On maximally totally real embeddings

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Abstract

We consider complex structures with totally real zero section of the tangent bundle. We assume that the complex structure tensor is real-analytic along the fibers of the tangent bundle. This assumption is quite natural in view of a well known result by Bruhart and Whitney [Br-Wh]. We provide explicit integrability equations for such complex structures in terms of the fiberwise Taylor expansion. For any torsion free complex covariant derivative operator acting on the smooth sections of the complexified tangent bundle we provide very simple and very explicit fiberwise Taylor expansion of an associated canonical complex structure of the type described above.

1 Introduction and statement of the main result

Let (E, π_E, M) be a smooth vector bundle over a manifold M. Let E_p be the fiber of E over a point $p \in M$ and let $\eta \in E_p$. We consider the transition map $\tau_\eta(v) := \eta + v$ acting over E_p and we consider its differential

$$d_0\tau_\eta : T_{E_p,0} \longrightarrow T_{E_p,\eta}\,,$$

at the point 0. Composing $d_0\tau_\eta$ with the canonical isomorphism $E_p \simeq T_{E_p,0}$ we obtain an isomorphism map

$$T_\eta : E_p \longrightarrow T_{E_p,\eta}\,. \tag{1.1}$$

We denote by 0_M the zero section of E. Differentiating the identity $\mathrm{id}_M = \pi_E \circ 0_M$ we obtain $\mathbb{I}_{T_{M,p}} = d_{0_p}\pi_E \circ d_p 0_M$. This implies the decomposition

$$T_{E,0_p} = d_p\, 0_M\,(T_{M,p}) \oplus \mathrm{Ker}\, d_{0_p}\pi_E\,.$$

We notice also the obvious equalities $\mathrm{Ker}\, d_\eta\pi_E = d_0\tau_\eta\left(T_{E_p,0}\right) = T_\eta\,(E_p) \simeq E_p$, for any $\eta \in E_p$. Now applying this to $\eta = 0_p$, using the previous decomposition and the canonical isomorphism $d_p\, 0_M\,(T_{M,p}) \simeq T_{M,p}$, we infer the existence of the canonical isomorphism $T_{E,0_p} \simeq T_{M,p} \oplus E_p$, that we rewrite as

$$T_{E|M} \simeq T_M \oplus E\,. \tag{1.2}$$

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Definition 1 *A real sub-manifold M of an almost complex manifold (X, J) is called totally real if $T_{M,p} \cap J(T_{M,p}) = 0_p$ for all $p \in M$. A totally real sub-manifold M of an almost complex manifold (X, J) is called maximally totally real if $\dim_{\mathbb{R}} M = \dim_{\mathbb{C}} X$.*

1.1 M-totally real almost complex structures over T_M

We consider M included inside T_M via the zero section. We know by the isomorphism (1.2) with $E = T_M$, that this embedding induces the canonical isomorphism $T_{T_M|M} \simeq T_M \oplus T_M$. The vector bundle $T_{T_M|M}$ is a complex one with the canonical complex structure $J^{\mathrm{can}} : (u, v) \longmapsto (-v, u)$ acting on the fibers.

Any almost complex structure which is a continuous extension of J^{can} in a neighborhood of M inside T_M makes M a maximally totally real sub-manifold of T_M.

Over an arbitrary small neighborhood of M inside T_M the complex distribution $T^{0,1}_{T_M}$ is horizontal with respect to the natural projection $\pi : T_M \longrightarrow M$.

We remind that the data of a smoth complex horizontal distribution over T_M coincides with the one of section

$$A \in C^\infty\left(T_M, \pi^* \mathbb{C}T^*_M \otimes_{\mathbb{C}} \mathbb{C}T_{T_M}\right),$$

such that $d\pi \cdot A = \mathbb{I}_{\pi^* \mathbb{C}T_M}$.

For any complex vector field $\xi \in C^\infty(M, \mathbb{C}T_M)$ we will denote by abuse of notations $A(\xi) \equiv A \cdot (\xi \circ \pi)$. The section A evaluated at the point $\eta \in T_M$ will be denoted by A_η.

We notice that we can write $A = \alpha + i\beta$, with

$$\alpha, \beta \in C^\infty\left(T_M, \pi^* T^*_M \otimes_{\mathbb{R}} T_{T_M}\right),$$

such that $d\pi \cdot \alpha = \mathbb{I}_{\pi^* T_M}$ and $\beta_\eta = T_\eta B_\eta$, with $B \in C^\infty(T_M, \pi^* \operatorname{End}(T_M))$. The section A determines an almost complex structure J_A over T_M such that

$$T^{0,1}_{T_M, J_A, \eta} = A_\eta\left(\mathbb{C}T_{M,\pi(\eta)}\right) \subset \mathbb{C}T_{T_M,\eta},$$

if and only if

$$A_\eta\left(\mathbb{C}T_{M,\pi(\eta)}\right) \cap \overline{A_\eta\left(\mathbb{C}T_{M,\pi(\eta)}\right)} = 0. \tag{1.3}$$

This condition is equivalent to the property:

$$\overline{A_\eta\left(\bar{\xi}_1\right)} = A_\eta(\xi_2), \tag{1.4}$$

implies $\xi_1 = \xi_2 = 0$. Taking $d_\eta \pi$ in the equality (1.4) we infer $\xi_1 = \xi_2$. Thus equality (1.4) is equivalent to $\left(\overline{A} - A\right)(\xi_1) = 0$ and the previous property is equivalent to $\operatorname{Ker}\left(\overline{A} - A\right) = 0$, i.e.

$$B \in C^\infty\left(T_M, \pi^* \operatorname{GL}(T_M)\right).$$

We notice that with respect to the canonical complex structure of $T_{T_M|M}$ we have the equality $(u, v)^{0,1} = (\xi, i\xi)$, with $\xi := (u - iv)/2$. Then J_A is an extension of this complex structure over an open neighborhood $U \subseteq T_M$ of M if and only if for any point $p \in M$ we have $\alpha_{0_p} = d_p 0_M$ and $B_{0_p} = \mathbb{I}_{T_{M,p}}$. We denote by

$$T \in C^\infty\left(T_M, \pi^* T^*_M \otimes_{\mathbb{R}} T_{T_M}\right),$$

the canonical section which at the point $\eta \in T_M$ takes the value T_η.

Definition 2 *Let M be a smooth manifold. An M-totally real almost complex structure over an open neighborhood $U \subseteq T_M$ of the zero section is a couple (α, B) with*

$$\alpha \in C^\infty \left(U, \pi^* T_M^* \otimes_{\mathbb{R}} T_{T_M}\right),$$

and

$$B \in C^\infty \left(U, \pi^* \operatorname{GL}(T_M)\right),$$

such that $d\pi \cdot \alpha = \mathbb{I}_{\pi^* T_M}$ over U and such that $\alpha_{0_p} = d_p 0_M$, $B_{0_p} = \mathbb{I}_{T_{M,p}}$, for all $p \in M$. The almost complex structure J_A, with $A = \alpha + iTB$ associated to (α, B) is the one which satisfies

$$T^{0,1}_{T_M, J_A, \eta} = A_\eta \left(\mathbb{C}T_{M,\pi(\eta)}\right) \subset \mathbb{C}T_{T_M,\eta},$$

for all $\eta \in U \subseteq T_M$.

Every almost complex continuous extension of the canonical complex structure $J^{\rm can}$ of $T_{T_M|M}$ over a neighborhood of M inside T_M writes as the almost complex structure associated to an M-totally real almost complex structure defined over a sufficiently small neighborhood of M.

We provide below an explicit formula for the almost complex structure J_A. For this purpose we notice first that for any vector $\xi \in T_{T_M,\eta}$,

$$\begin{aligned}
\xi^{0,1}_{J_A} &= \frac{1}{2} A_\eta \left[d_\eta\pi - iB_\eta^{-1} T_\eta^{-1} \left(\mathbb{I}_{T_{T_M}} - \alpha_\eta\, d_\eta\pi \right) \right] \xi, \\
\xi^{1,0}_{J_A} &= \frac{1}{2} \overline{A}_\eta \left[d_\eta\pi + iB_\eta^{-1} T_\eta^{-1} \left(\mathbb{I}_{T_{T_M}} - \alpha_\eta\, d_\eta\pi \right) \right] \xi.
\end{aligned}$$

Indeed $\xi^{0,1}_{J_A} \in T^{0,1}_{T_M,J_A,\eta}$, $\xi^{1,0}_{J_A} \in T^{1,0}_{T_M,J_A,\eta}$ and $\xi = \xi^{1,0}_{J_A} + \xi^{0,1}_{J_A}$. We deduce the expression

$$J_{A,\eta} = -\alpha_\eta\, B_\eta^{-1} T_\eta^{-1} \left(\mathbb{I}_{T_{T_M}} - \alpha_\eta\, d_\eta\pi \right) + T_\eta\, B_\eta\, d_\eta\pi. \tag{1.5}$$

This shows that for any α-horizontal vector $\xi \in T_{T_M,\eta}$, i.e. $\xi = \alpha_\eta\, d_\eta\pi\, \xi$, we have

$$J_{A,\eta}\, \xi = T_\eta\, B_\eta\, d_\eta\pi\, \xi.$$

In equivalent terms

$$J_{A,\eta}\, \alpha_\eta\, v = T_\eta\, B_\eta\, v, \tag{1.6}$$

for any $\eta \in U \subset T_M$ and any $v \in T_{M,\pi(\eta)}$. Moreover (1.5) implies

$$J_{A,\eta|\operatorname{Ker} d_\eta\pi} = -\alpha_\eta\, B_\eta^{-1} T_\eta^{-1}. \tag{1.7}$$

A well known theorem by Bruhat and Whitney [Br-Wh] states that for any compact real-analytic manifold M there exist a complex manifold (X, J) and a real-analytic embedding of M in X such that as a sub-manifold of X, M is maximally totally real. In addition one can arrange that X is an open neighborhood $U \subseteq T_M$ of the zero section and $J_{|M} = J^{\rm can}$.

Moreover Bruhat and Whitney show [Br-Wh] that if X is a real-analytic manifold equipped with two different real-analytic complex structures J_1 and J_2 which contains a real analytic sub-manifold M which is maximally totally

real with respect to both J_1 and J_2, then there exist neighborhoods U_1 and U_2 of M inside X and a real-analytic diffeomorphism $\kappa : U_1 \longrightarrow U_2$ which is the identity on M and is a holomorphic mapping of (U_1, J_1) onto (U_2, J_2).

In other terms the structure J constructed by Bruhat and Whitney in [Br-Wh] is unique up to complex isomorphisms.

In a long series of celebrated papers inspired by the work of Grauert [Gra], Guillemin-Stenzel [Gu-St], Lempert [Lem], Lempert-Szöke [Le-Sz1, Le-Sz2], Szöke [Szo1, Szo2], Burns [Bu1, Bu2], Burns-Halverscheid-Hind [BHH], Aslam-Burns-Irvine [ABI] as well as Bielawski [Bie] put pluri-potential and metric constraints on J. Some of their results will be reminded in great detail in the next section.

Their results are needed in a crucial way in analytic micro-local analysis, in pluri-potential theory (see the impressive work by Zelditch [Zel]) as well as in Hamiltonian dynamics and in geometric quantization (see the work by Morao-Nunes [Mo-Nu] and Hall-Kirwin [Ha-Ki]).

We state below our results on the integrability conditions for J.

1.2 The integrability equations for M-totally real almost complex structures

Let (E, π_E, M) be a vector bundle over a manifold M. For an arbitrary section $B \in C^\infty(E, \pi_E^* (T_M^* \otimes E))$, we define the derivative along the fiber

$$DB \in C^\infty \left(E, \pi_E^* \left(E^* \otimes T_M^* \otimes E\right)\right),$$

by the formula

$$D_\eta B \left(v\right) := \frac{d}{dt}_{|_{t=0}} B_{\eta + tv} \in T_{M,p}^* \otimes E_p \,,$$

for any $\eta, v \in E_p$. We denote by Alt_2 the alternating operator (without normalizing coefficient!) which acts on the first two entries of a tensor. For any morphism $A : T_M \longrightarrow E$ and any bilinear form $\beta : E \times T_M \longrightarrow E$ we define the contraction operation

$$A \neg \beta := \mathrm{Alt}_2 \left(\beta \circ A\right),$$

where the composition operator $\circ$ act on the first entry of β.

Theorem 1 *Let M be a smooth manifold and let J_A with $A = \alpha + iTB$ be a M-totally real almost complex structure over an open neighborhood $U \subseteq T_M$ of the zero section. Let also ∇ be a covariant derivative operator acting on the smooth sections of T_M and let $\Gamma \in C^\infty \left(U, \pi^* \operatorname{End}\left(T_M\right)\right)$ such that $\alpha := H^\nabla - T\Gamma$.*

Then J_A is integrable over U if and only if the complex section $S := \Gamma + iB$ satisfies the equation

$$H_\eta^\nabla \neg \left(\nabla^{\mathrm{End}(T_M),\pi} S\right)_\eta - S_\eta \neg D_\eta S + S_\eta \tau^\nabla + R^\nabla \cdot \eta = 0 \,, \tag{1.8}$$

for any point $\eta \in U$, where $\nabla^{\mathrm{End}(T_M),\pi}$ is the covariant derivative operator acting on the smooth sections of $\pi^* \operatorname{End}\left(T_M\right)$ induced by ∇ and where τ^∇ and R^∇ are respectively the torsion and curvature forms of ∇.

We notice that $S_{|M} = i\mathbb{I}_{T_M}$ by the conditions $\alpha_{0_p} = H^{\nabla}_{0_p} = d_p\, 0_M$ and $B_{0_p} = \mathbb{I}_{T_{M,p}}$.

Notations for the statement of the main theorem.

For any $A \in T^{*,\otimes p}_M \otimes \mathrm{End}_{\mathbb{C}}\left(\mathbb{C}T_M\right)$ and for any $\theta \in T^{*,\otimes q}_M \otimes \mathbb{C}T_M$, the product operations of tensors $A \cdot \theta, A \neg \theta \in T^{*,\otimes(p+q)}_M \otimes \mathbb{C}T_M$ are defined by

$$\begin{aligned}
\left(A \cdot \theta\right)(u_1, \ldots, u_p, v_1, \ldots, v_q) &:= A(u_1, \ldots, u_p) \cdot \theta(v_1, \ldots, v_q)\,, \\
\left(A \neg \theta\right)(u_1, \ldots, u_p, v_1, \ldots, v_q) &:= \sum_{j=1}^{q} \theta(v_1, \ldots, A(u_1, \ldots, u_p) \cdot v_j, \ldots, v_q)\,.
\end{aligned}$$

We will denote for notation simplicity $R^{\nabla}.\theta := R^{\nabla} \cdot \theta - R^{\nabla} \neg \theta$. We will denote by Circ the circular operator

$$\left(\mathrm{Circ}\,\theta\right)(v_1, v_2, v_3, \bullet) = \theta\left(v_1, v_2, v_3, \bullet\right) + \theta\left(v_2, v_3, v_1, \bullet\right) + \theta\left(v_3, v_1, v_2, \bullet\right),$$

acting on the first three entries of any q-tensor θ, with $q \geqslant 3$. We define also the permutation operation $\theta_2\left(v_1, v_2, \bullet\right) := \theta\left(v_2, v_1, \bullet\right)$.

For any covariant derivative ∇ acting on the smooth sections of $\mathbb{C}T_M$ we define the operator

$$d^{\nabla}_1 : C^{\infty}\left(M, T^{*,\otimes k}_M \otimes_{\mathbb{R}} \mathbb{C}T_M\right) \longrightarrow C^{\infty}\left(M, \Lambda^2 T^*_M \otimes_{\mathbb{R}} T^{*,\otimes(k-1)}_M \otimes_{\mathbb{R}} \mathbb{C}T_M\right),$$

with $k \geqslant 1$ as follows

$$d^{\nabla}_1 A\left(\xi_1, \xi_2, \mu\right) := \nabla_{\xi_1} A\left(\xi_2, \mu\right) - \nabla_{\xi_2} A\left(\xi_1, \mu\right),$$

with $\xi_1, \xi_2 \in T_M$ and with $\mu \in T^{\oplus(k-1)}_M$. Moreover for any

$$\begin{aligned}
A &\in C^{\infty}\left(M, T^{*,\otimes(k+1)}_M \otimes_{\mathbb{R}} \mathbb{C}T_M\right), \\
B &\in C^{\infty}\left(M, T^{*,\otimes(l+1)}_M \otimes_{\mathbb{R}} \mathbb{C}T_M\right),
\end{aligned}$$

we define the exterior product

$$A \wedge_1 B \in C^{\infty}\left(M, \Lambda^2 T^*_M \otimes_{\mathbb{R}} T^{*,\otimes(k+l-1)}_M \otimes_{\mathbb{R}} \mathbb{C}T_M\right),$$

as

$$\left(A \wedge_1 B\right)(\xi_1, \xi_2, \eta, \mu) := A\left(\xi_1, B\left(\xi_2, \eta\right), \mu\right) - A\left(\xi_2, B\left(\xi_1, \eta\right), \mu\right),$$

with $\xi_1, \xi_2 \in T_M$, $\eta \in T^{\oplus l}_M$ and $\mu \in T^{\oplus(k-1)}_M$. We denote by $\mathrm{Sym}_{r_1,\ldots,r_s}$ the symmetrizing operator (without normalizing coefficient!) acting on the entries $r_1, \ldots, r_s$ of a multi-linear form. We use in this paper the common convention that a sum and a product running over an empty set is equal respectively to 0 and 1.

