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ASYMPTOTICS OF THE DISTRIBUTION AND HARMONIC MOMENTS FOR A SUPERCRITICAL BRANCHING PROCESS IN A RANDOM ENVIRONMENT

ION GRAMA, QUANSHENG LIU, AND ERIC MIQUEU

Abstract. Let (Z_n) be a supercritical branching process in an independent and identically distributed random environment ξ. We deduce the exact decay rate of the probability $\mathbb{P}(Z_n = j|Z_0 = k)$ as $n \to \infty$, for each $j \geqslant k$, assuming that $\mathbb{P}(Z_1 = 0) = 0$. We also study the existence of harmonic moments of the random variable $W = \lim_{n\to\infty} \frac{Z_n}{\mathbb{E}(Z_n|\xi)}$ under a simple moment condition.

Résumé. Soit (Z_n) un processus de branchement surcritique en environnement aléatoire ξ indépendant et identiquement distribué. Nous donnons un équivalent de la probabilité $\mathbb{P}(Z_n = j|Z_0 = k)$ lorsque $n \to \infty$, pour tout $j \geqslant k$, sous la condition $\mathbb{P}(Z_1 = 0) = 0$. Nous étudions également l'existence des moments harmoniques de la variable aléatoire limite $W = \lim_{n\to\infty} \frac{Z_n}{\mathbb{E}(Z_n|\xi)}$, sous une hypothèse simple d'existence de moments.

1. Introduction

A branching process in a random environment (BPRE) is a natural and important generalisation of the Galton-Watson process, where the reproduction law varies according to a random environment indexed by time. It was introduced in Smith and Wilkinson [19] to model the growth of a population in an unknown exogenous environment. For background concepts and basic results concerning a BPRE we refer to Athreya and Karlin [5, 4]. In the critical and subcritical regime the branching process gets extinct and the research interest is mostly concentrated on the survival probability and conditional limit theorems, see e.g. Afanasyev, Böinghoff, Kersting, Vatutin [1, 2], Vatutin [21], Vatutin and Zheng [22], and the references therein. In the supercritical case, a great deal of current research has been focused on large deviations, see Bansaye and Berestycki [7], Bansaye and Böinghoff [8, 9, 10], Böinghoff and Kersting [12], Huang and Liu [15], Nakashima [18]. In the particular case when the offspring distribution is geometric, precise asymptotics can be found in Böinghoff [11], Kozlov [16].

An important closely linked issue is the asymptotic behavior of the distribution of a BPRE (Z_n), i.e. the limit of $\mathbb{P}(Z_n = j|Z_0 = k)$ as $n \to \infty$, for fixed $j \geqslant 1$

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when the process starts with $k \geqslant 1$ initial individuals. For the Galton-Watson process, the asymptotic is well-known and can be found in the book by Athreya and Ney [6]. Concerning the lower large deviation principle of a BPRE, Bansaye and Böinghoff have shown in [10] that, for any fixed $j \geqslant 1$ and $k \geqslant 1$ it holds $n^{-1}\log \mathbb{P}(Z_n = j|Z_0 = k) \to -\rho$ as $n \to \infty$, where $\rho > 0$ is a constant. This result characterizes the exponential decrease of the probability $\mathbb{P}(Z_n = j|Z_0 = k)$ for the general supercritical case, when extinction can occur. However, it stands only on a logarithmic scale, and the constant ρ is not explicit, except when the reproduction law is fractional linear, for which ρ is explicitly computed in [10]. Sharper asymptotic results for the fractional linear case can be found in [11]. In the present paper, we improve the results of [10] and extend those of [11] by giving an asymptotic equivalent of the probability $\mathbb{P}(Z_n = j|Z_0 = k)$ as $n \to \infty$, provided that each individual gives birth to at least one child. These results are important to understand the asymptotics of the process, and are useful to obtain sharper asymptotic large deviation results. We also improve the result of [15] about the existence of the harmonic moment of the limit variable $W = \lim_{n\to\infty} \frac{Z_n}{\mathbb{E}(Z_n|\xi)}$, and closes a gap in [13] on this topic.

Let us explain briefly the findings of the paper. Assume that $\mathbb{P}(Z_1 = 0) = 0$. From Theorem 2.4 of the paper it follows that when $Z_0 = 1$,

$$\mathbb{P}\left(Z_n = j\right) \underset{n\to\infty}{\sim} \gamma^n q_j \quad \text{with} \quad \gamma = \mathbb{P}(Z_1 = 1) > 0, \tag{1.1}$$

where $q_j \in [0, +\infty)$ can be computed as the unique solution of some recurrence equations, and $q_j > 0$ if and only if $\mathbb{P}\left(Z_n = j\right) > 0$ for some $n \geqslant 0$; moreover, the generating function $Q(t) = \sum_{j=1}^{\infty} q_j t^j$ has the radius of convergence equal to 1 and is characterized by the functional equation

$$\gamma Q(t) = \mathbb{E}Q(f_0(t)), \quad t \in [0, 1), \tag{1.2}$$

where $f_0(t) = \sum_{i=1}^{\infty} p_i(\xi_0)t^i$ is the conditional generating function of Z_1 given the environment. These results extend the corresponding results for the Galton-Watson process (see [6]). They also improve and complete the results in [10] and [11]: it was proved in [10] that $\frac{1}{n}\log \mathbb{P}\left(Z_n = j\right) \to \log \gamma$, and in [11] that $\mathbb{P}\left(Z_n = 1\right) \underset{n\to\infty}{\sim} \gamma^n q_1$ in the fractional linear case.

In the proofs of the above results we make use of Theorem 2.1 on the harmonic moments of W, which shows that: a) for any integer $k \geqslant 1$, $\mathbb{E}[W^{-a}|Z_0 = k] < \infty$ for some $a > 0$ under a simple moment condition on $m_0^{(p)} = \sum_{i=1}^{\infty} i^p p_i(\xi_0)$ for some $p > 1$, b) for any fixed $a > 0$,

$$\mathbb{E}[W^{-a}|Z_0 = k] < \infty \quad \text{if and only if} \quad \mathbb{E}\left[p_1^k(\xi_0)m_0^a\right] < 1, \tag{1.3}$$

under a boundedness condition on $m_0 = \sum_{i=1}^{\infty} i p_i(\xi_0)$ and $m_0^{(p)}$. Part a) corrects an error in an earlier work [13, Theorem 1.2] where the case $Z_0 = 1$ was considered, and improves the corresponding result in [15] where a boundedness condition on m_0 and $m_0^{(p)}$ was required; part b) extends Theorem 1.4 in [15] where the case $k = 1$ was treated.

The proof of Theorem 2.1 is based on the method developed in [17, Lemma 4.1] which enables to obtain the decay rate of the Laplace transform $\phi_k(t) = \mathbb{E}[e^{-tW}|Z_0 = k]$ as $t \to \infty$, from a functional inequality of the form

$$\phi(t) \leqslant q\mathbb{E}\phi(Yt) + Ct^{-a}, \tag{1.4}$$

where Y is a positive random variable. Such an equality is obtained by a careful analysis using a recursive procedure and a bound of the quenched p-th moment $\mathbb{E}[W^p|\xi]$.

In the proof of Theorem 2.4, the equivalence relation (1.1) and the recursive equations for the limit values (q_j) come from simple monotonicity arguments. The difficulty is to characterize the sequence (q_j) by its generating function Q. To this end, we first calculate the radius of convergence of Q by determining the asymptotic behavior of the normalized harmonic moments $\mathbb{E}Z_n^{-r}/\gamma^n$ as $n \to \infty$ for some $r > 0$ large enough and by using the fact that $\sum_{j=1}^{\infty} j^{-r} q_j = \lim_{n\to\infty} \mathbb{E}Z_n^{-r}/\gamma^n$. We then show that the functional equation (1.2) has a unique solution subject to an initial condition.

The rest of the paper is organized as follows. The main results, Theorems 2.1 and 2.4, are presented in Section 2. Their proofs are given respectively in Sections 3 and 4.

2. Main results

A BPRE (Z_n) can be described as follows. The random environment is represented by a sequence $\xi = (\xi_0, \xi_1, ...)$ of independent and identically distributed random variables (i.i.d. r.v.'s), whose realizations determine the probability generating functions

$$f_n(t) = f(\xi_n, t) = \sum_{i=0}^{\infty} p_i(\xi_n) t^i, \quad t \in [0,1], \quad p_i(\xi_n) \geqslant 0, \quad \sum_{i=0}^{\infty} p_i(\xi_n) = 1. \tag{2.1}$$

The branching process $(Z_n)_{n\geqslant 0}$ is defined by the relations

$$Z_{n+1} = \sum_{i=1}^{Z_n} N_{n,i}, \quad \text{for} \quad n \geqslant 0, \tag{2.2}$$

where $N_{n,i}$ is the number of children of the i-th individual of the generation n. Conditionally on the environment ξ, the r.v.'s $Z_0, N_{n,i}$ $(n \geqslant 0, i \geqslant 1)$ are all independent of each other, each $N_{n,i}$ has the probability generating function f_n. For simplicity, we only consider the case where Z_0 is a constant.

