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# Almost global attitude stabilisation of an underactuated axially symmetric 3-D pendulum <sup>★</sup>

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## Abstract

We investigate the problem of stabilising the attitude of an axially symmetric 3-D pendulum. The system is assumed to be actuated by two torques acting orthogonally to the symmetry axis. In the process of stabilising the system, two main difficulties are encountered: first, the presence of a well-known topological obstruction to the global stabilisation by means of continuous feedback laws, and second, the fact that the linearised system around the target equilibrium is not controllable due to the strong underactuation of the system. To cope with the first issue, in this paper we look for feedback laws capable of stabilising the system starting from almost all initial conditions. The second issue entails the impossibility of stabilising the system exponentially by means of a smooth feedback. The contributions of this paper are (1) to propose a family of smooth feedback laws capable of almost-globally stabilising the system with a polynomial rate of convergence, and (2) to define a second family of feedback laws presenting a discontinuity at the target attitude configuration and continuous elsewhere inducing almost global exponential stabilisation.

*Key words:* Attitude control, 3-D spherical pendulum, stabilisation, nonlinear control systems

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## 1 Introduction

The 3-D spherical pendulum is a benchmark mechanical system providing a simplified model for robotic and spacecraft systems [12,21,24,28,11] as well as for the human stance [13]. The space of its configurations is characterised by 3 rotational degrees of freedom (DOFs) and by 3 or 2 translational DOFs, depending on whether the pivot of the pendulum is free or constrained to move on a plane. When the pendulum presents a symmetry axis it is usually referred to as axially symmetric pendulum. Often, in this case, a reduced attitude is considered by ignoring rotations around the symmetry axis. The ensuing system, usually named 2-D spherical pendulum, requires only two rotational DOFs.

Despite its deceiving simplicity, the 3-D pendulum is a source of many challenging control problems (see for instance [8,9,23] for a glimpse of the recent literature on the topic). If the goal is to stabilise the full attitude of

a 3-D pendulum, for instance in the upright position with a given angle around the vertical axis, then usually three control inputs are used, but it is not always a trivial task. We recall, indeed, that, while a locally stabilising, time invariant smooth feedback can be defined in the case of three independent inputs [5], topological obstructions prevent the construction of a global feedback with the same characteristics [26,3]. In [9] it has been shown that three torques allow the almost global, asymptotic stabilisation of the complete attitude in the upright equilibrium with a smooth control law.

In the case of a 2-D spherical pendulum, two inputs are sufficient not only to stabilise the pendulum to the unstable equilibrium and the cart (planar position of the pivot) to a desired position on a large domain of attraction (see [4,18]), but also to impart even more complex motions to the system: the pivot can follow a circular path, while keeping the attitude confined in a cone close to the upright position (see [17]).

The stabilisation, even in a local sense, becomes tougher when a stronger under-actuation is present, i.e., in the case of a 3-D pendulum, when only two control inputs

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are available. Indeed, the complete attitude cannot be locally asymptotically stabilised to an equilibrium by any time-invariant continuous state feedback control law [21]. In the case the two controlled torques are directed orthogonally to the symmetry axis, the linearised system about the upright equilibrium is not even controllable. Which fact implies that no smooth feedback can ensure the stabilisation with an exponential rate. Therefore, in [12] a discontinuous feedback and in [24,11] time-varying smooth feedbacks have been proposed to locally stabilise the attitude of a spacecraft, essentially a 3-D pendulum without gravity, by means of two inputs. In [8] two smooth inputs (torques) are used to almost globally, asymptotically stabilise a 2-D spherical pendulum in the upright position. We stress that the reduced attitude only is stabilised here. The full attitude is globally asymptotically stabilised in [21] via a discontinuous control law based on sequential manoeuvres and in [6,27] by means of hybrid feedbacks. We remark that in [27] the asymptotic stability is achieved in a practical sense. In [28] the dynamics of an axially symmetric spacecraft is considered. Two torques are used to stabilise the complete attitude, but the control law depends on the initial conditions, which have to belong to a compact annular subset of the state space not containing the target equilibrium. The feedback is smooth except in the origin, where it is singular.

In this paper we focus on an axially symmetric 3-D pendulum actuated by two torques acting on a plane orthogonal to the symmetry axis, with the pivot constrained to move on a plane. We address the problem of stabilising the complete attitude in the upright position, assuming zero angular velocity along the symmetry axis. The stabilisation problem is tackled in two steps. First, we define a virtual feedback for the angular velocity guaranteeing the attitude stabilisation for almost every initial condition. Second, we look for a couple of control torques ensuring the convergence of the actual angular velocity to the virtual feedback. The virtual feedback is defined in terms of two weighting functions ( $\gamma_1$  and  $\gamma_2$ ), whose choice may produce a smooth as well as a non-smooth control law. We present here two distinct results. In the first one, whose preliminary version can be found in [22], the virtual feedback is smooth and the control torques are chosen to ensure an exponential convergence of the angular velocity to the virtual feedback. To the best of our knowledge, this is the first example in literature of a family of smooth feedback laws that almost globally asymptotically stabilise the system to the target configuration. The smoothness of the controller, however, comes with the limitation of a polynomial rate of convergence. To overcome such drawback, we present a second result where the virtual feedback is smooth everywhere except on a zero measure set and the control torques are chosen to ensure the convergence of the angular velocity to the virtual feedback in finite time. The resulting overall control law is capable of almost globally stabilising the full attitude with an exponential rate. The non-smooth

feedback, however, could potentially lead to a high value control signal when the state is close to the discontinuity set. We stress that both feedbacks have merits and limitations and it is up to the designer to choose the trade-off (speed of convergence vs low control values) that better fits his/her needs.

The paper is organised as follows. In Section 2 we briefly present the model and state the stabilisation problem, further discussing the main technical difficulties and limitations. In Section 3 we show our main stabilisation results: first, in Section 3.1, we identify a general family of virtual feedback laws for the angular velocity guaranteeing the almost global attitude stabilisation and then, in Sections 3.2.1 and 3.2.2, we show the almost global attitude stabilisation of the complete system with a smooth feedback law and with a non-smooth one, respectively. Finally, in Section 4, the effectiveness and characteristics of the proposed feedback laws are investigated via some simulations.

### Notation

The unit 3-sphere, i.e. the set of unit vectors in  $\mathbb{R}^4$ , is denoted by  $S^3$ . We denote by  $SO(3)$  the group of matrices  $R \in \mathbb{R}^{3 \times 3}$  satisfying  $R^{-1} = R^T$  and  $\det(R) = 1$  (special orthogonal group). The symbol  $\wedge$  denotes the usual cross product in  $\mathbb{R}^3$ . With each vector  $w = (w_1, w_2, w_3)^T$  we associate a skew-symmetric matrix

$$\hat{w} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}.$$

Recall that  $\hat{w}x = w \wedge x$  for any  $w, x \in \mathbb{R}^3$ . Given a function  $F$  defined on a manifold  $\mathcal{M}$  we use  $\dot{F}$  to denote the (Lie) derivative of  $F$  along the flow of a differential equation on  $\mathcal{M}$ , assuming that the differential equation is clear from the context.

## 2 Problem formulation

The derivation of the mathematical model for the mechanical system considered in this paper is fairly standard. A thorough analysis of its features and dynamical properties is conducted in [7]. We consider here a simplified model of a 3-D pendulum of mass  $m$ , whose pivot is constrained on the horizontal plane. An inertial frame is centred in the pivot with the first two axes lying in the horizontal plane and the third one pointing opposite to the gravity vector. A body-fixed frame is centred in the pivot, with the third axis aligned with the vector from the pivot to the centre of mass (the symmetry axis). We denote with  $J_{\text{piv}} = \text{diag}(J_1, J_2, J_3)$  the inertia matrix with respect to the pivot in the body-fixed frame. Due to

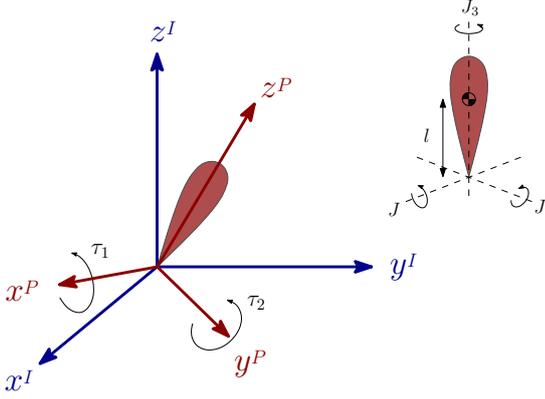


Fig. 1. Representation of a controlled axially symmetric pendulum; the torques  $\tau_1, \tau_2$  acting on the pendulum are directed along the axes  $x^P, y^P$  of the body-fixed frame.

symmetry, in the following we assume that  $J_1 = J_2 = J$  and  $J \neq J_3$ . We use a rotation matrix  $R \in SO(3)$  to describe the state of the 3-D pendulum:  $R$  describes the orientation of the body-fixed frame with respect to the inertial frame. The angular velocity vector is represented, in the body-fixed frame, by  $\omega = (\omega_1, \omega_2, \omega_3)^T \in \mathbb{R}^3$ .

