Uncertainty Theories, Degrees of Truth and Epistemic States

Didier Dubois
IRIT - Université Paul Sabatier -
31062 Toulouse Cedex (France)
e-mail: dubois@irit.fr

Introduction: Agents in Uncertainty

- AN AGENT IS UNCERTAIN ABOUT A PROPOSITION IF (S)HE DOES NOT KNOW ITS TRUTH VALUE
- **Origins of uncertainty**
  1. The **variability** of observed natural phenomena : randomness.
  2. The lack of information: **incompleteness**
  3. Conflicting testimonies or reports: **inconsistency**
- How to model partial belief of agents due to uncertainty?

Competing traditions

- **Probability theory**: frequencies and betting rates
- **Epistemic (modal) logics**: logical characterisations of belief
- **Many-valued logics** (Kleene, Belnap): modelling incomplete knowledge and inconsistency using truth-tables.

Can these approaches be reconciled or clarified ?

Aim of the talk

1. Point some limitations of unique probability distributions when information is lacking
2. Outline set-based representations of incomplete knowledge (possibility theory, evidence theory, imprecise probability)
3. Criticize some attempts at using 3 and 4-valued propositional logics for reasoning about epistemic states and inconsistency.
4. Position epistemic logics with respect to possibility theory, Kleene and Belnap logics
5. Uncertainty theories bridge the gap between modal logics and probability.

Limitations of single probability representations

A single subjective probability distribution cannot properly account for incomplete information

- *Ambiguity* : A uniform probability cannot tell pure randomness from a lack of knowledge.
- *Inconsistency*: Representations by single probability distributions are language- (or scale-) sensitive
- *Against empirical evidence*: When information is missing, decision-makers do not always choose according to a single subjective probability (Ellsberg paradox).

What is questionable is the idea that any state of information can be represented by a single probability distribution.

Set-Valued Representations of Partial Knowledge

- Uncertainty due to ignorance can be represented as a *disjunctive* set, i.e. a subset E of mutually exclusive states, one of which is the real one.
- Examples:
  - **Intervals** $x \in E = [a, b]$: good for representing incomplete numerical information to be propagated by interval analysis methods
  - **Sets of possible worlds** : good for symbolic (Boolean) information in Classical Logic : $E =$ Models of a set $\mathcal{B}$ of propositions considered true.
- A totally unbiased approach, that is not ambiguous, is language-insensitive, but poorly expressive

Epistemic states and possibility theory

Definition : An epistemic state E is a non-empty set of mutually exclusive possible worlds, one of which is the right one.

If $E \subseteq S$ represents the epistemic state of an agent, the characteristic function of E is understood as a possibility distribution on possible worlds: **if** $\pi_E(s) = 1$**:** s **is possible, impossible otherwise**

- The Boolean possibility function is Π:

  $\Pi(A) = 1$ **if and only if** $E \cap A \neq \emptyset$**, 0 otherwise.**

  $\Pi(A) = 1$ *if and only if* A *is logically consistent with* E

- The Boolean necessity function is N:

  $N(A) = 1$ **if and only if** $E \subseteq A$**, 0 otherwise.**

  $N(A) = 1$ *if and only if* A *is logically entailed by* E

- **Main properties** :

  $N(A \cap B) = \min(N(A), N(B))$; $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$;

  $N(A) = 1 - \Pi(A^c)$.

Blending possibility, necessity and probability

Boolean possibility theory is a set-function rendering of doxastic / epistemic modalities
One may consider more general settings encompassing both probability and possibility.
In increasing order of generality :

- Consider some states can be more likely than others and replace an epistemic state E by a plausibility ranking $\geq_E$ or a numerical possibility distribution π_E: **graded possibility theory** (Lewis, Shackle, Zadeh)
- Randomise epistemic states E with probabilities $m(E)$ attached to their correctness: **evidence theory** (Dempster, Shafer, Smets).
- Consider disjunctive convex sets of probabilities $\mathcal{P}$: **imprecise probability theory** (Walley).

Uncertainty theories

In each theory a proposition A is attached two degrees:

- A **degree of certainty** $Cr(A)$ (generalising necessity and probability)
- A **degree of of plausibility** $Pl(A) \geq Cr(A)$ (generalising possibility and probability), such that
- These degrees satisfy
  - $Pl(A) = 1 - Cr(A^c)$ (like doxastic modalities)
  - $Pl(A) - Cr(A)$ measures ignorance about A
  - Total ignorance : $Pl(A) = 1; Cr(A) = 0$
  - if $Pl(A) = Cr(A)$ then it is a probability degree.
- If knowledge is a probability family $\mathcal{P}$:
  - $Cr(A) = \inf_{P \in \mathcal{P}} P(A)$.
  - $Pl(A) = \sup_{P \in \mathcal{P}} P(A)$.