In the sequel we denote by $\mathbb{P}_\xi$ the *quenched law*, i.e. the conditional probability when the environment ξ is given, and by τ the law of the environment ξ. Then $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$ is the total law of the process, called *annealed law*. The corresponding quenched and annealed expectations are denoted respectively by $\mathbb{E}_\xi$ and $\mathbb{E}$. When $Z_0 = 1$, for $n \in \mathbb{N}$, the quenched and annealed probability generating function of Z_n are respectively

$$g_n(t) := f_0 \circ \ldots \circ \ f_{n-1}(t) = \mathbb{E}_\xi(t^{Z_n} |\ Z_0 = 1),$$

$$G_n(t) = \mathbb{E}[t^{Z_n} |\ Z_0 = 1] = \mathbb{E}\left[f_0 \circ \ldots \circ f_{n-1}(t)\right] = \mathbb{E}\left[g_n(t)\right], \tag{2.3}$$

with the convention that $g_0(t) = t$. When the process starts with k individuals, we denote by $\mathbb{P}_k$ and $\mathbb{E}_k$ the annealed probability and expectation, that is,

$$\mathbb{P}_k(\cdot) = \mathbb{P}(\cdot|Z_0 = k) \quad \text{and} \quad \mathbb{E}_k(\cdot) = \mathbb{E}(\cdot|Z_0 = k), \quad k \in \mathbb{N}^*.$$

It follows from (2.2) that the annealed probability generating function of Z_n starting with k individuals is

$$G_{k,n}(t) = \mathbb{E}_k t^{Z_n} = \mathbb{E}\left[g_n^k(t)\right], \qquad k \in \mathbb{N}^*,\ n \in \mathbb{N}. \tag{2.4}$$

Define, for any environment sequence $\xi = (\xi_0, \xi_1, \cdots)$, any integer $n \geqslant 0$ and any real number $p > 0$,

$$m_n^{(p)} = m_n^{(p)}(\xi) = \sum_{i=0}^{\infty} i^p p_i(\xi_n), \quad m_n = m_n(\xi) = m_n^{(1)}(\xi),$$
$$\Pi_0 = 1 \quad \text{and} \quad \Pi_n = \Pi_n(\xi) = \Pi_{k=0}^{n-1} \quad \text{if } n \geqslant 1.$$

Then $m_n^{(p)} = \mathbb{E}_\xi[N_{n,i}^p]$ is the conditional p-th moment of the offspring distribution at generation n, when the environment ξ is given, and $\Pi_n = \Pi_n(\xi) = \mathbb{E}_\xi[Z_n|Z_0 = 1]$. Let

$$W_n = \frac{Z_n}{\Pi_n}, \quad n \geqslant 0, \tag{2.5}$$

be the normalized population size. It is well known that under $\mathbb{P}_\xi$, as well as under $\mathbb{P}$, the sequence $(W_n)_{n\geqslant 0}$ is a non-negative martingale with respect to the filtration

$$\mathcal{F}_n = \sigma\left(\xi, N_{j,i}, 0 \leqslant j \leqslant n-1, i = 1, 2 \ldots\right),$$

where by convention $\mathcal{F}_0 = \sigma(\xi)$. Then the limit $W = \lim_{n\to\infty} W_n$ exists $\mathbb{P}$ - a.s. and $\mathbb{E}W \leqslant 1$.

We shall assume that

$$\mu := \mathbb{E}\log m_0 \in (0, \infty),$$

which implies that the BPRE is supercritical (see e.g. [19, 5, 6]) and that

$$\gamma := \mathbb{P}(Z_1 = 1) \in [0, 1). \tag{2.6}$$

With the extra condition $\mathbb{E}|\log(1-p_0(\xi_0))| < \infty$, the population size tends to infinity with positive probability (see [19]). We also assume in the whole paper that each individual gives birth to at least one child, i.e.

$$p_0(\xi_0) = 0 \quad a.s. \tag{2.7}$$

In particular, $m_0 \geqslant 1$ a.s., and $\mathbb{P}(m_0 = 1) < 1$. Consequently, under the condition

$$\mathbb{E}\frac{Z_1}{m_0}\log^+ Z_1 < \infty, \tag{2.8}$$

the martingale (W_n) converges to W in $L^1(\mathbb{P})$ (see e.g. [20]) and

$$\mathbb{P}(W > 0) = \mathbb{P}(Z_n \to \infty) = 1.$$

Our first result concerns the harmonic moments of the r.v. W. As usual, we write $\|p_1\|_\infty = \operatorname{ess\,sup} p_1$ for the essential supremum of $p_1 = p_1(\xi_0)$.

Theorem 2.1. *a) Assume that there are some constants $p > 1$ and $\varepsilon > 0$ such that*

$$\mathbb{E}m_0^{\varepsilon} < \infty \quad \text{and} \quad \mathbb{E}\Big(\frac{m_0^{(p)}}{m_0^p}\Big)^{\varepsilon} < \infty \tag{2.9}$$

Then there exists $a > 0$ such that

$$\mathbb{E}_k W^{-a} < \infty.$$

b) Assume that there are some constants $\bar{p}_1 \in (0,1)$ and $c_1, p, c_p > 1$ such that

$$p_1(\xi_0) \leqslant \bar{p}_1, \quad c_1 \leqslant m_0 \quad \text{and} \quad m_0(p) \leqslant c_p \quad a.s. \tag{2.10}$$

Then for any $a > 0$,

$$\mathbb{E}_k W^{-a} < \infty \quad \text{if and only if} \quad \mathbb{E}\left[p_1^k(\xi_0) m_0^a\right] < 1.$$

Notice that the two moments conditions in (2.9) are implied by the single one $\mathbb{E}(m_0^{(p)})^{\varepsilon} < \infty$; in practice it is in general easier to verify the condition on the normalized p-th conditional moment $\frac{m_0^{(p)}}{m_0^p}$ of the offspring distribution, rather than on the non-normalized one $m_0^{(p)}$.

Part a) improves Theorem 2.2(i) of [15], which states that $\mathbb{E}[W^{-a}|Z_0 = 1] < \infty$ for some $a > 0$ if the boundedness conditions on m_0 and $m_0^{(p)}$ in (2.10) hold. Instead of these boundedness conditions, here only a simple moment condition is used. Part b) gives the critical value for the existence of harmonic moments of W under the boundedness condition 2.10; it extends the corresponding result in [15, Theorem 1.4] where the case $k = 1$ was treated. Proving the critical value without the boundedness condition 2.10 seems very delicate. Fortunately, for the usual study of large and moderate deviations as we studied in [15, 13, 23], the existence of the harmonic moment of some order is enough.

Remark 2.2. It is stated in Theorem 1.2 (or Theorem 3.1) of [13] that $\mathbb{E}[W^{-a}|Z_0 = 1] < \infty$ for all $0 < a < a_0$, for an explicitly calculated $a_0 > 0$, provided that $\mathbb{E}m_0^{\varepsilon} < \infty$ for some $\varepsilon > 0$. Unfortunately, the proof of this claim in [13] contains an error: on p.1261, the inequality of the last line cannot be obtained from the inequality of line 3 from the bottom, because there is the common term $\phi_{T_n\xi}$ in the factors of the product therein (so that these factors are not independent); consequently, Eq. (2.7) of that paper is not proved. We are grateful to the referee who pointed out the error in an earlier version of the present paper, originated from [13]. Part a) of Theorem 2.1 is a corrected version of Theorem 1.2 (or Theorem 3.1) of [13], with the conditional moment condition $\mathbb{E}\frac{(m_0^{(p)})^{\varepsilon}}{m_0^{p\varepsilon}} < \infty$ (which holds for $\varepsilon = 1$ when (A2) of [13] holds, and for $\varepsilon = 1/p$ when (A4) of [13] holds), and the slightly modified conclusion that $\mathbb{E}[W^{-a}|Z_0 = 1] < \infty$ for some $a > 0$ (which is enough for the purposes in [13, 23]). Due to this result, the gap in the proof of Theorem 1.2 [13] does not have impact on the results on Berry Esseen's bound and large and moderate deviations stated in [13, 23].

From Theorem 2.1 we get the following corollary.

Corollary 2.3. *Assume (2.10). If $\mathbb{P}(p_1 = 0) = 1$, then $\mathbb{E}_k W^{-a} < \infty$ for all $a > 0$. If $\mathbb{P}(p_1 = 0) < 1$, then the equation*

$$\mathbb{E}[p_1^k m_0^{a_k}] = 1 \tag{2.11}$$

has a unique solution a_k on $(0, \infty)$, and

$$\begin{cases} \mathbb{E}_k W^{-a} < \infty & \text{for} \quad a \in [0, a_k), \\ \mathbb{E}_k W^{-a} = \infty & \text{for} \quad a \in [a_k, \infty). \end{cases}$$

This indicates that the solution a_k of the equation (2.11) is the critical value for the existence of harmonic moments of the r.v. W.

