamma(s)w^2\rho(y)\mathrm{d}y\mathrm{d}s}_{A_2}\underbrace{-\frac{a}{p-1}\int_{s_1}^{s_2}\int_B \frac{1}{s}\partial_s w w\rho(y)\mathrm{d}y\mathrm{d}s}_{A_3} \tag{3.42}$$
$$+\underbrace{\int_{s_1}^{s_2}\frac{1}{s\sqrt{s}}\int_B w\partial_s w\rho(y)\mathrm{d}y\mathrm{d}s}_{A_4}+\underbrace{\frac{p+1}{2}\int_{s_1}^{s_2}\int_B e^{-\frac{2(p+1)s}{p-1}}s^{\frac{2a}{p-1}}F_1(\phi w)\rho(y)\mathrm{d}y\mathrm{d}s}_{A_5}$$
$$+\underbrace{\frac{p+1}{2}\int_{s_1}^{s_2}\int_B e^{-\frac{2(p+1)s}{p-1}}s^{\frac{2a}{p-1}}F_2(\phi w)\rho(y)\mathrm{d}y\mathrm{d}s}_{A_6}.$$

We claim that Proposition 3.6 follows from the following Lemma where we control the space-time integral of the nonlinear term of w and all the terms on the right-hand side of the relation (3.42) in terms of the left-hand side:

Lemma 3.7. *For all $s \geq 1-\log(T^*(x)-t_3(x_0))$, for some $t_3(x_0)\in[t_2(x_0),T(x_0))$, for all $\nu_0>0$, for all $\varepsilon\in(0,1)$,*

$$\int_B |w|^{p+1-\varepsilon}\rho(y)\mathrm{d}y \leq K_{19}+C\int_B e^{-\frac{2(p+1)s}{p-1}}s^{\frac{2a}{p-1}}F(\phi w)\rho(y)\mathrm{d}y, \tag{3.43}$$

$$\int_B e^{-\frac{2(p+1)s}{p-1}}s^{\frac{2a}{p-1}}F(\phi w)\rho(y)\mathrm{d}y \leq K_{19}+C\int_B |w|^{p+1+\varepsilon}\rho(y)\mathrm{d}y, \tag{3.44}$$

$$\int_{s_1}^{s_2}\int_B |y.\nabla w\partial_s w|\rho(y)\mathrm{d}y\mathrm{d}s \leq \frac{K_{19}}{\nu_0}+K_{19}\nu_0\int_{s_1}^{s_2}\int_B e^{-\frac{2(p+1)s}{p-1}}s^{\frac{2a}{p-1}}F(\phi w)\rho(y)\mathrm{d}y\mathrm{d}s, \tag{3.45}$$

$$\sup_{s\in[s_1,s_2]}\int_B w^2(y,s)\rho(y)\mathrm{d}y \leq \frac{K_{19}}{\nu_0}+K_{19}\nu_0\int_{s_1}^{s_2}\int_B e^{-\frac{2(p+1)s}{p-1}}s^{\frac{2a}{p-1}}F(\phi w)\rho(y)\mathrm{d}y\mathrm{d}s, \tag{3.46}$$

$$\int_{s_1}^{s_2}\int_B w\partial_s w\frac{|y|^2\rho(y)}{1-|y|^2}\mathrm{d}y\mathrm{d}s \leq \frac{K_{19}}{\nu_0}+K_{19}\nu_0\int_{s_1}^{s_2}\int_B e^{-\frac{2(p+1)s}{p-1}}s^{\frac{2a}{p-1}}F(\phi w)\rho(y)\mathrm{d}y\mathrm{d}s, \tag{3.47}$$

$$\int_B |w\partial_s w|\rho(y)\mathrm{d}y \leq \int_B (\partial_s w)^2\rho(y)\mathrm{d}y+\frac{K_{19}}{\nu_0}$$
$$+K_{19}\nu_0\int_{s_1}^{s_2}\int_B e^{-\frac{2(p+1)s}{p-1}}s^{\frac{2a}{p-1}}F(\phi w)\rho(y)\mathrm{d}y\mathrm{d}s, \tag{3.48}$$

