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Multi-parametric families solutions to the Burgers equation

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Abstract

We construct $2N$ real parameter solutions to the Burgers' equation in terms of determinant of order N and we call these solutions, N order solutions.

We deduce general expressions of these solutions in terms of exponentials and study the patterns of these solutions in functions of the parameters for $N = 1$ until $N = 4$.

Key Words : Burgers equation.

PACS numbers :

33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

1 Introduction

The Burgers' equation is considered in the following normalization

$$u_t = \alpha u_{xx} - uu_x \tag{1}$$

where the subscripts x and t denote partial derivatives, and α an arbitrary real number.

Bateman [1] introduced this equation (1) in 1915 in a quite similar form. This equation appears in gas dynamics [2], acoustics [3], heat conduction [4], in soil water [5], in hydrodynamics turbulence [6, 7, 8], in shock waves [9],...

Bateman [1] constructed the first solutions in 1915. A lot of methods have been used to solve this equation : the exp-function method [10], the tanh-coth method [11], the groups actions on coset bundles [12], the Cole-Hopf method [13, 14, 15], the homotopy perturbation method [16],...

Solutions to the Burgers' equation are constructed in this paper. We give solutions in terms of determinant of order N depending on $2N$ real parameters. We study some patterns of the solutions in the (x, t) plane for orders 1 to 4.

2 Multi-parameter solutions to the Burger's equation

2.1 A first representation

Theorem 2.1 *The function v defined by*

$$v(x, t) = -2\alpha\partial_x \ln \det(f_{ij})_{1 \leq i \leq N, 1 \leq j \leq N}, \quad (2)$$

where

$$\begin{aligned} f_{i1} &= 1 + a_i b_i \exp(-a_i x + \alpha a_i^2 t), \\ f_{ij} &= a_i^j \end{aligned} \quad (3)$$

is a solution to the Burgers' equation (1) depending on $2N$ real parameters $a_i, b_i, 1 \leq i \leq N$.

2.2 Another expression of the solutions of the Burgers' equation

We can expand the determinant in the preceding formula along the first column to give another expression of the solutions of the Burgers' equation. It is given in the following statement :

Proposition 2.1 *The function v defined by*

$$v(x, t) = 2\alpha \frac{\sum_{i=1}^N (-1)^{i+1} a_i^2 b_i \exp(-a_i x + \alpha a_i^2 t) \prod_{1 \leq l < k \leq N, k \neq i, l \neq i} (a_k - a_l) \prod_{j \neq i} a_j}{\sum_{i=1}^N (-1)^{i+1} (1 + a_i b_i \exp(-a_i x + \alpha a_i^2 t)) \prod_{1 \leq l < k \leq N, k \neq i, l \neq i} (a_k - a_l) \prod_{j \neq i} a_j}, \quad (4)$$

is a solution to the Burgers' equation (1) depending on $2N$ real parameters $a_i, b_i, 1 \leq i \leq N$.

Remark 2.1 *By convention, for $N = 1$ or $N = 2$, $\prod_{1 \leq l < k \leq N, k \neq i, l \neq i} (a_k - a_l)$ will be equal to 1.*

For $N = 1$, $\prod_{j \neq i} a_j$ will be equal also to 1.

3 Study of the patterns of solutions for the first orders

Unlike other equations like the NLS equation for example, the figures in the plane of the coordinates vary little according to the orders and the values of the

parameters.

In all the following study, to construct solutions in the (x, t) plane, we choose the parameter α equal to 1.

We present for orders 1 until 4 some figures depending on parameters. Unfortunately, no specific feature appears from this study.

The following expressions of the solutions to the Burgers' equation are given for the coefficient α equal to 1. These results are a consequence of the previous section. But it is also possible to replace the expression of the solution given in the proposition in the equation (1) and check that the relation is verified.

3.1 First order solutions

Proposition 3.1 *The function v defined by*

$$v(x, t) = \frac{-2 a_1^2 b_1 e^{a_1(-x+ta_1)}}{a_1 b_1 e^{(-x+ta_1)a_1} + 1}, \quad (5)$$

is a solution to the Burgers' equation (1).

