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NONPARAMETRIC ESTIMATION FOR I.I.D. PATHS OF A MARTINGALE DRIVEN MODEL WITH APPLICATION TO NON-AUTONOMOUS FRACTIONAL SDE

NICOLAS MARIE[†]

ABSTRACT. This paper deals with a projection least square estimator of the function J_0 computed from multiple independent observations on $[0, T]$ of the process Z defined by $dZ_t = J_0(t)d\langle M \rangle_t + dM_t$, where M is a centered, continuous and square integrable martingale vanishing at 0. Risk bounds are established on this estimator and on an associated adaptive estimator. An appropriate transformation allows to rewrite the differential equation $dX_t = V(X_t)(b_0(t)dt + \sigma(t)dB_t)$, where B is a fractional Brownian motion of Hurst parameter $H \in (1/2, 1)$, as a model of the previous type. So, the second part of the paper deals with risk bounds on a nonparametric estimator of b_0 derived from the results on the projection least square estimator of J_0 . In particular, our results apply to the estimation of the drift function in a non-autonomous extension of the fractional Black-Scholes model introduced in Hu et al. [15].

CONTENTS

1. Introduction	1
2. A projection least square estimator of the map J_0	3
2.1. The objective function	3
2.2. The projection least square estimator	3
3. Risk bound and model selection	4
4. Application to differential equations driven by the fractional Brownian motion	5
4.1. Auxiliary model	5
4.2. An estimator of Q_0	6
4.3. Example: drift estimation in a non-autonomous fractional Black-Scholes model	7
5. Numerical experiments	8
6. Proofs	9
6.1. Proof of Proposition 3.1	9
6.2. Proof of Corollary 3.2	11
6.3. Proof of Theorem 3.4	11
6.4. Proof of Proposition 4.1	14
6.5. Proof of Proposition 4.2	14
6.6. Proof of Proposition 4.3	15
References	16

1. INTRODUCTION

Since the 1980's, the statistical inference for stochastic differential equations (SDE) driven by a Brownian motion has been widely investigated by many authors in the parametric and in the nonparametric frameworks. Classically (see Kutoyants [18]), the estimators of the drift function are computed from one path of the solution to the SDE and converge when the time horizon $T > 0$ goes to infinity. The existence and the uniqueness of the stationary solution to the SDE are then required, and obtained thanks to

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restrictive conditions on the drift function.

Since few years, a new type of parametric and nonparametric estimators is investigated ; those computed from multiple independent observations on $[0, T]$ of the SDE solution. Indeed, this functional data analysis problem is already studied in the parametric framework (see Ditlevsen and De Gaetano [14], Overgaard et al. [21], Picchini, De Gaetano and Ditlevsen [22], Picchini and Ditlevsen [23], Comte, Genon-Catalot and Samson [6], Delattre and Lavielle [10], Delattre, Genon-Catalot and Samson [9], Dion and Genon-Catalot [13], Delattre, Genon-Catalot and Larédo [8], etc.) and more recently in the nonparametric framework (see Comte and Genon-Catalot [4, 5], Della Maestra and Hoffmann [11], and Marie and Rosier [19]). In [4, 5], the authors extend to the diffusion processes framework the projection least squares estimators already well studied in the regression framework (see Cohen et al. [2] and Comte and Genon-Catalot [3]). Our paper deals with a nonparametric estimation problem close to this last one.

Consider the stochastic process $Z = (Z_t)_{t \in [0, T]}$, defined by

$$(1) \quad Z_t = \int_0^t J_0(s) d\langle M \rangle_s + M_t ; \forall t \in [0, T],$$

where $M = (M_t)_{t \in [0, T]} \neq 0$ is a centered, continuous and square integrable martingale vanishing at 0, and J_0 is an unknown function which belongs to $\mathbb{L}^2([0, T], d\langle M \rangle_t)$. By assuming that the quadratic variation $\langle M \rangle_t$ of M is deterministic for every $t \in [0, T]$, our paper deals with the estimator $\hat{J}_{m, N}$ of J_0 minimizing the objective function

$$J \longmapsto \gamma_{m, N}(J) := \frac{1}{N} \sum_{i=1}^N \left(\int_0^T J(s)^2 d\langle M^i \rangle_s - 2 \int_0^T J(s) dZ_s^i \right)$$

on a m -dimensional function space \mathcal{S}_m , where M^1, \dots, M^N (resp. Z^1, \dots, Z^N) are $N \in \mathbb{N}^*$ independent copies of M (resp. Z) and $m \in \{1, \dots, N\}$. Precisely, risk bounds are established on $\hat{J}_{m, N}$ and on the adaptive estimator $\hat{J}_{\hat{m}, N}$, where

$$\hat{m} = \arg \min_{m \in \mathcal{M}_N} \{ \gamma_{m, N}(\hat{J}_{m, N}) + \text{pen}(m) \} \quad \text{with} \quad \text{pen}(\cdot) := \frac{\mathfrak{c}_{\text{cal}}}{N} \quad \text{and} \quad \mathcal{M}_N \subset \{1, \dots, N\}.$$

Now, consider the differential equation

$$(2) \quad X_t = x_0 + \int_0^t V(X_s)(b_0(s)ds + \sigma(s)dB_s) ; t \in [0, T],$$

where $x_0 \in \mathbb{R}^*$, B is a fractional Brownian motion of Hurst parameter $H \in (1/2, 1)$, the stochastic integral with respect to B is taken pathwise (in Young's sense), and $V : \mathbb{R} \rightarrow \mathbb{R}$, $\sigma : [0, T] \rightarrow \mathbb{R}^*$ and $b_0 : [0, T] \rightarrow \mathbb{R}$ are *regular enough*. An *appropriate* transformation (see Subsection 4.1) allows to rewrite Equation (2) as a model of type (1) driven by the Molchan martingale which quadratic variation is t^{2-2H} for every $t \in [0, T]$. Our paper also deals with a risk bound on an estimator of b_0/σ derived from $\hat{J}_{m, N}$. Finally, let us consider a financial market model in which the prices of the risky asset are modeled by the following equation of type (2):

$$(3) \quad S_t = S_0 + \int_0^t S_u \left(\left(\bar{b}_0(u) - \frac{\sigma^2}{2} u^{2H-1} \right) du + \sigma dB_u \right) ; t \in [0, T]$$

where $S_0, \sigma \in (0, \infty)$ and $\bar{b}_0 \in C^0([0, T]; \mathbb{R})$. This is a non-autonomous extension of the fractional Black-Scholes model defined in Hu et al. [15]. An estimator of \bar{b}_0 is derived from $\hat{J}_{m, N}$ at Subsection 4.3.

Up to our knowledge, only Comte and Marie [7] deals with a nonparametric estimator of the drift function computed from multiple independent observations on $[0, T]$ of the solution to a fractional SDE.

At Section 2, a detailed definition of the projection least square estimator of J_0 is provided. Section 3 deals with risk bounds on $\hat{J}_{m, N}$ and on the adaptive estimator $\hat{J}_{\hat{m}, N}$. At Section 4, the results of Section 3 on the estimator of J_0 are applied to the estimation of b_0 in Equation (2) and then of \bar{b}_0 in Equation (3). Finally, at Section 5, some numerical experiments on Model (1) are provided when M is the Molchan martingale.

2. A PROJECTION LEAST SQUARE ESTIMATOR OF THE MAP J_0

In the sequel, the quadratic variation $\langle M \rangle = (\langle M \rangle_t)_{t \in [0, T]}$ of M fulfills the following assumption.

Assumption 2.1. *The (nonnegative, increasing and continuous) process $\langle M \rangle$ is a deterministic function.*

For some results, $\langle M \rangle$ fulfills the following stronger assumption.

Assumption 2.2. *There exists $\mu \in C^0((0, T]; \mathbb{R}_+)$ such that $1/\mu$ is continuous from $[0, T]$ into \mathbb{R}_+ , and such that*

$$\langle M \rangle_t = \int_0^t \mu(s) ds ; \forall t \in [0, T].$$

2.1. The objective function. In order to define a least square projection estimator of J_0 , let us consider $N \in \mathbb{N}^*$ independent copies M^1, \dots, M^N (resp. Z^1, \dots, Z^N) of M (resp. Z), and the objective function $\gamma_{m, N}$ defined by

$$\gamma_{m, N}(J) := \frac{1}{N} \sum_{i=1}^N \left(\int_0^T J(s)^2 d\langle M^i \rangle_s - 2 \int_0^T J(s) dZ_s^i \right)$$

for every $J \in \mathcal{S}_m$, where $m \in \{1, \dots, N\}$, $\mathcal{S}_m := \text{span}\{\varphi_1, \dots, \varphi_m\}$ and $\varphi_1, \dots, \varphi_m$ are continuous functions from $[0, T]$ into \mathbb{R} such that $(\varphi_1, \dots, \varphi_m)$ is an orthonormal family in $\mathbb{L}^2([0, T], dt)$.