The next result gives an asymptotic equivalent as $n \to \infty$ of the probability $\mathbb{P}_k(Z_n = j) = \mathbb{P}(Z_n = j | Z_0 = k)$, for integers $j, k \geqslant 1$. Assume $\mathbb{P}(p_1 = 0) < 1$. Then for $k \geqslant 1$,

$$\gamma_k = \mathbb{P}_k(Z_1 = k) = \mathbb{E} p_1^k > 0. \tag{2.12}$$

Define r_k as the unique solution on $(0, \infty)$ of the equation

$$\gamma_k = \mathbb{E} m_0^{-r_k}. \tag{2.13}$$

Notice that if $\mathbb{P}(0 < p_1 < 1) > 0$, then γ_k is strictly decreasing, and r_k is strictly increasing. As usual, we write $a_n \uparrow a$ or $a = \lim_{n\to\infty} \uparrow a_n$ to mean that (a_n) is increasing and its limit is a.

Theorem 2.4. *Assume $\mathbb{P}(p_1 = 0) < 1$, and let $k \geqslant 1$ be a fixed integer.*

a) For any $j \geqslant k$, we have, as $n \to \infty$,

$$q_{k,j} := \lim_{n\to\infty} \uparrow \frac{\mathbb{P}_k(Z_n = j)}{\gamma_k^n} \in [0, \infty]. \tag{2.14}$$

Moreover, $q_{k,k} = 1$, and for each $j > k$, $q_{k,j} > 0$ if and only if the state j is accessible in the sense that $\mathbb{P}_k(Z_n = j) > 0$ for some $n \geqslant 0$.

If additionally $\mathbb{P}(0 < p_1 < 1) > 0$, then $q_{k,j} < \infty$, whose values are uniquely determined by the recurrence relation

$$\gamma_k q_{k,j} = \sum_{i=k}^{j} q_{k,i}\, p(i,j) \qquad \text{with } p(i,j) = \mathbb{P}(Z_1 = j | Z_0 = i), \quad j > k, \tag{2.15}$$

with the initial condition $q_{k,k} = 1$; the recurrence relation reads also

$$q_{k,j} = \frac{1}{\gamma_k - \gamma_j} \sum_{i=k}^{j-1} q_{k,i}\, p(i,j), \quad j > k. \tag{2.16}$$

b) Assume (2.10). Then

$$\sum_{j=k}^{\infty} q_{k,j} = \infty \quad \text{and} \quad \sum_{j=k}^{\infty} j^{-r} q_{k,j} < \infty \quad \forall r > r_k. \tag{2.17}$$

In particular the radius of convergence of the power series

$$Q_k(t) = \sum_{j=k}^{+\infty} q_{k,j} t^j \tag{2.18}$$

is equal to 1.

c) For all $t \in [0,1)$, we have,

$$\frac{\mathbb{E}_k t^{Z_n}}{\gamma_k^n} \uparrow Q_k(t) \quad as \quad n \to \infty, \tag{2.19}$$

d) The power series $Q_k(t)$ satisfies the functional equation

$$\gamma_k Q_k(t) = \mathbb{E}\left[Q_k(f_0(t))\right] \tag{2.20}$$

for $t \in [0,1)$, with the initial condition $Q_k^{(k)}(0) = k!$. Moreover, under the condition $\mathbb{P}(0 < p_1 < 1) > 0$, the functional equation (2.20) together with the initial condition $Q_k^{(k)}(0) = k!$ characterizes the power series $Q_k(t)$ in the following sense: if $\hat{Q}_k(t) = \sum_{j=0}^{\infty} \hat{q}_{k,j} t^j$ is a power series with $\hat{q}_{k,j} \in \mathbb{C}$ and $\hat{q}_{k,k} = 1$, which converges and satisfies

$$\gamma_k \hat{Q}_k(t) = \mathbb{E}\left[\hat{Q}_k(f_0(t))\right]$$

for $t > 0$ small enough, then $\hat{Q}_k$ coincide with Q_k.

Part a) sheds light on the bound $\mathbb{P}(Z_n \leqslant j) \leqslant n^j \gamma^n$ obtained in [7] (Lemma 7) for a BPRE with $\mathbb{P}(Z_1 = 0) = 0$. Furthermore, Theorem 2.4 extends the results of [6] for the Galton-Watson process, with some significant differences. Indeed, when the environment is random and non-degenerate, we have, for $k \geqslant 2$, $G_{k,1}(t) = \mathbb{E} f_0^k(t) \neq G_1^k(t)$ in general, which implies that $Q_k(t) \neq Q^k(t)$, whereas we have the relation $Q_k(t) = Q^k(t)$ for the Galton-Watson process.

Theorem 2.4 also improves the results of [10] (Theorem 2.1), where it has been proved that for a general supercritical BPRE

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_k(Z_n = j) = -\rho < 0. \tag{2.21}$$

In the case where $\mathbb{P}(Z_1 = 0) = 0$, our result is sharper by giving an equivalent of $\mathbb{P}_k(Z_n = j)$. Moreover, also in the case where $\mathbb{P}(Z_1 = 0) = 0$, it has been stated mistakenly in [10] that $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_k(Z_n = j) = k \log \gamma$, whereas the correct asymptotic is

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_k(Z_n = j) = \log \gamma_k.$$

Now we discuss the particular fractional linear case. The reproduction law of a BPRE is said to be fractional linear if

$$p_0(\xi_0) = a_0, \quad p_k(\xi_0) = \frac{(1-a_0)(1-b_0)}{b_0} b_0^k \quad \text{for} \quad k \geqslant 1, \tag{2.22}$$

that is, when the generating function of the offspring distribution $\{p_k(\xi_0) : k \in \mathbb{N}\}$ is

$$f_0(t) = a_0 + \frac{(1-a_0)(1-b_0)t}{1-b_0 t},$$

where $a_0 \in [0,1)$, $b_0 \in (0,1)$, with $a_0 + b_0 \leqslant 1$, are random variables depending on the environment ξ_0. In this case, the mean of the offspring distribution is given by

$$m_0 = \frac{1-a_0}{1-b_0}.$$

The constant ρ in (2.21) was computed in [10]: with $X = \log m_0$,

$$\rho = \begin{cases} -\log \mathbb{E}[e^{-X}] & \text{if} \quad \mathbb{E}[Xe^{-X}] \geqslant 0 \quad \text{(intermediately and strongly supercritical case)}, \\ -\log \inf_{\lambda \geqslant 0} \mathbb{E}[e^{-X}] & \text{if} \quad \mathbb{E}[Xe^{-X}] < 0 \quad \text{(weakly supercritical case)}. \end{cases}$$

Moreover, precise asymptotic results for the strongly and intermediately supercritical case can be found in [11], where the following assertions are proved:

(1) if $\mathbb{E}[Xe^{-X}] > 0$ (strongly supercritical case),

$$\mathbb{P}(Z_n = 1) \sim \nu \left(\mathbb{E}[e^{-X}]\right)^n;$$

(2) if $\mathbb{E}[Xe^{-X}] = 0$ (intermediately supercritical case),

$$\mathbb{P}(Z_n = 1) \sim \theta \left(\mathbb{E}[e^{-X}]\right)^n l(n) n^{-(1-s)},$$

with θ, ν, s positive constants and $l(\cdot)$ a slowly varying function. In the particular case where $a_0 = 0$, Theorem 2.4 recovers Theorem 2.1.1 of [11] with $p_1(\xi_0) = 1/m_0$, $X = \log m_0 > 0$ and $\mathbb{E}\left[Xe^{-X}\right] > 0$. Therefore the process is strongly supercritical and $\mathbb{P}(Z_n = 1) \sim \nu \left(\mathbb{E}[e^{-X}]\right)^n = \gamma^n$. However, since we assume $\mathbb{P}(Z_1 = 0) = 0$, our result does not highlight the previous two asymptotic regimes stated in the particular case when the distribution is fractional linear. The study of the general case is a challenging problem which still remains open.

3. Harmonic moments of W

In this section we prove Theorem 2.1. Denote the quenched Laplace transform of W under the environment ξ by

$$\phi_\xi(t) = \mathbb{E}_\xi \left[e^{-tW} | Z_0 = 1\right] \tag{3.1}$$

when the process starts with one initial particle, and the annealed Laplace transform of W by

$$\phi_k(t) = \mathbb{E}\left[\phi_\xi^k(t)\right] = \mathbb{E}_k \left[e^{-tW}\right], \tag{3.2}$$

when the process starts with k individuals.

The following lemma is the key technical tool to study the exact decay rate of the Laplace transform of the limit variable W.

Lemma 3.1 ([17], Lemma 4.1)**.** *Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a bounded function and let Y be a positive random variable such that for some constants $q \in (0,1)$, $a \in (0,\infty)$, $C > 0$ and $t_0 \geqslant 0$ and all $t > t_0$,*

$$\phi(t) \leqslant q\mathbb{E}\phi(Yt) + Ct^{-a}.$$

If $q\mathbb{E}\left(Y^{-a}\right) < 1$, then $\phi(t) = O(t^{-a})$ as $t \to \infty$.

We will use the following bound for the quenched p-th moments of W.

