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A remark on weak-strong uniqueness for suitable weak solutions of the Navier–Stokes equations

Pierre Gilles Lemarié–Rieusset*

Abstract

We extend Barker’s weak-strong uniqueness results for the Navier–Stokes equations and consider a criterion involving Besov spaces and weighted Lebesgue spaces.

Keywords : Navier–Stokes equations, weak-strong uniqueness, Besov spaces, uniformly locally square integrable functions, weighted Lebesgue spaces

AMS classification : 35K55, 35Q30, 76D05.

1 The Prodi–Serrin criterion for weak-strong uniqueness

In this paper, we are interested in extensions of the Prodi–Serrin weak-strong uniqueness for (suitable) weak Leray solutions of the Navier–Stokes equations. We consider solutions of the Navier–Stokes equations

$$\begin{cases} \partial_t \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} = \Delta \vec{u} - \vec{\nabla} p \\ \operatorname{div} \vec{u} = 0 \\ \vec{u}(0, \cdot) = \vec{u}_0 \end{cases}$$

where \vec{u}_0 is a square-integrable divergence-free vector field on the space \mathbb{R}^3 .

Looking for weak solutions, where the derivatives are taken in the sense of distributions, it is better to write the first line of the system as

$$\partial_t \vec{u} + \operatorname{div} (\vec{u} \otimes \vec{u}) = \Delta \vec{u} - \vec{\nabla} p.$$

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If \vec{u} is a solution on $(0, T) \times \mathbb{R}^3$ such that $\vec{u} \in L^\infty((0, T), L^2)$, then the pressure p can be eliminated through the formula

$$\operatorname{div}(\vec{u} \otimes \vec{u}) + \vec{\nabla} p = \mathbb{P}(\operatorname{div}(\vec{u} \otimes \vec{u}))$$

where \mathbb{P} is the Leray projection operator on solenoidal vector fields:

$$\mathbb{P}\vec{f} = -\frac{1}{\Delta}\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{f}).$$

Moreover, \vec{u} can be represented as a distribution which depends continuously on the time t [LR 5] as

$$\vec{u} = \vec{u}_0 + \int_0^t \Delta \vec{u} - \mathbb{P}(\operatorname{div}(\vec{u} \otimes \vec{u})) ds.$$

Leray [LER] proved existence of solutions \vec{u} on $(0, +\infty) \times \mathbb{R}^3$ such that :

- $\vec{u} \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$
- $\lim_{t \rightarrow 0^+} \|\vec{u}(t, \cdot) - \vec{u}_0\|_2 = 0$
- we have the Leray energy inequality

$$\|\vec{u}(t, \cdot)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{u}\|_2^2 ds \leq \|\vec{u}_0\|_2^2 \quad (1)$$

Such solutions are called *Leray solutions*¹. His proof is based on a compactness criterion, it does not provide any clue on the uniqueness of the solution to the Cauchy initial value problem.

A classical case of uniqueness of Leray weak solutions is the weak-strong uniqueness criterion described by Prodi and Serrin [PRO, SERR]: if $\vec{u}_0 \in L^2$ and if the Navier-Stokes equations have a solution \vec{u} on $(0, T)$ such that

$$\vec{u} \in L_t^p L_x^q \text{ with } \frac{2}{p} + \frac{3}{q} \leq 1 \text{ and } 2 \leq p \leq +\infty$$

then, if \vec{v} is a Leray solution with the same initial value \vec{u}_0 , we have $\vec{u} = \vec{v}$ on $(0, T)$. Let us remark that the existence of such a solution \vec{u} restricts the range of the initial value \vec{u}_0 : as a matter of fact, when $2 < p < +\infty$, existence of a time $T > 0$ and of a solution $\vec{u} \in L_t^p L_x^q$ is equivalent to the fact that \vec{u}_0 belongs to the Besov space $B_{q,p}^{-\frac{2}{p}}$ (see Theorem 6 below).

¹Remark that the continuity at $t = 0$ of $t \mapsto \vec{u}(t, \cdot)$ in L^2 norm is a consequence of the Leray inequality (1)

We will see that a corollary of Barker's theorem [BAR] shows the following extension of the criterion: if $\vec{u}_0 \in L^2$ and if the Navier-Stokes equations have a solution \vec{u} on $(0, T)$ such that

$$\sup_{0 < t < T} t^{\frac{1}{p}} \|\vec{u}\|_q < +\infty \text{ with } \frac{2}{p} + \frac{3}{q} \leq 1 \text{ and } 2 < p < +\infty$$

and with

$$\lim_{t \rightarrow 0} t^{\frac{1}{p}} \|\vec{u}\|_q = 0 \text{ if } \frac{2}{p} + \frac{3}{q} \leq 1$$

then, if \vec{v} is a Leray solution with the same initial value \vec{u}_0 , we have $\vec{u} = \vec{v}$ on $(0, T)$. Let us remark again that the existence of such a time T and such a solution \vec{u} is equivalent to the fact that \vec{u}_0 belongs to the Besov space $B_{q, \infty}^{-\frac{2}{p}} \cap \text{bmo}_0^{-1}$ (see Definition 1 and Theorem 7 below).

The space bmo^{-1} was introduced in 2001 by Koch and Tataru [KOT] for the study of mild solutions to the Navier-Stokes problem. Let us recall the characterization of bmo^{-1} through the heat kernel [KOT, LR 1]:

Proposition 1.

For $0 < T < \infty$, define

$$\|\vec{u}\|_{X_T} = \sup_{0 < t < T} \sqrt{t} \|\vec{u}(t, \cdot)\|_{\infty} + \sup_{0 < t < T, x_0 \in \mathbb{R}^3} (t^{-3/2} \int_0^t \int_{B(x_0, \sqrt{t})} |\vec{u}(s, y)|^2 dy ds)^{1/2}.$$

Then $\vec{u}_0 \in \text{bmo}^{-1}$ if and only if $(e^{t\Delta} \vec{u}_0)_{0 < t < T} \in X_T$ (with equivalence of the norms $\|\vec{u}_0\|_{\text{bmo}^{-1}}$ and $\|e^{t\Delta} \vec{u}_0\|_{X_T}$).

Recall that the differential Cauchy problem for Navier-Stokes equations reads as

$$\begin{cases} \partial_t \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} = \Delta \vec{u} - \vec{\nabla} p \\ \text{div } \vec{u} = 0 \\ \vec{u}(0, \cdot) = \vec{u}_0 \end{cases}$$

Under reasonable assumptions, the problem is equivalent to the following integro-differential problem :

$$\vec{u} = e^{t\Delta} \vec{u}_0 - B(\vec{u}, \vec{u})(t, x)$$

where

$$B(\vec{u}, \vec{v}) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \text{div}(\vec{u} \otimes \vec{v}) ds \quad (2)$$

and \mathbb{P} is the Leray projection operator. (See [LR 1, LR 5] for details).

Koch and Tataru's theorem is then the following one:

Theorem 1.

There exists C_0 (which does not depend on T) such that, if \vec{u} and \vec{v} are defined on $(0, T) \times \mathbb{R}^3$, then

$$\|B(\vec{u}, \vec{v})\|_{X_T} \leq C_0 \|\vec{u}\|_{X_T} \|\vec{v}\|_{X_T}.$$

Corollary 1.

Let $\vec{u}_0 \in \text{bmo}^{-1}$ with $\text{div } \vec{u}_0 = 0$. If $\|e^{t\Delta} \vec{u}_0\|_{X_T} < \frac{1}{4C_0}$, then the integral Navier–Stokes equations have a solution on $(0, T)$ such that $\|\vec{u}\|_{X_T} \leq 2\|e^{t\Delta} \vec{u}_0\|_{X_T}$.

This is the unique solution such that $\|\vec{u}\|_{X_T} \leq \frac{1}{2C_0}$.

The solution \vec{u} can be computed through Picard iteration as the limit of \vec{U}_n , where $\vec{U}_0 = e^{t\Delta} \vec{u}_0$ and $\vec{U}_{n+1} = e^{t\Delta} \vec{u}_0 - B(\vec{U}_n, \vec{U}_n)$. In particular, we have, by induction,

$$\|\vec{U}_{n+1} - \vec{U}_n\|_{X_T} \leq (4C_0 \|e^{t\Delta} \vec{u}_0\|_{X_T})^{n+1} \|e^{t\Delta} \vec{u}_0\|_{X_T}.$$

Thus, Corollary 1 grants local existence of a solution for the Navier–Stokes equations when the initial value belongs to the space bmo_0^{-1} :

Definition 1.

$\vec{u}_0 \in \text{bmo}_0^{-1}$ if $\vec{u} \in \text{bmo}^{-1}$ and $\lim_{T \rightarrow 0} \|e^{t\Delta} \vec{u}_0\|_{X_T} = 0$.

We may now recall Barker’s theorem [BAR]:

Theorem 2.

Let \vec{u}_0 be a divergence-free vector field with $\vec{u}_0 \in L^2$. Assume moreover

$$\vec{u}_0 \in \text{bmo}_0^{-1} \cap B_{q,\infty}^{-s} \text{ with } 3 < q < +\infty \text{ and } s < 1 - \frac{2}{q}$$

and let \vec{u} be the mild solution of the Navier–Stokes equations with initial value \vec{u}_0 such that $\|\vec{u}\|_{X_T} \leq \frac{1}{2C_0}$. If \vec{v} is a weak Leray solution of the Navier–Stokes equations with the same initial value \vec{u}_0 , then $\vec{u} = \vec{v}$ on $(0, T)$.

Again, we remark that, if $0 < s < 1 - \frac{2}{q}$ and if $\vec{u}_0 \in \text{bmo}_0^{-1}$, if \vec{u} is the mild solution with $\|\vec{u}\|_{X_T} \leq \frac{1}{2C_0}$, then $\vec{u}_0 \in B_{q,\infty}^{-s}$ is equivalent to

$$\sup_{0 < t < T} t^{s/2} \|\vec{u}(t, \cdot)\|_q < +\infty.$$

In the following theorems, we shall state the assumptions in terms of the mild solution \vec{u} instead of the initial value \vec{u}_0 . In Theorem 5, we shall give the

equivalence between the assumption on the solution \vec{u} and the assumption on the initial value \vec{u}_0 .

We aim to generalize Barker's result to a larger class of mild solutions. Barker's result is based on an interpolation lemma which states that, if $\vec{u}_0 \in \text{bmo}_0^{-1} \cap L^2 \cap B_{q,\infty}^{-s}$ with $3 < q < +\infty$ and $-s > -1 + \frac{2}{q}$, then $\vec{u}_0 \in [L^2, B_{\infty,\infty}^{-\delta}]_{\theta,\infty}$ for some $\theta \in (0, 1)$ and some $\delta \in (0, 1)$. (Those conditions are in a way equivalent, as we shall see in Corollary 2.) Then the comparison between the Leray solution \vec{v} and the mild solution \vec{u} is performed through an estimation of both $\|\vec{u} - \vec{w}_\epsilon\|_2$ and $\|\vec{v} - \vec{w}_\epsilon\|_2$, where \vec{w}_ϵ is the solution of the Navier–Stokes problem with initial value $\vec{w}_{0,\epsilon}$ such that $\|\vec{w}_{0,\epsilon} - \vec{u}_0\|_2 \leq C_1 \epsilon^\theta$ and $\|\vec{w}_\epsilon\|_{B_{\infty,\infty}^{-\delta}} < C_1 \epsilon^{\theta-1}$ (with C_1 depending on \vec{u}_0 but not on ϵ).

Our idea is to replace the space L^2 by the larger space $L_w^2 = L^2(w \, dx)$ with $w(x) = \frac{1}{(1+|x|)^2}$, and use the interpolation space $[L_w^2, B_{\infty,\infty}^{-\delta}]_{\theta,\infty}$ for some $\theta \in (0, 1)$ and some $\delta \in (0, 1)$. As we shall no longer deal with the L^2 norm, the Leray inequality on $\|\vec{v}\|_2$ will not be sufficient. Instead, we shall consider a stricter class of weak solutions, namely the suitable weak Leray solutions [CKN]:

Definition 2.

A Leray solution is suitable on $(0, T)$ if it fulfills the local energy inequality: there exists a non-negative locally finite measure μ on $(0, T) \times \mathbb{R}^3$ such that we have

$$\partial_t(|\vec{u}|^2) + 2|\vec{\nabla} \otimes \vec{u}|^2 = \Delta(|\vec{u}|^2) - \text{div}((2p + |\vec{u}|^2)\vec{u}) - \mu. \quad (3)$$

We may now state our main results. The first one (stated in [LR 6]) weakens the integrability requirement on the solution \vec{u} from the Lebesgue space L^q to the Morrey space $M^{p,q}$. Recall that the Morrey space $M^{p,q}$, $1 < p \leq q < +\infty$, is defined by

$$\|f\|_{M^{p,q}} = \sup_{x_0 \in \mathbb{R}^3} \sup_{0 < r \leq 1} r^{\frac{3}{q} - \frac{3}{p}} \left(\int_{B(x_0, r)} |f(x)|^p \, dx \right)^{\frac{1}{p}} < +\infty.$$

For $p = 1$, one replaces the requirement $f \in L_{\text{loc}}^p$ by the assumption that f is a locally finite Borel measure μ with

$$\|f\|_{M^{1,q}} = \sup_{x_0 \in \mathbb{R}^3} \sup_{0 < r \leq 1} r^{\frac{3}{q} - 3} \int_{B(x_0, r)} d|\mu|(x) < +\infty.$$

For $1 < p \leq +\infty$, we have the continuous embeddings

$$L^q \subset M^{q,q} \subset M^{p,q} \subset M^{1,q}.$$

The idea of considering Morrey spaces instead of Lebesgue spaces is quite natural. Indeed, in the direct proof of the Prodi–Serrin criterion, a key estimate is the inequality

$$\int |uv| |\vec{\nabla} w| dx \leq C \|u\|_q \|v\|_2^{1-\theta} \|\vec{\nabla} v\|_2^\theta \|\vec{\nabla} w\|_2$$

for $0 \leq \theta \leq 1$ and $\frac{1}{q} = \frac{\theta}{3}$. This inequality still holds when the L^q norm is replaced by the norm in the homogeneous Morrey space $\dot{M}^{2,q}$ with $0 < \theta < 1$ and $\frac{1}{q} = \frac{\theta}{3}$ [LR 2].

Theorem 3.

Let \vec{u}_0 be a divergence-free vector field with $\vec{u}_0 \in L^2 \cap \text{bmo}_0^{-1}$. Assume moreover that the mild solution \vec{u} of the Navier–Stokes equations with initial value \vec{u}_0 such that $\|\vec{u}\|_{X_T} \leq \frac{1}{2C_0}$ is such that

$$\sup_{0 < t < T} t^{s/2} \|\vec{u}(t, \cdot)\|_{\dot{M}^{p,q}} < +\infty \text{ with } 2 < p \leq q < +\infty \text{ and } 0 \leq s < 1 - \frac{2}{p}.$$

If \vec{v} is a suitable weak Leray solution of the Navier–Stokes equations with the same initial value \vec{u}_0 , then $\vec{u} = \vec{v}$ on $(0, T)$.

Let us remark that the statement and proof of Theorem 3 we gave in [LR 6] was false (we assumed only that $s < 1 - \frac{2}{q}$).

The second one weakens the integrability requirement on the solution \vec{u} from the Lebesgue space L^q to the weighted Lebesgue space $L^q(\frac{1}{(1+|x|)^N} dx)$ for some $N \geq 0$.

Theorem 4.

Let \vec{u}_0 be a divergence-free vector field with $\vec{u}_0 \in L^2 \cap \text{bmo}_0^{-1}$. Assume moreover that the mild solution \vec{u} of the Navier–Stokes equations with initial value \vec{u}_0 such that $\|\vec{u}\|_{X_T} \leq \frac{1}{2C_0}$ is such that

$$\sup_{0 < t < T} t^{s/2} \|\vec{u}\|_{L^q(\frac{1}{(1+|x|)^N} dx)} < +\infty \text{ with } N \geq 0, 2 < q < +\infty \text{ and } 0 \leq s < 1 - \frac{2}{q}.$$

If \vec{v} is a suitable weak Leray solution of the Navier–Stokes equations with the same initial value \vec{u}_0 , then $\vec{u} = \vec{v}$ on $(0, T)$.

Of course, Theorem 3 is a corollary of Theorem 4, as $M^{p,q} \subset L^p(\frac{1}{(1+|x|)^N} dx)$ for $N > 3 - \frac{3p}{q}$.