Remark. Note that since $t \in [0, T] \mapsto \langle M \rangle_t$ is nonnegative, increasing and continuous, and since the φ_j 's are continuous from $[0, T]$ into \mathbb{R} , the objective function $\gamma_{m, N}$ is well-defined.

For any $J \in \mathcal{S}_m$,

$$\begin{aligned} \mathbb{E}(\gamma_{m, N}(J)) &= \int_0^T J(s)^2 d\langle M \rangle_s - 2 \int_0^T J(s) J_0(s) d\langle M \rangle_s - 2\mathbb{E} \left(\int_0^T J(s) dM_s \right) \\ &= \int_0^T (J(s) - J_0(s))^2 d\langle M \rangle_s - \int_0^T J_0(s)^2 d\langle M \rangle_s. \end{aligned}$$

Then, the more J is close to J_0 , the more $\mathbb{E}(\gamma_{m, N}(J))$ is small. For this reason, the estimator of J_0 minimizing $\gamma_{m, N}$ is studied in this paper.

2.2. The projection least square estimator. Consider the estimator

$$(4) \quad \widehat{J}_{m, N} := \arg \min_{J \in \mathcal{S}_m} \gamma_{m, N}(J)$$

of J_0 . Since $\mathcal{S}_m = \text{span}\{\varphi_1, \dots, \varphi_m\}$, there exist m square integrable random variables $\widehat{\theta}_1, \dots, \widehat{\theta}_m$ such that

$$\widehat{J}_{m, N} = \sum_{j=1}^m \widehat{\theta}_j \varphi_j.$$

Then,

$$\nabla \gamma_{m, N}(\widehat{J}_{m, N}) = \left(\frac{1}{N} \sum_{i=1}^N \left(2 \sum_{k=1}^m \widehat{\theta}_k \int_0^T \varphi_j(s) \varphi_k(s) d\langle M^i \rangle_s - 2 \int_0^T \varphi_j(s) dZ_s^i \right) \right)_{j \in \{1, \dots, m\}}.$$

Therefore, by (4), necessarily

$$\widehat{\theta}_{m, N} := (\widehat{\theta}_1, \dots, \widehat{\theta}_m)^* = \Psi_m^{-1} \mathbf{z}_{m, N},$$

where

$$\Psi_m := \left(\int_0^T \varphi_j(s) \varphi_k(s) d\langle M \rangle_s \right)_{j, k \in \{1, \dots, m\}} \quad \text{and} \quad \mathbf{z}_{m, N} := \left(\frac{1}{N} \sum_{i=1}^N \int_0^T \varphi_j(s) dZ_s^i \right)_{j \in \{1, \dots, m\}}.$$

3. RISK BOUND AND MODEL SELECTION

In the sequel, the space $\mathbb{L}^2([0, T], d\langle M \rangle_t)$ is equipped with the scalar product $\langle \cdot, \cdot \rangle_{\langle M \rangle}$ defined by

$$\langle \varphi, \psi \rangle_{\langle M \rangle} := \int_0^T \varphi(s) \psi(s) d\langle M \rangle_s$$

for every $\varphi, \psi \in \mathbb{L}^2([0, T], d\langle M \rangle_t)$. The associated norm is denoted by $\|\cdot\|_{\langle M \rangle}$.

First, the following proposition provides a risk bound on $\widehat{J}_{m,N}$ for a fixed $m \in \{1, \dots, N\}$.

Proposition 3.1. *Under Assumption 2.1,*

$$(5) \quad \mathbb{E}(\|\widehat{J}_{m,N} - J_0\|_{\langle M \rangle}^2) \leq \min_{J \in \mathcal{S}_m} \|J - J_0\|_{\langle M \rangle}^2 + \frac{2m}{N}.$$

Note that Inequality (5) says first that the bound on the variance of our least square estimator of J_0 is of order m/N , as in the usual nonparametric regression framework. Under Assumption 2.2, the following corollary provides a more understandable expression of the bound on the bias in Inequality (5).

Corollary 3.2. *Under Assumption 2.2,*

$$\mathbb{E}(\|\widehat{J}_{m,N} - J_0\|_2^2) \leq \|\mu^{-1}\|_{\infty, T} \|p_{\mathcal{S}_m}^\perp(\mu^{1/2} J_0) - \mu^{1/2} J_0\|_2^2 + 2\|\mu^{-1}\|_{\infty, T} \frac{m}{N}$$

where

$$\mathcal{S}_m(\mu) := \{\iota \in \mathbb{L}^2([0, T], dt) : \exists \varphi \in \mathcal{S}_m, \forall t \in (0, T], \iota(t) = \mu(t)^{1/2} \varphi(t)\}.$$

For instance, assume that $\mathcal{S}_m = \text{span}\{\bar{\varphi}_1, \dots, \bar{\varphi}_m\}$, where

$$\bar{\varphi}_1(t) := \sqrt{\frac{1}{\mu(t)T}}, \quad \bar{\varphi}_{2j}(t) := \sqrt{\frac{2}{\mu(t)T}} \cos\left(2\pi j \frac{t}{T}\right) \quad \text{and} \quad \bar{\varphi}_{2j+1}(t) := \sqrt{\frac{2}{\mu(t)T}} \sin\left(2\pi j \frac{t}{T}\right)$$

for every $t \in [0, T]$ and $j \in \mathbb{N}^*$ satisfying $2j + 1 \leq m$. The basis $(\varphi_1, \dots, \varphi_m)$ of \mathcal{S}_m , orthonormal in $\mathbb{L}^2([0, T], dt)$, is obtained from $(\bar{\varphi}_1, \dots, \bar{\varphi}_m)$ via the Gram-Schmidt process. Consider also the Sobolev space

$$\mathbb{W}_2^\beta([0, T]) := \left\{ \iota \in C^{\beta-1}([0, T]; \mathbb{R}) : \int_0^T \iota^{(\beta)}(t)^2 dt < \infty \right\}; \quad \beta \in \mathbb{N}^*,$$

and assume that there exists $\iota_0 \in \mathbb{W}_2^\beta([0, T])$ such that $\iota_0(t) = \mu(t)^{1/2} J_0(t)$ for every $t \in (0, T]$. Then, by DeVore and Lorentz [12], Theorem 2.3 p. 205, there exists a constant $\mathbf{c}_{\beta, T} > 0$, not depending on m , such that

$$\|p_{\mathcal{S}_m}^\perp(\mu^{1/2} J_0) - \mu^{1/2} J_0\|_2^2 = \|p_{\mathcal{S}_m}^\perp(\iota_0) - \iota_0\|_2^2 \leq \mathbf{c}_{\beta, T} m^{-2\beta}.$$

Therefore, by Corollary 3.2,

$$\mathbb{E}(\|\widehat{J}_{m,N} - J_0\|_2^2) \leq \|\mu^{-1}\|_{\infty, T} \left(\mathbf{c}_{\beta, T} m^{-2\beta} + \frac{2m}{N} \right).$$

Finally, consider $m_N \in \{1, \dots, N\}$, $\mathcal{M}_N := \{1, \dots, m_N\}$ and

$$\widehat{m} = \arg \min_{m \in \mathcal{M}_N} \{\gamma_{m,N}(\widehat{J}_{m,N}) + \text{pen}(m)\} \quad \text{with} \quad \text{pen}(\cdot) := \mathbf{c}_{\text{cal}} \frac{\cdot}{N},$$

where $\mathbf{c}_{\text{cal}} > 0$ is a constant to calibrate in practice via, for instance, the *slope heuristic*. In the sequel, the φ_j 's fulfill the following assumption.

Assumption 3.3. *For every $m, m' \in \{1, \dots, N\}$, if $m > m'$, then $\mathcal{S}_{m'} \subset \mathcal{S}_m$.*

The following theorem provides a risk bound on the adaptive estimator $\widehat{J}_{\widehat{m}, N}$.

