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## INFINITARY NOETHERIAN CONSTRUCTIONS I. INFINITE WORDS

BY

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**Abstract.** We define and study Noetherian topologies for spaces of infinite sets and for spaces of infinite words. In each case, we also obtain S-representations, namely, computable presentations of the sobrifications of those spaces.

**1. Introduction.** A quasi-ordering, or a preordering, is a reflexive and transitive relation. A *well-quasi-order* (*wqo*) is a quasi-ordered set in which every infinite sequence  $(x_n)_{n \in \mathbb{N}}$  must contain two elements  $x_m \leq x_n$  with  $m < n$ . Well-quasi-orders are a fundamental tool in mathematics and computer science; however, they are not closed under several infinitary constructions: e.g., the set of all subsets of a wqo is not in general wqo [12], and a similar problem plagues sets of infinite words, and of infinite trees, over a well-quasi-ordered alphabet. Nash-Williams discovered that a strengthening of the notion of wqo, the notion of *better quasi-orders* (*bqo*), was closed under the usual finitary constructions that preserved being wqo (finite words, finite trees, etc.), and also under their infinitary variants [11].

A *Noetherian space* is a topological space in which every open set is compact, i.e., in which every open cover of an open set contains a finite subcover—we do not assume any separation axiom here. It was observed in [3] that Noetherian spaces formed a natural topological generalization of the order-theoretic notion of wqo. Noetherian spaces are closed under the same finitary constructions as wqos (finite words under embedding, finite trees under homeomorphic embedding, etc., see [4, Section 9.7]), but also under some infinitary constructions. Notably, in [3] we remarked that the so-called Hoare powerdomain of a Noetherian space—equivalently, its powerset under the so-called lower Vietoris topology—is Noetherian.

Our main purpose is to show that Noetherianness is preserved under some of the usual infinitary constructions that spurred the invention of bqos.

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A secondary purpose is to design those infinitary constructions in such a way that the closed subsets have finite representations suitable for an implementation on a computer. We do not mean to develop this here, but one should note that infinite words are pervasive in verification, mostly as infinite runs in various forms of automata, such as Büchi automata (see [14] for a survey). Downwards-closed subsets of runs, a very closely related notion, were also instrumental in the pioneering paper by Leroux and Schmitz [9] on the complexity of VASS reachability.

Let us illustrate our goal by an example of a finitary construction, taken from [4, Section 9.7] and [2, Section 7]. Let  $X^*$  denote the set of finite words over an alphabet  $X$  (not necessarily finite). For every quasi-ordering  $\leq$  on  $X$ , the (scattered) word embedding quasi-ordering  $\leq^*$  on  $X^*$  is defined by  $w \leq^* w'$  if  $w'$  can be obtained from  $w$  by increasing some letters from  $w$  and by inserting arbitrarily many new letters at arbitrary positions. Higman's Lemma [6] states that  $\leq^*$  is a well-quasi-ordering if and only if  $\leq$  is. Similarly, the *word topology* on  $X^*$ , where  $X$  is now a topological space, is generated by basic open sets of the form  $\langle U_1; \dots; U_n \rangle := X^*U_1X^*U_2 \cdots X^*U_nX^*$ , where  $n \in \mathbb{N}$  and each  $U_i$  is open in  $X$ —those are the sets of words that contain a letter in  $U_1$ , then a letter in  $U_2$  to the right of the previous one, and so on, until we find a letter in  $U_n$ . (Note that they form a base, not just a subbase.) Then the following hold (all the required notions will be introduced in Section 2):

- (A)  $X$  is Noetherian if and only if  $X^*$  (with the word topology) is Noetherian.
- (B) The specialization preordering of  $X^*$  is  $\leq^*$ , where  $\leq$  denotes the specialization preordering of  $X$ .
- (C) If  $X$  is wqo, then so is  $X^*$ .
- (D) If  $X$  has an S-representation (a certain computable way of representing the irreducible closed subsets of  $X$ , and therefore all closed subsets of  $X$ ), then  $X^*$  has an S-representation, too.

We wish to obtain similar results for infinitary constructions, e.g., spaces of infinite words. Our proposals will allow us to obtain equivalents of (A) and (D); (B) will only hold if  $X$  is wqo, and (C) will hold if and only if  $X$  is essentially finite (see below); however negative the latter result seems, one should note that we define a topological space as *wqo* if its specialization preordering is a wqo *and* its topology is Aleksandrov—that is a pretty strong requirement.

*Outline.* After some preliminaries in Section 2, we examine the case of the powerset  $\mathbb{P}(X)$  for Noetherian  $X$ . That  $\mathbb{P}(X)$  is Noetherian in that case is not new [3, 4, 2], but it seems important to understand why. This will occupy Section 3.1, in which we will deal with properties (A) and (D) in

that case. In Section 3.2, we examine properties (B) and (C). That is new. As promised, property (B) will hold only when  $X$  is wqo, and (C) only when  $X$  is essentially finite. We then make a small detour and introduce a few useful results pertaining to so-called initial maps in Section 4. With all that in hand, we will proceed to show that the space  $X^\omega$  of all infinite words over  $X$ , with a natural topology, enjoys properties (A) through (D)—in the case of (B) and (C), exactly with the same restrictions on  $X$  as above.

**2. Preliminaries.** We have already defined well-quasi-orders. They can be defined in many equivalent ways. Notably, as already observed by Higman [6, Theorem 2.1], it is equivalent to require any of the following properties, for a quasi-ordered set  $X$ : (i) every upwards-closed subset of  $X$  is the upward-closure of a finite set; (ii) the lattice of upwards-closed subsets of  $X$  has the ascending chain condition, namely: every ascending sequence  $(U_n)_{n \in \mathbb{N}}$  of upwards-closed subsets is stationary, in other words, there is a rank  $n_0$  such that  $U_n = U_{n_0}$  for every  $n \geq n_0$  (in general, we say that a quasi-ordering  $\leq$  has the *ascending chain condition* if it has no strictly ascending infinite sequence  $x_0 < x_1 < \dots < x_n < \dots$ , where  $x < y$  means  $x \leq y$  and  $y \not\leq x$ ); (iv) every infinite sequence of elements of  $X$  has an infinite ascending subsequence; (v)  $X$  is wqo. (We omit characterizations (iii) and (vi), which we will not require.)

Let us turn to topology, for which we refer the reader to [4]. Noetherian spaces are specifically covered in Section 9.7 there. Note that none of the topologies we will consider are Hausdorff. In fact, a Hausdorff topological space is Noetherian if and only if it is finite.

A *subbase* of a topology is any family of open sets that generates the family. A *base* of a topology is a family of open sets such that every open set can be written as a union of basic open sets. We write  $cl(A)$  for the closure of a subset  $A$  of a topological space. We will often use the fact that  $cl(A)$  intersects an open set  $U$  if and only if  $A$  intersects  $U$ .

Noetherian spaces have many equivalent characterizations (compare with the equivalent characterizations (i)–(vi) of wqos mentioned earlier). Those are also the spaces in which every ascending sequence  $(U_n)_{n \in \mathbb{N}}$  of open subsets is stationary; or also the spaces in which every descending sequence  $(C_n)_{n \in \mathbb{N}}$  of closed subsets is stationary. The first of those characterizations shows that Noetherianness is a property that depends only on the lattice of open subsets of the space, not on its points.

Noetherian spaces are closed under finite products, finite coproducts, subspaces, under the process of replacing the topology by a coarser one, under images by continuous maps, and various other constructions, such as the  $X^*$  construction.

Every topological space has a *specialization* preordering  $\leq$ , defined by  $x \leq y$  if and only if every open neighborhood of  $x$  contains  $y$ . We then say that  $x$  is less than or equal to  $y$ , or *below*  $y$ , or that  $y$  is larger than or equal to  $x$ , or *above*  $x$ . The closure of  $\{x\}$  is the *principal ideal*  $\downarrow x$ , the set of all points below  $x$  in that quasi-ordering. (Symmetrically, we write  $\uparrow x$  for the set of all points above  $x$ .) An *Aleksandrov topology* is a topology in which every intersection of open subsets is open, or equivalently, in which the open subsets are exactly the upwards-closed subsets in the specialization preordering  $\leq$ . The Aleksandrov topology of a given quasi-ordering  $\leq$  is, correspondingly, the collection of all its upwards-closed sets. Among the topologies with a given specialization preordering  $\leq$ , the Aleksandrov topology is the finest, and the coarsest is the *upper topology*, whose closed sets are intersections of sets of the form  $\downarrow E$ ,  $E$  finite; the notation  $\downarrow E$  denotes  $\bigcup_{x \in E} \downarrow x$ .

There is some degree of ambiguity in a notation such as  $\downarrow E$ , which will be particularly apparent when we work in spaces of subsets. For  $E \in \mathbb{P}(X)$ , where  $\mathbb{P}(X)$  is equipped with the inclusion ordering, say,  $\downarrow E$  might denote  $\{E' \in \mathbb{P}(X) \mid E' \subseteq E\}$  or  $\{x \in X \mid \exists y \in E, x \leq y\}$ . In such cases, we will disambiguate by writing  $\downarrow_{\mathbb{P}(X)} E$  for the first set, and  $\downarrow_X E$  for the second one.

It turns out that a quasi-ordering  $\leq$  is a well-quasi-ordering if and only if its Aleksandrov topology is Noetherian [4, Proposition 9.7.17]. For short, we will say that a topological space is a *wqo* if it is Noetherian and its topology is the Aleksandrov topology, or equivalently if its topology is the Aleksandrov topology of a well-quasi-ordering.

A subset  $C$  of a topological space  $X$  is *irreducible* if it is non-empty, and for all closed subsets  $C_1, C_2$  of  $X$  such that  $C \subseteq C_1 \cup C_2$ , we have  $C \subseteq C_1$  or  $C \subseteq C_2$ . Equivalently,  $C$  is non-empty, and for all open subsets  $U_1, U_2$  of  $X$  that intersect  $C$ ,  $U_1 \cap U_2$  also intersects  $C$ .

A *sober space* is a topological space in which every irreducible closed subset is the closure  $cl(\{x\}) = \downarrow x$  of a unique point  $x$ . (Chapter 8 of [4] is all about sober spaces.) The (standard) *sobrification*  $\mathcal{S}X$  of a topological space  $X$  is its set of irreducible closed subsets, with the *hull-kernel topology*, whose open subsets are (exactly) the sets of the form  $\diamond U := \{C \in \mathcal{S}X \mid C \cap U \neq \emptyset\}$ , where  $U$  ranges over the open subsets of  $X$ . The specialization (quasi-)ordering of  $\mathcal{S}X$  is inclusion. The space  $\mathcal{S}X$  is always sober, there is a continuous map  $\eta_X: X \rightarrow \mathcal{S}X: x \mapsto \downarrow x$ , and for every continuous map  $f: X \rightarrow Y$  where  $Y$  is sober, there is a unique continuous map  $\hat{f}: \mathcal{S}X \rightarrow Y$  such that  $\hat{f} \circ \eta_X = f$ .

$\mathcal{S}$  defines an endofunctor on the category of topological spaces, and its action on morphisms is defined by  $\mathcal{S}(f)(C) := cl(f[C])$ , where  $f[C]$  denotes

the image of  $C$  under  $f$ . In particular,  $cl(f[C])$  is irreducible closed for every irreducible closed set  $C$  and every continuous map  $f$ .

Sober spaces are closed under arbitrary topological products. Furthermore, the sobrification of any product of spaces is homeomorphic to the product of their sobrifications. Explicitly, and in the binary case, given any two irreducible closed subsets  $C$  of  $X$  and  $C'$  of  $Y$ ,  $C \times C'$  is irreducible closed in  $X \times Y$ . Moreover, all irreducible closed subsets of  $X \times Y$  are of this form:  $(C, C') \mapsto C \times C'$  is a homeomorphism from  $\mathcal{S}(X) \times \mathcal{S}(Y)$  to  $\mathcal{S}(X \times Y)$ .

A space is Noetherian if and only if its sobrification is Noetherian. Indeed, the map  $U \mapsto \diamond U$  is an order-isomorphism, hence the lattice of open sets of  $X$  has the ascending chain condition if and only if the lattice of open sets of  $\mathcal{S}X$  does.

We say that a quasi-ordered set (resp., a topological space) is *essentially finite* if it has only finitely many upwards-closed subsets (resp., open subsets). Note that the topology of an essentially finite topological space is Aleksandrov, and trivially Noetherian. A topological space  $X$  is essentially finite if and only if its  $T_0$  quotient, namely the quotient  $X/\equiv$  where  $\equiv := \leq \cap \geq$ , is finite.

The sober Noetherian spaces are particularly interesting, as they can be characterized entirely in terms of their specialization preordering. Explicitly, the sober Noetherian spaces are exactly the sets  $X$  with a well-founded quasi-ordering  $\leq$  such that every finite intersection of principal ideals can be expressed as a finite union of principal ideals (a quasi-ordering  $\leq$  is *well-founded* if every strictly descending chain is finite); furthermore, the topology of  $X$  is uniquely determined as the upper topology of  $\leq$ . In that case, the closed subsets are exactly the sets of the form  $\downarrow E$  with  $E$  finite, which makes them suitable for a representation on a computer—provided all the elements of  $E$  are themselves representable.

As a corollary, the closed subsets  $C$  of a Noetherian space  $X$  are exactly the finite unions of irreducible closed subsets  $C_1, \dots, C_n$  of  $X$ . Indeed, given any closed subset  $C$  of  $X$ , the set  $\downarrow_{\mathcal{S}X} C$  of all irreducible closed subsets of  $X$  below (included in)  $C$  is equal to  $\mathcal{S}X \setminus \diamond(X \setminus C)$ , hence is closed in  $\mathcal{S}X$ . Also,  $\eta_X^{-1}(\downarrow\{C\}) = C$ . Since  $\mathcal{S}X$  is Noetherian, one can write  $\downarrow\{C\}$  as  $\downarrow\{C_1, \dots, C_n\}$  for finitely many points  $C_1, \dots, C_n$  of  $\mathcal{S}X$ , and then  $C = \eta_X^{-1}(\downarrow\{C\})$  is the union of the finitely many irreducible closed subsets  $\eta_X^{-1}(\downarrow\{C_i\}) = C_i$ ,  $1 \leq i \leq n$ .

We will be interested in computer representations of irreducible closed subsets of  $X$  (i.e., of elements of  $\mathcal{S}(X)$ ), and this will immediately allow us to represent all closed subsets  $C$  as finite sets  $\{C_1, \dots, C_n\}$ , where each  $C_i$  is in  $\mathcal{S}(X)$ . If we can decide inclusion of irreducible closed subsets, one can also decide the inclusion of arbitrary closed subsets: if  $C$  is represented by the

finite set  $\{C_1, \dots, C_m\}$  and  $C'$  is represented by  $\{C'_1, \dots, C'_n\}$ , then  $C \subseteq C'$  if and only if for every  $i$ , there is a  $j$  such that  $C_i \subseteq C'_j$ . This is a simple consequence of the fact that each  $C_i$  is irreducible. We will also require to be able to compute the intersection  $C \cap C'$  of any two irreducible closed subsets of  $X$  as a finite union  $C_1 \cup \dots \cup C_n$  of irreducible closed subsets.

