Multi-parametric families solutions to the Burgers equation

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Abstract

We construct $2N$ real parameter solutions to the Burgers' equation in terms of determinant of order N and we call these solutions, N order solutions.
We deduce general expressions of these solutions in terms of exponentials and study the patterns of these solutions in functions of the parameters for $N = 1$ until $N = 4$.

Key Words : Burgers equation.

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1 Introduction

The Burgers' equation is considered in the following normalization

$$u_t = \alpha u_{xx} - u u_x \tag{1}$$

where the subscripts x and t denote partial derivatives, and α an arbitrary real number.
Bateman [1] introduced this equation (1) in 1915 in a quite similar form. This equation appears in gas dynamics [2], acoustics [3], heat conduction [4], in soil water [5], in hydrodynamics turbulence [6, 7, 8], in shock waves [9],...
Bateman [1] constructed the first solutions in 1915. A lot of methods have been used to solve this equation : the exp-function method [10], the tanh-coth method [11], the groups actions on coset bundles [12], the Cole-Hopf method [13, 14, 15], the homotopy perturbation method [16],...

Solutions to the Burgers' equation are constructed in this paper. We give solutions in terms of determinant of order N depending on $2N$ real parameters. We study some patterns of the solutions in the (x,t) plane for orders 1 to 4.

2 Multi-parameter solutions to the Burger's equation

2.1 A first representation

Theorem 2.1 *The function v defined by*

$$v(x,t) = -2\alpha\partial_x \ln\det(f_{ij})_{1\leq i\leq N,\,1\leq j\leq N}, \tag{2}$$

where

$$\begin{aligned} f_{i1} &= 1 + a_i b_i \exp(-a_i x + \alpha a_i^2 t), \\ f_{ij} &= a_i^j \end{aligned} \tag{3}$$

is a solution to the Burgers' equation (1) depending on $2N$ real parameters a_i, b_i, $1\leq i\leq N$.

2.2 Another expression of the solutions of the Burgers' equation

We can expand the determinant in the preceding formula along the first column to give another expression of the solutions of the Burgers' equation. It is given in the following statement :

Proposition 2.1 *The function v defined by*

$$v(x,t) = 2\alpha\frac{\sum_{i=1}^{N}(-1)^{i+1}a_i^2 b_i \exp(-a_i x + \alpha a_i^2 t)\prod_{1\leq l<k\leq N,\,k\neq i,\,l\neq i}(a_k - a_l)\prod_{j\neq i} a_j}{\sum_{i=1}^{N}(-1)^{i+1}(1 + a_i b_i \exp(-a_i x + \alpha a_i^2 t))\prod_{1\leq l<k\leq N,\,k\neq i,\,l\neq i}(a_k - a_l)\prod_{j\neq i} a_j}, \tag{4}$$

is a solution to the Burgers' equation (1) depending on $2N$ real parameters a_i, b_i, $1\leq i\leq N$.

Remark 2.1 *By convention, for $N=1$ or $N=2$, $\prod_{1\leq l<k\leq N,\,k\neq i,\,l\neq i}(a_k - a_l)$ will be equal to 1.*
For $N=1$, $\prod_{j\neq i} a_j$ will be equal also to 1.

3 Study of the patterns of solutions for the first orders

Unlike other equations like the NLS equation for example, the figures in the plane of the coordinates vary little according to the orders and the values of the

parameters.
In all the following study, to construct solutions in the (x,t) plane, we choose the parameter α equal to 1.
We present for orders 1 until 4 some figures depending on parameters. Unfortunately, no specific feature appears from this study.
The following expressions of the solutions to the Burgers' equation are given for the coefficient α equal to 1. These results are a consequence of the previous section. But it is also possible to replace the expression of the solution given in the proposition in the equation (1) and check that the relation is verified.

3.1 First order solutions

Proposition 3.1 *The function v defined by*

$$v(x,t) = \frac{-2\,{a_1}^2 b_1 \mathrm{e}^{a_1(-x+ta_1)}}{a_1 b_1 \mathrm{e}^{(-x+ta_1)a_1} + 1}, \tag{5}$$

is a solution to the Burgers' equation (1).