The paper is then organized in the following manner:

²The mistake was due to an incorrect equality $\rho = \eta\gamma$ while it should have been $\gamma = \eta\rho$; as $\eta < 1$, the equality turned to be incorrect.

- In Section 2, we define stable spaces and collect some technical results on generalized Besov spaces based on stable spaces.
- In Section 3, we define potential spaces based on stable spaces and prove some interpolation estimates.
- In section 4, we give some remarks on the Koch and Tataru solutions for the Navier–Stokes problem.
- In section 5, we study stability estimates for suitable weak Leray solutions with initial data in $L^2 \cap [L^2(\frac{1}{(1+|x|)^2}, B_{\infty,\infty}^{-\delta})]_{\theta,\infty}$ (see Theorem 8).
- In section 6, we prove the uniqueness theorem (Theorem 4).
- In section 7, we pay some further comments on Barker’s conjecture on the uniqueness problem.

2 Stable spaces and Besov spaces.

We define the convolutor space \mathbb{K} by the following convention:

- a **suitable kernel** is a function $K \in L^1(\mathbb{R}^3)$ such that K is radial and radially non-increasing (in particular, K is nonnegative); this is noted as $K \in \mathbb{K}_0$
- f is a convolutor if $f \in L^1$ and if there exists $K \in \mathbb{K}_0$ such that $|f| \leq K$ almost everywhere
- the norm of f in \mathbb{K} is defined as

$$\|f\|_{\mathbb{K}} = \inf\{\|K\|_1 / K \in \mathbb{K}_0 \text{ and } |f| \leq K \text{ a.e.}\}.$$

One easily checks that $\|\cdot\|_{\mathbb{K}}$ is a norm and that $(\mathbb{K}, \|\cdot\|_{\mathbb{K}})$ is a Banach space.

Definition 3.

A **stable space** of measurable functions on \mathbb{R}^3 is a Banach space E such that

- $E \subset L^1_{\text{loc}}(\mathbb{R}^3)$
- if $f \in E$ and $g \in L^\infty$, $fg \in E$ and $\|fg\|_E \leq C\|f\|_E\|g\|_\infty$ (where C does not depend on f nor g)

- if $f \in E$ and $g \in \mathbb{K}$, $f * g \in E$ and $\|f * g\|_E \leq C\|f\|_E\|g\|_{\mathbb{K}}$ (where C does not depend on f nor g).

Examples of stable spaces

- a) $E = L^p$, $1 \leq p \leq +\infty$.
- b) $E = L^p(w dx)$ where w belongs to the Muckenhoupt class \mathcal{A}_p for some $1 < p < +\infty$: if $g \in \mathbb{K}_0$, then

$$|f * g(x)| \leq \|g\|_1 \mathcal{M}_f(x)$$

where \mathcal{M}_f is the Hardy–Littlewood maximal function of f ; recall that the Hardy–Littlewood maximal function is a bounded sublinear operator on $L^p(w dx)$ when $w \in \mathcal{A}_p$ [STE 2].

- c) $E = L^p_{\text{uloc}}$ for some $1 \leq p \leq +\infty$, where

$$\|f\|_{L^p_{\text{uloc}}} = \sup_{x_0 \in \mathbb{R}^3} \left(\int_{B(x_0,1)} |f(x)|^p dx \right)^{\frac{1}{p}}.$$

By Minkowski's inequality, we have

$$\|f * g\|_E \leq \int |g(y)| \|f(\cdot - y)\|_{L^p_{\text{uloc}}} dy = \|g\|_1 \|f\|_{L^p_{\text{uloc}}}.$$

- d) This example can be generalized to other shift-invariant spaces (for which the norms $\|f\|_E$ and $\|f(\cdot - y)\|_E$ are equal). For instance, we may take E as the Morrey space $M^{p,q}$, $1 < p \leq q < +\infty$.

Our next step is to introduce Besov-like Banach spaces based on stable spaces and to describe the regularity of Koch–Tataru solutions when the initial value belongs moreover to the Besov space.

Definition 4.

Let $T \in (0, +\infty)$. Let E be a stable space of measurable functions on \mathbb{R}^3 . For $s > 0$ and $1 \leq q \leq +\infty$, we define the Besov-like Banach space $B_{E,q}^{-s}$ as the space of tempered distributions such that

$$t^{\frac{s}{2}} \|e^{t\Delta} f\|_E \in L^q((0, T), \frac{dt}{t}).$$

The norms $\|t^{\frac{s}{2}} \|e^{t\Delta} f\|_E\|_{L^q((0,T), \frac{dt}{t})}$ are all equivalent, so that $B_{E,q}^{-s}$ does not depend on T .

Proof. Assume that $t^{\frac{\sigma}{2}} \|e^{t\Delta} f\|_E \in L^q((0, T), \frac{dt}{t})$ for some $T > 0$ and consider $t \geq T$. We have

$$e^{t\Delta} f = \frac{2}{T} \int_{T/2}^T e^{(t-\theta)\Delta} e^{\theta\Delta} f d\theta$$

so that

$$\begin{aligned} \|e^{t\Delta} f\|_E &\leq C \frac{2}{T} \int_{T/2}^T \|e^{\theta\Delta} f\|_E d\theta \\ &\leq C \frac{2}{T} \|\theta^{\sigma/2} \|e^{\theta\Delta} f\|_E\|_{L^q((0, T), \frac{d\theta}{\theta})} \|\mathbb{1}_{T/2 < \theta} \theta^{1-\frac{\sigma}{2}}\|_{L^{\frac{q}{q-1}}((0, T), \frac{d\theta}{\theta})}. \end{aligned}$$

Equivalence of the norms is proved. \square

Remark: this proofs shows as well that, if $1 \leq q \leq r \leq +\infty$, then $B_{E,q}^{-s} \subset B_{E,r}^{-s}$. Another obvious property of Besov spaces is that, if $0 < s < \sigma$, then $B_{E,\infty}^{-s} \subset B_{E,1}^{-\sigma}$.

The main result in this section is the following theorem:

Theorem 5.

Let E be a stable space of measurable functions on \mathbb{R}^3 . Let $0 < T < +\infty$, and let $\vec{u}_0 \in \text{bmo}^{-1}$ with $\text{div } \vec{u}_0 = 0$ and $\|e^{t\Delta} \vec{u}_0\|_{X_T} < \frac{1}{4C_0}$. Let \vec{u} be the solution of the integral Navier–Stokes equations on $(0, T)$ such that $\|\vec{u}\|_{X_T} \leq \frac{1}{2C_0}$. Then the following assertions are equivalent for $0 < \sigma < 1$ and $2 < q \leq +\infty$:

- (A) $\vec{u}_0 \in B_{E,q}^{-\sigma}$
- (B) $t^{\frac{\sigma}{2}} \|\vec{u}\|_E \in L^q((0, T), \frac{dt}{t})$.

Proof. Let us remark that the operator $e^{(t-s)\Delta} \mathbb{P} \text{div}$ is a matrix of convolution operators whose kernels are bounded by $C \frac{1}{(\sqrt{t-s} + |x-y|)^4}$, hence are controlled in the convolutor norm $\|\cdot\|_{\mathbb{K}}$ by $C \frac{1}{\sqrt{t-s}}$. We thus have the inequality

$$\begin{aligned} \|B(\vec{u}, \vec{v})\|_E &\leq C \int_0^t \frac{1}{\sqrt{t-s}} \|\vec{u} \otimes \vec{v}\|_E ds \\ &\leq C' \sup_{0 < s < t} \sqrt{s} \|\vec{u}(s, \cdot)\|_{\infty} \int_0^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}} \|\vec{v}(s, \cdot)\|_E ds \end{aligned}$$

(and a similar estimate interchanging \vec{u} and \vec{v} in the last line). We thus want to estimate $J(t) = t^{-\frac{1}{q} + \frac{\sigma}{2}} \int_0^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}} s^{\frac{1}{q} - \frac{\sigma}{2}} L(s) ds$ with $L \in L^q((0, T), dt)$.

- if $q = +\infty$, we easily check that $\|J\|_{\infty} \leq C_{\sigma} \|L\|_{\infty}$ (since $\sigma < 1$).

- if $\sigma \leq \frac{2}{q}$, we have $s^{\frac{1}{q}-\frac{\sigma}{2}} \leq t^{\frac{1}{q}-\frac{\sigma}{2}}$, so that $J(t) \leq \int_0^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}} L(s) ds$. If $2 < q < +\infty$, as $\frac{1}{\sqrt{s}}$ belongs to the Lorentz space $L^{2,\infty}$, we use the product laws and convolution laws in Lorentz spaces to get that, if $L \in L^q$, $\frac{1}{\sqrt{s}} L \in L^{r,q}$ with $\frac{1}{r} = \frac{1}{q} + \frac{1}{2}$ and $\frac{1}{\sqrt{s}} * (\frac{1}{\sqrt{s}} L) \in L^{q,q} = L^q$. Thus, $\|J\|_q \leq C\|L\|_q$.
- if $\sigma > \frac{2}{q}$, we write

$$J(t) \leq C \left(\int_0^t \frac{(t-s)^{-\frac{1}{q}+\frac{\sigma}{2}}}{\sqrt{t-s}} \frac{1}{\sqrt{s}} s^{\frac{1}{q}-\frac{\sigma}{2}} L(s) ds + \int_0^t \frac{1}{\sqrt{t-s}} \frac{s^{-\frac{1}{q}+\frac{\sigma}{2}}}{\sqrt{s}} s^{\frac{1}{q}-\frac{\sigma}{2}} L(s) ds \right)$$

and we use again the product laws and convolution laws in Lorentz spaces to get that, if $L \in L^q$, $\frac{1}{s^{\frac{1+\sigma}{2}-\frac{1}{q}}} L \in L^{r,q}$ with $\frac{1}{r} = \frac{1+\sigma}{2}$ and $\frac{1}{s^{\frac{1-\sigma}{2}+\frac{1}{q}}} * (\frac{1}{s^{\frac{1+\sigma}{2}-\frac{1}{q}}} L) \in L^{q,q} = L^q$. We find again $\|J\|_q \leq C\|L\|_q$.

We may now easily check that $(B) \implies (A)$: we just write $e^{t\Delta} \vec{u}_0 = \vec{u} + B(\vec{u}, \vec{u})$ and

$$\left\| t^{\frac{\sigma}{2}} \|B(\vec{u}, \vec{u})\|_E \right\|_{L^q((0,T), \frac{dt}{t})} \leq C \sup_{0 < t < T} \sqrt{t} \|\vec{u}(t, \cdot)\|_\infty \left\| t^{\frac{\sigma}{2}} \|\vec{u}\|_E \right\|_{L^q((0,T), \frac{dt}{t})}.$$

In order to prove $(A) \implies (B)$, we write \vec{u} as the limit of \vec{U}_n , where $\vec{U}_0 = e^{t\Delta} \vec{u}_0$ and $\vec{U}_{n+1} = e^{t\Delta} \vec{u}_0 - B(\vec{U}_n, \vec{U}_n)$. By induction, \vec{U}_n satisfies

$$\left\| t^{\frac{\sigma}{2}} \|\vec{U}_n\|_E \right\|_{L^q((0,T), \frac{dt}{t})} < +\infty$$

and

$$\begin{aligned} & \left\| t^{\frac{\sigma}{2}} \|\vec{U}_{n+1} - \vec{U}_n\|_E \right\|_{L^q((0,T), \frac{dt}{t})} \\ & \leq C \sup_{0 < t < T} \sqrt{t} \|\vec{U}_n - \vec{U}_{n-1}\|_\infty \left(\left\| t^{\frac{\sigma}{2}} \|\vec{U}_n\|_E \right\|_{L^q((0,T), \frac{dt}{t})} + \left\| t^{\frac{\sigma}{2}} \|\vec{U}_{n-1}\|_E \right\|_{L^q((0,T), \frac{dt}{t})} \right). \end{aligned}$$

If

$$A_N = \left\| t^{\frac{\sigma}{2}} \|\vec{U}_0\|_E \right\|_{L^q((0,T), \frac{dt}{t})} + \sum_{n=0}^{N-1} \left\| t^{\frac{\sigma}{2}} \|\vec{U}_{n+1} - \vec{U}_n\|_E \right\|_{L^q((0,T), \frac{dt}{t})}$$

and $\epsilon = 4C_0 \|\vec{U}_0\|_{X_T}$, we have

$$\left\| t^{\frac{\sigma}{2}} \|\vec{U}_N\|_E \right\|_{L^q((0,T), \frac{dt}{t})} \leq A_N$$

and

$$A_{N+1} \leq A_N (1 + 2C\epsilon^{N+1}) \leq A_0 \prod_{j=1}^{N+1} (1 + 2C\epsilon^j).$$

This proves that $\left\| t^{\frac{\sigma}{2}} \|\vec{u}\|_E \right\|_{L^q((0,T), \frac{dt}{t})} < +\infty$. □

Let us remark that the assumption $\vec{u}_0 \in \text{bmo}^{-1}$ can be dropped in some cases, as for example the solutions \vec{u} in the Serrin class $L^q((0, T), L^r)$ with $\frac{2}{q} + \frac{3}{r} \leq 1$ and $3 < r < +\infty$. In analogy with L^r , we define r -stable spaces in the following way:

Definition 5.

For $2 < r < +\infty$, a **r -stable space** of measurable functions on \mathbb{R}^3 is a stable space E such that

- E is contained in $B_{\infty, \infty}^{-\frac{3}{r}}$ and, for $f \in E$, $\|f\|_{B_{\infty, \infty}^{-\frac{3}{r}}} \leq C\|f\|_E$.
- E is contained in L_{loc}^2
- If $f, g \in E$ then $fg \in B_{E, \infty}^{-\frac{3}{r}}$ and $\|fg\|_{B_{E, \infty}^{-\frac{3}{r}}} \leq C\|f\|_E\|g\|_E$.

The Morrey space $M^{2, r}$ is a r -stable space; it is more precisely the largest r -stable space:

Lemma 1.

Let E be a r -stable space of measurable functions on \mathbb{R}^3 , where $r \in (2, +\infty)$. Then $E \subset M^{2, r}$ and $\|f\|_{M^{2, r}} \leq C\|f\|_E$.

Proof. Let $\rho < 1$ and $x_0 \in \mathbb{R}^3$. We have

$$e^{\rho^2 \Delta}(f^2)(x_0) \geq \int_{B(x_0, \rho)} f^2(y) dy \inf_{y \in B(x_0, \rho)} W_{\rho^2}(x_0 - y) = \frac{e^{-\frac{1}{4}}}{(4\pi\rho^2)^{3/2}} \int_{B(x_0, \rho)} f^2(y) dy$$

where $W_t(x) = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|x|^2}{4t}}$. On the other hand, we have

$$e^{\rho^2 \Delta}(f^2)(x_0) \leq C\rho^{-\frac{3}{r}} \|e^{\rho^2 \Delta}(f^2)\|_{B_{\infty, \infty}^{-\frac{3}{r}}} \leq C'\rho^{-\frac{3}{r}} \|e^{\frac{\rho^2}{2} \Delta}(f^2)\|_E \leq C''\rho^{-\frac{6}{r}} \|f\|_E^2.$$

This gives

$$\int_{B(x_0, \rho)} f^2(y) dy \leq C\rho^{3-\frac{6}{r}} \|f\|_E^2$$

and thus $f \in M^{2, r}$. □

Theorem 6.

Let E be a r -stable space of measurable functions on \mathbb{R}^3 . Let $\vec{u}_0 \in E$ with $\text{div } \vec{u}_0 = 0$. Let $0 < \sigma < 1$ and $2 < q < +\infty$, with

$$\frac{2}{q} \leq \sigma \leq 1 - \frac{3}{r}$$

and $q < +\infty$ if $\sigma = 1 - \frac{3}{r}$. Then the following assertions are equivalent:

(A) $\vec{u}_0 \in B_{E,q}^{-\sigma}$

(B) There exists $T > 0$ and a solution \vec{u} of the integral Navier–Stokes equations on $(0, T)$ with initial value \vec{u}_0 such that $t^{\frac{\sigma}{2}} \|\vec{u}\|_E \in L^q((0, T), \frac{dt}{t})$.

(This theorem thus holds for solutions $\vec{u} \in L^q((0, T), E)$ under the Serrin condition $\frac{2}{q} + \frac{3}{r} \leq 1$.)

