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Analysis of Least square estimator for simple Linear Regression with a uniform distribution error

M.R. Jlibene, S. Taoufik,
S. Benjelloun

MSDA, Université Mohammed VI Polytechnique, Hay Moulay Rachid Benguerir, Morocco.

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Abstract

We study the least square estimator, in the framework of simple linear regression, when the deviance term ϵ with respect to the linear model is modeled by a uniform distribution. In particular, we give the law of this estimator, and prove some convergence properties.

1 Introduction

we consider the standard linear regression problem $Y = \beta_0 + \beta_1 X + \epsilon$ where X is a random or deterministic variable in \mathbb{R} , $\beta \in \mathbb{R}$ the vector of model coefficients, and ϵ a scalar centered random vector, $\mathbb{E}(\epsilon) = 0$, modeling a random perturbation term.

Given a n -sample of data points we adopt the following matrix notation :

$$Y = [1, x]\beta + \epsilon$$

where :

- x is the data vector in \mathbb{R}^n . Furthermore we suppose that the vector x is not co-linear with $[1, 1, \dots, 1]^T \in \mathbb{R}^n$. We note X the matrix $[1, x]$.
- $\beta = [\beta_0, \beta_1]^T \in \mathbb{R}^2$ are the coefficients vector of the regression model.
- $\epsilon = (\epsilon_i)_i \in \mathbb{R}^n$ is the vector for the disturbance samples. We suppose that the n identically distributed realisations are non-correlated, or eventually independent.

The Gauss-Markov theorem apply in our setting and the least square estimator (LSE) $\hat{\beta}_n = (X^T X)^{-1} X^T Y = \beta + (X^T X)^{-1} X^T \epsilon = M\epsilon$ is the Best Linear Unbiased Estimator (BLUE) for β . The least square estimator $\hat{\beta}_n$ can be also written as :

$$\begin{aligned}\hat{\beta}_n &= \hat{X} P Y = \hat{X} P (X \beta + \epsilon) \\ \hat{\beta}_n &= \beta + \hat{X} P \epsilon\end{aligned}$$

with

- $P = X(X^T X)^{-1} X^T$ is the projection in \mathbb{R}^n on $\text{Im}(X)$.

- $\hat{X} : \text{Im}(X) \subset \mathbb{R}^n \rightarrow \mathbb{R}^d$ is the inverse of X restricted to $\text{Im}(X)$.

Hence, computing the distribution law of $\hat{\beta}_n$, boils down to the computation of the distribution law of $P\epsilon$. When the disturbance term ϵ has a centred Gaussian distribution $\mathcal{N}(0, \sigma^2)$, of known variance σ^2 , with independent realisations making $(\epsilon_i)_i$ a Gaussian vector, the computation of this law is straightforward once the invert $(X^T X)^{-1}$ is computed : $\hat{\beta}_n \sim \mathcal{N}(\beta, (X^T X)^{-1} \sigma^2)$.

We have :

$$X = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad X^T X = \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}$$

$$\det(X^T X) = n(\sum x_i^2) - (\sum x_i)^2 = \sum_{i \neq j} (x_j - x_i)^2$$

$$(X^T X)^{-1} = \frac{1}{\det(X^T X)} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix}.$$

$$P_{i,j} = \frac{1}{\det(X^T X)} \left[\sum x_k^2 + n x_i x_j - (x_j + x_i) \sum x_k \right]$$

$$\hat{X} = \frac{1}{\det(X^T X)} \begin{bmatrix} \sum_{k \neq 1} x_k^2 - x_1 \sum_{k \neq 1} x_k & \dots & \sum_{k \neq j} x_k^2 - x_j \sum_{k \neq j} x_k & \dots & \sum_{k \neq n} x_k^2 - x_n \sum_{k \neq n} x_k \\ n x_1 - \sum x_k & \dots & n x_j - \sum x_k & \dots & n x_n - \sum x_k \end{bmatrix}$$