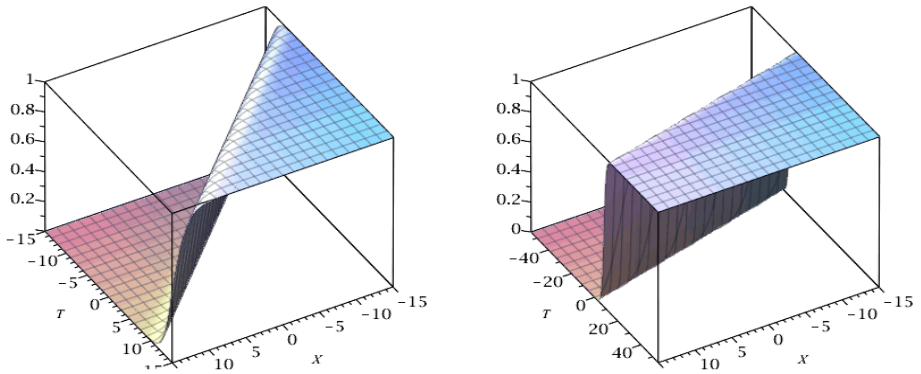

Figure 1. Solution of order 1 to (1); on the left, $a_1 = 1$, $b_1 = 1$; on the right, $a_1 = 1$, $b_1 = 10^3$.

3.2 Solutions of order two

Proposition 3.2 *The function v defined by*

$$v(x,t) = \frac{n(x,t)}{d(x,t)}, \tag{6}$$

with
$n(x,t) = -2\,a_2 a_1(-a_1 b_1 \mathrm{e}^{a_1(-x+ta_1)} + a_2 b_2 \mathrm{e}^{a_2(-x+ta_2)})$

$d(x,t) = -a_1 a_2 b_1 \mathrm{e}^{(-x+ta_1)a_1} + a_1 a_2 b_2 \mathrm{e}^{(-x+ta_2)a_2} + a_1 - a_2$
is a solution to the Burgers' equation (1).

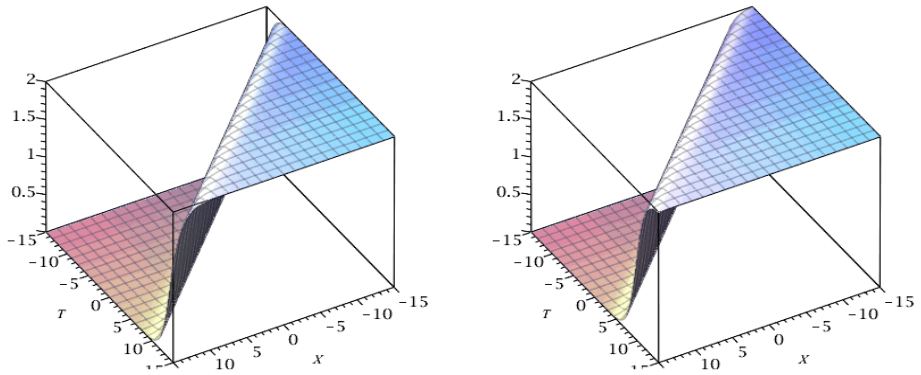

Figure 2. Solution of order 2 to (1); on the left, $a_1 = 1$, $a_2 = 2$, $b_1 = 2$, $b_2 = 0$; on the right, $a_1 = 1$, $a_2 = 2$, $b_1 = 100$, $b_2 = 0$.

3.3 Solutions of order three

Proposition 3.3 *The function v defined by*

$$v(x,t) = \frac{n(x,t)}{d(x,t)^2}, \tag{7}$$

with
$n(x,t) = -2\,a_1 a_2 a_3(-a_1 b_1 \mathrm{e}^{a_1(-x+ta_1)}a_3 + a_1 b_1 \mathrm{e}^{a_1(-x+ta_1)}a_2 + a_2 b_2 \mathrm{e}^{a_2(-x+ta_2)}a_3$
$-a_2 b_2 \mathrm{e}^{a_2(-x+ta_2)}a_1 - a_3 b_3 \mathrm{e}^{a_3(-x+ta_3)}a_2 + a_3 b_3 \mathrm{e}^{a_3(-x+ta_3)}a_1)$

$d(x,t) = -{a_1}^2 a_2 a_3 b_2 \mathrm{e}^{(-x+ta_2)a_2} + {a_1}^2 a_2 a_3 b_3 \mathrm{e}^{(-x+ta_3)a_3} + a_1 {a_2}^2 a_3 b_1 \mathrm{e}^{(-x+ta_1)a_1} -$
$a_1 {a_2}^2 a_3 b_3 \mathrm{e}^{(-x+ta_3)a_3} - a_1 a_2 {a_3}^2 b_1 \mathrm{e}^{(-x+ta_1)a_1} + a_1 a_2 a_3^2 b_2 \mathrm{e}^{(-x+ta_2)a_2} + {a_1}^2 a_2 -$
${a_1}^2 a_3 - a_1 {a_2}^2 + a_1 {a_3}^2 + {a_2}^2 a_3 - a_2 {a_3}^2$
is a solution to the Burgers' equation (1).