We assume that the pendulum is actuated by a pair of torques  $\tau_1, \tau_2$  acting on a plane orthogonal to the symmetry axis (see Figure 1). We set  $\tau = (\tau_1, \tau_2, 0)^T$ . The dynamics of the pendulum is given by

$$J_{\text{piv}} \dot{\omega} = (J_{\text{piv}} \omega) \wedge \omega + mg(R^T e_3) \wedge w_{\text{cm}} + \tau, \quad (1)$$

where  $g$  is the gravity acceleration,  $e_3 = (0, 0, 1)^T$ ,  $w_{\text{cm}} = l e_3$  is the centre of mass of the pendulum in the body-fixed frame. The rotational kinematics equation is

$$\dot{R} = R \hat{\omega}. \quad (2)$$

From (1) it is easy to see that  $\dot{\omega}_3 = 0$ , which implies that the system is not completely controllable. Therefore, we assume that  $\omega_3 \equiv 0$  and we focus on the dynamics of the remaining variables. The fact that  $J_1 = J_2$  and  $\omega_3 \equiv 0$  implies that  $(J_{\text{piv}} \omega) \wedge \omega \equiv 0$  in (1).

We choose

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = mglP\hat{e}_3R^T e_3 + Ju \quad (3)$$

where

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (4)$$

and  $u = (u_1, u_2)^T$  are the new control variables. By

using (3), equation (1) reduces to

$$\begin{aligned} \dot{\omega}_1 &= u_1 \\ \dot{\omega}_2 &= u_2. \end{aligned} \quad (5)$$

In order to analyse the rotational kinematics (2) it is convenient to rewrite rotations in terms of quaternions. Recall that any rotation matrix may be identified with a rotation axis, represented by a unit vector  $p$ , and an angle  $\alpha$ , that is  $R = \exp(\alpha \hat{p})$ . This allows us to define the associated unit quaternion as

$$\begin{aligned} \mathbf{q} &= (q_0, q) \in S^3 \\ q_0 &= \cos \frac{\alpha}{2}, \quad q = (q_1, q_2, q_3) = p \sin \frac{\alpha}{2}. \end{aligned}$$

Note that the quaternions  $\mathbf{q}$  and  $-\mathbf{q}$  identify the same rotation. The kinematics (2) in the quaternion setting takes the form [10]

$$\begin{aligned} \dot{q}_0 &= -\frac{1}{2} q^T \omega \\ \dot{q} &= \frac{1}{2} q \wedge \omega + \frac{1}{2} q_0 \omega. \end{aligned} \quad (6)$$

We remark that the coupled system (5)-(6) is an equivalent formulation of the one considered in [28].

The purpose of this work is to define a suitable feedback law capable of asymptotically steering the body-fixed frame to the inertial frame. In terms of quaternions, setting  $\mathbf{q}_d = (1, 0, 0, 0)$ , this amounts to finding a feedback control  $(u_1, u_2)$  for the system (5)-(6) capable of asymptotically steering  $(\omega_1, \omega_2, \mathbf{q}) \in \mathbb{R}^2 \times S^3$  to  $(0, 0, \mathbf{q}_d)$ .

When tackling this problem one has to deal with two main issues. First of all, a well-known topological obstruction impedes the global stabilisation of the system to the equilibrium. More precisely, the manifold  $\mathbb{R}^2 \times S^3$  turns out to be not contractible as a consequence of the non-contractibility of  $S^3$ . Then [26, Corollary 5.9.13] implies that no globally stabilising feedback exists for (5)-(6). A similar reasoning applies also to the original system (1)-(2). Thus, in the following we will focus on the almost global stabilisation of the system, that is, we will look for feedback laws capable of stabilising the system except for a zero measure set of initial conditions  $(\omega_1, \omega_2, \mathbf{q}) \in \mathbb{R}^2 \times S^3$ . A further important issue is illustrated in the following. By using the components  $q$  as a local set of coordinates to describe the quaternion variables around  $(0, 0, \mathbf{q}_d)$ , the equilibrium becomes the origin in  $\mathbb{R}^5$ . The linearised system is simply given by  $\dot{\omega}_1 = u_1, \dot{\omega}_2 = u_2, \dot{q} = \frac{1}{2} \omega$  and it is therefore not controllable (recall that  $\omega_3 = 0$ ). As a consequence, independently of the chosen feedback law, the linearisation of the closed loop system (5)-(6) is necessarily associated with a singular matrix. We immediately deduce the following result.

**Proposition 1** *There exists no feedback law  $(u_1, u_2)$  for (5)-(6) which is continuously differentiable at  $(0, 0, \mathbf{q}_d)$  and that makes  $(0, 0, \mathbf{q}_d)$  a (locally) exponentially stable equilibrium for the closed loop system.*

The previous proposition motivates the study of the stabilisation problem by considering separately smooth and non-smooth feedback laws. In the first case one should expect a regular behaviour of the closed loop system on a neighbourhood of the target equilibrium, at the cost of a slow convergence towards it. In the second case, one may look for non-smooth feedback laws guaranteeing exponential convergence to the target state, however, as a counterpart, it is reasonable to expect complicated expressions for the control laws and abrupt changes or high values of the control laws on a neighbourhood of the equilibrium.

**Remark 1** *The representation of the attitude in terms of quaternions has some drawbacks. Namely, the space of quaternions  $S^3$  provides a double covering of the space of rotation matrices  $SO(3)$ , hence any feedback depending on the quaternion variables gives rise to two (a priori) distinct actions associated with the same rotation matrix. With any measured rotation  $R(t)$  are associated two possible quaternion representation  $\pm \mathbf{q}$ . From a practical point of view, in order to ensure the continuity of the solution of (5)-(6) with a given feedback law, the controller must select a continuous representation  $\mathbf{q}(t)$  of  $R(t)$ , whose existence is guaranteed by the continuity of  $R(t)$ .*

Another drawback of the quaternion representation is that the system dynamics may exhibit the unwinding phenomenon (see [3]). For instance, while  $(0, 0, \mathbf{q}_d)$  and  $(0, 0, -\mathbf{q}_d)$  correspond to the same physical configuration, they are perceived as different points from the controller. In particular, in our case  $(0, 0, \mathbf{q}_d)$  is the stable equilibrium of the controlled dynamics, while  $(0, 0, -\mathbf{q}_d)$  is not necessarily an equilibrium of the system. As a consequence, the evolution of the system may pass through the point  $(0, 0, -\mathbf{q}_d)$  without realising that it corresponds to the target configuration. An appropriate initialisation of the quaternion  $\mathbf{q}(0)$  may alleviate this issue, for instance  $\mathbf{q}(0)$  may be chosen so that  $q_0 \geq 0$ . Note, in any case, that the unwinding phenomenon does not prevent the stabilisation of the system, it can just render it slightly inefficient.

### 3 Main results

We divide the stabilisation problem in two steps. First, we consider the system (6) with  $\omega$  as a control variable, and we identify a rather large family of feedbacks  $\omega_{\text{ref}}$  ensuring that  $q_0$  goes to 1 for almost every initial condition. Second, under additional technical assumptions on  $\omega_{\text{ref}}$ , we show that a feedback  $u$  for (5), guaranteeing

that the solution  $\omega$  converges to  $\omega_{\text{ref}}$ , also almost globally stabilises the full system (5)-(6). The smooth case, with a polynomial rate of convergence, and the non-smooth one, with exponential convergence, are studied separately.

#### 3.1 Stabilisation of the rotation kinematics

We look for functions  $\omega = (\omega_1, \omega_2, 0)$  of the form

$$\omega(\mathbf{q}) = \gamma_1(\mathbf{q})(e_3 \wedge q) + \gamma_2(\mathbf{q})(e_3 \wedge (e_3 \wedge q)), \quad (7)$$

for some functions  $\gamma_1, \gamma_2$ . Note that in the feedback above,  $\omega = 0$  whenever  $q$  is parallel to  $e_3$ , that is whenever  $q_1 = q_2 = 0$ . In other words, for any choice of the functions  $\gamma_1, \gamma_2$  the set of unit quaternions  $Q_0 = \{\mathbf{q} \in S^3 : q_1 = q_2 = 0\}$  is made up of equilibria of the system<sup>1</sup>. We would like to find  $\gamma_1, \gamma_2$  such that, whenever we start outside  $Q_0$ , the trajectory always converges to  $\mathbf{q}_d$ .

The advantage of the form (7) is that the dynamics of  $q_0$  and  $q_3$  are described by very simple equations in terms of  $\gamma_1, \gamma_2$ . Indeed, setting  $f(\mathbf{q}) = q_1^2 + q_2^2 = 1 - q_0^2 - q_3^2$  we have

$$\begin{aligned} \dot{q}_0 &= -\frac{1}{2}q^T \omega(\mathbf{q}) \\ &= -\frac{1}{2}\gamma_2(\mathbf{q})q^T(e_3 \wedge (e_3 \wedge q)) \\ &= \frac{1}{2}\gamma_2(\mathbf{q})|e_3 \wedge q|^2 \\ &= \frac{1}{2}\gamma_2(\mathbf{q})f(\mathbf{q}) \end{aligned} \quad (8)$$

and

$$\begin{aligned} \dot{q}_3 &= e_3^T \dot{q} \\ &= \frac{1}{2}e_3^T(q \wedge \omega(\mathbf{q})) \\ &= \frac{1}{2}\gamma_1(\mathbf{q})e_3^T(q \wedge (e_3 \wedge q)) + \frac{1}{2}\gamma_2(\mathbf{q})e_3^T(q \wedge (e_3 \wedge (e_3 \wedge q))) \\ &= \frac{1}{2}\gamma_1(\mathbf{q})|e_3 \wedge q|^2 - \frac{1}{2}\gamma_2(\mathbf{q})q^T \hat{e}_3^3 q \\ &= \frac{1}{2}\gamma_1(\mathbf{q})f(\mathbf{q}), \end{aligned} \quad (9)$$

which also imply

$$\begin{aligned} \dot{f}(\mathbf{q}) &= \frac{d}{dt}(1 - q_0^2 - q_3^2) \\ &= -(q_0\gamma_2(\mathbf{q}) + q_3\gamma_1(\mathbf{q}))f(\mathbf{q}). \end{aligned} \quad (10)$$

<sup>1</sup> The set  $Q_0 \setminus \{\mathbf{q}_d\}$  corresponds to the configurations which differ from the target configuration only by a rotation about the symmetry axis.