Lemma 3.2. *For $p \in (1,2]$,*

$$\mathbb{E}_\xi W^p \leqslant 1 + \sum_{k=0}^{\infty} \frac{1}{\Pi_k^{p-1}} \frac{m_k^{(p)}}{m_k^p}. \tag{3.3}$$

Proof. We shall use the following elementary inequality due to Assmussen and Hering [3, p.41]: if $S = \sum_{i=1}^n X_i$ is the sum of N independent random variables $X_i \geqslant 0$, and $h : [0,\infty) \to [0,\infty)$ is concave and non-decreasing, then

$$\mathbb{E}[Sh(S)] \leqslant \sum_{i=1}^{n} \mathbb{E}[X_i h(X_i)] + (\mathbb{E}S)h(\mathbb{E}S). \tag{3.4}$$

From the definition of the branching process, we have the decomposition

$$W_{n+1} := \frac{Z_{n+1}}{\Pi_{n+1}} = \frac{1}{\Pi_n} \sum_{i=1}^{Z_n} \frac{N_{n,i}}{m_n}, \quad n \geqslant 0.$$

Applying inequality (3.4) with this sum and the conditional expectation $\mathbb{E}_\xi(\cdot|\ \mathcal{F}_n)$, together with $h(t) = t^{p-1}$, and using the fact that $\mathbb{E}_\xi(W_{n+1}|\mathcal{F}_n) = W_n$, we get

$$\mathbb{E}_\xi[W_{n+1}^p|\mathcal{F}_n] \leqslant \frac{Z_n}{\Pi_n^p} \frac{m_n^{(p)}}{m_n^p} + W_n^p.$$

Taking expectation with respect to $\mathbb{P}_\xi$, we obtain for $n \geqslant 0$,

$$\mathbb{E}_\xi[W_{n+1}^p] \leqslant \frac{1}{\Pi_n^{p-1}} \frac{m_n^{(p)}}{m_n^p} + \mathbb{E}_\xi W_n^p.$$

By induction, this gives for $n \geqslant 0$,

$$\mathbb{E}_\xi W_{n+1}^p \leqslant 1 + \sum_{k=0}^{n} \frac{1}{\Pi_k^{p-1}} \frac{m_k^{(p)}}{m_k^p}, \tag{3.5}$$

which implies (3.3). □

We remark that (3.3) remains valid if $\frac{m_n^{(p)}}{m_n^p}$ is replaced by the p-th centered moments $\sum_{k=0}^{\infty} |\frac{k}{m_n} - 1|^p p_k(\xi_n)$, multiplied by a constant B_p depending on p. This can be seen from inequality (2.1) of [15], and can be checked using the Bukholder, Davis and Gundy inequality for martingales.

Proof of Theorem 2.1. We can suppose that the moment condition $\mathbb{E}(\frac{m_0^{(p)}}{m_0^p})^\varepsilon < \infty$ holds for some $p \in (1,2]$, since the normalized conditional L^p-norm $m_0(p)^{1/p}/m_0$ is increasing in p. In the following, we assume $p \in (1,2]$. We proceed in six steps.

Step 1: we first establish a bound of $\phi_\xi(t)$ uniformly in ξ for which $\mathbb{E}_\xi W^p$ is bounded. Notice that the function $t \mapsto (e^{-t} - 1 + t)/t^p$ is bounded on $(0,\infty)$, so that there is a constant $C > 0$ such that for all $t \geqslant 0$,

$$e^{-t} \leqslant 1 - t + Ct^p/p. \tag{3.6}$$

Using this inequality with t replaced by tW and taking expectation with respect to $\mathbb{P}_\xi$, together with the fact that $\mathbb{E}_\xi W = 1$, we obtain, for all $t > 0$,

$$\phi_\xi(t) \leqslant 1 - t + C(\mathbb{E}_\xi W^p)t^p/p. \tag{3.7}$$

Therefore, if $\mathbb{E}_\xi W^p \leqslant K$ for some constant $K > 0$, then for all $t > 0$,

$$\phi_\xi(t) \leqslant 1 - t + CKt^p/p =: g(t).$$

At $t_K := (CK)^{-1/(p-1)}$, the function g attains its minimum

$$\beta_K := \min_{t>0} g(t) = g(t_K) = 1 - (1 - 1/p)t_K < 1.$$

Therefore

$$\phi_\xi(t) \leqslant \phi_\xi(t_K) \leqslant g(t_K) = \beta_K \quad \text{ if } t \geqslant t_K \text{ and } \mathbb{E}_\xi W^p \leqslant K. \tag{3.8}$$

Step 2: we next obtain a recurrence relation on $\phi_\xi(t)$. It is well-known that $\phi_\xi(t)$ satisfies the functional relation

$$\phi_\xi(t) = f_0\left(\phi_{T\xi}\left(\frac{t}{m_0}\right)\right), \tag{3.9}$$

where $f_0(t) = \sum_{k=1}^\infty p_k(\xi_0)t^k$ is the generating function of the offspring distribution $\{p_k(\xi_0) : k \in \mathbb{N}\}$, T is the shift operator of the environment sequence defined by $T(\xi_0, \xi_1, \ldots) = (\xi_1, \xi_2, \ldots)$. Using (3.9) and the fact that $\phi^k_{T\xi}\left(\frac{t}{m_0}\right) \leqslant \phi^2_{T\xi}\left(\frac{t}{m_0}\right)$ for all $k \geqslant 2$, we obtain

$$\phi_\xi(t) \leqslant p_1(\xi_0)\phi_{T\xi}\left(\frac{t}{m_0}\right) + (1 - p_1(\xi_0))\phi^2_{T\xi}\left(\frac{t}{m_0}\right). \tag{3.10}$$

In particular, $\phi_\xi(t) \leqslant \phi_{T\xi}\left(\frac{t}{m_0}\right)$. By iteration, this implies, for any $t > 0$,

$$\phi_\xi(t) \leqslant \phi_{T\xi}\left(\frac{t}{m_0}\right) \leqslant \phi_{T^2\xi}\left(\frac{t}{m_0 m_1}\right) \leqslant \cdots \leqslant \phi_{T^n\xi}\left(\frac{t}{\Pi_n}\right), \quad n \geqslant 1, \tag{3.11}$$

where T^n denotes the n-fold iteration of T. Taking the k-th power in (3.10), using the binomial expansion and the fact that $\phi^{2k-i}_{T\xi}\left(\frac{t}{m_0}\right) \leqslant \phi^{k+1}_{T\xi}\left(\frac{t}{m_0}\right)$ for all $i \in \{0, \ldots, k-1\}$, we get

$$\begin{aligned}
\phi^k_\xi(t) &= p^k_1(\xi_0)\phi^k_{T\xi}\left(\frac{t}{m_0}\right) + \sum_{i=0}^{k-1}\binom{k}{i} p_1(\xi_0)^i(1 - p_1(\xi_0))^{k-i}\phi^{2(k-i)+i}_{T\xi}\left(\frac{t}{m_0}\right) \\
&\leqslant p^k_1(\xi_0)\phi^k_{T\xi}\left(\frac{t}{m_0}\right) + (1 - p^k_1(\xi_0))\phi^{k+1}_{T\xi}\left(\frac{t}{m_0}\right) \\
&= \phi^k_{T\xi}\left(\frac{t}{m_0}\right)\left[p^k_1(\xi_0) + (1 - p^k_1(\xi_0))\phi_{T\xi}\left(\frac{t}{m_0}\right)\right].
\end{aligned} \tag{3.12}$$

By iteration, this together with (3.11) leads to

$$\phi^k_\xi(t) \leqslant \phi^k_{T^n\xi}\left(\frac{t}{\Pi_n}\right)\prod_{j=0}^{n-1}\left(p^k_1(\xi_j) + (1 - p^k_1(\xi_j))\phi_{T^n\xi}\left(\frac{t}{\Pi_n}\right)\right), \quad k, n \geqslant 1. \tag{3.13}$$

Step 3: we establish a recurrence relation on $\phi_k(t) = \mathbb{E}\left[\phi_\xi^k(t)\right] = \mathbb{E}_k\left[e^{-tW}\right]$. From (3.13) and (3.8) applied to $\phi_{T_n\xi}(t/\Pi_n)$, we obtain, for any $t > 0$ and $n \geqslant 1$,

$$\phi_\xi^k(t) \leqslant \phi_{T^n\xi}^k\left(\frac{t}{\Pi_n}\right) \prod_{j=0}^{n-1}\left(p_1^k(\xi_j) + (1-p_1^k(\xi_j))\beta_K\right) \tag{3.14}$$
$$+ 1_{\{\frac{t}{\Pi_n} < t_K\}} + \phi_{T^n\xi}^k\left(\frac{t}{\Pi_n}\right) 1_{\{\mathbb{E}_{T^n\xi}W^p > K\}},$$

where $1_{\{\cdot\}}$ denotes the indicator function of the set $\{\cdot\}$. Taking expectation and using the independence between $T_n\xi$ and $(\xi_0, \cdots, \xi_{n-1})$ and the fact that $1_{\{\mathbb{E}_{T^n\xi}W^p > K\}} \leqslant \mathbb{E}_{T^n\xi}W^p/K$, we obtain, for any $\eta > 0, t > 0$ and $n \geqslant 1$,

$$\phi_k(t) \leqslant \mathbb{E}\Big[\phi_k\left(\frac{t}{\Pi_n}\right) \prod_{j=0}^{n-1}\left(p_1^k(\xi_j) + (1-p_1^k(\xi_j))\beta_K\right)\Big]$$
$$+ \frac{1}{K^\eta}\mathbb{E}\Big[\phi_{T^n\xi}^k\left(\frac{t}{\Pi_n}\right)(\mathbb{E}_{T^n\xi}W^p)^\eta\Big] + \mathbb{P}(\frac{t}{\Pi_n} < t_K). \tag{3.15}$$