Proof. (A) \implies (B) is a direct consequence of Theorem 5 and of the embedding $B_{M^{2,r},q}^{-\sigma} \subset \text{bmo}_0^{-1}$ for $\sigma \leq 1 - \frac{3}{r}$ and $(\sigma, q) \neq (1 - \frac{3}{r}, \infty)$. Indeed, we have, for $0 < t < 1$,

$$\begin{aligned} \|e^{t\Delta} f\|_\infty &\leq \frac{2}{t} \int_{t/2}^t \|e^{\theta\Delta} f\|_\infty d\theta \\ &\leq C \frac{2}{t} t^{-\frac{3}{2r}} \int_{t/2}^t \|e^{\frac{\theta}{2}\Delta} f\|_{M^{2,r}} d\theta \\ &\leq C' t^{-1-\frac{3}{2r}} \|\theta^{\sigma/2} \|e^{\theta\Delta} f\|_{M^{2,r}}\|_{L^q((0,t),d\theta)} \|\theta^{-\frac{\sigma}{2}}\|_{L^{\frac{q}{q-1}}((t/2,t),d\theta)} \\ &\leq C'' t^{-1-\frac{3}{2r}} t^{\frac{1}{q}} \|\theta^{\sigma/2} \|e^{\theta\Delta} f\|_{M^{2,r}}\|_{L^q((0,t),\frac{d\theta}{\theta})} t^{1-\frac{1}{q}} t^{-\frac{\sigma}{2}} \\ &\leq C''' t^{-1/2} t^{\frac{1-\sigma-3/r}{2}} \left(\int_0^t (\theta^{\sigma/2} \|e^{\theta\Delta} f\|_{M^{2,r}})^q \frac{d\theta}{\theta} \right)^{1/q} \end{aligned}$$

and

$$\begin{aligned} \int_0^t \int_{B(x_0, \sqrt{t})} |e^{s\Delta} f|^2 dy ds &\leq C \int_0^t \|e^{s\Delta} f\|_{M^{2,r}}^2 t^{3/2-3/r} ds \\ &\leq C' t^{3/2-3/r} \|s^{\frac{\sigma}{2}} \|e^{s\Delta} f\|_{M^{2,r}}\|_{L^q((0,t),\frac{ds}{s})}^2 \|\mathbf{1}\|_{L^{\frac{q}{q-2}}((0,t),\frac{ds}{s})} \\ &\leq C'' t^{3/2-3/r} t^{(1-\sigma)\frac{2}{q}} \|s^{\frac{\sigma}{2}} \|e^{s\Delta} f\|_{M^{2,r}}\|_{L^q((0,t),\frac{ds}{s})}^2 t^{(1-\sigma)(1-\frac{2}{q})} \\ &\leq C''' t^{3/2} t^{1-\sigma-\frac{3}{r}} \left(\int_0^t (s^{\sigma/2} \|e^{s\Delta} f\|_{M^{2,r}})^q \frac{ds}{s} \right)^{2/q}. \end{aligned}$$

We now prove (B) \implies (A). We use again the identity

$$e^{t\Delta} \vec{u}_0 = \frac{2}{t} \int_{t/2}^t e^{(t-s)\Delta} e^{s\Delta} \vec{u}_0 ds$$

and get

$$e^{2t\Delta} \vec{u}_0 = \frac{2}{t} \int_{t/2}^t e^{(2t-s)\Delta} \vec{u}(s, \cdot) ds + \frac{2}{t} \int_{t/2}^t e^{(2t-s)\Delta} B(\vec{u}, \vec{u}) ds = \vec{v}(t, \cdot) + \vec{w}(t, \cdot).$$

We want to estimate $\|t^{\sigma/2}\|e^{2t\Delta}\vec{u}_0\|_E\|_{L^q((0,T),\frac{dt}{t})} = \|t^{\sigma/2-1/q}\|e^{2t\Delta}\vec{u}_0\|_E\|_{L^q((0,T),dt)}$.

We have

$$\begin{aligned} t^{\sigma/2-1/q}\|\vec{v}(t, \cdot)\|_E &\leq Ct^{\sigma/2-1/q}\frac{2}{t}\int_{t/2}^t\|\vec{u}\|_E ds \\ &\leq C\frac{2}{t}\int_{t/2}^t\|s^{\sigma/2-1/q}\|\vec{u}\|_E\| ds \\ &\leq 4C\mathcal{M}_{s^{\sigma/2-1/q}\|\vec{u}\|_E}(t) \end{aligned}$$

and thus $t^{\sigma/2-1/q}\|\vec{v}(t, \cdot)\|_E \in L^q((0, T), dt)$.

On the other hand, we have

$$\begin{aligned} \|\vec{w}(t, \cdot)\|_E &\leq \sup_{t/2 \leq s \leq t} \left\| \int_0^s e^{(\frac{3t}{2}-\tau)\Delta} \mathbb{P} \operatorname{div} e^{\frac{t}{2}\Delta} (\vec{u} \otimes \vec{u}) d\tau \right\|_E \\ &\leq C \int_0^t \frac{1}{\sqrt{\frac{3t}{2}-\tau}} \|e^{\frac{t}{2}\Delta} (\vec{u} \otimes \vec{u})\|_E d\tau \\ &\leq C' \int_0^t \frac{t^{\frac{1}{2}-\sigma+\frac{1}{q}}}{(t-\tau)^{1-\sigma+\frac{1}{q}}} \|e^{t\Delta} |\vec{u}|^2\|_E d\tau \\ &\leq C'' t^{\frac{1}{2}-\sigma+\frac{1}{q}-\frac{3}{2r}} \int_0^t \frac{1}{(t-\tau)^{1-\sigma+\frac{1}{q}}} \|\vec{u}\|_E^2 d\tau \end{aligned}$$

and thus

$$\begin{aligned} t^{\sigma/2-1/q}\|\vec{w}(t, \cdot)\|_E &\leq CT^{\frac{1}{2}-\frac{\sigma}{2}-\frac{3}{2r}} \int_0^t \frac{1}{(t-\tau)^{1-\sigma+\frac{1}{q}}} \|\vec{u}\|_E^2 d\tau \\ &= CT^{\frac{1}{2}-\frac{\sigma}{2}-\frac{3}{2r}} \int_0^t \frac{1}{(t-\tau)^{1-\sigma+\frac{1}{q}}} \tau^{-\sigma+\frac{2}{q}} (\tau^{\sigma/2-\frac{1}{q}} \|\vec{u}\|_E)^2 d\tau. \end{aligned}$$

If $J(\tau) = \tau^{\sigma/2-\frac{1}{q}} \|\vec{u}\|_E$, we have $J(\tau) \in L^q((0, T), d\tau)$, hence $J^2 \in L^{q/2}((0, T), d\tau)$, $\tau^{-\sigma+\frac{2}{q}} J^2 \in L^{p_0, q/2}((0, T), dt)$ with $\frac{1}{p_0} = \frac{2}{q} + \sigma - \frac{2}{q} = \sigma$ and $\frac{1}{\tau^{1-\sigma+\frac{1}{q}}} * (\tau^{-\sigma+\frac{2}{q}} J^2) \in L^{p_1, q/2}((0, T), dt)$ with $\frac{1}{p_1} = \frac{1}{p_0} + 1 - \frac{1}{\sigma} + \frac{1}{q} - 1 = \frac{1}{q}$.

Thus $t^{\sigma/2-1/q}\|e^{2t\Delta}\vec{u}_0(t, \cdot)\|_E \in L^q((0, T), dt)$ and $\vec{u}_0 \in B_{E, q}^{-\sigma}$. \square

The case $(\sigma, q) = (1 - \frac{3}{r}, +\infty)$ can be treated in a similar way:

Theorem 7.

Let E be a r -stable space of measurable functions on \mathbb{R}^3 with $3 < r < +\infty$. Let $\vec{u}_0 \in E$ with $\operatorname{div} \vec{u}_0 = 0$. Then the following assertions are equivalent:

- (A) $\vec{u}_0 \in B_{E,\infty}^{-1+\frac{3}{r}}$ and $\lim_{t \rightarrow 0} t^{\frac{1}{2}-\frac{3}{2r}} \|e^{t\Delta} \vec{u}_0\|_E = 0$.
 (B) There exists $T > 0$ and a solution \vec{u} of the integral Navier–Stokes equations on $(0, T)$ with initial value \vec{u}_0 such that $\sup_{0 < t < T} t^{\frac{1}{2}-\frac{3}{2r}} \|\vec{u}\|_E < +\infty$ and $\lim_{t \rightarrow 0} t^{\frac{1}{2}-\frac{3}{2r}} \|\vec{u}\|_E = 0$.

Remark: We have the embedding $B_{E,\infty}^{-1+\frac{3}{r}} \subset \text{bmo}^{-1}$, but this does not grant existence of a solution. The extra condition $\lim_{t \rightarrow 0} t^{\frac{1}{2}-\frac{3}{2r}} \|e^{t\Delta} \vec{u}_0\|_E = 0$ is used to get $\vec{u}_0 \in \text{bmo}_0^{-1}$, and thus to have existence of a local solution.

3 Potential spaces and interpolation

If E is a stable space, we define, for $s \in \mathbb{R}$, the potential space H_E^s as $H_E^s = (\text{Id} - \Delta)^{-s/2} E$, normed with $\|f\|_{H_E^s} = \|(\text{Id} - \Delta)^{s/2} f\|_E$. For positive s , we have an obvious comparison of the potential space H_E^{-s} with the Besov spaces:

Lemma 2. *Let E be a stable space, and $s > 0$. Then,*

$$B_{E,1}^{-s} \subset H_E^{-s} \subset B_{E,\infty}^{-s}.$$

Proof. Indeed, we have

$$(\text{Id} - \Delta)^{-s/2} = \frac{1}{\Gamma(s/2)} \int_0^{+\infty} e^{-t} e^{t\Delta} t^{s/2} \frac{dt}{t}.$$

If f belongs to $B_{E,1}^{-s}$, then $t^{s/2} \|e^{t\Delta} f\|_E \in L^1((0, 1), \frac{dt}{t})$ while $\|e^{\Delta} f\|_1 \leq \|f\|_{B_{E,\infty}^{-s}} \leq C \|f\|_{B_{E,1}^{-s}}$ so that

$$\|f\|_{H_E^{-s}} \leq \frac{1}{\Gamma(s/2)} \left(\int_0^1 t^{s/2} \|f\|_E \frac{dt}{t} + C \|e^{\Delta} f\|_E \int_0^{+\infty} e^{-t} t^{s/2} \frac{dt}{t} \right) \leq C' \|f\|_{B_{E,1}^{-s}}.$$

Conversely, if $f \in H_E^{-s}$, $f = (\text{Id} - \Delta)^{s/2} g$ and if $0 < \theta < 1$, then we pick $N \in \mathbb{N}$ with $N > s/2$ and write

$$\begin{aligned} e^{\theta\Delta} f &= e^{\theta\Delta} (\text{Id} - \Delta)^N (\text{Id} - \Delta)^{s/2-N} g \\ &= \frac{1}{\Gamma(N - s/2)} \int_0^{+\infty} e^{-t} (\text{Id} - \Delta)^N e^{(t+\theta)\Delta} g t^{N-\frac{s}{2}} \frac{dt}{t}. \end{aligned}$$

For $\alpha \in \mathbb{N}^3$, with $0 \leq |\alpha| \leq 2N$, we have

$$\|\partial^\alpha e^{(t+\theta)\Delta} g\|_E \leq C_\alpha (t + \theta)^{-\frac{|\alpha|}{2}} \|g\|_E \leq C_\alpha (1 + (t + \theta)^{-N}) \|g\|_E$$

so that

$$\begin{aligned}
\|e^{\theta\Delta}f\|_E &\leq C\|g\|_E \int_0^{+\infty} e^{-t}(1+(t+\theta)^{-N}) t^{N-\frac{s}{2}} \frac{dt}{t} \\
&\leq C\|g\|_E (\Gamma(N-s/2) + \int_0^\theta t^{N-\frac{s}{2}} \frac{dt}{t} + \int_\theta^{+\infty} \frac{dt}{t^{1+s/2}}) \\
&\leq C'\|g\|_E \theta^{-s/2}.
\end{aligned}$$

The lemma is proved. \square

Let us recall the definition of Calderón's interpolation spaces $[A_0, A_1]_\theta$ and $[A_0, A_1]^\theta$ [CAL]. We assume that A_0 and A_1 are subspaces of \mathcal{S}' , so that $A_0 \cap A_1$ and $A_0 + A_1$ are well-defined.

We begin with the definition of the first interpolate $[A_0, A_1]_\theta$. Let Ω be the open complex strip $\Omega = \{z \in \mathbb{C} / 0 < \Re z < 1\}$. $\mathcal{F}(A_0, A_1)$ is the space of functions F defined on the closed complex strip $\overline{\Omega}$ such that :

1. F is continuous and bounded from $\overline{\Omega}$ to $A_0 + A_1$
2. F is analytic from Ω to $A_0 + A_1$
3. $t \mapsto F(it)$ is continuous from \mathbb{R} to A_0 , and $\lim_{|t| \rightarrow +\infty} \|F(it)\|_{A_0} = 0$
4. $t \mapsto F(1+it)$ is continuous from \mathbb{R} to A_1 , and $\lim_{|t| \rightarrow +\infty} \|F(1+it)\|_{A_0} = 0$

Then

$$f \in [A_0, A_1]_\theta \Leftrightarrow \exists F \in \mathcal{F}(A_0, A_1), f = F(\theta)$$

and

$$\|f\|_{[A_0, A_1]_\theta} = \inf_{f=F(\theta)} \max(\sup_{t \in \mathbb{R}} \|F(it)\|_{A_0}, \sup_{t \in \mathbb{R}} \|F(1+it)\|_{A_1}).$$

Now, let us recall the definition of the second interpolate $[A_0, A_1]^\theta$. $\mathcal{G}(A_0, A_1)$ is the space of functions G defined on the closed complex strip $\overline{\Omega}$ such that :

1. $\frac{1}{1+|z|}G$ is continuous and bounded from $\overline{\Omega}$ to $A_0 + A_1$
2. G is analytic from Ω to $A_0 + A_1$
3. $t \mapsto G(it) - G(0)$ is Lipschitz from \mathbb{R} to A_0
4. $t \mapsto G(1+it) - G(1)$ is Lipschitz from \mathbb{R} to A_1

Then

$$f \in [A_0, A_1]^\theta \Leftrightarrow \exists G \in \mathcal{G}(A_0, A_1), f = G'(\theta)$$

and

$$\|f\|_{[A_0, A_1]^\theta} = \inf_{f=G'(\theta)} \max \left(\sup_{t_1, t_2 \in \mathbb{R}} \left\| \frac{G(it_2) - G(it_1)}{t_2 - t_1} \right\|_{A_0}, \sup_{t_1, t_2 \in \mathbb{R}} \left\| \frac{G(1+it_2) - G(1+it_1)}{t_2 - t_1} \right\|_{A_1} \right).$$

Three important properties of those complex interpolation functors are:

- the equivalence theorem: if A_0 (or A_1) is reflexive, then $[A_0, A_1]^\theta = [A_0, A_1]_\theta$ for $0 < \theta < 1$;
- the duality theorem: if $A_0 \cap A_1$ is dense in A_0 and A_1 , then $([A_0, A_1]_\theta)' = [A_0', A_1']_\theta$ for $0 < \theta < 1$.
- the density theorem: $A_0 \cap A_1$ is dense in $[A_0, A_1]_\theta$

An easy classical example of interpolation concerns the Lebesgue spaces L^p on a measured space (X, μ) : $[L^{p_0}, L^{p_1}]_\theta = L^p$ with $1 < p_0 < +\infty$, $1 < p_1 < +\infty$, $0 < \theta < 1$ and $\frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1}$. Indeed, if $f \in L^p$, we write $f = F_\theta$ where $F_z(x) = |f(x)|^{(1-z)\frac{p}{p_0} + z\frac{p}{p_1}} \frac{f(x)}{|f(x)|^{\frac{p}{p_1}}}$. If $p_0 \leq p_1$, we have $|F_z(x)| \leq |f(x)|^{\frac{p}{p_0}}$ if $|f(x)| \geq 1$. and $|F_z(x)| \leq |f(x)|^{\frac{p}{p_1}}$ if $|f(x)| < 1$. By dominated convergence, this gives the continuity of F from $\overline{\Omega}$ to $L^{p_0} + L^{p_1}$. For the holomorphy, we use the equivalence between (strong) holomorphy and weak-* holomorphy; thus, it is enough to check that $z \in \Omega \mapsto \int F_z(x)g(x) d\mu$ is holomorph if $g \in L^{q_0} \cap L^{q_1}$, where $\frac{1}{q_i} + \frac{1}{p_i} = 1$. Thus, we obtain that $L^p \subset [L^{p_0}, L^{p_1}]_\theta$. As

$$[L^{p_0}, L^{p_1}]_\theta = [L^{p_0}, L^{p_1}]^\theta = ([L^{q_0}, L^{q_1}]_\theta)'$$

and as L^q is dense in $[L^{q_0}, L^{q_1}]_\theta$ (where $\frac{1}{q} + \frac{1}{p} = 1$), we obtain from the embedding $L^q \subset [L^{q_0}, L^{q_1}]_\theta$ that $[L^{p_0}, L^{p_1}]_\theta \subset L^p$.