Theorem 3.4. *Under Assumptions 2.1 and 3.3, there exists a deterministic constant $\mathfrak{c}_{3.4} > 0$, not depending on N , such that*

$$\mathbb{E}(\|\widehat{J}_{\widehat{m},N} - J_0\|_{(M)}^2) \leq \mathfrak{c}_{3.4} \left(\min_{m \in \mathcal{M}_N} \{\mathbb{E}(\|\widehat{J}_{m,N} - J_0\|_{(M)}^2) + \text{pen}(m)\} + \frac{1}{N} \right).$$

Moreover, under Assumption 2.2,

$$\mathbb{E}(\|\widehat{J}_{\widehat{m},N} - J_0\|_2^2) \leq \mathfrak{c}_{3.4} \|\mu^{-1}\|_{\infty, T} \left(\min_{m \in \mathcal{M}_N} \left\{ \|p_{\mathcal{S}_m(\mu)}^\perp(\mu^{1/2} J_0) - \mu^{1/2} J_0\|_2^2 + (2 + \mathfrak{c}_{\text{cal}}) \frac{m}{N} \right\} + \frac{1}{N} \right).$$

As in the usual nonparametric regression framework, since $\text{pen}(m)$ is of same order than the bound on the variance term of $\widehat{J}_{m,N}$ for every $m \in \mathcal{M}_N$, Theorem 3.4 says that the risk of our adaptive estimator is controlled by the minimal risk of $\widehat{J}_{\cdot, N}$ on \mathcal{M}_N up to a multiplicative constant not depending on N .

4. APPLICATION TO DIFFERENTIAL EQUATIONS DRIVEN BY THE FRACTIONAL BROWNIAN MOTION

Throughout this section, $V : \mathbb{R} \rightarrow \mathbb{R}$ is twice continuously differentiable with bounded derivatives, $\sigma : [0, T] \rightarrow \mathbb{R}^*$ is γ -Hölder continuous with $\gamma \in (1 - H, 1]$, and $b_0 : [0, T] \rightarrow \mathbb{R}$ is continuous. Under these conditions on V , σ and b_0 , Equation (2) has a unique solution which paths are α -Hölder continuous from $[0, T]$ into \mathbb{R} for every $\alpha \in (1/2, H)$ (see Kubilius et al. [17], Theorem 1.42). The maps V and σ are known and our purpose is to provide a nonparametric estimator of b_0 .

4.1. Auxiliary model. The model transformation used in the sequel has been introduced in Kleptsyna and Le Breton [16] in the parametric estimation framework. Consider the function space

$$\mathcal{Q} := \{Q : \text{the function } t \mapsto t^{1/2-H} Q(t) \text{ belongs to } \mathbb{L}^1([0, T], dt)\},$$

let $Q_0 : [0, T] \rightarrow \mathbb{R}$ be the map defined by

$$Q_0(t) := \frac{b_0(t)}{\sigma(t)} ; \forall t \in [0, T],$$

and assume that $Q_0 \in \mathcal{Q}$. Consider also the Molchan martingale M defined by

$$M_t := \int_0^t \ell(t, s) dB_s ; \forall t \in [0, T],$$

where

$$\ell(t, s) := \mathfrak{c}_H s^{1/2-H} (t-s)^{1/2-H} \mathbf{1}_{(0,t)}(s) ; \forall s, t \in [0, T]$$

with

$$\mathfrak{c}_H = \left(\frac{\Gamma(3-2H)}{2H\Gamma(3/2-H)^3\Gamma(H+1/2)} \right)^{1/2},$$

and the process Z defined by

$$Z_t := \int_0^t \ell(t, s) dY_s \quad \text{with} \quad Y_t := \int_0^t \frac{dX_s}{V(X_s)\sigma(s)}$$

for every $t \in [0, T]$. Then, Equation (2) leads to

$$(6) \quad \begin{aligned} Z_t &= j(Q_0)(t) + M_t \\ &= \int_0^t J(Q_0)(s) d\langle M \rangle_s + M_t, \end{aligned}$$

where

$$j(Q)(t) := \int_0^t \ell(t, s) Q(s) ds \quad \text{and} \quad J(Q)(t) := (2-2H)^{-1} t^{2H-1} j(Q)'(t)$$

for every $Q \in \mathcal{Q}$ and almost every $t \in [0, T]$. Note that the Molchan martingale M fulfills Assumption 2.2 with $\mu(t) = t^{1-2H}$ for every $t \in (0, T]$.

4.2. An estimator of Q_0 . In Model (6), for any $m \in \{1, \dots, N\}$, the solution $\widehat{J}_{m,N}$ to Problem (4) is a nonparametric estimator of $J(Q_0)$. So, this subsection deals with an estimator of Q_0 solving the inverse problem

$$(7) \quad J(\widehat{Q}) = \widehat{J}_{m,N}.$$

Let us consider the function space

$$\mathcal{J} := \left\{ \iota : \text{the function } t \in [0, T] \mapsto \int_0^t s^{1-2H} \iota(s) ds \text{ belongs to } \mathcal{I}_{0+}^{3/2-H}(\mathbb{L}^1([0, T], dt)) \right\},$$

where $\mathcal{I}_{0+}^{3/2-H}(\cdot)$ is the Riemann-Liouville left-sided fractional integral of order $3/2 - H$. The reader can refer to Samko et al. [25] on fractional calculus.

In order to provide an estimator of Q_0 with a closed-form expression, let us establish first the following technical proposition.

Proposition 4.1. *The map $J : Q \mapsto J(Q)$ is one-to-one from \mathcal{Q} into \mathcal{J} . Moreover, for every $\iota \in \mathcal{J}$ and almost every $t \in [0, T]$,*

$$J^{-1}(\varphi)(t) = \bar{c}_H t^{H-1/2} \int_0^t (t-s)^{H-3/2} s^{1-2H} \iota(s) ds$$

with

$$\bar{c}_H = \frac{2-2H}{c_H \Gamma(3/2-H) \Gamma(H-1/2)}.$$

By Proposition 4.1, if $\varphi_1, \dots, \varphi_m \in \mathcal{J}$, then

$$\widehat{Q}_{m,N}(t) := \bar{c}_H t^{H-1/2} \int_0^t (t-s)^{H-3/2} s^{1-2H} \widehat{J}_{m,N}(s) ds ; t \in [0, T]$$

is the solution to Problem (7) in \mathcal{Q} . Note that even if the φ_j 's don't belong to \mathcal{J} , since these functions are continuous from $[0, T]$ into \mathbb{R} , $\widehat{Q}_{m,N}$ is well-defined but not necessarily a solution to Problem (7). A simple vector subspace of \mathcal{J} is provided at the end of this subsection.

The following proposition provides risk bounds on $\widehat{Q}_{m,N}$, $m \in \{1, \dots, N\}$, and on the adaptive estimator $\widehat{Q}_{\widehat{m},N}$.

Proposition 4.2. *If the φ_j 's belong to \mathcal{J} , then there exists a deterministic constant $c_{4.2,1} > 0$, not depending on N , such that*

$$\mathbb{E}(\|\widehat{Q}_{m,N} - Q_0\|_2^2) \leq c_{4.2,1} \left(\min_{\iota \in \mathcal{S}_m} \|\iota - J(Q_0)\|_{\langle M \rangle}^2 + \frac{m}{N} \right) ; \forall m \in \{1, \dots, N\}.$$

If in addition the φ_j 's fulfill Assumption 3.3, then there exists a deterministic constant $c_{4.2,2} > 0$, not depending on N , such that

$$\mathbb{E}(\|\widehat{Q}_{\widehat{m},N} - Q_0\|_2^2) \leq c_{4.2,2} \left(\min_{m \in \mathcal{M}_N} \left\{ \min_{\iota \in \mathcal{S}_m} \|\iota - J(Q_0)\|_{\langle M \rangle}^2 + \frac{m}{N} \right\} + \frac{1}{N} \right).$$

Proposition 4.2 says that the MISE of $\widehat{Q}_{m,N}$, $m \in \{1, \dots, N\}$, (resp. $\widehat{Q}_{\widehat{m},N}$) has at most the same bound than the MISE of $\widehat{J}_{m,N}$ (resp. $\widehat{J}_{\widehat{m},N}$).

Finally, the following proposition provides a simple vector subspace of \mathcal{J} .

Proposition 4.3. *The function space*

$$\mathbb{J} := \left\{ \iota \in C^1([0, T]; \mathbb{R}) : \lim_{t \rightarrow 0^+} t^{-2H} \iota(t) \text{ and } \lim_{t \rightarrow 0^+} t^{1-2H} \iota'(t) \text{ exist and are finite} \right\}$$

is a subset of \mathcal{J} .