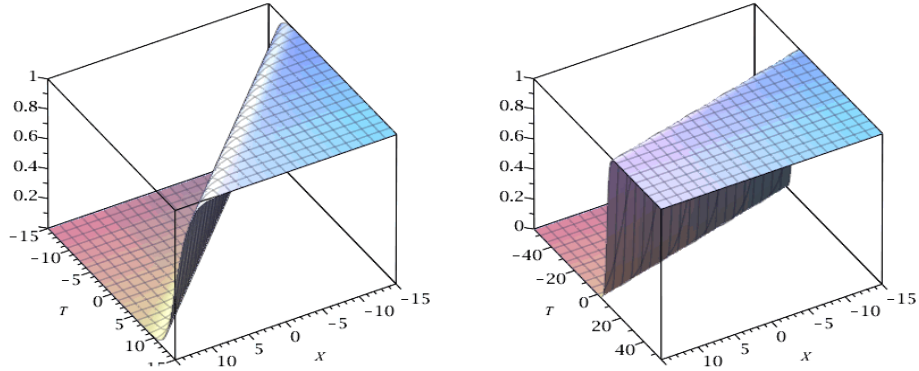


Figure 1. Solution of order 1 to (1); on the left, $a_1 = 1$, $b_1 = 1$; on the right, $a_1 = 1$, $b_1 = 10^3$.

3.2 Solutions of order two

Proposition 3.2 *The function v defined by*

$$v(x, t) = \frac{n(x, t)}{d(x, t)}, \quad (6)$$

with

$$n(x, t) = -2 a_2 a_1 (-a_1 b_1 e^{a_1(-x+ta_1)} + a_2 b_2 e^{a_2(-x+ta_2)})$$

$$d(x, t) = -a_1 a_2 b_1 e^{(-x+ta_1)a_1} + a_1 a_2 b_2 e^{(-x+ta_2)a_2} + a_1 - a_2$$

is a solution to the Burgers' equation (1).

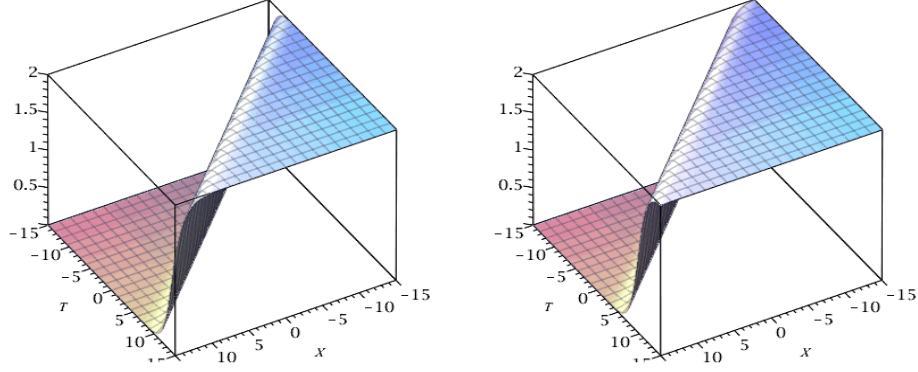


Figure 2. Solution of order 2 to (1); on the left, $a_1 = 1, a_2 = 2, b_1 = 2, b_2 = 0$; on the right, $a_1 = 1, a_2 = 2, b_1 = 100, b_2 = 0$.

3.3 Solutions of order three

Proposition 3.3 *The function v defined by*

$$v(x, t) = \frac{n(x, t)}{d(x, t)^2}, \quad (7)$$

with

$$n(x, t) = -2 a_1 a_2 a_3 (-a_1 b_1 e^{a_1(-x+ta_1)} a_3 + a_1 b_1 e^{a_1(-x+ta_1)} a_2 + a_2 b_2 e^{a_2(-x+ta_2)} a_3 - a_2 b_2 e^{a_2(-x+ta_2)} a_1 - a_3 b_3 e^{a_3(-x+ta_3)} a_2 + a_3 b_3 e^{a_3(-x+ta_3)} a_1)$$

$$d(x, t) = -a_1^2 a_2 a_3 b_2 e^{(-x+ta_2)a_2} + a_1^2 a_2 a_3 b_3 e^{(-x+ta_3)a_3} + a_1 a_2^2 a_3 b_1 e^{(-x+ta_1)a_1} - a_1 a_2^2 a_3 b_3 e^{(-x+ta_3)a_3} - a_1 a_2 a_3^2 b_1 e^{(-x+ta_1)a_1} + a_1 a_2 a_3^2 b_2 e^{(-x+ta_2)a_2} + a_1^2 a_2 - a_1^2 a_3 - a_1 a_2^2 + a_1 a_3^2 + a_2^2 a_3 - a_2 a_3^2$$

is a solution to the Burgers' equation (1).