A similar result holds for weighted Lebesgue spaces $L^p(w d\mu)$:

$$[L^{p_0}(w_0 d\mu), L^{p_1}(w_1 d\mu)]_\theta = L^p(w d\mu)$$

with $1 < p_0 < +\infty$, $1 < p_1 < +\infty$, $0 < \theta < 1$ and $\frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1}$ and $w = w_0^{1-\theta} w_1^\theta$. If $f \in L^p(w d\mu)$, one defines

$$F_z(x) = \left(\frac{w(x)}{w_0(x)} \right)^{(1-z)\frac{1}{p_0}} \left(\frac{w(x)}{w_1(x)} \right)^{z\frac{1}{p_1}} |f(x)|^{(1-z)\frac{p}{p_0} + z\frac{p}{p_1}} \frac{f(x)}{|f(x)|^{\frac{p}{p_1}}}.$$

We have

$$|F_z(x)| \leq \max \left(\left(\frac{w(x)}{w_0(x)} \right)^{\frac{1}{p_0}} |f(x)|^{\frac{p}{p_0}}, \left(\frac{w(x)}{w_1(x)} \right)^{\frac{1}{p_1}} |f(x)|^{\frac{p}{p_1}} \right)$$

The proof then is similar to the case of Lebesgue spaces.

If we want to interpolate Morrey spaces $M^{p_0, q_0}(\mathbb{R}^3)$ and $M^{p_1, q_1}(\mathbb{R}^3)$ and obtain a Morrey space, then it is necessary to assume that $\frac{p_0}{q_0} = \frac{p_1}{q_1}$ [LR 3, LR 4]. We then obtain:

$$[M^{p_0, q_0}, M^{p_1, q_1}]^\theta = M^{p, q}$$

when $1 < p_0 \leq q_0 < +\infty$, $1 < p_1 \leq q_1 < +\infty$, $\frac{p_0}{q_0} = \frac{p_1}{q_1}$, $0 < \theta < 1$, $\frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1}$ and $\frac{1}{q} = (1-\theta)\frac{1}{q_0} + \theta\frac{1}{q_1}$. As $M^{p_0, q_0} \cap M^{p_1, q_1}$ is not dense in $M^{p, q}$ and is dense in $[M^{p_0, q_0}, M^{p_1, q_1}]_\theta$, we can see that we must use the second interpolation functor. The embedding $[M^{p_0, q_0}, M^{p_1, q_1}]^\theta \subset M^{p, q}$ is obvious: for a ball B with radius $r \leq 1$, we have that the map $f \mapsto f\mathbf{1}_B$ is bounded from M^{p_0, q_0} to L^{p_0} with norm less or equal to $r^{3(\frac{1}{p_0} - \frac{1}{q_0})}$ and from M^{p_1, q_1} to L^{p_1} with norm less or equal to $r^{3(\frac{1}{p_1} - \frac{1}{q_1})}$, hence from $[M^{p_0, q_0}, M^{p_1, q_1}]^\theta$ to $[L^{p_0}, L^{p_1}]^\theta$ with norm less or equal to $r^{3(\frac{1}{p} - \frac{1}{q})}$. As $[L^{p_0}, L^{p_1}]^\theta = L^p$, we obtain the desired estimates.

If f belongs to $M^{p, q}$, we define $F_z(x) = |f(x)|^{(1-z)\frac{p}{p_0} + z\frac{p}{p_1}} \frac{f(x)}{|f(x)|}$. As $|F_z(x)| \leq \max(|f(x)|^{\frac{p}{p_0}}, |f(x)|^{\frac{p}{p_1}})$, we find that $z \mapsto F_z$ is bounded from $\overline{\Omega}$ to $M^{p_0, q_0} + M^{p_1, q_1}$ and holomorph on the open strip Ω (again by equivalence between analyticity and weak-* analyticity). But it is no longer continuous, and we cannot apply the first functor of Calderón. Instead, we follow Cwikel and Janson [CWJ] and define $G_z = \int_{1/2}^z F_w dw$. We may then apply the definition of the second functor and find that $f \in [M^{p_0, q_0}, M^{p_1, q_1}]^\theta$. Thus, $[M^{p_0, q_0}, M^{p_1, q_1}]^\theta = M^{p, q}$.

Now, we are going to describe complex interpolation of potential spaces on weighted Lebesgue spaces when varying both the regularity exponents and the weights³:

Proposition 2.

Let $\theta \in (0, 1)$, s_0, s_1 be real numbers, $1 < p_0, p_1 < +\infty$ and $s = (1-\theta)s_0 + \theta s_1$ and $\frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1}$. Then, if w_0 is a weight in the Muckenhoupt class \mathcal{A}_{p_0} and w_1 is a weight in the Muckenhoupt class \mathcal{A}_{p_1} ,

$$(\text{Id} - \Delta)^{-s} L^p(w_0^{1-\theta} w_1^\theta dx) = [(\text{Id} - \Delta)^{-s_0} L^{p_0}(w_0 dx), (\text{Id} - \Delta)^{-s_1} L^{p_1}(w_1 dx)]_\theta.$$

Proof. Let $f = (\text{Id} - \Delta)^{-s} g$ where $g \in L^p(w dx)$ We define

$$H_z(x) = \left(\frac{w(x)}{w_0(x)} \right)^{(1-z)\frac{1}{p_0}} \left(\frac{w(x)}{w_1(x)} \right)^{z\frac{1}{p_1}} |f(x)|^{(1-z)\frac{p}{p_0} + z\frac{p}{p_1}} \frac{f(x)}{|f(x)|}$$

³This can be seen as a variation on Stein's interpolation theorem [STE 1, CWJ].

and

$$F_{z,\epsilon}(\cdot) = \left(\frac{2-\theta}{2-z} \right)^4 e^{\epsilon\Delta} (\text{Id} - \Delta)^{-(1-z)s_0 - zs_1} H_z.$$

We first remark that, for $\epsilon > 0$ fixed, the operators $e^{\epsilon\Delta} (\text{Id} - \Delta)^{-\tau}$ with $\tau \in [s_0, s_1]$ are equicontinuous from $L^{p_i}(w_i dx)$ to $(\text{Id} - \Delta)^{-s_i} L^{p_i}(w_i dx)$ (it is enough to check that the norms of the convolutors $e^{\epsilon\Delta} (\text{Id} - \Delta)^{s_i - \tau}$ in \mathbb{K} are uniformly bounded).

Moreover, the operators $\left(\frac{2-\theta}{2-it} \right)^4 (\text{Id} - \Delta)^{-it}$, $t \in \mathbb{R}$, are uniformly bounded on $L^{p_i}(w_i dx)$. Let us recall the definition of Calderón–Zygmund convolutors. A Calderón–Zygmund convolutor is a distribution $K \in \mathcal{S}'(\mathbb{R}^3)$ such that $\hat{K} \in L^\infty$ (so that convolution with K is a bounded operator on L^2) and such that, when restricted to $\mathbb{R}^3 \setminus \{0\}$, K is defined by a locally Lipschitz function such that $\sup_{x \neq 0} |x|^3 |K(x)| + |x|^4 |\vec{\nabla} x| < +\infty$. The space CZ of Calderón–Zygmund convolutors is normed by

$$\|K\|_{CZ} = \|\hat{K}\|_\infty + \sup_{x \neq 0} |x|^3 |K(x)| + |x|^4 |\vec{\nabla} x|.$$

If $1 < p < +\infty$ and $w \in \mathcal{A}_p$ and $K \in \text{CZ}$, we have $\|f * K\|_{L^p(w dx)} \leq C_{w,p} \|f\|_{L^p(w dx)} \|K\|_{CZ}$. Since we have

$$\|K\|_{CZ} \leq C \sum_{|\alpha| \leq 4} \| |\xi|^{|\alpha|} \partial_\xi^\alpha \hat{K} \|_\infty,$$

it is clear that $\left(\frac{2-\theta}{2-it} \right)^4 (\text{Id} - \Delta)^{-it} f = K_t * f$ with $\sup_{t \in \mathbb{R}} \|K_t\|_{CZ} < +\infty$.

We may apply the second interpolation functor and find that $e^{\epsilon\Delta} f = F_{\theta,\epsilon} \in [(\text{Id} - \Delta)^{-s_0} L^{p_0}(w_0 dx), (\text{Id} - \Delta)^{-s_1} L^{p_1}(w_1 dx)]_\theta$ if $g \in L^p(w dx)$. Moreover its norm is controlled independently from $\epsilon > 0$ as, for $\alpha = 0$ or $\alpha = 1$, the functions $H_{\alpha+it}$ are bounded in $L^{p_\alpha}(w_\alpha dx)$, the operators $\left(\frac{2-\theta}{2-it} \right)^4 (\text{Id} - \Delta)^{-it}$ are equicontinuous on $L^{p_\alpha}(w_\alpha dx)$ and the operators $e^{\epsilon\Delta}$ are equicontinuous on $L^{p_\alpha}(w_\alpha dx)$. One then writes

$$F_{\alpha+it,\epsilon} = (\text{Id} - \Delta)^{-(1-\alpha)s_0 - \alpha s_1} \left(e^{\epsilon\Delta} \left(\frac{2-\theta}{2-it} \right)^4 (\text{Id} - \Delta)^{-it} H_{\alpha+it} \right).$$

To conclude, we remark that $L^{p_i}(w_i dx)$ is the dual of $L^{q_i}(w_i^{-\frac{q_i}{p_i}} dx)$ and that \mathcal{S} is dense in this predual. Thus $e^{\epsilon\Delta} f$ is bounded in

$$[H_{L^{p_0}(w_0 dx)}^{s_0}, H_{L^{p_1}(w_1 dx)}^{s_1}]^\theta = ([(\text{Id} - \Delta)^{s_0} L^{q_0}(w_0^{-\frac{q_0}{p_0}} dx), (\text{Id} - \Delta)^{s_1} L^{q_1}(w_1^{-\frac{q_1}{p_1}} dx)]_\theta)'$$

if $(\text{Id} - \Delta)^s f \in L^p(w dx)$. As ϵ goes to 0, $e^{\epsilon\Delta} f$ is weak- * convergent to f . Thus,

$$(\text{Id} - \Delta)^{-s} L^p(w dx) \subset [(\text{Id} - \Delta)^{-s_0} L^{p_0}(w_0 dx), (\text{Id} - \Delta)^{-s_1} L^{p_1}(w_1 dx)]^\theta$$

and we can switch the second and the first interpolation functors as $H_{L^{p_0}(w_0 dx)}^{s_0}$ is reflexive.

Conversely, assume that $f \in [(\text{Id} - \Delta)^{-s_0} L^{p_0}(w_0 dx), (\text{Id} - \Delta)^{-s_1} L^{p_1}(w_1 dx)]^\theta$ and pick $F \in \mathcal{F}((\text{Id} - \Delta)^{-s_0} L^{p_0}(w_0 dx), (\text{Id} - \Delta)^{-s_1} L^{p_1}(w_1 dx))$ such that $f = F(\theta)$. Define

$$H_{z,\epsilon} = \left(\frac{2 - \theta}{2 - z} \right)^4 e^{\epsilon\Delta} (\text{Id} - \Delta)^{(1-z)s_0 + zs_1} F_z.$$

We easily check that $H_{z,\epsilon} \in \mathcal{A}(L^{p_0}(w_0 dx), L^{p_1}(w_1 dx))$ with $H_{\theta,\epsilon} = e^{\epsilon\Delta} (\text{Id} - \Delta)^s f$. Thus, we find that $e^{\epsilon\Delta} (\text{Id} - \Delta)^s f$ is bounded in $[L^{p_0}(w_0 dx), L^{p_1}(w_1 dx)]_\theta = L^p(w dx)$, and finally $f \in (\text{Id} - \Delta)^{-s} L^p(w dx)$. \square

Corollary 2.

Let $2 < q < +\infty$ and $s < 1 - \frac{2}{q}$. Then there exists such that :

a) There exists $\gamma > 0$ and $2 < r < +\infty$ such that $\gamma + \frac{3}{r} < 1$ and $\theta \in (0, 1)$ such that

$$B_{q,\infty}^{-s} \subset [L^2, H_r^{-\gamma}]_{\theta,\infty} \subset [L^2, B_{\infty,\infty}^{-\sigma}]_{\theta,\infty}.$$

b) For $0 \leq N < \frac{4}{q}$, there exists $\gamma > 0$ and $2 < r < +\infty$ such that $\gamma + \frac{3}{r} < 1$ and $\theta \in (0, 1)$ such that

$$B_{L^q(1+|x|^{-N}dx),\infty}^{-s} \subset [L^2(\frac{dx}{(1+|x|)^2}), H_r^{-\gamma}]_{\theta,\infty} \subset [L^2(\frac{dx}{(1+|x|)^2}), B_{\infty,\infty}^{-\sigma}]_{\theta,\infty}.$$

Proof. If $s < \sigma < 1 - \frac{2}{q}$, we have $B_{q,\infty}^{-s} \subset H_{L^q}^{-\sigma}$ and $B_{L^q(1+|x|^{-N}dx),\infty}^{-s} \subset H_{L^q(1+|x|^{-N}dx)}^{-\sigma}$. Thus, if $r > q$, we have, for $\theta \in (0, 1)$, $\gamma > \sigma$ such that $(1 - \theta)\frac{1}{2} + \theta\frac{1}{r} = \frac{1}{q}$ and $\theta\gamma = \sigma$,

$$B_{q,\infty}^{-s} \subset [L^2, H_r^{-\gamma}]_\theta \subset [L_2, H_r^{-\gamma}]_{\theta,\infty} \subset [L_2, B_{\infty,\infty}^{-\gamma-\frac{3}{r}}]_{\theta,\infty}.$$

As $\gamma + \frac{3}{r} = (1 - \frac{2}{r})\frac{\sigma}{1-\frac{2}{q}} + \frac{3}{r} = \frac{\sigma}{1-\frac{2}{q}} + O(\frac{1}{r})$, we have $\gamma + \frac{3}{r} < 1$ for r large enough. Similarly, if $(1 - \theta)M = N$ and $M < 2$ (so that in particular $\frac{1}{(1+|x|)^M} \in \mathcal{A}_2$), we have

$$\begin{aligned} B_{L^q(\frac{1}{(1+|x|)^N}dx),\infty}^{-s} &\subset [L^2(\frac{1}{(1+|x|)^M}dx), H_r^{-\gamma}]_\theta \\ &\subset [L_2(\frac{1}{(1+|x|)^2}dx), H_r^{-\gamma}]_{\theta,\infty} \subset [L_2(\frac{1}{(1+|x|)^2}dx), B_{\infty,\infty}^{-\gamma-\frac{3}{r}}]_{\theta,\infty}. \end{aligned}$$

As $M = \frac{N}{1-\theta} = N\frac{\frac{1}{2}-\frac{1}{r}}{\frac{1}{2}-\frac{1}{r}} = \frac{qN}{2} + O(\frac{1}{r})$, we have $M < 2$ for r large enough. \square

4 Mild solutions for the Navier–Stokes equation.

In this section, we develop some remarks on the solutions provided by Koch and Tataru’s theorem (Theorem 1 and Corollary 1).