Those computability requirements are formalized by the notion of an  $S$ -representation [2, Definition 5.1]. An  $S$ -representation of a Noetherian space  $X$  is a 5-tuple  $(S, \llbracket \_ \rrbracket, \preceq, \tau, \wedge)$  where

- (1)  $S$  is a recursively enumerable set of so-called *codes* (of irreducible closed subsets);
- (2)  $\llbracket \_ \rrbracket$  is a surjective map from  $S$  to  $\mathcal{S}X$ ;
- (3)  $\preceq$  is a decidable relation such that, for all codes  $a, b \in S$ ,  $a \preceq b$  iff  $\llbracket a \rrbracket \leq \llbracket b \rrbracket$ ;
- (4)  $\tau$  is a finite subset of  $S$  such that  $X = \bigcup_{a \in \tau} \llbracket a \rrbracket$ ;
- (5)  $\wedge$  is a computable map (the *intersection map*) from  $S \times S$  to the collection  $\mathbb{P}_{\text{fin}}(S)$  of finite subsets of  $S$  (and we write  $a \wedge b$  for  $\wedge(a, b)$ ) such that  $\llbracket a \rrbracket \cap \llbracket b \rrbracket = \bigcup_{c \in a \wedge b} \llbracket c \rrbracket$ .

Let us take  $X^*$ , with the word topology, as an example. We use standard notations for certain regular languages on  $X$ : for every  $C \subseteq X$ ,  $C^?$  denotes the set of words of at most one letter, and that letter is in  $C$ ; for every  $F \subseteq X$ ,  $F^*$  is the set of words whose letters are all in  $F$ ; for all  $A, B \subseteq X^*$ ,  $AB$  denotes the set of all concatenations of one word from  $A$  and one from  $B$ ;  $\epsilon$  denotes both the empty word and the language  $\{\epsilon\}$ . A *word product* is a language of the form  $P := A_1 \cdots A_N$ , where each  $A_i$  is an *atom*, i.e., a language of the form  $C^?$  with  $C \in \mathcal{S}X$  or  $F^*$  where  $F$  is a closed subset of  $X$ . When  $X$  is Noetherian, the irreducible closed subsets of  $X^*$  are exactly the word products [2, Proposition 7.14]. One can also decide inclusion of word products in polynomial time with an oracle deciding inclusion in  $\mathcal{S}X$  [2, Lemma 7.10, Corollary 7.11], and compute intersections of word products as finite unions of word products in polynomial time with an oracle computing binary intersections in  $\mathcal{S}X$  as finite unions of irreducible closed subsets [2, Lemma 7.13]. Formally:

PROPOSITION 2.1 ([2, Theorem 7.15]). *Given an  $S$ -representation  $(S, \llbracket \_ \rrbracket, \preceq, \tau, \wedge)$  of a Noetherian space  $X$ , the following tuple  $(S', \llbracket \_ \rrbracket', \preceq', \tau', \wedge')$  is an  $S$ -representation of  $X^*$ :*

- (1)  $S'$  is the collection of all (syntactic) word products over the alphabet  $S$ , namely all regular expressions  $A_1 \cdots A_N$  where each  $A_i$  is either an expression of the form  $a^?$  with  $a \in S$ , or  $u^*$  where  $u$  is a finite subset of  $S$  (we write  $\epsilon$  when  $N = 0$ ).
- (2)  $\llbracket A_1 \cdots A_N \rrbracket' := \llbracket A_1 \rrbracket' \cdots \llbracket A_N \rrbracket'$ , where  $\llbracket a^? \rrbracket' := \llbracket a \rrbracket'$  and  $\llbracket \{a_1, \dots, a_n\}^* \rrbracket' := (\llbracket a_1 \rrbracket' \cup \dots \cup \llbracket a_n \rrbracket')^*$ .

(3)  $\leq'$  is defined inductively by:

$\varepsilon \leq' Q$  is always true,

$P \leq' \varepsilon$  is false if  $P \neq \varepsilon$ ,

$$a^? P \leq' b^? Q \quad \text{iff} \quad \begin{cases} P \leq' Q & \text{if } a \leq b, \\ a^? P \leq' Q & \text{otherwise,} \end{cases}$$

$$a^? P \leq' v^* Q \quad \text{iff} \quad \begin{cases} P \leq' v^* Q & \text{if } \exists b \in v, a \leq b, \\ a^? P \leq' Q & \text{otherwise,} \end{cases}$$

$$u^* P \leq' b^? Q \quad \text{iff} \quad \begin{cases} P \leq' b^? Q & \text{if } u = \emptyset, \\ u^* P \leq' Q & \text{otherwise,} \end{cases}$$

$$u^* P \leq' v^* Q \quad \text{iff} \quad \begin{cases} P \leq' v^* Q & \text{if } \forall a \in u, \exists b \in v, a \leq b, \\ u^* P \leq' Q & \text{otherwise.} \end{cases}$$

(4)  $\tau'$  is  $\{\tau^*\}$ .

(5)  $\wedge'$  is implemented by the following clauses (together with the obvious symmetric clauses):

$$(2.1) \quad \varepsilon \wedge' Q := \{\varepsilon\},$$

$$(2.2) \quad a^? P \wedge' b^? Q := (a^? P \wedge' Q) \cup (P \wedge' b^? Q) \\ \cup \{c^? R \mid c \in a \wedge b, R \in P \wedge' Q\},$$

$$(2.3) \quad a^? P \wedge' v^* Q := \begin{cases} \{c^? R \mid c \in \bigcup_{b \in v} (a \wedge b), R \in P \wedge' v^* Q\} \cup (a^? P \wedge' Q) & \text{if } a \wedge b \neq \emptyset \text{ for some } b \in v, \\ (P \wedge' v^* Q) \cup (a^? P \wedge' Q) & \text{otherwise} \end{cases}$$

$$(2.4) \quad u^* P \wedge' v^* Q := \left\{ \left( \bigcup_{a \in u, b \in v} a \wedge b \right)^* R \mid R \in (P \wedge' v^* Q) \cup (u^* P \wedge' Q) \right\}.$$

REMARK 2.2. One can optimize the procedures above in a number of ways. In the definition of  $\wedge'$ , one can remove any subsumed word product in the result. A word product  $P$  is *subsumed* by another one,  $Q$ , in a given set, if and only if

$$P \leq' Q, \quad \text{or equivalently} \quad \llbracket P \rrbracket' \subseteq \llbracket Q \rrbracket'.$$

As a special case, in (2.3), if  $Q = \varepsilon$ , then we can remove  $a^? P \wedge' Q (= \{\varepsilon\})$ , which is subsumed by some other word product, since  $\llbracket \varepsilon \rrbracket' = \{\varepsilon\}$  is included in the notation of the remaining word products (the union of the sets  $\llbracket c^? R \rrbracket'$  where  $c \in \bigcup_{b \in v} (a \wedge b)$  and  $R \in P \wedge' v^* Q$  if  $a \wedge b \neq \emptyset$  for some  $b \in v$ , and the union of the sets  $\llbracket R \rrbracket'$  where  $R \in P \wedge' v^* Q$  otherwise).



### 3. Powersets

**3.1. Properties (A) and (D).** Let  $\mathbb{P}(X)$  denote the powerset of a space  $X$ , with the *lower Vietoris topology*, generated by subbasic open sets of the form  $\diamond U := \{A \in \mathbb{P}(X) \mid A \cap U \neq \emptyset\}$ . By that we mean that the open subsets of  $\mathbb{P}(X)$  are the unions of finite intersections  $\bigcap_{i=1}^n \diamond U_i$ . (Note the similarity of that notation with the open subsets  $\diamond U$  of  $\mathcal{S}X$ . They are defined the same way, but the sets  $\diamond U$  only form a subbase of the lower Vietoris topology, whereas the sets  $\diamond U$  range over all the open sets in the hull-kernel topology on  $\mathcal{S}X$ . Also,  $\diamond$  and  $\diamond$  commute with arbitrary unions, but  $\diamond$  additionally commutes with finite intersections.)

The subset of  $\mathbb{P}(X)$  consisting of all closed subsets of  $X$  is called the *Hoare powerspace* of  $X$ , and will be written as  $\mathcal{H}(X)$ . We again write  $\diamond U$  for the open set  $\{C \in \mathcal{H}(X) \mid C \cap U \neq \emptyset\}$ . Those sets generate the subspace topology on  $\mathcal{H}(X)$ , and we will also call it the lower Vietoris topology. For any set  $A$ ,  $A$  intersects an open set  $U$  if and only if  $cl(A)$  intersects  $U$ , and this implies that the function that maps every open subset of  $\mathbb{P}(X)$  to its intersection with  $\mathcal{H}(X)$  is an order-isomorphism. The following lemma, which is of independent interest, shows that  $\mathcal{H}(X)$  is homeomorphic to  $\mathcal{S}(\mathbb{P}(X))$ . We also deal with  $\mathbb{P}^*(X)$ , the subspace of *non-empty* subsets of  $X$ , and with  $\mathcal{H}^*(X)$ , the subspace of non-empty closed subsets of  $X$ .

LEMMA 3.1 ([2, Lemma 5.10]). *The map  $F \mapsto \downarrow F$  is a homeomorphism from  $\mathcal{H}(X)$  onto  $\mathcal{S}(\mathbb{P}(X))$ , resp. from  $\mathcal{H}^*(X)$  onto  $\mathcal{S}(\mathbb{P}^*(X))$ . ■*

It follows that for every space  $X$ ,  $\mathbb{P}(X)$  is Noetherian if and only if  $\mathcal{H}(X)$  is Noetherian, and similarly for  $\mathbb{P}^*(X)$  and  $\mathcal{H}^*(X)$ . It is easy to see that every subspace and every homeomorph of a Noetherian space is Noetherian, so any of those properties implies that  $X$ , which is homeomorphic to the subspace of points  $\{x\}$  in  $\mathbb{P}(X)$  (resp.,  $\mathbb{P}^*(X)$ ), is Noetherian. Conversely, if  $X$  is Noetherian, then  $\subseteq$  is well-founded on  $\mathcal{H}(X)$ . Any finite intersection of principal ideals  $\downarrow_{\mathcal{H}(X)} F_i$ ,  $1 \leq i \leq n$ , in  $\mathcal{H}(X)$  can be expressed as a finite union of principal ideals, in fact just as  $\downarrow_{\mathcal{H}(X)} (F_1 \cap \dots \cap F_n)$ . It follows that  $\mathcal{H}(X)$  is Noetherian, and sober, with the upper topology of inclusion. Since the complement of  $\downarrow_{\mathcal{H}(X)} \{F_1, \dots, F_n\}$  is equal to  $\diamond U_1 \cap \dots \cap \diamond U_n$ , where each  $U_i$  is the complement of  $F_i$  in  $X$ , that upper topology is none other than the lower Vietoris topology.

The next proposition follows easily, and is a reformulation of Theorem 5.11(A)–(C), of [2]; that theorem actually gives a full description of an S-representation for  $\mathbb{P}(X)$  and for  $\mathbb{P}^*(X)$ , while Theorem 5.8 of [2] gives the analogous S-representation for  $\mathcal{H}(X)$  and for  $\mathcal{H}^*(X)$ .

PROPOSITION 3.2. *For every topological space  $X$ ,  $X$  is Noetherian if and only if  $\mathbb{P}(X)$  (resp.,  $\mathcal{H}(X)$ ,  $\mathbb{P}^*(X)$ ,  $\mathcal{H}^*(X)$ ) is.*

Letting  $Y := \mathbb{P}(X)$  (resp.,  $\mathbb{P}^*(X)$ ), the irreducible closed subsets of  $Y$  are exactly the sets of the form  $\downarrow_Y F = \{A \in Y \mid A \subseteq F\}$ , where  $F \in \mathcal{H}(X)$  (resp.,  $\mathcal{H}^*(X)$ ).

In particular, if  $X$  is Noetherian, then the irreducible closed subsets of  $Y$  can be represented as finite sets  $\{C_1, \dots, C_n\}$  (resp., with  $n \geq 1$ ), denoting  $\downarrow_Y(C_1 \cup \dots \cup C_n)$ , where each  $C_i \in \mathcal{S}X$ ; if inclusion is decidable on  $\mathcal{S}X$ , then inclusion is decidable on  $\mathcal{S}Y$ : if  $F$  is represented by the finite set  $\{C_1, \dots, C_m\}$  and  $F'$  is represented by the finite set  $\{C'_1, \dots, C'_n\}$ , then  $F \subseteq F'$  if and only if for every  $i$ , there is a  $j$  such that  $C_i \subseteq C'_j$ . ■

The properties stated in Proposition 3.2 match properties (A) and (D) mentioned in the introduction, as promised.

**3.2. Properties (B) and (C).** As for property (B), the specialization preordering on  $\mathbb{P}(X)$  (resp.,  $\mathbb{P}^*(X)$ ) is given by  $A \leq^b B$  if and only if  $cl(A) \subseteq cl(B)$ . When  $X$  is a wqo,  $cl(A) = \downarrow_X A$ , so  $A \leq^b B$  if and only if for every  $a \in A$ , there is a  $b \in B$  such that  $a \leq b$ , and we retrieve the usual *domination* (or *Hoare*) quasi-ordering.

We now inquire about property (C). One may wonder when  $\mathbb{P}(X)$  is wqo, in the sense that its topology is both Aleksandrov and Noetherian. One might think that this would be the case if and only if  $X$  is  $\omega^2$ -wqo (see [8, 10] or [7] for example). This is wrong, as we will see in Proposition 3.4 below. If  $X$  is  $\omega^2$ -wqo, what we obtain is that the domination quasi-ordering on  $\mathbb{P}(X)$  is a well-quasi-ordering (this can be taken as a definition of an  $\omega^2$ -wqo), *not* that the lower Vietoris topology is Aleksandrov. We will use the following lemma.

LEMMA 3.3.