We have the following result.

**Proposition 2** *Assume that the function  $\gamma_2$  is  $C^1$  and strictly positive outside  $Q_0$ . Moreover, set  $\gamma_1(\mathbf{q}) = -\frac{1}{1-q_0}k(\mathbf{q})q_3$ , where  $k$  is a  $C^1$  function satisfying  $k(\mathbf{q}) \geq \gamma_2(\mathbf{q})/2$  for  $\mathbf{q}$  belonging to a neighbourhood  $U$  of  $Q_0$ . Then for every initial condition  $\mathbf{q} \notin Q_0$  the corresponding trajectory of (6) with the feedback law (7) converges asymptotically to  $\mathbf{q}_d$ , without reaching  $\mathbf{q}_d$  in finite time.*

*Proof.* We first show that if the initial condition is different from  $\mathbf{q}_d$  then the trajectory cannot reach  $\mathbf{q}_d$  in finite time. Let us define

$$V(\mathbf{q}) = (1 - q_0)^2.$$

We observe that

$$f(\mathbf{q}) \leq 1 - q_0^2 \leq 2(1 - q_0) \quad (11)$$

for any  $\mathbf{q} \in S^3$ , so that

$$\dot{V}(\mathbf{q}) = -\gamma_2(\mathbf{q})(1 - q_0)f(\mathbf{q}) \geq -2\gamma_2(\mathbf{q})V(\mathbf{q}).$$

Since  $\gamma_2$  is continuous, and thus bounded, on the compact set  $S^3$  we deduce from Gronwall lemma that  $V(\mathbf{q}(t)) > 0$  for all  $t \geq 0$ , and in particular  $\mathbf{q}_d$  cannot be reached in finite time.

On the other hand, if  $\gamma_2$  is chosen to be positive outside  $Q_0$  we have that  $\dot{V}(\mathbf{q}) \leq 0$  with equality holding true only on  $Q_0$ . Although the feedback law may be discontinuous at the point  $\mathbf{q}_d$ , a trivial adaptation of the LaSalle invariance principle shows that any trajectory of the system whose initial condition is different from  $\mathbf{q}_d$  must necessarily converge to  $Q_0$ .

Note that the convergence to  $Q_0$  is solely a consequence of the fact that  $\gamma_2 > 0$  outside  $Q_0$ , and is obtained independently of the choice of  $\gamma_1$ . Next, by exploiting the extra condition on  $\gamma_1$ , we show that any trajectory starting outside  $Q_0$  converges precisely to  $\mathbf{q}_d$ . Let us consider the function

$$W(\mathbf{q}) = \frac{1 - q_0}{f(\mathbf{q})},$$

which is well defined outside  $Q_0$ . Note that  $f$  goes to zero as a consequence of what precedes. Hence, if we show that  $\dot{W} \leq 0$  on a neighbourhood of  $Q_0$ , we can conclude that the function  $1 - q_0$  is dominated by a multiple of  $f$

and thus must also converge to 0. We have

$$\begin{aligned} \dot{W} &= \frac{-\dot{q}_0 f - (1 - q_0)\dot{f}}{f^2} \\ &= \frac{-\frac{1}{2}\gamma_2 f^2 + (1 - q_0)(q_0\gamma_2 + q_3\gamma_1)f}{f^2} \\ &= \frac{-\frac{1}{2}\gamma_2(1 - q_0^2 - q_3^2) + (1 - q_0)q_0\gamma_2 - q_3^2 k}{f} \\ &= -\frac{\gamma_2(1 - q_0)^2}{2f} - (k - \gamma_2/2)\frac{q_3^2}{f} \leq 0 \end{aligned} \quad (12)$$

on the neighbourhood  $U$  of  $Q_0$ , which concludes the proof of the asymptotic convergence to  $\mathbf{q}_d$ .  $\square$

**Remark 2** *We stress that under the assumptions of Proposition 2 the feedback law (7) may be discontinuous at  $\mathbf{q}_d$ . For instance, this happens when one considers constant functions  $\gamma_2, k$ .*

As in Proposition 1, one can show that no smooth feedback control law can stabilise (6) exponentially. We show below that exponential stabilisation is possible under slightly stronger assumptions than those in Proposition 2. In this case the feedback law becomes discontinuous at  $\mathbf{q}_d$ . Furthermore, despite the discontinuity at  $\mathbf{q}_d$ , the control  $\omega$  converges to zero along trajectories of the system.

**Proposition 3** *In addition to the hypotheses of Proposition 2, assume that  $k(\mathbf{q}_d) > \frac{1}{2}\gamma_2(\mathbf{q}_d) > 0$ . Then, for every solution  $\mathbf{q}(\cdot)$  of (6) with the feedback law (7) and such that  $\mathbf{q}(0) \notin Q_0$*

- *the trajectory  $\mathbf{q}(\cdot)$  converges exponentially to  $\mathbf{q}_d$  as  $t$  goes to infinity,*
- *$\omega(\mathbf{q}(t))$  converges exponentially to zero as  $t$  goes to infinity.*

*Proof.* Letting  $V, W$  as in the proof of Proposition 2 we have  $1 - q_0(t) \leq W(\mathbf{q}(0))f(\mathbf{q}(t))$  for any trajectory  $\mathbf{q}(\cdot)$  of (6)-(7) and hence

$$\dot{V}(\mathbf{q}(t)) = -\gamma_2(\mathbf{q}(t))(1 - q_0)f(\mathbf{q}(t)) \leq -\frac{\gamma_2(\mathbf{q}(t))}{W(\mathbf{q}(0))}V(\mathbf{q}(t)).$$

Being  $\gamma_2$  bounded from below by a positive constant on a neighbourhood of  $\mathbf{q}_d$  we have the exponential convergence of  $V$  to zero, i.e. the exponential convergence of  $\mathbf{q}(\cdot)$  to  $\mathbf{q}_d$ .

In order to estimate the limit of  $\omega(\mathbf{q}(t))$  as  $t$  goes to infinity, we first note that the convergence of  $\mathbf{q}(\cdot)$  to  $\mathbf{q}_d$  is equivalent to the convergence of  $q(\cdot)$  to zero. This implies that the second term in the right-hand side of (7) converges to zero. As for the first term, using (11) we

have

$$|\gamma_1(\mathbf{q})(e_3 \wedge q)| = |\gamma_1(\mathbf{q})| \sqrt{f(\mathbf{q})} \leq \frac{2k(\mathbf{q})|q_3|}{\sqrt{f(\mathbf{q})}}. \quad (13)$$

In the following we show that

$$\lim_{s \rightarrow +\infty} \frac{q_3^2(s)}{f(\mathbf{q}(s))} = 0, \quad (14)$$

which implies that (13) converges to zero as well.

In view of Proposition 2, we assume without loss of generality that the trajectory  $\mathbf{q}(\cdot)$  is contained in a small enough neighbourhood  $B$  of  $\mathbf{q}_d$  so that  $C_1 := \inf_{\mathbf{q} \in B} (k(\mathbf{q}) - \gamma_2(\mathbf{q})/2)$  is strictly positive. We have from (12) that

$$-W(\mathbf{q}(0)) \leq W(\mathbf{q}(t)) - W(\mathbf{q}(0)) \leq -C_1 \int_0^t \frac{q_3^2(s)}{f(\mathbf{q}(s))} ds,$$

that is, letting  $t$  go to infinity,

$$\int_0^\infty \frac{q_3^2(s)}{f(\mathbf{q}(s))} ds \leq \frac{W(\mathbf{q}(0))}{C_1}. \quad (15)$$

In  $B \setminus \{\mathbf{q}_d\}$  we have that  $q_3 \gamma_1 \leq 0$ , hence

$$\frac{d}{dt} \left( \frac{q_3^2}{f} \right) = q_3 \gamma_1 + \frac{q_3^2 (q_0 \gamma_2 + q_3 \gamma_1)}{f} \leq C_2 \frac{q_3^2}{f}, \quad (16)$$

with  $C_2 = \max_{\mathbf{q} \in S^3} \gamma_2(\mathbf{q})$ . By (15) and (16) for every  $\varepsilon > 0$  there exists  $t > 0$  large enough so that, for  $s > t$ ,

$$\frac{q_3^2(s)}{f(\mathbf{q}(s))} - \frac{q_3^2(t)}{f(\mathbf{q}(t))} \leq C_2 \int_t^s \frac{q_3^2(\tau)}{f(\mathbf{q}(\tau))} d\tau \leq \frac{\varepsilon}{2}$$

and, moreover, we can assume  $\frac{q_3^2(t)}{f(\mathbf{q}(t))} \leq \frac{\varepsilon}{2}$  by (15).