$$\hat{\beta} = \beta + M \epsilon$$

with

$$M = (X^T X)^{-1} X^T = \frac{1}{\det(X^T X)} \begin{bmatrix} \sum_{k \neq 1} x_k^2 - x_1 \sum_{k \neq 1} x_k & \dots & \sum_{k \neq j} x_k^2 - x_j \sum_{k \neq j} x_k & \dots & \sum_{k \neq n} x_k^2 - x_n \sum_{k \neq n} x_k \\ n x_1 - \sum x_k & \dots & n x_j - \sum x_k & \dots & n x_n - \sum x_k \end{bmatrix}$$

$$= \frac{1}{\det(X^T X)} \begin{bmatrix} \sum_{k \neq 1} x_k^2 - x_1 \sum_{k \neq 1} x_k & \dots & \sum_{k \neq j} x_k^2 - x_j \sum_{k \neq j} x_k & \dots & \sum_{k \neq n} x_k^2 - x_n \sum_{k \neq n} x_k \\ n x_1 - \sum x_k & \dots & n x_j - \sum x_k & \dots & n x_n - \sum x_k \end{bmatrix}$$

We note that

$$\sum_j M_{1,j} = \frac{1}{\det(X^T X)} \left[(n-1) \sum_i x_i^2 - \sum_{i \neq j} x_i x_j \right] = \frac{1}{\det(X^T X)} \sum_{i \neq j} (x_i - x_j)^2 = 1$$

$$\sum_j M_{2,j} = 0$$

When σ is unknown, from the maximum likelihood estimator one can construct the unbiased estimator for σ^2 is given by

$$\hat{\sigma}^2 = \frac{1}{n-2} \|Y - \hat{Y}\|^2$$

with $\hat{Y} = X \hat{\beta}$.

We summarize here in the results for the Gaussian case :

- Law for $\hat{\beta} : \mathcal{N}(\beta, (X^T X)^{-1} \sigma^2)$. In our case (simple linear regression, $d = 1.5$) we have:

$$- \hat{\beta}_0 \sim \mathcal{N}\left(\beta_0, \frac{S_2}{n S_2 - S_1^2} \sigma^2\right) \text{ where } S_1 = \sum x_i \text{ and } S_2 = \sum x_i^2$$

- $\hat{\beta}_1 \sim \mathcal{N}\left(\beta_1, \frac{n}{nS_2 - S_1^2} \sigma^2\right)$
- $\text{cov}(\hat{\beta}_0, \hat{\beta}_1) = \frac{-S_1}{nS_2 - S_1^2} \sigma^2$
- Hence, if we suppose σ^2 to be known, confidence intervals can be built using the normal laws:

$$\frac{(\hat{\beta}_0 - \beta_0) \sqrt{nS_2 - S_1^2}}{\sigma \sqrt{S_2}} \sim \mathcal{N}(0, 1)$$

$$\frac{(\hat{\beta}_1 - \beta_1) \sqrt{nS_2 - S_1^2}}{\sigma \sqrt{n}} \sim \mathcal{N}(0, 1)$$

- The (symetric) confidence intervals at level $1 - \alpha$ for β_0 and β_1 are respectively :

$$\left[\hat{\beta}_0 - q_{1-\frac{\alpha}{2}} \frac{\sigma \sqrt{S_2}}{\sqrt{nS_2 - S_1^2}}, \hat{\beta}_0 + q_{1-\frac{\alpha}{2}} \frac{\sigma \sqrt{S_2}}{\sqrt{nS_2 - S_1^2}} \right]$$

$$\left[\hat{\beta}_1 - q_{1-\frac{\alpha}{2}} \frac{\sigma \sqrt{n}}{\sqrt{nS_2 - S_1^2}}, \hat{\beta}_1 + q_{1-\frac{\alpha}{2}} \frac{\sigma \sqrt{n}}{\sqrt{nS_2 - S_1^2}} \right]$$

where $q_{1-\frac{\alpha}{2}}$ is the quantile at level $1 - \frac{\alpha}{2}$ for the normal law $\mathcal{N}(0, 1)$.