- (1) *A topological space whose lattice of open subsets is well-founded under inclusion has the Aleksandrov topology of its specialization preordering, and that quasi-ordering has the ascending chain condition.*
- (2) *A well-quasi-ordering with the ascending chain condition is essentially finite.*
- (3) *A Noetherian space whose lattice of open subsets is well-founded is essentially finite.*

*Proof.* (1) Let us assume that the lattice of open subsets of  $X$  is well-founded. Given any  $x \in X$ , there is a minimal open neighborhood  $U_x$  of  $x$ . By definition of the specialization preordering  $\leq$ , for every point  $y \in X$  such that  $x \not\leq y$ , there is an open subset  $U$  of  $X$  that contains  $x$  but not  $y$ . Since  $U \cap U_x$  is an open neighborhood of  $x$ , the minimality of  $U_x$  entails that  $U \cap U_x = U_x$ , that is,  $U_x \subseteq U$ . It follows that  $y$  is not in  $U_x$ . We have shown the implication  $x \not\leq y \Rightarrow y \notin U_x$ , from which we deduce  $U_x \subseteq \uparrow x$ . Every open set is upwards-closed in the specialization preordering, so  $U_x = \uparrow x$ .

From this, we deduce that  $\uparrow x$  is open for every  $x \in X$ . Every upwards-closed subset  $A$  is equal to  $\bigcup_{x \in A} \uparrow x$ , hence is open. Hence the topology of  $X$  is the Aleksandrov topology of  $\leq$ .

We now consider any strictly increasing sequence  $x_0 < x_1 < \dots$ . Then the sets  $\uparrow x_n$  form a strictly descending sequence of open subsets, contradicting our well-foundedness assumption. Hence  $\leq$  has the ascending chain condition.

(2) For contradiction, let us assume that there is an infinite set  $A$  whose elements are pairwise inequivalent with respect to  $\equiv := \leq \cap \geq$ . We extract a countable infinite subset  $(x_n)_{n \in \mathbb{N}}$  of  $A$ . In a well-quasi-ordering, every infinite sequence has an infinite ascending subsequence, so we may assume without loss of generality that  $x_0 \leq x_1 \leq \dots$ . By the ascending chain condition, only finitely many of those inequalities can be strict, hence  $x_n \equiv x_{n+1}$  for at least one  $n$ . That is absurd.

(3) If  $X$  is Noetherian with a well-founded lattice of open subsets, by (1) it is Aleksandrov, hence wqo, and satisfies the ascending chain condition. We conclude by (2). ■

**PROPOSITION 3.4.** *Let  $X$  be a Noetherian space. The lower Vietoris topology on  $\mathbb{P}(X)$  (resp.,  $\mathcal{H}(X)$ ,  $\mathbb{P}^*(X)$ ,  $\mathcal{H}^*(X)$ ) is Aleksandrov if and only if  $X$  is essentially finite.*

*Proof.* The “if” direction is clear. Let  $Y$  be  $\mathbb{P}(X)$  (resp.,  $\mathcal{H}(X)$ ,  $\mathbb{P}^*(X)$ ,  $\mathcal{H}^*(X)$ ), and let us assume that  $Y$  is Aleksandrov. We use Lemma 3.3(3), first showing that the lattice of open subsets of  $X$  is well-founded, or equivalently that there is no infinite strictly ascending sequence of closed subsets of  $X$ .

For the sake of contradiction, we assume that there is such an infinite strictly ascending sequence  $F_0 \subsetneq F_1 \subsetneq \dots$ . Up to the removal of  $F_0$ , we may assume that every  $F_n$  is non-empty: this is needed for the cases where  $Y$  is  $\mathbb{P}^*(X)$  or  $\mathcal{H}^*(X)$ . For each  $n \in \mathbb{N}$ ,  $\downarrow_Y F_n = Y \setminus \diamond(X \setminus F_n)$  is closed. Since  $Y$  is Aleksandrov, any union of closed subsets of  $Y$  is closed, so  $\mathcal{F} := \bigcup_{n \in \mathbb{N}} \downarrow_Y F_n$  is closed.

Let  $F_\infty$  be the closure of  $\bigcup_{n \in \mathbb{N}} F_n$  in  $X$ . We claim that  $F_\infty$  is in  $\mathcal{F}$ . Otherwise, by the definition of the lower Vietoris topology,  $F_\infty$  would be in some finite intersection  $\bigcap_{i=1}^m \diamond U_i$ , disjoint from  $\mathcal{F}$ , where each  $U_i$  is open in  $X$ . For each  $i$ ,  $U_i$  would then intersect  $F_\infty$ , hence  $\bigcup_{n \in \mathbb{N}} F_n$ , hence  $F_{n_i}$  for some  $n_i \in \mathbb{N}$ . Let  $n \in \mathbb{N}$  be larger than every  $n_i$ ,  $1 \leq i \leq m$ . Then  $F_n$  intersects every  $U_i$ ,  $1 \leq i \leq m$ , as well, so  $F_n \in \bigcap_{i=1}^m \diamond U_i$ . However,  $F_n$  is in  $\mathcal{F}$ , by definition of  $\mathcal{F}$ , which is impossible since  $\mathcal{F}$  is disjoint from  $\bigcap_{i=1}^m \diamond U_i$  by assumption.

Since  $F_\infty$  is in  $\mathcal{F}$ , it is in some  $\downarrow_Y F_n$ . In particular,  $F_{n+1} \subseteq F_\infty \subseteq F_n$ , which is impossible. Hence there is no strictly ascending sequence of closed subsets of  $X$ , and we conclude the proof by invoking Lemma 3.3(3). ■

**4. Initial maps.** We will use the following additional facts about Noetherian spaces. An *initial* map  $f: Y \rightarrow Z$  between topological spaces is one such that the open subsets of  $Y$  are exactly the sets of the form  $f^{-1}(W)$ ,  $W$  open in  $Z$ . All initial maps are continuous.

REMARK 4.1. Given a subbase of the topology of  $Y$ , a practical way of checking that  $f: Y \rightarrow Z$  is initial consists in verifying that  $f$  is continuous, and that every subbasic open subset  $V$  of  $Y$  can be written as  $f^{-1}(W)$  for some open subset  $W$  of  $Z$ . In other words, we do not need to check the latter for every open subset of  $Y$ , just for subbasic open sets. Indeed, every open subset  $V$  of  $Y$  can be written as  $\bigcup_{i \in I} \bigcap_{j=1}^{n_i} V_{ij}$  where each  $V_{ij}$  is subbasic, and if we can write each  $V_{ij}$  as  $f^{-1}(W_{ij})$  with  $W_{ij}$  open in  $Z$ , then  $V$  is equal to  $f^{-1}(\bigcup_{i \in I} \bigcap_{j=1}^{n_i} W_{ij})$ .

A general way of finding initial maps is as follows. Let  $Z$  be a topological space and  $f$  be a map from a set  $Y$  to  $Z$ . With the coarsest topology on  $Y$  that makes  $f$  continuous,  $f$  is initial. This is notably the case of topological embeddings, which are those initial maps that are injective.

LEMMA 4.2. *Let  $f: Y \rightarrow Z$  be an initial map between topological spaces. If  $Z$  is Noetherian, then  $Y$  is Noetherian.*

*Proof.* The open subsets of  $Y$  are the sets  $f^{-1}(W)$ ,  $W$  open in  $Z$ . Let  $(f^{-1}(W_n))_{n \in \mathbb{N}}$  be a monotonic sequence of open subsets in  $Y$ . Replacing  $W_n$  by  $W_0 \cup W_1 \cup \dots \cup W_n$ , we may assume that  $(W_n)_{n \in \mathbb{N}}$  is also a monotonic sequence. Since  $Z$  is Noetherian, all sets  $W_n$  are equal for  $n$  large enough. Hence all sets  $f^{-1}(W_n)$  are equal for  $n$  large enough. ■

LEMMA 4.3. *Let  $f: Y \rightarrow Z$  be an initial map. The irreducible closed subsets  $D$  of  $Y$  are all of the form  $f^{-1}(C)$  where  $C$  is some irreducible closed in  $Z$ . More precisely, one can always choose  $C := cl(f[D])$ .*

*Proof.* Let  $D$  be irreducible closed in  $Y$ , and consider  $C := cl(f[D])$ . Recall that  $C = \mathcal{S}(f)(D)$  is irreducible closed.

Since  $f$  is initial,  $D = f^{-1}(C')$  for some closed subset  $C'$  of  $Z$ . Then  $f[f^{-1}(C')]$  is included in  $C'$ , so its closure  $C := cl(f[D])$  is also included in  $C'$ . In particular,  $f^{-1}(C)$  is included in  $f^{-1}(C') = D$ . Conversely, for every  $y \in D$ ,  $f(y)$  is in  $f[D]$  hence in  $C$ , so  $D$  is included in  $f^{-1}(C)$ . Therefore  $D = f^{-1}(C)$ , where  $C$  is irreducible closed. ■

Note that Lemma 4.3 does not say that every set  $f^{-1}(C)$ ,  $C \in \mathcal{S}Z$ , is irreducible closed in  $Y$ , just that every irreducible closed subset of  $Y$  must be of that form. We have a complete characterization when the image of  $f$  is open or closed:

LEMMA 4.4. *Let  $f: Y \rightarrow Z$  be an initial map.*

- (1) *If the image of  $f$  is open, then the irreducible closed subsets of  $Y$  are exactly those sets of the form  $f^{-1}(C)$  where  $C$  ranges over the irreducible closed subsets of  $Z$  that intersect the image of  $f$ .*
- (2) *If the image of  $f$  is closed, then the irreducible closed subsets of  $Y$  are exactly those sets of the form  $f^{-1}(C)$  where  $C$  ranges over the irreducible closed subsets of  $Z$  that are included in the image of  $f$ .*

*Proof.* (1) Let us assume that the image  $\text{Im } f$  of  $f$  is open. By Lemma 4.3, every irreducible closed subset of  $Y$  must be of the form  $f^{-1}(C)$ , with  $C$  irreducible closed in  $Z$ . Necessarily,  $f^{-1}(C)$  must be non-empty, and that implies that  $C$  intersects the image  $\text{Im } f$  of  $f$ .

In the converse direction, let  $C$  be irreducible closed in  $Z$ , and let us assume that  $C$  intersects  $\text{Im } f$ . Then  $f^{-1}(C)$  is non-empty. Let us now consider two open subsets of  $Y$  that intersect  $f^{-1}(C)$ . Since  $f$  is initial, they must be of the form  $f^{-1}(U)$  and  $f^{-1}(V)$ , where  $U$  and  $V$  are open in  $Z$ . Since  $f^{-1}(C)$  intersects  $f^{-1}(U)$ , there is a point  $y \in Y$  such that  $f(y)$  is in  $C$  and in  $U$ , and therefore  $\text{Im } f \cap U \cap C$  is non-empty. In other words,  $C$  intersects  $\text{Im } f \cap U$ . Similarly,  $C$  also intersects  $\text{Im } f \cap V$ . Both  $\text{Im } f \cap U$  and  $\text{Im } f \cap V$  are open in  $Z$ . Since  $C$  is irreducible, it must intersect their intersection, which is  $\text{Im } f \cap U \cap V$ . Hence there is a point  $f(y)$  (with  $y \in Y$ ) in  $\text{Im } f$  which is also in  $C$ ,  $U$ , and  $V$ . Then  $y$  is in  $f^{-1}(C)$ ,  $f^{-1}(U)$  and  $f^{-1}(V)$ , so  $f^{-1}(C)$  intersects  $f^{-1}(U) \cap f^{-1}(V)$ . We conclude that  $f^{-1}(C)$  is irreducible.

(2) We now assume that  $\text{Im } f$  is closed. By Lemma 4.3, every irreducible closed subset  $D$  of  $Y$  must be of the form  $f^{-1}(C)$ , where  $C := \text{cl}(f[D])$  is irreducible closed in  $Z$ . Since  $f[D] \subseteq \text{Im } f$  and  $\text{Im } f$  is closed,  $\text{cl}(f[D])$  is also included in  $\text{Im } f$ , so  $C$  is included in the image of  $f$ .

In the converse direction, let  $C$  be irreducible closed in  $Z$ , and let us assume that  $C \subseteq \text{Im } f$ . Since  $C$  is non-empty,  $f^{-1}(C)$  is non-empty. Let us now consider two closed subsets of  $Y$  whose union contains  $f^{-1}(C)$ . Since  $f$  is initial, they must be of the form  $f^{-1}(C_1)$  and  $f^{-1}(C_2)$ , where  $C_1$  and  $C_2$  are closed in  $Z$ . For every  $z \in C$ , we can write  $z$  as  $f(y)$  for some  $y \in Y$  since  $C \subseteq \text{Im } f$ . Then  $y$  is in  $f^{-1}(C)$ , hence in  $f^{-1}(C_1)$  or in  $f^{-1}(C_2)$ . It follows that  $z$  is in  $C_1$  or in  $C_2$ . This shows that  $C$  is included in  $C_1 \cup C_2$ , hence in  $C_1$  or in  $C_2$ , by irreducibility. In the first case,  $f^{-1}(C)$  is included in  $f^{-1}(C_1)$ , otherwise in  $f^{-1}(C_2)$ . We conclude that  $f^{-1}(C)$  is irreducible. ■

**5. Infinite words.** Let  $X$  be an alphabet, by which we simply mean a topological space, not necessarily finite. An *infinite word* on  $X$  is an infinite sequence of elements of  $X$ , i.e., a function from  $\mathbb{N}$  to  $X$ . We let  $X^\omega$  denote the set of all infinite words on  $X$ . We write every  $w \in X^\omega$  as  $w_0 w_1 \cdots w_n \cdots$ ,

where  $w_n \in X$ . We also write  $w_{<n}$  for the length  $n$  prefix  $w_0w_1 \cdots w_{n-1}$  of  $w$ , and  $w_{\geq n}$  for the remainder  $w_nw_{n+1} \cdots$ .

We will also consider the set of *finite-or-infinite words*  $X^{\leq\omega} := X^\omega \cup X^*$ . Those can be defined as the functions  $w$  from an initial segment  $\text{dom } w$  of  $\mathbb{N}$  to  $X$ .

There is a standard quasi-ordering  $\leq^\omega$  on  $X^{\leq\omega}$ , defined by  $w \leq^\omega w'$  if and only if  $w$  is a *subword* of  $w'$ , namely if there is a monotonic, injective map  $f: \text{dom } w \rightarrow \text{dom } w'$  such that  $w_n \leq w'_{f(n)}$  for every  $n \in \text{dom } w$ . (As usual,  $\leq$  is the specialization preordering of  $X$ .)

The topology we will be interested in is the following. We reuse the notation  $\langle U_1; \dots; U_n \rangle$  to denote the set of *finite or infinite words* that have a (finite) subword in  $U_1 \cdots U_n$ . The context should make it clear whether we reason in  $X^*$  or in  $X^{\leq\omega}$ . The notation  $\langle U_1; \dots; U_n; (\infty)U \rangle$  denotes the set of (necessarily infinite) words that can be written as a concatenation  $uw$  where  $u$  is a finite word in  $\langle U_1; \dots; U_n \rangle$  and  $w$  contains infinitely many letters from  $U$ .