Let $\vec{u}_0 \in \text{bmo}_0^{-1}$ with $\text{div } \vec{u}_0 = 0$. If $\|e^{t\Delta}\vec{u}_0\|_{X_T} < \frac{1}{4C_0}$, then the integral Navier–Stokes equations have a solution on $(0, T)$ such that $\|\vec{u}\|_{X_T} \leq 2\|e^{t\Delta}\vec{u}_0\|_{X_T}$. This solution is computed through Picard iteration as the limit of \vec{U}_n , where $\vec{U}_0 = e^{t\Delta}\vec{u}_0$ and $\vec{U}_{n+1} = e^{t\Delta}\vec{u}_0 - B(\vec{U}_n, \vec{U}_n)$. In particular, we have, by induction,

$$\|\vec{U}_{n+1} - \vec{U}_n\|_{X_T} \leq (4C_0\|e^{t\Delta}\vec{u}_0\|_{X_T})^{n+1}\|e^{t\Delta}\vec{u}_0\|_{X_T}$$

and

$$\|\vec{U}_n\|_{X_T} \leq 2\|e^{t\Delta}\vec{u}_0\|_{X_T}.$$

It is easy to check that \vec{u} is smooth: if $X_\alpha = L^\infty$ if $\alpha = 0$ and $\dot{B}_{\infty,\infty}^\alpha$ if $\alpha > 0$, we have

$$\|uv\|_{X_\alpha} \leq C_\alpha(\|u\|_\infty\|v\|_{X_\alpha} + \|v\|_\infty\|u\|_{X_\alpha})$$

and $\|\vec{u}\|_\infty \leq 2\|e^{t\Delta}\vec{u}_0\|_{X_T} \frac{1}{\sqrt{t}}$ for $0 < t < T$ while

$$\vec{u}(t, \cdot) = e^{\frac{t}{2}\Delta}\vec{u}(t/2, \cdot) - \int_0^{t/2} e^{(\frac{t}{2}-s)\Delta} \mathbb{P} \text{div}(\vec{u}(\frac{t}{2} + s, \cdot) \otimes \vec{u}(\frac{t}{2} + s, \cdot)) ds$$

so that

$$\|\vec{u}(t, \cdot)\|_{X_{(n+1)/2}} \leq C \frac{1}{t^{1/4}} \|\vec{u}(t/2, \cdot)\|_{X_{n/2}} + C \|e^{t\Delta}\vec{u}_0\|_{X_T} \int_0^{t/2} \frac{1}{(\frac{t}{2} - s)^{3/4}} \frac{1}{\sqrt{s}} \|\vec{u}(\frac{t}{2} + s, \cdot)\|_{X_{n/2}} ds$$

and, by induction on n ,

$$\|\vec{u}(t, \cdot)\|_{X_{n/2}} \leq C_n t^{-\frac{1}{2} - \frac{n}{4}}.$$

Thus, for $0 < t < T$, \vec{u} is smooth with respect to the space variable x . So is $\vec{\nabla} p$, by hypoellipticity of the Laplacian (as $\Delta p = -\sum_{i=1}^3 \sum_{j=1}^3 \partial_i u_j \partial_j u_i$). Then we have smoothness with respect to the time variable by controlling the time derivatives through the Navier–Stokes equations.

Proposition 3.

Let $\vec{u}_0 \in \text{bmo}_0^{-1}$ with $\text{div } \vec{u}_0 = 0$. Let $E \subset \mathcal{S}'$ be a stable space. If moreover \vec{u}_0 belongs to E , then the small solution \vec{u} to the integral Navier–Stokes equations with initial value \vec{u}_0 , i.e. the solution on $(0, T)$ such that $\|\vec{u}\|_{X_T} \leq$

$2\|e^{t\Delta}\vec{u}_0\|_{X_T}$, satisfies $\sup_{0 < t < T} \|\vec{u}(t, \cdot)\|_E < +\infty$ and $\lim_{t \rightarrow 0} \|\vec{u}(t, \cdot) - e^{t\Delta}\vec{u}_0\|_E = 0$. In particular, if \mathcal{S} is dense in E , then $\lim_{t \rightarrow 0} \|\vec{u}(t, \cdot) - \vec{u}_0\|_E = 0$.

Moreover, if $E \subset \mathcal{S}'$ is the dual of a space E_0 where \mathcal{S} is dense,

$$\sup_{0 < t < T} \sqrt{t} \|\vec{\nabla} \otimes \vec{u}\|_E < +\infty.$$

Proof. We have

$$\|B(\vec{u}, \vec{v})(t, \cdot)\|_E \leq C_E \int_0^t \frac{1}{\sqrt{t-s}} \min(\|\vec{u}\|_\infty \|\vec{v}\|_E, \|\vec{u}\|_E \|\vec{v}\|_\infty) ds.$$

By induction we have $\vec{U}_n \in L^\infty((0, T), E)$ with, for $n \geq 0$ [and $\vec{U}_{-1} = 0$]

$$\begin{aligned} & \|\vec{U}_{n+1}(t, \cdot) - \vec{U}_n(t, \cdot)\|_E \\ & \leq C \int_0^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}} \sqrt{s} \|\vec{U}_n(s, \cdot) - \vec{U}_{n-1}(s, \cdot)\|_\infty (\|\vec{U}_n(s, \cdot)\|_E + \|\vec{U}_{n-1}(s, \cdot)\|_E) ds \\ & \leq C' (4C_0 \|e^{t\Delta}\vec{u}_0\|_{X_T})^n \sum_{k=0}^n \|\vec{U}_k - \vec{U}_{k-1}\|_{L^\infty((0, T), E)}. \end{aligned}$$

Thus, we have

$$\sum_{k=0}^{+\infty} \|\vec{U}_k - \vec{U}_{k-1}\|_{L^\infty((0, T), E)} \leq \|\vec{U}_0\|_{L^\infty((0, T), E)} \prod_{n=0}^{\infty} (1 + C(4C_0 \|e^{t\Delta}\vec{u}_0\|_{X_T})^n).$$

Thus, $\sup_{0 < t < T} \|\vec{u}(t, \cdot)\|_E < +\infty$.

We have $\sup_{t > 0} \sqrt{t} \|\vec{\nabla} \otimes \vec{U}_0\|_E < +\infty$. We will show by induction that $\sup_{t > 0} \sqrt{t} \|\vec{\nabla} \otimes \vec{U}_n\|_E < +\infty$. Indeed, for $\eta \in (0, 1)$ and $0 < t < T$, we have

$$\vec{U}_{n+1}(t, \cdot) = e^{\eta t \Delta} \vec{U}_{n+1}((1-\eta)t, \cdot) - \int_0^{\eta t} e^{(\eta t-s)\Delta} \mathbb{P} \operatorname{div}(\vec{U}_n((1-\eta)t+s, \cdot) \otimes \vec{U}_n((1-\eta)t+s, \cdot)) ds$$

and, since $\operatorname{div}(\vec{u} \otimes \vec{v}) = \vec{u} \cdot \vec{\nabla} \vec{v}$,

$$\partial_j \vec{U}_{n+1}(t, \cdot) = e^{\eta t \Delta} \vec{U}_{n+1}((1-\eta)t, \cdot) - \int_0^{\eta t} e^{(\eta t-s)\Delta} \mathbb{P} \partial_j(\vec{U}_n((1-\eta)t+s, \cdot) \cdot \vec{\nabla} \vec{U}_n((1-\eta)t+s, \cdot)) ds.$$

This gives

$$\begin{aligned} & \|\vec{\nabla} \vec{U}_{n+1}(t, \cdot)\|_E \\ & \leq C \frac{1}{\sqrt{\eta t}} \|\vec{U}_{n+1}\|_{L^\infty((0, T), E)} \\ & \quad + C \int_0^{\eta t} \frac{1}{\sqrt{\eta t-s}} \frac{1}{(1-\eta)t+s} ds \sup_{0 < s < T} \sqrt{s} \|\vec{\nabla} \otimes \vec{U}_n(s, \cdot)\|_E \sqrt{s} \|\vec{U}_n(s, \cdot)\|_\infty \\ & \leq C_1 \frac{1}{\sqrt{\eta t}} + C_1 \frac{\sqrt{\eta}}{1-\eta} \frac{1}{\sqrt{t}} \sup_{0 < s < T} \sqrt{s} \|\vec{\nabla} \otimes \vec{U}_n(s, \cdot)\|_E \end{aligned}$$

where C_1 does not depend on n nor on η . For η small enough, we have $C_1 \frac{\sqrt{\eta}}{1-\eta} < \frac{1}{4}$ and $\sup_{0 < s < T} \sqrt{s} \|\vec{\nabla} \otimes \vec{U}_0(s, \cdot)\|_E \leq 2C_1 \frac{1}{\sqrt{\eta}}$. By induction, we get $\sup_{0 < s < T} \sqrt{s} \|\vec{\nabla} \otimes \vec{U}_n(s, \cdot)\|_E \leq 2C_1 \frac{1}{\sqrt{\eta}}$ for every $n \in \mathbb{N}$. If $E \subset \mathcal{S}'$ is the dual of a space E_0 where \mathcal{S} is dense, we conclude that $\sup_{0 < s < T} \sqrt{s} \|\vec{\nabla} \otimes \vec{u}(s, \cdot)\|_E < +\infty$. \square

Proposition 4.

Let $\vec{u}_0 \in \text{bmo}_0^{-1}$ with $\text{div } \vec{u}_0 = 0$. Let $w = \frac{1}{(1+|x|)^N}$ where $0 \leq N < 3$. If moreover \vec{u}_0 belongs to $L^2(w dx)$, then the small solution \vec{u} to the integral Navier–Stokes equations with initial value \vec{u}_0 , i.e. the solution on $(0, T)$ such that $\|\vec{u}\|_{X_T} \leq 2\|e^{t\Delta}\vec{u}_0\|_{X_T}$, satisfies $\vec{u} \in L^\infty((0, T), L^2(w dx))$ and $\vec{\nabla} \otimes \vec{u} \in L^2((0, T), L^2(w dx))$

Proof. Let $\phi_R = \theta(\frac{x}{R}) \frac{1}{(1+|x|)^{N/2}}$ where $\theta \in \mathcal{D}$ is equal to 1 on a neighborhood of 0. We know that \vec{u} is smooth, so that, for $0 < t_0 \leq t < T$,

$$\partial_t(|\vec{u}|^2) + 2|\vec{\nabla} \otimes \vec{u}|^2 = \Delta(|\vec{u}|^2) - \text{div}((2p + |\vec{u}|^2)\vec{u})$$

and thus

$$\begin{aligned} & \int \phi_R(x) |\vec{u}(t, x)|^2 dx + 2 \int_{t_0}^t \int \phi_R(x) |\vec{\nabla} \otimes \vec{u}(s, x)|^2 dx ds \\ &= \int \phi_R(x) |\vec{u}(t_0, x)|^2 dx + \int_{t_0}^t \int \Delta(\phi_R(x)) |\vec{u}(t, x)|^2 dx ds \\ & \quad + \int_{t_0}^t \int (2p + |\vec{u}|^2) \vec{u} \cdot \vec{\nabla}(\phi_R(x)) dx ds. \end{aligned}$$

We have, for $|\alpha| \leq 2$, $|\partial^\alpha(\phi_R)| \leq Cw$. On the other hand, we know that $\vec{u} \in L^\infty(L^2(w dx))$, that $\sqrt{t}u_i u_j \in L^\infty(L^2(w dx))$, and thus $\sqrt{t}(2p + |\vec{u}|^2) \in L^\infty(L^2(w dx))$ (as $w \in \mathcal{A}_2$ and $p = -\sum_{1 \leq i \leq 3} \sum_{j=1}^3 \frac{\partial_i \partial_j}{\Delta}(u_i u_j)$), thus we get that

$$\begin{aligned} & \int \phi_R(x) |\vec{u}(t, x)|^2 dx + 2 \int_{t_0}^t \int \phi_R(x) |\vec{\nabla} \otimes \vec{u}(s, x)|^2 dx ds \\ & \leq C \sup_{0 < s < T} \int |\vec{u}(s, x)|^2 w(x) dx + C \int_0^T \int |\vec{u}(t, x)|^2 w(x) dx ds \\ & \quad + \int_0^T \int \sqrt{s} |2p + |\vec{u}|^2| |\vec{u}| w(x) dx \frac{ds}{\sqrt{s}} < +\infty. \end{aligned}$$

We then let R go to $+\infty$ and t_0 go to 0. \square

5 Barker's stability theorem

In this section, we extend a lemma of Barker on Leray weak solutions with initial values in $L^2 \cap [L^2, \dot{B}_{\infty, \infty}^{-\delta}]_{\theta, \infty}$ (for some $\delta < 1$ and $\theta \in (0, 1)$) to the case of some solutions with initial values in $L^2(w \, dx) \cap [L^2(w \, dx), H_r^{-\gamma}]_{\theta, \infty}$ where $w = \frac{1}{(1+|x|)^N}$ and $0 \leq N \leq 2$, and $\gamma + \frac{3}{r} < 1$.

Definition 6.

A weighted Leray weak solution for the Navier–Stokes equations with divergence-free initial value $\vec{u}_0 \in L^2(w \, dx)$, where $w = \frac{1}{(1+|x|)^N}$ and $0 \leq N \leq 2$, is a divergence-free vector field \vec{u} defined on $(0, T) \times \mathbb{R}^3$ such that

- $\vec{u} \in L^\infty((0, T), L^2(w \, dx))$ and $\vec{\nabla} \otimes \vec{u} \in L^2((0, T), L^2(w \, dx))$
- there exists $p \in \mathcal{D}'((0, T) \times \mathbb{R}^3)$ such that

$$\partial_t \vec{u} = \Delta \vec{u} - \vec{u} \cdot \vec{\nabla} \vec{u} - \vec{\nabla} p$$

- $\lim_{t \rightarrow 0} \|\vec{u}(t, \cdot) - \vec{u}_0\|_{L^2(w \, dx)} = 0$
- \vec{u} fulfills the weighted Leray inequality: for $0 < t < T$,

$$\begin{aligned} & \int |\vec{u}(t, x)|^2 w(x) \, dx + 2 \int_0^t \int |\vec{\nabla} \otimes \vec{u}(s, x)|^2 w(x) \, dx \, ds \\ & \leq \int |\vec{u}_0(t, x)|^2 w(x) \, dx - 2 \sum_{i=1}^3 \int_0^t \int \partial_i w(s, x) \vec{u}(s, x) \cdot \partial_i \vec{u}(s, x) \, dx \, ds \\ & \quad + \int_0^t \int (|\vec{u}(s, x)|^2 + 2p(s, x)) \vec{u}(s, x) \cdot \vec{\nabla} w(x) \, dx \, ds \end{aligned}$$

The Navier–Stokes problem in $L^2(w \, dx)$ has recently been studied by Bradshaw, Kukavica and Tsai [BKT], and Fernández-Dalgo and Lemarié-Rieusset [FLR 1]. As $|\vec{\nabla} w| \leq Nw$, we find that $\sqrt{w} \vec{u} \in L^2((0, T), H^1)$. In particular, we have $w u_i u_j \in L^4((0, T), L^{6/5})$. The pressure p is determined by the equation $\Delta p = - \sum_{i=1}^3 \sum_{j=1}^3 u_i u_j$ (see [FLR 2]) and, as $w^{6/5} \in \mathcal{A}_{6/5}$, we have $p \in L^4((0, T), L^{6/5}(w^{6/5} \, dx))$. As $|\vec{\nabla} w| \leq Nw^{3/2}$, we see that the right-hand side of the weighted Leray inequality is well-defined. As in the case of Leray solutions, the strong continuity at $t = 0$ of $t \in [0, T) \mapsto \vec{u}(t, \cdot) \in L^2(w \, dx)$ (which is only weakly continuous for $t > 0$) is a consequence of the weighted Leray inequality.

Theorem 8.

Let \vec{u}_0 be a divergence-free vector field such that $\vec{u}_0 \in L^2(w dx)$, where $w = \frac{1}{(1+|x|)^N}$ and $0 \leq N \leq 2$. Let \vec{u}_1, \vec{u}_2 be two weighted Leray weak solutions for the Navier–Stokes equations with initial value \vec{u}_0 . If moreover \vec{u}_0 belongs to $[L^2(w dx), H_r^{-\gamma}]_{\theta, \infty}$ for some $\gamma > 0$, $2 < r < +\infty$ with $\gamma + \frac{3}{r} < 1$ and $\theta \in (0, 1)$, then there exists $T_0 > 0$, $C \geq 0$ and $\eta > 0$ such that, for $0 \leq t \leq T_0$,

$$\|\vec{u}_1(t, \cdot) - \vec{u}_2(t, \cdot)\|_{L^2(w dx)} \leq Ct^\eta.$$

Proof. This theorem was proved by Barker [BAR] in the case $N = 0$. Our proof will follow the same lines as Barker’s proof.