We thus get  $\frac{q_3^2(s)}{f(\mathbf{q}(s))} \leq \varepsilon$  for any  $s \geq t$ , that is  $\limsup_{s \rightarrow +\infty} \frac{q_3^2(s)}{f(\mathbf{q}(s))} \leq \varepsilon$ . By arbitrariness of  $\varepsilon > 0$  we get (14).

To conclude the proof it is enough to show that the convergence in (14) occurs with an exponential rate. From the equality in (16) and the fact that  $q_3 \gamma_1 \leq 0$  we have

$$\begin{aligned} \frac{d}{dt} \left( \frac{q_3^2}{f} \right) &\leq q_3 \gamma_1 + \frac{q_3^2}{f} q_0 \gamma_2 \\ &= -\frac{q_3^2}{f} \left( \frac{f}{1 - q_0} k - q_0 \gamma_2 \right). \end{aligned} \quad (17)$$

Writing  $\frac{f}{1 - q_0} k = (1 + q_0) \frac{1}{1 + q_0^2/f} k$  and using (14) and the convergence of  $\mathbf{q}(t)$  to  $\mathbf{q}_d$ , we obtain that  $\frac{f(\mathbf{q}(t))}{1 - q_0(t)} k(\mathbf{q}(t))$

converges to  $2k(\mathbf{q}_d)$  as  $t$  goes to infinity. Furthermore,  $q_0(t) \gamma_2(\mathbf{q}(t))$  converges to  $\gamma_2(\mathbf{q}_d)$ . Hence the expression within parentheses in (17) converges to the positive value  $2k(\mathbf{q}_d) - \gamma_2(\mathbf{q}_d)$  as  $t$  goes to infinity and, for  $\mu \in (0, 2k(\mathbf{q}_d) - \gamma_2(\mathbf{q}_d))$ , the inequality

$$\frac{d}{dt} \left( \frac{q_3^2}{f} \right) \leq -\mu \frac{q_3^2}{f}$$

holds true for  $t$  large enough. We deduce that the convergence in (14) occurs with an exponential rate.  $\square$

**Remark 3** *Note that, by (14), the action of the controller forces the variable  $q_3$  to converge to zero faster than  $q_1, q_2$ . This fact may be interpreted as an attempt to counteract the lack of controllability of the linearised system at the equilibrium in the direction of the component  $q_3$ .*

We have shown in Proposition 2 that the reduced system (6), accounting for the kinematics only, can be (almost globally) asymptotically stabilised by the virtual control (7). Proposition 3 extends this result to exponential stability at the cost of a discontinuity of the control feedback at  $\mathbf{q}_d$ . In the next section we will focus on the entire system (5)-(6) with the goal of finding a stabilising feedback  $u$ .

### 3.2 Almost global stabilisation of the complete system

Recall that, by assumption, we have  $\omega_3 \equiv 0$  along the dynamics. Hence, we write  $\omega = (\omega^p, 0)$  where  $\omega^p \in \mathbb{R}^2$  (here and in the following the superscript  $p$  denotes the projection of a three dimensional vector onto its first two components). For simplicity, we also denote by  $G(\omega^p, \mathbf{q})$  the right-hand side of (6). Motivated by the previous section, we set

$$\omega_{\text{ref}}(\mathbf{q}) = \gamma_1(\mathbf{q})(e_3 \wedge q) + \gamma_2(\mathbf{q})(e_3 \wedge (e_3 \wedge q)). \quad (18)$$

The basic idea is to choose a feedback control law for (5)-(6) forcing  $\omega(t)$  to converge to the function  $\omega_{\text{ref}}(\mathbf{q}(t))$ . This problem will be studied by considering separately the case in which  $\omega_{\text{ref}}(\mathbf{q})$  is smooth and the case in which it is discontinuous at  $\mathbf{q}_d$ . While in the first case one cannot expect exponential convergence to the equilibrium as a consequence of Proposition 1, in the second one we will show that the trajectories converge to  $\mathbf{q}_d$  exponentially for almost every initial condition  $(\omega^p, \mathbf{q})$  in  $\mathbb{R}^2 \times S^3$  (where the measure in  $\mathbb{R}^2 \times S^3$  is the product between the Lebesgue measure in  $\mathbb{R}^2$  and the spherical measure in  $S^3$ ).

In the following sections we will exploit the previous results to devise a (almost globally) stabilising feedback law for the complete system (5)-(6). In particular, Theorem 4 makes use of Proposition 2 to provide a smooth

control law with polynomial convergence, while Theorem 9 exploits the results in Proposition 3 to provide a non-smooth control feedback ensuring exponential convergence.

### 3.2.1 Stabilisation by smooth feedback law

In addition to the assumptions of Proposition 2, we suppose here that  $\gamma_1$  is smooth (as an example, taking  $k = \gamma_2 = 1 - q_0$ , one has  $\gamma_1 = -q_3$ ). In particular, this guarantees that  $\omega_{\text{ref}}$  is a smooth function.

In order to stabilise (5)-(6) we define  $\tilde{\omega} = \omega - \omega_{\text{ref}}(\mathbf{q})$  and we impose  $\dot{\tilde{\omega}}^p = -K\tilde{\omega}^p$  for some  $K > 0$ . This corresponds to choosing the feedback control

$$\begin{aligned} u(\omega^p, \mathbf{q}) &= \dot{\omega}_{\text{ref}}^p(\mathbf{q}) - K(\omega^p - \omega_{\text{ref}}^p(\mathbf{q})) \\ &= \frac{d\omega_{\text{ref}}^p}{d\mathbf{q}}(\mathbf{q})G(\omega^p, \mathbf{q}) - K\omega^p + K\omega_{\text{ref}}^p(\mathbf{q}). \end{aligned} \quad (19)$$

**Theorem 4** *Let the assumptions of Proposition 2 on  $\gamma_1, \gamma_2$  be satisfied and assume in addition that  $\gamma_2$  is strictly positive on  $Q_0 \setminus \{\mathbf{q}_d\}$  and  $\gamma_1$  is smooth. Then, the feedback control (19) almost globally stabilises system (5)-(6) to the equilibrium  $(0, 0, \mathbf{q}_d)$ .*

In order to prove Theorem 4, it is convenient to rewrite system (5)-(6) in terms of the error variable  $\tilde{\omega}$ :

$$\dot{\tilde{\omega}}^p = -K\tilde{\omega}^p \quad (20)$$

$$\dot{\mathbf{q}} = \tilde{G}(\tilde{\omega}^p, \mathbf{q}) \quad (21)$$

where  $\tilde{G}(\tilde{\omega}^p, \mathbf{q}) = G(\tilde{\omega}^p + \omega_{\text{ref}}^p(\mathbf{q}), \mathbf{q})$ . Note that the map  $(\omega^p, \mathbf{q}) \mapsto (\tilde{\omega}^p, \mathbf{q})$  is a diffeomorphism from  $\mathbb{R}^2 \times S^3$  to itself.

We need some preliminary results. First, we show the following non-smooth extension of the classical LaSalle invariance theorem.<sup>2</sup>

**Lemma 5** *Consider a differential equation*

$$\dot{x} = F(x), \quad (22)$$

where  $x$  belongs to a manifold  $\mathcal{M} \subset \mathbb{R}^N$  and  $F$  is Lipschitz continuous. Let  $\Omega$  be a compact invariant subset of  $\mathcal{M}$ ,  $D$  be a compact subset of  $\Omega$  such that both  $D$  and  $\Omega \setminus D$  are positively invariant. Moreover assume that there exists a continuous function  $V : \Omega \rightarrow \mathbb{R}$  strictly decreasing along the flow of (22) on  $\Omega \setminus D$ . Then for any trajectory  $x(\cdot)$  in  $\Omega \setminus D$  there exists  $c \in \mathbb{R}$  such that  $x(\cdot)$  converges to a connected component of  $D \cap V^{-1}(c)$ .