DEFINITION 5.1. The *asymptotic subword topology* on  $X^{\leq\omega}$  is generated by the subbasic open sets  $\langle U_1; \dots; U_n \rangle$  and  $\langle U_1; \dots; U_n; (\infty)U \rangle$ , where  $n \in \mathbb{N}$ , and  $U_1, \dots, U_n, U$  are open in  $X$ .

Note that  $X^\omega$  is an open subset of  $X^{\leq\omega}$ , since  $X^\omega = \langle (\infty)X \rangle$ . It follows that  $X^*$  occurs as a closed subset of  $X^{\leq\omega}$ .

We equip each subspace of  $X^{\leq\omega}$  with the subspace topology. In particular, we will call the subspace topology on  $X^\omega$  the *asymptotic subword topology*. The subspace topology on  $X^*$  happens to coincide with the word topology.

FACT 5.2. *Every open (resp., closed) subset of  $X^{\leq\omega}$  in the asymptotic subword topology is upwards-closed (resp., downwards-closed) with respect to  $\leq^\omega$ . ■*

We will now show that if  $X$  is Noetherian, then the asymptotic subword topology is the join of two simpler topologies, the prefix and the suffix topology. (The *join* of two topologies is the coarsest topology that is finer than both. It has a base of open sets of the form  $U_1 \cap U_2$ , where  $U_1$  is open in the first topology and  $U_2$  is open in the other one.)

**5.1. The prefix topology.** For every  $w \in X^{\leq\omega}$ , the set  $\text{pref}(w) := \text{cl}(\{w_{<n} \mid n \in \text{dom } w\})$  is irreducible in  $X^*$ : if  $\mathcal{U}_1$  and  $\mathcal{U}_2$  are two open subsets of  $X^*$  that intersect  $\text{pref}(w)$ , then  $w_{<m} \in \mathcal{U}_1$  and  $w_{<n} \in \mathcal{U}_2$  for some  $m, n \in \text{dom } w$ ; since all open subsets of  $X^*$  are upwards-closed in  $\leq^*$ ,  $w_{<\max(m,n)}$  is both in  $\mathcal{U}_1$  and in  $\mathcal{U}_2$ , showing that  $\text{pref}(w)$  intersects  $\mathcal{U}_1 \cap \mathcal{U}_2$ .

DEFINITION 5.3. The *prefix map*  $\text{pref}: X^{\leq\omega} \rightarrow \mathcal{S}(X^*)$  is defined by  $\text{pref}(w) := \text{cl}(\{w_{<n} \mid n \in \text{dom } w\})$  for every  $w \in X^{\leq\omega}$ .

The *prefix topology* on  $X^{\leq\omega}$  is the coarsest topology that makes  $\text{pref}$  continuous. In other words, a subbase of the prefix topology is given by sets of the form  $\text{pref}^{-1}(\diamond U)$ , where  $U$  is open in  $X^*$ .

REMARK 5.4. Let us take  $X$  Noetherian. Since  $\text{pref}(w)$  is in  $\mathcal{S}(X^*)$ , one must be able to write it as a word product. The closed subsets  $F_n := \text{cl}(\{w_m \mid m \in \text{dom } w, m \geq n\})$ ,  $n \in \mathbb{N}$ , form a descending sequence, so there is an index  $n_0$  such that for every  $n \geq n_0$ ,  $F_n = F_{n_0}$ . Although we will not use it, one can show that  $\text{pref}(w) = (\downarrow w_0)^?(\downarrow w_1)^? \cdots (\downarrow w_{n-1})^? F_n^*$  for every  $n \geq n_0$ . We leave this as an exercise to the reader. As a hint, first, show that the right-hand side  $\mathcal{C} := (\downarrow w_0)^?(\downarrow w_1)^? \cdots (\downarrow w_{n-1})^? F_n^*$  does not depend on  $n \geq n_0$ . Second, show that  $w_{<n} \in \mathcal{C}$  for every  $n \in \mathbb{N}$ , and deduce that  $\text{pref}(w) \subseteq \mathcal{C}$ . In the reverse direction, consider any basic open set  $\mathcal{U} := \langle U_1; \dots; U_k \rangle$  that intersects  $\mathcal{C}$ , say at  $u := u_0 a_1 u_1 a_2 u_2 \cdots u_{k-1} a_k u_k$ , where each  $u_i$  is in  $X^*$ , each  $a_i$  is in  $U_i$ . By picking  $n$  larger than the length of  $u$  in the definition of  $\mathcal{C}$ , observe that  $a_1 \cdots a_k$  is a subword of  $w_{<n}$ , and conclude that  $\mathcal{U}$  intersects  $\text{pref}(w)$  at  $w_{<n}$ .

For the next lemma, we recall that the sets of the form  $\langle U_1; \dots; U_n \rangle$ , where each  $U_i$  is open in  $X$ , form a base, not just a subbase of the asymptotic subword topology on  $X^*$ .

LEMMA 5.5. *The prefix map  $\text{pref}$  is continuous from  $X^{\leq\omega}$  with its asymptotic subword topology to  $\mathcal{S}(X^*)$ . A base of the prefix topology is given by the sets  $\langle U_1; \dots; U_n \rangle$ , where  $U_1, \dots, U_n$  are open in  $X$ .*

*The prefix topology is coarser than the asymptotic subword topology.*

*Proof.* Every open subset of  $X^*$  is a union of basic open subsets of the form  $\langle U_1; \dots; U_n \rangle$  where each  $U_i$  is open in  $X$ . In order to show that  $\text{pref}$  is continuous, since  $\diamond$  commutes with arbitrary unions, it is enough to show that  $\text{pref}^{-1}(\diamond(\langle U_1; \dots; U_n \rangle))$  is open in the asymptotic subword topology. That is the set of finite or infinite words  $w$  such that  $\text{cl}(\{w_{<m} \mid m \in \text{dom } w\})$  intersects the open set  $\langle U_1; \dots; U_n \rangle$ ; equivalently, such that some prefix  $w_{<m}$  belongs to  $\langle U_1; \dots; U_n \rangle$ . The set  $\text{pref}^{-1}(\diamond(\langle U_1; \dots; U_n \rangle))$  is therefore equal to the open subset  $\langle U_1; \dots; U_n \rangle$  of  $X^{\leq\omega}$ .

This also shows that the sets  $\langle U_1; \dots; U_n \rangle = \text{pref}^{-1}(\diamond(\langle U_1; \dots; U_n \rangle))$  form a subbase of the prefix topology.

Since  $\text{pref}^{-1}$  and  $\diamond$  commute with finite intersections, and since the sets  $\langle U_1; \dots; U_n \rangle$  form a base of the topology on  $X^*$ , every finite intersection  $U$  of sets of the form  $\text{pref}^{-1}(\diamond(\langle U_1; \dots; U_n \rangle))$  can be written as  $\text{pref}^{-1}(\diamond \mathcal{U})$  where  $\mathcal{U}$  is a union of basic open sets of  $X^*$ . Since  $\text{pref}^{-1}$  and  $\diamond$  commute with all unions,  $\mathcal{U}$  is also a union of subbasic open sets of the form  $\langle U_1; \dots; U_n \rangle$ . This shows that the given subbase is a base.

The final claim is an immediate consequence of the first one. ■

**5.2. The suffix topology.** In any complete lattice  $L$ , the *limit superior* of a sequence  $(u_n)_{n \in \mathbb{N}}$  is  $\limsup_{n \in \mathbb{N}} u_n = \bigwedge_{n \in \mathbb{N}} \bigvee_{m \geq n} u_m$ . We will use that notion in lattices of closed sets. Then  $\limsup_{n \in \mathbb{N}} C_n = \bigcap_{n \in \mathbb{N}} cl(\bigcup_{m \geq n} C_m)$ . On a Noetherian space, descending families of closed sets are stationary, so  $\limsup_{n \in \mathbb{N}} C_n = cl(\bigcup_{m \geq n} C_m)$  for large enough  $n$ . (The quantifier “for large enough  $n$ ” means “for some  $n_0$ , for every  $n \geq n_0$ ”.)

Let the *suffix map*  $\text{suf}: X^{\leq \omega} \rightarrow \mathcal{H}(X)$  be defined by

$$\begin{aligned} \text{suf}(w) &:= \limsup_{n \in \mathbb{N}} \downarrow \{w_m \mid m \in \text{dom } w, m \geq n\} \\ &= \bigcap_{n \in \mathbb{N}} cl(\{w_m \mid m \in \text{dom } w, m \geq n\}). \end{aligned}$$

Note that  $\text{suf}(w)$  is empty for every finite word  $w$ . When  $X$  is Noetherian (or more generally, compact), this is an equivalence:  $\text{suf}(w) = \emptyset$  if and only if  $w \in X^*$ .

The *suffix topology* on  $X^{\leq \omega}$  is the coarsest topology that makes  $\text{suf}$  continuous.

LEMMA 5.6. *Let  $X$  be a Noetherian space. The suffix map  $\text{suf}$  is continuous from  $X^{\leq \omega}$  with its asymptotic subword topology to  $\mathcal{H}(X)$ . A subbase of the suffix topology is given by the sets  $\langle (\infty)U \rangle$ ,  $U$  open in  $X$ .*

*The suffix topology is coarser than the asymptotic subword topology.*

*Proof.* A subbase of the suffix topology is given by the sets  $\text{suf}^{-1}(\diamond U)$ ,  $U$  open in  $X$ . We claim that  $\text{suf}^{-1}(\diamond U) = \langle (\infty)U \rangle$ . This readily implies the first and the second claim, and the third one will be an immediate consequence.

Let  $w \in X^{\leq \omega}$ , and let  $n_0$  be such that  $\limsup_{n \in \mathbb{N}} \downarrow \{w_m \mid m \in \text{dom } w, m \geq n\} = cl(\{w_m \mid m \in \text{dom } w, m \geq n\})$  for every  $n \geq n_0$ . If  $w \in \text{suf}^{-1}(\diamond U)$ , then for every  $n \geq n_0$ ,  $cl(\{w_m \mid m \in \text{dom } w, m \geq n\})$  intersects  $U$ , so  $w_m \in U$  for some  $m \in \text{dom } w$  such that  $m \geq n$ . Hence  $w \in \langle (\infty)U \rangle$ . Conversely, if  $w \in \langle (\infty)U \rangle$ , then  $\text{dom } w = \mathbb{N}$  and there are infinitely many positions  $n \geq n_0$  where  $w_n$  is in  $U$ . Take one. Then  $cl(\{w_m \mid m \in \text{dom } w, m \geq n_0\})$  intersects  $U$  at  $w_n$ , showing that  $w$  is in  $\text{suf}^{-1}(\diamond U)$ . ■

PROPOSITION 5.7. *Let  $X$  be a Noetherian space. The asymptotic subword topology on  $X$  is the join of the prefix and the suffix topologies. The function  $\langle \text{pref}, \text{suf} \rangle: X^{\leq \omega} \rightarrow \mathcal{S}(X^*) \times \mathcal{H}(X)$  that maps  $w$  to  $(\text{pref}(w), \text{suf}(w))$  is initial.*

*Proof.* The asymptotic subword topology is finer than both the prefix and the suffix topologies, by Lemmas 5.5 and 5.6. Conversely, every subbasic open set  $\langle U_1; \dots; U_k \rangle$  is prefix-open, and every subbasic open set  $\langle U_1; \dots; U_k; (\infty)U \rangle$  is the intersection of the prefix-open set  $\langle U_1; \dots; U_k \rangle$  with the suffix-open set  $\langle (\infty)U \rangle$ .

By the first part of the lemma, the asymptotic subword topology on  $X^{\leq \omega}$  is the coarsest topology that makes both  $\text{pref}$  and  $\text{suf}$  continuous, hence the



coarsest topology that makes  $\langle \text{pref}, \text{suf} \rangle$  continuous. It follows that  $\langle \text{pref}, \text{suf} \rangle$  is initial. ■

Property (A) follows:

**THEOREM 5.8.** *For every space  $X$ ,  $X^{\leq\omega}$  (with the asymptotic subword topology) is Noetherian if and only if  $X$  is Noetherian, and similarly for  $X^\omega$ .*

*Proof.* If  $X$  is Noetherian, then so are  $X^*$ ,  $\mathcal{S}(X^*)$ ,  $\mathcal{H}(X)$  and their product  $\mathcal{S}(X^*) \times \mathcal{H}(X)$ , as we have already seen in Sections 2 and 3. Since  $f := \langle \text{pref}, \text{suf} \rangle$  is initial (Proposition 5.7), Lemma 4.2 ensures that  $X^{\leq\omega}$  and its subspace  $X^\omega$  are Noetherian in the asymptotic subword topology.

In the converse direction, we use the following argument, which works in both the  $X^{\leq\omega}$  and  $X^\omega$  cases. Let  $g$  be the function that maps every  $x \in X$  to the infinite word  $x^\omega := xx \cdots$  (in  $X^{\leq\omega}$ , resp.,  $X^\omega$ ). This is continuous since  $g^{-1}(\langle U_1; \dots; U_k \rangle) = U_1 \cap \cdots \cap U_k$  and  $g^{-1}(\langle U_1; \dots; U_k; (\infty)U \rangle) = U_1 \cap \cdots \cap U_k \cap U$ . With  $k := 0$ , we also find that every open subset  $U$  of  $X$  is obtained as  $g^{-1}(\langle (\infty)U \rangle)$ , so  $g$  is initial. By Lemma 4.2, if  $X^{\leq\omega}$  (resp.,  $X^\omega$ ) is Noetherian, then so is  $X$ . ■

From now on, and unless noted otherwise,  $X^{\leq\omega}$  (and  $X^\omega$ ) is endowed with the asymptotic subword topology.

**5.3. A few useful auxiliary results.** We pause for a moment, and establish two useful results.

**PROPOSITION 5.9.** *The concatenation map  $\text{cat}: X^* \times X^{\leq\omega} \rightarrow X^{\leq\omega}$  is continuous.*

*Proof.* Let  $(u, w)$  be any point in  $\text{cat}^{-1}(W)$ , where  $W = \langle U_1; \dots; U_k \rangle$  (resp.,  $W = \langle U_1; \dots; U_k; (\infty)U \rangle$ ), where  $U_1, \dots, U_k$ , and  $U$  are open in  $X$ . There is an index  $j$ ,  $0 \leq j \leq k$  such that  $u$  is in  $\langle U_1; \dots; U_j \rangle$ , and  $w$  is in  $\langle U_{j+1}; \dots; U_k \rangle$  (resp.,  $\langle U_{j+1}; \dots; U_k; (\infty)U \rangle$ ). Then  $\langle U_1; \dots; U_j \rangle \times \langle U_{j+1}; \dots; U_k \rangle$  (resp.,  $\langle U_1; \dots; U_j \rangle \times \langle U_{j+1}; \dots; U_k; (\infty)U \rangle$ ) is an open neighborhood of  $(u, w)$  that is included in  $\text{cat}^{-1}(W)$ . ■

The sets  $\langle U_1; \dots; U_k \rangle$  and  $\langle U_1; \dots; U_k; (\infty)U \rangle$  only form a subbase of  $X^\omega$ . We obtain a base as follows.  $\langle U_1; \dots; U_k; (\infty)V_1 \cap \cdots \cap \cdots (\infty)V_\ell \rangle$  denotes the set of finite-or-infinite words that contain letters from  $U_1, \dots, U_k$  in that order, followed by a suffix that contains infinitely many letters from  $V_1$ , infinitely many from  $V_2$ ,  $\dots$ , and infinitely many from  $V_\ell$ . We allow  $\ell$  to be equal to 0; if  $\ell \neq 0$ , then that set only contains infinite words.