Consider $m \in \{1, \dots, N\}$ and $\psi_1, \dots, \psi_m \in C^1([0, T]; \mathbb{R})$ such that (ψ_1, \dots, ψ_m) is an orthonormal family of $\mathbb{L}^2([0, T], dt)$. In particular, note that ψ_1, \dots, ψ_m are linearly independent. Moreover, assume that $\mathcal{S}_m = \text{span}\{\bar{\varphi}_1, \dots, \bar{\varphi}_m\}$ with $\bar{\varphi}_j(t) := t^{2H}\psi_j(t)$ for every $j \in \{1, \dots, m\}$ and $t \in [0, T]$. The basis $(\varphi_1, \dots, \varphi_m)$ of \mathcal{S}_m , orthonormal in $\mathbb{L}^2([0, T], dt)$, is obtained from $(\bar{\varphi}_1, \dots, \bar{\varphi}_m)$ via the Gram-Schmidt process, and the φ_j 's belong to $\mathbb{J} \subset \mathcal{J}$. For every $\iota \in \mathcal{S}_m$, there exist $\alpha_1(\iota), \dots, \alpha_m(\iota) \in \mathbb{R}$ such that

$$\iota = \sum_{j=1}^m \alpha_j(\iota) \bar{\varphi}_j = v\mu^{-1/2}\bar{\iota},$$

where $v(t) := t^{H+1/2}$ for every $t \in [0, T]$, and

$$\bar{\iota} := \sum_{j=1}^m \alpha_j(\iota) \psi_j \in \mathbb{S}_m \quad \text{with} \quad \mathbb{S}_m := \text{span}\{\psi_1, \dots, \psi_m\}.$$

So, by assuming that $J(Q_0) \in \mathbb{J}$, the bound on the bias term of $\widehat{J}_{m,N}$ in the inequalities of Proposition 4.2 can be controlled the following way:

$$\begin{aligned} \min_{\iota \in \mathcal{S}_m} \|\iota - J(Q_0)\|_{\langle M \rangle}^2 &= \min_{\bar{\iota} \in \mathbb{S}_m} \|v(\bar{\iota} - v^{-1}\mu^{1/2}J(Q_0))\|_2^2 \\ &\leq T^{2H+1} \min_{\bar{\iota} \in \mathbb{S}_m} \|\bar{\iota} - \bar{\mu}J(Q_0)\|_2^2 = \|p_{\mathbb{S}_m}^\perp(\bar{\mu}J(Q_0)) - \bar{\mu}J(Q_0)\|_2^2 \end{aligned}$$

where, for every $t \in (0, T]$,

$$\bar{\mu}(t) := v(t)^{-1}\mu(t)^{1/2} = t^{-2H}.$$

If (ψ_1, \dots, ψ_m) is the m -dimensional trigonometric basis, and if there exists $\iota_0 \in \mathbb{W}_2^\beta([0, T])$ such that $\iota_0(t) = t^{-2H}J(Q_0)(t)$ for every $t \in (0, T]$, then

$$\min_{\iota \in \mathcal{S}_m} \|\iota - J(Q_0)\|_{\langle M \rangle}^2 \leq T^{2H+1} \|p_{\mathbb{S}_m}^\perp(\iota_0) - \iota_0\|_2^2 \leq \mathbf{c}_{\beta, T} T^{2H+1} m^{-2\beta}.$$

4.3. Example: drift estimation in a non-autonomous fractional Black-Scholes model. Let us consider a financial market model in which the prices process $S = (S_t)_{t \in [0, T]}$ of the risky asset is defined by

$$S_t := S_0 \exp \left(\int_0^t \left(\bar{b}_0(u) - \frac{\sigma^2}{2} u^{2H-1} \right) du + \sigma B_t \right); \quad \forall t \in [0, T],$$

where $S_0, \sigma \in (0, \infty)$ and $\bar{b}_0 \in C^0([0, T]; \mathbb{R})$. This is a non-autonomous extension of the fractional Black-Scholes model defined in Hu et al. [15]. Thanks to the change of variable formula for Young's integral,

$$S_t = S_0 + \int_0^t S_u \left(\left(\bar{b}_0(u) - \frac{\sigma^2}{2} u^{2H-1} \right) du + \sigma dB_u \right); \quad \forall t \in [0, T].$$

Then, S is the solution to Equation (2) with $V = \text{Id}_{\mathbb{R}}$, $\sigma(\cdot) \equiv \sigma > 0$ and

$$b_0(t) = \bar{b}_0(t) - \frac{\sigma^2}{2} t^{2H-1}; \quad \forall t \in [0, T].$$

Consider N independent copies S^1, \dots, S^N of S . For every $i \in \{1, \dots, N\}$ and $t \in [0, T]$, consider also

$$Z_t^i := \frac{\mathbf{c}_H}{\sigma} \int_0^t \frac{u^{1/2-H}(t-u)^{1/2-H}}{S_u^i} dS_u^i.$$

If the volatility constant σ is known, thanks to Subsection 4.2, a nonparametric estimator of \bar{b}_0 is given by

$$\widehat{\bar{b}}_0(m, N; t) := \frac{\sigma^2}{2} t^{2H-1} + \sigma \widehat{Q}_{m, N}(t); \quad t \in [0, T],$$

where

$$\widehat{Q}_{m, N}(t) := \bar{\mathbf{c}}_H t^{H-1/2} \int_0^t (t-s)^{H-3/2} s^{1-2H} \widehat{J}_{m, N}(s) ds; \quad t \in [0, T]$$

and

$$\widehat{J}_{m,N} = \arg \min_{J \in \mathcal{S}_m} \left\{ \frac{1}{N} \sum_{i=1}^N \left(\int_0^T J(s)^2 s^{1-2H} ds - 2 \int_0^T J(s) dZ_s^i \right) \right\}.$$

Since

$$\|\widehat{b}_0(m, N; \cdot) - \bar{b}_0\|_2^2 = \sigma^2 \|\widehat{Q}_{m,N} - Q_0\|_2^2 \quad \text{with} \quad Q_0 = \frac{\bar{b}_0}{\sigma},$$

if the φ_j 's belong to \mathcal{J} , then Proposition 4.2 provides risk bounds on $\widehat{b}_0(m, N; \cdot)$, $m \in \{1, \dots, N\}$, and on the adaptive estimator $\widehat{b}_0(\widehat{m}, N; \cdot)$.

5. NUMERICAL EXPERIMENTS

Some numerical experiments on our estimation method of J_0 in Equation (1) are presented in this subsection when M is the Molchan martingale:

$$M_t = \int_0^t \ell(t, s) dB_s = (2 - 2H)^{1/2} \int_0^t s^{1/2-H} dW_s ; t \in [0, 1]$$

with $H \in \{0.6, 0.9\}$ and W the Brownian motion driving the Mandelbrot-Van Ness representation of the fractional Brownian motion B . The estimation method investigated on the theoretical side at Section 3 is implemented here for the three following examples of functions J_0 :

$$J_{0,1} : t \in [0, 1] \mapsto 10t^2, \quad J_{0,2} : t \in (0, 1] \mapsto 10(-\log(t))^{1/2} \quad \text{and} \quad J_{0,3} : t \in (0, 1] \mapsto 20t^{-0.05}.$$

These functions belong to $\mathbb{L}^2([0, 1], d\langle M \rangle_t)$ as required. Indeed, on the one hand, $J_{0,1}$ is continuous on $[0, 1]$ and

$$\begin{aligned} - \int_0^1 \log(t) d\langle M \rangle_t &= - \int_0^1 \log(t) t^{1-2H} dt \\ &= \frac{1}{2-2H} \left(\lim_{\varepsilon \rightarrow 0^+} \log(\varepsilon) \varepsilon^{2-2H} + \int_0^1 t^{1-2H} dt \right) = \frac{1}{(2-2H)^2} < \infty. \end{aligned}$$

On the other hand, for every $\alpha \in (0, 1/2)$ such that $H \in (1/2, 1 - \alpha)$,

$$\begin{aligned} \int_0^1 t^{-2\alpha} d\langle M \rangle_t &= \int_0^1 t^{1-2\alpha-2H} dt \\ &= \frac{1}{2(1-\alpha-H)} \left(1 - \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{2(1-\alpha-H)} \right) = \frac{1}{2(1-\alpha-H)} < \infty. \end{aligned}$$

Since for every $t \in (0, 1]$, $J_{0,3}(t) = 20t^{-\alpha}$ with $\alpha = 0.05$, and since $H \in \{0.6, 0.9\} \subset (0.5, 0.95)$ in our numerical experiments, $J_{0,3}$ is square-integrable with respect to $d\langle M \rangle_t$.