Example : the unreliable testimony

- Suppose and agent $\mathcal{E}$ declares to an agent $\mathcal{R}$ that $x \in E$, but $\mathcal{R}$ thinks there is a chance (p) that $\mathcal{E}$ knows but lies and a chance (q) that $\mathcal{E}$ is ignorant and makes it up.
- What does $\mathcal{R}$ knows about x ?
  1. With probability $(1-p)(1-q)$ the information is correct :
     $m(E) = (1-p)(1-q)$
  2. With probability $p(1-q)$ the real information is $x \notin E$: $m(E^c) = p(1-q)$
  3. With probability q the information is useless : $m(S) = q$ (tautology)
- Then $Cr(E) = \sum_{A \subseteq E} m(A) = (1-p)(1-q)$, and
- $Pl(E) = \sum_{E \cap A \neq \emptyset} m(A) = (1-p)(1-q) + q = 1 - p + pq$.

Many-valued Logics and Uncertainty

- From the inception of many-valued logics, there has been a temptation to attach an epistemic flavor to truth degrees.
- Well-known cases
  - Łukasiewicz 's 3d truth-value interpreted as "possible"
  - Kleene's 3-valued logic, capturing the idea of "undefined", "unknown" (too difficult to compute).
  - Reichenbach probabilistic logic, where degrees of probabilities are dubbed truth-values.
  - Partial logic extending satisfaction to partial models
  - Belnap's four-valued logic capturing the idea of possibly conflicting information received about atomic propositions from several sources

Conceptual confusion lead to misusing many-valued logics

Frequent confusion between many-valued logics and uncertainty theories (in to-day works).

- Elkan (AAAI 1991) criticising the usual fuzzy connectives $\max, \min, 1-$, as leading to an inconsistent approach to *uncertainty* handling in a logic accepting the classical tautologies.
- Attempts to use Kleene's 3-valued logic for incompleteness in databases, formal concept analysis
- Attempts to use Belnap's bilattice logics for inconsistency handling.

Kleene logic

- **Idea:** capturing the idea of a proposition with unknown status by means of an additional truth-value $\mathbf{U}$ = unknown
  - Encoding $\mathbf{U}$ as a value $1/2$ lying between 0 and 1
  - Keeping the language of propositional logic PL
  - Using truth tables for negation : $t(p) = 1 - t(\neg p)$ and $t(p \wedge q) = \min(t(p), t(q))$
  - Deriving $t(p \vee q) = \max(t(p), t(q))$ and $t(p \to q) = t(\neg p \vee q) = \max(1 - t(p), t(q))$
- **This logic has no tautologies** (no formulas with $t(p) \equiv 1$): if $t(a_i) = 1/2$ for all atoms in p then $t(p) = 1/2$

This feature is questionable : where is the bug?

Epistemic view of propositional logic

A set $\mathcal{B}$ of consistent Boolean formulae understood as a set of propositions *accepted as true* by an agent (data, belief or knowledge base):

- The set $Cons(\mathcal{B})$ is understood as a *belief set* (Gärdenfors),
- and its set of models $E = [\mathcal{B}]$ form an epistemic state E.

One can attach one of THREE epistemic valuations **T, F, U** to propositions

1. p is accepted (believed, or known), if $\mathcal{B}$ implies p : attach **T**;
2. its negation is accepted (believed, or known), if $\mathcal{B}$ implies $\neg p$: attach **F** ;
3. neither p nor $\neg p$ is accepted (ignorance), if $\mathcal{B}$ implies neither p nor $\neg p$: attach **U**.

Such epistemic valuations then refer to the notion of validity of propositions in the face of $\mathcal{B}$ *: a matter of consequencehood, not truth-values, contrary to what Kleene logic suggests*

Epistemic states as *subsets of possible truth-values*

One can represent epistemic valuations in PL by means of *subsets* $E(p)$ *of possible truth-values* attached to propositions.

- Attaching **T** to p corresponds to the singleton $E(p) = \{1\}$ (when $E \subset [p]$, only 'true" is left)
- Attaching **F** to p corresponds to the singleton $E(p) = \{0\}$ (when $E \subset [\neg p]$, only 'false" is left)
- Attaching **U** to p for the agent corresponds to the set $E(p) = \{0, 1\}$ (both 0 and 1 are possible if none of the above inclusions hold)

Claim We cannot consistently reason under incomplete or conflicting information about classical propositions by augmenting *true = 1* and *false = 0* with additional values, like Kleene logic does.