**LEMMA 5.10.** *Let  $X$  be a Noetherian space. A base of the asymptotic subword topology on  $X^{\leq\omega}$  is given by the subsets  $\langle U_1; \dots; U_k; (\infty)V_1 \cap \cdots \cap \cdots (\infty)V_\ell \rangle$  where  $U_1, \dots, U_n, V_1, \dots, V_\ell$  are open in  $X$ .*

*Proof.* As a consequence of Proposition 5.7, a base is given by intersections of one element  $\langle U_1; \dots; U_k \rangle$  of the base of the prefix topology given in Lemma 5.5, and one element of a base of the suffix topology. For the latter, one can take finite intersections  $\langle (\infty)V_1 \rangle \cap \dots \cap \langle (\infty)V_\ell \rangle = \langle (\infty)V_1 \cap \dots \cap (\infty)V_\ell \rangle$ . ■

**5.4. The sobrification of  $X^{\leq\omega}$ .** We extend the notion of word product to *finite-or-infinite word products*:  $\omega$ -regular expressions of the form  $PF^{\leq\omega}$ , where  $P$  is a word product and  $F$  is a closed subset of  $X$ .  $PF^{\leq\omega}$  denotes the sets of finite or infinite words that are obtained as the concatenation of a finite word in  $P$  with a finite-or-infinite word whose letters are all in  $F$ . We also write  $PF^\omega$  for  $PF^{\leq\omega} \cap X^\omega$ : this is the set of infinite words obtained as the concatenation of a finite word in  $P$  with an infinite word whose letters are all in  $F$ . This is empty if  $F$  is empty. *Infinite word products* are expressions  $PF^\omega$  where  $P$  is a word product and  $F$  is a *non-empty* closed subset of  $X$ .

PROPOSITION 5.11. *Let  $X$  be a Noetherian space. Every irreducible closed subset of  $X^{\leq\omega}$  is an finite-or-infinite word product.*

*Proof.* Let  $D$  be an irreducible closed subset of  $X^{\leq\omega}$ . The map  $f := \langle \text{pref}, \text{suf} \rangle$  is initial by Proposition 5.7. We can therefore apply Lemma 4.3 to  $f$ , and we find that  $D$  is the inverse image of some irreducible closed subset of  $\mathcal{S}(X^*) \times \mathcal{H}(X)$  by  $f$ . Since the latter space is already sober ( $\mathcal{H}(X)$  is sober by Lemma 3.1, or by [13, Proposition 1.7]), an irreducible subset is just the downward closure of a point  $(P, F)$  of  $\mathcal{S}(X^*) \times \mathcal{H}(X)$ . Here  $P$  must be a word product  $A_1 \cdots A_N$ , and  $F$  must be a closed subset of  $X$ .

Hence  $D$  is the set of words  $w \in X^{\leq\omega}$  such that  $\text{pref}(w) \subseteq P$  and  $\text{suf}(w) \subseteq F$ , that is, all the finite prefixes of  $w$  are in  $P$  and the letters  $w_n$  are in  $F$  for  $n \in \text{dom } w$  large enough. There may be different choices of the pair  $(P, F)$ , and we choose one such that the number  $N$  of atoms in  $P$  is minimal, and such that given that  $N$ ,  $F$  is minimal with respect to inclusion. This is possible since  $X$  is Noetherian.

If  $F$  is empty, then  $D$  is the set of finite words  $w$  such that  $\text{pref}(w) \subseteq P$ , or equivalently such that every prefix of  $w$  is in  $P$ . Therefore  $D = P$ , and that can also be written as  $PF^{\leq\omega}$ , since  $F = \emptyset$ . Henceforth we assume that  $F$  is non-empty.

We note that

(\*)  $D$  is not included in  $X^*$ .

Indeed, otherwise, for every  $w \in D$ ,  $w$  would be finite, so  $\text{suf}(w)$  would be empty. It would follow that  $D \subseteq f^{-1}(\downarrow(P, \emptyset))$ , contrary to the minimality of  $F$ .

Let us write  $P$  as a product  $A_1 \cdots A_N$  of atoms, with  $N$  minimal. Since  $\emptyset^* = \{\epsilon\}$ , we may simply erase all atoms of the form  $A^*$  with  $A$  empty: since  $N$  is minimal, no  $A_i$  is of the form  $\emptyset^*$ . We claim that  $N \geq 1$  and that  $A_N$  is

of the form  $F'^*$  for some (necessarily non-empty) closed set  $F'$ . We cannot have  $N = 0$ , since that would imply that for every  $w \in D$ ,  $\text{pref}(w) = \{\epsilon\}$ , hence  $D = \{\epsilon\}$ ; this is impossible since  $\text{suf}(\{\epsilon\}) = \emptyset$ , contradicting  $F \neq \emptyset$ . Hence let us write  $P$  as  $P'A_N$ . We now assume that  $A_N$  is of the form  $C^2$  with  $C$  irreducible closed, and we aim for a contradiction. For every infinite word  $w \in D$ , we note that  $\text{pref}(w)$  is included in  $P'$ : for every finite prefix  $w_{<n}$  of  $w$ , the finite prefix  $w_{<n+1} = w_{<n}w_n$  is in  $\text{pref}(w)$ , hence in  $P = P'C^2$ , and that implies that  $w_{<n}$  is in  $P'$ ; since  $n$  is arbitrary,  $\{w_{<n} \mid n \in \text{dom } w\}$  is included in  $P'$ , and therefore  $\text{pref}(w) \subseteq P'$ , since  $P'$  is closed. Hence every infinite word in  $D$  is included in  $\text{pref}^{-1}(P')$ . Alternatively,  $D$  is included in the union of the set  $X^*$  of finite words and of  $\text{pref}^{-1}(P')$ . Since  $D$  is irreducible, and both  $X^*$  and  $\text{pref}^{-1}(P')$  are closed (the latter by Lemma 5.5),  $D$  must be included in one of them. By (\*),  $D$  is not included in  $X^*$ , so it is included in  $\text{pref}^{-1}(P')$ . It follows that  $D$  is the set of words  $w \in X^{\leq\omega}$  such that  $\text{pref}(w) \subseteq P'$  (not just  $P$ ) and  $\text{suf}(w) \subseteq F$ , and this contradicts the minimality of  $N$ .

We have shown that  $P$  is of the form  $P'F'^*$  for some non-empty closed subset  $F'$  of  $X$ . We now claim that  $F$  must be included in  $F'$ . For every infinite word  $w$  in  $D$ ,  $\text{pref}(w) \subseteq P$ , so every finite prefix of  $w$  is in  $P$ . Then, either every finite prefix of  $w$  is in  $P'$ —in which case  $\text{pref}(w) \subseteq P'$ —or there is a largest  $n_0 \in \mathbb{N}$  such that  $w_{<n_0}$  is in  $P'$ . In the latter case, every letter  $w_n$  with  $n \geq n_0$  must be in  $F'$ . For  $n_1$  large enough,  $\text{suf}(w) = \text{cl}(\{w_m \mid m \geq n_1\})$ . By picking  $n_1$  larger than  $n_0$ , every letter  $w_n$  with  $n \geq n_1$  is also in  $F$ . Hence  $w_n$  is in  $F \cap F'$  for every  $n \geq n_1$ , from which we deduce that  $\text{suf}(w) \subseteq F \cap F'$ . We have shown that every infinite word  $w$  in  $D$  is in  $\text{pref}^{-1}(\downarrow P') \cup \text{suf}^{-1}(\downarrow(F \cap F'))$ , or equivalently, that  $D$  is included in the union of  $X^*$ ,  $\text{pref}^{-1}(\downarrow P')$ , and  $\text{suf}^{-1}(\downarrow(F \cap F'))$ . Those three sets are closed, by Lemma 5.5 in the case of the second one, and Lemma 5.6 for the third one. Since  $D$  is irreducible, it must be included in one of them. It is not included in  $X^*$  by (\*). It is not included in  $\text{pref}^{-1}(\downarrow P')$ , otherwise  $D$  would be included in  $f^{-1}(\downarrow(P', F))$ , contradicting the minimality of  $N$ . Therefore  $D$  is included in  $\text{suf}^{-1}(\downarrow(F \cap F'))$ . This entails that  $D$  is included in  $f^{-1}(\downarrow(P, F \cap F'))$ . Since  $F$  is minimal,  $F = F \cap F'$ , hence  $F \subseteq F'$ .

Now that we know that  $D = f^{-1}(\downarrow(P'F'^*, F))$  with  $F \subseteq F'$ , we verify that  $D = P'F'^*F^{\leq\omega}$ . For every  $w \in D$ ,  $\text{suf}(w) \subseteq F$  so there is an index  $n_0$  such that, for every  $n \in \text{dom } w$  with  $n \geq n_0$ ,  $w_n$  is in  $F$ . If  $w$  is a finite word, we may take  $n_0$  equal to 1 plus the length of  $w$ . Whatever the case,  $w_{>n}$  is in  $F^{\leq\omega}$ . Since  $\text{pref}(w) \subseteq P'F'^*$ ,  $w_{<n}$  is in  $P'F'^*$ , so  $w$  is in  $P'F'^*F^{\leq\omega}$ . Conversely, let  $w \in P'F'^*F^{\leq\omega}$ . If  $w$  is finite, then it is in  $P'F'^*F^* \subseteq P'F'^*$  (since  $F \subseteq F'$ ), so  $\text{pref}(w) \subseteq P'F'^*$ ; also,  $\text{suf}(w) = \emptyset \subseteq F$ , thus  $w$  is in  $f^{-1}(\downarrow(P'F'^*, F)) = D$ . Hence let us assume that  $w$  is infinite. There is an

$n_0 \in \mathbb{N}$  such that  $w_{<n_0} \in P'F'^*$  and  $w_n \in F$  for every  $n \geq n_0$ . In particular,  $\text{suf}(w)$  is included in  $\text{cl}(\{w_n \mid n \in \text{dom } w, n \geq n_0\})$ , hence in  $F$ . For every finite prefix  $w_{<n}$  of  $w$ , either  $n \leq n_0$ , in which case  $w_{<n}$  is a subword of  $w_{<n_0}$ , hence is in  $P'F'^*$ , or  $n > n_0$ , in which case we can write  $w_{<n}$  as  $w_{<n_0}w'$  where  $w' \in F^*$ . Since  $F \subseteq F'$ ,  $w_{<n}$  is then again in  $P'F'^*$ . It follows that  $\text{pref}(w) \subseteq P'F'^*$ , hence  $f(w) \in \downarrow(P'F'^*, F)$ , so  $w \in D$ . ■

The last part of the previous proof shows the useful fact that for every word product  $P$ , for all closed subsets  $F$  and  $F'$  of  $X$  such that  $F \subseteq F'$ ,  $PF'^*F^{\leq\omega} = \langle \text{pref}, \text{suf} \rangle^{-1}(\downarrow(PF'^*, F))$ . When  $F = F'$ , and since  $PF^*F^{\leq\omega} = PF^{\leq\omega}$  (resp.,  $PF^\omega = PF^{\leq\omega} \cap X^\omega$ ), we obtain:

**FACT 5.12.** *Let  $X$  be a Noetherian space. For every word product  $P$ , and every closed subset  $F$  of  $X$ ,  $PF^{\leq\omega} = \langle \text{pref}, \text{suf} \rangle^{-1}(\downarrow(PF^*, F))$ . In particular, every finite-or-infinite word product  $PF^{\leq\omega}$  is closed in  $X^{\leq\omega}$  (resp., every infinite word product  $PF^\omega$  is closed in  $X^\omega$ ). ■*

Recall that we have required the closed set  $F$  to be non-empty in infinite products  $PF^\omega$ . This is unimportant in Fact 5.12, since  $P\emptyset^\omega = \emptyset$  is closed anyway, but it matters in the following.

**THEOREM 5.13.** *Let  $X$  be a Noetherian space. The irreducible closed subsets of  $X^{\leq\omega}$  (resp.,  $X^\omega$ ) are the finite-or-infinite word products (resp., the infinite word products).*

*Proof.* We deal with  $X^{\leq\omega}$  first. Considering Proposition 5.11 and Fact 5.12, it remains to show that every finite-or-infinite word product  $PF^{\leq\omega}$  is irreducible (where  $P$  is a word product, and  $F$  is closed in  $X$ ).

We start by showing that  $F^{\leq\omega}$  is irreducible in  $X^{\leq\omega}$ . Actually, it is *directed* with respect to the quasi-ordering  $\leq^\omega$ . In other words, we show that  $F^{\leq\omega}$  is non-empty (which is clear, since it contains the empty word  $\epsilon$ ), and that any two elements  $w_1$  and  $w_2$  of  $F^{\leq\omega}$  have an upper bound in  $F^{\leq\omega}$  with respect to  $\leq^\omega$ . For that upper bound, we can simply take the concatenation  $w_1w_2$  if  $w_1$  is finite (or  $w_2w_1$  if  $w_2$  is finite), and the one-for-one interleaving of  $w_1$  and  $w_2$  if both are infinite (i.e., the letters at even positions are those from  $w_1$ , the letters at odd positions are those from  $w_2$ ). Every set that is directed with respect to  $\leq^\omega$  is irreducible: if it intersects two open sets  $U_1$  (say at  $w_1$ ) and  $U_2$  (say at  $w_2$ ), then it intersects  $U_1 \cap U_2$  (at the chosen upper bound of  $w_1$  and  $w_2$  in the directed set), by Fact 5.2.

Given any word product  $P$ , we know that  $P$  is irreducible closed in  $X^*$ , so  $P \times F^{\leq\omega}$  is irreducible closed in  $X^* \times X^{\leq\omega}$ . Then, using the fact that  $\text{cat}$  is continuous (Proposition 5.9),  $\text{cl}(\text{cat}[P \times F^{\leq\omega}])$  is irreducible closed. (Recall from Section 2 that  $\text{cl}(f[C]) = \mathcal{S}(f)(C)$  is irreducible closed for every irreducible closed subset  $C$  and every continuous map  $f$ .) Evidently,

$\text{cat}[P \times F^{\leq \omega}] = PF^{\leq \omega}$ , and  $\text{cl}(PF^{\leq \omega}) = PF^{\leq \omega}$  by Fact 5.12, so  $PF^{\leq \omega}$  is irreducible closed in  $X^{\leq \omega}$ .