Our adaptive estimator is computed for $J_0 = J_{0,1}$, $J_{0,2}$ or $J_{0,3}$ on $N = 300$ paths of the process Z observed along the dissection $\{k/n ; k = 1, \dots, n\}$ of $[1/n, 1]$ with $n = 100$, when $(\varphi_1, \dots, \varphi_m)$ is the m -dimensional trigonometric basis for every $m \in \{1, \dots, N\}$. This experiment is repeated 100 times, and the means and the standard deviations of the MISE of $\widehat{J}_{\widehat{m},N}$ are stored in Table 1. Moreover, only for $H = 0.9$, 10 estimations (dashed black curves) of $J_{0,1}$, $J_{0,2}$ and $J_{0,3}$ (red curves) are respectively plotted on Figures 1, 2 and 3.

On average, the MISE of our adaptive estimator is higher for the three examples of functions J_0 when $H = 0.6$ than when $H = 0.9$. This is related to the fact that the more the Hurst parameter H is close to 1, the more the paths of B are regular. The standard deviation of the MISE of $\widehat{J}_{\widehat{m},N}$ is also higher when

$J_0; H$	$J_{0,1}; 0.6$	$J_{0,2}; 0.6$	$J_{0,3}; 0.6$	$J_{0,1}; 0.9$	$J_{0,2}; 0.9$	$J_{0,3}; 0.9$
Mean MISE	0.084	0.162	0.079	0.028	0.036	0.022
StD MISE	0.023	0.029	0.022	0.006	0.006	0.005

TABLE 1. Means and StD of the MISE of $\widehat{J}_{\widehat{m},N}$ (100 repetitions).

$H = 0.6$.

For both $H = 0.6$ and $H = 0.9$, on average, the MISE of $\widehat{J}_{\widehat{m},N}$ is higher for $J_{0,2}$ than for $J_{0,1}$ and $J_{0,3}$. This is probably related to the fact that $J_{0,2}(t)$ goes faster to infinity when $t \rightarrow 0^+$ than $J_{0,3}(t)$, and of course than $J_{0,1}(t)$ which goes not.

Our estimation method seems stable in the sense that for a fixed H (0.6 or 0.9), the standard deviation of the MISE of our adaptive estimator is almost the same for the three examples of functions J_0 . This can be observed on Figures 1, 2 and 3.

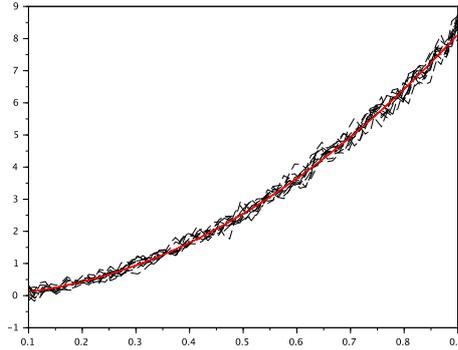


FIGURE 1. Plots of $J_{0,1}$ and of 10 adaptive estimations ($\overline{\widehat{m}} = 51.28$).

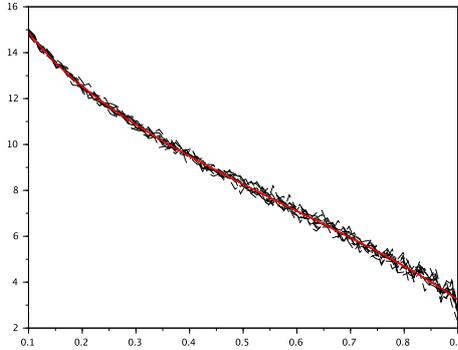


FIGURE 2. Plots of $J_{0,2}$ and of 10 adaptive estimations ($\overline{\widehat{m}} = 72.86$).

6. PROOFS

6.1. Proof of Proposition 3.1. For every $J, K \in \mathcal{S}_m$,

$$\begin{aligned} \gamma_{m,N}(J) - \gamma_{m,N}(K) &= \|J\|_{\langle M \rangle}^2 - \|K\|_{\langle M \rangle}^2 - \frac{2}{N} \sum_{i=1}^N \int_0^T (J(s) - K(s)) dZ_s^i \\ &= \|J - J_0\|_{\langle M \rangle}^2 - \|K - J_0\|_{\langle M \rangle}^2 - \frac{2}{N} \sum_{i=1}^N \int_0^T (J(s) - K(s)) dM_s^i. \end{aligned}$$

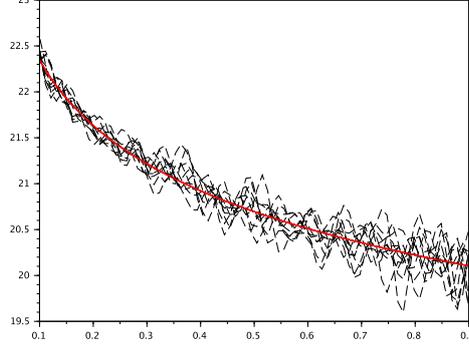


FIGURE 3. Plots of $J_{0,3}$ and of 10 adaptive estimations ($\widehat{m} = 39.64$).

Moreover,

$$\gamma_{m,N}(\widehat{J}_{m,N}) \leq \gamma_{m,N}(J) ; \forall J \in \mathcal{S}_m.$$

So,

$$\|\widehat{J}_{m,N} - J_0\|_{\langle M \rangle}^2 \leq \|J - J_0\|_{\langle M \rangle}^2 + \frac{2}{N} \sum_{i=1}^N \int_0^T (\widehat{J}_{m,N}(s) - J(s)) dM_s^i$$

for any $J \in \mathcal{S}_m$, and then

$$\mathbb{E}(\|\widehat{J}_{m,N} - J_0\|_{\langle M \rangle}^2) \leq \|J - J_0\|_{\langle M \rangle}^2 + 2\mathbb{E} \left(\frac{1}{N} \sum_{i=1}^N \int_0^T \widehat{J}_{m,N}(s) dM_s^i \right).$$

Consider $\mathbf{j}_0 = (\langle \varphi_j, J_0 \rangle_{\langle M \rangle})_{j=1, \dots, m}$, and $\mathbf{e} = (\mathbf{e}_1, \dots, \mathbf{e}_m)^*$ such that

$$\mathbf{e}_j := \frac{1}{N} \sum_{i=1}^N \int_0^T \varphi_j(s) dM_s^i ; \forall j \in \{1, \dots, m\}.$$

Since \mathbf{e} is a centered random vector,

$$\begin{aligned} \mathbb{E} \left(\frac{1}{N} \sum_{i=1}^N \int_0^T \widehat{J}_{m,N}(s) dM_s^i \right) &= \sum_{j=1}^m \mathbb{E} \left(\widehat{\theta}_j \cdot \frac{1}{N} \sum_{i=1}^N \int_0^T \varphi_j(s) dM_s^i \right) \\ &= \mathbb{E}(\langle \widehat{\theta}, \mathbf{e} \rangle_{2,m}) = \mathbb{E}(\mathbf{e}^* \Psi_m^{-1} (\mathbf{j}_0 + \mathbf{e})) = \mathbb{E}(\mathbf{e}^* \Psi_m^{-1} \mathbf{e}). \end{aligned}$$

Moreover, since M_1, \dots, M_N are independent copies of M , and since Ψ_m is a symmetric matrix,

$$\begin{aligned} \mathbb{E}(\mathbf{e}^* \Psi_m^{-1} \mathbf{e}) &= \sum_{j,k=1}^m [\Psi_m^{-1}]_{j,k} \mathbb{E}(\mathbf{e}_j \mathbf{e}_k) = \frac{1}{N} \sum_{j,k=1}^m [\Psi_m^{-1}]_{j,k} \int_0^T \varphi_j(s) \varphi_k(s) d\langle M \rangle_s \\ &= \frac{1}{N} \sum_{k=1}^m \sum_{j=1}^m [\Psi_m]_{k,j} [\Psi_m^{-1}]_{j,k} = \frac{1}{N} \sum_{k=1}^m [\Psi_m \Psi_m^{-1}]_{k,k} = \frac{m}{N}. \end{aligned}$$

Therefore,

$$\mathbb{E}(\|\widehat{J}_{m,N} - J_0\|_{\langle M \rangle}^2) \leq \min_{J \in \mathcal{S}_m} \|J - J_0\|_{\langle M \rangle}^2 + \frac{2m}{N}.$$