<sup>2</sup> for similar results in a much more general context, see e.g. [1,25]

*Proof.* Let  $\phi(x_0, t)$  denote the trajectory of the system (22) at time  $t$  starting from  $x_0 \in \Omega \setminus D$ . Then  $V(\phi(x_0, t))$  is decreasing with respect to  $t$  by assumption and, since the continuous function  $V$  admits a minimum in the compact set  $\Omega$ ,  $V(\phi(x_0, t))$  must converge to a constant  $c \in \mathbb{R}$ . Since, by the positive invariance of  $\Omega$ , the  $\omega$ -limit set of  $x_0$  is connected, it remains to show that  $\phi(x_0, t)$  necessarily converges to the set  $D$ . Let  $\varepsilon > 0$  and define  $K_\varepsilon$  as the compact set formed by the points of  $\Omega$  whose distance from  $D$  is larger or equal than  $\varepsilon$ . From the assumptions on  $V$  and by compactness of  $K_\varepsilon$  and continuity of the map  $x \mapsto \phi(x, 1)$ , we have  $V(\phi(x, 1)) - V(x) \leq -\delta$  for every  $x \in K_\varepsilon$ , for some  $\delta > 0$ . Then  $V(\phi(x_0, t+1)) - V(\phi(x_0, t)) \leq -\delta$  as long as  $\phi(x_0, t) \in K_\varepsilon$ . But we also know that there exists  $T > 0$  such that  $V(\phi(x_0, t)) < c + \delta$  for any  $t > T$ . Hence

$$\begin{aligned} c &< V(\phi(x_0, t+1)) < V(\phi(x_0, t)) < c + \delta \\ &\Rightarrow V(\phi(x_0, t+1)) - V(\phi(x_0, t)) > -\delta \end{aligned}$$

for any  $t > T$ , implying that  $\phi(x_0, t) \notin K_\varepsilon$ . Being  $\varepsilon$  arbitrary this concludes the proof.  $\square$

We apply Lemma 5 to the system (20)-(21). We define  $\Omega = [-1, 1] \times [-1, 1] \times S^3$ ,  $D = \{(0, 0)\} \times Q_0$  and

$$V(\tilde{\omega}^p, \mathbf{q}) = (1 - q_0)^2 + 3|\tilde{\omega}^p|/K.$$

It is immediate to see that both  $\Omega$  and  $D$  are positively invariant for (20)-(21). Furthermore, we have

$$\begin{aligned} \dot{V} &= -\gamma_2(1 - q_0)f(\mathbf{q}) + (1 - q_0)q^T \tilde{\omega} - 3|\tilde{\omega}^p| \\ &\leq -\gamma_2(1 - q_0)f(\mathbf{q}) - |\tilde{\omega}^p| \end{aligned}$$

which is well-defined and strictly negative outside  $\{(0, 0)\} \times Q_0$ . It is easy to see that for any  $c \in \mathbb{R}$  there exists at most two distinct points in  $D \cap V^{-1}(c)$ . Then we deduce from Lemma 5 that any trajectory of the system (20)-(21) starting in  $\Omega \setminus D$  asymptotically converges to a single point of  $D = \{(0, 0)\} \times Q_0$ . Since the set  $\Omega$  defined above is globally attractive in finite time the result extends to  $(\mathbb{R}^2 \times S^3) \setminus D$ . Summing up we get the following.

**Lemma 6** *Under the assumptions of Theorem 4, any trajectory of the system (20)-(21) asymptotically converges to a single point of  $\{(0, 0)\} \times Q_0$ .*

It remains to show that almost every trajectory of the system (20)-(21) converges exactly to  $(0, 0, \mathbf{q}_d)$ .

The lemma below allows a characterisation of the set of initial points such that the corresponding trajectories do not converge to  $(0, 0, \mathbf{q}_d)$ .

**Lemma 7** *Consider the system (20)-(21). Let  $\mathbf{q}^* = (q_0^*, 0, 0, q_3^*) \in Q_0 \setminus \{\mathbf{q}_d\}$ . Then, under the assumptions*

of Theorem 4, the linearised system at the equilibrium  $(0, 0, \mathbf{q}^*)$  (on the five-dimensional tangent space  $T_{(0,0,\mathbf{q}^*)}(\mathbb{R}^2 \times S^3)$ ) is associated with a Jacobian matrix having two eigenvalues equal to  $-K$  and one eigenvalue equal to 0. The remaining two eigenvalues have strictly positive real part.

*Proof.* To simplify the computations we embed  $\mathbb{R}^2 \times S^3$  in  $\mathbb{R}^6$ , using the coordinates  $(\tilde{\omega}_1, \tilde{\omega}_2, q_0, q_1, q_2, q_3)$ .

Equation (20) immediately implies the existence of two eigenvalues of the linearised system equal to  $-K$ . The remaining ones are eigenvalues of the four dimensional square matrix

$$\frac{\partial \tilde{G}}{\partial \mathbf{q}}(\tilde{\omega}^p, \mathbf{q})|_{(0,0,\mathbf{q}^*)} = \frac{\partial G}{\partial \omega^p}(0, 0, \mathbf{q}^*) \frac{d\omega_{\text{ref}}^p}{d\mathbf{q}}(\mathbf{q}^*) + \frac{\partial G}{\partial \mathbf{q}}(0, 0, \mathbf{q}^*),$$

where we have used the fact that  $\omega_{\text{ref}}(\mathbf{q}^*) = 0$ . A direct computation shows that the kernel of this matrix is generated by  $e_1, e_4$  and therefore contains both  $\mathbf{q}^*$  and the vectors tangent to  $Q_0$  at  $\mathbf{q}^*$ . The zero eigenvalue corresponding to the radial direction  $\mathbf{q}^*$  must be neglected, being  $\mathbf{q}^*$  orthogonal to the tangent space  $TS^3$ .

The two remaining eigenvalues of the matrix may be easily computed as

$$\frac{1}{2}(-\gamma_1 q_3^* - \gamma_2 q_0^*) \pm \frac{1}{2}i(-\gamma_2 q_3^* + \gamma_1 q_0^*),$$

and, by replacing the expression of  $\gamma_1$  in Proposition 2, we have

$$-\gamma_1 q_3^* - \gamma_2 q_0^* = k(1+q_0^*) - \gamma_2 q_0^* \geq \frac{\gamma_2}{2}(1-q_0^*) > 0 \quad \text{if } q_0^* \neq 1.$$

This concludes the proof of the lemma.  $\square$

The previous result allows us to cast our dynamical model in the well established framework of normally hyperbolic invariant manifolds, first developed in [14–16, 19, 20], which generalises classical results on hyperbolic equilibrium points. A normally hyperbolic manifold  $\mathcal{V}$  is an invariant compact submanifold of the state space such that the linearised dynamics around  $\mathcal{V}$  may be decoupled into three parts: a stable dynamics and an unstable one, both of which are transverse to  $\mathcal{V}$ , and a dynamics tangent to the manifold  $\mathcal{V}$ . In addition, it is assumed that, roughly speaking, the rates of contraction and expansion of the flow respectively in the direction of the stable and unstable subspaces are larger than those along  $\mathcal{V}$ . The latter condition is automatically satisfied if  $\mathcal{V}$  is made of equilibrium points.

We are now ready to prove Theorem 4.

*Proof of Theorem 4.* From Lemma 6, in order to characterise the trajectories that do not converge to  $(0, 0, \mathbf{q}_d)$

it is enough to study the family of all trajectories converging to the equilibria  $(0, 0, \mathbf{q}^*)$  with  $\mathbf{q}^* \in Q_0 \setminus \mathbf{q}_d$ . For this purpose, let us fix an arbitrary small  $\varepsilon > 0$  and consider the compact manifold (with boundary)

$$P_0^\varepsilon = \{(\tilde{\omega}^p, \mathbf{q}) \in \{(0, 0)\} \times Q_0 \mid q_0 \leq 1 - \varepsilon\}.$$

According to Lemma 7, for the linearised dynamics, the tangent space at any equilibrium point  $\mathbf{p} \in P_0^\varepsilon$  splits into the sum of a two-dimensional stable subspace  $E_{\mathbf{p}}^s$ , a two-dimensional unstable subspace  $E_{\mathbf{p}}^u$ , and the one-dimensional space  $T_{\mathbf{p}}P_0^\varepsilon$  (which coincides with the kernel of the linearised system). Thus, in the setting of e.g. [20],  $P_0^\varepsilon$  is a normally hyperbolic invariant manifold. Hence, by classical results, there exists a local invariant manifold  $\mathcal{W}_\varepsilon^s$  tangent to the direct sum  $E_{\mathbf{p}}^s \oplus T_{\mathbf{p}}P_0^\varepsilon$  at any  $\mathbf{p} \in P_0^\varepsilon$ , and which is therefore of dimension 3.

An interesting and helpful characterisation of  $\mathcal{W}_\varepsilon^s$  is given, in a very general setting, in [2]. In that paper the authors show that for a small enough smooth tubular neighbourhood  $\mathcal{N}$  of  $P_0^\varepsilon$  one can write

$$\mathcal{W}_\varepsilon^s = \{\mathbf{p} \in \mathcal{N} \mid \phi^t(\mathbf{p}) \in \mathcal{N}, \forall t \geq 0 \text{ and } \lim_{t \rightarrow \infty} \phi^t(\mathbf{p}) \in P_0^\varepsilon\},$$

where  $\phi^t(\mathbf{p})$  is the flow of the system at time  $t$  applied to  $\mathbf{p}$ . Let us further define the set

$$\overline{\mathcal{W}}_\varepsilon^s = \{\mathbf{p} \in \mathbb{R}^2 \times S^3 \mid \lim_{t \rightarrow \infty} \phi^t(\mathbf{p}) \in P_0^\varepsilon\}.$$

Since  $\mathcal{W}_\varepsilon^s$  is a three-dimensional manifold, it has zero measure. Recall that the flow at a (positive or negative) time  $t$  is a diffeomorphism, hence we deduce that the set  $\phi^{-t}(\mathcal{W}_\varepsilon^s)$  has zero measure as well. Then  $\overline{\mathcal{W}}_\varepsilon^s = \cup_{n \geq 0} \phi^{-n}(\mathcal{W}_\varepsilon^s)$  is a countable union of zero measure sets and thus it has zero measure.