In the case of  $X^\omega$ , the inclusion map  $i$  of  $X^\omega$  into  $X^{\leq \omega}$  is a topological embedding (hence initial), and its image  $X^\omega = \langle (\infty)X \rangle$  is open. By Lemma 4.4(1), the irreducible closed subsets of  $X^\omega$  are the sets of the form  $i^{-1}(PF^{\leq \omega}) = PF^{\leq \omega} \cap X^\omega$  ( $P$  word product,  $F$  closed) such that  $PF^{\leq \omega}$  intersects  $X^\omega$ . Those are the sets of the form  $PF^\omega$  where, additionally,  $F$  is non-empty. ■

### 5.5. The specialization preordering on $X^{\leq \omega}$ and property (B).

Our first step in characterizing the specialization preordering on  $X^{\leq \omega}$  is to characterize the downward closure, equivalently, the closure, of its points.

LEMMA 5.14. *Let  $X$  be a Noetherian space. Given a fixed word  $w \in X^{\leq \omega}$ , let  $n_0$  be such that  $\text{suf}(w) = \text{cl}(\{w_m \mid m \in \text{dom } w, m \geq n\})$  for every  $n \geq n_0$ . For every  $n \geq n_0$ ,*

- (1) *the closure of  $w$  in  $X^{\leq \omega}$  is  $(\downarrow w_0)^?(\downarrow w_1)^? \cdots (\downarrow w_{n-1})^?(\text{suf}(w))^{\leq \omega}$ ;*
- (2) *if  $w \in X^\omega$ , then its closure in  $X^\omega$  is  $(\downarrow w_0)^?(\downarrow w_1)^? \cdots (\downarrow w_{n-1})^?(\text{suf}(w))^\omega$ .*

*Proof.* Let  $P := (\downarrow w_0)^?(\downarrow w_1)^? \cdots (\downarrow w_{n-1})^?$ ,  $F := \text{suf}(w)$ , and  $C := PF^{\leq \omega}$ .

(1)  $C$  certainly contains  $w$ , and is closed by Fact 5.12, so  $\text{cl}(\{w\}) \subseteq C$ . (We write  $\text{cl}$  for closure in  $X^{\leq \omega}$  here.)

We turn to the converse implication. Since  $\text{cl}(\{w\})$  is irreducible closed (the closures of points are always irreducible closed), it must be a finite-or-infinite word product  $P'F'^{\leq \omega}$  by Theorem 5.13. Further,  $P'F'^{\leq \omega}$  is equal to  $(\text{pref}, \text{suf})^{-1}(\downarrow(P'F'^*, F'))$  by Fact 5.12. Since  $w$  is in its closure  $P'F'^{\leq \omega}$ ,  $\text{pref}(w)$  is included in  $P'F'^*$  and  $\text{suf}(w)$  is included in  $F'$ . The latter means that  $F \subseteq F'$ . Using the former, we claim that  $PF^* \subseteq P'F'^*$ . Since  $F \subseteq F'$ , it is equivalent to show that  $P \subseteq P'F'^*$ . That is obvious, since every element of  $P$  is a subword of  $w_{<n}$ , hence a subword of a finite prefix of  $w$ , hence belongs to  $\text{pref}(w)$ , which is included in  $P'F'^*$ .

Now that  $PF^* \subseteq P'F'^*$  and  $F \subseteq F'$ ,  $C = (\text{pref}, \text{suf})^{-1}(\downarrow(PF^*, F))$  (Fact 5.12) is included in  $(\text{pref}, \text{suf})^{-1}(\downarrow(P'F'^*, F')) = \text{cl}(\{w\})$ .

(2) The closure of  $w$  in  $X^\omega$  is  $\text{cl}(\{w\}) \cap X^\omega = PF^{\leq \omega} \cap X^\omega = PF^\omega$ . ■

Lemma 5.14 yields a description of the specialization preordering of  $X^{\leq \omega}$  and of  $X^\omega$ , since  $w'$  is below  $w$  in that ordering if and only if  $w'$  is in the closure of  $w$ . That is far from explicit.

We can improve on that situation when  $X$  is a wqo, obtaining an analogue of property (B) for  $X^{\leq \omega}$  and  $X^\omega$ .

LEMMA 5.15. *If  $X$  is a wqo, then for every  $w \in X^{\leq \omega}$ ,  $\text{suf}(w)$  is the set of letters that are below infinitely many letters from  $w$ , and is equal to  $\bigcup_{m \in \text{dom } w, m \geq n} \downarrow w_m$  for  $n$  large enough.*

*Proof.* If  $w$  is finite, then  $\text{suf}(w)$  is empty, and the claim is clear. Let us assume that  $w$  is an infinite word. Since  $X$  is a wqo, for every  $n \in \mathbb{N}$ ,  $\text{cl}(\{w_m \mid m \geq n\})$  is equal to  $\downarrow\{w_m \mid m \geq n\} = \bigcup_{m \geq n} \downarrow w_m$ . Hence  $\text{suf}(w) = \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} \downarrow w_m$  is the set of letters that are below infinitely many letters from  $w$ . Since  $\text{suf}(w) = \downarrow\{w_m \mid m \geq n\}$  for  $n$  large enough, it is also equal to  $\bigcup_{m \geq n} \downarrow w_m$  for  $n$  large enough. ■

**PROPOSITION 5.16.** *If  $X$  is a wqo, then the specialization preordering on  $X^{\leq \omega}$  is the subword preordering  $\leq^\omega$ .*

*Proof.* Let us fix  $w \in X^{\leq \omega}$ . It suffices to show that the closure of  $w$  is exactly the set of finite-or-infinite subwords of  $w$ . By Fact 5.2, every subword of  $w$  is in the closure of  $w$ . Conversely, let  $w'$  be any element of the closure of  $w$ . By Lemmas 5.14 and 5.15, there is a natural number  $n_0$  such that for every  $n \geq n_0$ ,  $\text{suf}(w) = \bigcup_{m \in \text{dom } w, m \geq n} \downarrow w_m$ , and then  $w'$  is in  $(\downarrow w_0)^?(\downarrow w_1)^? \cdots (\downarrow w_{n_0-1})^?(\text{suf}(w))^{\leq \omega}$ .

We first use this formula for  $\text{suf}(w)$  with  $n := n_0$ . Then  $w' = us$  where  $u$  is in  $(\downarrow w_0)^?(\downarrow w_1)^? \cdots (\downarrow w_{n_0-1})^?$ , hence is a subword of  $w_{<n_0}$ , and  $s$  is in  $(\text{suf}(w))^{\leq \omega}$ . The latter means that  $s$  is an (infinite) word whose letters are all in  $\text{suf}(w)$ . We claim that  $s$  is a subword of  $w_{\geq n_0}$ , namely there is a monotonic injective map  $f: \text{dom } s \rightarrow \text{dom } w_{\geq n_0}$  such that  $s_i \leq w_{\geq n_0}(f(i)) = w_{f(i)+n_0}$  for every  $i \in \text{dom } s$ . We define  $f(i)$  by induction on  $i \in \text{dom } s$  as follows. If  $i = 0$  is in  $\text{dom } s$ , then  $s_0$  is in  $\text{suf}(w)$ . Using the formula for  $\text{suf}(w)$  with  $n := n_0$ ,  $s_0$  is in  $\downarrow w_m$  for some  $m \geq n_0$ . We define  $f(0)$  as  $m - n_0$ . For every non-zero  $i \in \text{dom } s$ , and remembering that  $f(i-1)$  is already defined by induction hypothesis, we use the formula for  $\text{suf}(w)$  with  $n := f(i-1) + n_0 + 1$ . Then  $s_{i+1}$  is in  $\downarrow w_m$  for some  $m > f(i-1) + n_0$ , and we let  $f(i)$  be  $m - n_0$ , so that  $f(i) > f(i-1)$  and  $s_i \leq w_{f(i)+n_0}$ .

This establishes that  $s$  is a subword of  $w_{\geq n_0}$ . It follows that  $w' = us$  is a subword of  $w_{<n_0}w_{\geq n_0} = w$ . ■

**5.6. Property (C).** We now investigate when  $X^{\leq \omega}$  and  $X^\omega$  are themselves wqos. In particular, we ask when their topology is Aleksandrov. As with powersets, this is a different question from asking when  $\leq^\omega$  is a well-quasi-ordering on  $X^\omega$ , which is equivalent to  $\leq$  being an  $\omega^2$ -wqo. (That equivalence is the special case  $\alpha = \omega^2$  of [10, Theorem 2.8], but be aware that what Marcone calls  $\alpha$ -wqo is what we call  $\omega^\alpha$ -wqo—some authors also use the term  $\omega^\alpha$ -bqo.)

**PROPOSITION 5.17.** *If  $X$  is essentially finite, then the asymptotic subword topology on  $X^{\leq \omega}$  (resp.,  $X^\omega$ ) is the Aleksandrov topology of  $\leq^\omega$ .*

*Proof.* We only deal with  $X^{\leq \omega}$ . The case of  $X^\omega$  will follow, because the subspace topology of a space with the Aleksandrov topology of a preordering  $\preceq$  is the Aleksandrov topology of the restriction of  $\preceq$  to the subspace.

Considering Proposition 5.16, it suffices to show that every upwards-closed subset of  $X^{\leq\omega}$ , with respect to  $\leq^\omega$ , is open in the asymptotic subword topology. To that end, it suffices to show that the upward closure  $\uparrow^\omega w$  of any  $w \in X^{\leq\omega}$  with respect to  $\leq^\omega$  is open, since every upwards-closed set is a union of such upward closures.

If  $w$  is finite, then  $\uparrow^\omega w = \langle \uparrow w_0; \uparrow w_1; \dots; \uparrow w_{n-1} \rangle$  where  $n$  is the length of  $w$ . Note that each set  $\uparrow w_i$  is open in  $X$ , because the topology of an essentially finite space is always Aleksandrov.

Let us assume  $w$  infinite. Since  $X$  is essentially finite, there are only finitely many distinct sets of the form  $\uparrow w_n$ ,  $n \in \mathbb{N}$ . Some of them occur at only finitely many positions  $n$  in  $w$ ; let  $n_0$  be any index exceeding all those positions. Then every  $\uparrow w_n$ ,  $n \geq n_0$ , is also equal to  $\uparrow w_m$  for infinitely many indices  $m \geq n_0$ . Let  $\{V_1, \dots, V_\ell\}$  be the (finite, non-empty) set  $\{\uparrow w_n \mid n \geq n_0\}$ , and let  $U := \langle \uparrow w_0; \uparrow w_1; \dots; \uparrow w_{n_0-1}; (\infty)V_1 \cap \dots \cap (\infty)V_\ell \rangle$ . This is an open set in the asymptotic subword topology. It contains  $w$ , by construction. In view of Fact 5.2,  $\uparrow^\omega w$  is entirely included in  $U$ . Conversely, let  $w'$  be any element of  $U$ . Then  $w' = us$  where  $u$  is a finite word that contains a letter above  $w_0$ , a later letter above  $w_1, \dots$ , and a letter above  $w_{n_0-1}$ , and  $s \in X^\omega$  contains infinitely many letters from each of  $V_1, \dots, V_\ell$ —in other words, for every  $n \geq n_0$ ,  $s$  contains infinitely many letters above  $w_n$ . Hence  $s$  contains a letter above  $w_{n_0}$ , then a later letter above  $w_{n_0+1}$ , etc., so  $w_{\geq n_0} \leq^\omega s$ . It follows that  $w \leq^\omega w'$ . Therefore  $U \subseteq \uparrow^\omega w$ , whence equality follows. ■

**PROPOSITION 5.18.** *Let  $X$  be a Noetherian space. The asymptotic subword topology on  $X^{\leq\omega}$  (resp.,  $X^\omega$ ) is Aleksandrov if and only if  $X$  is essentially finite.*

*Proof.* One direction is by Proposition 5.17. In the converse direction, we assume that  $X^\omega$  is Aleksandrov, and we wish to show that  $X$  is essentially finite. The case of  $X^{\leq\omega}$  reduces to that case: if  $X^{\leq\omega}$  is Aleksandrov, so is its subspace  $X^\omega$ .

Let  $C_0 \subsetneq C_1 \subsetneq \dots$  be a strictly ascending sequence of closed subsets of  $X$ , and let  $x_n$  be a point of  $C_{n+1} \setminus C_n$  for every  $n \in \mathbb{N}$ . For each  $n \in \mathbb{N}$ ,  $C_n^\omega$  is closed in  $X^\omega$ , by Theorem 5.13. Let  $\mathcal{C} := \bigcup_{n \in \mathbb{N}} C_n^\omega$ : since the topology of  $X^\omega$  is Aleksandrov, this is again closed.

Note that  $w := x_0 x_1 \dots x_n \dots$  is in no  $C_n^\omega$ , hence not in  $\mathcal{C}$ . By Lemma 5.10, there is a basic open subset  $W := \langle U_1; \dots; U_k; (\infty)V_1 \cap \dots \cap (\infty)V_\ell \rangle$  of  $X^\omega$  that contains  $w$  and is disjoint from  $\mathcal{C}$ . Since it contains  $w$ , we can write  $w$  as  $w_{<n_0} w_{\geq n_0}$  where  $w_{<n_0} \in \langle U_1; \dots; U_k \rangle$  and  $w_{\geq n_0}$  contains infinitely many letters from each of  $V_1, \dots, V_\ell$ . Let us pick one letter  $x_{n_1}$  from  $w_{\geq n_0}$  in  $V_1, \dots$ , one letter  $x_{n_\ell}$  from  $w_{\geq n_0}$  in  $V_\ell$ . Then the infinite word  $w_{<n_0}(x_{n_1} \dots x_{n_\ell})^\omega$  is in  $W$ , but it is also in  $C_{n+1}^\omega$ , where  $n$  is any natural number exceeding  $\max(n_0, n_1, \dots, n_\ell)$ . In particular,  $W$  intersects  $\mathcal{C}$ , which is impossible.

We conclude that there cannot be any infinite strictly ascending sequence of closed subsets of  $X$ . By Lemma 3.3,  $X$  must be essentially finite. ■

### 5.7. An S-representation on $X^{\leq\omega}$ and on $X^\omega$ , and property (D).

Testing inclusion of finite-or-infinite word products is as easy as testing inclusion of finite word products.