As $\vec{u}_0 \in [L^2(w dx), H_r^{-\gamma}]_{\theta, \infty}$, for every $\epsilon \in (0, 1)$ we may split \vec{u}_0 in $\vec{u}_0 = \vec{v}_{0, \epsilon} + \vec{w}_{0, \epsilon}$ with $\|\vec{v}_{0, \epsilon}\|_{H_r^{-\gamma}} \leq C_1 \epsilon^{\theta-1}$ and $\|\vec{w}_{0, \epsilon}\|_{L^2(w dx)} \leq C_1 \epsilon^\theta$, where C_1 depends only on \vec{u}_0 . As $\vec{u}_0 = \mathbb{P}\vec{u}_0$ and as \mathbb{P} is continuous on $H_r^{-\gamma}$ and on $L^2(w dx)$, we may assume (changing the value of the constant C_1) that $\vec{v}_{0, \epsilon}$ and $\vec{w}_{0, \epsilon}$ are divergence free. Let $\delta = \gamma + \frac{3}{r} < 1$. Since $H_r^{-\gamma} \subset B_{\infty, \infty}^{-\delta}$, we have for $0 < t \leq 1$, $\|e^{t\Delta}\vec{v}_{0, \epsilon}\|_\infty \leq C_2 t^{-\delta/2} \epsilon^{\theta-1}$. If $0 < T_1 < 1$, we have

$$\sup_{0 < t < T_1} \sqrt{t} \|e^{t\Delta}\vec{v}_{0, \epsilon}\|_\infty \leq C_2 \epsilon^{\theta-1} T_1^{\frac{1-\delta}{2}}$$

and

$$\sup_{0 < t < T_1, x \in \mathbb{R}^3} \sqrt{\frac{1}{t^{3/2}} \int_0^t \int_{B(x, \sqrt{t})} |e^{t\Delta}\vec{v}_{0, \epsilon}|^2 dx} \leq C_3 \epsilon^{\theta-1} T_1^{\frac{1-\delta}{2}}$$

so that $\|e^{t\Delta}\vec{v}_{0, \epsilon}\|_{X_{T_1}} \leq (C_2 + C_3) \epsilon^{\theta-1} T_1^{\frac{1-\delta}{2}} < \frac{1}{8C_0}$ if $T_1 < \min(1, C_4 \epsilon^{\frac{2}{1-\delta}(1-\theta)})$.

By (the proof of) Theorem 5, we know that the Navier–Stokes equations with initial value $\vec{v}_{0, \epsilon}$ will have a solution \vec{v}_ϵ on $(0, T_1)$ such that $\|\vec{v}_\epsilon(t, \cdot)\|_\infty \leq C_5 t^{-\delta/2} \epsilon^{\theta-1}$. Moreover, by Proposition 3, \vec{v}_ϵ is a weighted Leray weak solution.

Let \vec{u} be a weighted Leray solution on $(0, T)$ for the Navier–Stokes equations with initial value \vec{u}_0 . We are going to compare \vec{u} and \vec{v}_ϵ . We know that \vec{v}_ϵ is smooth, so that $\partial_t(\vec{u} \cdot \vec{v}_\epsilon) = \vec{u} \cdot \partial_t \vec{v}_\epsilon + \vec{v}_\epsilon \cdot \partial_t \vec{u}$. If p_ϵ is the pressure associated to \vec{v}_ϵ , we have on $(0, T_2)$ where $T_2 = \min(T, T_1)$

$$\begin{aligned} \partial_t(\vec{u} \cdot \vec{v}_\epsilon) &= \vec{u} \cdot \Delta \vec{v}_\epsilon + \vec{v}_\epsilon \cdot \Delta \vec{u} - \operatorname{div}(p_\epsilon \vec{u} + p \vec{v}_\epsilon) - \vec{u} \cdot (\vec{v}_\epsilon \cdot \vec{\nabla} \vec{v}_\epsilon) - \vec{v}_\epsilon \cdot (\vec{u} \cdot \vec{\nabla} \vec{u}) \\ &= \vec{u} \cdot \Delta \vec{v}_\epsilon + \vec{v}_\epsilon \cdot \Delta \vec{u} - \operatorname{div}(p_\epsilon \vec{u} + p \vec{v}_\epsilon) \\ &\quad - (\vec{u} - \vec{v}_\epsilon) \cdot (\vec{v}_\epsilon \cdot \vec{\nabla} \vec{v}_\epsilon) - \vec{v}_\epsilon \cdot (\vec{u} \cdot \vec{\nabla} (\vec{u} - \vec{v}_\epsilon)) - \operatorname{div}\left(\frac{|\vec{v}_\epsilon|^2}{2} (\vec{u} + \vec{v}_\epsilon)\right) \\ &= \vec{u} \cdot \Delta \vec{v}_\epsilon + \vec{v}_\epsilon \cdot \Delta \vec{u} - \vec{v}_\epsilon \cdot ((\vec{u} - \vec{v}_\epsilon) \cdot \vec{\nabla} (\vec{u} - \vec{v}_\epsilon)) \\ &\quad - \operatorname{div}\left(p_\epsilon \vec{u} + p \vec{v}_\epsilon + \frac{|\vec{v}_\epsilon|^2}{2} (\vec{u} + \vec{v}_\epsilon) + (\vec{v}_\epsilon \cdot (\vec{u} - \vec{v}_\epsilon)) \vec{v}_\epsilon\right). \end{aligned}$$

As $\vec{v}_\epsilon \in L^2((0, T_2), L^\infty)$, this can be integrated on $(0, t) \times \mathbb{R}^3$ against the measure $w(x) dx ds$ and gives

$$\begin{aligned}
& \int \vec{u} \cdot \vec{v}_\epsilon w(x) dx - \int \vec{u}_0 \cdot \vec{v}_{0,\epsilon} w(x) dx \\
&= - \int_0^t \int \sum_{i=1}^3 \partial_i w(x) (\vec{u}(s, x) \cdot \partial_i \vec{v}_\epsilon(s, x) + \vec{v}_\epsilon(s, x) \cdot \partial_i \vec{u}(s, x)) dx ds \\
&\quad - 2 \int_0^t \int (\vec{\nabla} \otimes \vec{u}(s, x) \cdot \vec{\nabla} \otimes \vec{v}_\epsilon(s, x)) w(x) dx ds \\
&\quad - \int_0^t \int \vec{v}_\epsilon(s, x) \cdot ((\vec{u}(s, x) - \vec{v}_\epsilon(s, x)) \cdot \vec{\nabla}(\vec{u}(s, x) - \vec{v}_\epsilon(s, x))) w(x) dx ds \\
&\quad + \int_0^t \int p(s, x) \vec{v}_\epsilon(s, x) \cdot \vec{\nabla} w(x) + p_\epsilon(s, x) \vec{u}(s, x) \cdot \vec{\nabla} w(x) dx ds \\
&+ \int_0^t \int \frac{|\vec{v}_\epsilon(s, x)|^2}{2} (\vec{u}(s, x) - \vec{v}_\epsilon(s, x)) \cdot \vec{\nabla} w(x) + (\vec{v}_\epsilon(s, x) \cdot \vec{u}(s, x)) \vec{v}_\epsilon(s, x) \cdot \vec{\nabla} w(x) dx ds.
\end{aligned}$$

Together with

$$\begin{aligned}
& \int |\vec{u}(t, x)|^2 w(x) dx + 2 \int_0^t \int |\vec{\nabla} \otimes \vec{u}(s, x)|^2 w(x) dx ds \\
&\leq \int |\vec{u}_0(t, x)|^2 w(x) dx - 2 \sum_{i=1}^3 \int_0^t \int \partial_i w(s, x) \vec{u}(s, x) \cdot \partial_i \vec{u}(s, x) dx ds \\
&\quad + \int_0^t \int (|\vec{u}(s, x)|^2 + 2p(s, x)) \vec{u}(s, x) \cdot \vec{\nabla} w(x) dx ds
\end{aligned}$$

and

$$\begin{aligned}
& \int |\vec{v}_\epsilon(t, x)|^2 w(x) dx + 2 \int_0^t \int |\vec{\nabla} \otimes \vec{v}_\epsilon(s, x)|^2 w(x) dx ds \\
&= \int |\vec{v}_{0,\epsilon}(t, x)|^2 w(x) dx - 2 \sum_{i=1}^3 \int_0^t \int \partial_i w(s, x) \vec{v}_\epsilon(s, x) \cdot \partial_i \vec{v}_\epsilon(s, x) dx ds \\
&\quad + \int_0^t \int (|\vec{v}_\epsilon(s, x)|^2 + 2p_\epsilon(s, x)) \vec{v}_\epsilon(s, x) \cdot \vec{\nabla} w(x) dx ds,
\end{aligned}$$

this gives

$$\begin{aligned}
& \int |\vec{v}_\epsilon(t, x) - \vec{u}(t, x)|^2 w(x) dx + 2 \int_0^t \int |\vec{\nabla} \otimes (\vec{v}_\epsilon - \vec{u})|^2 w(x) dx ds \\
& \leq \int |\vec{v}_{0,\epsilon} - \vec{u}_0|^2 w(x) dx - 2 \sum_{i=1}^3 \int_0^t \int \partial_i w (\vec{v}_\epsilon - \vec{u}) \cdot \partial_i (\vec{v}_\epsilon - \vec{u}) dx ds \\
& + 2 \int_0^t \int (p_\epsilon - p) (\vec{v}_\epsilon - \vec{u}) \cdot \vec{\nabla} w dx ds - 2 \int_0^t \int \vec{v}_\epsilon \cdot ((\vec{u} - \vec{v}_\epsilon) \cdot \vec{\nabla} (\vec{u} - \vec{v}_\epsilon)) w dx ds \\
& + \int_0^t \int |\vec{v}_\epsilon - \vec{u}|^2 \vec{v}_\epsilon \cdot \vec{\nabla} w + (|\vec{u}|^2 - |\vec{v}_\epsilon|^2) (\vec{u} - \vec{v}_\epsilon) \cdot \vec{\nabla} w dx ds.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
& \int |\vec{v}_\epsilon(t, x) - \vec{u}(t, x)|^2 w(x) dx + 2 \int_0^t \int |\vec{\nabla} \otimes (\vec{v}_\epsilon - \vec{u})|^2 w(x) dx ds \\
& \leq \int |\vec{v}_{0,\epsilon} - \vec{u}_0|^2 w(x) dx + C_6 \int_0^t \|\sqrt{w}(\vec{u} - \vec{v}_\epsilon)\|_2 \|\sqrt{w} \vec{\nabla}(\vec{u} - \vec{v}_\epsilon)\|_2 ds \\
& + C_6 \int_0^t \|(p - p_\epsilon) w\|_{L^{6/5}} \|\sqrt{w}(\vec{v}_\epsilon - \vec{u})\|_6 ds \\
& + C_6 \int_0^t \|\vec{v}_\epsilon\|_\infty \|\sqrt{w}(\vec{u} - \vec{v}_\epsilon)\|_2 \|\sqrt{w} \vec{\nabla} \otimes (\vec{u} - \vec{v}_\epsilon)\|_2 ds \\
& + C_6 \int_0^t \|\sqrt{w}(\vec{u} - \vec{v}_\epsilon)\|_3^2 (\|\sqrt{w} \vec{u}\|_3 + \|\sqrt{w} \vec{v}_\epsilon\|_3) ds.
\end{aligned}$$

We have

$$\|w(p - p_\epsilon)\|_{6/5} \leq C_7 \|w(\vec{u} \otimes \vec{u} - \vec{v}_\epsilon \otimes \vec{v}_\epsilon)\|_{6/5} \leq C_7 \|\sqrt{w}(\vec{u} - \vec{v}_\epsilon)\|_2 (\|\sqrt{w} \vec{u}\|_3 + \|\sqrt{w} \vec{v}_\epsilon\|_3)$$

$$\|\sqrt{w}(\vec{u} - \vec{v}_\epsilon)\|_3^2 \leq \|\sqrt{w}(\vec{u} - \vec{v}_\epsilon)\|_2 \|\sqrt{w}(\vec{u} - \vec{v}_\epsilon)\|_6$$

and

$$\|\sqrt{w}(\vec{u} - \vec{v}_\epsilon)\|_6 \leq C_8 \left(\|\sqrt{w}(\vec{u} - \vec{v}_\epsilon)\|_2 + \|\sqrt{w} \vec{\nabla} \otimes (\vec{u} - \vec{v}_\epsilon)\|_2 \right),$$

so that

$$\begin{aligned}
& \|\sqrt{w}(\vec{u}(t, \cdot) - \vec{v}_\epsilon(t, \cdot))\|_2^2 + 2 \int_0^t \int \|\sqrt{w} \vec{\nabla} \otimes (\vec{v}_\epsilon - \vec{u})\|_2^2 ds \\
& \leq \|\sqrt{w}(\vec{v}_{0,\epsilon} - \vec{u}_0)\|_2^2 + C_9 \int_0^t \|\sqrt{w}(\vec{u} - \vec{v}_\epsilon)\|_2 \|\sqrt{w} \vec{\nabla}(\vec{u} - \vec{v}_\epsilon)\|_2 ds \\
& + C_9 \int_0^t (\|\sqrt{w}(\vec{u} - \vec{v}_\epsilon)\|_2 + \|\sqrt{w} \vec{\nabla} \otimes (\vec{u} - \vec{v}_\epsilon)\|_2) \|\sqrt{w}(\vec{u} - \vec{v}_\epsilon)\|_2 (\|\sqrt{w} \vec{u}\|_3 + \|\sqrt{w} \vec{v}_\epsilon\|_3) ds \\
& \quad + C_9 \int_0^t \|\vec{v}_\epsilon\|_\infty \|\sqrt{w}(\vec{u} - \vec{v}_\epsilon)\|_2 \|\sqrt{w} \vec{\nabla} \otimes (\vec{u} - \vec{v}_\epsilon)\|_2 ds \\
& \leq \|\sqrt{w}(\vec{v}_{0,\epsilon} - \vec{u}_0)\|_2^2 + \int_0^t \|\sqrt{w} \vec{\nabla} \otimes (\vec{v}_\epsilon - \vec{u})\|_2^2 ds \\
& \quad + C_{10} \int_0^t \|\sqrt{w}(\vec{u} - \vec{v}_\epsilon)\|_2^2 (1 + \|\sqrt{w} \vec{u}\|_3^2 + \|\sqrt{w} \vec{v}_\epsilon\|_3^2 + \|\vec{v}_\epsilon\|_\infty^2) ds.
\end{aligned}$$

By Gonwall's lemma, we find that, for $0 < t < T_2$, we have

$$\|\sqrt{w}(\vec{u}(t, \cdot) - \vec{v}_\epsilon(t, \cdot))\|_2^2 \leq \|\sqrt{w}(\vec{v}_{0,\epsilon} - \vec{u}_0)\|_2^2 e^{\int_0^{T_2} C_{10}(1 + \|\sqrt{w} \vec{u}\|_3^2 + \|\sqrt{w} \vec{v}_\epsilon\|_3^2 + \|\vec{v}_\epsilon\|_\infty^2) ds}.$$

We know that $T_2 \leq T$, $\int_0^{T_2} \|\sqrt{w} \vec{u}\|_3^2 ds \leq \int_0^T \|\sqrt{w} \vec{u}\|_3^2 ds < +\infty$, and, by Propositions 3 and 4, $\int_0^{T_2} \|\sqrt{w} \vec{v}_\epsilon\|_3^2 ds \leq \int_0^{T_1} \|\sqrt{w} \vec{v}_\epsilon\|_3^2 ds \leq C_{11} \|\vec{u}_0\|_{L^2(w dx)}^2$. Finally, we have

$$\int_0^{T_2} \|\vec{v}_\epsilon\|_\infty^2 ds \leq C_{12} \int_0^{T_1} t^{-\delta} \|\vec{v}_{0,\epsilon}\|_{B_{\infty,\infty}^{-\delta}}^2 dt \leq C_{13} T_1^{1-\delta} \epsilon^{2(\theta-1)} \leq C_{14}.$$

Thus, we have

$$\|\sqrt{w}(\vec{u}(t, \cdot) - \vec{v}_\epsilon(t, \cdot))\|_2^2 \leq C_{15} \epsilon^{2\theta}.$$

C_{15} depends only on \vec{u} and \vec{u}_0 .

We may now estimate $\|\vec{u}_1(t, \cdot) - \vec{u}_2(t, \cdot)\|_{L^2(w dx)}$ for two weighted Leray weak solutions defined on $(0, T)$. If $t \in (0, T)$, we define $\epsilon = (\frac{4}{C_4} t)^{\frac{1-\delta}{2(1-\theta)}}$ and $T_3 = \frac{1}{2} C_4 \epsilon^{\frac{2}{1-\delta}(1-\theta)} = 2t$. If t is small enough, we have $0 < \epsilon < 1$ and $T_3 < \min(1, T)$. Thus, we know that, for a constant C that depends only on \vec{u}_1 , \vec{u}_2 and \vec{u}_0 ,

$$\begin{aligned}
\|\vec{u}_1(t, \cdot) - \vec{u}_2(t, \cdot)\|_{L^2(w dx)} & \leq \|\vec{u}_1(t, \cdot) - \vec{v}_\epsilon(t, \cdot)\|_{L^2(w dx)} + \|\vec{v}_\epsilon(t, \cdot) - \vec{u}_2(t, \cdot)\|_{L^2(w dx)} \\
& \leq C \epsilon^\theta = C \left(\frac{4}{C_4} t\right)^{\theta \frac{1-\delta}{2(1-\theta)}}
\end{aligned}$$

The theorem is proved. \square

6 Weak-strong uniqueness

We may now prove Theorem 4.