Finally, the set of initial points in  $\mathbb{R}^2 \times S^3$  such that the corresponding trajectories converge to a point of  $Q_0 \setminus \mathbf{q}_d$  coincides with  $\cup_{m \geq 1} \overline{\mathcal{W}}_{1/m}^s$  and thus it has zero measure.  $\square$

### 3.2.2 Stabilisation by non-smooth feedback law

We analyse here the case in which the feedback law is discontinuous at the equilibrium  $\mathbf{q}_d$ . For the complete system (5)-(6) it is natural to require that the control law  $u$  converges to zero as  $\mathbf{q}$  converges to the equilibrium  $\mathbf{q}_d$ . For this reason it is important to guarantee that not only the virtual control  $\omega$  but also its derivative converges to zero along the solutions of the reduced system (6). This is formalised in the following result.

**Proposition 8** *Assume that the hypotheses of Proposition 3 are satisfied and, in addition, that  $k(\mathbf{q}_d) > \frac{3}{4}\gamma_2(\mathbf{q}_d)$ . Then  $\dot{\omega}$  converges to zero as time goes to infinity along the trajectories of (6) with the feedback law (7) starting from  $\mathbf{q}(0) \notin Q_0$ .*

*Proof.* We have

$$\dot{\omega} = \dot{\gamma}_1 e_3 \wedge q + \dot{\gamma}_2 e_3 \wedge (e_3 \wedge q) + \gamma_1 e_3 \wedge \dot{q} + \gamma_2 e_3 \wedge (e_3 \wedge \dot{q}), \quad (23)$$

which is well-defined since, by Proposition 2,  $\mathbf{q}(t) \neq \mathbf{q}_d$  for every positive  $t$ . The second term in the sum (23) converges to zero since  $q(t)$  goes to zero as  $t$  tends to infinity and  $\gamma_2$  is  $\mathcal{C}^1$  on  $S^3$ , so that  $\dot{\gamma}_2$  is uniformly bounded. Since  $\omega$  converges to zero by Proposition 3, we have that  $\dot{q}$  tends to zero, and so does the last term in the sum (23). We are left to estimate the first and the third terms in (23), which are more delicate since  $\gamma_1, \dot{\gamma}_1$  may grow unbounded along the trajectory.

Since

$$\dot{\gamma}_1 = -\frac{(\dot{k}q_3 + k\dot{q}_3)(1 - q_0) + \dot{q}_0 k q_3}{(1 - q_0)^2},$$

and

$$\begin{aligned} \frac{|\dot{k}q_3|}{1 - q_0} |e_3 \wedge q| &\leq 2|\dot{k}| \frac{|q_3|}{f} \sqrt{f} = 2|\dot{k}| \frac{|q_3|}{\sqrt{f}}, \\ \frac{|\dot{q}_0 q_3|}{(1 - q_0)^2} |e_3 \wedge q| &= \frac{\gamma_2 f |q_3|}{2(1 - q_0)^2} \sqrt{f} \leq 2\gamma_2 \frac{|q_3|}{\sqrt{f}}, \\ \frac{|\dot{q}_3|}{1 - q_0} |e_3 \wedge q| &= \frac{k f |q_3|}{2(1 - q_0)^2} \sqrt{f} \leq 2k \frac{|q_3|}{\sqrt{f}} \end{aligned}$$

go to zero by (14), it is immediate to see that the first term in the sum (23) goes to zero.

To show that the third term in (23) goes to zero, we notice, by (6), that

$$\frac{|q_3|}{1 - q_0} |e_3 \wedge \dot{q}| \leq \frac{|q_3|}{1 - q_0} |e_3 \wedge (q \wedge \omega)| + \frac{|q_3|}{1 - q_0} |e_3 \wedge \omega|,$$

so that it is enough to show that the right-hand side converges to zero. We have that  $\frac{|q_3|}{1 - q_0} |e_3 \wedge (q \wedge \omega)| \leq \frac{|q_3||q|}{1 - q_0} |\omega| = \frac{|q_3||q|}{|q|^2} (1 + q_0) |\omega| \leq 2|\omega|$  tends to zero. We are left to prove that  $\frac{|q_3|}{1 - q_0} |e_3 \wedge \omega| = \frac{|q_3|}{1 - q_0} |\omega|$  goes to zero as well. By substituting the expression (7) we have

$$\begin{aligned} \frac{|q_3|}{1 - q_0} |\omega| &\leq \frac{|q_3|}{1 - q_0} |\gamma_1| |e_3 \wedge q| + \frac{|q_3|}{1 - q_0} |\gamma_2| |e_3 \wedge (e_3 \wedge q)| \\ &= |k| \frac{q_3^2 \sqrt{f}}{(1 - q_0)^2} + |\gamma_2| \frac{|q_3| \sqrt{f}}{1 - q_0} \\ &\leq |k| \frac{q_3^2 \sqrt{f}}{(1 - q_0)^2} + 2|\gamma_2| \frac{|q_3|}{\sqrt{f}}, \end{aligned}$$

where the second term in the last estimate goes to zero by (14). Hence, to show that  $\frac{|q_3|}{1 - q_0} |\omega|$  converges to zero it is left to prove that  $\Gamma = \frac{q_3^2 \sqrt{f}}{(1 - q_0)^2}$  goes to zero. A direct

computation shows that

$$\dot{\Gamma} = \Gamma \left( - (k - \gamma_2) \frac{f}{1 - q_0} - \frac{1}{2} q_0 \gamma_2 + k \frac{q_3^2}{2(1 - q_0)} \right).$$

Note that the last term inside the parenthesis goes to zero as  $t$  goes to infinity because of (14). Since  $q_0$  converges to 1 and  $\frac{f}{1 - q_0}$  converges to 2 (again, as a consequence of (14)), the sum of the first two terms converges to  $-(2k(\mathbf{q}_d) - 3\gamma_2(\mathbf{q}_d)/2)$ . This shows that  $\Gamma$  converges exponentially towards zero whenever  $k(\mathbf{q}_d) > 3\gamma_2(\mathbf{q}_d)/4$ .

The convergence of  $\dot{\omega}$  to zero is therefore demonstrated.  $\square$

**Remark 4** *The additional condition  $k(\mathbf{q}_d) > \frac{3}{4}\gamma_2(\mathbf{q}_d)$  is crucial. Indeed, by following the same lines of the previous proof one has that, if  $k(\mathbf{q}_d) < \frac{3}{4}\gamma_2(\mathbf{q}_d)$ , then  $\Gamma$  grows unbounded. In this case the dominant term in  $\dot{\omega}$  along the direction  $e_3 \wedge (e_3 \wedge q)$  goes to infinity, and thus  $\dot{\omega}$  grows unbounded as well.*

The following result concerns the stabilisation of the complete system (5)-(6) by using a control law continuous everywhere except at  $(\omega^p, \mathbf{q}_d)$  for  $\omega^p \in \mathbb{R}^2$ . As  $\omega_{\text{ref}}(\mathbf{q})$  is undefined at the equilibrium  $\mathbf{q}_d$ , in order to ensure that the reference dynamics is well-defined we will impose conditions on the feedback law  $u$  forcing the  $\mathbf{q}$  components of the trajectories of (5)-(6) not to reach  $\mathbf{q}_d$  in finite time.

**Theorem 9** *Assume that the hypotheses of Proposition 8 are satisfied and let  $\tilde{\omega} = \omega - \omega_{\text{ref}}(\mathbf{q})$ . Then:*

- *If  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a continuous function smooth outside the origin and such that  $x^T \varphi(x) \leq 0$  for  $x \in \mathbb{R}^2$ , then the feedback law defined by*

$$u(\omega, \mathbf{q}) = \begin{cases} \varphi(\tilde{\omega}^p) - \frac{|q^T \omega|}{4(1 - q_0)} \tilde{\omega}^p + \frac{d\omega_{\text{ref}}^p}{d\mathbf{q}}(\mathbf{q}) G(\omega^p, \mathbf{q}) & \text{if } \mathbf{q} \neq \mathbf{q}_d \\ 0 & \text{if } \mathbf{q} = \mathbf{q}_d \end{cases} \quad (24)$$

- *is such that the solutions of (5)-(6) with  $\mathbf{q}(0) \neq \mathbf{q}_d$  are such that  $\mathbf{q}(t) \neq \mathbf{q}_d$  for every positive  $t$ .*
- *If  $\varphi$  is defined componentwise as*

$$\varphi(x)_i = -\text{sign}(x_i) \sqrt{|x_i|}$$

*then the feedback control (24) almost globally exponentially stabilises system (5)-(6) to the equilibrium  $(0, 0, \mathbf{q}_d)$ . Furthermore, the control variables converge to zero along all trajectories converging to  $(0, 0, \mathbf{q}_d)$ .*

*Proof.* We set  $V(\mathbf{q}) = (1 - q_0)^2$ , and we have

$$\begin{aligned}\dot{V}(\mathbf{q}) &= q^T \omega (1 - q_0) \\ &= q^T \omega_{\text{ref}}(\mathbf{q})(1 - q_0) + q^T \tilde{\omega} (1 - q_0) \\ &= -\gamma_2(\mathbf{q})f(\mathbf{q})(1 - q_0) + q^T \tilde{\omega} (1 - q_0) \\ &\geq -2\|\gamma_2\|_\infty V - \sqrt{2}V^{3/4}|\tilde{\omega}|,\end{aligned}$$

where the last inequality is a consequence of (11) and of the Cauchy-Schwarz inequality. In order to guarantee that  $V$  does not vanish in finite time it is enough to find a feedback control law such that, along each trajectory,  $|\tilde{\omega}| \leq cV^{1/4}$ , with  $c > 0$  possibly depending on the initial condition  $\mathbf{q}(0)$ . It is therefore enough to ensure that

$$\frac{d}{dt} \left( \frac{|\tilde{\omega}|^2}{\sqrt{V}} \right) = \frac{\tilde{\omega}^T (2(1 - q_0)\dot{\tilde{\omega}} - \frac{1}{2}(q^T \omega)\tilde{\omega})}{(1 - q_0)^2} \leq 0.$$

This inequality is true in particular whenever  $\dot{\tilde{\omega}}$  satisfies

$$\dot{\tilde{\omega}}^{\text{P}} = \varphi(\tilde{\omega}^{\text{P}}) - \frac{|q^T \omega|}{4(1 - q_0)} \tilde{\omega}^{\text{P}},$$

where  $(\tilde{\omega}^{\text{P}})^T \varphi(\tilde{\omega}^{\text{P}}) \leq 0$ . Since  $\dot{\tilde{\omega}}^{\text{P}} = u(\omega, \mathbf{q}) - \frac{d\omega_{\text{ref}}^{\text{P}}}{d\mathbf{q}}(\mathbf{q})G(\omega^{\text{P}}, \mathbf{q})$ , the first part of the theorem is proved.