- The statistical tests at significance level α for the hypothesis $H_0 : \beta_0 = 0$ and $H_0 : \beta_1 = 0$ have respectively the critical regions defined as :

$$|\hat{\beta}_1| > q_{1-\frac{\alpha}{2}} \frac{\sigma \sqrt{n}}{\sqrt{nS_2 - S_1^2}}$$

$$|\hat{\beta}_0| > q_{1-\frac{\alpha}{2}} \frac{\sigma \sqrt{S_2}}{\sqrt{nS_2 - S_1^2}}$$

When the perturbation ϵ is not Gaussian, the above properties are still verified asymptotically (large sample size n).

we are interested in the case where the error is uniform, $\epsilon \sim U(-\theta, \theta)$, and when the sample is small. This may arise in many cases related to physical and experimental measures, where the sensibility of measurement equipment may be better modeled by random perturbations with a uniform law.

In the case where ϵ is uniform in $[-\theta, \theta]$, with independent realisations, the Gauss Markov theorem still apply in this case and the *LSE* is the BLUE. We are interested by studying other properties of the LSE estimator in this case (law, confidence intervals, regression tests). In the case where the error law parameters θ is known, a closed formula for the estimator distribution law can be derived by slightly generalising the Irwin-Hall distribution such in [Sadooghi, Nematollahi and Habibi \(2007\)](#), or [Bradley and Gupta \(2002\)](#).

In section 2 we give this closed formula for the estimator law, in the case where the uniform law parameter are known $\epsilon \sim \mathcal{U}(-\theta, \theta)$ with known θ . In section ?? we show a numerical test for illustration.

2 Least Square estimator

We start by introducing a generalisation of the Erwin-Hall density function as follows :

Theorem 2.1. Assume that U_1, U_2, \dots, U_n are independents variables with U_k uniformly distributed on $[0, p_k]$ where $p_k > 0$, and let $U = \sum_{k=1}^n U_k$. The probability density function for U is given by

$$f_U(x; n) = \frac{1}{P_n(n-1)!} \sum_{k=0}^n (-1)^k \sum_{l=1}^{n_k} (x - S_{nkl})_+^{n-1}$$

for $x \in [0, \sum_{k=1}^n p_k]$.

where $n_k = \binom{n}{k}$ is the number of k -combinations, $l \in [0, n_k]$ is the index over the n_k combinations, $(S_{nkl})_l$ are the sums over each k -combination of (p_1, \dots, p_n) , and $P_n = \prod_{k=1}^n p_k$ and $x_+ = \max(0, x)$.

Proof. See [Sadooghi, Nematollahi and Habibi \(2007\)](#). \square

The Erwin-Hall distribution corresponds to the special case $p_k = 1$, so $S_{nkl} = k$ and

$$f(x; n) = \frac{1}{(n-1)!} \sum_{k=0}^n (-1)^k n_k (x - k)_+^{n-1}$$

Generalisation: let $V = \sum_{k=1}^n V_k$, where V_1, V_2, \dots, V_n are independents variables and $V_k \sim U(-p_k, p_k)$ and independents.

Let $U_k = \frac{V_k + p_k}{2}$ and $U = \sum_{k=1}^n U_k$. Clearly U_k satisfies the Theorem 2.1 conditions. We have $V = \sum_{k=1}^n (2U_k - p_k) = 2U - \sum_{k=1}^n p_k$ so the probability density function for V is given by:

$$f_V(x; n) = \frac{1}{2} f_U\left(\frac{x + \sum_{k=1}^n p_k}{2}; n\right) = \frac{1}{2P_n(n-1)!} \sum_{k=0}^n (-1)^k \sum_{l=1}^{n_k} \left(\frac{x}{2} + \frac{(\sum_{i=1}^n p_i)}{2} - S_{nkl}\right)_+^{n-1}$$

Generalisation: let $W = P\epsilon = \sum_{k=1}^n p_k \epsilon_k$, $\epsilon_k \sim U(-\theta, \theta)$.