At the mathematical level, viewing epistemic attitudes as additional truth-values comes down to confusing elements and subsets of a set.

Justifying the truth-tables for Kleene logic

Interpreting unknown (**U**) as the set $\{0,1\}$, one may be tempted to extend the standard Boolean connectives to such three-valued sets by means of interval computation: for instance $\{0\} \vee \{0,1\} = \{0 \vee 0, 0 \vee 1\} = \{0,1\} = \{0,1\} \vee \{0,1\}$; etc.

$\wedge$	$\{0\}$	$\{0,1\}$	$\{1\}$
$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
$\{0,1\}$	$\{0\}$	$\{0,1\}$	$\{0,1\}$
$\{1\}$	$\{0\}$	$\{0,1\}$	$\{1\}$

Table 1: Conjunction for Boolean propositions

- Such truth-tables are the same as Kleene three-valued logic truth-tables (encoding $\{0,1\}$ as $1/2$) using $(\{0, \frac{1}{2}, 1\}, \min, \max, 1 - \cdot)$.
- However, we cannot compute the epistemic valuation of p compositionally by using the truth tables on epistemic valuations of atoms!!!

Computing epistemic valuations is not compositional

- The canonical extension of the calculus of truth-values of propositions from the truth-values of atoms takes the following form: if $p = f(a_1, a_2, \ldots, a_n)$, and $t(a_i) \in E_i \subseteq \{0, 1\}, i = 1, \ldots n$, then

$$E(p) := \{t(p), t(a_i) \in E_i, i = 1, \ldots n\}$$

- As a consequence, if p is a Boolean tautology, then obviously $E(p) = \{1\}$
- For instance, $E(p \vee \neg p) = \{1\}$, while if $E(p) = \{0, 1\}$, then $E(\neg p) = \{0, 1\}$ and using the truth-table of disjunction, $E(p \vee \neg p) = \max(E(p), E(\neg p)) = \{0, 1\}$
- *Applying Kleene logic to handle incomplete information on Boolean propositions is losing information conveyed by by the axioms of the underlying Boolean logic*

A Kleenean logic of Incompleteness: Partial Logic

Partial logic has the same syntax as propositional logic, but deviates from PL semantics: from interpretation to partial interpretations.

- A partial interpretation $\sigma \in \mathcal{S}$ can be represented as any conjunction of literals pertaining to distinct propositional variables in $Prop = \{a, b, c, \ldots\}$.
- The truth-assignment $t_\sigma(a)$ is a partial function from $Prop$ to $\{0, 1\}$ such that $t_\sigma(a) = 1$ if a is true in σ, 0 if a is false in σ, and is undefined otherwise.

A Kleenean logic of Incompleteness: Partial Logic

Two relations are defined for the semantics of connectives, namely *satisfies* ($\models_T$) and *falsifies* ($\models_F$):

$\sigma \models_T a$ if and only if $t_\sigma(a) = 1$; $\sigma \models_F a$ if and only if $t_\sigma(a) = 0$;

$\sigma \models_T \neg p$ if and only if $\sigma \models_F p$; $\sigma \models_F \neg p$ if and only if $\sigma \models_T p$;

$\sigma \models_T p \wedge q$ if and only if $\sigma \models_T p$ and $\sigma \models_T q$,

$\sigma \models_F p \wedge q$ **if and only if** $\sigma \models_F p$ **or** $\sigma \models_F q$

$\sigma \models_T p \vee q$ **if and only if** $\sigma \models_T p$ **or** $\sigma \models_T q$

$\sigma \models_F p \vee q$ if and only if $\sigma \models_F p$ and $\sigma \models_F q$.

A Kleenean logic of Incompleteness: Partial Logic

- a partial interpretation = a complete truth-assignment t_σ mapping each propositional variable to the set $\{0, \frac{1}{2}, 1\}$, understood as a partial Boolean truth-assignment in $\{0, 1\}$: $t_\sigma(a) = \frac{1}{2}$ if $t_\sigma(a)$ is undefined.
- The semantics corresponds to Kleene 3-valued logic using $(\{0, \frac{1}{2}, 1\}, \min, \max, 1 - \cdot)$.