With theese notations we can state our main theorem.

Theorem 2 (***Integrability in the fiberwise real analytic case***).

Let M be a smooth manifold, let $U \subseteq T_M$ be an open neighborhood of the zero section with connected fibers and let J_A with $A = \alpha + iTB$ be a M-totally real almost complex structure over U, which is real-analytic along the fibers of U. Let ∇ be a torsion free covariant derivative operator acting on the smooth sections of T_M and let $\Gamma \in C^\infty\left(U, \pi^* \operatorname{End}\left(T_M\right)\right)$ such that $\alpha = H^\nabla - T\Gamma$.

Consider the fiberwise Taylor expansion of the complex section $S := \Gamma + iB$ at the origin, given by

$$S_\eta \cdot \xi = \sum_{k \geqslant 0} S_k\left(\xi, \eta^k\right),$$

for any $\eta \in T_M$ in a neighborhood of the zero section and any $\xi \in T_{M,\pi(\eta)}$, with $S_0 = i\mathbb{I}_{T_M}$ and with

$$S_k \in C^\infty\left(M, T_M^* \otimes_{\mathbb{R}} S^k T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M\right),$$

for all $k \geqslant 1$, (here we denote by $\eta^k := \eta^{\times k} \in T^{\oplus k}_{M,\pi(\eta)}$), and let ∇^{S_1} be the complex covariant derivative operator acting on the smooth sections of $\mathbb{C}T_M$ defined by

$$\nabla^{S_1}_\xi \eta := \nabla_\xi \eta + S_1\left(\xi, \eta\right).$$

Then J_A is integrable over U if and only if $S_1 \in C^\infty\left(M, S^2 T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M\right)$, (i.e. ∇^{S_1} is torsion free) and for all $k \geqslant 2$,

$$S_k = \frac{i}{k} \nabla^{S_1} \sigma_{k-1} + \frac{i}{(k+1)!} \operatorname{Sym}_{2,\ldots,k+1} \beta_{k-1}\left(\sigma_{k-2}\right) + \sigma_k,$$

$$\sigma_k \in C^\infty\left(M, S^{k+1} T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M\right),$$

$$\operatorname{Circ} \beta_{k+1}\left(\sigma_k\right) = 0,$$

where $\sigma_1 := 0$, $\beta_1\left(\sigma_0\right) := R^{\nabla^{S_1}}$, $\beta_2\left(\sigma_1\right) := -\frac{i}{3}(\nabla^{S_1} R^{\nabla^{S_1}})_2$ and for all $k \geqslant 3$,

$$\begin{aligned}
\beta_k\left(\sigma_{k-1}\right) &:= \frac{i}{k} R^{\nabla^{S_1}} . \sigma_{k-1} + \frac{1}{(k+1)!k!} \operatorname{Sym}_{3,\ldots,k+2} \theta_k\left(\sigma_{k-1}\right), \\
\theta_k\left(\sigma_{k-1}\right) &:= i \sum_{r=2}^{k-2} \frac{(r+2)!}{r+1} (i\, d_1^{\nabla^{S_1}})^{k-r-1} (R^{\nabla^{S_1}} . \sigma_r) \\
&- 2\, i\, (i\, d_1^{\nabla^{S_1}})^{k-2} (\nabla^{S_1} R^{\nabla^{S_1}})_2 \\
&+ \sum_{r=3}^{k} (r+1)! \sum_{p=2}^{r-1} (i\, d_1^{\nabla^{S_1}})^{k-r} \left(p S_p \wedge_1 S_{r-p+1}\right).
\end{aligned}$$

In more explicit terms

$$S_2 = S_2^0 + \sigma_2, \tag{1.9}$$

$$S_2^0\left(\xi_1, \xi_2, \xi_3\right) := \frac{i}{6}\left[R^{\nabla^{S_1}}\left(\xi_1, \xi_2\right)\xi_3 + R^{\nabla^{S_1}}\left(\xi_1, \xi_3\right)\xi_2\right], \tag{1.10}$$

$$\sigma_2 \in C^\infty\left(M, S^3 T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M\right), \tag{1.11}$$

$$\operatorname{Circ}\beta_3(\sigma_2) = 0\,, \tag{1.12}$$

$$\beta_3(\sigma_2) := \frac{i}{3}R^{\nabla^{S_1}}.\,\sigma_2 + \frac{1}{4!3!}\operatorname{Sym}_{3,4,5}\theta_3(\sigma_2)\,, \tag{1.13}$$

$$\theta_3(\sigma_2) := 2d_1^{\nabla^{S_1}}(\nabla^{S_1}R^{\nabla^{S_1}})_2 + 4!2S_2\wedge_1 S_2\,. \tag{1.14}$$

The assumption that the complex structure tensor is real-analytic along the fibers of the tangent bundle is quite natural. Indeed in the case M is real analytic then the M-totally real complex structure constructed by Bruhat and Whitney [Br-Wh] is also real analytic with respect to the real analytic structure of the tangent bundle induced by M.

In this paper we request from the readers some knowledge of the geometric theory of linear connections. Basics of such theory can be found in the appendix.

2 Some old and new facts

2.1 The almost complex structure associated to a connection over the tangent bundle

It is well known (see [Dom]) that we can construct an M-totally real almost complex structure over T_M by using the horizontal distribution $\mathcal{H}\subset T_M$ associated to a linear connection ∇ acting on the sections of T_M. Indeed in this case we set $\alpha_\eta := H_\eta$ and $B_\eta := \mathbb{I}_{T_M,\pi(\eta)}$, where $\eta\longmapsto H_\eta$ is the horizontal map associated to $\mathcal{H}$. We will denote $J_{\mathcal{H}} := J_A$. If we define for any $\eta\in T_{M,p}$ the vertical projection $\operatorname{Vert}_\eta : T_{T_M,\eta}\longrightarrow T_{T_{M,p},\eta}$ as

$$\operatorname{Vert}_\eta := \mathbb{I}_{T_{T_M,\eta}} - H_\eta d_\eta\pi\,,$$

where $\pi : T_M\longrightarrow M$ is the canonical projection, then

$$J_{\mathcal{H},\eta} := -H_\eta\, T_\eta^{-1}\operatorname{Vert}_\eta + T_\eta\, d_\eta\pi\,.$$

If we decompose any vector $\xi\in\mathbb{C}T_{T_M,\eta}$ in its horizontal and vertical parts $\xi=\xi^h+\xi^v$ with $\xi^v := \operatorname{Vert}_\eta(\xi)$ then we have the expressions

$$\begin{aligned} J_{\mathcal{H},\eta}\xi &= -H_\eta\, T_\eta^{-1}\xi^v + T_\eta\, d_\eta\pi\,\xi^h,\\ (J_{\mathcal{H},\eta}\xi)^h &= -H_\eta\, T_\eta^{-1}\xi^v,\\ (J_{\mathcal{H},\eta}\xi)^v &= T_\eta\, d_\eta\pi\,\xi^h. \end{aligned}$$

We infer

$$\begin{aligned} \xi^{0,1}_{J_{\mathcal{H}}}(\eta) &= \frac{1}{2}\left[\xi^h - i\,H_\eta\, T_\eta^{-1}\xi^v + \xi^v + i\,T_\eta\, d_\eta\pi\,\xi^h\right]\\ &= \frac{1}{2}\left[\xi^h + H_\eta\,\mu + i\,T_\eta\left(d_\eta\pi\xi^h+\mu\right)\right], \end{aligned}$$

with $\mu := -i\,T_\eta^{-1}\xi^v$. We notice also the identity

$$T^{0,1}_{T_M,J_{\mathcal{H}},\eta} = \frac{1}{2}\left(H_\eta + i\,T_\eta\right)\mathbb{C}T_{M,p}\,, \tag{2.1}$$

for any any $\eta \in T_{X,p}$. The distribution $T^{0,1}_{T_M,J_{\mathcal{H}}}$ is horizontal, but the associated map does not satisfies the condition (8.5) of linear connections thanks to the identity (8.4). Therefore this distribution does not identify a linear connection. However its integrability implies that the vector bundle T_M is flat. Indeed we have the following well known lemma due to Dombrowsky [Dom].

Lemma 1 *The torsion form $\tau^{J_{\mathcal{H}}}$ of the almost complex structure $J_{\mathcal{H}}$ satisfies at the point $\eta \in T_M$ in the directions $V_1, V_2 \in T^{0,1}_{T_M,J_{\mathcal{H}},\eta}$ the identity*

$$\begin{aligned} 8\tau^{J_{\mathcal{H}}}(V_1,V_2)(\eta) &= -H_\eta\left[\tau^{\nabla}(v_1,v_2) + i\,R^{\nabla}(v_1,v_2)\,\eta\right] \\ &+ T_\eta\left[i\,\tau^{\nabla}(v_1,v_2) - R^{\nabla}(v_1,v_2)\,\eta\right], \end{aligned}$$

where $R^{\nabla} := \nabla^2$ is the complex linear extension of the curvature tensor of ∇, where τ^{∇} is the torsion of the complex connection ∇ and where $v_j := d_\eta\pi V_j$, $j = 1, 2$. In particular $J_{\mathcal{H}}$ is a complex structure if and only if the linear connection ∇ is flat and torsion free.

Proof Let ξ_j be vector field local extensions of v_j such that $[\xi_1, \xi_2]\pi(\eta) = 0$. Then

$$\Xi_j := \frac{1}{2}(H + i\,T)\,\xi_j\,,$$

are local vector field extensions of V_j. We expand the bracket

$$\begin{aligned} 4[\Xi_1,\Xi_2](\eta) &= \Big([H\xi_1,H\xi_2] + i\,[H\xi_1,T\xi_2] + i\,[T\xi_1,H\xi_2] - [T\xi_1,T\xi_2]\Big)(\eta) \\ &= H_\eta[\xi_1,\xi_2] - T_\eta[R^{\nabla}(v_1,v_2)\,\eta] + i\,T_\eta[\nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1]\,. \end{aligned}$$

The last equality follows from to the computation at the end of the proof of lemma 19 and thanks to the identity (8.8) in the appendix. (We notice that $[T\xi_1, T\xi_2] \equiv 0$, since the vector fields $T\xi_1$ are tangent constant along the fibers). Thanks to the assumption $[\xi_1, \xi_2]\pi(\eta) = 0$, we infer the equality

$$4[\Xi_1,\Xi_2](\eta) = T_\eta\left[i\,\tau^{\nabla}(v_1,v_2) - R^{\nabla}(v_1,v_2)\,\eta\right].$$

The required formula follows from the identity

$$\xi^{1,0}_{J_{\mathcal{H}}}(\eta) = \frac{1}{2}\left[\xi^h + i\,H_\eta\,T_\eta^{-1}\xi^v + \xi^v - i\,T_\eta\,d_\eta\pi\,\xi^h\right].$$

The fact that that the distribution $T^{1,0}_{T_M,J_{\mathcal{H}}}$ is horizontal implies that $\tau^{J_{\mathcal{H}}}(V_1,V_2)(\eta)$ vanishes for all V_j if and only if the quantity

$$\tau^{\nabla}(v_1,v_2) + i\,R^{\nabla}(v_1,v_2)\,\eta\,,$$

vanishes for all v_j. In particular for real vectors v_j this implies that R^{∇} and τ^{∇} vanish at the point $\pi(\eta)$. □

We observe that a connection over T_M is flat and torsion free if and only if there exist local parallel frames with vanishing Lie brackets.

3 The symplectic approach

Let M be a smooth manifold and let $\theta \in C^{\infty}(T^*_M, T^*_{T^*_M})$ be the canonical 1-form on the total space of the cotangent bundle defined as $\theta_{\lambda} := \lambda \cdot d_{\lambda}\pi_{T^*_M}$, for any $\lambda \in T^*_M$. The canonical symplectic form over the total space T^*_M is defined as $\Omega := -d\theta$. Let now g be a Riemann metric over M viewed as a vector bundle map $g : T_M \longrightarrow T^*_M$. We define also the forms $\theta^g := g^*\theta$ and $\Omega^g := g^*\Omega = -d\theta^g$ over the total space of the tangent bundle. In explicit terms $\theta^g_{\eta} = g(\eta) \cdot d_{\eta}\pi_{T_M}$, for all $\eta \in T_M$, i.e.

$$\theta^g_{\eta}(\xi) = g_{\pi_{T_M}(\eta)}(\eta, d_{\eta}\pi_{T_M} \cdot \xi),$$

for all $\xi \in T_{T_M,\eta}$. Let ∇^g be the Levi-Civita connection, defined as

$$2\,\nabla^g_{\xi}\eta := g^{-1}\left[\xi \neg d(g\eta) + \eta \neg d(g\xi) + d\langle \xi, \eta\rangle_g\right] + [\xi, \eta],$$

for any $\xi, \eta \in C^{\infty}(M, T_M)$. Let also $\gamma^g \in C^{\infty}(T_M, T^*_{T_M} \otimes T_{T_M})$ be the Levi-Civita 1-form, which is determined along any section $\eta \in C^{\infty}(M, T_M)$, by the identity $\gamma^g_{\eta} \cdot d\eta = T_{\eta}\nabla^g\eta$.

For any curve $\eta : t \longmapsto \eta_t \in T_M$, we define the covariant derivative

$$\frac{\nabla^g \eta}{dt} := T^{-1}_{\eta_t}\gamma^g_{\eta_t}\dot{\eta}_t \in T_{M,\pi(\eta_t)}.$$

We consider now two curves $\eta_j : t \longmapsto \eta_{j,t} \in T_M$, $j = 1, 2$, such that $\pi_{T_M}(\eta_{1,t}) = \pi_{T_M}(\eta_{2,t}) = x_t$. Then

$$\frac{d}{dt}g_{|x_t}(\eta_{1,t}\,,\,\eta_{2,t}) = g_{|x_t}\left(\frac{\nabla^g\eta_1}{dt}\,,\,\eta_{2,t}\right) + g_{|x_t}\left(\eta_{1,t}\,,\,\frac{\nabla^g\eta_2}{dt}\right).$$

With the previous notations hold the following well known lemma (see also Klingenberg's book [Kli] for a proof using local coordinates).

Lemma 2 *The formula*

$$\Omega^g_{\eta}(\xi_1, \xi_2) \quad = \quad g_p(d_{\eta}\pi_{T_M}\xi_1, T^{-1}_{\eta}\gamma^g_{\eta}\xi_2) - g_p(d_{\eta}\pi_{T_M}\xi_2, T^{-1}_{\eta}\gamma^g_{\eta}\xi_1),$$

hold for any $\eta \in T_M$, $p = \pi_{T_M}(\eta)$ and for any $\xi_1, \xi_2 \in T_{T_M,\eta}$.

Proof With respect to a local coordinate trivialization of the tangent bundle we can extend in a linear way the vectors ξ_1, ξ_2 in to vector fields Ξ_1, Ξ_2 in a neighborhood of $T_{M,p}$ inside T_M. In this way $[\Xi_1, \Xi_2] = 0$ and thus $\Omega^g(\Xi_1, \Xi_2) = \Xi_2.\theta^g(\Xi_1) - \Xi_1.\theta^g(\Xi_2)$. We denote by $\eta_{j,t}$, $j = 1, 2$ the corresponding flow lines starting from η. Then

$$\begin{aligned}\Omega^g_{\eta}(\xi_1, \xi_2) \quad &= \quad \frac{d}{dt}_{|_{t=0}}\left[g_{\pi_{T_M}(\eta_{2,t})}\left(\eta_{2,t}\,, d_{\eta_{2,t}}\pi_{T_M} \cdot \Xi_1(\eta_{2,t})\right)\right] \\ &- \quad \frac{d}{dt}_{|_{t=0}}\left[g_{\pi_{T_M}(\eta_{1,t})}\left(\eta_{1,t}\,, d_{\eta_{1,t}}\pi_{T_M} \cdot \Xi_2(\eta_{1,t})\right)\right].\end{aligned}$$

We distinguish two cases.

• In the case when $d_\eta \pi_{T_M} \xi_j = 0$ for some j, say $j = 1$, then $d_{\eta_{2,t}} \pi_{T_M} \Xi_1 (\eta_{2,t}) = 0$ and

$$\frac{d}{dt} d_{\eta_{1,t}} \pi_{T_M} \Xi_2 (\eta_{1,t}) = 0 \,,$$

by the linear nature of the local extension. Then

$$\Omega^g_\eta (\xi_1, \xi_2) = -g_p(T^{-1}_\eta \gamma^g_\eta \xi_1, d_\eta \pi_{T_M} \xi_2) \,.$$

The case $j = 2$ is quite similar.