In this case $W = \sum_{k=1}^n W_k$, $W_k = p_k \epsilon_k \sim U(-p_k \theta, p_k \theta)$, and

$$f_W(x; n) = \frac{1}{2P_n \theta^n (n-1)!} \sum_{k=0}^n (-1)^k \sum_{l=1}^{n_k} \left(\frac{x}{2} + \theta \cdot \frac{(\sum_{i=1}^n p_i)}{2} - \theta \cdot S_{nkl}\right)_+^{n-1}$$

In our case, as θ is known, we know that: $\hat{\beta} = \beta + (X^T X)^{-1} X^T \epsilon$.

Hence, with :

$$d = \det(X^T X) = nS_2 - S_1^2,$$

$$p_k = S_2 - x_k S_1,$$

$$p_k' = x_k n - S_1,$$

we have

$$\hat{\beta}_0 = \beta_0 + \frac{1}{d} \sum_{k=1}^n p_k \epsilon_k = \beta_0 + \frac{1}{d} W$$

$$\hat{\beta}_1 = \beta_1 + \frac{1}{d} \sum_{k=1}^n p'_k \epsilon_k = \beta_1 + \frac{1}{d} W'$$

$$f_{\hat{\beta}_0}(x; n) = d \cdot f_W(d(x - \beta_0); n)$$

$$f_{\hat{\beta}_1}(x; n) = d \cdot f_{W'}(d(x - \beta_1); n)$$

$$f_{\hat{\beta}_0}(x; n) = d \cdot \frac{1}{P_n \theta^n (n-1)!} \sum_{k=0}^n (-1)^k \sum_{l=1}^{n_k} (d(x - \beta_0) + \theta \cdot d - \theta \cdot S_{nkl})_+^{n-1}$$

$$f_{\hat{\beta}_1}(x; n) = d \cdot \frac{1}{P'_n \theta^n (n-1)!} \sum_{k=0}^n (-1)^k \sum_{l=1}^{n_k} (d(x - \beta_1) + -\theta \cdot S'_{nkl})_+^{n-1}$$

Theorem 2.2. If $\frac{\max_{1 \leq i \leq n} (|p_i|)}{\sqrt{dS_2}} \rightarrow 0$ when $n \rightarrow +\infty$, we have : $\sqrt{\frac{d}{S_2}}(\hat{\beta}_0 - \beta_0)$ converges in law to the normal distribution $N(0, \frac{\theta^2}{3})$. If $\frac{\max_{1 \leq i \leq n} (|p'_i|)}{\sqrt{dn}} \rightarrow 0$ when $n \rightarrow +\infty$, we have $\sqrt{\frac{d}{n}}(\hat{\beta}_1 - \beta_1)$ that converges in law to the normal distribution $N(0, \frac{\theta^2}{3})$

Proof. We have

$$\hat{\beta}_0 - \beta_0 = \frac{1}{d} \sum_{k=1}^n p_k \epsilon_k.$$

$$\text{Hence } \sqrt{\frac{d}{S_2}}(\hat{\beta}_0 - \beta_0) = \frac{1}{\sqrt{dS_2}} \sum_{k=1}^n p_k \epsilon_k.$$

The characteristic function for $\sqrt{\frac{d}{S_2}}(\hat{\beta}_0 - \beta_0)$ is :

$$\phi_n(t) = \prod \frac{\sin(\frac{p_i \theta t}{\sqrt{dS_2}})}{\frac{p_i \theta t}{\sqrt{dS_2}}}$$

Let t be a real number, as $n \rightarrow +\infty$ and given that $\frac{p_i \theta t}{\sqrt{dS_2}} \rightarrow 0$ we have

$$\phi_n(t) = \prod (1 - \frac{1}{6}(\frac{p_i \theta t}{\sqrt{dS_2}})^2 + o((\frac{p_i \theta t}{\sqrt{dS_2}})^2))$$