We will now find a suitable bound of $\mathbb{E}\Big[\phi_{T^n\xi}^k\left(\frac{t}{\Pi_n}\right)(\mathbb{E}_{T^n\xi}W^p)^\eta\Big]$. To this end the difficulty is that there is a dependence between $\phi_{T^n\xi}^k$ and $(\mathbb{E}_{T^n\xi}W^p)$. To overcome this difficulty we will use an explicit bound of $(\mathbb{E}_{T^n\xi}W^p)$ and the recurrence relation (3.11). Let $\eta \in (0,1]$ be small enough such that $\mathbb{E}\big(\frac{m_0^{(p)}}{m_0^p}\big)^\eta < \infty$. By Lemma 3.2 and the sub-additivity of the function $x \mapsto x^\eta$, we have

$$(\mathbb{E}_{T^n\xi}W^p)^\eta \leqslant 1 + \sum_{k=0}^{\infty}\frac{1}{\Pi_k^{(p-1)\eta}(T_n\xi)}\Big(\frac{m_k^{(p)}(T_n\xi)}{m_k^p(T_n\xi)}\Big)^\eta. \tag{3.16}$$

From (3.11), we have $\phi_{T^n\xi}(x) \leqslant \phi_{T^{k+1}T^n\xi}\left(\frac{x}{\Pi_{k+1}(T^n\xi)}\right)$. Therefore from the preceding bound of $(\mathbb{E}_{T^n\xi}W^p)^\eta$ we obtain

$$\phi_{T^n\xi}^k\left(\frac{t}{\Pi_n}\right)(\mathbb{E}_{T^n\xi}W^p)^\eta \leqslant \phi_{T^n\xi}^k\left(\frac{t}{\Pi_n}\right) + \sum_{k=0}^{\infty}\phi_{T^n\xi}^k\left(\frac{t}{\Pi_n}\right)\frac{1}{\Pi_k^{(p-1)\eta}(T_n\xi)}\Big(\frac{m_k^{(p)}(T_n\xi)}{m_k^p(T_n\xi)}\Big)^\eta$$
$$\leqslant \phi_{T^n\xi}^k\left(\frac{t}{\Pi_n}\right) + \sum_{k=0}^{\infty}\phi_{T^{k+1+n}\xi}^k\left(\frac{t}{\Pi_{k+1}(T^n\xi)\Pi_n(\xi)}\right)\frac{1}{\Pi_k^{(p-1)\eta}(T_n\xi)}\Big(\frac{m_k^{(p)}(T_n\xi)}{m_k^p(T_n\xi)}\Big)^\eta. \tag{3.17}$$

Since $T^{k+1+n}\xi$ is independent of $(\xi_i)_{i\leqslant k+n}$ (this is why we passed from $T^n\xi$ to $T^{k+1+n}\xi$), taking expectation in the above inequality leads to

$$\mathbb{E}\Big[\phi_{T^n\xi}^k\left(\frac{t}{\Pi_n}\right)(\mathbb{E}_{T^n\xi}W^p)^\eta\Big] \leqslant \mathbb{E}\phi_k\left(\frac{t}{\Pi_n}\right) + \sum_{k=0}^{\infty}\mathbb{E}\Big[\frac{\phi_k\left(\frac{t}{\Pi_{k+1}(T^n\xi)\Pi_n(\xi)}\right)}{\Pi_k^{(p-1)\eta}(T_n\xi)}\Big(\frac{m_k^{(p)}(T_n\xi)}{m_k^p(T_n\xi)}\Big)^\eta\Big]. \tag{3.18}$$

Therefore, coming back to (3.15) we obtain the following inequality on ϕ_k:

$$\phi_k(t) \leqslant \mathbb{E}\Big[\phi_k\left(\frac{t}{\Pi_n}\right)\prod_{j=0}^{n-1}\left(p_1^k(\xi_j)+(1-p_1^k(\xi_j))\beta_K\right)\Big] + \mathbb{P}(\frac{t}{\Pi_n}<t_K) + \frac{1}{K^\eta}\Big\{\mathbb{E}\phi_k\left(\frac{t}{\Pi_n}\right)+\sum_{k=0}^{\infty}\mathbb{E}\Big[\frac{\phi_k\left(\frac{t}{\Pi_{k+1}(T^n\xi)\Pi_n(\xi)}\right)}{\Pi_k^{(p-1)\eta}(T_n\xi)}\left(\frac{m_k^{(p)}(T_n\xi)}{m_k^p(T_n\xi)}\right)^\eta\Big]\Big\}. \tag{3.19}$$

Step 4: we prove that if $\mathbb{E}m_0^\varepsilon<\infty$ and $\mathbb{E}\left(\frac{m_0^{(p)}}{m_0^p}\right)^\varepsilon<\infty$ for some $\varepsilon>0$, then

$$\phi_k(t)\leqslant Ct^{-a} \tag{3.20}$$

for some constants $C,a>0$ and all $t>0$, which implies that $\mathbb{E}_kW^{-b}<\infty$ for all $b\in(0,a)$ (the implication is a standard result, and can be easily checked e.g. by using the formula $\mathbb{E}_kW^{-b}=\frac{1}{\Gamma(b)}\int_0^{+\infty}\phi_k(t)t^{b-1}dt$, where $\Gamma(b)=\int_0^\infty e^{-t}t^{b-1}dt$).

As in Step 3, let $\eta\in(0,1]$ be small enough such that $\mathbb{E}\left(\frac{m_0^{(p)}}{m_0^p}\right)^\eta<\infty$. Let Y be a positive random variable whose distribution is determined by

$$\mathbb{E}\left[g(Y)\right] = \frac{1}{q}\Big\{\mathbb{E}\Big[g\left(\frac{1}{\Pi_n}\right)\prod_{j=0}^{n-1}\left(p_1^k(\xi_j)+(1-p_1^k(\xi_j))\beta_K\right)\Big] + \frac{1}{K^\eta}\mathbb{E}g\left(\frac{1}{\Pi_n}\right)+\frac{1}{K^\eta}\sum_{k=0}^{\infty}\mathbb{E}\Big[\frac{g\left(\frac{1}{\Pi_{k+1}(T^n\xi)\Pi_n(\xi)}\right)}{\Pi_k^{(p-1)\eta}(T_n\xi)}\left(\frac{m_k^{(p)}(T_n\xi)}{m_k^p(T_n\xi)}\right)^\eta\Big]\Big\} \tag{3.21}$$

for all bounded and measurable function g, where q is the norming constant (to make $\mathbb{E}\left[g(Y)\right]=1$ when $g=1$) defined by

$$q = \mathbb{E}\Big[\prod_{j=0}^{n-1}\left(p_1^k(\xi_j)+(1-p_1^k(\xi_j))\beta_K\right)\Big]+\frac{1}{K^\eta}\Big\{1+\sum_{k=0}^{\infty}\mathbb{E}\Big[\frac{1}{\Pi_k^{(p-1)\eta}(T_n\xi)}\left(\frac{m_k^{(p)}(T_n\xi)}{m_k^p(T_n\xi)}\right)^\eta\Big]\Big\}$$
$$= \Big[\mathbb{E}\Big(\left(p_1^k(\xi_0)+(1-p_1^k(\xi_0))\beta_K\right)\Big)\Big]^n+\frac{1}{K^\eta}\Big\{1+\frac{1}{1-\mathbb{E}m_0^{-(p-1)\eta}}\cdot\mathbb{E}\Big(\frac{(m_0^{(p)})^\eta}{m_0^{p\eta}}\Big)\Big\}.$$

Then from (3.15) and Markov's inequality we obtain for all $a>0$ and $t>0$,

$$\phi_k(t)\leqslant q\mathbb{E}\phi_k\left(Yt\right)+\frac{t_K^a(\mathbb{E}m_0^a)^n}{t^a}. \tag{3.22}$$