• In the case when $d_\eta \pi_{T_M} \xi_j$, do not vanish for $j = 1, 2$, then the vector fields $\zeta_j := d\pi_{T_M} \Xi_j$ are well defined and $[\zeta_1, \zeta_2] = 0$. Then

$$\begin{aligned}
\Omega^g_\eta (\xi_1, \xi_2) &= g_p(T^{-1}_\eta \gamma^g_\eta \xi_2, d_\eta \pi_{T_M} \xi_1) + g_p(\eta, \nabla^g_{\zeta_2(p)} \zeta_1 - \nabla^g_{\zeta_1(p)} \zeta_2) \\
&- g_p(T^{-1}_\eta \gamma^g_\eta \xi_1, d_\eta \pi_{T_M} \xi_2) \\
&= g_p(T^{-1}_\eta \gamma^g_\eta \xi_2, d_\eta \pi_{T_M} \xi_1) + g_p(\eta, [\zeta_1, \zeta_2] (p)) \\
&- g_p(T^{-1}_\eta \gamma^g_\eta \xi_1, d_\eta \pi_{T_M} \xi_2) \,,
\end{aligned}$$

which implies the required conclusion. □

We need to remind in detail also the following very well known lemma (see also [Kli]).

Lemma 3 *Let* $2\,\zeta^g := \Omega^{g,-1} d\,|\cdot|^2_g$ *and let* Φ^g_t *be the corresponding* 1*-parameter sub-group of transformations of* T_M*. Then for any* $\eta \in T_M$ *the curve* $c_t := \pi_{T_M} \circ \Phi^g_t (\eta)$ *is the geodesic with initial speed* $\dot{c}_0 = \eta$ *and* $\dot{c}_t = \Phi^g_t (\eta)$*.*

Proof For any $\eta \in T_M$ and for any $\xi \in T_{T_M,\eta}$, let $t \longmapsto \eta_t \in T_M$ be the curve such that $\dot{\eta}_0 = \xi$. Then

$$\begin{aligned}
\xi \,.\, |\cdot|^2_g &= \frac{d}{dt}_{|_{t=0}} \left[g_{\pi_{T_M}(\eta_t)} (\eta_t \,, \eta_t) \right] \\
&= 2\, g_p \left(\eta \,, T^{-1}_\eta \gamma^g_\eta \xi \right) ,
\end{aligned}$$

and thus

$$\Omega^g_\eta \left(\zeta^g_\eta \,,\, \xi \right) = g_p \left(\eta \,, T^{-1}_\eta \gamma^g_\eta \xi \right) ,$$

by the definition of the vector field ζ^g_η. Using lemma 2 we infer

$$g_p(d_\eta \pi_{T_M} \zeta^g_\eta, T^{-1}_\eta \gamma^g_\eta \xi) - g_p(d_\eta \pi_{T_M} \xi, T^{-1}_\eta \gamma^g_\eta \zeta^g_\eta) = g_p \left(\eta, T^{-1}_\eta \gamma^g_\eta \xi \right) . \tag{3.1}$$

In the case $d_\eta \pi_{T_M} \xi = 0$, the identity (3.1) yields

$$g_p(d_\eta \pi_{T_M} \zeta^g_\eta, T^{-1}_\eta \xi) = g_p \left(\eta, T^{-1}_\eta \xi \right) ,$$

and thus $d_\eta \pi_{T_M} \zeta^g_\eta = \eta$. In the case $\gamma^g_\eta \, \xi = 0$, the identity (3.1) yields

$$g_p(d_\eta \pi_{T_M} \xi \,, T^{-1}_\eta \gamma^g_\eta \zeta^g_\eta) = 0 \,,$$

and thus $\gamma_\eta^g\,\zeta_\eta^g = 0$. We deduce the formula

$$\zeta_\eta^g = H_\eta^g \cdot \eta\,. \tag{3.2}$$

Thus the flow line $\eta_t := \Phi_t^g\,(\eta)$ satisfies the identity

$$\dot{\eta}_t = H_{\eta_t}^g \cdot \eta_t\,. \tag{3.3}$$

We deduce

$$\begin{aligned}\dot{c}_t &= d_{\eta_t}\pi_{T_M} \cdot \dot{\eta}_t \\ &= d_{\eta_t}\pi_{T_M} \cdot H_{\eta_t}^g \cdot \eta_t \\ &= \eta_t\,,\end{aligned}$$

and $\ddot{c}_t = H_{\dot{c}_t}^g \cdot \dot{c}_t$, which is the geodesic equation. □

We provide now a proof of the following well known result due to Lempert-Szöke [Le-Sz1]. See also Guillemin-Stenzel [Gu-St], Burns [Bu1, Bu2] and Burns-Halverscheid-Hind [BHH].

Corollary 1 *Let (M, g) be a smooth Riemannian manifold. A complex structure J over the total space of the tangent bundle T_M satisfies the conditions*

$$J_{|M} = J^{\mathrm{can}}, \tag{3.4}$$

$$2\theta^g = d|\cdot|_g^2 \cdot J. \tag{3.5}$$

if and only if for any $\eta \in T_M$, the complex curve $\psi_\eta : t + is \longmapsto s\Phi_t^g\,(\eta)$, defined in a neighborhood of $0 \in \mathbb{C}$, is J-holomorphic.

Proof We define the Reeb vector field $\Xi := \Omega^{g,-1}\theta^g$. This vector field is independent of the metric g. Indeed by lemma 2 hold the identity

$$g_p(\eta\,, d_\eta\pi_{T_M}\xi) = g_p(d_\eta\pi_{T_M}\Xi_\eta\,, T_\eta^{-1}\gamma_\eta^g\xi) - g_p(d_\eta\pi_{T_M}\xi\,, T_\eta^{-1}\gamma_\eta^g\Xi_\eta)\,, \tag{3.6}$$

for any $\xi \in T_{T_M,\eta}$. Thus if $d_\eta\pi_{T_M}\xi = 0$ we deduce the equality

$$g_p(d_\eta\pi_{T_M}\Xi_\eta\,, T_\eta^{-1}\xi) = 0\,,$$

and thus $d_\eta\pi_{T_M}\Xi_\eta = 0$. Then the identity (3.6) reduces as

$$g_p(\eta\,, d_\eta\pi_{T_M}\xi) = -\,g_p(d_\eta\pi_{T_M}\xi\,, T_\eta^{-1}\Xi_\eta)\,,$$

for any $\xi \in T_{T_M,\eta}$. We infer the formula

$$\Xi_\eta = -\,T_\eta \cdot \eta\,, \tag{3.7}$$

for all $\eta \in T_M$. We notice now that the identity (3.5) is equivalent to the identity

$$\Omega^g\,(2\,\Xi\,, \xi) = d\,|\cdot|_g^2\,J\xi\,,$$

and is also equivalent to the identity $\theta^g = -d^c_{_J} |\cdot|^2_g$. Thus

$$\Omega^g = dd^c_{_J} |\cdot|^2_g = i\partial_{_J}\overline{\partial}_{_J} |\cdot|^2_g \,,$$

thanks to the fact that J^g is integrable. We infer that the symplectic form Ω^g is J-invariant. Thus

$$\Omega^g \left(2\, J\,\Xi\,, J\xi\right) = d\,|\cdot|^2_g\, J\xi\,,$$

i.e.

$$J\,\Xi = \zeta^g. \tag{3.8}$$

This combined with (3.7) and with (3.2) implies that (3.5) is equivalent to the identity

$$J_\eta\, H^g_\eta \cdot \eta = T_\eta \cdot \eta\,. \tag{3.9}$$

We show now that the later combined with (3.4) is equivalent to the J-holomorphy of the maps ψ_η. For this purpose we observe that the differential of such maps is given by

$$d_{t_0+is_0}\psi_\eta \left(a\frac{\partial}{\partial t} + b\frac{\partial}{\partial s}\right) = a\, d\left(s_0 \mathbb{I}_{T_M}\right) \dot{\Phi}^g_{t_0}(\eta) + b\, T_{s_0\Phi^g_{t_0}(\eta)}\Phi^g_{t_0}(\eta)\,.$$

But

$$\begin{aligned}\dot{\Phi}^g_{t_0}(\eta) &= \zeta^g \circ \Phi^g_{t_0}(\eta)\\ &= H^g_{\Phi^g_{t_0}(\eta)} \cdot \Phi^g_{t_0}(\eta)\,,\end{aligned}$$

thanks to (3.2). Then using the property (8.5) of the linear connection ∇^g we infer

$$d_{t_0+is_0}\psi_\eta \left(a\frac{\partial}{\partial t} + b\frac{\partial}{\partial s}\right) = \left(aH_{s_0\Phi^g_{t_0}(\eta)} + bT_{s_0\Phi^g_{t_0}(\eta)}\right)\cdot \Phi^g_{t_0}(\eta)\,. \tag{3.10}$$

The complex curve ψ_η is J-holomorphic if and only if

$$d_{t_0+is_0}\psi_\eta \left(-b\frac{\partial}{\partial t} + a\frac{\partial}{\partial s}\right) = Jd_{t_0+is_0}\psi_\eta \left(a\frac{\partial}{\partial t} + b\frac{\partial}{\partial s}\right),$$

thus, if and only if

$$\left(-bH_{s_0\Phi^g_{t_0}(\eta)} + aT_{s_0\Phi^g_{t_0}(\eta)}\right)\cdot \Phi^g_{t_0}(\eta) = J\left(aH_{s_0\Phi^g_{t_0}(\eta)} + bT_{s_0\Phi^g_{t_0}(\eta)}\right)\cdot \Phi^g_{t_0}(\eta)\,.$$

For $s_0 \neq 0$ this is equivalent to (3.9). For $s_0 = 0$ this is equivalent to (3.4). We deduce the required conclusion. □

The condition (3.4) implies that J is an M-totally real complex structure. We show now the following corollary of the main theorem 2.

Corollary 2 *Let (M, g) be a smooth Riemannian manifold and let $J \equiv J_A$ with $A = \alpha + iTB$ be an M-totally real almost complex structure over an open neighborhood U of M inside T_M which is real analytic along the fibers of U. Then J is*

integrable over U and for any $\eta \in U$, the complex curve $\psi_\eta : t + is \longmapsto s\Phi_t^g(\eta)$, defined in a neighborhood of $0 \in \mathbb{C}$, is J-holomorphic if and only if the fiberwise Taylor expansion of J at the origin given by the expansion of the complex section $S := T^{-1}(H^g - \alpha) + iB$,

$$S_\eta \cdot \xi \quad = \quad \sum_{k \geqslant 0} S_k\left(\xi, \eta^k\right),$$

for any $\eta \in T_M$ in a neighborhood of the zero section and any $\xi \in T_{M,\pi(\eta)}$, with $S_0 = i\mathbb{I}_{T_M}$ and with

$$S_k \in C^\infty(M, T_M^* \otimes_{\mathbb{R}} S^k T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M),$$

for all $k \geqslant 1$, (here we denote by $\eta^k := \eta^{\times k} \in T^{\oplus k}_{M,\pi(\eta)}$) satisfies $S_1 = 0$,

$$S_k \quad := \quad \frac{i}{(k+1)!k!} \operatorname{Sym}_{2,\ldots,k+1} \Theta_k(g),$$

for all $k \geqslant 2$, with $\Theta_2(g) := 2R^g$ and

$$\begin{aligned} \Theta_k(g) \quad &:= \quad -2i\left(id_1^{\nabla^g}\right)^{k-3}(\nabla^g R^g)_2 \\ &+ \quad \sum_{r=3}^{k-1}(r+1)!\sum_{p=2}^{r-1}\left(id_1^{\nabla^g}\right)^{k-1-r}(pS_p \wedge_1 S_{r-p+1}), \end{aligned}$$

for all $k \geqslant 3$ and the metric g satisfies the equations $\operatorname{Circ}\operatorname{Sym}_{3,\ldots,k+1}\Theta_k(g) = 0$ for all $k \geqslant 4$.

Proof If we write $\alpha = H^g - T\Gamma$ then the complex section S rewrites as $S := \Gamma + iB$. We set $S_k = \Gamma_k + iB_k$. From the proof of corollary 1 we know that in the case J is integrable over U, the curve ψ_η is J-holomorphic if and only if hold (3.9). The later rewrites as

$$H_\eta^g \cdot \eta = -J_\eta T_\eta \cdot \eta.$$

Using (1.7) we infer that the previous identity is equivalent to

$$H_\eta^g \cdot \eta = \alpha_\eta B_\eta^{-1} \cdot \eta. \tag{3.11}$$

Taking $d_\eta\pi$ on both sides of (3.11) we deduce $\eta = B_\eta^{-1} \cdot \eta$. Therefore (3.11) is equivalent to the system

$$\begin{cases} B_\eta \cdot \eta = \eta, \\ \\ H_\eta^g \cdot \eta = \alpha_\eta \cdot \eta. \end{cases} \tag{3.12}$$

Then the system (3.12) rewrites as

$$\begin{cases} \sum_{k \geqslant 1} B_k\left(\eta^{k+1}\right) = 0, \\ \\ \sum_{k \geqslant 1} \Gamma_k\left(\eta^{k+1}\right) = 0. \end{cases}$$

and thus as $S_k\left(\eta^{k+1}\right) = 0$ for all $k \geqslant 1$. We remind now that, according to theorem 2, the integrability of the structure J implies the condition $S_1 \in$

$C^\infty(M, S^2T^*_M \otimes_{\mathbb{R}} \mathbb{C}T_M)$. We infer $S_1 = 0$. We notice that with the notations of the statement of theorem 2, the identity

$$\beta_k\left(\eta^{k+2}\right) = 0\,, \tag{3.13}$$

hold for all $k \geqslant 1$, since every term in the definition of β_k contains two alternating entries. So if we use the definition of S_2 in the statement of theorem 2 we infer $S_2\left(\eta^3\right) = \sigma_2\left(\eta^3\right)$, which implies $\sigma_2 = 0$. We show now by induction that $\sigma_k = 0$ for all $k \geqslant 2$. Indeed by the inductive assumption

$$S_{k+1} = \frac{i}{(k+2)!}\,\mathrm{Sym}_{2,\ldots,k+2}\,\beta_k + \sigma_{k+1}\,.$$

Using the identity (3.13), we infer $S_{k+1}\left(\eta^{k+2}\right) = \sigma_{k+1}\left(\eta^{k+2}\right)$. We deduce $\sigma_{k+1} = 0$. Using the identity

$$\mathrm{Sym}_{2,\ldots,k+1}\,\mathrm{Sym}_{3,\ldots,k+1} = (k-1)!\,\mathrm{Sym}_{2,\ldots,k+1}\,, \tag{3.14}$$

we infer from the statement of theorem 2 and with the notations there

$$S_k \;=\; \frac{i}{(k+1)!k!}\,\mathrm{Sym}_{2,\ldots,k+1}\,\theta_{k-1},$$

for $k \geqslant 2$, with $\theta_1 := 2R^g$ and

$$\begin{aligned}\theta_k \;&:=\; -2i\left(id_1^{\nabla^g}\right)^{k-2}\left(\nabla^g R^g\right)_2\\ &+\; \sum_{r=3}^{k}(r+1)!\sum_{p=2}^{r-1}\left(id_1^{\nabla^g}\right)^{k-r}\left(pS_p \wedge_1 S_{r-p+1}\right),\end{aligned}$$

for all $k \geqslant 2$. Moreover we observe that the equation $\mathrm{Circ}\,\beta_k = 0$, $k \geqslant 3$ rewrites as

$$\mathrm{Circ}\,\mathrm{Sym}_{3,\ldots,k+2}\,\theta_k = 0\,.$$

If we set $\Theta_k(g) := \theta_{k-1}$, for all $k \geqslant 2$ we obtain the required expansion.

On the other hand if the expansion in the statement of the lemma under consideration hold then J is integrable thanks to theorem 2 and $S_k\left(\eta^{k+1}\right) = 0$, for all $k \geqslant 1$, which is equivalent to (3.11) and so to the fact that the curves ψ_η are J-holomorphic. □

We notice in particular that the equation $\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}\,\Theta_4(g) = 0$, writes as

$$\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}\left[3d_1^{\nabla^g}\left(\nabla^g R^g\right)_2 - \,2\tilde{R}^g \wedge_1 \tilde{R}^g\right] = 0\,, \tag{3.15}$$

with $\tilde{R}^g := \mathrm{Sym}_{2,3}\,R^g$. We will show below that the previous equation is an identity.

We remind first the following elementary and well known fact.