Hence

$$\ln(\phi_n(t)) = \sum \ln(1 - \frac{1}{6}(\frac{p_i \theta t}{\sqrt{dS_2}})^2 + o((\frac{p_i \theta t}{\sqrt{dS_2}})^2))$$

$$\ln(\phi_n(t)) = \sum (-\frac{1}{6}(\frac{p_i \theta t}{\sqrt{dS_2}})^2 + o((\frac{p_i \theta t}{\sqrt{dS_2}})^2))$$

As $\sum p_i^2 = dS_2$ we have $\sum o((\frac{p_i \theta t}{\sqrt{dS_2}})^2) \rightarrow 0$ and hence

$$\ln(\phi_n(t)) = -\frac{1}{6}(\theta t)^2 + o(1)$$

We proved then that $\phi_n(t) \rightarrow \exp(-\frac{1}{6}(\theta t)^2)$ when $n \rightarrow +\infty$ which gives the result for β_0 . The same can be done for β_1 using $\sum p_i'^2 = dn$. \square

A similar result can be given for the limit law for the couple $\hat{\beta}$.

Theorem 2.3. If $\frac{\max_{1 \leq i \leq n} (|p_i|)}{\sqrt{dS_2}} \rightarrow 0$, $\frac{\max_{1 \leq i \leq n} (|p'_i|)}{\sqrt{dn}} \rightarrow 0$ and $\frac{S_1}{nS_2} \rightarrow 0$ when $n \rightarrow +\infty$, then $(\sqrt{\frac{d}{S_2}}(\hat{\beta}_0 - \beta_0), \sqrt{\frac{d}{n}}(\hat{\beta}_1 - \beta_1))$ converges in law to $N_2\left(0, \frac{\theta^2}{3} \mathbf{I}_2\right)$

Proof. The demonstration is similar to the previous theorem. It suffices to justify the convergence to 0 for $\sum (\frac{p_i p'_i}{d\sqrt{n}S_2}) = -\frac{S_1}{nS_2}$ which is satisfied under the two conditions of the theorem. \square

Let us now consider the question whether the conditions for the last theorems hold in practical situations.

If we consider the example where the x_k are uniform over $[a, b]$:

$$x_k = (b - a) \frac{k - 1}{n - 1} + a.$$

We can see that the two conditions of the previous theorems are indeed satisfied.

If the x_k are iid random variables, uniformly distributed over an interval $[a, b]$, then using the strong law of large numbers we demonstrate that the conditions for the two theorems are satisfied almost surely

$$\frac{p_i}{\sqrt{dS_2}} = \frac{1}{\sqrt{n}} \cdot \frac{\frac{S_2}{n} - x_k \frac{S_1}{n}}{\sqrt{\left(\frac{S_2}{n}\right)^2 - \left(\frac{S_1}{n}\right)^2 \cdot \frac{S_2}{n}}},$$

and

$$\frac{S_1}{nS_2} = \frac{1}{n} \cdot \frac{S_1}{n} \cdot \frac{1}{\frac{S_2}{n}}.$$

2.1 Estimator for θ

We give an unbiased estimator for θ .

Theorem 2.4. When θ is unknown, an unbiased estimator for θ^2 is given by

$$\hat{\theta}^2 = \frac{3}{n - 2} \|Y - \hat{Y}\|^2$$

with $\hat{Y} = X\hat{\beta} = PY$

Proof. we have

$$\|Y - \hat{Y}\|^2 = (Y - \hat{Y})^T (Y - \hat{Y}) = (Y - PY)^T (Y - PY) = Y^T (\mathbf{I}_n - P) Y$$

$$\begin{aligned} \mathbb{E}(\|Y - \hat{Y}\|^2) &= \mathbb{E} [\text{trace} (Y^T (\mathbf{I}_n - P) Y)] \\ &= \mathbb{E} [\text{trace} ((\mathbf{I}_n - P) Y Y^T)] \\ &= \text{trace} [(\mathbf{I}_n - P) \mathbb{E} (Y Y^T)] \\ &= \text{trace} \left[(\mathbf{I}_n - P) \left(\frac{\theta^2}{3} \mathbf{I}_n + X \beta \beta^T X^T \right) \right] \\ &= \text{trace} \left[\frac{\theta^2}{3} (\mathbf{I}_n - P) \right] \\ &= \frac{\theta^2}{3} (n - 2). \square \end{aligned}$$

3 Numerical simulation

??