THE FLAW: Interpreting a partial model σ as an epistemic state E, the disjunctive set of interpretations I_σ completing σ, the equivalence $\sigma \models_T p \vee q$ if and only if $\sigma \models_T p$ or $\sigma \models_T q$ cannot hold under classical model semantics

$E \subseteq [p \vee q]$ *holds whenever* $E \subseteq [p]$ *or* $E \subseteq [q]$ *holds, while the converse is invalid!*

Hence :

- **Partial logic handles special types of epistemic states (partial models)**
- **tautologies of PL are lost for no clear reason!**

A Kleenean logic of Incompleteness: Partial Logic

Solutions:

- Supervaluations (Van Fraassen) : p is "super-true" in σ if and only if it is true in all its completions : $E_\sigma \subseteq [p]$.

  Even if p and $\neg p$ are unknown ($t_\sigma(p) = t_\sigma(\neg p) = \frac{1}{2}$), $p \vee \neg p$ is always super true.

- Possibility theory : Use the characteristic function of E_σ as a possibility distribution and compute $\Pi(p) = 1$ if $E_\sigma \cap [p] \neq \emptyset$. Then, $\Pi(p) = \Pi(\neg p) = 1$ when neither $\sigma \models_T p$ nor $\sigma \models_F p$, but $\Pi(p \wedge \neg p) = 0$.

- A modal logic : a language where it can be syntactically expressed that a proposition is unknown.

The case of Belnap logic: a logic of reported conflicting information

The notion of *epistemic set-up* is defined as an assignment $\mathcal{E}$, of one of four values denoted $\mathbf{T}, \mathbf{F}, \mathbf{U}, \mathbf{C}$, to each atomic BOOLEAN proposition $a, b, \ldots$:

1. Assigning $\mathbf{T}$ to a means the computer has only been told that a is true.
2. Assigning $\mathbf{F}$ to a means the computer has only been told that a is false.
3. Assigning $\mathbf{C}$ to a means the computer has been told at least that a is true by one source and false by another.
4. Assigning $\mathbf{U}$ to a means the computer has been told nothing about a.

So this is like an agent receiving information from other agents on atoms of a propositional Boolean logic.

The Four Valuations

- The ontological truth-set of Belnap logic is $\{0, 1\}$ is (propositions are true or false in the world)
- The “epistemic” truth-set is **4** $= \{\mathbf{T}, \mathbf{F}, \mathbf{U}, \mathbf{C}\}$ (there are four situations considered when receiving information from sources)
- The “epistemic” truth-set coincides with the power set of $\{0, 1\}$:
  - Our convention : $\mathbf{C} = \emptyset$ (contradiction : no truth-values left);
  - $\mathbf{U} = \{0, 1\}$ (ignorance: all truth-values left)
- Belnap interprets subsets of $\{0,\ 1\}$ in an accumulative way
  - $\mathbf{U} = \emptyset$ that is, told neither true nor false (Belnap uses **NONE**)
  - $\mathbf{C} = \{0, 1\}$: told both true and false (Belnap uses **BOTH**).

The case of Belnap logic : truth-tables

Truth tables extend the epistemic truth assignment $\mathcal{E} : Prop \to$ **4** to PL :

$\cap$	$\mathbf{F}$	$\mathbf{U}$	$\mathbf{C}$	$\mathbf{T}$
$\mathbf{F}$	$\mathbf{F}$	$\mathbf{F}$	$\mathbf{F}$	$\mathbf{F}$
$\mathbf{U}$	$\mathbf{F}$	$\mathbf{U}$	$\mathbf{F}$	$\mathbf{U}$
$\mathbf{C}$	$\mathbf{F}$	$\mathbf{F}$	$\mathbf{C}$	$\mathbf{C}$
$\mathbf{T}$	$\mathbf{F}$	$\mathbf{U}$	$\mathbf{C}$	$\mathbf{T}$

Table 2: Belnap conjunction

- Belnap connectives restricted to $\{\mathbf{T}, \mathbf{F}, \mathbf{U}\}$ and $\{\mathbf{T}, \mathbf{F}, \mathbf{C}\}$ coincide with connectives in Kleene logic.
- Belnap's logic inherits all difficulties of partial logic regarding the truth-value **U** because, ontologically, propositions are Boolean.

The case of Belnap logic : $\mathbf{C} \wedge \mathbf{U} = \mathbf{F}$

Explanation: Suppose two sources S_1 and S_2 and two atoms a and b.

- S_1 says a is true and S_2 says it is false : a is $\mathbf{C}$
- Both sources say nothing about b, so b is $\mathbf{U}$.
- Hence S_1 has nothing to say about $a \wedge b$, but S_2 would say $a \wedge b$ is false
- So $a \wedge b$ should be assigned $E(a \wedge b) = \mathbf{F}$

This reasoning makes sense for independent propositions not for any pair thereof.