Lemma 4 *For any covariant derivative operator ∇ acting on the smooth sections of $\mathbb{C}T_M$ and for any tensor $\theta \in C^\infty\left(X, T^{*,\otimes q}_M \otimes \mathbb{C}T_M\right)$ holds the identity*

$$\mathrm{Alt}_2\,\nabla^2\theta = R^\nabla.\,\theta. \tag{3.16}$$

Proposition 1 *Let ∇ be a torsion free complex covariant derivative operator acting on the smooth sections of the bundle $\mathbb{C}T_M$ with curvature operator $R^{\nabla}(\cdot,\cdot)\cdot \equiv R^{\nabla}(\cdot,\cdot,\cdot)$. Let $\tilde{R}^{\nabla} := \mathrm{Sym}_{2,3} R^{\nabla}$. Then holds the identity*

$$\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}\left[3d_1^{\nabla}\left(\nabla R^{\nabla}\right)_2 - 2\tilde{R}^{\nabla}\wedge_1 \tilde{R}^{\nabla}\right] = 0\,. \tag{3.17}$$

Proof We expand first the term

$$\begin{aligned}
& d_1^{\nabla}\left(\nabla R^{\nabla}\right)_2(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)\\
= \;& \nabla_{\xi_1}\left(\nabla R^{\nabla}\right)_2(\xi_2,\xi_3,\xi_4,\xi_5) - \nabla_{\xi_2}\left(\nabla R^{\nabla}\right)_2(\xi_1,\xi_3,\xi_4,\xi_5)\\
= \;& \nabla^2 R^{\nabla}(\xi_1,\xi_3,\xi_2,\xi_4,\xi_5) - \nabla^2 R^{\nabla}(\xi_2,\xi_3,\xi_1,\xi_4,\xi_5)\\
= \;& \nabla^2 R^{\nabla}(\xi_1,\xi_3,\xi_2,\xi_4,\xi_5) + \nabla^2 R^{\nabla}(\xi_2,\xi_3,\xi_4,\xi_1,\xi_5)\\
= \;& \nabla^2 R^{\nabla}(\xi_3,\xi_1,\xi_2,\xi_4,\xi_5) + \nabla^2 R^{\nabla}(\xi_3,\xi_2,\xi_4,\xi_1,\xi_5)\\
+ \;& (R^{\nabla}.R^{\nabla})(\xi_1,\xi_3,\xi_2,\xi_4,\xi_5) + (R^{\nabla}.R^{\nabla})(\xi_2,\xi_3,\xi_4,\xi_1,\xi_5)\,,
\end{aligned}$$

thanks to formula (3.16). Using the differential Bianchi identity we infer

$$\begin{aligned}
& d_1^{\nabla}\left(\nabla R^{\nabla}\right)_2(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)\\
= \;& -\nabla^2 R^{\nabla}(\xi_3,\xi_4,\xi_1,\xi_2,\xi_5)\\
+ \;& (R^{\nabla}.R^{\nabla})(\xi_1,\xi_3,\xi_2,\xi_4,\xi_5) + (R^{\nabla}.R^{\nabla})(\xi_2,\xi_3,\xi_4,\xi_1,\xi_5)\,.
\end{aligned}$$

In order to simplify the notations in the computations that will follow we will use from now on the identification

$$\theta(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5) \equiv \theta(12345)\,,$$

for any tensor θ. We expand now the term

$$\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}\, d_1^{\nabla}\left(\nabla R^{\nabla}\right)_2\,.$$

We let

$$\theta(12345) := \nabla^2 R^{\nabla}(34125)\,,$$

and we observe the identities

$$\begin{aligned}
\left(\mathrm{Sym}_{3,4,5}\,\theta\right)(12345) &= \nabla^2 R^{\nabla}(34125) + \nabla^2 R^{\nabla}(35124) + \nabla^2 R^{\nabla}(43125) \\
&+ \nabla^2 R^{\nabla}(45123) + \nabla^2 R^{\nabla}(53124) + \nabla^2 R^{\nabla}(54123)\,, \\
\left(\mathrm{Sym}_{3,4,5}\,\theta\right)(23145) &= \nabla^2 R^{\nabla}(14235) + \nabla^2 R^{\nabla}(15234) + \nabla^2 R^{\nabla}(41235) \\
&+ \nabla^2 R^{\nabla}(45231) + \nabla^2 R^{\nabla}(51234) + \nabla^2 R^{\nabla}(54231)\,, \\
\left(\mathrm{Sym}_{3,4,5}\,\theta\right)(31245) &= \nabla^2 R^{\nabla}(24315) + \nabla^2 R^{\nabla}(25314) + \nabla^2 R^{\nabla}(42315) \\
&+ \nabla^2 R^{\nabla}(45312) + \nabla^2 R^{\nabla}(52314) + \nabla^2 R^{\nabla}(54312)\,,
\end{aligned}$$

Summing up we obtain

$$\begin{aligned}
&\left(\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}\,\theta\right)(12345) \\
= \;& \nabla^2 R^{\nabla}(34125) + \nabla^2 R^{\nabla}(14235) + \nabla^2 R^{\nabla}(24315) \\
+ \;& \nabla^2 R^{\nabla}(35124) + \nabla^2 R^{\nabla}(15234) + \nabla^2 R^{\nabla}(25314) \\
+ \;& \nabla^2 R^{\nabla}(43125)_{1} + \nabla^2 R^{\nabla}(41235)_{1} + \nabla^2 R^{\nabla}(42315)_{1} \\
+ \;& \nabla^2 R^{\nabla}(45123)_{2} + \nabla^2 R^{\nabla}(45231)_{2} + \nabla^2 R^{\nabla}(45312)_{2} \\
+ \;& \nabla^2 R^{\nabla}(53124)_{3} + \nabla^2 R^{\nabla}(51234)_{3} + \nabla^2 R^{\nabla}(52314)_{3} \\
+ \;& \nabla^2 R^{\nabla}(54123)_{4} + \nabla^2 R^{\nabla}(54231)_{4} + \nabla^2 R^{\nabla}(54312)_{4}\,,
\end{aligned}$$

where we denote by $\nabla^2 R^{\nabla}(\cdot\cdot\cdot\cdot\cdot)_{j}$ the terms that summed up together equal zero thanks to the differential Bianchi identity for $j = 1, 3$ and thanks to the algebraic Bianchi identity for $j = 2, 4$. Using formula (3.16) we infer

$$\begin{aligned}
&\left(\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}\,\theta\right)(12345) \\
= \;& \nabla^2 R^{\nabla}(43125)_{1} + \nabla^2 R^{\nabla}(41235)_{1} + \nabla^2 R^{\nabla}(42315)_{1} \\
+ \;& \nabla^2 R^{\nabla}(53124)_{2} + \nabla^2 R^{\nabla}(51234)_{2} + \nabla^2 R^{\nabla}(52314)_{2} \\
+ \;& (R^{\nabla}.R^{\nabla})(34125) + (R^{\nabla}.R^{\nabla})(14235) + (R^{\nabla}.R^{\nabla})(24315) \\
+ \;& (R^{\nabla}.R^{\nabla})(35124) + (R^{\nabla}.R^{\nabla})(15234) + (R^{\nabla}.R^{\nabla})(25314)\,,
\end{aligned}$$

where as before we denote by $\nabla^2 R^\nabla\left(\cdot\cdot\cdot\cdot\right)_j$ the terms that summed up together equal zero thanks to the differential Bianchi identity. We deduce the expression

$$\begin{aligned}
&\left(\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}\,\theta\right)(12345)\\
=\;&\left(R^\nabla.R^\nabla\right)(34125)+\left(R^\nabla.R^\nabla\right)(14235)+\left(R^\nabla.R^\nabla\right)(24315)\\
+\;&\left(R^\nabla.R^\nabla\right)(35124)+\left(R^\nabla.R^\nabla\right)(15234)+\left(R^\nabla.R^\nabla\right)(25314)\ . \quad (3.18)
\end{aligned}$$

We set now for notation simplicity $\rho := R^\nabla.R^\nabla$ and let

$$\Theta\,(12345) := \rho\,(13245)+\rho\,(23415)\ .$$

We observe that, by definition, the tensor

$$\rho \in C^\infty\left(M, \Lambda^2 T^*_M \otimes_{\mathbb{R}} \Lambda^2 T^*_M \otimes_{\mathbb{R}} T^*_M \otimes_{\mathbb{R}} \mathbb{C}T_M\right)\ ,$$

satisfies the circular identity with respect to its last three entries. We expand now the term

$$\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}\,\Theta\,.$$

We observe the identities

$$\begin{aligned}
\left(\mathrm{Sym}_{3,4,5}\,\Theta\right)(12345) &= \rho\,(13245)+\rho\,(23415)\\
&+ \rho\,(13254)+\rho\,(23514)\\
&+ \rho\,(14235)+\rho\,(24315)\\
&+ \rho\,(14253)+\rho\,(24513)\\
&+ \rho\,(15234)+\rho\,(25314)\\
&+ \rho\,(15243)+\rho\,(25413)\,,\\
\left(\mathrm{Sym}_{3,4,5}\,\Theta\right)(23145) &= \rho\,(21345)+\rho\,(31425)\\
&+ \rho\,(21354)+\rho\,(31524)\\
&+ \rho\,(24315)+\rho\,(34125)\\
&+ \rho\,(24351)+\rho\,(34521)\\
&+ \rho\,(25314)+\rho\,(35124)\\
&+ \rho\,(25341)+\rho\,(35421)\,,
\end{aligned}$$

$$
\begin{aligned}
\left(\mathrm{Sym}_{3,4,5}\,\Theta\right)(31245) &= \rho\,(32145) + \rho\,(12435) \\
&+ \rho\,(32154) + \rho\,(12534) \\
&+ \rho\,(34125) + \rho\,(14235) \\
&+ \rho\,(34152) + \rho\,(14532) \\
&+ \rho\,(35124) + \rho\,(15234) \\
&+ \rho\,(35142) + \rho\,(15432)\,.
\end{aligned}
$$

Summing up we obtain

$$
\begin{aligned}
&\left(\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}\,\Theta\right)(12345) \\
= \;& \rho\,(13245)_1 + \rho\,(23415)_2 + \rho\,(21345)_3 + \rho\,(31425)_1 \\
+ \;& \rho\,(32145)_2 + \rho\,(12435)_3 \\
+ \;& \rho\,(13254)_4 + \rho\,(23514)_5 + \rho\,(21354)_6 + \rho\,(31524)_4 \\
+ \;& \rho\,(32154)_5 + \rho\,(12534)_6 \\
+ \;& \rho\,(14235)_7 + \rho\,(24315)_8 + \rho\,(24315)_8 + \rho\,(34125)_9 \\
+ \;& \rho\,(34125)_9 + \rho\,(14235)_7 \\
+ \;& \rho\,(14253)_7 + \rho\,(24513)_8 + \rho\,(24351)_8 + \rho\,(34521)_9 \\
+ \;& \rho\,(34152)_9 + \rho\,(14532)_7 \\
+ \;& \rho\,(15234)_{10} + \rho\,(25314)_{11} + \rho\,(25314)_{11} + \rho\,(35124)_{12} \\
+ \;& \rho\,(35124)_{12} + \rho\,(15234)_{10} \\
+ \;& \rho\,(15243)_{10} + \rho\,(25413)_{11} + \rho\,(25341)_{11} + \rho\,(35421)_{12} \\
+ \;& \rho\,(35142)_{12} + \rho\,(15432)_{10}\ ,
\end{aligned}
$$

where we denote by $\rho\,(\cdot\cdot\cdot\cdot\cdot)_j$ the terms that we sum up together using the

symmetries of ρ. We obtain

$$\begin{aligned}
&\left(\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}\,\Theta\right)(12345)\\
=\;& 2\rho\,(13245)+2\rho\,(23415)+2\rho\,(12435)\\
+\;& 2\rho\,(13254)+2\rho\,(23514)+2\rho\,(12534)\\
+\;& 3\rho\,(14235)+3\rho\,(24315)+3\rho\,(34125)\\
+\;& 3\rho\,(15234)+3\rho\,(25314)+3\rho\,(35124)\ .
\end{aligned}$$

We conclude the expression

$$\begin{aligned}
&\left[\mathrm{Circ}\ \mathrm{Sym}_{3,4,5}\,d_1^{\nabla}\left(\nabla R^{\nabla}\right)_2\right](12345)\\
=\;& 2\rho\,(13245)+2\rho\,(23415)+2\rho\,(12435)\\
+\;& 2\rho\,(13254)+2\rho\,(23514)+2\rho\,(12534)\\
+\;& 2\rho\,(14235)+2\rho\,(24315)+2\rho\,(34125)\\
+\;& 2\rho\,(15234)+2\rho\,(25314)+2\rho\,(35124)\ . \qquad (3.19)
\end{aligned}$$

We expand now the term

$$\mathrm{Circ}\ \mathrm{Sym}_{3,4,5}(\tilde{R}^{\nabla}\wedge_1\tilde{R}^{\nabla})\,.$$

From now on we will denote for notation simplicity $(123)\equiv R^{\nabla}(123)$ and

$$[123]:=(123)+(132)\ .$$

We observe the identities

$$\begin{aligned}
\left[\mathrm{Sym}_{3,4,5}(\tilde{R}^{\nabla}\wedge_1\tilde{R}^{\nabla})\right](12345) \;=\;& [1\,[234]\,5]-[2\,[134]\,5]\\
+\;& [1\,[235]\,4]-[2\,[135]\,4]\\
+\;& [1\,[243]\,5]-[2\,[143]\,5]\\
+\;& [1\,[245]\,3]-[2\,[145]\,3]\\
+\;& [1\,[253]\,4]-[2\,[153]\,4]\\
+\;& [1\,[254]\,3]-[2\,[154]\,3]\ ,
\end{aligned}$$

$$
\begin{aligned}
\left[\mathrm{Sym}_{3,4,5}(\tilde{R}^{\nabla} \wedge_1 \tilde{R}^{\nabla})\right](23145) &= [2\,[314]\,5] - [3\,[214]\,5] \\
&+ [2\,[315]\,4] - [3\,[215]\,4] \\
&+ [2\,[341]\,5] - [3\,[241]\,5] \\
&+ [2\,[345]\,1] - [3\,[245]\,1] \\
&+ [2\,[351]\,4] - [3\,[251]\,4] \\
&+ [2\,[354]\,1] - [3\,[254]\,1]\ , \\
\left[\mathrm{Sym}_{3,4,5}(\tilde{R}^{\nabla} \wedge_1 \tilde{R}^{\nabla})\right](31245) &= [3\,[124]\,5] - [1\,[324]\,5] \\
&+ [3\,[125]\,4] - [1\,[325]\,4] \\
&+ [3\,[142]\,5] - [1\,[342]\,5] \\
&+ [3\,[145]\,2] - [1\,[345]\,2] \\
&+ [3\,[152]\,4] - [1\,[352]\,4] \\
&+ [3\,[154]\,2] - [1\,[354]\,2]\ .
\end{aligned}
$$

Summing up using the symmetries of $[\cdots]$ and $(\cdots)$ we obtain

$$
\begin{aligned}
&\left[\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}(\tilde{R}^{\nabla} \wedge_1 \tilde{R}^{\nabla})\right](12345) \\
= \;& 6\,[1\,(234)\,5] + 6\,[2\,(314)\,5] + 6\,[3\,(124)\,5] \\
+ \;& 6\,[1\,(235)\,4] + 6\,[2\,(315)\,4] + 6\,[3\,(125)\,4] \\
+ \;& 2\,[1\,[245]\,3]_1 - 2\,[2\,[145]\,3]_2 \\
+ \;& 2\,[2\,[345]\,1]_3 - 2\,[3\,[245]\,1]_1 \\
+ \;& 2\,[3\,[145]\,2]_2 - 2\,[1\,[345]\,2]_3\ .
\end{aligned}
$$

We combine now the terms $[\cdot[\cdots]\cdot]_j$ for each $j = 1, 2, 3$ and we explicit and

simplify them by using the algebraich Bianchi identity. We obtain

$$\begin{aligned}
&\left[\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}(\tilde{R}^{\nabla} \wedge_1 \tilde{R}^{\nabla})\right](12345) \\
= \;& 6\,[1\,(234)\,5] + 6\,[2\,(314)\,5] + 6\,[3\,(124)\,5] \\
+ \;& 6\,[1\,(235)\,4] + 6\,[2\,(315)\,4] + 6\,[3\,(125)\,4] \\
+ \;& 6\,(13\,[245]) + 6\,(32\,[145]) + 6\,(21\,[345])\ .
\end{aligned}$$

Expanding further we obtain the complete expansion

$$\begin{aligned}
&\left[\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}(\tilde{R}^{\nabla} \wedge_1 \tilde{R}^{\nabla})\right](12345) \\
= \;& 6\,(1\,(234)\,5) + 6\,(15\,(234)) \\
+ \;& 6\,(2\,(314)\,5) + 6\,(25\,(314)) \\
+ \;& 6\,(3\,(124)\,5) + 6\,(35\,(124)) \\
+ \;& 6\,(1\,(235)\,4) + 6\,(14\,(235)) \\
+ \;& 6\,(2\,(315)\,4) + 6\,(24\,(315)) \\
+ \;& 6\,(3\,(125)\,4) + 6\,(34\,(125)) \\
+ \;& 6\,(13\,(245)) + 6\,(13\,(254)) \\
+ \;& 6\,(32\,(145)) + 6\,(32\,(154)) \\
+ \;& 6\,(21\,(345)) + 6\,(21\,(354))\ .
\end{aligned}$$