We choose $a>0$ and $\eta\in(0,1)$ small enough such that $\mathbb{E}m_0^{2a}<\infty$ and $\mathbb{E}\Big(\frac{(m_0^{(p)})}{m_0^p}\Big)^{2\eta}<\infty$, so that $\mathbb{E}\Big(\frac{m_0^a(m_0^{(p)})^\eta}{m_0^{p\eta}}\Big)<\infty$ by Cauchy-Schwartz's inequality. Notice that

$$\begin{aligned}
q\mathbb{E}Y^{-a} &= \mathbb{E}\Big[\Pi_n^a\prod_{j=0}^{n-1}\Big(p_1^k(\xi_j)+(1-p_1^k(\xi_j))\beta_K\Big)\Big]\\
&\quad+\frac{1}{K^\eta}\mathbb{E}\Pi_n^a+\frac{1}{K^\eta}\sum_{k=0}^{\infty}\mathbb{E}\Big[\frac{\Pi_{k+1}^a(T^n\xi)\Pi_n^a(\xi)}{\Pi_k^{(p-1)\eta}(T_n\xi)}\Big(\frac{m_k^{(p)}(T_n\xi)}{m_k^p(T_n\xi)}\Big)^\eta\Big]\\
&=\Big[\mathbb{E}\Big(m_0^a\Big(p_1^k(\xi_0)+(1-p_1^k(\xi_0))\beta_K\Big)\Big)\Big]^n\\
&\quad+\frac{(\mathbb{E}m_0^a)^n}{K^\eta}\Big\{1+\frac{1}{1-\mathbb{E}m_0^{-(p-1)\eta+a}}\cdot\mathbb{E}\Big(\frac{m_0^a(m_0^{(p)})^\eta}{m_0^{p\eta}}\Big)\Big\}.
\end{aligned}$$

By the dominated convergence theorem,

$$q\mathbb{E}Y^{-a}\xrightarrow{a\downarrow 0}q\xrightarrow{n\to\infty}\frac{1}{K^\eta}\Big\{1+\frac{1}{1-\mathbb{E}m_0^{-(p-1)\eta}}\cdot\mathbb{E}\Big(\frac{(m_0^{(p)})^\eta}{m_0^{p\eta}}\Big)\Big\}\xrightarrow{K\to\infty}0.$$

So we can choose K,n large enough and $a>0$ small enough, such that $q<1$ and $q\mathbb{E}Y^{-a}<1$. Therefore, from (3.22) and Lemma 3.1, we get (3.20).

Step. 5: prove that under the boundedness condition (2.10),

$$\mathbb{E}p_1^km_0^a<1\quad\Rightarrow\quad\phi_k(t)=O(t^{-a})\quad\text{as } t\to\infty. \tag{3.23}$$

We follow the argument in [15] where the case $k=1$ was treated. Notice that when $m_0\geqslant c_1>1$ and $m_0^{(p)}\leqslant c_p$, from Lemma 3.2 we see that there is a constant $K>0$ such that $\mathbb{E}_\xi W^p\leqslant K$ for almost every environment ξ. With such a K, from (3.14) we get for all $t>0$,

$$\phi_\xi(t)\leqslant b_K^n+1_{\{\frac{t}{\Pi_n}<t_K\}},\qquad\text{where } b_K:=\bar p_1+(1-\bar p_1)\beta_K.$$

From (3.12) and this bound applied to $\phi_{T\xi}$, we get

$$\begin{aligned}
\phi_\xi^k(t) &\leqslant \phi_{T\xi}^k\Big(\frac{t}{m_0}\Big)\Big[p_1^k(\xi_0)+(1-p_1^k(\xi_0))(b_K^n+1_{\{\frac{t}{\Pi_{n+1}}\leqslant t_K\}})\Big]\\
&\leqslant \phi_{T\xi}^k\Big(\frac{t}{m_0}\Big)\Big(p_1^k(\xi_0)+b_K^n\Big)+1_{\{\frac{t}{\Pi_{n+1}}<t_K\}}.
\end{aligned}\tag{3.24}$$

Taking expectation, we get for all $n\geqslant 1$ and $t>0$,

$$\begin{aligned}
\phi_k(t) &\leqslant \mathbb{E}\phi_k\Big(\frac{t}{m_0}\Big)\Big[p_1^k(\xi_0)+b_K^n\Big]+\mathbb{P}(\frac{t}{\Pi_{n+1}}<t_K)\\
&\leqslant q_1\mathbb{E}\phi(Y_1t)+\frac{t_K^a(\mathbb{E}m_0^a)^{n+1}}{t^a},
\end{aligned}\tag{3.25}$$

where $q_1 = \mathbb{E}\left[p_1^k(\xi_0) + b_K^n\right]$, Y_1 is a positive random variable whose distribution is determined by

$$\mathbb{E}g(Y_1) = \frac{1}{q_1}\mathbb{E}\Big[g(\frac{1}{m_0})\left(p_1^k(\xi_0) + b_K^n\right)\Big]$$

for any bounded and measurable function g. Notice that $q_1 \to \mathbb{E}\Big[p_1^k(\xi_0)\Big] < 1$ as $n \to \infty$, and

$$q_1\mathbb{E}Y_1^{-a} = \mathbb{E}\Big[m_0^a\left(p_1^k(\xi_0) + b_K^n\right)\Big] \xrightarrow{n\to\infty} \mathbb{E}\Big[m_0^a p_1^k(\xi_0)\Big] < 1.$$

Therefore we can choose n large enough such that $q_1 < 1$ and $q_1\mathbb{E}Y_1^{-a} < 1$. So from (3.25) and Lemma 3.1, we conclude that $\phi_k(t) \leqslant Ct^{-a}$ for some constant $C > 0$ and all $t > 0$. This ends the proof of the implication (3.23).

Notice that when $\mathbb{E}p_1^k m_0^a < 1$, we can take $a_0 > a$ close to a such that $\mathbb{E}p_1^k m_0^{a_0} < 1$. Applying the implication (3.23) with a replaced by a_0, we get $\phi_k(t) \leqslant Ct^{-a_0}$ for some constant $C > 0$ and all $t > 0$, which implies that $\mathbb{E}_k W^{-a} < \infty$. So we have proved the implication

$$\mathbb{E}p_1^k m_0^a < 1 \quad \Rightarrow \quad \mathbb{E}_k W^{-a} < \infty, \tag{3.26}$$

under the boundedness condition (2.10).

Step 6: prove that for any $a > 0$,

$$\mathbb{E}_k W^{-a} < \infty \Rightarrow \mathbb{E}p_1^k(\xi_0)m_0^a < 1. \tag{3.27}$$

Assume that $\mathbb{E}_k W^{-a} < \infty$. Note that the r.v. W admits the well-known decomposition

$$W = \frac{1}{m_0}\sum_{i=1}^{Z_1} W(i),$$

where the r.v.'s $W(i)$ $(i \geqslant 1)$ are i.i.d. and independent of Z_1 under $\mathbb{P}_\xi$, and are also independent of Z_1 and ξ_0 under $\mathbb{P}$. The decomposition can be easily seen from the definition of the branching process, by considering the sub branching processes starting from the particles i of the first generation, and by taking $W(i)$ as the corresponding limit variable of the fundamental martingale of the branching process starting from i. The conditional probability law of $W(i)$ satisfies $\mathbb{P}_\xi(W(i) \in \cdot) = \mathbb{P}_{T\xi}(W \in \cdot)$. Since $\mathbb{P}_k(Z_1 \geqslant k+1) > 0$, we have

$$\mathbb{E}_k W^{-a} > \mathbb{E}_k m_0^a\Big(\sum_{i=1}^{Z_1} W(i)\Big)^{-a}\mathbb{1}\{Z_1 = k\} = \mathbb{E}p_1^k(\xi_0)m_0^a\,\mathbb{E}_k W^{-a}, \tag{3.28}$$

which implies that $\mathbb{E}p_1^k(\xi_0)m_0^a < 1$.

Combing the results proved in Steps 4-6, we obtain the conclusions of Theorem 2.1 : in fact, Part a) is proved in Step 4, while Part b) is proved in Steps 5 and 6.

4. Small value probability in the non-extinction case

In this section we prove Theorem 2.4.

a) We start with the proof of part a). For $k \geqslant 1$ and $j \geqslant k$, define

$$q_{k,j,n} := \frac{\mathbb{P}\left(Z_n = j | Z_0 = k\right)}{\gamma_k^n}, \quad \text{with} \quad \gamma_k = \mathbb{P}_k(Z_1 = k) = \mathbb{E}p_1^k. \tag{4.1}$$

By the Markov property, we have

$$\mathbb{P}_k\left(Z_{n+1} = j\right) \geqslant \mathbb{P}_k\left(Z_1 = k\right)\mathbb{P}_k\left(Z_n = j\right).$$

Dividing by γ_k^{n+1} leads to

$$q_{k,j,n+1} \geqslant q_{k,j,n}. \tag{4.2}$$

Therefore, the limit of $q_{k,j,n}$ exists as $n \to \infty$:

$$q_{k,j} := \lim_{n\to\infty} \uparrow q_{k,j,n} \in [0, \infty].$$

We shall prove that $q_{k,j}$ satisfies the properties claimed in the theorem.