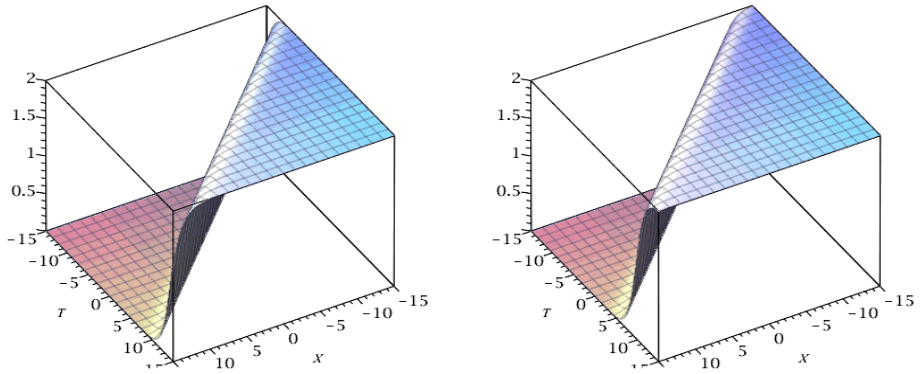

Figure 3. Solution of order 3 to (1); on the left, $a_1 = 1$, $a_2 = 2$, $b_1 = 2$, $b_2 = 0$; on the right, $a_1 = 1$, $a_2 = 2$, $b_1 = 100$, $b_2 = 0$.

3.4 Solutions of order four

Proposition 3.4 *The function v defined by*

$$v(x,t) = \frac{n(x,t)}{d(x,t)^2}, \tag{8}$$

with

$$n(x,t) = -2\,a_1a_2a_3a_4(a_1b_1e^{a_1(-x+ta_1)}a_3a_4{}^2 - a_1b_1e^{a_1(-x+ta_1)}a_3{}^2a_4 - a_1b_1e^{a_1(-x+ta_1)}a_2a_4{}^2 + a_1b_1e^{a_1(-x+ta_1)}a_2{}^2a_4 + a_1b_1e^{a_1(-x+ta_1)}a_2a_3{}^2 - a_1b_1e^{a_1(-x+ta_1)}a_2{}^2a_3 - a_2b_2e^{a_2(-x+ta_2)}a_3a_4{}^2 + a_2b_2e^{a_2(-x+ta_2)}a_3{}^2a_4 + a_2b_2e^{a_2(-x+ta_2)}a_1a_4{}^2 - a_2b_2e^{a_2(-x+ta_2)}a_1{}^2a_4 - a_2b_2e^{a_2(-x+ta_2)}a_1a_3{}^2 + a_2b_2e^{a_2(-x+ta_2)}a_1{}^2a_3 + a_3b_3e^{a_3(-x+ta_3)}a_2a_4{}^2 - a_3b_3e^{a_3(-x+ta_3)}a_2{}^2a_4 - a_3b_3e^{a_3(-x+ta_3)}a_1a_4{}^2 + a_3b_3e^{a_3(-x+ta_3)}a_1{}^2a_4 + a_3b_3e^{a_3(-x+ta_3)}a_1a_2{}^2 - a_3b_3e^{a_3(-x+ta_3)}a_1{}^2a_2 - a_4b_4e^{a_4(-x+ta_4)}a_2a_3{}^2 + a_4b_4e^{a_4(-x+ta_4)}a_2{}^2a_3 + a_4b_4e^{a_4(-x+ta_4)}a_1a_3{}^2 - a_4b_4e^{a_4(-x+ta_4)}a_1{}^2a_3 - a_4b_4e^{a_4(-x+ta_4)}a_1a_2{}^2 + a_4b_4e^{a_4(-x+ta_4)}a_1{}^2a_2)$$