Expanding the terms ρ present in the expression (3.19) we obtain the complete expansion of the term

$$\left\{\mathrm{Circ}\,\mathrm{Sym}_{3,4,5}\left[3d_1^{\nabla}\left(\nabla R^{\nabla}\right)_2 - \,2\tilde{R}^{\nabla} \wedge_1 \tilde{R}^{\nabla}\right]\right\}(12345)\ ,$$

given by

$$
\begin{aligned}
& \left\{ \mathrm{Circ}\,\mathrm{Sym}_{3,4,5} \left[3d_1^{\nabla} \left(\nabla R^{\nabla} \right)_2 - \, 2\tilde{R}^{\nabla} \wedge_1 \tilde{R}^{\nabla} \right] \right\} (12345) \\
= \;& 6\,(13\,(245))_{1} - 6\,((132)\,45)_{2} - 6\,(2\,(134)\,5)_{3} - 6\,(24\,(135))_{4} \\
+ \;& 6\,(23\,(415))_{5} - 6\,((234)\,15)_{6} - 6\,(4\,(231)\,5)_{2} - 6\,(41\,(235))_{7} \\
+ \;& 6\,(12\,(435))_{8} - 6\,((124)\,35)_{9} - 6\,(4\,(123)\,5)_{2} - 6\,(43\,(125))_{10} \\
+ \;& 6\,(13\,(254))_{11} - 6\,((132)\,54)_{12} - 6\,(2\,(135)\,4)_{13} - 6\,(25\,(134))_{14} \\
+ \;& 6\,(23\,(514))_{15} - 6\,((235)\,14)_{16} - 6\,(5\,(231)\,4)_{12} - 6\,(51\,(234))_{17} \\
+ \;& 6\,(12\,(534))_{18} - 6\,((125)\,34)_{19} - 6\,(5\,(123)\,4)_{12} - 6\,(53\,(124))_{20} \\
+ \;& 6\,(14\,(235))_{7} - 6\,((142)\,35)_{9} - 6\,(2\,(143)\,5)_{3} - 6\,(23\,(145))_{5} \\
+ \;& 6\,(24\,(315))_{4} - 6\,((243)\,15)_{6} - 6\,(3\,(241)\,5)_{9} - 6\,(31\,(245))_{1} \\
+ \;& 6\,(34\,(125))_{10} - 6\,((341)\,25)_{3} - 6\,(1\,(342)\,5)_{6} - 6\,(12\,(345))_{8} \\
+ \;& 6\,(15\,(234))_{17} - 6\,((152)\,34)_{19} - 6\,(2\,(153)\,4)_{13} - 6\,(23\,(154))_{15} \\
+ \;& 6\,(25\,(314))_{14} - 6\,((253)\,14)_{16} - 6\,(3\,(251)\,4)_{19} - 6\,(31\,(254))_{11} \\
+ \;& 6\,(35\,(124))_{20} - 6\,((351)\,24)_{13} - 6\,(1\,(352)\,4)_{16} - 6\,(12\,(354))_{18} \\
- \;& 12\,(1\,(234)\,5)_{6} - 12\,(15\,(234))_{17} \\
- \;& 12\,(2\,(314)\,5)_{3} - 12\,(25\,(314))_{14} \\
- \;& 12\,(3\,(124)\,5)_{9} - 12\,(35\,(124))_{20} \\
- \;& 12\,(1\,(235)\,4)_{16} - 12\,(14\,(235))_{7} \\
- \;& 12\,(2\,(315)\,4)_{13} - 12\,(24\,(315))_{4} \\
- \;& 12\,(3\,(125)\,4)_{19} - 12\,(34\,(125))_{10} - 12\,(13\,(245))_{1} - 12\,(13\,(254))_{11} \\
- \;& 12\,(32\,(145))_{5} - 12\,(32\,(154))_{15} - 12\,(21\,(345))_{8} - 12\,(21\,(354))_{18} \;,
\end{aligned}
$$

where as before we denote by $(\cdots\cdots)_j$ the terms that we sum up together using the symmetries of the curvature tensor R^∇. All the terms summed up together cancel up. This is obvoious for all the sub indexes j with the exeption of $j = 3, 6, 9, 13, 16, 19$ for which me must provide the detail of the computation. Indeed for $j = 3$ we have

$$\begin{aligned}
& -6\left(\left(341\right)25\right) - 6\left(2\left(134\right)5\right) - 6\left(2\left(143\right)5\right) - 12\left(2\left(314\right)5\right) \\
= \;& 6\left(2\left(341\right)5\right) + 6\left(2\left(413\right)5\right) - 6\left(2\left(314\right)5\right) \\
= \;& -6\left(2\left(134\right)5\right) - 6\left(2\left(314\right)5\right) \\
= \;& 0\,.
\end{aligned}$$

For $j = 6$ we have

$$\begin{aligned}
& -12\left(1\left(234\right)5\right) - 6\left(\left(234\right)15\right) - 6\left(\left(243\right)15\right) - 6\left(1\left(342\right)5\right) \\
= \;& 6\left(1\left(243\right)5\right) + 6\left(1\left(432\right)5\right) - 6\left(1\left(234\right)5\right) \\
= \;& -6\left(1\left(324\right)5\right) - 6\left(1\left(234\right)5\right) \\
= \;& 0\,.
\end{aligned}$$

For $j = 9$ we have

$$\begin{aligned}
& -6\left(\left(124\right)35\right) - 6\left(\left(142\right)35\right) - 6\left(3\left(241\right)5\right) - 12\left(3\left(124\right)5\right) \\
= \;& 6\left(3\left(142\right)5\right) + 6\left(3\left(421\right)5\right) - 6\left(3\left(124\right)5\right) \\
= \;& -6\left(3\left(214\right)5\right) - 6\left(3\left(124\right)5\right) \\
= \;& 0\,.
\end{aligned}$$

For $j = 13$ we have

$$\begin{aligned}
& -6\left(\left(351\right)24\right) - 6\left(2\left(135\right)4\right) - 6\left(2\left(153\right)4\right) - 12\left(2\left(315\right)4\right) \\
= \;& 6\left(2\left(351\right)4\right) + 6\left(2\left(513\right)4\right) - 6\left(2\left(315\right)4\right) \\
= \;& -6\left(2\left(135\right)4\right) - 6\left(2\left(315\right)4\right) \\
= \;& 0\,.
\end{aligned}$$

For $j = 16$ we have

$$\begin{aligned} & -12\,(1\,(235)\,4) - 6\,((235)\,14) - 6\,((253)\,14) - 6\,(1\,(352)\,4) \\ = \;& 6\,(1\,(253)\,4) + 6\,(1\,(532)\,4) - 6\,(1\,(235)\,4) \\ = \;& -6\,(1\,(325)\,4) - 6\,(1\,(235)\,4) \\ = \;& 0\,. \end{aligned}$$

For $j = 19$ we have

$$\begin{aligned} & -6\,((125)\,34) - 6\,((152)\,34) - 6\,(3\,(251)\,4) - 12\,(3\,(125)\,4) \\ = \;& 6\,(3\,(152)\,4) + 6\,(3\,(521)\,4) - 6\,(3\,(125)\,4) \\ = \;& -6\,(3\,(215)\,4) - 6\,(3\,(125)\,4) \\ = \;& 0\,. \end{aligned}$$

We infer the required identity (3.17). □

4 General connections over vector bundles

4.1 Basic definitions

Definition 3 *Let (E, π_E, M) be a smooth vector bundle over a manifold M. A connection form over E is a section $\gamma \in C^\infty(E, T_E^* \otimes T_E)$ such that $d\pi_E \cdot \gamma = 0$ and $\gamma_{|\mathrm{Ker}\, d\pi_E} = \mathbb{I}_{\mathrm{Ker}\, d\pi_E}$.*

We will denote by γ_η the connection form γ evaluated at the point $\eta \in E$.

Lemma 5 *For any connection $\gamma \in C^\infty(E, T_E^* \otimes T_E)$ the map*

$$d_\eta \pi_{E|\mathrm{Ker}\,\gamma_\eta} : \mathrm{Ker}\,\gamma_\eta \longrightarrow T_{M,\pi_E(\eta)}\,, \tag{4.1}$$

is an isomorphism for all $\eta \in E$.

Proof The assumption $\gamma_{|\mathrm{Ker}\, d\pi_E} = \mathbb{I}_{\mathrm{Ker}\, d\pi_E}$ implies $\gamma \cdot (\mathbb{I}_{T_E} - \gamma) = 0$. Thus $\mathrm{Im}\,(\mathbb{I}_{T_E} - \gamma) \subseteq \mathrm{Ker}\,\gamma$. Then $\mathrm{Im}\,(\mathbb{I}_{T_E} - \gamma) = \mathrm{Ker}\,\gamma$. Indeed if $\gamma(u) = 0$ then $u = (\mathbb{I}_{T_E} - \gamma)\,u$. On the other hand we notice that the condition $d\pi_E \cdot \gamma = 0$ implies $d\pi_E \cdot (\mathbb{I}_{T_E} - \gamma) = d\pi_E$ and thus

$$d_\eta \pi_{E|\mathrm{Ker}\,\gamma_\eta} \cdot (\mathbb{I}_{T_E} - \gamma) = d\pi_E\,. \tag{4.2}$$

This equality shows that the map (4.1) is surjective. The injectivity follows from the fact that if $u, v \in \mathrm{Ker}\,\gamma_\eta$ and $d_\eta \pi_E\,(u - v) = 0$ then $u - v = \gamma\,(u - v) = 0$ by the assumption $\gamma_{|\mathrm{Ker}\, d\pi_E} = \mathbb{I}_{\mathrm{Ker}\, d\pi_E}$. □

We denote by $H^{\gamma}_{\eta} := (d_{\eta}\pi_{E|\mathrm{Ker}\,\gamma_{\eta}})^{-1}$ the horizontal map. We deduce the existence of a section

$$H^{\gamma} = C^{\infty}\left(E, \pi_E^* T_M^* \otimes T_E\right),$$

such that $d\pi_E \cdot H^{\gamma} = \mathbb{I}_{\pi_E^* T_M}$. (We notice that $d\pi_E \in C^{\infty}\left(E, T_E^* \otimes \pi_E^* T_M\right)$). Composing both sides of (4.2) with H^{γ}_{η} we infer

$$\gamma = \mathbb{I}_{T_E} - H^{\gamma} \cdot d\pi_E\,,$$

and the smooth vector bundle decomposition $T_E = \mathrm{Ker}\, d\pi_E \oplus \mathrm{Ker}\,\gamma$.

The data of a connection form γ is equivalent with the data of a horizontal form H^{γ}. The connection form is called linear if the horizontal form H^{γ} satisfies

$$\begin{aligned} d_{(\eta_1,\eta_2)}\left(sm_{_E}\right)\cdot\left(H^{\gamma}_{\eta_1} \oplus H^{\gamma}_{\eta_2}\right) &= H^{\gamma}_{\eta_1+\eta_2}\,, \\ H^{\gamma}_{\lambda\eta} &= d_{\eta}\left(\lambda \mathbb{I}_E\right)\cdot H^{\gamma}_{\eta}\,, \end{aligned}$$

where $sm_{_E} : E \oplus E \longrightarrow E$ is the sum bundle map where $\eta_1, \eta_2, \eta \in E$ with $\pi_E(\eta_1) = \pi_E(\eta_2)$, and λ is a scalar.

Definition 4 *The **curvature form** $\theta^{\gamma} \in C^{\infty}\left(E, \Lambda^2 T_E^* \otimes T_E\right)$ of a connection form γ is defined as*

$$\theta^{\gamma}(\xi_1, \xi_2) := -\gamma[\left(\mathbb{I}_{T_E} - \gamma\right)\xi_1\,, \left(\mathbb{I}_{T_E} - \gamma\right)\xi_2]\,,$$

for all $\xi_1, \xi_2 \in C^{\infty}(E, T_E)$.

The definition is tensorial. Indeed if $f \in C^{\infty}(E, \mathbb{R})$ then

$$\begin{aligned} [\left(\mathbb{I}_{T_E} - \gamma\right) f\xi_1\,, \left(\mathbb{I}_{T_E} - \gamma\right)\xi_2] &= f[\left(\mathbb{I}_{T_E} - \gamma\right)\xi_1\,, \left(\mathbb{I}_{T_E} - \gamma\right)\xi_2] \\ &- \left[\left(\mathbb{I}_{T_E} - \gamma\right)\xi_2 \,.\, f\right]\left(\mathbb{I}_{T_E} - \gamma\right)\xi_1\,. \end{aligned}$$

The conclusion follows from the fact that $\gamma \cdot \left(\mathbb{I}_{T_E} - \gamma\right) = 0$. We notice that

$$\theta^{\gamma} \in C^{\infty}\left(E, \Lambda^2\left(\mathrm{Ker}\,\gamma\right)^* \otimes \mathrm{Ker}\, d\pi\right),$$

and such element is uniquely determined by the **curvature field** Θ^{γ} defined as

$$\Theta^{\gamma}(\xi_1, \xi_2)(\eta) := T_{\eta}^{-1}\theta^{\gamma}_{\eta}(H^{\gamma}_{\eta}\xi_1, H^{\gamma}_{\eta}\xi_2)\,,$$

for all $\xi_1, \xi_2 \in T_{M,\pi_E(\eta)}$. In the case γ is linear then

$$\Theta^{\gamma} \in C^{\infty}\left(M, \Lambda^2 T_M^* \otimes \mathrm{End}\left(E\right)\right),$$

is called the **curvature operator**. The terminology is consistent with the fact that if we denote by ∇^{γ} the covariant derivative associated to γ then the identity $R^{\nabla^{\gamma}} = \Theta^{\gamma}$ holds, thanks to lemma 19 in the appendix.

4.2 Parallel transport

Given any horizontal form $\alpha \in C^{\infty}\left(E, \pi^{*}T_{M}^{*} \otimes_{\mathbb{R}} T_{E}\right)$ over a vector bundle E, the parallel transport with respect to α is defined as follows. We consider a smooth curve $c : (-\varepsilon, \varepsilon) \longrightarrow M$ and the section $\sigma \in C^{1}\left((-\varepsilon, \varepsilon), c^{*}E\right)$ which satisfies the equation

$$\dot{\sigma} = (\alpha \circ \sigma) \cdot \dot{c},$$

over $(-\varepsilon, \varepsilon)$ with $\sigma(0) = \eta \in E_{c(0)}$. We define the parallel transport map $\tau_{c,t}^{\alpha} : E_{c(0)} \longrightarrow E_{c(t)}$, $t \in (-\varepsilon, \varepsilon)$ along c with respect to α as $\tau_{c,t}^{\alpha}(\eta) = \sigma(t)$.

We consider now a C^{1}-vector field ξ over M and let $\varphi_{\xi,t}$ be the associated 1-parameter sub-group of transformations of M. Let $\Phi_{\xi,t}^{\alpha} : E \longrightarrow E$ be the parallel transport map along the flow lines of $\varphi_{\xi,t}$. In equivalent terms the map $\Phi_{\xi,t}^{\alpha}$ is determined by the ODE

$$\dot{\Phi}_{\xi,t}^{\alpha} = \left(\alpha \circ \Phi_{\xi,t}^{\alpha}\right) \cdot \left(\xi \circ \varphi_{\xi,t} \circ \pi_{E}\right),$$

with initial condition $\Phi_{\xi,0}^{\alpha} \equiv \mathbb{I}_{E}$. We observe that by definition of parallel transport, the map $\Phi_{\xi,t}^{\alpha}$ satisfies $\pi_{E} \circ \Phi_{\xi,t}^{\alpha} = \varphi_{\xi,t} \circ \pi_{E}$. This follows also from the equalities

$$\begin{aligned} \left(d\pi_{E} \circ \Phi_{\xi,t}^{\alpha}\right) \cdot \dot{\Phi}_{\xi,t}^{\alpha} &= \xi \circ \varphi_{\xi,t} \circ \pi_{E} \\ &= \dot{\varphi}_{\xi,t} \circ \pi_{E}. \end{aligned}$$

Moreover the vector field $\Xi^{\alpha} := \alpha \cdot (\xi \circ \pi_{E})$ over E satisfies $\dot{\Phi}_{\xi,t}^{\alpha} = \Xi^{\alpha} \circ \Phi_{\xi,t}^{\alpha}$. Indeed

$$\begin{aligned} \Xi^{\alpha} \circ \Phi_{\xi,t}^{\alpha} &= \left(\alpha \circ \Phi_{\xi,t}^{\alpha}\right) \cdot \left(\xi \circ \pi_{E} \circ \Phi_{\xi,t}^{\alpha}\right) \\ &= \left(\alpha \circ \Phi_{\xi,t}^{\alpha}\right) \cdot \left(\xi \circ \varphi_{\xi,t} \circ \pi_{E}\right). \end{aligned}$$

We deduce that $t \longmapsto \Phi_{\xi,t}^{\alpha}$ is also a 1-parameter sub-group of transformations of E.