a1) We first remark that $q_{k,k} = 1$ and that for any $j > k$, $q_{k,j} > 0$ if and only if the state j is accessible. In fact, when $Z_0 = k$, then $Z_n = k$ if and only if $Z_j = k$ for all $j = 0, \cdots, n$, so that

$$\mathbb{P}(Z_n = k | Z_0 = k) = \prod_{j=1}^{n} \mathbb{P}(Z_j = k | Z_{j-1} = k) = \gamma_k^n,$$

which implies $q_{k,k} = 1$. If j is accessible, that is, if there exists $n \geqslant 0$ such that $\mathbb{P}_k(Z_n = j) > 0$, then $q_{k,j} \geqslant q_{k,j,n} = \mathbb{P}_k(Z_n = j)/\gamma_k^n > 0$; if j is not accessible, that is, if $\mathbb{P}_k(Z_n = j) = 0$ for all $n \geqslant 0$, then $q_{k,j,n} = 0$ for all $n \geqslant 0$, so that $q_{k,j} = \lim_{n\to\infty} q_{k,j,n} = 0$.

a2) We next show that if $\mathbb{P}(0 < p_1 < 1) > 0$, then for all $j \geqslant k$, we have

$$H(j): \quad q_{k,j} < \infty.$$

We do the proof by induction. For $j = k$, we have $q_{k,k} = 1$, so that $H(k)$ holds. Assume that $j \geqslant k+1$ and that $H(i)$ is true for all $k \leqslant i \leqslant j-1$. By the total probability formula, we obtain

$$\frac{\mathbb{P}_k\left(Z_{n+1} = j\right)}{\gamma_k^{n+1}} = \frac{1}{\gamma_k}\sum_{i=k}^{j} \mathbb{P}\left(Z_{n+1} = j | Z_n = i\right) \frac{\mathbb{P}_k\left(Z_n = i\right)}{\gamma_k^n},$$

which is equivalent to

$$q_{k,j,n+1} = \frac{1}{\gamma_k}\Big[\sum_{i=k}^{j-1} p(i,j) q_{k,i,n} + \gamma_j q_{k,j,n}\Big], \tag{4.3}$$

with $p(i,j) = \mathbb{P}\left(Z_1 = j | Z_0 = i\right)$. Using the fact that $q_{k,j,n} \leqslant q_{k,j,n+1}$, we get by the induction hypothesis that

$$\sup_{n\in\mathbb{N}} \; q_{k,j,n+1}(\gamma_k - \gamma_j) \;\; \leqslant \;\; \sum_{i=k}^{j-1} q_{k,i}\, p(i,j) < \infty.$$

Since $\mathbb{P}(0 < p_1 < 1) > 0$, we have $\gamma_k - \gamma_j > 0$. Thus $q_{k,j} < \infty$, so that $H(j)$ remains true. By induction $H(j)$ holds for all $j \geqslant k$.

a3) We then prove the recurrent relation for $q_{k,j}$. Taking the limit as $n \to \infty$ in (4.3), leads to

$$q_{k,j} = \frac{1}{\gamma_k}\Big[\sum_{i=k}^{j-1} q_{k,i}\, p(i,j) + \gamma_j q_{k,j}\Big]. \tag{4.4}$$

This gives the recurrent relation (2.15). Notice that this relation holds even if $\mathbb{P}(0 < p_1 < 1) = 0$, although in this case we do not know whether $q_{k,j} < \infty$.

When $\mathbb{P}(0 < p_1 < 1) > 0$, then $\gamma_j < \gamma_k$ for $j > k$, so that the recurrence relation reads also

$$q_{k,j} = \frac{1}{\gamma_k - \gamma_j}\sum_{i=k}^{j-1} q_{k,i}\, p(i,j), \qquad j > k. \tag{4.5}$$

We have therefore finished the proof of part a) of Theorem 2.4.

b) Now we prove part b) of Theorem 2.4 about the moments and radius of convergence of the power series $Q_k(t)$, assuming $\mathbb{P}(p_1 = 0) < 1$ and (2.10). Notice that these conditions imply that $\mathbb{P}(0 < p_1 < 1) > 0$.

The method that we developpe here is new even in the case of the Galton-Watson process.

We start with a lemma about the harmonic moments of Z_n.

Lemma 4.1. *Assume $\mathbb{P}(p_1 = 0) < 1$ and (2.10). Let $k \geqslant 1$ and let r_k be the unique solution on $(0,\infty)$ of the equation $\gamma_k = \mathbb{E}m_0^{-r_k}$. Then, for any $r > r_k$, we have*

$$\lim_{n\to\infty} \uparrow \frac{\mathbb{E}_k Z_n^{-r}}{\gamma_k^n} < \infty. \tag{4.6}$$

Proof. By the Markov property,

$$\mathbb{E}_k\left[Z_{n+1}^{-r}\right] \geqslant \mathbb{E}_k\left[Z_{n+1}^{-r}|Z_1 = k\right]\mathbb{P}_k(Z_1 = k) = \gamma_k \mathbb{E}_k\left[Z_n^{-r}\right],$$

which proves that the sequence $(\mathbb{E}_k\left[Z_n^{-r}\right]/\gamma_k^n)_{n\in\mathbb{N}}$ is increasing. We show that it is bounded. For $n \geqslant 1$ and $m \geqslant 0$, we have the following well-known branching property for Z_n:

$$Z_{n+m} = \sum_{i=1}^{Z_m} Z_{n,i}^{(m)}, \tag{4.7}$$

where, under $\mathbb{P}_\xi$, the random variables $Z_{n,i}^{(m)}$ $(i \geqslant 1)$ are i.i.d., independent of Z_m, whose conditional probability law satisfies $\mathbb{P}_\xi\left(Z_{n,i}^{(m)} \in \cdot\right) = \mathbb{P}_{T^m\xi}\left(Z_n \in \cdot\right)$. Intuitively, relation (4.7) shows that, conditionally on $Z_m = i$, the annealed law of the process Z_{n+m} is the same as that of a new process Z_n starting with i individuals.

Using (4.7) with $m = 1$, the independence between Z_1 and $Z_{n,i}^{(1)}$ $(i \geqslant 1)$ and the fact that $\mathbb{E}_i Z_n^{-r} \leqslant \mathbb{E}_{k+1} Z_n^{-r}$ for all $i \geqslant k+1$, we have

$$\begin{aligned}\mathbb{E}_k\left[Z_{n+1}^{-r}\right] &= \mathbb{E}_k\left[Z_{n+1}^{-r}|Z_1=k\right]\mathbb{P}_k(Z_1=k)\\ &\quad+\sum_{i=k+1}^{\infty}\mathbb{E}\left[\left(\sum_{h=1}^{i}Z_{n,h}^{(1)}\right)^{-r}\middle|Z_1=i\right]\mathbb{P}_k(Z_1=i)\\ &\leqslant \gamma_k\mathbb{E}_k\left[Z_n^{-r}\right]+\mathbb{E}_{k+1}\left[Z_n^{-r}\right]. \end{aligned}\tag{4.8}$$

Therefore, by iteration,

$$\begin{aligned}\frac{\mathbb{E}_k\left[Z_{n+1}^{-r}\right]}{\gamma_k^{n+1}} &\leqslant \frac{\mathbb{E}_k\left[Z_n^{-r}\right]}{\gamma_k^{n}}+\frac{\mathbb{E}_{k+1}\left[Z_n^{-r}\right]}{\gamma_k^{n+1}}\leqslant\cdots\\ &\leqslant 1+\sum_{j=0}^{n}\frac{\mathbb{E}_{k+1}\left[Z_j^{-r}\right]}{\gamma_k^{j+1}}\end{aligned}\tag{4.9}$$

We shall use a change of measure, by modifying the distribution of the environment sequence. Let τ_0 be the common law of ξ_n. Recall that the annealed law is $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$, where $\tau = \tau_0^{\otimes\mathbb{N}}$ is the product measure. For $r > 0$, let τ be a new probability mesure on the environment space defined by

$$\tau_r(dt) = \frac{m(t)^{-r}}{c_r}\tau_0(dt) \quad \text{with} \quad c_r = \mathbb{E}m_0^{-r},$$

where $m(t) = \sum_{i\geqslant 0} ip_i(t)$ is the conditional mean of the offspring distribution $\{p_k(t)\}$ at 0 given the environment $\xi_0 = t$. Define the new annealed measure

$$\mathbb{P}^{(r)}(dx, d\xi) = \mathbb{P}_\xi(dx)\tau_r(d\xi).$$

Denote the corresponding expectation by $\mathbb{E}^{(r)}$. For $k \geqslant 1$, set

$$\mathbb{P}_k^{(r)}(\cdot) = \mathbb{P}^{(r)}(\cdot|Z_0 = k) \quad \text{and} \quad \mathbb{E}_k^{(r)}(\cdot) = \mathbb{E}^{(r)}(\cdot|Z_0 = k).$$

Then

$$\mathbb{E}_k^{(r)}[X] = \frac{\mathbb{E}_k\left[\Pi_n^{-r}X\right]}{c_r^n}\tag{4.10}$$

for any $\mathcal{F}_n$-measurable random variable X. It can be easily seen that under the new probability measure $\mathbb{P}^{(r)}$, (Z_n) is still a superritical branching process in a random environment. By (4.10), we obtain

$$\mathbb{E}_{k+1}\left[Z_n^{-r}\right] = \mathbb{E}_{k+1}^{(r)}[W_n^{-r}]c_r^n.\tag{4.11}$$