6.2. **Proof of Corollary 3.2.** Under Assumption 2.2,

$$\min_{J \in \mathcal{S}_m} \|J - J_0\|_{\langle M \rangle}^2 = \min_{J \in \mathcal{S}_m} \|\mu^{1/2}(J - J_0)\|_2^2 = \min_{J \in \mathcal{S}_m(\mu)} \|J - \mu^{1/2}J_0\|_2^2$$

with

$$\mathcal{S}_m(\mu) = \{\iota \in \mathbb{L}^2([0, T], dt) : \exists \varphi \in \mathcal{S}_m, \forall t \in (0, T], \iota(t) = \mu(t)^{1/2}\varphi(t)\}.$$

Since $\mathcal{S}_m(\mu)$ is a closed vector subspace of $\mathbb{L}^2([0, T], dt)$,

$$(8) \quad \min_{J \in \mathcal{S}_m(\mu)} \|J - \mu^{1/2}J_0\|_2^2 = \|p_{\mathcal{S}_m(\mu)}^\perp(\mu^{1/2}J_0) - \mu^{1/2}J_0\|_2^2.$$

Moreover, since $1/\mu$ is continuous from $[0, T]$ into \mathbb{R}_+ under Assumption 2.2,

$$(9) \quad \begin{aligned} \|\widehat{J}_{m,N} - J_0\|_2^2 &= \|\mu^{-1/2}(\widehat{J}_{m,N} - J_0)\|_{\langle M \rangle}^2 \\ &\leq \|\mu^{-1}\|_{\infty, T} \|\widehat{J}_{m,N} - J_0\|_{\langle M \rangle}^2. \end{aligned}$$

Equality (8) together with Inequality (9) allow to conclude.

6.3. **Proof of Theorem 3.4.** The proof of Theorem 3.4 relies on the following lemma, which is a straightforward consequence of a Bernstein type inequality established in Revuz and Yor [24], p. 153, for continuous local martingales vanishing at 0.

Lemma 6.1. *For every $\varepsilon > 0$ and every $\varphi \in \mathbb{L}^2([0, T], d\langle M \rangle_t)$,*

$$\mathbb{P}\left(\frac{1}{N} \sum_{i=1}^N \int_0^T \varphi(s) dM_s^i \geq \varepsilon\right) \leq \exp\left(-\frac{N\varepsilon^2}{2\|\varphi\|_{\langle M \rangle}^2}\right).$$

The proof of Theorem 3.4 is dissected in three steps.

Step 1. As established in the proof of Proposition 3.1, for every $J, K \in \mathcal{S}_m$,

$$\gamma_{m,N}(J) - \gamma_{m,N}(K) = \|J - J_0\|_{\langle M \rangle}^2 - \|K - J_0\|_{\langle M \rangle}^2 - \frac{2}{N} \sum_{i=1}^N \int_0^T (J(s) - K(s)) dM_s^i.$$

Moreover,

$$\gamma_{\widehat{m},N}(\widehat{J}_{\widehat{m},N}) + \text{pen}(\widehat{m}) \leq \gamma_{m,N}(\widehat{J}_{m,N}) + \text{pen}(m)$$

for any $m \in \mathcal{M}_N$, and then

$$\gamma_{\widehat{m},N}(\widehat{J}_{\widehat{m},N}) - \gamma_{m,N}(\widehat{J}_{m,N}) \leq \text{pen}(m) - \text{pen}(\widehat{m}).$$

So, since $2ab \leq a^2 + b^2$ for every $a, b \in \mathbb{R}$, and since $\mathcal{S}_m + \mathcal{S}_{\widehat{m}} \subset \mathcal{S}_{m \vee \widehat{m}}$ under Assumption 3.3,

$$\begin{aligned} \|\widehat{J}_{\widehat{m},N} - J_0\|_{\langle M \rangle}^2 &\leq \|\widehat{J}_{m,N} - J_0\|_{\langle M \rangle}^2 + \frac{2}{N} \sum_{i=1}^N \int_0^T (\widehat{J}_{\widehat{m},N}(s) - \widehat{J}_{m,N}(s)) dM_s^i + \text{pen}(m) - \text{pen}(\widehat{m}) \\ &\leq \|\widehat{J}_{m,N} - J_0\|_{\langle M \rangle}^2 + \frac{1}{4} \|\widehat{J}_{\widehat{m},N} - \widehat{J}_{m,N}\|_{\langle M \rangle}^2 \\ &\quad + 4 \left(\left[\sup_{\varphi \in \mathcal{B}_{m, \widehat{m}}} |\nu_N(\varphi)| \right]^2 - p(m, \widehat{m}) \right)_+ + \text{pen}(m) + 4p(m, \widehat{m}) - \text{pen}(\widehat{m}), \end{aligned}$$

where

$$\mathcal{B}_{m, m'} := \{\varphi \in \mathcal{S}_{m \vee m'} : \|\varphi\|_{\langle M \rangle} \leq 1\} \quad \text{and} \quad p(m, m') := \frac{\mathbf{c}_{\text{cal}}}{4} \cdot \frac{m \vee m'}{N}$$

for every $m' \in \mathcal{M}_N$, and

$$\nu_N(\varphi) := \frac{1}{N} \sum_{i=1}^N \int_0^T \varphi(s) dM_s^i$$

for every $\varphi \in \mathbb{L}^2([0, T], d\langle M \rangle_t)$. Therefore, since $(a + b)^2 \leq 2a^2 + 2b^2$ for every $a, b \in \mathbb{R}$, and since $4p(m, \widehat{m}) \leq \text{pen}(m) + \text{pen}(\widehat{m})$,

$$(10) \quad \|\widehat{J}_{\widehat{m}, N} - J_0\|_{\langle M \rangle}^2 \leq 3\|\widehat{J}_{m, N} - J_0\|_{\langle M \rangle}^2 + 4\text{pen}(m) + 8 \left(\left[\sup_{\varphi \in \mathcal{B}_{m, \widehat{m}}} |\nu_N(\varphi)| \right]^2 - p(m, \widehat{m}) \right)_+.$$

Step 2. By using Lemma 6.1, and by following the pattern of the proof of Baraud et al. [1], Proposition 6.1, the purpose of this step is to find a suitable bound on

$$\mathbb{E} \left[\left(\left[\sup_{\varphi \in \mathcal{B}_{m, m'}} |\nu_N(\varphi)| \right]^2 - p(m, m') \right)_+ \right]; m' \in \mathcal{M}_N.$$

Consider $\delta_0 \in (0, 1)$ and let $(\delta_n)_{n \in \mathbb{N}^*}$ be the real sequence defined by

$$\delta_n := \delta_0 2^{-n}; \forall n \in \mathbb{N}^*.$$

Since $\mathcal{S}_{m \vee m'}$ is a vector subspace of $\mathbb{L}^2([0, T], d\langle M \rangle_t)$ of dimension $m \vee m'$, for any $n \in \mathbb{N}$, there exists $T_n \subset \mathcal{B}_{m, m'}$ such that $|T_n| \leq (3/\delta_n)^{m \vee m'}$ and, for any $\varphi \in \mathcal{B}_{m, m'}$,

$$\exists f_n \in T_n : \|\varphi - f_n\|_{\langle M \rangle} \leq \delta_n.$$

In particular, note that

$$\varphi = f_0 + \sum_{n=1}^{\infty} (f_n - f_{n-1}).$$

Then, for any $v > 0$ and any sequence $(\Delta_n)_{n \in \mathbb{N}}$ of elements of $(0, \infty)$ such that $\Delta := \sum_{n \in \mathbb{N}} \Delta_n < \infty$,

$$\begin{aligned} & \mathbb{P} \left(\left[\sup_{\varphi \in \mathcal{B}_{m, m'}} \nu_N(\varphi) \right]^2 > v^2 \Delta^2 \right) \\ &= \mathbb{P} \left(\exists (f_n)_{n \in \mathbb{N}} \in \prod_{n=0}^{\infty} T_n : \nu_N(f_0) + \sum_{n=1}^{\infty} \nu_N(f_n - f_{n-1}) > v\Delta \right) \\ &\leq \mathbb{P} \left(\exists (f_n)_{n \in \mathbb{N}} \in \prod_{n=0}^{\infty} T_n : \nu_N(f_0) > v\Delta_0 \text{ or } [\exists n \in \mathbb{N}^* : \nu_N(f_n - f_{n-1}) > v\Delta_n] \right) \\ &\leq \sum_{f_0 \in T_0} \mathbb{P}(\nu_N(f_0) > v\Delta_0) + \sum_{n=1}^{\infty} \sum_{(f_{n-1}, f_n) \in \mathbb{T}_n} \mathbb{P}(\nu_N(f_n - f_{n-1}) > v\Delta_n) \end{aligned}$$