Concerning the second part of the theorem, defining  $\varphi(\tilde{\omega}^{\text{P}})$  componentwise as  $\varphi(\tilde{\omega}^{\text{P}})_i = -\text{sign}(\tilde{\omega}_i)\sqrt{|\tilde{\omega}_i|}$ , enforces  $\tilde{\omega}$  to reach the origin in finite time. This, together with Proposition 3, implies that all the trajectories of the complete system that do not reach the equilibria in  $\{(0, 0)\} \times (Q_0 \setminus \{\mathbf{q}_d\})$  in finite time necessarily converge exponentially to  $(0, 0, \mathbf{q}_d)$ . Indeed, for any such trajectory, the value of  $\tilde{\omega}$  vanishes in finite time so that, from that moment on, the  $\mathbf{q}$  components satisfy the equation (6) with  $\omega = \omega_{\text{ref}}(\mathbf{q})$ , hence Proposition 3 applies. On the other hand, the trajectories reaching the equilibria in  $\{(0, 0)\} \times (Q_0 \setminus \{\mathbf{q}_d\})$  in finite time form a three-dimensional manifold (with boundary) locally parametrised by the point reached in  $Q_0 \setminus \{\mathbf{q}_d\}$  and the times  $t_i$  such that the  $i$ -th component of  $\tilde{\omega}$  vanishes. Note that the time  $t$  to reach  $Q_0 \setminus \{\mathbf{q}_d\}$  coincides with the largest of the times  $t_i$ . In particular all the trajectories of the complete system starting in the complement of this manifold, a set of full measure, converge exponentially to  $(0, 0, \mathbf{q}_d)$ . Finally, the convergence of the control variables follows from Proposition 8.  $\square$

**Remark 5** *As observed in the introduction, several results concerning attitude stabilisation of the dynamics of a rigid body (with two or three torques as control inputs) are available in the literature. Some of these results, such as those in [8], focus on the reduced attitude stabilisation or, equivalently, the stabilisation of a 2-D spherical pendulum. In our setting, the stabilisation of the reduced attitude corresponds to the stabilisation of the system (5)-(6) to the manifold  $\{(0, 0)\} \times Q_0$ . It is easy to see that the*

*feedback law (7) with  $\gamma_1 = q_3$  and  $\gamma_2 = q_0$  stabilises the kinematic subsystem (6) to  $Q_0$  with an exponential rate (unless the initial position of the pendulum corresponds to the downward position, i.e.  $q_0(0) = q_3(0) = 0$ ). Then it can be shown, by a standard Lyapunov analysis, that a feedback control defined as in Section 3.2.1 stabilises exponentially (almost globally) the complete system (5)-(6). Note that in this case exponential stabilisation is achieved without resorting to a non-smooth control law, while this is impossible for the complete attitude stabilisation problem, due to Proposition 1.*

*Concerning the complete attitude stabilisation problem, to the best of our knowledge Theorem 4 and Theorem 9 provide the first instances of almost globally stabilising control laws in purely feedback form for the considered model. The hybrid strategy developed in [6] to construct a globally stabilising control law applies to the case in which the rigid body is non-symmetric, although it is reasonable to expect that it can be adapted to the axially symmetric case considered in this paper. Compared to this method, our control laws appear to have a simpler form and, under the assumptions of Theorem 9, they ensure exponential convergence to the target state. Still for the non-symmetric case, a hybrid control law stabilising the system around the target state, but only in a practical sense, has been proposed in [27]. Closer in spirit to our approach is [28], which studies the axially symmetric case. In that work, the authors define a family of feedback laws, each one of which exponentially stabilises the system to the target state, provided that the initial condition belongs to a compact annular subset of the state space not containing the target equilibrium. Note that, in this case, the choice of the stabilising feedback law may vary depending on the initial condition.*

## 4 Simulations

In this section we are going to show some numerical simulations of the controlled pendulum evolving from two sets of initial conditions.

As a first set, we choose the following initial conditions:  $\omega(0) = (0, 0)$  and  $\mathbf{q}(0) = (0.8, 0, 0.06, 0.597)$ . It is an interesting case to test the effectiveness of the presented control feedbacks, since it is close to an equilibrium of  $\{(0, 0)\} \times Q_0$  different from  $(0, 0, \mathbf{q}_d)$ . Notice that the use of two control torques acting on a plane orthogonal to the symmetry axis, impedes a direct rotation about that axis. Hence, the attitude stabilisation would require the third axis to first move sensibly away from the initial configuration before coming back.

We consider first the smooth control law (19). According to Proposition 2 and Theorem 4 we choose  $\gamma_1 = -5q_3$ ,  $\gamma_2 = 2(1 - q_0)$  and  $K = 1$ . In Figure 2 the trajectory of the tip of the pendulum is shown in red and its projections onto the planes in grey for a simulation of 1500 s.

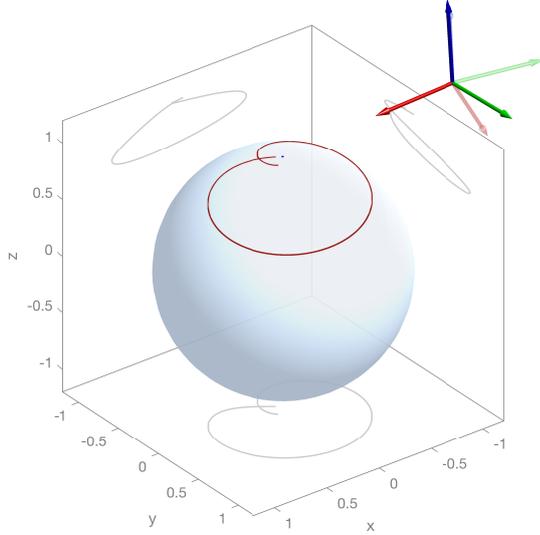


Fig. 2. Evolution of the pendulum attitude from the initial position (fainter axes) to the final one (stronger axes). The red trace represents the trajectory of the tip of the pendulum, while the grey ones their projections onto the planes. Case with smooth feedback and initial conditions:  $\omega(0) = (0, 0)$  and  $\mathbf{q}(0) = (0.8, 0, 0.06, 0.597)$ .

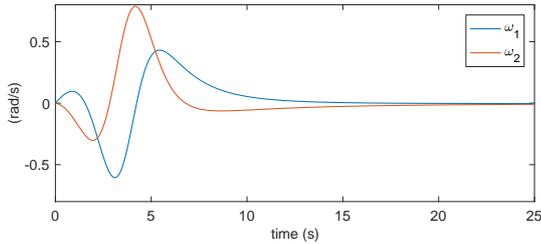


Fig. 3. Evolution of  $\omega_1$  and  $\omega_2$ .

In the upper right corner two frames are shown with the standard colour/axis association (R=x, G=y, B=z). The fainter one represents the initial pendulum configuration, while the stronger one represents the final configuration attained by the pendulum. As expected, the pendulum quickly moves away from the unstable equilibrium to slowly approach the final equilibrium  $(0, 0, \mathbf{q}_d)$ . Figures 3 and 4 show how, after some oscillations,  $\omega$  and  $u$  quickly converge to  $(0, 0)$ . Figure 5 depicts the evolution of the quaternions  $\mathbf{q}$ , which is representative of the slow convergence of the pendulum attitude to the desired equilibrium.

While the smooth control law always ensures bounded control values, its polynomial convergence is clearly very slow. A faster (exponential) convergence can be ensured to the system by the non-smooth control law (24) with  $\varphi$  defined component-wise as  $\varphi(x)_i = -\text{sign}(x_i)\sqrt{|x_i|}$ . In order to compare the two control feedbacks, we perform a

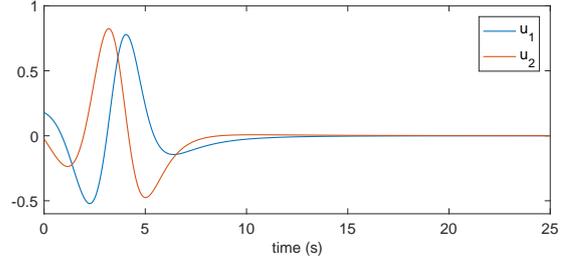


Fig. 4. Evolution of the control signals  $u_1$  and  $u_2$ .