Proof. Recall that we consider two solutions \vec{u}, \vec{v} of the Navier–Stokes equations on $(0, T)$ with the same initial value \vec{u}_0 such that:

- \vec{u}_0 be a divergence-free vector field with $\vec{u}_0 \in L^2 \cap \text{bmo}_0^{-1}$
- $\|e^{t\Delta}\vec{u}_0\|_{X_T} < \frac{1}{4C_0}$
- \vec{u} is the mild solution of the Navier–Stokes equations with initial value \vec{u}_0 such that $\|\vec{u}\|_{X_T} \leq \frac{1}{2C_0}$
- for some $N \geq 0$, $2 < q < +\infty$ and $0 \leq s < 1 - \frac{2}{q}$,

$$\sup_{0 < t < T} t^{s/2} \|\vec{u}\|_{L^q(\frac{1}{(1+|x|)^N} dx)} < +\infty$$

- \vec{v} is a suitable weak Leray solution of the Navier–Stokes equations.

We know, by Propositions 3 and 4, that the mild solution \vec{u} is a suitable weak Leray solution. In particular, we have $\sup_{0 < t < T} \|\vec{u}(t, \cdot)\|_2 < +\infty$, while $\sup_{0 < t < T} t^{1/2} \|\vec{u}(t, \cdot)\|_\infty \leq \|\vec{u}\|_{X_T} < +\infty$. Thus,

$$\sup_{0 < s < T} t^{\frac{1}{2} - \frac{1}{q}} \|\vec{u}\|_q < +\infty.$$

If $0 \leq \alpha \leq 1$, we find that

$$\sup_{0 < t < T} (\sqrt{t})^{(1-\alpha)(1-\frac{2}{q})+\alpha s} \|\vec{u}\|_{L^q(\frac{1}{(1+|x|)^{\alpha N}} dx)} < +\infty.$$

By Theorem 5, we find that

$$\vec{u}_0 \in B_{L^q(\frac{1}{(1+|x|)^{\alpha N}} dx, \infty)}^{-s_\alpha} \text{ with } s_\alpha = (1-\alpha)(1-\frac{2}{q}) + \alpha s.$$

For $0 < \alpha < \min(1, \frac{4}{Nq})$, $0 < s_\alpha < 1 - \frac{2}{q}$ and $\alpha N < \frac{4}{q}$, so that we may apply Corollary 2.

The next step is to check that \vec{u} and \vec{v} , that are suitable Leray weak solutions, are weighted Leray weak solutions, for the weight $w(x) = \frac{1}{(1+|x|)^2}$. It means that we must check that \vec{v} (and \vec{u}) fulfills the weighted Leray energy inequality. We consider a non-negative function $\theta \in \mathcal{D}(\mathbb{R}^3)$ equal to 1 on a neighborhood of 0 and 0 for $|x| \geq 1$ and a function α smooth on \mathbb{R} , such that $0 \leq \alpha \leq 1$, with $\alpha(t)$ equal to 0 on $(\infty, 0)$ and 1 on $(1, +\infty)$. For

$0 < t_0 < t_1 < T$, $R > 0$ and $0 < \epsilon < \min(t_1 - t_0, T - t_1)$, we define the test function

$$\varphi_{t_0, t_1, \epsilon, R}(t, x) = \alpha\left(\frac{t - t_0}{\epsilon}\right)(1 - \alpha\left(\frac{t - t_1}{\epsilon}\right)) \frac{1}{(1 + \sqrt{\frac{1}{R^2} + x^2})^2} \theta\left(\frac{x}{R}\right) = \alpha_{t_0, t_1, \epsilon}(t) \theta_R(x)$$

which is non-negative and supported in $[t_0, t_1 + \epsilon] \times \overline{B(0, R)}$. We have, by suitability of \vec{v} , if q is the pressure associated to the solution \vec{v} ,

$$\iint \varphi_{t_0, t_1, \epsilon, R} \left(\partial_t(|\vec{v}|^2) + 2|\vec{\nabla} \otimes \vec{v}|^2 - \Delta(|\vec{v}|^2) + \operatorname{div}((2q + |\vec{v}|^2)\vec{v}) \right) dx dt \leq 0.$$

As, for $R \geq 1$, $|\theta_R| \leq Cw$ and $|\vec{\nabla}\theta_R| \leq Cw^{3/2}$, dominated convergence when R goes to $+\infty$ gives us

$$\begin{aligned} & \iint \left(\frac{1}{\epsilon} \alpha'\left(\frac{t - t_1}{\epsilon}\right) - \frac{1}{\epsilon} \alpha'\left(\frac{t - t_0}{\epsilon}\right) \right) |\vec{v}|^2 w(x) dx dt + 2 \iint \alpha_{t_0, t_1, \epsilon} |\vec{\nabla} \otimes \vec{v}|^2 w(x) dx dt \\ & \leq -2 \sum_{i=1}^3 \iint \alpha_{t_0, t_1, \epsilon} \partial_i w(\vec{v} \cdot \partial_i \vec{v}) dx dt + \iint \alpha_{t_0, t_1, \epsilon} (|\vec{v}|^2 + 2q) \vec{v} \cdot \vec{\nabla} w dx dt \end{aligned}$$

If ϵ goes to 0, we get

$$\begin{aligned} & \limsup_{\epsilon \rightarrow 0} \int \left(\frac{1}{\epsilon} \alpha'\left(\frac{s - t_1}{\epsilon}\right) - \frac{1}{\epsilon} \alpha'\left(\frac{s - t_0}{\epsilon}\right) \right) \left(\int |\vec{v}(s, x)|^2 w(x) dx \right) ds \\ & + 2 \int_{t_0}^{t_1} \int |\vec{\nabla} \otimes \vec{v}|^2 w(x) dx ds \\ & \leq -2 \sum_{i=1}^3 \int_{t_0}^{t_1} \int \partial_i w(\vec{v} \cdot \partial_i \vec{v}) dx ds + \int_{t_0}^{t_1} \int (|\vec{v}|^2 + 2q) \vec{v} \cdot \vec{\nabla} w dx ds \end{aligned}$$

For almost every t_0, t_1 , we have that t_0 and t_1 are Lebesgue points of the map $s \mapsto \int |\vec{v}(s, x)|^2 w(x) dx$, so that

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int \left(\frac{1}{\epsilon} \alpha'\left(\frac{s - t_1}{\epsilon}\right) - \frac{1}{\epsilon} \alpha'\left(\frac{s - t_0}{\epsilon}\right) \right) \left(\int |\vec{v}(s, x)|^2 w(x) dx \right) ds \\ & = \int |\vec{v}(t_1, x)|^2 w(x) dx - \int |\vec{v}(t_0, x)|^2 w(x) dx. \end{aligned}$$

If t_0 goes to 0 and t_1 goes to t , we have $\|\vec{v}(t_0, \cdot) - \vec{u}_0\|_{L^2(w dx)} \leq \|\vec{v}(t_0, \cdot) - \vec{u}_0\|_2 \rightarrow 0$, so that

$$\lim_{t_0 \rightarrow 0} \int |\vec{v}(t_0, x)|^2 w(x) dx = \int |\vec{u}_0(x)|^2 w(x) dx$$

while $\vec{v}(t_1, \cdot)$ is weakly convergent to $\vec{v}(t, \cdot)$ so that

$$\int |\vec{v}(t, x)|^2 w(x) dx \leq \liminf_{t_1 \rightarrow t} \int |\vec{v}(t_1, x)|^2 w(x) dx.$$

Thus, we get the weighted Leray energy inequality.

By Theorem 8, we then know that there exists $T_0 > 0$, $C \geq 0$ and $\eta > 0$ such that, for $0 \leq t \leq T_0$,

$$\|\vec{u}(t, \cdot) - \vec{v}(t, \cdot)\|_{L^2(w dx)} \leq Ct^\eta.$$

Moreover, we can do the same computations as in the proof of Theorem 8 in order to estimate $\partial_t(\vec{u} \cdot \vec{v})$ (since \vec{u} is smooth) and write, if p is the pressure associated to \vec{u} and q the pressure associated to \vec{v} ,

$$\begin{aligned} \partial_t(\vec{u} \cdot \vec{v}) &= \vec{u} \cdot \Delta \vec{v} + \vec{v} \cdot \Delta \vec{u} - \vec{u} \cdot ((\vec{u} - \vec{v}) \cdot \vec{\nabla}(\vec{u} - \vec{v})) \\ &\quad - \operatorname{div}(q\vec{u} + p\vec{v}) + \frac{|\vec{u}|^2}{2}(\vec{u} + \vec{v}) + (\vec{u} \cdot (\vec{v} - \vec{u}))\vec{u}. \end{aligned}$$

As $\vec{u} \in L^2((\epsilon, T), L^\infty)$ for every $\epsilon > 0$, this can be integrated on $(\epsilon, t) \times \mathbb{R}^3$ against the measure $w(x) dx ds$ and gives

$$\begin{aligned} &\int \vec{u}(t, x) \cdot \vec{v}(t, x) w(x) dx - \int \vec{u}(\epsilon, x) \cdot \vec{v}(\epsilon, x) w(x) dx \\ &= - \int_\epsilon^t \int \sum_{i=1}^3 \partial_i w (\vec{u} \cdot \partial_i \vec{v} + \vec{v} \cdot \partial_i \vec{u}) dx ds \\ &\quad - 2 \int_\epsilon^t \int (\vec{\nabla} \otimes \vec{u} \cdot \vec{\nabla} \otimes \vec{v}) w(x) dx ds \\ &\quad - \int_\epsilon^t \int \vec{u} \cdot ((\vec{u} - \vec{v}) \cdot \vec{\nabla}(\vec{u} - \vec{v})) w(x) dx ds \\ &\quad + \int_\epsilon^t \int p\vec{v} \cdot \vec{\nabla} w + q\vec{u} \cdot \vec{\nabla} w dx ds \\ &+ \int_\epsilon^t \int \frac{|\vec{u}|^2}{2} (\vec{v} - \vec{u}) \cdot \vec{\nabla} w + (\vec{v} \cdot \vec{u}) \vec{u} \cdot \vec{\nabla} w(x) dx ds. \end{aligned}$$

As $\vec{u}(\epsilon, \cdot)$ and $\vec{v}(\epsilon, \cdot)$ are strongly convergent to \vec{u}_0 in $L^2(w \, dx)$, we find

$$\begin{aligned}
& \int \vec{u}(t, x) \cdot \vec{v}(t, x) w(x) \, dx - \int \vec{u}_0 \cdot \vec{u}_0 w(x) \, dx \\
&= - \int_0^t \int \sum_{i=1}^3 \partial_i w (\vec{u} \cdot \partial_i \vec{v} + \vec{v} \cdot \partial_i \vec{u}) \, dx \, ds \\
&\quad - 2 \int_0^t \int (\vec{\nabla} \otimes \vec{u} \cdot \vec{\nabla} \otimes \vec{v}) w \, dx \, ds \\
&- \lim_{\epsilon \rightarrow 0} \int_\epsilon^t \int \vec{u} \cdot ((\vec{u} - \vec{v}) \cdot \vec{\nabla}(\vec{u} - \vec{v})) w \, dx \, ds \\
&\quad + \int_0^t \int p \vec{v} \cdot \vec{\nabla} w + q \vec{u} \cdot \vec{\nabla} w \, dx \, ds \\
&+ \int_0^t \int \frac{|\vec{u}|^2}{2} (\vec{v} - \vec{u}) \cdot \vec{\nabla} w + (\vec{v} \cdot \vec{u}) \vec{u} \cdot \vec{\nabla} w \, dx \, ds.
\end{aligned}$$

We have

$$\lim_{\epsilon \rightarrow 0} \int_\epsilon^t \int \vec{u} \cdot ((\vec{u} - \vec{v}) \cdot \vec{\nabla}(\vec{u} - \vec{v})) w \, dx \, ds = \int_0^t \int s^\eta \vec{u} \cdot s^{-\eta} ((\vec{u} - \vec{v}) \cdot \vec{\nabla}(\vec{u} - \vec{v})) w \, dx \, ds$$

as $s^\eta \vec{u} \in L^2 L^\infty$, $s^{-\eta}(\vec{u} - \vec{v}) \in L^\infty(L^2(w \, dx))$ and $\vec{\nabla} \otimes (\vec{u} - \vec{v}) \in L^2(L^2(w \, dx))$.

Using now the weighted Leray inequalities on \vec{v} and on \vec{u} , we get

$$\begin{aligned}
& \int |\vec{v}(t, x) - \vec{u}(t, x)|^2 w(x) \, dx + 2 \int_0^t \int |\vec{\nabla} \otimes (\vec{v} - \vec{u})|^2 w \, dx \, ds \\
&\leq - 2 \sum_{i=1}^3 \int_0^t \int \partial_i w (\vec{v} - \vec{u}) \cdot \partial_i (\vec{v} - \vec{u}) \, dx \, ds \\
&+ 2 \int_0^t \int (q - p) (\vec{v} - \vec{u}) \cdot \vec{\nabla} w \, dx \, ds - 2 \int_0^t \int \vec{u} \cdot ((\vec{u} - \vec{v}) \cdot \vec{\nabla}(\vec{u} - \vec{v})) w \, dx \, ds \\
&\quad + \int_0^t \int |\vec{v} - \vec{u}|^2 \vec{u} \cdot \vec{\nabla} w + (|\vec{u}|^2 - |\vec{v}|^2) (\vec{u} - \vec{v}) \cdot \vec{\nabla} w \, dx \, ds,
\end{aligned}$$

and thus

$$\begin{aligned}
& \int |\vec{v}(t, x) - \vec{u}(t, x)|^2 w(x) dx + 2 \int_0^t \int |\vec{\nabla} \otimes (\vec{v} - \vec{u})|^2 w dx ds \\
& \leq C \int_0^t \|\sqrt{w}(\vec{u} - \vec{v})\|_2 \|\sqrt{w} \vec{\nabla}(\vec{u} - \vec{v})\|_2 ds \\
& \quad + C \int_0^t \|(p - q) w\|_{L^{6/5}} \|\sqrt{w}(\vec{v} - \vec{u})\|_6 ds \\
& \quad + C \int_0^t \|\vec{u}\|_\infty \|\sqrt{w}(\vec{u} - \vec{v})\|_2 \|\sqrt{w} \vec{\nabla} \otimes (\vec{u} - \vec{v})\|_2 ds \\
& \quad + C \int_0^t \|\sqrt{w}(\vec{u} - \vec{v})\|_3^2 (\|\sqrt{w} \vec{u}\|_3 + \|\sqrt{w} \vec{v}\|_3) ds.
\end{aligned}$$

At this point, we get

$$\begin{aligned}
& \|\sqrt{w}(\vec{u}(t, \cdot) - \vec{v}(t, \cdot))\|_2^2 + 2 \int_0^t \int \|\sqrt{w} \vec{\nabla} \otimes (\vec{v} - \vec{u})\|_2^2 ds \\
& \leq \int_0^t \|\sqrt{w} \vec{\nabla} \otimes (\vec{v} - \vec{u})\|_2^2 ds \\
& \quad + C \int_0^t \|\sqrt{w}(\vec{u} - \vec{v})\|_2^2 (1 + \|\sqrt{w} \vec{u}\|_3^2 + \|\sqrt{w} \vec{v}\|_3^2 + \|\vec{u}\|_\infty^2) ds.
\end{aligned}$$

Let

$$A(t) = t^{-2\eta} \|\sqrt{w}(\vec{u}(t, \cdot) - \vec{v}(t, \cdot))\|_2^2$$

and

$$B(t) = 1 + \|\sqrt{w} \vec{u}\|_3^2 + \|\sqrt{w} \vec{v}\|_3^2.$$

We have

$$A(t) \leq C \int_0^t A(s) B(s) ds + C t^{-2\eta} \int_0^t A(s) s^{2\eta} \|\vec{u}\|_\infty^2 ds.$$

Thus, for $0 < t < \tau < T$,

$$A(t) \leq C \sup_{0 < s < \tau} A(s) \left(\int_0^\tau B(s) ds + \frac{1}{2\eta} \sup_{0 < s < \tau} s \|\vec{u}(s, \cdot)\|_\infty^2 \right).$$

For τ small enough, we have

$$C \left(\int_0^\tau B(s) ds + \frac{1}{2\eta} \sup_{0 < s < \tau} s \|\vec{u}(s, \cdot)\|_\infty^2 \right) < 1$$

and thus $\sup_{0 < t < \tau} A(t) = 0$. We conclude that $\vec{u} = \vec{v}$ on $[0, \tau]$. Since \vec{u} is bounded on $[\tau, T]$, then uniqueness is easily extended to the whole interval $[0, T]$. \square

7 Further comments on Barker's conjecture

In his paper [BAR], Barker raised the following question :

Question 1. *If \vec{u}_0 belongs to $L^2 \cap \text{bmo}_0^{-1}$, does there exists a positive time T such that every weak Leray solution of the Cauchy problem for the Navier–Stokes equations with \vec{u}_0 as initial value coincide with the mild solution in X_T ?*

This can be seen as the endpoint case of the Prodi–Serrin weak-strong uniqueness criterion, as the assumption of Prodi–Serrin's criterion, i.e. existence of a solution \vec{u} such that

$$\vec{u} \in L_t^p L_x^q \text{ with } \frac{2}{p} + \frac{3}{q} \leq 1 \text{ and } 2 \leq p \leq +\infty,$$

is equivalent, if $2 < p < +\infty$, to the fact that \vec{u}_0 belongs to $B_{q,p}^{-1+\frac{3}{q}} \subset \text{bmo}_0^{-1}$. Existence of a mild solution when \vec{u}_0 belongs to $B_{q,p}^{-1+\frac{3}{q}}$ goes back to the paper of Fabes, Jones and Rivière [FJR]. Existence of mild solutions has been extended by Cannone [CAN] to the case of $B_{q,\infty}^{-1+\frac{3}{q}} \cap \text{bmo}_0^{-1}$, and Koch and Tataru's theorem [KOT] can be seen as the endpoint case of the theory for existence of mild solutions.