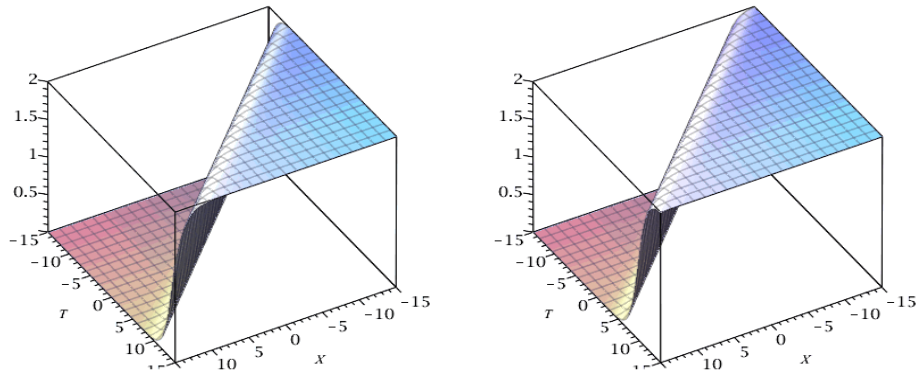


Figure 3. Solution of order 3 to (1); on the left, $a_1 = 1, a_2 = 2, b_1 = 2, b_2 = 0$; on the right, $a_1 = 1, a_2 = 2, b_1 = 100, b_2 = 0$.

3.4 Solutions of order four

Proposition 3.4 *The function v defined by*

$$v(x, t) = \frac{n(x, t)}{d(x, t)^2}, \quad (8)$$

with

$$\begin{aligned} n(x, t) = & -2a_1a_2a_3a_4(a_1b_1e^{a_1(-x+ta_1)}a_3a_4^2 - a_1b_1e^{a_1(-x+ta_1)}a_3^2a_4 - a_1b_1e^{a_1(-x+ta_1)}a_2a_4^2 + \\ & a_1b_1e^{a_1(-x+ta_1)}a_2^2a_4 + a_1b_1e^{a_1(-x+ta_1)}a_2a_3^2 - a_1b_1e^{a_1(-x+ta_1)}a_2^2a_3 - a_2b_2e^{a_2(-x+ta_2)}a_3a_4^2 + \\ & a_2b_2e^{a_2(-x+ta_2)}a_3^2a_4 + a_2b_2e^{a_2(-x+ta_2)}a_1a_4^2 - a_2b_2e^{a_2(-x+ta_2)}a_1^2a_4 - a_2b_2e^{a_2(-x+ta_2)}a_1a_3^2 + \\ & a_2b_2e^{a_2(-x+ta_2)}a_1^2a_3 + a_3b_3e^{a_3(-x+ta_3)}a_2a_4^2 - a_3b_3e^{a_3(-x+ta_3)}a_2^2a_4 - a_3b_3e^{a_3(-x+ta_3)}a_1a_4^2 + \\ & a_3b_3e^{a_3(-x+ta_3)}a_1^2a_4 + a_3b_3e^{a_3(-x+ta_3)}a_1a_2^2 - a_3b_3e^{a_3(-x+ta_3)}a_1^2a_2 - a_4b_4e^{a_4(-x+ta_4)}a_2a_3^2 + \\ & a_4b_4e^{a_4(-x+ta_4)}a_2^2a_3 + a_4b_4e^{a_4(-x+ta_4)}a_1a_3^2 - a_4b_4e^{a_4(-x+ta_4)}a_1^2a_3 - a_4b_4e^{a_4(-x+ta_4)}a_1a_2^2 + \\ & a_4b_4e^{a_4(-x+ta_4)}a_1^2a_2) \end{aligned}$$