However $\mathbf{C}$ and $\mathbf{U}$ are invariant under negation:
$E(\neg a \wedge b) = E(a \wedge \neg b) = E(\neg a \wedge \neg b) = \mathbf{F}$.

Hence $E((a \wedge b) \vee (\neg a \wedge b) \vee (a \wedge \neg b) \vee (\neg a \wedge \neg b)) = \mathbf{F}$

That is, acknowledging Boolean atoms, $\mathcal{E}(\top) = \mathbf{F}$ which is hardly acceptable again.

Degrees of truth and information states

- A truth degree defines the nature of a proposition p (ontological), and evaluates the extent to which it applies to a precise situation.
    1. The choice of the truth-set is a matter of convention (2 or more truth-values)
    2. The truth-degree is an abstract notion that measures the p-ness of a situation.
    3. The truth set is the range of propositional variables or logical functions
- Epistemic notions (knowledge, or belief) measure the extent to which a proposition is compatible *with an agent's information* about the situation
    1. The lack of knowledge is not a matter of convention.
    2. Degrees of belief, quality of knowledge, reliability of a report are meta-notions
    3. Elementary models of epistemic notions on Boolean propositions are basically ternary: being informed about p, about $\neg p$, and not being informed.

Reasoning about incomplete information : the need for two levels of language

Basic remark: We cannot reason about another agent's beliefs in propositional logic PL.

In a propositional **belief base**, only conjunctions of beliefs are allowed:

- The semantics is overloaded : ontological truth-values (0 or 1) vs. epistemic values.
- You cannot distinguish between *not believing* p and *believing* $\neg p$: you can only write $\neg p$.
- You cannot distinguish between believing either p or q and believing $p \vee q$: you can only write $p \vee q$.

So, the formal language must handle:

- atomic formulae of the form $\Box p$, that mean "an agent believes p", where p is expressible in propositional logic,
- Their negations, conjunctions, disjunctions;

A simple Doxastic Modal Language : Meta-Epistemic Logic

Given a propositional language $\mathcal{L}$ with formulas $p, q, r, ...$

Atomic formulas of MEL are $\{\Box p, p \in \mathcal{L}\}$.

The set of MEL-formulae, denoted $\phi, \psi...$ is generated from the set of atomic formulae, with the help of the Boolean connectives $\neg$, $\wedge$

- $MEL := \ \Box p \mid \neg\phi \mid \phi \wedge \psi$
- $\phi \vee \psi = \neg(\neg\phi \wedge \neg\psi)$
- $\Diamond p := \neg\Box\neg p$, where $p \in PL$.

Modalities $\Box$, $\Diamond$ apply only on PL-formulae.

Remark Contrary to epistemic logics,

- iteration of the 'modal' operators $\Box$, $\Diamond$ is not allowed.
- Propositional formulas in $\mathcal{L}$ and their combinations with ϕ's are not allowed in MEL.

What agent $\mathcal{R}$ can know from another agent $\mathcal{R}$

Any set Γ of formulas in this language is interpreted as what a receiver agent $\mathcal{R}$ knows about what an emitter agent $\mathcal{E}$ can believe based on the latter's testimony.

- $\Box p \in \Gamma$ means agent $\mathcal{R}$ knows that $\mathcal{E}$ believes p is true.
- $\Diamond p \in \Gamma$ means agent $\mathcal{R}$ knows that $\mathcal{E}$ considers p is not impossible (she has no argument as to the falsity of p). (*All that $\mathcal{R}$ can conclude is that either $\mathcal{E}$ believes p is true, or ignores whether p is true or not).*
- $\Diamond p \wedge \Diamond\neg p \in \Gamma$ means $\mathcal{R}$ knows that $\mathcal{E}$ ignores whether p is true or not.
- $\Box p \vee \Box\neg p \in \Gamma$ means $\mathcal{R}$ knows that $\mathcal{E}$ is not ignorant about p ($\mathcal{E}$ believes either p or not, but $\mathcal{R}$).

The language allows to express that $\mathcal{R}$ believes that $\mathcal{E}$ ignores whether a proposition p is true or not, but it cannot express that $\mathcal{R}$ ignores if agent $\mathcal{E}$ believes p .

The logic MEL : Axioms and inference rules

Axioms:

- $(K\wedge)$: $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$.
- (N) : $\Box\top$.
- (D) : $\Box p \to \Diamond p$
- (PL) : PL-axioms for all MEL-formulae.

Rules:

- (MP) : If $\phi, \phi \to \psi$ then ψ.
- (N') : If $\vdash p \to q$ then $\vdash \Box p \to \Box q$.