It is known (see [15], Lemma 2.1) that

$$\sup_{n\in\mathbb{N}}\mathbb{E}_{k+1}^{(r)}[W_n^{-r}] = \mathbb{E}_{k+1}^{(r)}[W^{-r}].\tag{4.12}$$

Notice that

$$\mathbb{E}^{(r)}[p_1^{k+1}(\xi_0)m_0^r] = \frac{\gamma_{k+1}}{\mathbb{E}m_0^{-r}} = \frac{\mathbb{E}m_0^{-r_{k+1}}}{\mathbb{E}m_0^{-r}} < 1 \quad \text{for any } r < r_{k+1}.$$

So by Theorem 2.1 we get for any $r < r_{k+1}$,

$$C(r) := \mathbb{E}_{k+1}^{(r)}[W^{-r}] < \infty. \tag{4.13}$$

Coming back to (4.9) and using (4.11) - (4.13) , we get for $r \in (r_k, r_{k+1})$,

$$\frac{\mathbb{E}_k\left[Z_{n+1}^{-r}\right]}{\gamma_k^{n+1}} \leqslant 1 + \frac{C(r)}{\gamma_k}\sum_{j=0}^{n}\left(\frac{c_r}{\gamma_k}\right)^j < \frac{C(r)}{\gamma_k - c_r}, \tag{4.14}$$

using the fact that $\gamma_k = \mathbb{E}m_0^{-r_k} > \mathbb{E}m_0^{-r} = c_r$. Thus the sequence $(\mathbb{E}_k\left[Z_n^{-r}\right]/\gamma_k^n)_{n\in\mathbb{N}}$ is bounded and (4.6) holds for any $r \in (r_k, r_{k+1})$. Using the fact that $\mathbb{E}_k\left[Z_{n+1}^{-r'}\right] \leqslant \mathbb{E}_k\left[Z_{n+1}^{-r}\right]$ for any $r' > r$, the result follows for any $r > r_k$, which ends the proof of the lemma. □

Remark 4.2. From the results stated above, with some additional analysis one can obtain an equivalent of the harmonic moments $\mathbb{E}Z_n^{-r}$ for any $r > 0$. However, it is delicate to have an expression of the concerned constant in the equivalence. See [14] on this topic.

We can now show the moments results (2.17) about $\{q_{k,j}\}$, which imply that the radius of convergence R of the power series $Q_k(t) = \sum_{j=k}^{\infty} q_{k,j}t^j$ is equal to 1. Since $\sum_{j=k}^{\infty}\mathbb{P}_k\left(Z_n = j\right) = 1$, we have

$$\sum_{j=k}^{\infty}\gamma_k^{-n}\mathbb{P}_k\left(Z_n = j\right) = \gamma_k^{-n}.$$

Passing to the limit as $n \to \infty$ and using part a) of Theorem 2.4 and the monotone convergence theorem, we obtain:

$$\sum_{j=k}^{\infty} q_{k,j} = +\infty,$$

which proves that $R \leqslant 1$. We prove that $R \geqslant 1$ by showing that

$$\sum_{j=k}^{+\infty} j^{-r}q_{k,j} < \infty$$

for $r > r_k$. Using again part a) of Theorem 2.4, the monotone convergence theorem and Lemma 4.1, we have, for any $r > r_k$,

$$\sum_{j=k}^{+\infty} j^{-r}q_{k,j} = \sum_{j=k}^{+\infty} j^{-r}\lim_{n\to\infty}\uparrow\frac{\mathbb{P}_k(Z_n = j)}{\gamma_k^n} = \lim_{n\to\infty}\uparrow\frac{\mathbb{E}_k Z_n^{-r}}{\gamma_k^n} < \infty. \tag{4.15}$$

This ends the proof of part b).

c) We now prove part c) of Theorem 2.4. Since

$$\frac{\mathbb{E}_k t^{Z_n}}{\gamma_k^n} = \sum_{j=k}\frac{\mathbb{P}_k(Z_n = j)}{\gamma_k^n}t^j,$$

using part a) and the monotone convergence theorem, we get (2.19).

d) We finally prove that the power series $Q_k(t)$ satisfies the functional equation (2.20), and that this equation characterizes the power series $Q_k(t)$. Recall that $\mathbb{E}_k t^{Z_1} = \sum_{j=k}^{\infty} p(k,j)t^j = \mathbb{E} f_0^k(t)$. By Fubini's theorem, we have

$$\begin{aligned}
\mathbb{E}\left[Q_k(f_0(t))\right] &= \mathbb{E}\left[\sum_{i=k}^{\infty} q_{k,i} f_0^i(t)\right] = \sum_{i=k}^{\infty} q_{k,i}\mathbb{E}\left[f_0^i(t)\right] \\
&= \sum_{i=k}^{\infty} q_{k,i}\mathbb{E}\left[t^{Z_1}|\ Z_0 = i\right] = \sum_{i=k}^{\infty} q_{k,i}\sum_{j=i}^{\infty} p(i,j)t^j \\
&= \sum_{j=k}^{\infty}[\sum_{i=k}^{j} q_{k,i}\ p(i,j)]t^j
\end{aligned}$$

On the other hand, $\gamma_k Q_k(t) = \sum_{j=k}^{\infty} \gamma_k q_{k,j} t^j$. Therefore, by the recursive relation (2.15), for all $t \in [0,1)$,

$$\gamma_k Q_k(t) = \mathbb{E}\left[Q_k(f_0(t))\right].$$

The above argument also applies to prove that this functional equation characterizes the power series $Q_k(t)$. Assume $\mathbb{P}(0 < p_0 < 1) > 0$. Suppose that $\hat{Q}_k(t) = \sum_{j=0}^{\infty} \hat{q}_{k,j} t^j$ is a power series with $\hat{q}_{k,j} \in \mathbb{C}$ (a priori not necessarily real or positive) and $\hat{q}_{k,k} = 1$, whose radius of convergence is $\rho > 0$, which satisfies

$$\gamma_k \hat{Q}_k(t) = \mathbb{E}\left[\hat{Q}_k(f_0(t))\right]$$

for $t > 0$ small enough. We will prove that $\hat{Q}_k$ coincide with Q_k. In fact, the same argument used in the calculation of $\mathbb{E}\left[Q_k(f_0(t))\right]$ applies for $\mathbb{E}\left[\hat{Q}_k(f_0(t))\right]$, yielding that for $0 \leqslant t < \rho$,

$$\mathbb{E}\left[\hat{Q}_k(f_0(t))\right] = \sum_{j=0}^{\infty}[\sum_{i=0}^{j} \hat{q}_{k,i} p(i,j)]t^j;$$

in the argument we can still use Fubini's theorem, due to the fact that $f_0(t) \leqslant t$ and that the series $\hat{Q}(t) = \sum_{j=0}^{\infty} \hat{q}_{k,j} t^j$ converges absolutely for $t \in [0,\rho)$. Therefore, by comparing the coefficients, we see that $\gamma_k \hat{Q}_k(t) = \mathbb{E}\left[\hat{Q}_k(f_0(t))\right]$ for all $t > 0$ small enough if and only if

$$\gamma_k \hat{q}_{k,j} = \sum_{i=0}^{j} \hat{q}_{k,i} p(i,j), \quad \forall j \geqslant 0. \tag{4.16}$$

From this relation, we can easily cheque by reduction on j that

$$\hat{q}_{k,j} = 0 \quad \forall j = 0, \cdots, k-1. \tag{4.17}$$

Actually, for $j = 0$, Eq. (4.16) gives $\gamma_k \hat{q}_{k,0} = \hat{q}_{k,0} p(0,0)$; as $\gamma_k = \mathbb{E} p_1^k < 1 = p(0,0)$, this implies $q_{k,0} = 0$. Assume that $\hat{q}_{k,j} = 0\ \forall j = 0, \cdots, n$ with $n+1 < k$, then Eq. (4.16) with $j = n+1$ gives $\gamma_k \hat{q}_{k,j} = \hat{q}_{k,n+1} p(n+1, n+1)$; as $\gamma_k = \mathbb{E} p_1^k > \mathbb{E} p_1^{n+1} = p(n+1, n+1)$, this implies $\hat{q}_{k,n+1} = 0$. By induction, this proves (4.17).

With (4.17) at hand, (4.16) implies

$$\gamma_k \hat{q}_{k,j} = \sum_{i=k}^{j} \hat{q}_{k,i} p(i,j), \quad \forall j \geqslant k, \tag{4.18}$$

which reads also

$$\hat{q}_{k,j} = \frac{1}{\mathbb{E}(p_1^k - p_1^j)} \sum_{i=k}^{j-1} p(i,j) \hat{q}_{k,i}, \quad j > k. \tag{4.19}$$

Since $\hat{q}_{k,k} = q_{k,k} = 1$, from this relation and the same relation satisfied by $(q_{k,j})$, it follows by an easy reduction argument that $\hat{q}_{k,j} = q_{k,j}$ for all $j \geqslant k$. We have therefore roved that $\hat{Q}(t)$ coincides with $Q(t)$.

This ends the proof of Theorem 2.4.

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