$$\begin{aligned} d(x, t) = & -a_1^3a_2^2a_3a_4b_3e^{(-x+ta_3)a_3} + a_1^3a_2^2a_3a_4b_4e^{(-x+ta_4)a_4} + a_1^3a_2a_3^2a_4b_2e^{(-x+ta_2)a_2} - \\ & a_1^3a_2a_3^2a_4b_4e^{(-x+ta_4)a_4} - a_1^3a_2a_3a_4^2b_2e^{(-x+ta_2)a_2} + a_1^3a_2a_3a_4^2b_3e^{(-x+ta_3)a_3} + \\ & a_1^2a_2^3a_3a_4b_3e^{(-x+ta_3)a_3} - a_1^2a_2^3a_3a_4b_4e^{(-x+ta_4)a_4} - a_1^2a_2a_3^3a_4b_2e^{(-x+ta_2)a_2} + \\ & a_1^2a_2a_3^3a_4b_4e^{(-x+ta_4)a_4} + a_1^2a_2a_3a_4^3b_2e^{(-x+ta_2)a_2} - a_1^2a_2a_3a_4^3b_3e^{(-x+ta_3)a_3} - \\ & a_1a_2^3a_3^2a_4b_1e^{(-x+ta_1)a_1} + a_1a_2^3a_3^2a_4b_4e^{(-x+ta_4)a_4} + a_1a_2^3a_3a_4^2b_1e^{(-x+ta_1)a_1} - \\ & a_1a_2^3a_3a_4^2b_3e^{(-x+ta_3)a_3} + a_1a_2^2a_3^3a_4b_1e^{(-x+ta_1)a_1} - a_1a_2^2a_3^3a_4b_4e^{(-x+ta_4)a_4} - \\ & a_1a_2^2a_3a_4^3b_1e^{(-x+ta_1)a_1} + a_1a_2^2a_3a_4^3b_3e^{(-x+ta_3)a_3} - a_1a_2a_3^3a_4^2b_1e^{(-x+ta_1)a_1} + \\ & a_1a_2a_3^3a_4^2b_2e^{(-x+ta_2)a_2} + a_1a_2a_3^2a_4^3b_1e^{(-x+ta_1)a_1} - a_1a_2a_3^2a_4^3b_2e^{(-x+ta_2)a_2} + \\ & a_1^3a_2^2a_3 - a_1^3a_2^2a_4 - a_1^3a_2a_3^2 + a_1^3a_2a_4^2 + a_1^3a_3^2a_4 - a_1^3a_3a_4^2 - a_1^2a_2^3a_3 + \\ & a_1^2a_2^3a_4 + a_1^2a_2a_3^3 - a_1^2a_2a_4^3 - a_1^2a_3^3a_4 + a_1^2a_3a_4^3 + a_1a_2^3a_3^2 - a_1a_2^3a_4^2 - \\ & a_1a_2^2a_3^3 + a_1a_2^2a_4^3 + a_1a_3^3a_4^2 - a_1a_3^2a_4^3 - a_2^3a_3^2a_4 + a_2^3a_3a_4^2 + a_2^2a_3^3a_4 - \\ & a_2^2a_3a_4^3 - a_2a_3^3a_4^2 + a_2a_3^2a_4^3 \end{aligned}$$

is a solution to the Burgers' equation (1).

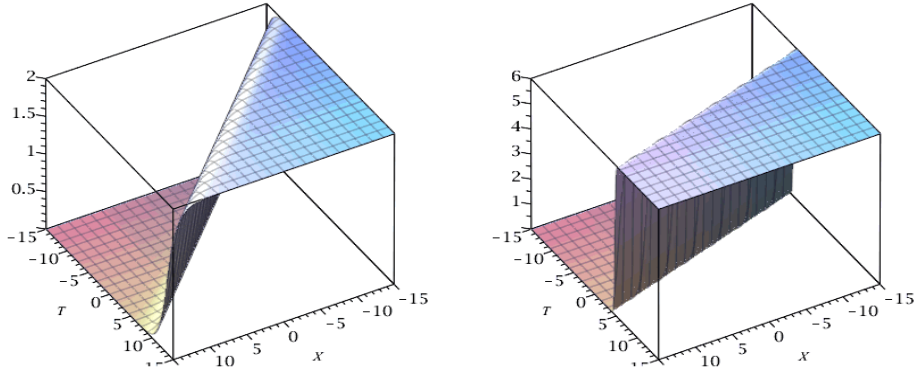


Figure 4. Solution of order 4 to (1); on the left, $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, b_1 = 1, b_2 = 0, b_3 = 0, b_4 = 0$; on the right, $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, b_1 = 0, b_2 = 0, b_3 = 1, b_4 = 0$.