Barker [BAR] extended weak-strong uniqueness to the case $B_{q,\infty}^{-1+\frac{3}{q}} \cap \text{bmo}_0^{-1}$, and could even relax the regularity exponent and consider the case $B_{q,\infty}^{-s} \cap \text{bmo}_0^{-1}$ with $s < 1 - \frac{2}{q}$. We have shown that the integrability could even be relaxed into $B_{L^q(\frac{1}{(1+|x|)^N} dx),\infty}^{-1+\frac{3}{q}} \cap \text{bmo}_0^{-1}$ with $N \geq 0$ and $s < 1 - \frac{2}{q}$. But under the sole assumption $\vec{u}_0 \in L^2 \cap \text{bmo}_0^{-1}$, weak-strong uniqueness remains an open question.

An alternative way to study the problem is to impose restrictions on the class of solutions, beyond the Leray energy inequality or the local Leray energy inequality. One may for instance consider an approximation process that provides weak Leray solutions when $\vec{u}_0 \in L^2$ and consider whether the solutions provided by this process coincide with the mild solution when moreover $\vec{u}_0 \in \text{bmo}_0^{-1}$. There are many processes that pave the way to Leray solutions (and in most cases to suitable weak Leray solutions); in [LR 5], we described fourteen different processes (including α -models, frequency cut-off, damping, artificial viscosity, hyperviscosity, ...).

The scheme is always the same. One approximates the Navier–Stokes equations (NS) by equations (NS $_\alpha$) depending on a small parameter $\alpha \in (0, 1)$. Equations (NS $_\alpha$) with initial value $\vec{u}_0 \in L^2$ have a unique solution

\vec{u}_α . One then establishes an energy (in)equality that allows to control \vec{u}_α uniformly on $L^\infty((0, T), L^2) \cap L^2((0, T), H^1)$. Moreover, one proves that $\partial_t \vec{u}_\alpha$ is controlled uniformly in $L^{6/5}((0, T), H^{-3})$. By the Aubin–Lions theorem, there exists a sequence $\alpha_k \rightarrow 0$ such that \vec{u}_{α_k} is weakly convergent in $L^2((0, T), H^1)$ and strongly convergent in $(L^2((0, T) \times \mathbb{R}^3))_{\text{loc}}$ to a limit \vec{v} . One then checks that \vec{v} is a weak Leray solution of the Navier–Stokes equations with initial value \vec{u}_0 .

Some of those processes behave well for initial values $\vec{u}_0 \in \text{bmo}_0^{-1}$, others don't seem to be well adapted to such initial values. More precisely, if one can prove that, when \vec{u}_0 belongs to $L^2 \cap \text{bmo}_0^{-1}$, there exists a time T_0 such that the solutions \vec{u}_α remain small in X_{T_0} ($\|e^{t\Delta} \vec{u}_0\|_{X_{T_0}} < \eta < \frac{1}{40}$ and $\sup_{\alpha \in (0,1)} \|\vec{u}_\alpha\|_{X_{T_0}} \leq 2\eta \leq \frac{1}{2C_0}$), then the weak limit \vec{v} will still remain controlled in X_{T_0} . But there is only one weak solution \vec{u} in X_{T_0} such that $\|\vec{u}\|_{X_{T_0}} \leq \frac{1}{2C_0}$. Thus, the process cannot create a Leray solution that would escape the weak–strong uniqueness.

Such processes can be found in processes that mimic Leray's mollification. Mollicication has been introduced by Leray [LER] in his seminal paper on weak solutions for the Navier–Stokes equations. The approximated problem he considered is the following one: solve

$$\partial_t \vec{u}_\alpha + (\varphi_\alpha * \vec{u}_\alpha) \cdot \vec{\nabla} \vec{u}_\alpha = \Delta \vec{u}_\alpha - \vec{\nabla} p_\alpha$$

with $\text{div } \vec{u}_\alpha = 0$ and $\vec{u}_\alpha(0, \cdot) = \vec{u}_0$. Here, $\varphi \in \mathcal{D}$, $\varphi \geq 0$, $\int \varphi dx = 1$ and $\varphi_\alpha(x) = \frac{1}{\alpha^3} \varphi(\frac{x}{\alpha})$. Solving the mollified problem amounts to solve the following integro-differential problem :

$$\vec{v} = e^{t\Delta} \vec{u}_0 - B(\varphi_\alpha * \vec{v}, \vec{v})(t, x)$$

where

$$B(\vec{v}, \vec{w}) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \text{div}(\vec{v} \otimes \vec{w}) ds.$$

Since $\|\varphi_\alpha * \vec{v}(t, \cdot)\|_\infty \leq \|\vec{v}(t, \cdot)\|_\infty$ and

$$\begin{aligned} & \left(\int_0^t \int_{B(x_0, \sqrt{t})} |\varphi_\alpha * \vec{v}(s, \cdot)(y)|^2 dy ds \right)^{1/2} \\ &= \left(\int_0^t \int_{B(x_0, \sqrt{t})} \left| \int \varphi_\alpha(z) \vec{v}(s, y - z) dz \right|^2 dy ds \right)^{1/2} \\ &\leq \left(\int_0^t \int_{B(x_0, \sqrt{t})} \int \varphi_\alpha(z) |\vec{v}(s, y - z)|^2 dz dy ds \right)^{1/2} \\ &= \left(\int \varphi_\alpha(z) \left(\int_0^t \int_{B(x_0+z, \sqrt{t})} |\vec{v}(s, y)|^2 dy ds \right) dz \right)^{1/2}, \end{aligned}$$

we find that $\|\varphi_\alpha * \vec{v}\|_{X_T} \leq \|\vec{v}\|_{X_T}$. Thus, the theorem of Koch and Tataru (Theorem 1 and Corollary 1) still applies :

- for every $\alpha > 0$ and every $T > 0$, we have

$$\|B(\varphi_\alpha \vec{v}, \vec{w})\|_{X_T} \leq C_0 \|\vec{v}\|_{X_T} \|\vec{w}\|_{X_T}.$$

- If $\|e^{t\Delta} \vec{u}_0\|_{X_T} < \frac{1}{4C_0}$, then the mollified Navier–Stokes equations have a solution on $(0, T)$ such that $\|\vec{u}_\alpha\|_{X_T} \leq 2\|e^{t\Delta} \vec{u}_0\|_{X_T}$.

Now, we may consider various other approximations of the Navier–Stokes equations of the form

$$\vec{v} = e^{t\Delta} \vec{u}_0 - \sum_{i=1}^N \varphi_{i,\alpha} * B_i(\psi_{i,\alpha} * \vec{v}, \chi_{i,\alpha} * \vec{v})(t, x) \quad (4)$$

where

- $\varphi_i, \psi_i, \chi_i$ are either the Dirac mass or functions in L^1
- $f_\alpha(x) = \frac{1}{\alpha^3} f(\frac{x}{\alpha})$ for $f \in \{\varphi_i, \psi_i, \chi_i, i = 1, \dots, N\}$
- $B_i(\vec{v}, \vec{w}) = \int_0^t e^{(t-s)\Delta} \sigma_i(D)(\vec{v} \otimes \vec{w}) ds$ where σ_i is given convolutions with smooth Fourier multipliers homogeneous of degree 1: if $\vec{z} = \sigma_i(D)(\vec{v} \otimes \vec{w})$, $z_k = \sum_{p,q \leq 3} K_{i,k,p,q} * (v_p w_q)$ where the Fourier transform of $K_{i,k,p,q}$ is and homogenous of degree 1 and is smooth on \mathbb{R}^3 .

The proof of the Koch and Tataru theorem asserts that operators of the form $B(\vec{v}, \vec{w}) = \int_0^t e^{(t-s)\Delta} \sigma(D)(\vec{v} \otimes \vec{w}) ds$ are bounded on X_T .

Writing $\|\delta\|_1 = 1$, we have

$$\begin{aligned} & \left\| \sum_{i=1}^N \varphi_{i,\alpha} * B(\psi_{i,\alpha} * \vec{v}, \chi_{i,\alpha} * \vec{v})(t, x) \right\|_{X_T} \\ & \leq \left(\sum_{i=1}^N \|B_i\|_{\text{op}} \|\varphi_i\|_1 \|\psi_i\|_1 \|\chi_i\|_1 \right) \|\vec{v}\|_{X_T} \|\vec{w}\|_{X_T} \\ & = C_1 \|\vec{v}\|_{X_T} \|\vec{w}\|_{X_T} \end{aligned}$$

If $\|e^{t\Delta} \vec{u}_0\|_{X_T} < \frac{1}{4C_1}$, then the modified equations (4) have a solution on $(0, T)$ such that $\|\vec{u}_\alpha\|_{X_T} \leq 2\|e^{t\Delta} \vec{u}_0\|_{X_T}$.

Remark that the equations (4) can be written as well

$$\partial_t \vec{v} = \Delta \vec{v} - \sum_{i=1}^N \varphi_{i,\alpha} * \sigma_i(D)((\psi_{i,\alpha} * \vec{v}) \otimes (\chi_{i,\alpha} * \vec{v}))$$

with initial value $\vec{v}(0, \cdot) = \vec{u}_0$. Among example of such approximations, we have the various α -models studied by Holm and Titi:

- The Leray- α model.

The Leray- α model has been discussed in 2005 by Cheskidov, Holm, Olson and Titi [CHOT]. The approximated problem is the following one: solve

$$\partial_t \vec{u}_\alpha + ((\text{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha) \cdot \vec{\nabla} \vec{u}_\alpha = \Delta \vec{u}_\alpha - \vec{\nabla} p_\alpha$$

with $\text{div } \vec{u}_\alpha = 0$ and $\vec{u}_\alpha(0, \cdot) = \vec{u}_0$. This is equivalent to write

$$\partial_t \vec{u}_\alpha = \Delta \vec{u}_\alpha - \mathbb{P} \text{div}(((\text{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha) \otimes \vec{u}_\alpha).$$

- The Navier-Stokes- α model.

The mathematical study of the Navier-Stokes- α model has been done by Foias, Holm and Titi in 2002[FHT]. The approximated problem is the following one: solve

$$\partial_t \vec{u}_\alpha + ((\text{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha) \cdot \vec{\nabla} \vec{u}_\alpha = \Delta \vec{u}_\alpha - \sum_{k=1}^3 u_{\alpha,k} \vec{\nabla} (\text{Id} - \alpha^2 \Delta)^{-1} u_{\alpha,k} - \vec{\nabla} p_\alpha$$

with $\text{div } \vec{u}_\alpha = 0$ and $\vec{u}_\alpha(0, \cdot) = \vec{u}_0$. We can rewrite the equation as

$$\begin{aligned} \partial_t \vec{u}_\alpha + ((\text{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha) \cdot \vec{\nabla} \vec{u}_\alpha = & \Delta \vec{u}_\alpha - \sum_{k=1}^3 (\alpha^2 \Delta (\text{Id} - \alpha^2 \Delta)^{-1} u_{\alpha,k}) \vec{\nabla} (\text{Id} - \alpha^2 \Delta)^{-1} u_{\alpha,k} \\ & - \vec{\nabla} (p_\alpha + \frac{|((\text{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha)|^2}{2}) \end{aligned}$$

This is equivalent to write

$$\begin{aligned} \partial_t \vec{u}_\alpha = & \Delta \vec{u}_\alpha - \mathbb{P} \text{div}(((\text{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha) \otimes \vec{u}_\alpha) \\ & - \sum_{j=1}^3 \sum_{k=1}^3 \mathbb{P} \partial_j ((\alpha \partial_j (\text{Id} - \alpha^2 \Delta)^{-1} u_{\alpha,k}) (\alpha \vec{\nabla} (\text{Id} - \alpha^2 \Delta)^{-1} u_{\alpha,k})). \end{aligned}$$

- The Clark- α model.

The Clark- α model has been discussed in 2005 by Cao, Holm and Titi [CHT]. The approximated problem is the following one: solve

$$\begin{aligned} \partial_t \vec{u}_\alpha + (\text{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha \cdot \vec{\nabla} \vec{u}_\alpha = & \Delta \vec{u}_\alpha + ((\text{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha - \vec{u}_\alpha) \cdot \vec{\nabla} (\text{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha \\ & + \alpha^2 \sum_{k=1}^3 (\partial_k (\text{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha) \cdot \vec{\nabla} (\partial_k (\text{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha) - \vec{\nabla} p_\alpha \end{aligned}$$

with $\operatorname{div} \vec{u}_\alpha = 0$ and $\vec{u}_\alpha(0, \cdot) = \vec{u}_0$. We can rewrite the equation as

$$\begin{aligned} \partial_t \vec{u}_\alpha + ((\operatorname{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha) \cdot \vec{\nabla} \vec{u}_\alpha \\ = \Delta \vec{u}_\alpha + \sum_{k=1}^3 \alpha^2 \partial_k \left((\partial_k (\operatorname{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha) \cdot \vec{\nabla} (\operatorname{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha \right) - \vec{\nabla} \cdot p_\alpha \end{aligned}$$

This is equivalent to write

$$\begin{aligned} \partial_t \vec{u}_\alpha = \Delta \vec{u}_\alpha - \mathbb{P} \operatorname{div}((\operatorname{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha) \otimes \vec{u}_\alpha \\ - \sum_{k=1}^3 \mathbb{P} \partial_j ((\alpha \partial_k (\operatorname{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha) \cdot (\alpha \vec{\nabla} (\operatorname{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha)). \end{aligned}$$

- The simplified Bardina model.

The simplified Bardina model is another α -model studied by Cao, Lunasin and Titi in 2006 [CLT]. This model is given by

$$\partial_t \vec{u}_\alpha + ((\operatorname{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha) \cdot \vec{\nabla} ((\operatorname{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha) = \Delta \vec{u}_\alpha - \vec{\nabla} p_\alpha$$

where we have again $\operatorname{div} \vec{u}_\alpha = 0$ and $\vec{u}_\alpha(0, \cdot) = \vec{u}_0$.

$$\partial_t \vec{u}_\alpha = \Delta \vec{u}_\alpha - \mathbb{P} \operatorname{div}((\operatorname{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha) \otimes ((\operatorname{Id} - \alpha^2 \Delta)^{-1} \vec{u}_\alpha).$$

Thus, when $\vec{u}_0 \in \operatorname{bmo}_0^{-1}$, all those α -models give back the mild solution $\vec{u} \in X_T$ when α goes to 0.

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