4.3 The geometric meaning of the curvature field

The following result provides a clear geometric meaning of the curvature field.

Lemma 6 *Let (E, π_{E}, M) be a smooth vector bundle over a manifold M and consider a horizontal form $\alpha \in C^{\infty}\left(E, \pi^{*}T_{M}^{*} \otimes_{\mathbb{R}} T_{E}\right)$ over bundle E. Then the curvature field Θ^{α} associated to α satisfies*

$$\Theta^{\alpha}(\xi_{1}, \xi_{2})(\eta) = T_{\eta}^{-1} \frac{\partial^{2}}{\partial t \partial s}_{|_{t=s=0}} \left(\Phi_{\xi_{1},-s}^{\alpha} \circ \Phi_{\xi_{2},-t}^{\alpha} \circ \Phi_{\xi_{1},s}^{\alpha} \circ \Phi_{\xi_{2},t}^{\alpha}(\eta)\right).$$

for any $\xi_{1}, \xi_{2} \in C^{\infty}(M, T_{M})$ such that $[\xi_{1}, \xi_{2}] \equiv 0$ and for any $\eta \in E$.

Proof We observe first that if we have a family of transformations $(\Psi_s)_s$ over a manifold with $\Psi_0 = \mathrm{id}$ and a curve c then

$$\begin{aligned}\frac{d}{ds}_{|_{s=0}} \Psi_s(c_s) &= \dot{\Psi}_0(c_0) + d\Psi_0(\dot{c}_0) \\ &= \dot{\Psi}_0(c_0) + \dot{c}_0.\end{aligned}$$

Applying the last equality to $\Psi_s = \varphi_{\xi_2,-s}$ and $c_s := \varphi_{\xi_1,-t} \circ \varphi_{\xi_2,s} \circ \varphi_{\xi_1,t}$, we infer

$$\frac{d}{ds}_{|_{s=0}} (\varphi_{\xi_2,-s} \circ \varphi_{\xi_1,-t} \circ \varphi_{\xi_2,s} \circ \varphi_{\xi_1,t}) = -\xi_2 + \frac{d}{ds}_{|_{s=0}} (\varphi_{\xi_1,-t} \circ \varphi_{\xi_2,s} \circ \varphi_{\xi_1,t}),$$

and thus

$$\begin{aligned}[\xi_1, \xi_2] &= \frac{d}{dt}_{|_{t=0}} \frac{d}{ds}_{|_{s=0}} (\varphi_{\xi_1,-t} \circ \varphi_{\xi_2,s} \circ \varphi_{\xi_1,t}) \\ &= \frac{d}{dt}_{|_{t=0}} \frac{d}{ds}_{|_{s=0}} (\varphi_{\xi_2,-s} \circ \varphi_{\xi_1,-t} \circ \varphi_{\xi_2,s} \circ \varphi_{\xi_1,t}).\end{aligned}$$

In a similar way

$$[\Xi_2^\alpha, \Xi_1^\alpha] = \frac{d}{dt}_{|_{t=0}} \frac{d}{ds}_{|_{s=0}} \left(\Phi^\alpha_{\xi_1,-s} \circ \Phi^\alpha_{\xi_2,-t} \circ \Phi^\alpha_{\xi_1,s} \circ \Phi^\alpha_{\xi_2,t}\right),$$

with $\Xi_j^\alpha := \alpha \cdot (\xi_j \circ \pi_E)$, $j = 1, 2$. Let $\eta \in E_p$ and observe that

$$\Phi^\alpha_{\xi_1,-s} \circ \Phi^\alpha_{\xi_2,-t} \circ \Phi^\alpha_{\xi_1,s} \circ \Phi^\alpha_{\xi_2,t}(\eta) \in E_p,$$

for all parameters t, s, since $\varphi_{\xi_1,-s} \circ \varphi_{\xi_2,-t} \circ \varphi_{\xi_1,s} \circ \varphi_{\xi_2,t}(p) = p$ thanks to the assumption $[\xi_1, \xi_2] \equiv 0$. We conclude the required geometric identity □

4.4 Comparison of the curvature fields of two connections

We consider now two connection forms γ_j, $j = 1, 2$ over E and let $\alpha_j := H^{\gamma_j}$ be the corresponding horizontal forms. The fact that $d\pi_E(\alpha_1 - \alpha_2) = 0$ implies that there exist a section

$$B := T^{-1}(\alpha_1 - \alpha_2) \in C^\infty\left(E, \pi_E^*(T_M^* \otimes E)\right),$$

which satisfies

$$\gamma_1 = \gamma_2 - TB \cdot d\pi_E.$$

We want to compare the curvature fields $\Theta_j := \Theta^{\gamma_j}$. We will denote by abuse of notations $\alpha_j \xi \equiv \alpha_j \cdot (\xi \circ \pi_E)$ and $B\xi \equiv B \cdot (\xi \circ \pi_E)$ for any $\xi \in C^\infty(M, T_M)$.

Lemma 7 *In the above set up, the identity*

$$\begin{aligned}\Theta_1(\xi_1, \xi_2) &= (\Theta_2 - B\neg DB)(\xi_1, \xi_2) \\ &- T^{-1}\left(\left[\alpha_2\xi_1, TB\,\xi_2\right] - [\alpha_2\xi_2, TB\xi_1]\right) + B[\xi_1, \xi_2], \quad (4.3)\end{aligned}$$

holds for any $\xi_1, \xi_2 \in C^\infty(M, T_M)$.

Proof We notice first the equalities

$$\begin{aligned} T\Theta_1(\xi_1,\xi_2) &= \theta^{\gamma_1}(\alpha_1\xi_1,\alpha_1\xi_2) \\ &= -\gamma_1\left[\alpha_1\xi_1,\alpha_1\,\xi_2\right] \\ &= -\gamma_2\left[\alpha_1\xi_1,\alpha_1\,\xi_2\right]+TB\cdot d\pi_E\left[\alpha_1\xi_1,\alpha_1\,\xi_2\right] \\ &= T\Theta_2(\xi_1,\xi_2) \\ &- \gamma_2\left(\left[\alpha_2\xi_1,TB\,\xi_2\right]+[TB\xi_1,\alpha_2\xi_2]+[TB\xi_1,TB\xi_2]\right) \\ &+ TB[\xi_1,\xi_2]. \end{aligned}$$

In the last line we use the well known identity $d\pi_E[\alpha_1\xi_1,\alpha_1\xi_2]=[\xi_1,\xi_2]\circ\pi_E$, which follows from the fact that $d\pi_E\alpha_1\xi_j=\xi_j\circ\pi_E$, $j=1,2$. Let now $\Phi_{TB\xi_2,t}$ be the 1-parameter sub-group of transformations of E associated to the vertical vector field $TB\xi_2$. It satisfies $\pi_E\circ\Phi_{TB\xi_2,t}=\pi_E$. Using the standard expression of the Lie bracket

$$[\alpha_2\xi_1,TB\xi_2] = \frac{d}{dt}_{|t=0}\frac{d}{ds}_{|s=0}\left(\Phi^{\alpha_2}_{\xi_1,-t}\circ\Phi_{TB\xi_2,s}\circ\Phi^{\alpha_2}_{\xi_1,t}\right),$$

we deduce that this vector field is vertical. In the same way $[TB\xi_1,\alpha_2\xi_2]$ is vertical. It is obvious that the vector field $\left[TB\xi_1,TB\,\xi_2\right]$ is also vertical. We infer the identity

$$\begin{aligned} T\Theta_1(\xi_1,\xi_2) &= T\Theta_2(\xi_1,\xi_2)-\left[TB\xi_1,TB\,\xi_2\right] \\ &- \left[\alpha_2\xi_1,TB\,\xi_2\right]-[TB\xi_1,\alpha_2\xi_2]+TB[\xi_1,\xi_2]. \end{aligned}$$

The required formula (4.3) follows from the identity

$$[TB\xi_1,TB\xi_2] = T(B\neg DB)(\xi_1,\xi_2), \tag{4.4}$$

that we show now. We remind first that for any vector space V, the canonical translation operator $T:C^\infty(V,V)\longrightarrow C^\infty(V,T_V)$ defined as $(T\xi)(v):=T_v\xi_v$ is a Lie algebra isomorphism, where the Lie algebra structure over $C^\infty(V,V)$ is defined by $[\xi,\eta]_v:=D_v\eta\cdot\xi_v-D_v\xi\cdot\eta_v$. Indeed if we define the action of $C^\infty(V,V)$ over $C^\infty(V,\mathbb{R})$ as

$$\begin{aligned} (\xi.f)(v) &:= D_vf\cdot\xi_v \\ &= \frac{d}{dt}_{|t=0}f(v+t\xi_v) \\ &= [(T\xi).f](v), \end{aligned}$$

then

$$\begin{aligned}(\xi.\eta.f)(v) &= \frac{d}{dt}_{|t=0}(\eta.f)(v+t\xi_v)\\ &= \frac{d}{dt}_{|t=0}\left(D_{v+t\xi_v}f\cdot\eta_{v+t\xi_v}\right)\\ &= D_v^2 f(\xi_v,\eta_v)+D_v f\cdot D_v\eta\cdot\xi_v.\end{aligned}$$

The fact that the bilinear form $D_v^2 f$ is symmetric implies

$$\xi.\eta.f-\eta.\xi.f = [\xi,\eta].f.$$

On the other hand by definition

$$\begin{aligned}T\xi.T\eta.f-T\eta.T\xi.f &= \xi.\eta.f-\eta.\xi.f,\\ [\xi,\eta].f &= T[\xi,\eta].f.\end{aligned}$$

We conclude the required identity $[T\xi,T\eta]=T[\xi,\eta]$. We apply this remark to our set-up. For any point $p\in M$, we denote by $B\xi(p)\in C^\infty(E_p,E_p)$ the map $\eta\in E_p\longmapsto B_\eta\xi(p)\in E_p$ and we denote by $TB\xi(p)\in C^\infty\left(E_p,T_{E_p}\right)$ the section $\eta\in E_p\longmapsto T_\eta B_\eta\xi(p)\in T_{E_p,\eta}$. Then for any $\eta\in E_p$

$$\begin{aligned}[TB\xi_1,TB\xi_2]_\eta &= \left[TB\xi_1(p),TB\xi_2(p)\right]_\eta\\ &= T_\eta\left[B\xi_1(p),B\xi_2(p)\right]_\eta\\ &= T_\eta\left[D_\eta B(B_\eta\xi_1(p))\xi_2(p)-D_\eta B(B_\eta\xi_2(p))\xi_1(p)\right],\end{aligned}$$

which shows (4.4). □

We notice now that for any covariant derivative ∇ over E, the identity (8.8) rewrites as

$$\left[H^\nabla\xi,T\pi_E^*s\right]=T\pi_E^*(\nabla_\xi s), \tag{4.5}$$

for any vector field $\xi\in C^\infty(M,T_M)$ and any section $s\in C^\infty(M,E)$. We need to show the following more general formula.

Lemma 8 *Let (E,π_E,M) be a smooth vector bundle over a manifold M and let ∇ be a covariant derivative operator acting on the smooth sections of E. Then the equality hold*

$$\left[H^\nabla\xi,T\sigma\right]=T\nabla^{\pi_E}_{H^\nabla\xi}\sigma, \tag{4.6}$$

for any vector field $\xi\in C^\infty(M,T_M)$ and for any section $\sigma\in C^\infty(E,\pi_E^*E)$.

We observe that (4.6) implies (4.5), since $\nabla^{\pi_E}_{H^\nabla\xi}\sigma=\pi_E^*(\nabla_\xi s)$, thanks to the functorial property (8.6).

Proof In order to show the identity (4.6) we notice first that the assumption $\sigma\in C^\infty(E,\pi_E^*E)$ means that σ is a map $\sigma:E\longrightarrow E$ such that $\pi_E\circ\sigma=\pi_E$.

Then the 1-parameter subgroup of transformations of E associated to the vector field $T\sigma$ satisfies $\Phi_{T\sigma,t}(\eta) = \eta + t\sigma(\eta)$. Moreover with the notations in the proof of identity (8.8)

$$\left[H^{\nabla}\xi, T\sigma\right] = \frac{d}{dt}_{|t=0}\frac{d}{ds}_{|s=0}\left(\Phi_{\xi,-t}\circ\Phi_{T\sigma,s}\circ\Phi_{\xi,t}\right).$$

The fact that $\Phi_{\xi,-t}$ is linear on the fibers of E implies

$$\begin{aligned}\Phi_{\xi,-t}\circ\Phi_{\Sigma,s}\circ\Phi_{\xi,t} &= \Phi_{\xi,-t}\left[\Phi_{\xi,t} + s\sigma\circ\Phi_{\xi,t}\right]\\ &= \mathbb{I}_E + s\Phi_{\xi,-t}\cdot\sigma\circ\Phi_{\xi,t}.\end{aligned}$$

We infer

$$\frac{d}{ds}_{|s=0}\left(\Phi_{\xi,-t}\circ\Phi_{T\sigma,s}\circ\Phi_{\xi,t}\right)(\eta) = T_{\eta}\Phi_{\xi,-t}\cdot\sigma\circ\Phi_{\xi,t}(\eta),$$

for any $\eta\in E_p$. We observe that $\sigma\circ\Phi_{\xi,t}(\eta)\in E_{\varphi_{\xi,t}(p)}$. Indeed using the property $\pi_E\circ\sigma = \pi_E$ we deduce

$$\begin{aligned}\pi_E\circ\sigma\circ\Phi_{\xi,t}(\eta) &= \pi_E\circ\Phi_{\xi,t}(\eta)\\ &= \varphi_{\xi,t}(p).\end{aligned}$$

We remind now that if $t\longmapsto\eta_t\in E$ is a smooth curve such that $c_t := \pi_E(\eta_t)$ then

$$T^{-1}_{\eta_0}\gamma^{\nabla}_{\eta_0}\dot{\eta}_0 = \frac{d}{dt}_{|t=0}\left(\tau^{-1}_{c,t}\eta_t\right),$$

thanks to formula (8.7). We apply the previous identity to the curve $\eta_t := \sigma\circ\Phi_{\xi,t}(\eta)\in E_{\varphi_{\xi,t}(p)}$. We obtain

$$\begin{aligned}T^{-1}_{\sigma(\eta)}\gamma^{\nabla}_{\sigma(\eta)}\frac{d}{dt}_{|t=0}\left[\sigma\circ\Phi_{\xi,t}(\eta)\right] &= \frac{d}{dt}_{|t=0}\left[\Phi_{\xi,-t}\cdot\sigma\circ\Phi_{\xi,t}(\eta)\right]\\ &= T^{-1}_{\eta}\left[H^{\nabla}\xi, T\sigma\right](\eta).\end{aligned}$$

Moreover

$$\begin{aligned}\frac{d}{dt}_{|t=0}\left[\sigma\circ\Phi_{\xi,t}(\eta)\right] &= d_{\eta}\sigma\cdot\dot{\Phi}_{\xi,0}(\eta)\\ &= d_{\eta}\sigma\cdot H^{\nabla}_{\eta}\xi(p).\end{aligned}$$

We conclude the equality

$$T^{-1}_{\sigma(\eta)}\gamma_{\sigma(\eta)}d_{\eta}\sigma\cdot H^{\nabla}_{\eta}\xi(p) = T^{-1}_{\eta}\left[H^{\nabla}\xi, T\sigma\right](\eta),$$

which represents the required formula (4.6). □

We can show now the following result.

Lemma 9 *Let (E, π_E, M) be a smooth vector bundle over a manifold M and let ∇ and ∇^{T_M} be covariant derivative operators acting respectively on the smooth sections of the bundles E and T_M.*

Then for any section $B \in C^\infty(E, \pi_E^*(T_M^* \otimes E))$ the curvature field Θ^α of the horizontal form $\alpha := H^\nabla + TB$ satisfies

$$\Theta^\alpha = -H^\nabla \neg \nabla^{T_M^* \otimes E, \pi_E} B - B \neg DB - B\tau^{\nabla^{T_M}} + R^\nabla, \tag{4.7}$$

where $\nabla^{T_M^* \otimes E, \pi_E}$ is the covariant derivative acting on the smooth sections of the bundle $\pi_E^*(T_M^* \otimes E)$, induced by ∇ and ∇^{T_M} and where $\tau^{\nabla^{T_M}}$ is the torsion form of ∇^{T_M}.