LEMMA 5.19. *Let  $X$  be a Noetherian space.*

- (1) *For all finite-or-infinite word products  $PF^{\leq\omega}$  and  $P'F'^{\leq\omega}$ ,  $PF^{\leq\omega} \subseteq P'F'^{\leq\omega}$  if and only if  $PF^* \subseteq P'F'^*$  and  $F \subseteq F'$ .*
- (2) *For all infinite word products  $PF^\omega$  and  $P'F'^\omega$ ,  $PF^\omega \subseteq P'F'^\omega$  if and only if  $PF^* \subseteq P'F'^*$  and  $F \subseteq F'$ .*

Recall that  $F$  and  $F'$  are required to be non-empty in infinite word products, but not in finite-or-infinite word products.

*Proof.* We first show that

- (i) if  $F$  is non-empty, then  $PF^\omega \subseteq P'F'^\omega$  implies  $PF^* \subseteq P'F'^*$  and  $F \subseteq F'$ .

Henceforth, let us assume  $PF^\omega \subseteq P'F'^\omega$ . We first show that  $F$  is included in  $F'$ . Since  $F$  is non-empty, we can pick an element  $x$  from  $F$ . Note that the empty word  $\epsilon$  is in  $P$ . Hence  $x^\omega (= \epsilon x^\omega)$  is in  $PF^\omega$ , and therefore in  $P'F'^\omega$ . This means that we can write  $x^\omega$  as  $uv$  where  $u \in P'$  and  $v \in F'^\omega$ ; the latter, together with the fact that  $v = x^\omega$ , implies  $x \in F'$ . It follows that  $F \subseteq F'$ . We now claim that  $PF^* \subseteq P'F'^*$ . Let us pick any finite word  $w$  from  $PF^*$ . The infinite word  $wx^\omega$  is in  $PF^\omega$ , hence in  $P'F'^\omega$ . It follows that  $wx^\omega$  is of the form  $uv$  where  $u \in P'$  and  $v \in F'^\omega$ . If  $u$  is a prefix of  $w$ , then  $w$  is equal to the concatenation of  $u$  with a prefix of  $v$ , hence is in  $P'F'^*$ . Otherwise,  $w$  is a prefix of  $u$ . Since  $u$  is in  $P'$ , it is also in  $P'F'^*$ , and since the latter is closed under  $\leq^*$ ,  $w$  is also in  $P'F'^*$ .

We now deduce that

- (ii)  $PF^{\leq\omega} \subseteq P'F'^{\leq\omega}$  implies  $PF^* \subseteq P'F'^*$  and  $F \subseteq F'$ .

If  $F$  is non-empty, then since  $PF^\omega = PF^{\leq\omega} \cap X^\omega \subseteq P'F'^{\leq\omega} \cap X^\omega = P'F'^\omega$ , we can use (i) and conclude. Otherwise, since  $F = \emptyset$ ,  $PF^{\leq\omega} = P$ , so  $P$  is included in  $P'F'^{\leq\omega}$ . Since all the elements of  $P$  are finite words,  $P$  is in fact included in  $P'F'^{\leq\omega} \cap X^* = P'F'^*$ . The inequality  $F \subseteq F'$  is trivial.

In the converse direction, we have

- (iii) if  $PF^* \subseteq P'F'^*$  and  $F \subseteq F'$ , then  $PF^{\leq\omega} \subseteq P'F'^{\leq\omega}$ .

Indeed,

$$\begin{aligned} PF^{\leq\omega} &= \langle \text{pref}, \text{suf} \rangle^{-1}(\downarrow(PF^*, F)) && \text{(Fact 5.12)} \\ &\subseteq \langle \text{pref}, \text{suf} \rangle^{-1}(\downarrow(P'F'^*, F')) = P'F'^{\leq\omega}. \end{aligned}$$

Finally,



(iv) if  $PF^* \subseteq P'F'^*$  and  $F \subseteq F'$ , then  $PF^\omega \subseteq P'F'^\omega$ .

This follows from (iii) by taking intersections with  $X^\omega$ .

Item (1) follows from (ii) and (iii). Item (2) follows from (i) and (iv). ■

Computing intersections of infinite word products also reduces to the case of finite word products, as we will see in Lemma 5.21 below. We notice the following. The point is that  $F_i$  is the same for every  $i$ .

LEMMA 5.20. *Let  $X$  be a Noetherian space. For all finite word products  $P, P'$ , and all closed subsets  $F, F'$  of  $X$ ,  $PF^* \cap P'F'^*$  is a finite union of word products  $P_i F_i^*$  where  $F_i = F \cap F'$  for every  $i$ .*

*Proof.* Since  $X^*$  is Noetherian,  $PF^* \cap P'F'^*$  is a finite union  $\bigcup_{i=1}^n P_i$  where each  $P_i$  is a word product. For every  $i$ ,  $1 \leq i \leq n$ , every  $w \in P_i$ , and every  $w' \in (F \cap F')^*$ ,  $ww'$  is in  $PF^*(F \cap F')^* \subseteq PF^*$  and in  $P'F'^*(F \cap F')^* \subseteq P'F'^*$ , hence in  $PF^* \cap P'F'^*$ , and therefore in some  $P_j$ ,  $1 \leq j \leq n$ . It follows that  $\bigcup_{i=1}^n P_i (F \cap F')^* \subseteq \bigcup_{i=1}^n P_i$ . The converse inclusion is obvious. ■

LEMMA 5.21. *Let  $X$  be a Noetherian space. Given any two finite-or-infinite word products  $PF^{\leq \omega}$  and  $P'F'^{\leq \omega}$ , one can write  $PF^* \cap P'F'^*$  as a finite union of finite word products of the form  $P_i (F \cap F')^*$ ,  $1 \leq i \leq n$ , and so  $PF^{\leq \omega} \cap P'F'^{\leq \omega} = \bigcup_{i=1}^n P_i (F \cap F')^{\leq \omega}$ .*

*Proof.* The fact that  $PF^* \cap P'F'^*$  can be written as a finite union of finite word products of the form  $P_i (F \cap F')^*$ ,  $1 \leq i \leq n$ , follows from Lemma 5.20.

Let  $\mathcal{F} := \{C \in \mathcal{S}(X^*) \mid C \subseteq PF^* \cap P'F'^*\}$ . We claim that  $\mathcal{F} = \bigcup_{i=1}^n \downarrow_{\mathcal{S}(X^*)}(P_i (F \cap F')^*)$ . For every  $C \in \mathcal{F}$ ,  $C$  is included in  $PF^* \cap P'F'^* = \bigcup_{i=1}^n P_i (F \cap F')^*$ , hence in some  $P_i (F \cap F')^*$  because  $C$  is irreducible. Hence  $C$  is also in  $\bigcup_{i=1}^n \downarrow_{\mathcal{S}(X^*)}(P_i (F \cap F')^*)$ . Conversely, every element  $C$  of  $\bigcup_{i=1}^n \downarrow_{\mathcal{S}(X^*)}(P_i (F \cap F')^*)$  is included in some  $P_i (F \cap F')^*$ , hence in  $PF^* \cap P'F'^*$ .

Then  $\downarrow(PF^*, F) \cap \downarrow(P'F'^*, F') = \mathcal{F} \times \downarrow_{\mathcal{H}(X)}(F \cap F')$ , so

$$\begin{aligned} PF^{\leq \omega} \cap P'F'^{\leq \omega} &= \langle \text{pref, suf} \rangle^{-1}(\downarrow(PF^*, F)) \cap \langle \text{pref, suf} \rangle^{-1}(\downarrow(P'F'^*, F')) \\ &\quad \text{(by Fact 5.12)} \\ &= \langle \text{pref, suf} \rangle^{-1}(\mathcal{F} \times \downarrow_{\mathcal{H}(X)}(F \cap F')) \\ &= \langle \text{pref, suf} \rangle^{-1}\left(\bigcup_{i=1}^n \downarrow(P_i (F \cap F')^*, F \cap F')\right) \\ &= \bigcup_{i=1}^n P_i (F \cap F')^{\leq \omega}, \end{aligned}$$

by Fact 5.12 again. ■

We have a similar result for infinite words. We only have to pay attention that  $P_i(F \cap F')^*$  is an infinite word product (an irreducible closed set) if and only if  $F \cap F'$  is non-empty.

LEMMA 5.22. *Let  $X$  be a Noetherian space. Given any two infinite word products  $PF^\omega$  and  $P'F'^\omega$ , either*

- $F \cap F'$  is empty and  $PF^\omega \cap P'F'^\omega = \emptyset$ ; or
- $F \cap F'$  is non-empty, and one can write  $PF^* \cap P'F'^*$  as a finite union of finite word products of the form  $P_i(F \cap F')^*$ ,  $1 \leq i \leq n$ ; in that case,  $PF^\omega \cap P'F'^\omega = \bigcup_{i=1}^n P_i(F \cap F')^\omega$ .

*Proof.* If  $F \cap F'$  is empty, then it is clear that  $PF^\omega \cap P'F'^\omega = \emptyset$ . Otherwise,

$$\begin{aligned} PF^\omega \cap P'F'^\omega &= PF^{\leq \omega} \cap P'F'^{\leq \omega} \cap X^\omega \\ &= \bigcup_{i=1}^n P_i(F \cap F')^{\leq \omega} \cap X^\omega \quad (\text{by Lemma 5.21}) \\ &= \bigcup_{i=1}^n P_i(F \cap F')^\omega. \blacksquare \end{aligned}$$

Let us turn to actual S-representations. We assume an S-representation  $(S, [\_], \leq, \tau, \wedge)$  of  $X$ . For every finite subset  $u := \{a_1, \dots, a_n\}$  of  $S$ , let  $\llbracket u \rrbracket$  denote  $\llbracket a_1 \rrbracket \cup \dots \cup \llbracket a_n \rrbracket$ . For all finite subsets  $u$  and  $v$  of  $S$ , we also write  $u \wedge v$  for  $\bigcup_{a \in u, b \in v} a \wedge b$ , so that  $\llbracket u \wedge v \rrbracket = (\bigcup_{a \in u} \llbracket a \rrbracket) \cap (\bigcup_{b \in v} \llbracket b \rrbracket) = \llbracket u \rrbracket \cap \llbracket v \rrbracket$ .

Lemma 5.20 has the following computable equivalent, which says that for syntactic word products  $Pu^*$  and  $Qv^*$ ,  $Pu^* \wedge' Qv^*$  computes the intersection  $\llbracket Pu^* \rrbracket' \cap \llbracket Qv^* \rrbracket' = \llbracket P \rrbracket' \llbracket u \rrbracket^* \cap \llbracket Q \rrbracket' \llbracket v \rrbracket^*$  as a finite set of syntactic word products of the form  $P_i(u \wedge v)^*$ . For this result, we need to use the optimized version of  $\wedge'$  described in Remark 2.2.

LEMMA 5.23. *Let  $X$  be a Noetherian space, and  $(S, [\_], \leq, \tau, \wedge)$  be an S-representation of  $X$ . For all (syntactic) word products of the form  $Pu^*$  and  $Qv^*$ , their intersection, as computed using Proposition 2.1(5), and removing subsumed word products as per Remark 2.2, is a finite set of word products of the form  $R(u \wedge v)^*$ .*

*Proof.* We use induction on the sum of the length  $n$  of  $P$  and the length  $n'$  of  $Q$ . This is a direct appeal to the induction hypothesis if  $n \geq 1$  and  $n' \geq 1$ . The interesting case is when  $n' = 0$  (or, symmetrically,  $n = 0$ ). If  $n = n' = 0$ , then  $Pu^* \wedge' Qv^* = u^* \wedge' v^* = \{(u \wedge v)^* R \mid R \in (\varepsilon \wedge' v^*) \cup (u^* \wedge' \varepsilon)\} = \{(u \wedge v)^*\}$ , by (2.4) and (2.1). If  $n \geq 1$  and  $n' = 0$ , then we need to show the claim for intersections of the form (1)  $a^? Pu^* \wedge' v^*$  and (2)  $u_0^* Pu^* \wedge' v^*$ .

In case (1), we use (2.3): the elements of  $a^? Pu^* \wedge' v^*$  are of the form  $c^? R$  (where  $c$  ranges over  $u \wedge v$  if  $u \wedge v \neq \emptyset$ ) or just  $R$  (if  $u \wedge v = \emptyset$ ), where

$R \in Pu^* \wedge' v^*$ , plus elements of  $a^?Pu^* \wedge' \varepsilon = \{\varepsilon\}$ . Since we remove subsumed word products, as per Remark 2.2, the latter elements do not occur. The elements that remain are of the form  $c^?R$  or just  $R$ , where  $R \in Pu^* \wedge' v^*$  has the required form by induction hypothesis.

In case (2), we use (2.4): the elements of  $u_0^*Pu^* \wedge' v^*$  are of the form  $(u_0 \wedge v)^*R$  where  $R \in (Pu^* \wedge' v^*) \cup (u_0^*Pu^* \wedge' \varepsilon) = (Pu^* \wedge' v^*) \cup \{\varepsilon\}$ . Let us enumerate  $Pu^* \wedge' v^*$ : by induction hypothesis, we can write its elements as  $R_1(u \wedge v)^*, \dots, R_n(u \wedge v)^*$ . We note that  $n \geq 1$ , because  $\llbracket Pu^* \rrbracket' \cap \llbracket v^* \rrbracket$  is non-empty: indeed, that intersection contains the empty word  $\varepsilon$ . Then the element  $(u_0 \wedge v)^*\varepsilon$  of the set  $u_0^*Pu^* \wedge' v^*$  is subsumed by, say,  $(u_0 \wedge v)^*R_1(u \wedge v)^*$ , and will be removed, following Remark 2.2. The only elements that remain are  $(u_0 \wedge v)^*R_i(u \wedge v)^*$  when  $i$  varies over some subset of  $\{1, \dots, n\}$ , and they are of the required form. ■

Instead of redesigning an S-representation for  $X^{\leq\omega}$  (or  $X^\omega$ ) from scratch, this allows us to reuse most of what we know for  $X^*$ . Item (3) below is justified by Lemma 5.19, and item (5) by Lemma 5.21 (resp., Lemma 5.22), refined using Lemma 5.23 (i.e., every element of  $Pu^* \wedge' Qv^*$  is of the form  $R(u \wedge v)^*$  for some  $R$ , where  $u \wedge v := \bigcup_{a \in u, b \in v} (a \wedge b)$ ).