$$\int_B \Big((\partial_s w(y,s_1))^2+(\partial_s w(y,s_2))^2\Big)\rho(y)\mathrm{d}y \leq K_{19}, \tag{3.49}$$

$$|A_1| \leq \frac{K_{19}}{\nu_0}+(K_{19}\nu_0+\frac{C}{s_1})\int_{s_1}^{s_2}\int_B e^{-\frac{2(p+1)s}{p-1}}s^{\frac{2a}{p-1}}F(\phi w)\rho(y)\mathrm{d}y\mathrm{d}s, \tag{3.50}$$

$$|A_2|+|A_3|+|A_4| \leq \frac{K_{19}}{\nu_0}+K_{19}\nu_0\int_{s_1}^{s_2}\int_B e^{-\frac{2(p+1)s}{p-1}}s^{\frac{2a}{p-1}}F(\phi w)\rho(y)\mathrm{d}y\mathrm{d}s, \tag{3.51}$$

$$|A_5|+|A_6| \leq C+\frac{C}{s_1}\int_{s_1}^{s_2}\int_B e^{-\frac{2(p+1)s}{p-1}}s^{\frac{2a}{p-1}}F(\phi w)\rho(y)\mathrm{d}y\mathrm{d}s. \tag{3.52}$$

Indeed, from (3.42) and this Lemma, we deduce that

$$\int_{s_1}^{s_2}\int_B e^{-\frac{2(p+1)s}{p-1}} s^{\frac{2a}{p-1}} F(\phi w)\rho(y)\mathrm{d}y\mathrm{d}s \leq \frac{K_{19}}{\nu_0} + (K_{19}\nu_0 + \frac{C}{s_1})\int_{s_1}^{s_2}\int_B e^{-\frac{2(p+1)s}{p-1}} s^{\frac{2a}{p-1}} F(\phi w)\rho(y)\mathrm{d}y\mathrm{d}s.$$

Now, we can use the fact that $s_1 \geq -1-\log(T^*(x)-t_3(x_0)) \geq -1-\log(T(x_0)-t_3(x_0))$ and we choose $T(x_0)-t_3(x_0)$ small enough, so that $\frac{C}{s_1} \leq \frac{1}{-1-\log(T(x_0)-t_3(x_0))} \leq \frac{1}{4}$. If we choose ν_0 small enough so that $K_{19}\nu_0 \leq \frac{1}{4}$, we obtain

$$\int_{s_1}^{s_2}\int_B e^{-\frac{2(p+1)s}{p-1}} s^{\frac{2a}{p-1}} F(\phi w)\rho(y)\mathrm{d}y\mathrm{d}s \leq K_{19}.$$

Since $[s, s+1] \subset [s_1, s_2]$, we derive from (A.4) that (3.38).

It remains to prove Lemma 3.7.

Proof of Lemma 3.7: By (A.9) and (A.10), we can write easily (3.43) and (3.44). Thanks to (3.43), we can adapt with no difficulty the proof in the unperturbed case [34, 35] (up to some very minor changes), in order to get the proof of the estimates (3.45), (3.46), (3.47), (3.48) and (3.49). Also, by using (3.43) and the Hardy inequality

$$\int_B w^2 \frac{|y|^2\rho(y)}{1-|y|^2}\mathrm{d}y \leq C\int_B |\nabla w|^2(1-|y|^2)\rho(y)\mathrm{d}y + C\int_B w^2\rho(y)\mathrm{d}y.$$

(see the appendix in [34] for a proof), we easily conclude (3.50) and (3.51).