Proof In the case $\alpha_2 = H^\nabla$ in the identity (4.3) we can apply the formula (4.6) to the sections $B\xi_j \in C^\infty(E, \pi_E^* E)$. We obtain

$$\begin{aligned} \Theta_1(\xi_1, \xi_2) &= \left(R^\nabla - B \neg DB\right)(\xi_1, \xi_2) \\ &- \nabla^{\pi_E}_{H^\nabla \xi_1}(B\xi_2) + \nabla^{\pi_E}_{H^\nabla \xi_2}(B\xi_1) + B[\xi_1, \xi_2]. \end{aligned}$$

Using functorial properties of the pull-back we have (with no abuse of the notations)

$$\begin{aligned} \nabla^{\pi_E}_{H^\nabla \xi_1}(B \cdot \pi_E^* \xi_2) &= \nabla^{T_M^* \otimes E, \pi_E}_{H^\nabla \xi_1} B \cdot \pi_E^* \xi_2 + B \cdot \nabla^{T_M, \pi_E}_{H^\nabla \xi_1}(\pi_E^* \xi_2) \\ &= \nabla^{T_M^* \otimes E, \pi_E}_{H^\nabla \xi_1} B \cdot \pi_E^* \xi_2 + B \cdot \pi_E^*(\nabla^{T_M}_{\xi_1} \xi_2). \end{aligned}$$

We conclude by (4.3) that if $\alpha_1 = \alpha = H^\nabla + TB$ then the curvature field Θ^α of α satisfies the identity

$$\begin{aligned} \Theta^\alpha(\xi_1, \xi_2) &= \left(R^\nabla - B \neg DB\right)(\xi_1, \xi_2) \\ &- \nabla^{T_M^* \otimes E, \pi_E}_{H^\nabla \xi_1} B\xi_2 + \nabla^{T_M^* \otimes E, \pi_E}_{H^\nabla \xi_2} B\xi_1 - B\tau^{\nabla^{T_M}}(\xi_1, \xi_2), \end{aligned}$$

We infer the required formula (4.7). □

5 First reduction of the integrability equations

Proof of theorem 1.

Proof Let γ^A be the connection form associated to the horizontal form A. Then the integrability of J_A is equivalent to the condition

$$\gamma^A[A\xi_1, A\xi_2] = 0, \tag{5.1}$$

for all smooth complex vector fields ξ_1, ξ_2 over M. (We remind here the use of the abusive notation $A\xi \equiv A(\xi \circ \pi)$). We denote respectively by Θ^A and Θ^α the curvature fields of the horizontal distributions A and α. The integrability condition (5.1) is equivalent to the condition $\Theta^A \equiv 0$. Then applying the identity

(4.3) with $\alpha_1 = A$, $\alpha_2 = \alpha$ and separating real and imaginary parts we deduce that the integrability of J_A is equivalent to the system

$$\left\{\begin{array}{l} \Theta^{\alpha} + B\neg DB \;=\; 0\,, \\ \\ TB[\xi_1,\xi_2] \;=\; [\alpha\xi_1, TB\xi_2] - [\alpha\xi_2, TB\xi_1]\,. \end{array}\right. \tag{5.2}$$

Using the formula (4.7) in the case $E = T_M$ and $\nabla = \nabla^{T_M}$ we can write the previous equation of the system (5.2) as

$$H^{\nabla}\neg\nabla^{\mathrm{End}(T_M),\pi}\Gamma - \Gamma\neg D\Gamma + \Gamma\tau^{\nabla} + B\neg DB + R^{\nabla} = 0\,.$$

We rewrite the second equation of the system (5.2) as

$$\begin{array}{rcl} TB[\xi_1,\xi_2] & = & \Big[H^{\nabla}\xi_1, TB\xi_2\Big] - [T\Gamma\xi_1, TB\xi_2] \\ \\ & - & \Big[H^{\nabla}\xi_2, TB\xi_1\Big] + [T\Gamma\xi_2, TB\xi_1]\,. \end{array}$$

Using formula (4.6) we infer

$$\begin{array}{rcl} B[\xi_1,\xi_2] & = & \nabla^{\mathrm{End}(T_M),\pi}_{H^{\nabla}\xi_1} B\xi_2 - \nabla^{\mathrm{End}(T_M),\pi}_{H^{\nabla}\xi_2} B\xi_1 + B\left(\nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1\right) \\ \\ & - & DB\left(\Gamma\xi_1\right)\xi_2 + D\Gamma\left(B\xi_2\right)\xi_1 \\ \\ & + & DB\left(\Gamma\xi_2\right)\xi_1 - D\Gamma\left(B\xi_1\right)\xi_2\,, \end{array}$$

which rewrites as

$$H^{\nabla}\neg\nabla^{\mathrm{End}(T_M),\pi}B - \Gamma\neg DB - B\neg D\Gamma + B\tau^{\nabla} = 0\,.$$

We conclude that the system (5.2) is equivalent to the system

$$\left\{\begin{array}{l} H^{\nabla}\neg\nabla^{\mathrm{End}(T_M),\pi}\Gamma - \Gamma\neg D\Gamma + \Gamma\tau^{\nabla} + B\neg DB + R^{\nabla} \;=\; 0\,, \\ \\ H^{\nabla}\neg\nabla^{\mathrm{End}(T_M),\pi}B - \Gamma\neg DB - B\neg D\Gamma + B\tau^{\nabla} \;=\; 0\,. \end{array}\right. \tag{5.3}$$

It follows that, using the identification $S = \Gamma + iB$, the system (5.3) is equivalent to the complex equation (1.8). □

Remark 1 We notice that in the case $(\alpha, B) = \left(H^{\nabla}, \mathbb{I}_{\pi^* T_M}\right)$, i.e. in the case $J_A = J_{H^{\nabla}}$, the system (5.3) reduces to

$$\left\{\begin{array}{l} R^{\nabla} \;=\; 0\,, \\ \\ \tau^{\nabla} \;=\; 0\,. \end{array}\right.$$

In this way we re-obtain the statement of lemma 1.

Lemma 10 *Under the assumptions of the theorem 2 the M-totally real almost complex structure J_A is integrable over U if and only if*

$$S_1 \in C^\infty \left(M, S^2 T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M \right),$$

(i.e. ∇^{S_1} is torsion free),

$$R^{\nabla^{S_1}} = -2i \operatorname{Alt}_2 S_2 , \tag{5.4}$$

$$\left[d_1^{\nabla^{S_1}} S_k + \sum_{p=2}^{k-1} p S_p \wedge_1 S_{k-p+1} + i\,(k+1) \operatorname{Alt}_2 S_{k+1} \right] \left(\xi_1, \xi_2, \eta^k \right) = 0 . \tag{5.5}$$

for all $k \geqslant 2$ and for all $\xi_1, \xi_2, \eta \in T_{M,\pi(\eta)}$.

Proof In the case the connection ∇ is torsion free the equation (1.8) reduces to

$$H^\nabla \neg \nabla^{\mathrm{End}(T_M),\pi} S - S \neg DS + R^\nabla = 0 . \tag{5.6}$$

The identification $\mathcal{S}_{k,\eta} \cdot \xi \equiv S_k \left(\xi, \eta^k \right)$ shows that $\mathcal{S}_{k,\eta} \in T^*_{M,\pi(\eta)} \otimes \mathbb{C}T_{M,\pi(\eta)}$, i.e.

$$\mathcal{S}_k \in C^\infty \left(T_M, \pi^* \left(T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M \right) \right),$$

and

$$S = \sum_{k \geqslant 0} \mathcal{S}_k . \tag{5.7}$$

We remind the formula

$$\nabla^\pi_{H^\nabla \xi_1} \left(\mathcal{S}_k \cdot \xi_2 \right) = \nabla^{\mathrm{End}(T_M),\pi}_{H^\nabla \xi_1} \mathcal{S}_k \cdot \xi_2 + \mathcal{S}_k \cdot \nabla_{\xi_1} \xi_2 ,$$

for any vector field ξ_1, ξ_2 over M. On the other hand, by definition

$$\begin{aligned}
& \nabla^\pi_{H^\nabla \xi_1} \left(\mathcal{S}_k \cdot \xi_2 \right)_{|\eta} \\
= \; & T^{-1}_{S_k(\xi_2,\eta^k)} \gamma^\nabla_{S_k(\xi_2,\eta^k)} d_\eta \left(\mathcal{S}_k \cdot \xi_2 \right) \left(H^\nabla \xi_1 \right) \\
= \; & T^{-1}_{S_k(\xi_2,\eta^k)} \gamma^\nabla_{S_k(\xi_2,\eta^k)} \frac{d}{dt}_{|t=0} \left[S_k \left(\xi_2 \circ \varphi_{\xi_1,t} \circ \pi(\eta) , \Phi_{\xi_1,t}(\eta)^k \right) \right] .
\end{aligned}$$

Let now η be the vector field over $\mathrm{Im}\left(\varphi_{\xi_1,\bullet} \circ \pi(\eta) \right)$ defined by

$$\eta \left(\varphi_{\xi_1,\mathbf{t}} \circ \pi(\eta) \right) = \mathbf{\Phi}_{\xi_1,\mathbf{t}}(\eta) .$$

Then

$$\begin{aligned}
\nabla^\pi_{H^\nabla \xi_1} \left(\mathcal{S}_k \cdot \xi_2 \right)_{|\eta} &= \nabla_{\xi_1} \left[S_k \left(\xi_2, \eta^k \right) \right]_{|\pi(\eta)} \\
&= \nabla_{\xi_1} S_k \left(\xi_2 \circ \pi(\eta) , \eta^k \right) + S_k \left(\nabla_{\xi_1} \xi_{2|\pi(\eta)}, \eta^k \right) ,
\end{aligned}$$

since $\nabla_{\xi_1} \eta = 0$. We conclude the identity

$$\left(\nabla^{\mathrm{End}(T_M),\pi}_{H^\nabla \xi_1} \mathcal{S}_k \right)_{|\eta} \cdot \xi_2 = \nabla_{\xi_1} S_k \left(\xi_2, \eta^k \right) ,$$

$\xi_1, \xi_2 \in T_{M,\pi(\eta)}$. We infer the formula

$$H^{\nabla} \neg \nabla^{\mathrm{End}(T_M),\pi} \mathcal{S}_k = d_1^{\nabla} S_k \,, \tag{5.8}$$

We notice now the equalities

$$\begin{aligned} D_\eta \mathcal{S}_k \left(v\right) \cdot \xi &= \frac{d}{dt}_{|t=0} \left[S_k \left(\xi \,, \left(\eta + tv\right)^k \right) \right] \\ &= \sum_{j=1}^{k} S_k \left(\xi, \eta^{j-1}, v, \eta^{k-j} \right) \\ &= k S_k \left(\xi, v, \eta^{k-1} \right) , \end{aligned}$$

and

$$\left(\mathcal{S}_l \neg D\mathcal{S}_k\right)_{|\eta} \left(\xi_1, \xi_2\right) = k S_k \left(\xi_2, \mathcal{S}_{l,\eta} \cdot \xi_1, \eta^{k-1} \right) - k S_k \left(\xi_1, \mathcal{S}_{l,\eta} \cdot \xi_2, \eta^{k-1} \right) .$$

We infer the equality

$$\left(\mathcal{S}_l \neg D\mathcal{S}_k\right)_{|\eta} \left(\xi_1, \xi_2\right) = -k \left(S_k \wedge_1 S_l \right) \left(\xi_1, \xi_2, \eta^{k+l-1} \right) . \tag{5.9}$$

Let $W \subset U$ be any set containing the zero section of T_M such that $W \cap T_{M,p}$ is a neighborhood of 0_p for any $p \in M$ and such that the fiberwise expansion (5.7) converges over $W \cap T_{M,p}$. The fact that by assumption $U \cap T_{M,p}$ is connected implies by the fiberwise real analyticity of S that S is a solution of (5.6) over U if and only if it satisfies (5.6) over W.

Using (5.8) we can write the equation (5.6) under the form

$$\sum_{k \geqslant 1} d_1^{\nabla} S_k - \sum_{l,p \geqslant 0} \left(\mathcal{S}_l \neg D\mathcal{S}_p \right) + R^{\nabla} = 0 \,, \tag{5.10}$$

over W. We decompose the sum

$$\begin{aligned} & \sum_{l,p \geqslant 0} \left(\mathcal{S}_l \neg D\mathcal{S}_p \right) \\ = & \sum_{l \geqslant 0, p \geqslant 1} \left(\mathcal{S}_l \neg D\mathcal{S}_p \right) \\ = & \sum_{l,p \geqslant 1} \left(\mathcal{S}_l \neg D\mathcal{S}_p \right) + i \sum_{k \geqslant 0} \left(\mathbb{I}_{T_M} \neg D\mathcal{S}_{k+1} \right) \\ = & \sum_{k \geqslant 1} \sum_{p=1}^{k} \left(\mathcal{S}_{k-p+1} \neg D\mathcal{S}_p \right) + i \sum_{k \geqslant 0} \left(\mathbb{I}_{T_M} \neg D\mathcal{S}_{k+1} \right) \\ = & -\sum_{k \geqslant 1} \sum_{p=1}^{k} p \left(S_p \wedge_1 S_{k-p+1} \right) - i \sum_{k \geqslant 0} \left(k+1 \right) \left(S_{k+1} \wedge_1 \mathbb{I}_{T_M} \right) , \end{aligned}$$

thanks to the equality (5.9). If we denote by $\deg_\eta$the degree with respect to the fibre variable $\eta \in E_{\pi(\eta)}$ we have

$$\begin{aligned} \deg_\eta d_1^\nabla S_k &= \deg_\eta (S_p \wedge_1 S_{k-p+1}) = k, \\ \deg_\eta (S_{k+1} \wedge_1 \mathbb{I}_{T_M}) &= k, \\ \deg_\eta R^\nabla &= 1. \end{aligned}$$

Thus by homogeneity the equation (5.10) is equivalent to the countable system

$$\begin{cases} S_1 \wedge_1 \mathbb{I}_{T_M} = 0\,, \\ d_1^\nabla S_1 + S_1 \wedge_1 S_1 + 2iS_2 \wedge_1 \mathbb{I}_{T_M} + R^\nabla = 0\,, \\ \left[d_1^\nabla S_k + \sum_{p=1}^k p\,(S_p \wedge_1 S_{k-p+1}) + i\,(k+1)\,S_{k+1} \wedge_1 \mathbb{I}_{T_M}\right](\xi_1, \xi_2, \eta^k) = 0\,, \\ \forall k \geqslant 2\,,\ \forall \xi_1, \xi_2, \eta \in T_M\,. \end{cases} \tag{5.11}$$

The first equation in the system means $S_1 \in C^\infty\left(M, S^2 T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M\right)$, i.e. the complex connection ∇^{S_1} is torsion free. The second equation in the system (5.11) rewrites as (5.4). We show now that the equation for $k \geqslant 2$ in the system (5.11) rewrites as (5.5). Indeed using the formula

$$\begin{aligned} \nabla_\xi^\Gamma \theta\,(v_1, \ldots, v_p) &= \nabla_\xi \theta\,(v_1, \ldots, v_p) + \Gamma\,(\xi, \theta\,(v_1, \ldots, v_p)) \\ &- \sum_{j=1}^p \theta\,(v_1, \ldots, v_{j-1}, \Gamma\,(\xi, v_j)\,, v_{j+1}, \ldots, v_p)\,, \end{aligned}$$

where $\Gamma \in C^\infty(M, T_M^{*,\otimes 2} \otimes_{\mathbb{R}} \mathbb{C}T_M)$, $\theta \in C^\infty\left(M, T_M^{*,\otimes p} \otimes_{\mathbb{R}} \mathbb{C}T_M\right)$ and $\xi, v_k \in T_M$, we infer

$$\begin{aligned} & d_1^{\nabla^{S_1}} S_k\left(\xi_1, \xi_2, \eta^k\right) \\ = \ & \nabla_{\xi_1} S_k\left(\xi_2, \eta^k\right) - \nabla_{\xi_2} S_k\left(\xi_1, \eta^k\right) \\ + \ & S_1\left(\xi_1, S_k\left(\xi_2, \eta^k\right)\right) - S_k\left(S_1\left(\xi_1, \xi_2\right), \eta^k\right) - k S_k\left(\xi_2, S_1\left(\xi_1, \eta\right), \eta^{k-1}\right) \\ - \ & S_1\left(\xi_2, S_k\left(\xi_1, \eta^k\right)\right) + S_k\left(S_1\left(\xi_2, \xi_1\right), \eta^k\right) + k S_k\left(\xi_1, S_1\left(\xi_2, \eta\right), \eta^{k-1}\right) \\ = \ & \left[d_1^\nabla S_k + S_1 \wedge_1 S_k + k\, S_k \wedge_1 S_1\right]\left(\xi_1, \xi_2, \eta^k\right), \end{aligned}$$

since S_1 is symmetric and S_k is symmetric in the last k variables. We conclude (5.5). □

Remark 2 In the case $S_k = 0$, for all $k \geqslant 2$, the previous system reduces to the equation

$$d_1^\nabla S_1 + S_1 \wedge_1 S_1 + R^\nabla = 0\,. \tag{5.12}$$

The equation (5.12) means that the complex connection ∇^{S_1} acting on sections of $\mathbb{C}T_M$ is flat. In the case $B = \mathbb{I}_{\pi^* T_M}$, the second equation in the system (5.11) implies

$$d_1^{\nabla}\Gamma_1 + \Gamma_1 \wedge_1 \Gamma_1 + R^{\nabla} = 0\,,$$

with $\Gamma_1 := S_1$. This means that the real connection ∇^{Γ_1} is flat.