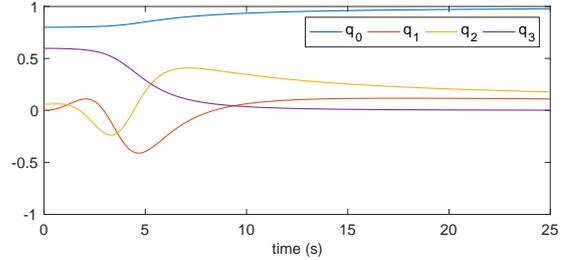


Fig. 5. Evolution of the pendulum attitude in quaternion representation.

simulation from the same initial conditions:  $\omega(0) = (0, 0)$  and  $\mathbf{q}(0) = (0.8, 0, 0.06, 0.597)$ . We fix the control parameters as  $\gamma_1 = -q_3/(1 - q_0)$  and  $\gamma_2 = 1$ , such that the condition  $k(\mathbf{q}_d) > \frac{3}{4}\gamma_2(\mathbf{q}_d)$  is verified. Figure 6 depicts the evolution of the tip of the pendulum (red curve) and the initial and final attitude (axes in the upper right corner) for a simulation of 25 s. The faster convergence is quite apparent, since in just 25 s the pendulum has reached its desired equilibrium (see also the evolution of  $\mathbf{q}$  in Figure 7). Here also, after some oscillations,  $\omega$  and  $u$  quickly converge to  $(0, 0)$  as shown in Figures 8 and 9.

As a last case, we consider the non-smooth feedback (with the same parameters as before) applied to the system with non-null initial velocities. In particular, we choose the following initial conditions  $\omega(0) = (-10, -12)$  and  $\mathbf{q}(0) = (-0.8, 0.267, -0.5, -0.197)$  and we perform a simulation for 25 s. In Figure 6 the evolution of the tip of the pendulum (red curve) and the initial and final attitude (axes in the upper right corner) are depicted. Here, the more ‘whimsical’ behaviour of the pendulum is due to the non-null initial velocities. The control feedback has to progressively zero them to ensure the convergence to the final equilibrium (see Figures 11 and 12). The fast convergence to the desired attitude is confirmed by the evolution of the quaternions in Figure 11.

## 5 Conclusion

In this paper we considered the problem of stabilising an axially symmetric 3-D pendulum to the upright ver-

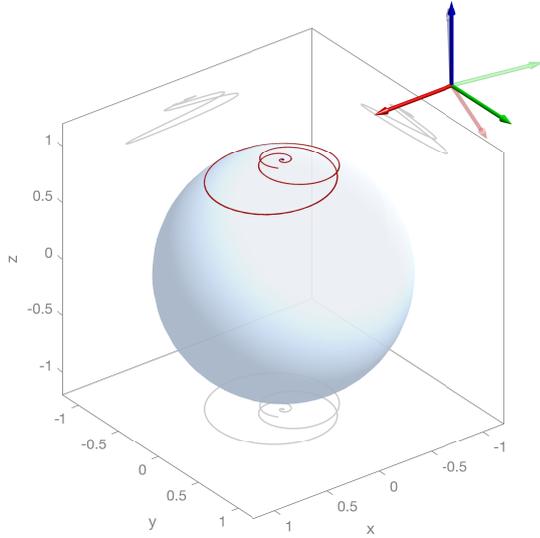


Fig. 6. Evolution of the pendulum attitude from the initial position (fainter axes) to the final one (stronger axes). The red trace represents the trajectory of the tip of the pendulum, while the grey ones their projections onto the planes. Case with non-smooth feedback and initial conditions:  $\omega(0) = (0, 0)$  and  $\mathbf{q}(0) = (0.8, 0, 0.06, 0.597)$ .

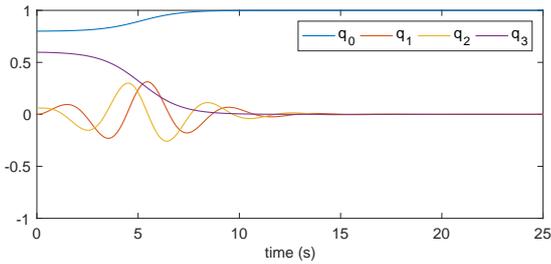


Fig. 7. Evolution of the pendulum attitude in quaternion representation.

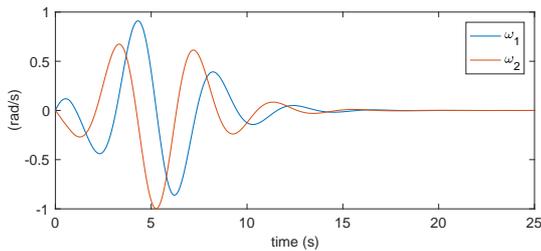


Fig. 8. Evolution of  $\omega_1$  and  $\omega_2$ .

tical position with a fixed orientation by means of two torques. The stabilisation was achieved in two steps. First, by viewing the angular velocity as a control parameter, we exhibited some general conditions on the control feedback enforcing stabilisation of the kinematic subsystem. Then, we designed control feedbacks for the

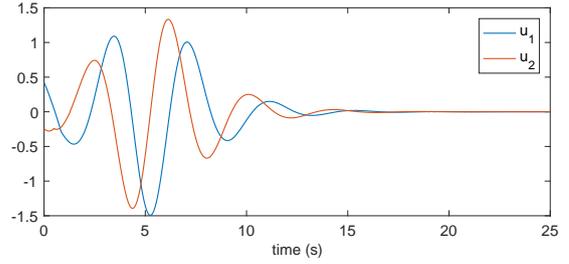


Fig. 9. Evolution of the control signals  $u_1$  and  $u_2$ .

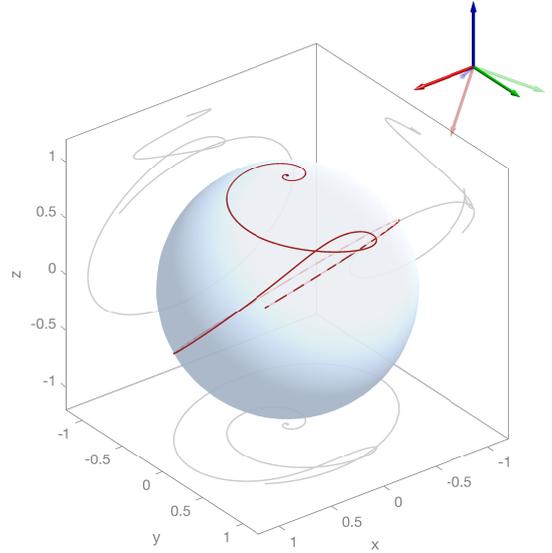


Fig. 10. Evolution of the pendulum attitude from the initial position (fainter axes) to the final one (stronger axes). The red trace represents the trajectory of the tip of the pendulum, while the grey ones their projections onto the planes. Case with non-smooth feedback and initial conditions:  $\omega(0) = (-10, -12)$  and  $\mathbf{q}(0) = (-0.8, 0.267, -0.5, -0.197)$ .

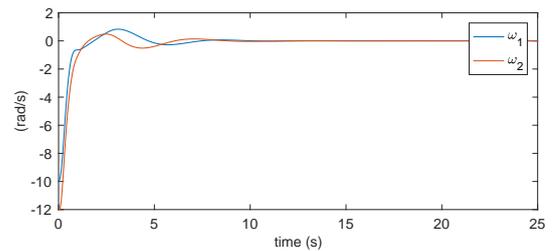


Fig. 11. Evolution of  $\omega_1$  and  $\omega_2$ .

original inputs to force the angular velocity to follow the reference computed in the previous step. By making different regularity assumptions on the reference feedback, we proved that this control is capable of steering the complete system to the desired equilibrium with polynomial or exponential rate of convergence. Future works will address the problem of stabilising the 3-D pendu-

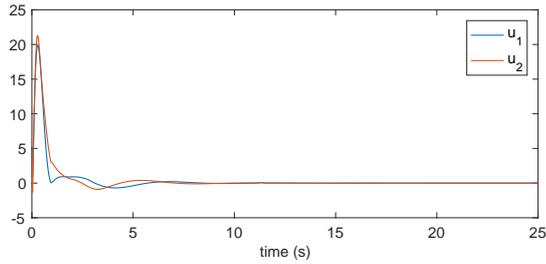


Fig. 12. Evolution of the control signals  $u_1$  and  $u_2$ .

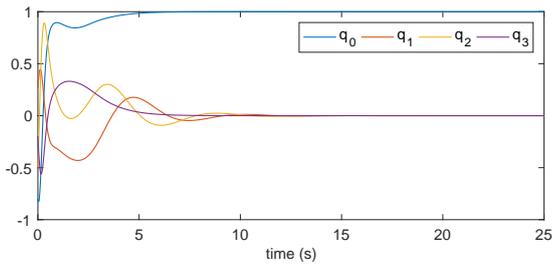


Fig. 13. Evolution of the pendulum attitude in quaternion representation.

lum in the case in which the pivot is constrained on a plane, by controlling two planar forces. The case in which the pivot satisfies a nonholonomic constraint will also be taken into account.

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