4 Conclusion

We have constructed multi-parametric families of solutions to the Burgers equation.

In particular, we have constructed explicit solutions to the Burgers equation of order 1 – 4 and give some patterns of these solutions in the $(x; t)$ plane.

But, unlike other equations, the characteristics of the solutions cannot be identified according to the type of parameters.

References

- [1] H. Bateman, Some recent researches on the motion of fluids, Monthly Weather Review, V. 43, 163-170, 1915
- [2] J.M. Burgers, A mathematical model illustrating the theory of turbulence, Adv. In Appl. Mech., V. 1, 171-199, 1948
- [3] N. Sugimoto, Burgers equation with a fractional derivative: hereditary effects on nonlinear acoustic waves, Jour. Of Fluid Mech., V. 225, 631-633, 1991
- [4] G.W. Bluman, J.D. Cole, The general similarity solution of the heat equation, Jour. Of Math. And Mech., V. 18, 1025-1042, 1969
- [5] N. Su, P.C. Jim, W.V. Keith, E.C. Murray, R. Mao, Some recent researches on the motion of fluids, Ausr. Jour. Soil Res., V. 42, 42-, 2004
- [6] J.M. Burgers, Mathematical examples illustrating the relations occurring in the theory of turbulent fluid motion, Trans. Roy. Neth. Acad. Sci. Amsterdam, V. 17, 1-53, 1939
- [7] J.M. Burgers, The non-linear diffusion equation: asymptotic solutions and statistical problems, Springer, 1974
- [8] J.D. Cole, On a quasi-linear parabolic equation occurring in aerodynamics, Quat. Of Appl. Math., V. 9, 225-236, 1951
- [9] G.W. Whitham, Linear and nonlinear waves, Pure and Appl. Math., V. 42, Wiley And Sons, 2011
- [10] A. Ebaid, Exact solitary wave solution for some nonlinear evolution equations via exp-function method, Phys. Lett. A, V. 365, N. 3, 213-219, 2007
- [11] A.M. Wazwaz, Multiple front solutions for the Burgers equation and the coupled Burgers equations, Appl. Math. And Comp., V. 190, N. 2, 1198-1206, 2007

- [12] K.B. Wolf, L. Hlavaty, S. Steinberg, Nonlinear differential equations as invariants under group action on coset bundles: Burgers and Korteweg de Vries equation families, *Jour. Of Math. Ana. And Appl.*, V. 114, 340-359, 1986
- [13] E. Hopf, The partial differential equation $u_t + uu_x = u_{xx}$, *Comm. On Pure And Appl. Math.*, V. 3, 201-230, 1950
- [14] T. Ohwada, Cole-Hopf transformation as numerical tool for the Burgers equation, *Appl. And Comp. Math.*, V. 8, N. 1, 107-113, 2009
- [15] B.M. Vaganan, Cole-Hopf transformation for higher dimensional Burgers equation with variable coefficients, *Stu. In Appl. Math.*, V. 129, N. 3, 300-308, 2012
- [16] E.R. Srivastava, G.W. Awasthi, (1+ n)-Dimensional Burgers' equation and its analytical solution: A comparative study of HPM, ADM and DTM, *Ain Shams Uni.*, V. 5, 625-629, 2014
- [17] P. Gaillard, V.B. Matveev, Wronskian addition formula and its applications, *Max-Planck-Institut für Mathematik, MPI 02-31*, V. **161**, 2002
- [18] P. Gaillard, A new family of deformations of Darboux-Pöschl-Teller potentials, *Lett. Math. Phys.*, V. **68**, 77-90, 2004
- [19] P. Gaillard, V.B. Matveev, New formulas for the eigenfunctions of the two-particle Calogero-Moser system, *Lett. Math. Phys.*, V. **89**, 1-12, 2009
- [20] P. Gaillard, V.B. Matveev, Wronskian and Casorai determinant representations for Darboux-Pöschl-Teller potentials and their difference extensions, *J. Phys A : Math. Theor.*, V. **42**, 404409-1-16, 2009
- [21] P. Dubard, P. Gaillard, C. Klein, V. B. Matveev, On multi-rogue wave solutions of the NLS equation and positon solutions of the KdV equation, *Eur. Phys. J. Spe. Top.*, V. **185**, 247-258, 2010
- [22] P. Gaillard, Families of quasi-rational solutions of the NLS equation and multi-rogue waves, *J. Phys. A : Meth. Theor.*, V. **44**, 435204-1-15, 2011
- [23] P. Gaillard, Wronskian representation of solutions of the NLS equation and higher Peregrine breathers, *J. Math. Sciences : Adv. Appl.*, V. **13**, N. 2, 71-153, 2012
- [24] P. Gaillard, Degenerate determinant representation of solution of the NLS equation, higher Peregrine breathers and multi-rogue waves, *J. Math. Phys.*, V. **54**, 013504-1-32, 2013
- [25] P. Gaillard, Wronskian representation of solutions of NLS equation and seventh order rogue waves, *J. Mod. Phys.*, V. **4**, N. 4, 246-266, 2013