Axioms $(K\wedge) + (N) + (PL)$: agent $\mathcal{E}$ is logically sophisticated, in the classical sense

This is a fragment of logic KD, the subjective fragment of KD45 or S5.

Epistemic states and Accessibility relations

- In usual epistemic logics, the set S of interpretations is equipped with a relation R;
  - an equivalence relation in the case of S5 for representing knowledge;
  - a transitive Euclidean serial relation in the case of KD45 for belief.
- The satisfaction relation says $(s, R) \models \Box p \iff R(s) = \{s' : sRs'\} \subseteq [p]$.

  Intuitions of this relation for handling incomplete information are questionable or unclear : sRs' is said to mean
  - s and s' are not distinguishable (Rough sets, Orlowska).
  - s' is a state of affairs considered possible by the agent (from the standpoint of what the agent knows) in state s.
- What does "agent (knows) in state s" mean ?

We must clearly enrich the propositional language with epistemic symbols, but we need a simpler semantics in terms of epistemic states.

The logic MEL : Semantics

A "model" of a MEL-formula is an epistemic state of agent $\mathcal{E}$: a nonempty subset E of propositional models. For clarity we call it a *meta-model*.

The satisfaction of MEL-formulae is defined recursively:

- $E \models \Box p$, if and only if $E \subseteq [p]$, where $\Box p \in At$;
- $E \models \neg\phi$, if and only if $E \not\models \phi$,
- $E \models \phi \wedge \psi$, if and only if $E \models \phi$ and $E \models \psi$,
  where ϕ, ψ are any MEL-formulae.

For any set $\Gamma \cup \{\phi\}$ of MEL-formulae, ϕ is a semantic consequence of Γ, written $\Gamma \models_{MEL} \phi$, provided for every epistemic state $E, E \models \Gamma$ implies $E \models \phi$.

The logic MEL : meta-models

Let $[\phi] \in 2^{\mathcal{V}}$ be the set of meta-models of $\phi \in MEL$. It is the *meta-epistemic state* of agent $\mathcal{R}$ representing what $\mathcal{R}$ knows about $\mathcal{E}$ if $\mathcal{E}$ said ϕ.

- $[\Box p] = \{E \neq \emptyset : E \subseteq [p]\}$
- $[\Diamond p] = \{E : E \cap [p] \neq \emptyset\}$
- $[\Box p \vee \Box q] = \{E \subseteq [p]\} \cup \{E \subseteq [q]\} \subset [\Box(p \vee q)]$

So MEL encodes the subset of $\mathcal{E}$'s beliefs known by $\mathcal{R}$, and it corresponds to a meta-epistemic state of $\mathcal{R}$, understood as a set of possible epistemic states of $\mathcal{E}$.

Theorem The Logic MEL is sound and complete w.r.t the epistemic meta-model semantics.

Is it a modal logic in the usual sense ?

- MEL is the same as the normal modal system KD with a restricted language.
- One can deduce axiom (K) from a system containing the axioms $(K\wedge)$, (N) and the rule (N').
- MEL does NOT allow for propositional ("objective") formulas
- The semantics of MEL is purposedly not defined via Kripke semantics (even if metamodels can be expressed by means of Kripke models)
- If Γ only contain atoms $\Box p$, and $B = \{p : \Box p \in \Gamma\}$, then $\Gamma \vdash_{MEL} \Box p$ if and only if $B \vdash p$ in the classical sense.

So propositional logic is *encapsulated* in MEL; MEL is not a modal extension of propositional logic : it is a **two-tiered logic**.

An alternative syntax : a Labelled Belief logic

One can encapsulate more explicitly propositional logic putting the Kleene's epistemic truth-values in the syntax, with $(p, \mathbf{T})$ standing for $E \subseteq [p]$, $\mathbf{T}$ for $\{1\}$.

- **Syntax**: $LBL := \ (p, \mathbf{T}) \mid \neg\phi \mid \phi \wedge \psi$ where $p \in PL$.
- Derived symbols: $\phi \vee \psi = \neg(\neg\phi \wedge \neg\psi)$; $(p, \mathbf{F}) := (\neg p, \mathbf{T})$, $(p, \mathbf{U}) := \neg(p, \mathbf{T}) \wedge \neg(p, \mathbf{F})$ (**U** stands for $\{0, 1\}$)
- **Axioms:**
  - $(K\wedge)$: $(p \wedge q, \mathbf{T}) \leftrightarrow (p, \mathbf{T}) \wedge (q, \mathbf{T})$.
  - (N) : $\ (\top, \mathbf{T})$.
  - (D) : $\ (p, \mathbf{T}) \to \neg(p, \mathbf{F})$
  - (PL) : PL-axioms for all LBL-formulae.
- **Inference rules** : Modus ponens and If If $\vdash p \to q$ then $\vdash (p, \mathbf{T}) \to (q, \mathbf{T})$
- **Theorem** : $\vdash (p, \mathbf{T}) \vee (p, \mathbf{F}) \vee (p, \mathbf{U})$