Finally, it remains only to control the terms A_5 and A_6. Note from (A.4), (A.5) and (A.6) that

$$|F_1(\phi w)| + |F_2(\phi w)| \leq C + C\frac{F(\phi w)}{s}. \tag{3.53}$$

The result (3.52) follows immediately from (3.53). This concludes the proof of Lemma 3.7 and Proposition 3.6 too. ■

Proof of Theorem 2 : Since the derivation of the boundedness of the $H^1_{loc,u}(\mathbb{R}^N)$ norm of solution of equation (1.18) from Proposition 3.6 is the same as in Proposition 2.1 (from the estimates (2.1), (2.2) (2.3) and (2.5) (up to some very minor changes). Moreover, thanks to the estimate (3.35), the boundedness of the $H^1_{loc,u}(\mathbb{R}^N)$ norm, we prove easily the boundedness of $L^2_{loc,u}(\mathbb{R}^N)$ norm of $\partial_s w$ with the ball $B(0,\frac{1}{2})$. Thanks to the covering technique (we refer the reader to Merle and Zaag [35] (pure power case) and Hamza and Zaag in Lemma 2.8 in [24]), we easily extend this estimate from $B(0,\frac{1}{2})$ to B. This concludes the proof of Theorem 2. ■

A Some elementary lemmas.

Let f, F, F_2 be the functions defined in (1.2), (1.25) and (3.11). Clearly, we have

Lemma A.1. *Let $q > 1$,*

$$\int_0^u |v|^{q-1} v \log^a(2+v^2)\mathrm{d}v \sim \frac{|u|^{q+1}}{q+1} \log^a(2+u^2), \quad as \quad |u| \to \infty, \tag{A.1}$$

$$F(u) \sim \frac{uf(u)}{p+1} \quad as \quad |u| \to \infty, \tag{A.2}$$

$$F_2(u) \sim \frac{Cuf(u)}{\log^2(2+u^2)} \quad as \quad |u| \to \infty. \tag{A.3}$$

Proof. See Lemma A.1 in [28]. ∎

Thanks to (A.1), (A.2) and (A.3), we can state and prove the following estimates:

Lemma A.2. *For all $s \geq 1$, for all $z \in \mathbb{R}$,*

$$C^{-1}\phi(s)zf(\phi(s)z)) \leq C + F\left(\phi(s)z\right) \leq C(1+\phi(s)zf(\phi(s)z))\,, \tag{A.4}$$

$$F_1(\phi(s)z) \leq C + C\frac{\phi(s)z}{s} f(\phi(s)z), \tag{A.5}$$

$$F_2(\phi(s)z) \leq C + C\frac{\phi(s)z}{s^2} f(\phi(s)z), \tag{A.6}$$

$$e^{-\frac{2ps}{p-1}} s^{\frac{a}{p-1}} |f(\phi(s)z)| \leq C + C(\varepsilon)|z|^{p+\varepsilon}, \qquad \forall \varepsilon > 0, \tag{A.7}$$

$$|z|^{p-\varepsilon} \leq C(\varepsilon) e^{-\frac{2ps}{p-1}} s^{\frac{a}{p-1}} |f(\phi(s)z)| + C, \qquad \forall \varepsilon \in (0,p), \tag{A.8}$$

$$e^{-\frac{2(p+1)s}{p-1}} s^{\frac{2a}{p-1}} F(\phi(s)z) \leq C + C(\varepsilon)|z|^{p+\varepsilon+1}, \qquad \forall \varepsilon > 0, \tag{A.9}$$

$$|z|^{p-\varepsilon+1} \leq C(\varepsilon) e^{-\frac{2(p+1)s}{p-1}} s^{\frac{2a}{p-1}} F(\phi(s)z) + C, \qquad \forall \varepsilon \in (0,p+1), \tag{A.10}$$

where ϕ, F, F_1 and F_2 are given in (1.21), (1.25), (3.10) *and* (3.11).

Proof. Note that (A.4) obviously follows from (A.2). In order to derive estimates (A.5) and (A.6), considering the first case $z^2\phi(s) \geq 4$, then the case $z^2\phi(s) \leq 4$, we would obtain (A.5) and (A.6) by using (A.1), (A.2) and(A.3). Similarly, by taking into account the inequality $\log^a(2+u^2) \leq C(\varepsilon) + |u|^\varepsilon$, we conclude easily (A.7), (A.8), (A.9) and (A.10). This ends the proof of Lemma A.2. ∎

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