**THEOREM 5.24.** *Given an S-representation  $(S, \llbracket \_ \rrbracket, \leq, \tau, \wedge)$  of a Noetherian space  $X$ , let  $(S', \llbracket \_ \rrbracket', \leq', \tau', \wedge')$  be the S-representation of  $X^*$  given in Proposition 2.1, with the optimization of Remark 2.2. Then the following tuple  $(S'', \llbracket \_ \rrbracket'', \leq'', \tau'', \wedge'')$  is an S-representation of  $X^{\leq\omega}$  (resp.,  $X^\omega$ ):*

- (1)  $S''$  is the collection of pairs  $(P, u)$  where  $P \in S'$  and  $u$  is a finite (resp., and non-empty) subset of  $S$ .
- (2)  $\llbracket (P, u) \rrbracket'' = \llbracket P \rrbracket' (\bigcup_{a \in u} \llbracket a \rrbracket)^{\leq\omega}$  (resp.,  $\llbracket P \rrbracket' (\bigcup_{a \in u} \llbracket a \rrbracket)^\omega$ ).
- (3)  $(P, u) \leq'' (Q, v)$  if and only if  $Pu^* \leq' Qv^*$  and for every  $a \in u$ , there is a  $b \in v$  such that  $a \leq b$ .
- (4)  $\tau''$  is  $(\varepsilon, \tau)$ .
- (5)  $(P, u) \wedge'' (Q, v)$  is defined as follows: writing  $u \wedge v$  for  $\bigcup_{a \in u, b \in v} (a \wedge b)$ ,  $Pu^* \wedge' Qv^*$  is a finite set of finite word products of the form  $R_i(u \wedge v)^*$ ,  $1 \leq i \leq n$ , and  $(P, u) \wedge'' (Q, v)$  is then the set of pairs  $(R_i, u \wedge v)$ ,  $1 \leq i \leq n$  (resp., the same formula if  $u \wedge v \neq \emptyset$ , otherwise  $\emptyset$ ). ■

## 6. Final notes

*Related work.* We must cite Simon Halfon's PhD thesis [5], and especially Section 9.1 there. Our study of  $X^\omega$  is very close to his. At first glance, it may seem that we add some generality to his study, in the sense that Halfon studies  $X^\omega$  (as a preordered set) in the special case where  $X$  is an  $\omega^2$ -wqo. In that case,  $X^\omega$  is wqo (as a set preordered by  $\leq^\omega$ ).

In a world of preorders, it is natural to replace sobrifications by ideal completions. Indeed, the ideal completion of a preordered set  $X$  coincides with the sobrification of  $X$ , provided that  $X$  is given its Aleksandrov topology. Halfon shows that the ideal completion of  $X^\omega$  (as a preordered set) is characterized in terms of  $\omega$ -regular expressions that are similar to the infinite word products we introduce in Section 5.4, although slightly more complicated, as the  $F^\omega$  part of  $\omega$ -regular expressions no longer involves elements  $F$  of  $\mathcal{H}(X)$  but ideals of  $\mathcal{H}(X)$ . The mismatch is due to the fact that our space  $X^\omega$  will almost never have an Aleksandrov topology (unless  $X$  is essentially finite, see Proposition 5.18), and therefore the ideal completion of  $X^\omega$  in general differs from  $\mathcal{S}(X^\omega)$  (where  $X^\omega$  is given the asymptotic subword topology, as in our case, not the Aleksandrov topology of  $\leq^\omega$ ), for every  $\omega^2$ -wqo  $X$  that is not essentially finite.

*Other initial maps.* Our study of  $X^{\leq\omega}$  (resp.,  $X^\omega$ ) proceeds by finding an initial map  $\langle \text{pref}, \text{suf} \rangle$  from  $X^{\leq\omega}$  to the more familiar space  $\mathcal{S}(X^*) \times \mathcal{H}(X)$ . This has notable advantages. For example, the fact that  $X^{\leq\omega}$  (and its subspace  $X^\omega$ ) is Noetherian if and only if  $X$  follows immediately from previously known results on sobrifications, on the Hoare powerspace, and on spaces of *finite* words. We took this further in the study of S-representations of  $X^{\leq\omega}$  (resp.,  $X^\omega$ ), where we insisted on *reducing* the question to S-representations for finite words (and powersets). We could have computed intersections of infinite word products directly, but we feel that would have been less interesting.

Remarkably, there are many other initial maps that we could have used instead of  $\langle \text{pref}, \text{suf} \rangle$ . The advantage of the latter is that it shows how the asymptotic subword topology splits into the study of finite chunks of information (prefixes) and infinite behaviors (suffixes). Here are two different initial maps that we could have used.

The first one is the composition

$$X^{\leq\omega} \xrightarrow{\langle \text{pref}, \text{suf} \rangle} \mathcal{S}(X^*) \times \mathcal{H}(X) \xrightarrow{\text{id} \times j} \mathcal{S}(X^*) \times \mathcal{S}(X^*) \\ \cong \mathcal{S}(X^* \times X^*) \xrightarrow{\mathcal{S}(c)} \mathcal{S}((X + X)^*)$$

where  $j: \mathcal{H}(X) \rightarrow \mathcal{S}(X^*)$  maps  $F$  to  $F^*$ , and  $c$  maps every pair of finite words  $(a_1 \cdots a_m, b_1 \cdots b_n)$  to the word  $\iota_1(a_1) \cdots \iota_1(a_m) \iota_2(b_1) \cdots \iota_2(b_n)$ , where  $\iota_1, \iota_2$  are the two canonical injections of  $X$  into  $X + X$ . Note that  $j^{-1}(\diamond \langle U_1; \dots; U_n \rangle) = \diamond U_1 \cap \dots \cap \diamond U_n$ , showing that  $j$  is initial. As for  $c$ , every open subset of  $X + X$  can be written as  $U + V$  where  $U$  and  $V$  are open in  $X$ , and we have  $c^{-1}(\langle U_1 + V_1; \dots; U_n + V_n \rangle) = \bigcup_{k=0}^n \langle U_1; \dots; U_k \rangle \times \langle V_{k+1}; \dots; V_n \rangle$ , showing that  $c$  is continuous, and as a special case (with  $n = 2$ ,  $V_1$  and  $U_2$  empty)  $\langle U_1 \rangle \times \langle V_2 \rangle = c^{-1}(\langle U_1 + \emptyset; \emptyset + V_2 \rangle)$ , which allows us to conclude that  $c$  is initial with the help of Remark 4.1. This implies that  $\mathcal{S}(c)$  is initial,

hence the whole composition shown above is initial, too. The point of using this, as an alternative to  $\langle \text{pref}, \text{suf} \rangle$ , is to realize that using the Hoare powerspace is not required at all, and the study of  $X^\omega$  reduces to the study of the sobrification of a space of finite words only, on the extended alphabet  $X + X$ .

A second alternative to  $\langle \text{pref}, \text{suf} \rangle$  is the map  $q: X^{\leq \omega} \rightarrow \mathcal{S}((X + X)^*)$  defined below (see Appendix 6 for a proof that  $q$  is initial when  $X$  is Noetherian). For short, let us write  $-a := (0, a)$  for every  $a \in X$ ,  $+a := (1, a)$ ,  $\pm A := \{-a, +a \mid a \in A\}$  for every  $A \subseteq X$ . For every  $w \in X^{\leq \omega}$  and every  $n \in \text{dom } w$ , let

$$q_n(w) := (\downarrow -w_0)^? (\downarrow -w_1)^? \cdots (\downarrow -w_n)^? (\text{cl}(\{\pm w_m \mid m \geq n + 1\}))^*.$$

The sequence  $(q_n(w))_{n \in \mathbb{N}}$  is a descending sequence of (irreducible) closed sets. When  $X$  is Noetherian, there must therefore be an index  $n_0 \in \mathbb{N}$  such that  $q_n(w) = q_{n_0}(w)$  for every  $n \geq n_0$ , and we define  $q(w)$  as  $q_{n_0}(w)$ . Notice the similarity with Remark 5.4. Note also that  $q$  is slightly different from our previous alternative, which maps  $w$  to

$$\begin{aligned} (\downarrow -w_0)^? (\downarrow -w_1)^? \cdots (\downarrow -w_n)^? (\text{cl}(\{-w_m \mid m \geq n + 1\}))^* \\ (\text{cl}(\{+w_m \mid m \geq n + 1\}))^* \end{aligned}$$

instead. A similar approach will turn out to be the right one in our study of infinite trees (which should be part III of this work).

*Transfinite sequences.* We have dealt with the space  $X^\omega$ , but what would be a proper, analogous treatment of spaces of sequences of length  $\alpha$ , for an arbitrary (or countable) indecomposable ordinal  $\alpha$ ? The bqo theory of such preordered sets is well-known [11]. We will deal with that aspect in part II.

**Appendix.  $q$  is initial.** We use an alternative definition of  $q$ . Given any  $w \in X^\omega$ , let  $A_n := \{w_m \mid m \geq n + 1\}$ . Then  $(\text{cl}(\pm A_n))_{n \in \mathbb{N}}$  is a descending sequence of closed subsets of  $X + X$ . If  $X$  is Noetherian, then there must be an index  $n_1$  such that for every  $n \geq n_1$ ,  $\text{cl}(\pm A_n) = \text{cl}(\pm A_{n_1})$ . We pick  $n_1$  larger than or equal to the  $n_0$  given in the definition of  $q$ . Then

$$q(w) = (\downarrow -w_0)^? (\downarrow -w_1)^? \cdots (\downarrow -w_n)^? (\text{cl}(\pm A_n))^*$$

for every  $n \geq n_1$ , by definition of  $q$ .

We now show that  $q$  is continuous. For that, we claim that

(\*)  $q^{-1}(\diamond \langle U_1 + V_1; \dots; U_\ell + V_\ell \rangle)$  is equal to

$$\bigcup_{k=0}^{\ell} \langle U_1; \dots; U_k; (\infty)(U_{k+1} \cup V_{k+1}) \cap \cdots \cap (\infty)(U_\ell \cup V_\ell) \rangle,$$

where  $U_1, V_1, \dots, U_\ell, V_\ell$  are arbitrary open subsets of  $X$ .

Let  $w \in X^{\leq \omega}$ , fix  $n := n_1$  in the definition of  $q(w)$ , and let us imagine that  $q(w)$  is in  $\diamond \langle U_1 + V_1; \dots; U_\ell + V_\ell \rangle$ . There are letters  $a_1 \in U_1 + V_1, \dots, a_\ell \in U_\ell + V_\ell$ , and indices  $k$  ( $0 \leq k \leq \ell$ ) and  $i_1 < \dots < i_k$  between 0 and  $n_1 - 1$  such that  $a_1 \leq -w_{i_1}, \dots, a_k \leq -w_{i_k}$ , and  $a_{k+1}, \dots, a_\ell$  are all in  $cl(\pm A_{n_1})$ . In particular, every  $U_i + V_i$  with  $i \geq k+1$  intersects  $cl(\pm A_{n_1})$ —which is equal to  $cl(\pm A_n)$  for every  $n \geq n_1$ —hence also  $\pm A_n$  for every  $n \geq n_1$ . This means that there are infinitely many indices  $n \geq n_1$  such that  $-w_n$  or  $+w_n$  is in  $U_i + V_i$ , in particular such that  $w_n \in U_i \cup V_i$ , and that holds for every  $i \geq k+1$ . Therefore  $w$  is in  $\langle U_1; \dots; U_k; (\infty)(U_{k+1} \cup V_{k+1}) \cap \dots \cap (\infty)(U_\ell \cup V_\ell) \rangle$ .

In the reverse direction, let  $w$  be in

$$\langle U_1; \dots; U_k; (\infty)(U_{k+1} \cup V_{k+1}) \cap \dots \cap (\infty)(U_\ell \cup V_\ell) \rangle$$

for some  $k$ ,  $0 \leq k \leq \ell$ . Let us write  $w$  as  $us$  where  $u \in \langle U_1; \dots; U_k \rangle$  and (if  $\ell > k$ )  $s$  contains infinitely many letters from each  $U_i \cup V_i$ ,  $k+1 \leq i \leq \ell$ . There are letters  $a_1 \in U_1, \dots, a_k \in U_k$  such that  $a_1 \dots a_k$  is a subword of  $u$ . We write again  $q(w)$  as

$$(\downarrow -w_0)^? (\downarrow -w_1)^? \dots (\downarrow -w_{n-1})^? (cl(\pm A_n))^*,$$

with  $n \geq n_1$  arbitrary. We pick such an  $n$  so that it exceeds the length of  $u$ . This way, the finite word  $(-a_1) \dots (-a_k)$  is in  $(\downarrow -w_0)^? (\downarrow -w_1)^? \dots (\downarrow -w_{n-1})^?$ . For each  $i$ ,  $k+1 \leq i \leq \ell$ , since  $s$  contains infinitely many letters from  $U_i \cup V_i$ , so does  $w$ , and therefore we can find at least one of the form  $w_m$  with  $m \geq n$ , hence in  $A_n$ . This implies that  $A_n$  intersects  $U_i \cup V_i$ . We pick a letter  $b_i$  in the intersection, for each  $i$  with  $k+1 \leq i \leq \ell$ . If  $b_i$  is in  $U_i$ , we let  $c_i := -b_i$ , otherwise  $c_i := +b_i$ , so that  $c_i$  is in  $U_i + V_i$ . Then the word  $(-a_1) \dots (-a_k) c_{k+1} \dots c_\ell$  is in  $q(w)$  and in  $\langle U_1; \dots; U_k; U_{k+1} \cup V_{k+1}; \dots; U_\ell \cup V_\ell \rangle$ . It follows that  $q(w)$  is in  $\diamond \langle U_1 + V_1; \dots; U_\ell + V_\ell \rangle$ .

That finishes the proof of (\*), hence  $q$  is continuous.

Specializing (\*) to the case where  $V_1, \dots, V_j, U_{j+1}, \dots, U_\ell$  are empty (for some  $j$ ,  $0 \leq j \leq \ell$ ), the terms  $\langle U_1; \dots; U_k; (\infty)(U_{k+1} \cup V_{k+1}) \cap \dots \cap (\infty)(U_\ell \cup V_\ell) \rangle$  with  $k \geq j+1$  are all empty (because  $U_k$  is empty). The same terms with  $k \leq j$  instead are of the form  $\langle U_1; \dots; U_k; (\infty)U_{k+1} \cap \dots \cap (\infty)U_j \cap (\infty)V_{j+1} \cap \dots \cap (\infty)V_\ell \rangle$ , and it is easy to see that they are all included in the term obtained when  $k = j$ , namely  $\langle U_1; \dots; U_j; (\infty)V_{j+1} \cap \dots \cap (\infty)V_\ell \rangle$ . It follows that  $q^{-1}(\diamond \langle U_1 + \emptyset; \dots; U_j + \emptyset; \emptyset + V_{j+1}; \dots; \emptyset + V_\ell \rangle)$  is equal to  $\langle U_1; \dots; U_j; (\infty)V_{j+1} \cap \dots \cap (\infty)V_\ell \rangle$ . The latter is the general form of the basic open subsets on  $X^{\leq \omega}$  given in Lemma 5.10. By Remark 4.1,  $q$  is initial.

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