Multiagent extensions of MEL : handling contradictions

Belnap setting could be captured and extended by the case of two agents $\mathcal{E}_1$ and $\mathcal{E}_2$ and providing information on Boolean propositions.

$1, 2$	$\mathbf{T_2}$	$\mathbf{U_2}$	$\mathbf{F_2}$
$\mathbf{T_1}$	$\mathbf{T}$	$\mathbf{T}$	$\mathbf{C}$
$\mathbf{U_1}$	$\mathbf{T}$	$\mathbf{U}$	$\mathbf{F}$
$\mathbf{F_1}$	$\mathbf{C}$	$\mathbf{F}$	$\mathbf{F}$

Table 3: Belnap Epistemic values from two sources

Each agent $\mathcal{E}_i$ either believes a proposition p is true $(\mathbf{T}_i)$, false $(\mathbf{F}_i)$ or unknown ($\mathbf{U}_i$), which can be written in MEL $\Box_i p$, $\Box_i \neg p$ and $\Diamond p \wedge \Diamond \neg p$, respectively.

Syntax

- Let $\Box p$ means "at least one source asserts p", formally as $\Box p \equiv \Box_1 p \vee \Box_2 p$.
- $\Diamond p \equiv \neg\Box\neg p$, that is $\Diamond p \equiv \Diamond_1 \neg p \wedge \Diamond_2 \neg p$, that is "no source asserts $\neg p$"
- Then :
  - $(p, \mathbf{T})$ is short for $\Box p \wedge \Diamond p$
  - $(p, \mathbf{F})$ is short for $\Box\neg p \wedge \Diamond\neg p$
  - $(p, \mathbf{U})$ is short for $\Diamond p \wedge \Diamond\neg p$ (as usual)
  - $(p, \mathbf{C})$ is short for $\Box p \wedge \Box\neg p$
- Again a higher level propositional calculus with atoms $\Box p$.

Axioms : a non-regular modal logic

(PL) : $(i)\ \phi \rightarrow (\psi \rightarrow \phi)$;
$(ii)\ (\phi \rightarrow (\psi \rightarrow \mu)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \mu))$;
$(iii)\ (\neg\phi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \phi)$.

(RE) : $\Box p \equiv \Box q$ if and only if $\vdash p \equiv q$.

(RM) : $\Box p \rightarrow \Box q$, whenever $\vdash p \rightarrow q$.

(N) : $\Box\top$.

(POS) : $\Diamond\top$

(MP) : If $\phi, \phi \rightarrow \psi$ then ψ.

It is a special case of the monotonic logic EMN

Its usual semantics is complex (neighborhood semantics).

Simple semantics

The set of (meta) interpretations for the new logic will be

$$\mathbb{I}_{12} = \{(E_1, E_2) : E_i \neq \emptyset, E_i \subseteq \Omega, i = 1, 2\}$$

where Ω is the set of interpretations of $\mathcal{L}$. Moreover we define satisfaction of $\Box p$ as follows:

$$(E_1, E_2) \models \Box p \quad \Longleftrightarrow \quad E_1 \subseteq [p] \text{ or } E_2 \subseteq [p].$$

- The receiver agent has imprecise information on the joint epistemic states of the two other agents.
- Axiom D : $\Box p \to \Diamond p$ fails.
- Axiom C: $(\Box p \wedge \Box q) \to \Box(p \wedge q)$ fails
- This system is the same as the modal logic of risky knowledge of Kyburg and Teng ($\Box p$ is then interpreted as $Prob(p) \geq \theta > 0.5$).

Handling uncertainty requires a two-tiered logic language

The MEL construction can be generalized for a proper logical handling of reasoning about uncertainty

- At the bottom level: an "objective" language $\mathcal{L}$ accounting for the real world.
- At level 1:
  - formulas in $\mathcal{L}$ encapsulated in doxastic modalities describing epistemic valuations attached to formulas (epistemically labelled objective formulas)
  - Conjunctions, disjunctions and negations thereof.
- The meta-language is at level 2 : inferring labelled formulas and combinations thereof
- Truth of formulas at level 1 return epistemic valuations of formulas at level 0.