We consider $n = 10$ points x_i uniformly and randomly sampled in $[-10, 10]$. We synthesize also Y data from a simple linear Regression :

$$Y = \beta_1 X + \beta_0 + \epsilon$$

The values of the true parameters are $\beta_0 = 7$ and $\beta_1 = 4$ where we consider a random perturbation ϵ that we will take either Gaussian $N(0, \sigma^2 = 3)$ or uniform on $[-3, 3]$ (so $\theta = 3$, $\text{var}(\epsilon) = \frac{\theta^2}{3} = 3$).

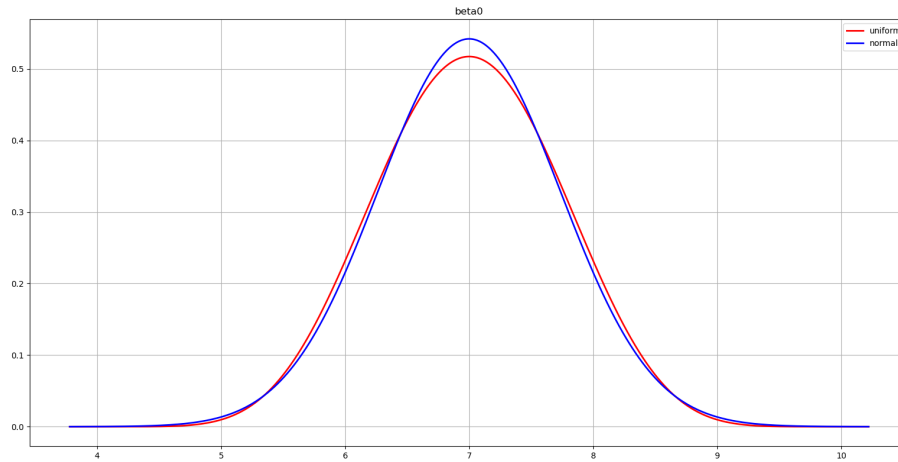


Figure 1: Distributions for the estimator $\hat{\beta}_0$

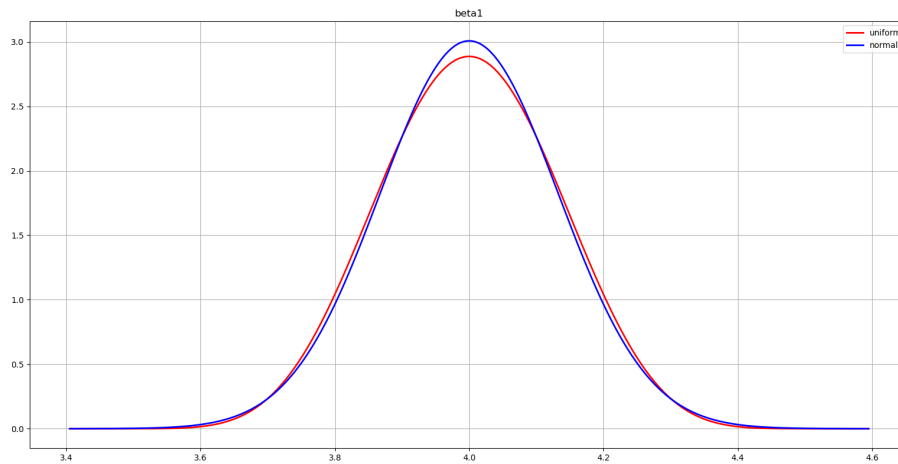


Figure 2: Distributions for the estimator $\hat{\beta}_1$

The 95% symmetric confidence intervals for $\hat{\beta}_0$ are of the form $[\beta_0 - h, \beta_0 + h]$. We show below the confidence intervals depending on n in the case of the normal and uniform case, for $\hat{\beta}_0$ and $\hat{\beta}_1$:

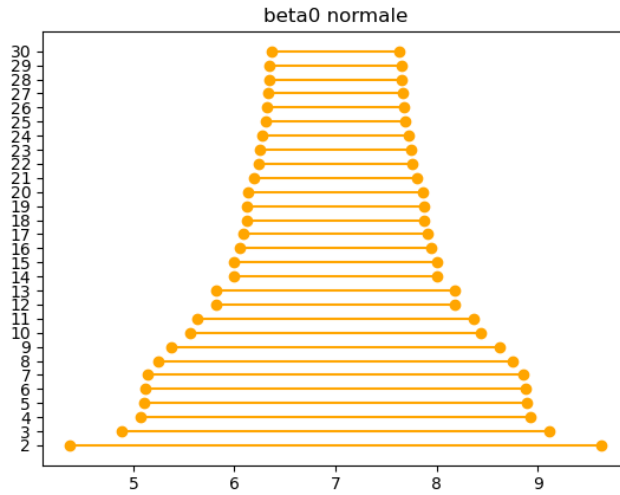


Figure 3: β_0 , 95% confidence interval with normal error

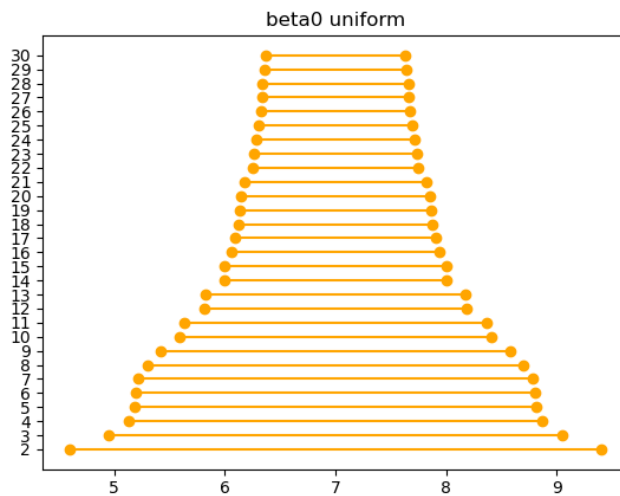


Figure 4: β_0 , 95% confidence interval with uniform error

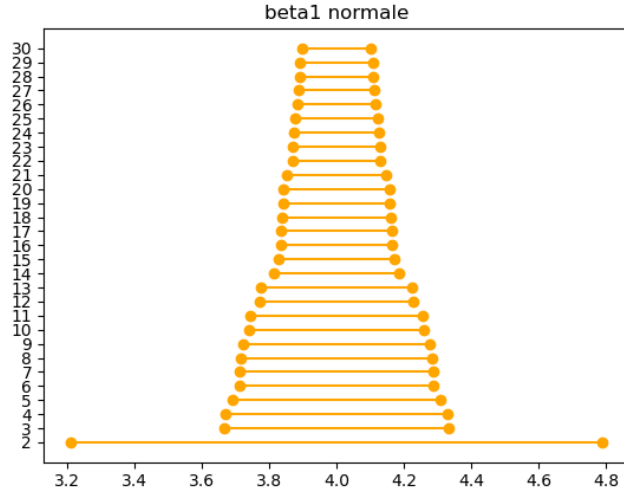


Figure 5: β_1 , 95% confidence interval with normal error

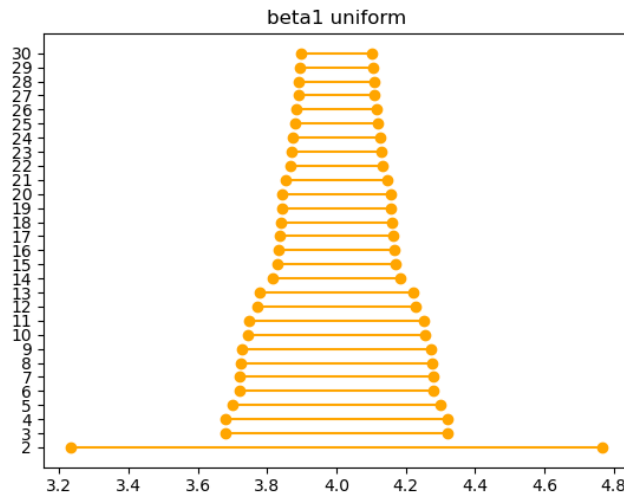


Figure 6: β_1 , 95% confidence interval with uniform error

We see that the even for $n = 10$, the distributions for the estimators are very close and that the confidence intervals are very close as well. However we note that we used the estimator from Theorem 2.4 to set $\sigma^2 = 3$. If we use the estimator assuming a Gaussian distribution, we get $\epsilon \approx N(0, 1)$:

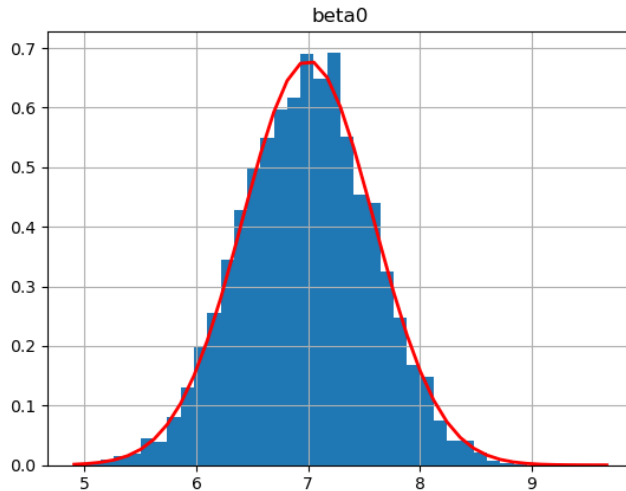


Figure 7: β_1 distribution, assuming normal error $\sigma^2 = 1$.

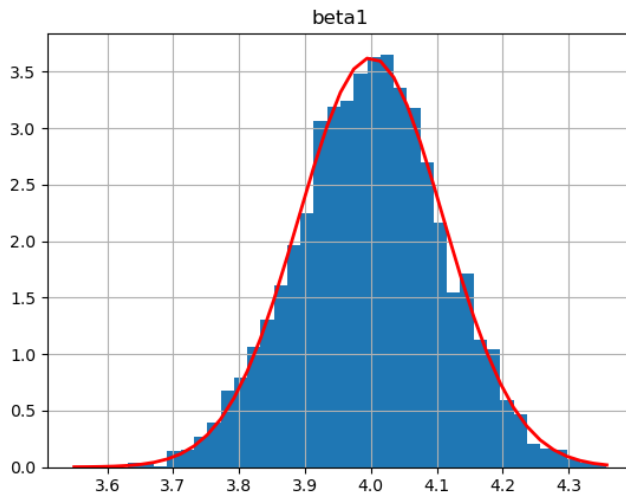


Figure 8: β_1 distribution, assuming normal error $\sigma^2 = 1$.

We see here that the confidence intervals will differ a lot with respect to the uniform case, because the variance σ^2 is estimated assuming a Gaussian distribution with $\frac{1}{n-2} \|Y - \hat{Y}\|^2$.

4 Conclusion

We gave above the law for the least square estimator in the case of simple linear regression model with a uniform distribution on the perturbation term ϵ . The uniform distribution hypothesis can be important to estimate the variance of the residual term. Visually it can be hard to assess if the model residuals are following a normal or a uniform distribution, however the variance parameter estimators differs significantly which leads to different confidence intervals for the parameters.

References

David M. Bradley and Ramesh C. Gupta. (2004). "On the distribution of the sum of n non-identically distributed uniform random variables" .

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