- [26] P. Gaillard, V.B. Matveev, Wronskian addition formula and Darboux-Pöschl-Teller potentials, *J. Math.*, V. **2013**, ID 645752, 1-10, 2013
- [27] P. Gaillard, Two-parameters determinant representation of seventh order rogue waves solutions of the NLS equation, *J. Theor. Appl. Phys.*, V. **7**, N. **45**, 1-6, 2013
- [28] P. Gaillard, Deformations of third order Peregrine breather solutions of the NLS equation with four parameters, *Phys. Rev. E*, V. **88**, 042903-1-9, 2013
- [29] P. Gaillard, Two parameters deformations of ninth Peregrine breather solution of the NLS equation and multi rogue waves, *J. Math.*, V. **2013**, 1-111, 2013
- [30] P. Gaillard, Six-parameters deformations of fourth order Peregrine breather solutions of the NLS equation, *J. Math. Phys.*, V. **54**, 073519-1-22, 2013
- [31] P. Gaillard, The fifth order Peregrine breather and its eight-parameters deformations solutions of the NLS equation, *Commun. Theor. Phys.*, V. **61**, 365-369, 2014
- [32] P. Gaillard, Ten parameters deformations of the sixth order Peregrine breather solutions of the NLS equation, *Phys. Scripta*, V. **89**, 015004-1-7, 2014
- [33] P. Gaillard, Higher order Peregrine breathers, their deformations and multi-rogue waves, *J. Of Phys. : Conf. Ser.*, V. **482**, 012016-1-7, 2014
- [34] P. Gaillard, M. Gastineau, Eighteen parameter deformations of the Peregrine breather of order ten solutions of the NLS equation, *Int. J. Mod. Phys. C*, V. **26**, N. 2, 1550016-1-14, 2014
- [35] P. Gaillard, Two parameters wronskian representation of solutions of nonlinear Schrödinger equation, eight Peregrine breather and multi-rogue waves, *J. Math. Phys.*, V. **5**, 093506-1-12, 2014
- [36] P. Gaillard, Hierarchy of solutions to the NLS equation and multi-rogue waves, *J. Phys. : Conf. Ser.*, V. **574**, 012031-1-5, 2015
- [37] P. Gaillard, Tenth Peregrine breather solution of the NLS, *Ann. Phys.*, V. **355**, 293-298, 2015
- [38] P. Gaillard, M. Gastineau, The Peregrine breather of order nine and its deformations with sixteen parameters solutions of the NLS equation *Phys. Lett. A*, V. **379**, 1309-1313, 2015
- [39] P. Gaillard, Other $2N-2$ parameters solutions to the NLS equation and $2N+1$ highest amplitude of the modulus of the N -th order AP breather, *J. Phys. A: Math. Theor.*, V. **48**, 145203-1-23, 2015