with $\mathbb{T}_n = T_{n-1} \times T_n$ for every $n \in \mathbb{N}^*$. Moreover, $\|f_0\|_{\langle M \rangle}^2 \leq \delta_0^2$ and

$$\|f_n - f_{n-1}\|_{\langle M \rangle}^2 \leq 2\delta_{n-1}^2 + 2\delta_n^2 = \frac{5}{2}\delta_{n-1}^2$$

for every $n \in \mathbb{N}^*$. So, by Lemma 6.1,

$$(11) \quad \begin{aligned} \mathbb{P} \left(\left[\sup_{\varphi \in \mathcal{B}_{m, m'}} \nu_N(\varphi) \right]^2 > v^2 \Delta^2 \right) &\leq \sum_{f_0 \in T_0} \exp \left(-\frac{Nv^2 \Delta_0^2}{2\|f_0\|_{\langle M \rangle}^2} \right) \\ &\quad + \sum_{n=1}^{\infty} \sum_{(f_{n-1}, f_n) \in \mathbb{T}_n} \exp \left(-\frac{Nv^2 \Delta_n^2}{2\|f_n - f_{n-1}\|_{\langle M \rangle}^2} \right) \\ &\leq \exp \left(h_0 - \frac{Nv^2 \Delta_0^2}{2\delta_0^2} \right) + \sum_{n=1}^{\infty} \exp \left(h_{n-1} + h_n - \frac{Nv^2 \Delta_n^2}{5\delta_{n-1}^2} \right) \end{aligned}$$

with $h_n = \log(|T_n|)$ for every $n \in \mathbb{N}$. Now, let us take Δ_0 such that

$$h_0 - \frac{Nv^2 \Delta_0^2}{2\delta_0^2} = -(m \vee m' + x) \quad \text{with } v, x > 0,$$

which leads to

$$\Delta_0 = \left[\frac{2\delta_0^2}{v^2 N} (m \vee m' + x + h_0) \right]^{1/2},$$

and for every $n \in \mathbb{N}^*$, let us take Δ_n such that

$$h_{n-1} + h_n - \frac{Nv^2\Delta_n^2}{5\delta_{n-1}^2} = -(m \vee m' + x + n),$$

which leads to

$$\Delta_n = \left[\frac{5\delta_{n-1}^2}{v^2 N} (m \vee m' + x + h_{n-1} + h_n + n) \right]^{1/2}.$$

For this appropriate sequence $(\Delta_n)_{n \in \mathbb{N}}$,

$$\mathbb{P} \left(\left[\sup_{\varphi \in \mathcal{B}_{m,m'}} \nu_N(\varphi) \right]^2 > v^2 \Delta^2 \right) \leq e^{-x} e^{-(m \vee m')} \left(1 + \sum_{n=1}^{\infty} e^{-n} \right) \leq 1.6 e^{-x} e^{-(m \vee m')}$$

by Inequality (11), and

$$\begin{aligned} \Delta^2 &\leq \frac{1}{v^2 N} \left[\sqrt{2}\delta_0[(m \vee m' + x)^{1/2} + h_0^{1/2}] + \sqrt{5} \sum_{n=1}^{\infty} \delta_{n-1}[(m \vee m' + x)^{1/2} + (h_{n-1} + h_n + n)^{1/2}] \right]^2 \\ &\leq \frac{\delta^{(1)}}{v^2 N} (m \vee m' + x) + \frac{\delta^{(2)}}{v^2 N} \leq \frac{\delta^{(1)} + \delta^{(2)}}{v^2 N} (m \vee m' + x) \end{aligned}$$

with

$$\delta^{(1)} = 2 \left(\sqrt{2}\delta_0 + \sqrt{5} \sum_{n=1}^{\infty} \delta_{n-1} \right)^2 \quad \text{and} \quad \delta^{(2)} = 2 \left(\sqrt{2}\delta_0 h_0^{1/2} + \sqrt{5} \sum_{n=1}^{\infty} \delta_{n-1} (h_{n-1} + h_n + n)^{1/2} \right)^2.$$

So,

$$\mathbb{P} \left(\left[\sup_{\varphi \in \mathcal{B}_{m,m'}} \nu_N(\varphi) \right]^2 - \frac{\delta^{(1)} + \delta^{(2)}}{v^2} p(m, m') > \frac{\delta^{(1)} + \delta^{(2)}}{v^2 N} x \right) \leq 1.6 e^{-x} e^{-(m \vee m')}$$

and then, by taking $v = (\delta^{(1)} + \delta^{(2)})^{1/2}$ and $y = x/N$,

$$\mathbb{P} \left(\left[\sup_{\varphi \in \mathcal{B}_{m,m'}} \nu_N(\varphi) \right]^2 - p(m, m') > y \right) \leq 1.6 e^{-Ny} e^{-(m \vee m')}.$$

Therefore,

$$\begin{aligned} \mathbb{E} \left[\left(\left[\sup_{\varphi \in \mathcal{B}_{m,m'}} \nu_N(\varphi) \right]^2 - p(m, m') \right)_+ \right] &= \int_0^{\infty} \mathbb{P} \left(\left[\sup_{\varphi \in \mathcal{B}_{m,m'}} \nu_N(\varphi) \right]^2 - p(m, m') > y \right) dy \\ (12) \quad &\leq 1.6 \frac{e^{-(m \vee m')}}{N}. \end{aligned}$$

Step 3. By Inequality (12), there exists a deterministic constant $\mathbf{c}_1 > 0$, not depending on m and N , such that

$$\begin{aligned} \mathbb{E} \left[\left(\left[\sup_{\varphi \in \mathcal{B}_{m,\hat{m}}} |\nu_N(\varphi)| \right]^2 - p(m, \hat{m}) \right)_+ \right] &\leq \sum_{m' \in \mathcal{M}_N} \mathbb{E} \left[\left(\left[\sup_{\varphi \in \mathcal{B}_{m,m'}} \nu_N(\varphi) \right]^2 - p(m, m') \right)_+ \right] \\ &\leq \frac{1.6}{N} \sum_{m' \in \mathcal{M}_N} e^{-(m \vee m')} \leq \frac{1.6}{N} \left(m e^{-m} + \sum_{m' > m} e^{-m'} \right) \leq \frac{\mathbf{c}_1}{N}. \end{aligned}$$

Therefore, by Inequality (10),

$$\mathbb{E}(\|\widehat{J}_{\widehat{m},N} - J_0\|_{\langle M \rangle}^2) \leq \min_{m \in \mathcal{M}_N} \{3\mathbb{E}(\|\widehat{J}_{m,N} - J_0\|_{\langle M \rangle}^2) + 4\text{pen}(m)\} + \frac{8\mathfrak{c}_1}{N}.$$

6.4. Proof of Proposition 4.1. First of all, recall that

$$\mathcal{Q} = \{Q : \text{the function } t \mapsto t^{1/2-H}Q(t) \text{ belongs to } \mathbb{L}^1([0, T], dt)\}$$

and

$$\mathcal{J} = \left\{ \iota : \text{the function } t \in [0, T] \mapsto \int_0^t s^{1-2H}\iota(s)ds \text{ belongs to } \mathcal{I}_{0+}^{3/2-H}(\mathbb{L}^1([0, T], dt)) \right\}.$$

In the sequel, $\mathcal{I}_{0+}^\alpha(\cdot)$ (resp. $\mathcal{D}_{0+}^\alpha(\cdot)$) is the Riemann-Liouville left-sided fractional integral (resp. derivative) of order $\alpha > 0$. Consider $Q \in \mathcal{Q}$. Then, the map $Q_H : t \mapsto t^{1/2-H}Q(t)$ belongs to $\mathbb{L}^1([0, T], dt)$,

$$\begin{aligned} J(Q)(t) &= \frac{\mathfrak{c}_H}{2-2H} t^{2H-1} \frac{\partial}{\partial t} \int_0^t s^{1/2-H} (t-s)^{1/2-H} Q(s) ds \\ &= \frac{\mathfrak{c}_H}{2-2H} \Gamma(3/2-H) t^{2H-1} \mathcal{I}_{0+}^{3/2-H} (Q_H)'(t) \end{aligned}$$

for almost every $t \in [0, T]$, and $J(Q) \in \mathcal{J}$. Therefore, the map J is one-to-one from \mathcal{Q} into \mathcal{J} , and

$$\begin{aligned} J^{-1}(\iota)(t) &= \frac{2-2H}{\mathfrak{c}_H \Gamma(3/2-H)} t^{H-1/2} \mathcal{D}_{0+}^{3/2-H} \left(\int_0^\cdot s^{1-2H} \iota(s) ds \right) (t) \\ &= \bar{\mathfrak{c}}_H t^{H-1/2} \int_0^t (t-s)^{H-3/2} s^{1-2H} \iota(s) ds \end{aligned}$$

for every $\iota \in \mathcal{J}$ and almost every $t \in [0, T]$.