$$d(x,t) = -a_1{}^3a_2{}^2a_3a_4b_3e^{(-x+ta_3)a_3} + a_1{}^3a_2{}^2a_3a_4b_4e^{(-x+ta_4)a_4} + a_1{}^3a_2a_3{}^2a_4b_2e^{(-x+ta_2)a_2} - a_1{}^3a_2a_3{}^2a_4b_4e^{(-x+ta_4)a_4} - a_1{}^3a_2a_3a_4{}^2b_2e^{(-x+ta_2)a_2} + a_1{}^3a_2a_3a_4{}^2b_3e^{(-x+ta_3)a_3} + a_1{}^2a_2{}^3a_3a_4b_3e^{(-x+ta_3)a_3} - a_1{}^2a_2{}^3a_3a_4b_4e^{(-x+ta_4)a_4} - a_1{}^2a_2a_3{}^3a_4b_2e^{(-x+ta_2)a_2} + a_1{}^2a_2a_3{}^3a_4b_4e^{(-x+ta_4)a_4} + a_1{}^2a_2a_3a_4{}^3b_2e^{(-x+ta_2)a_2} - a_1{}^2a_2a_3a_4{}^3b_3e^{(-x+ta_3)a_3} - a_1a_2{}^3a_3{}^2a_4b_1e^{(-x+ta_1)a_1} + a_1a_2{}^3a_3{}^2a_4b_4e^{(-x+ta_4)a_4} + a_1a_2{}^3a_3a_4{}^2b_1e^{(-x+ta_1)a_1} - a_1a_2{}^3a_3a_4{}^2b_3e^{(-x+ta_3)a_3} + a_1a_2{}^2a_3{}^3a_4b_1e^{(-x+ta_1)a_1} - a_1a_2{}^2a_3{}^3a_4b_4e^{(-x+ta_4)a_4} - a_1a_2{}^2a_3a_4{}^3b_1e^{(-x+ta_1)a_1} + a_1a_2{}^2a_3a_4{}^3b_3e^{(-x+ta_3)a_3} - a_1a_2a_3{}^3a_4{}^2b_1e^{(-x+ta_1)a_1} + a_1a_2a_3{}^3a_4{}^2b_2e^{(-x+ta_2)a_2} + a_1a_2a_3{}^2a_4{}^3b_1e^{(-x+ta_1)a_1} - a_1a_2a_3{}^2a_4{}^3b_2e^{(-x+ta_2)a_2} + a_1{}^3a_2{}^2a_3 - a_1{}^3a_2{}^2a_4 - a_1{}^3a_2a_3{}^2 + a_1{}^3a_2a_4{}^2 + a_1{}^3a_3{}^2a_4 - a_1{}^3a_3a_4{}^2 - a_1{}^2a_2{}^3a_3 + a_1{}^2a_2{}^3a_4 + a_1{}^2a_2a_3{}^3 - a_1{}^2a_2a_4{}^3 - a_1{}^2a_3{}^3a_4 + a_1{}^2a_3a_4{}^3 + a_1a_2{}^3a_3{}^2 - a_1a_2{}^3a_4{}^2 - a_1a_2{}^2a_3{}^3 + a_1a_2{}^2a_4{}^3 + a_1a_3{}^3a_4{}^2 - a_1a_3{}^2a_4{}^3 - a_2{}^3a_3{}^2a_4 + a_2{}^3a_3a_4{}^2 + a_2{}^2a_3{}^3a_4 - a_2{}^2a_3a_4{}^3 - a_2a_3{}^3a_4{}^2 + a_2a_3{}^2a_4{}^3$$

is a solution to the Burgers' equation (1).

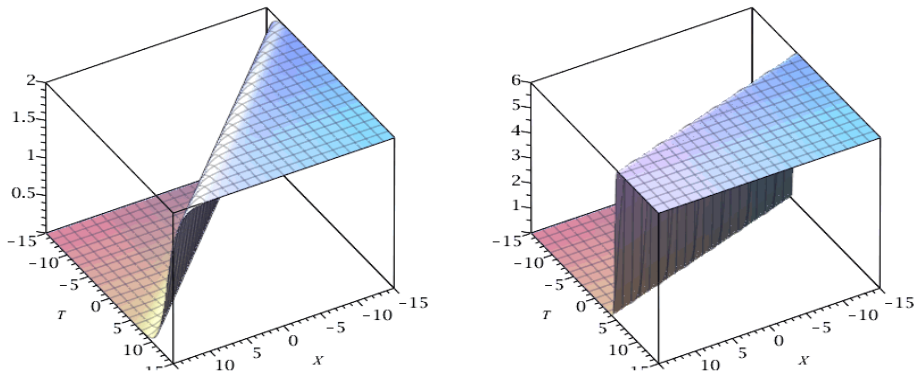

Figure 4. Solution of order 4 to (1); on the left, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$, $b_1 = 1$, $b_2 = 0$, $b_3 = 0$, $b_4 = 0$; on the right, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$, $b_1 = 0$, $b_2 = 0$, $b_3 = 1$, $b_4 = 0$.

4 Conclusion

We have constructed multi-parametric families of solutions to the Burgers equation.
In particular, we have constructed explicit solutions to the Burgers equation of order $1-4$ and give some patterns of these solutions in the $(x;t)$ plane.
But, unlike other equations, the characteristics of the solutions cannot be identified according to the type of parameters.

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