- [40] P. Gaillard, Multi-parametric deformations of the Peregrine breather of order N solutions to the NLS equation and multi-rogue waves, *Adv. Res.*, V. **4**, N. **5**, 346-364, 2015
- [41] P. Gaillard, Higher order Peregrine breathers solutions to the NLS equation, *Jour. Phys. : Conf. Ser.*, V. **633**, 012106-1-6, 2016
- [42] P. Gaillard, M. Gastineau Patterns of deformations of Peregrine breather of order 3 and 4, solutions to the NLS equation with multi-parameters, *Journal of Theoretical and Applied Physics*, V. **10**,1-7, 2016
- [43] P. Gaillard, M. Gastineau Twenty parameters families of solutions to the NLS equation and the eleventh Peregrine breather, *Commun. Theor. Phys*, V. **65**, N. **2**, 136-144, 2016
- [44] P. Gaillard, Rational solutions to the KPI equation and multi rogue waves, *Annals Of Physics*, V. **367**, 1-5, 2016
- [45] P. Gaillard, M. Gastineau Twenty two parameters deformations of the twelfth Peregrine breather solutions to the NLS equation, *Adv. Res.*, V. **10**, 83-89, 2016
- [46] P. Gaillard, Towards a classification of the quasi rational solutions to the NLS equation, *Theor. And Math. Phys.*, V. **189**, N. **1**, 1440-1449, 2016
- [47] P. Gaillard, Fredholm and Wronskian representations of solutions to the KPI equation and multi-rogue waves, *Jour. of Math. Phys.*, V. **57**, 063505-1-13, doi: 10.1063/1.4953383, 2016
- [48] P. Gaillard, M. Gastineau Families of deformations of the thirteenth Peregrine breather solutions to the NLS equation depending on twenty four parameters, *Jour. Of Bas. And Appl. Res. Int.*, V. **21**, N. 3, 130-139, 2017
- [49] P. Gaillard, From Fredholm and Wronskian representations to rational solutions to the KPI equation depending on $2N^2$ parameters, *Int. Jour. of Appl. Sci. And Math.*, V. **4**, N. 3, 60-70, 2017
- [50] P. Gaillard, Families of rational solutions of order 5 to the KPI equation depending on 8 parameters, *New Hor. in Math. Phys.*, V. 1, N. 1, 26-31, 2017
- [51] P. Gaillard, 6-th order rational solutions to the KPI Equation depending on 10 parameters, *Jour. Of Bas. And Appl. Res. Int.*, V. **21**, N. 2, 92-98, 2017
- [52] P. Gaillard, N -Order rational solutions to the Johnson equation depending on $2N - 2$ parameters, *Int. Jour. of Adv. Res. in Phys. Sci.*, V. **4**, N. 9, 19-37, 2017

- [53] P. Gaillard, Families of rational solutions to the KPI equation of order 7 depending on 12 parameters, *Int. Jour. of Adv. Res. in Phys. Sci.*, V. **4**, N. 11, 24-30, 2017
- [54] P. Gaillard, Rational solutions to the Johnson equation and rogue waves, *Int. Jour. of Inn. In Sci. and Math.*, V. **6**, N. 1, 14-19, 2018
- [55] P. Gaillard, Multiparametric families of solutions of the KPI equation, the structure of their rational representations and multi-rogue waves, *Theo. And Mat. Phys.*, V. **196**, N. 2, 1174-1199, 2018
- [56] P. Gaillard, The Johnson equation, Fredholm and wronskian representations of solutions and the case of order three, *Adv. In Math. Phys.*, V. **2018**, 1-18, 2018
- [57] P. Gaillard, Families of Solutions of Order 5 to the Johnson Equation Depending on 8 Parameters, *NHIMP*, V. **2**, N. 4, 53-61, 2018
- [58] P. Gaillard, Multiparametric families of solutions to the Johnson equation, *J. Phys. : Conf. Series*, V. **1141**, 012102-1-10, 2018
- [59] P. Gaillard, Rational solutions to the Boussinesq equation, *Fund. Jour. Of Math. And Appl.*, V. , 109-112, 2019
- [60] P. Gaillard, Differential relations for the solutions to the NLS equation and their different representations, *Comm. In Adv. Math. Sci.*, V. **2**, N. 4, 1-4, 2019
- [61] P. Gaillard, Multi-parametric families of solutions of order N to the Boussinesq and KP equations and the degenerate rational case, *UJMA*, V. **3**, N. 2, 44-52, 2020
- [62] P. Gaillard, The mKdV equation and multi-parameters rational solutions, *Wave Motion*, V. **100**, 102667-1-9, 2021
- [63] P. Gaillard, Degenerate Riemann theta functions, Fredholm and wronskian representations of the solutions to the KdV equation and the degenerate rational case, *Jour. Of Geom. And Phys.*, V. **161**, 104059-1-12, 2021