Many-valued extensions of two-tiered logics

- At the semantic level one can consider
  1. A non-Boolean ontological truth set (the unit interval) for describing the world (where 1/2 really means half-true).
  2. Many-valued extension of epistemic valuations (degrees of probability, possibility, certainty, etc.) : a many-valued truth set for epistemically labelled formulas.
- At the syntactic level one can consider
  - A fuzzy logic inside for graded propositions about the world.
  - A fuzzy logic outside for gradual modalities expressing possibility and necessity measures, belief functions...

    (Alternatively : generalized epistemic labels : uncertainty measures on the ontological truth set)

Basic ideas in a two-tiered logic

- The truth of $\Box p$ corresponds to the certainty of the proposition p.
- The truth degree of the many-valued proposition *Probable(p)* is the known degree of probability of proposition p.
- The truth degree of the many-valued proposition (p, α) in possibilistic logic is the degree of necessity $N(p)$ of a proposition p.

Examples of two-tiered systems

Possibilistic logics (Dubois, Lang, Prade):

- **Syntax** : Pairs (p, α) where p is in PL, and $\alpha \in [0, 1]$ is a degree of certainty; it stands for $N(p) \geq \alpha$ and the epistemic valuation corresponds to a possibility distribution $\{1 - \alpha/0, 1/1\}$,
  - Inside: classical logic
  - Outside: (a fragment of) Gödel logic : only the $\wedge$ connective between labelled formulas (for full-fledged Gödel logic : works by Sossai and Boldrin)
- **Semantics** : epistemic states are fuzzy sets of interpretations.

Examples of two-tiered systems

Signed multivalued logics (Pavelka, Rainer Haehnle)

- **Syntax** (p, α) where α enforces the truth-value of p to lie in $[\alpha, 1]$.
  - Inside: Łukasiewicz logic (many-valued propositions)
  - Outside: Classical logic (Boolean constraints on truth-values)
- **Semantics** : a constraint on the truth-value of p restricting the set of interpretations.

Possibilistic multivalued logics (Alsinet, Godo)

- **Syntax** (p, α) where $\alpha =$ the degree of certainty of a many-valued formula.
  - Inside:Gödel logic
  - Outside: (a fragment of) Gödel logic (only with conjunction = minimum)
- **Semantics** : epistemic states are fuzzy sets of many-valued interpretations.

Examples of two-tiered systems

Probability logic (Hajek, Godo, Flaminio)

- **Syntax** : Atoms of the form $Probable(p)$ where $Probable$ is a many-valued predicate
  - Inside : classical propositional logic (extended to multiple-valued logic recently)
  - Outside: a many-valued logic $Ł\Pi\frac{1}{2}$ capable of supporting axioms of probability theory
- **Semantics**: sets of probability distributions

Belief function logic

- **Syntax** : Atoms are many-valued modal formulas $Probable(\Box p)$ and $Bel(p) := P(\Box p)$ where $p \in PL$.
  - Inside : S5
  - Outside: the many-valued logic $Ł\Pi\frac{1}{2}$
- **Semantics**: Accessibility relations and a probability distribution on epistemic states.

Alternative approaches, open problems

- "Fuzzy truth-values" proposed by Zadeh : epistemic labels rather than ontological truth-values (see Lehmke).
- Avron's non-deterministic truth-tables may account for uncertainty about resulting truth-values. Compare with MEL....?
- Extending MEL or LBL to Belnap multisource information processing problem : links with A. Avron and J. Ben-Naim.
- Paraconsistent logics : do they suffer from similar problems for inconsistency handling as Kleene's for handling incompleteness.
- Jointly extend MEL and possibilistic logic
- Multiagent extensions : fusion of information coming from several agents and inconsistency handling.
- Relate MEL and the Belief function logic of Godo et al: simplify the semantics and restrict to subjective S5

Conclusion

- There is a close links between modal doxastic logics and uncertainty theories like imprecise probabilities, belief functions and possibility theory.
- It is important to separate truth-values and epistemic values expressing a state of knowledge: the former are supposed to be compositional, the latter cannot.
- Such difficulties appear in partial logic, Belnap logic, and interval-valued fuzzy logic, type 2 fuzzy sets.
- In logical approaches to incompleteness and contradiction, the goal of preserving tautologies of the underlying logic (classical or multivalued) should supersede the goal of maintaining a truth-functional setting.
- Laying the ground for simple logics for reasoning *about* uncertainty due to incomplete information that bridge the gap with uncertainty theories.