6.5. Proof of Proposition 4.2. Since $\langle M \rangle_t = t^{2-2H}$ for every $t \in [0, T]$ (see Norros et al. [20]), the Molchan martingale M fulfills Assumption 2.2 with $\mu(t) = t^{1-2H}$ for every $t \in (0, T]$. Consider $\iota \in \mathcal{J} \cap \mathbb{L}^2([0, T], d\langle M \rangle_t)$, and note that $J^{-1}(\iota) \in \mathcal{Q}$ thanks to Proposition 4.1. By Cauchy-Schwarz's inequality,

$$\begin{aligned} \|J^{-1}(\iota)\|_2^2 &= \bar{\mathfrak{c}}_H^2 \int_0^T t^{2H-1} \left(\int_0^t (t-s)^{H-3/2} s^{1-2H} \iota(s) ds \right)^2 dt \\ &\leq \bar{\mathfrak{c}}_H^2 \int_0^T t^{2H-1} \theta(t) \int_0^t (t-s)^{H-3/2} s^{1-2H} \iota(s)^2 ds dt \end{aligned}$$

with

$$\theta(t) := \int_0^t (t-s)^{H-3/2} s^{1-2H} ds ; \forall t \in (0, T].$$

Note that $\bar{\theta} : t \mapsto t^{2H-1}\theta(t)$ is bounded on $(0, T]$. Indeed, for every $t \in (0, T]$,

$$\begin{aligned} |\bar{\theta}(t)| &\leq t^{2H-1} \left[\left(\frac{t}{2}\right)^{H-3/2} \int_0^{t/2} s^{1-2H} ds + \left(\frac{t}{2}\right)^{1-2H} \int_{t/2}^t (t-s)^{H-3/2} ds \right] \\ &= t^{2H-1} \left[\frac{1}{2-2H} \left(\frac{t}{2}\right)^{H-3/2} \left(\frac{t}{2}\right)^{2-2H} + \frac{1}{H-1/2} \left(\frac{t}{2}\right)^{1-2H} \left(\frac{t}{2}\right)^{H-1/2} \right] \\ &= \frac{1}{2^{1/2-H}} \left(\frac{1}{2-2H} + \frac{1}{H-1/2} \right) t^{H-1/2} \xrightarrow{t \rightarrow 0^+} 0. \end{aligned}$$

So, by Fubini's theorem,

$$\begin{aligned} \|J^{-1}(\iota)\|_2^2 &\leq \bar{c}_H^2 \|\bar{\theta}\|_{\infty, T} \int_0^T \int_0^T (t-s)^{H-3/2} s^{1-2H} \iota(s)^2 \mathbf{1}_{(s, T]}(t) ds dt \\ &= \bar{c}_H^2 \|\bar{\theta}\|_{\infty, T} \int_0^T s^{1-2H} \iota(s)^2 \int_s^T (t-s)^{H-3/2} dt ds \\ &\leq (H-1/2)^{-1} \bar{c}_H^2 T^{H-1/2} \|\bar{\theta}\|_{\infty, T} \int_0^T s^{1-2H} \iota(s)^2 ds. \end{aligned}$$

Then, $J^{-1}(\iota) \in \mathcal{Q} \cap \mathbb{L}^2([0, T], dt)$ and

$$(13) \quad \|J^{-1}(\iota)\|_2^2 \leq \mathbf{c}_1 \|\iota\|_{\langle M \rangle}^2 \quad \text{with} \quad \mathbf{c}_1 = (H-1/2)^{-1} \bar{c}_H^2 T^{H-1/2} \|\bar{\theta}\|_{\infty, T}.$$

The conclusion comes from Inequality (13), Proposition 3.1 and Theorem 3.4.

6.6. Proof of Proposition 4.3. Thanks to Samko et al. [25], Theorem 2.3,

$$\mathcal{J} = \left\{ \iota : \mathcal{I}_{0+}^{H-1/2} \left(\int_0^\cdot s^{1-2H} \iota(s) ds \right) \text{ is absolutely continuous from } [0, T] \text{ into } \mathbb{R} \right\}.$$

So, since Lipschitz continuous functions are absolutely continuous,

$$\mathcal{J}_{\text{Lip}} := \left\{ \iota : \mathcal{I}_{0+}^{H-1/2} \left(\int_0^\cdot s^{1-2H} \iota(s) ds \right) \text{ is Lipschitz continuous from } [0, T] \text{ into } \mathbb{R} \right\}$$

is a subset of \mathcal{J} . Moreover, for every $\iota \in \mathcal{J}_{\text{Lip}}$ and $t \in [0, T]$, by Fubini's theorem,

$$\begin{aligned} \mathcal{I}_{0+}^{H-1/2} \left(\int_0^\cdot s^{1-2H} \iota(s) ds \right) (t) &= \frac{1}{\Gamma(H-1/2)} \int_0^t (t-u)^{H-3/2} \int_0^u s^{1-2H} \iota(s) ds du \\ &= \frac{1}{\Gamma(H-1/2)} \int_0^t s^{1-2H} \iota(s) \int_s^t (t-u)^{H-3/2} du ds \\ &= \frac{1}{(H-1/2)\Gamma(H-1/2)} \int_0^t (t-s)^{H-1/2} s^{1-2H} \iota(s) ds. \end{aligned}$$

Then,

$$\mathcal{J}_{\text{Lip}} = \left\{ \iota : \text{the function } t \in [0, T] \mapsto \int_0^t (t-s)^{H-1/2} s^{1-2H} \iota(s) ds \text{ is Lipschitz continuous} \right\}.$$

Now, recall that

$$\mathbb{J} = \left\{ \iota \in C^1([0, T]; \mathbb{R}) : \lim_{t \rightarrow 0^+} t^{-2H} \iota(t) \text{ and } \lim_{t \rightarrow 0^+} t^{1-2H} \iota'(t) \text{ exist and are finite} \right\}$$

and let us show that $\mathbb{J} \subset \mathcal{J}_{\text{Lip}} \subset \mathcal{J}$. For any $\iota \in \mathbb{J}$ and every $t \in [0, T]$,

$$\begin{aligned} \bar{\iota}(t) &:= \int_0^t (t-s)^{H-1/2} s^{1-2H} \iota(s) ds \\ &= - \lim_{\varepsilon \rightarrow 0} \left[s^{1-2H} \iota(s) \frac{(t-s)^{H+1/2}}{H+1/2} \right]_{\varepsilon}^t \\ &\quad + \int_0^t ((1-2H)s^{-2H} \iota(s) + s^{1-2H} \iota'(s)) \frac{(t-s)^{H+1/2}}{H+1/2} ds = \int_0^t \theta(s) (t-s)^{H+1/2} ds \end{aligned}$$

with

$$\theta(s) := \frac{1}{H+1/2} ((1-2H)s^{-2H} \iota(s) + s^{1-2H} \iota'(s)) ; \forall s \in (0, T].$$

Since $\iota \in \mathbb{J}$, the map $\theta : t \mapsto \theta(t)$ is bounded on $(0, T]$, and for every $t_1, t_2 \in [0, T]$ such that $t_2 > t_1$,

$$\begin{aligned} |\bar{\iota}(t_2) - \bar{\iota}(t_1)| &\leq \int_{t_1}^{t_2} \theta(s) (t_2-s)^{H+1/2} ds + \int_0^{t_1} \theta(s) |(t_2-s)^{H+1/2} - (t_1-s)^{H+1/2}| ds \\ &\leq (H+3/2) \|\theta\|_{\infty, T} T^{H+1/2} |t_2 - t_1|. \end{aligned}$$

So, the function $t \mapsto \bar{\iota}(t)$ is Lipschitz continuous from $[0, T]$ into \mathbb{R} , which leads to $\iota \in \mathcal{J}_{\text{Lip}}$.

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