6 Second reduction of the integrability equations

In this section we will prove the following result.

Proposition 2 *Under the assumptions of the theorem 2 the M-totally real almost complex structure J_A is integrable over U if and only if*

$$\begin{aligned}
S_1 &\in C^{\infty}\left(M, S^2 T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M\right), \text{ i.e. } \nabla^{S_1} \text{ is torsion free,}\\
S_2 &= S_2^0 + \sigma_2\,,\\
S_2^0(\xi_1,\xi_2,\xi_3) &:= \frac{i}{6}\left[R^{\nabla^{S_1}}(\xi_1,\xi_2)\,\xi_3 + R^{\nabla^{S_1}}(\xi_1,\xi_3)\,\xi_2\right],\\
\sigma_2 &\in C^{\infty}\left(M, S^3 T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M\right),\\
S_3 &= \frac{i}{3}\nabla^{S_1}\sigma_2 + \frac{1}{4!3}\,\mathrm{Sym}_{2,3,4}(\nabla^{S_1} R^{\nabla^{S_1}})_2 + \sigma_3\,,\\
\sigma_3 &\in C^{\infty}\left(M, S^4 T_M^* \otimes_{\mathbb{R}} \mathbb{C}T_M\right),
\end{aligned}$$

$(\nabla^{S_1} R^{\nabla^{S_1}})_2(\xi_1,\xi_2,\xi_3,\xi_4) := \nabla^{S_1}_{\xi_2} R^{\nabla^{S_1}}(\xi_1,\xi_3)\,\xi_4$, *for all* $\xi_1,\xi_2,\xi_3,\xi_4 \in T_{M,\pi(\xi_1)}$ *and for all* $k \geqslant 3$,

$$\left[d_1^{\nabla^{S_1}} S_k + \sum_{p=2}^{k-1} p S_p \wedge_1 S_{k-p+1} + i\,(k+1)\,\mathrm{Alt}_2\, S_{k+1}\right](\xi_1,\xi_2,\eta^k) = 0\,,$$

for all $\xi_1,\xi_2,\eta \in T_{M,\pi(\eta)}$.

We remind first that for any complex connection ∇ acting over the sections of $\mathbb{C}T_M$ its torsion τ^{∇} satisfies the identity

$$\tau^{\nabla} = d^{\nabla}\mathbb{I}_{T_M}\,,$$

where d^{∇} is the covariant exterior differentiation and $\mathbb{I}_{T_M} \in C^{\infty}(M, T_M^* \otimes T_M)$. Then

$$d^{\nabla}\tau^{\nabla} = R^{\nabla} \wedge \mathbb{I}_{T_M}\,,$$

and

$$(R^{\nabla} \wedge \mathbb{I}_{T_M})(\xi_1,\xi_2,\xi_3) = R^{\nabla}(\xi_1,\xi_2)\,\xi_3 + R^{\nabla}(\xi_2,\xi_3)\,\xi_1 + R^{\nabla}(\xi_3,\xi_1)\,\xi_2\,.$$

We conclude that if a connection is torsion free then then its curvature operator satisfies the algebraic Bianchi identity.

We denote by Alt_p the alternating operator (without normalizing coefficients!) acting on the first $p \geqslant 2$ entries of a tensor, counted from the left to the right. We notice the following very elementary fact.

Lemma 11 *Let V be a vector space over a field $\mathbb{K}$ of characteristic zero. Then for any integer $p \geqslant 2$, the sequence*

$$0 \longrightarrow S^{p+1}V^* \longrightarrow V^* \otimes S^pV^* \xrightarrow{\mathrm{Alt}_2} \Lambda^2V^* \otimes S^{p-1}V^* \xrightarrow{\mathrm{Alt}_3} \Lambda^3V^* \otimes S^{p-2}V^*,$$

is exact.

Proof The equality

$$S^{p+1}V^* \quad = \quad \mathrm{Ker}\left(V^* \otimes S^pV^* \xrightarrow{\mathrm{Alt}_2} \Lambda^2V^* \otimes S^{p-1}V^*\right),$$

is obvious. We show now the equality

$$\begin{aligned} & \mathrm{Im}\left(V^* \otimes S^pV^* \xrightarrow{\mathrm{Alt}_2} \Lambda^2V^* \otimes S^{p-1}V^*\right) \\ = \quad & \mathrm{Ker}\left(\Lambda^2V^* \otimes S^{p-1}V^* \xrightarrow{\mathrm{Alt}_3} \Lambda^3V^* \otimes S^{p-2}V^*\right). \end{aligned} \tag{6.1}$$

We show first the inclusion $\subseteq$ in (6.1). We notice the equality

$$\begin{aligned} & \left(\Lambda^2V^* \otimes S^{p-1}V^* \xrightarrow{\mathrm{Alt}_3} \Lambda^3V^* \otimes S^{p-2}V^*\right) \\ = \quad & \left(\Lambda^2V^* \otimes S^{p-1}V^* \xrightarrow{2\,\mathrm{Circ}} \Lambda^3V^* \otimes S^{p-2}V^*\right). \end{aligned}$$

Let now $\beta := \mathrm{Alt}_2\,\alpha$, with $\alpha \in V^* \otimes S^pV^*$. Then summing up the two equalities

$$\beta\,(v_1, v_2; v_3, v_4, \ldots, v_{p+1}) = \alpha\,(v_1; v_2, v_3, v_4, \ldots, v_{p+1}) - \alpha\,(v_2; v_1, v_3, v_4, \ldots, v_{p+1}),$$

$$-\beta\,(v_1, v_3; v_2, v_4, \ldots, v_{p+1}) = -\alpha\,(v_1; v_3, v_2, v_4, \ldots, v_{p+1}) + \alpha\,(v_3; v_1, v_2, v_4, \ldots, v_{p+1}),$$

we obtain

$$\begin{aligned} & \beta\,(v_1, v_2; v_3, v_4, \ldots, v_{p+1}) - \beta\,(v_1, v_3; v_2, v_4, \ldots, v_{p+1}) \\ = \quad & -\alpha\,(v_2; v_3, v_1, v_4, \ldots, v_{p+1}) + \alpha\,(v_3; v_2, v_1, v_4, \ldots, v_{p+1}) \\ = \quad & -\beta\,(v_2, v_3; v_1, v_4, \ldots, v_{p+1}), \end{aligned}$$

which rewrites as

$$\begin{aligned} & \beta\,(v_1, v_2; v_3, v_4, \ldots, v_{p+1}) \\ + \quad & \beta\,(v_2, v_3; v_1, v_4, \ldots, v_{p+1}) \\ + \quad & \beta\,(v_3, v_1; v_2, v_4, \ldots, v_{p+1}) \;=\; 0, \end{aligned}$$

i.e. $\operatorname{Circ}\beta = 0$, which shows the inclusion $\subseteq$ in (6.1). In order to show the reverse inclusion in (6.1) we consider $\beta \in \Lambda^2 V^* \otimes S^{p-1}V^*$ with $\operatorname{Circ}\beta = 0$ and we will prove that $\beta = C_p \operatorname{Alt}_2 \alpha$, with

$$\alpha := \operatorname{Sym}_{2,\dots,p+1} \beta \in V^* \otimes S^p V^* ,$$

and with $C_p := p/(p+1)!$. Indeed

$$\frac{1}{(p-1)!}\alpha(v_1; v_2, \dots, v_{p+1}) \quad = \quad \sum_{j=2}^{p+1} \beta(v_1, v_j; v_2, \dots, \hat{v}_j, \dots, v_{p+1})$$

and

$$\begin{aligned}
& \frac{1}{(p-1)!}(\operatorname{Alt}_2 \alpha)(v_1, v_2; \dots, v_{p+1}) \\
= \quad & \frac{1}{(p-1)!}\alpha(v_1; v_2, \dots, v_{p+1}) - \frac{1}{(p-1)!}\alpha(v_2; v_1, \hat{v}_2, \dots, v_{p+1}) \\
= \quad & \sum_{j=2}^{p+1} \beta(v_1, v_j; v_2, \dots, \hat{v}_j, \dots, v_{p+1}) \\
- \quad & \sum_{\substack{j=1 \\ j\neq 2}}^{p+1} \beta(v_2, v_j; v_1, \hat{v}_2, \dots, \hat{v}_j, \dots, v_{p+1}) \\
= \quad & \beta(v_1, v_2; v_3, \dots, v_{p+1}) + \sum_{j=3}^{p+1} \beta(v_1, v_j; v_2, v_3, \dots, \hat{v}_j, \dots, v_{p+1}) \\
+ \quad & \beta(v_1, v_2; v_3, \dots, v_{p+1}) + \sum_{j=3}^{p+1} \beta(v_j, v_2; v_1, \hat{v}_2, v_3, \dots, \hat{v}_j, \dots, v_{p+1}) .
\end{aligned}$$

Using the circular identity $\operatorname{Circ}\beta = 0$, we obtain

$$\begin{aligned}
& \frac{1}{(p-1)!}(\operatorname{Alt}_2 \alpha)(v_1, v_2; \dots, v_{p+1}) \\
= \quad & 2\beta(v_1, v_2; v_3, \dots, v_{p+1}) - \sum_{j=3}^{p+1} \beta(v_2, v_1; v_j, v_3, \dots, \hat{v}_j, \dots, v_{p+1}) .
\end{aligned}$$

This combined with the fact that $\beta \in \Lambda^2 V^* \otimes S^{p-1}V^*$ implies

$$\begin{aligned}
& \frac{1}{(p-1)!}(\operatorname{Alt}_2 \alpha)(v_1, v_2; \dots, v_{p+1}) \\
= \quad & 2\beta(v_1, v_2; v_3, \dots, v_{p+1}) + (p-1)\beta(v_1, v_2; v_3, \dots, v_{p+1}) \\
= \quad & (p+1)\beta(v_1, v_2; \dots, v_{p+1}) ,
\end{aligned}$$

which shows the required identity. □

A direct consequence of the proof of lemma 11 is the following fact.

Corollary 3 *Let $R \in C^\infty\left(M, \Lambda^2 T^*_M \otimes_{\mathbb{R}} T^*_M \otimes_{\mathbb{R}} \mathbb{C}T_M\right)$ satisfying the algebraic Bianchi identity. Then a tensor $S \in C^\infty\left(M, T^*_M \otimes_{\mathbb{R}} S^2 T^*_M \otimes_{\mathbb{R}} \mathbb{C}T_M\right)$ satisfies $3R = \mathrm{Alt}_2 S$ if and only if $S = \mathrm{Sym}_{2,3} R + \sigma$, with $\sigma \in C^\infty(M, S^3 T^*_M \otimes_{\mathbb{R}} \mathbb{C}T_M)$.*

We infer by corollary 3 that the equation (5.4) is satisfied by $S_2 = S_2^0 + \sigma_2$, with

$$S_2^0(\xi_1, \xi_2, \xi_3) = \frac{i}{6}\left[R^{\nabla^{S_1}}(\xi_1, \xi_2)\,\xi_3 + R^{\nabla^{S_1}}(\xi_1, \xi_3)\,\xi_2\right], \tag{6.2}$$

and with $\sigma_2 \in C^\infty\left(M, S^3 T^*_M \otimes_{\mathbb{R}} \mathbb{C}T_M\right)$. We consider now the equation (5.5) for $k = 2$, which writes as

$$\left[d_1^{\nabla^{S_1}} S_2 + 3i\,\mathrm{Alt}_2 S_3\right](\xi_1, \xi_2, \eta^2) = 0. \tag{6.3}$$

The fact that the tensor

$$d_1^{\nabla^{S_1}} S_2 + 3i\,\mathrm{Alt}_2 S_3\,,$$

is symmetric in the last two variables implies that the equation (6.3) is equivalent to the equation

$$d_1^{\nabla^{S_1}} S_2 + 3i\,\mathrm{Alt}_2 S_3 = 0\,,$$

that we can rewrite under the form

$$d_1^{\nabla^{S_1}} S_2^0 + 3i\,\mathrm{Alt}_2 \hat{S}_3 = 0\,, \tag{6.4}$$

with

$$\hat{S}_3 := S_3 - \frac{i}{3}\nabla^{S_1}\sigma_2\,.$$

Then using the expression (6.2) we can rewrite equation (6.4) in the explicit form

$$\begin{aligned}
& \nabla^{S_1}_{\xi_1} R^{\nabla^{S_1}}(\xi_2, \xi_3)\,\xi_4 + \nabla^{S_1}_{\xi_1} R^{\nabla^{S_1}}(\xi_2, \xi_4)\,\xi_3 \\
- \;& \nabla^{S_1}_{\xi_2} R^{\nabla^{S_1}}(\xi_1, \xi_3)\,\xi_4 - \nabla^{S_1}_{\xi_2} R^{\nabla^{S_1}}(\xi_1, \xi_4)\,\xi_3 \\
= \;& -18\left[\hat{S}_3(\xi_1, \xi_2, \xi_3, \xi_4) - \hat{S}_3(\xi_2, \xi_1, \xi_3, \xi_4)\right].
\end{aligned} \tag{6.5}$$

We notice that the fact that the complex connection ∇^{S_1} is torsion free implies that the tensor ρ given by $\rho(\xi_1, \xi_2, \xi_3, \xi_4) := \nabla^{S_1}_{\xi_1} R^{\nabla^{S_1}}(\xi_2, \xi_3)\,\xi_4$ satisfies the circular identity with respect to the first and last three entries. Moreover ρ is obviously skew-symmetric with respect to the variables ξ_2, ξ_3.

Lemma 12 *Let ρ be a 4-linear form which satisfies the circular identity with respect to the first and last three entries and which is skew-symmetric with respect to the second and third variables. Then a 4-linear form S which is symmetric with respect to the last three entries satisfies the equation*

$$\mathrm{Alt}_2[8\,\mathrm{Sym}_{3,4}\,\rho - S] = 0\,, \tag{6.6}$$

if and only if

$$\begin{aligned}
S &= -2\,\mathrm{Sym}_{2,3,4}\,\rho_2 + \sigma \\
&= 2\,\mathrm{Sym}_{2,3,4}\,\rho_3 + \sigma\,,
\end{aligned}$$

with $\rho_2(\xi_1,\xi_2,\xi_3,\xi_4) := \rho(\xi_2,\xi_1,\xi_3,\xi_4)$, *with* $\rho_3(\xi_1,\xi_2,\xi_3,\xi_4) := \rho(\xi_2,\xi_3,\xi_1,\xi_4)$, *for all* $\xi_1,\xi_2,\xi_3,\xi_4 \in T_{M,\pi(\xi_1)}$ *and with* σ *a 4-linear form which is symmetric with respect to all its entries.*

Proof We observe first that the assumptions on ρ imply $\operatorname{Circ}\operatorname{Alt}_2\operatorname{Sym}_{3,4}\rho = 0$. Indeed

$$\begin{aligned}(\operatorname{Alt}_2\operatorname{Sym}_{3,4}\rho)(\xi_1,\xi_2,\xi_3,\xi_4) &= \rho(\xi_1,\xi_2,\xi_3,\xi_4)+\rho(\xi_1,\xi_2,\xi_4,\xi_3)\\ &- \rho(\xi_2,\xi_1,\xi_3,\xi_4)-\rho(\xi_2,\xi_1,\xi_4,\xi_3),\end{aligned}$$

and

$$\begin{aligned}&(\operatorname{Circ}\operatorname{Alt}_2\operatorname{Sym}_{3,4}\rho)(\xi_1,\xi_2,\xi_3,\xi_4)\\ =\ &\rho(\xi_1,\xi_2,\xi_3,\xi_4)_1+\rho(\xi_1,\xi_2,\xi_4,\xi_3)_4-\rho(\xi_2,\xi_1,\xi_3,\xi_4)_2-\rho(\xi_2,\xi_1,\xi_4,\xi_3)_5\\ +\ &\rho(\xi_2,\xi_3,\xi_1,\xi_4)_2+\rho(\xi_2,\xi_3,\xi_4,\xi_1)_5-\rho(\xi_3,\xi_2,\xi_1,\xi_4)_3-\rho(\xi_3,\xi_2,\xi_4,\xi_1)_6\\ +\ &\rho(\xi_3,\xi_1,\xi_2,\xi_4)_3+\rho(\xi_3,\xi_1,\xi_4,\xi_2)_6-\rho(\xi_1,\xi_3,\xi_2,\xi_4)_1-\rho(\xi_1,\xi_3,\xi_4,\xi_2)_4,\end{aligned}$$

where we denote by $\rho(\cdot,\cdot,\cdot,\cdot)_j$ the terms we group together. Using the assumption ρ is skew-symmetric with respect to the second and third variables we infer

$$\begin{aligned}&(\operatorname{Circ}\operatorname{Alt}_2\operatorname{Sym}_{3,4}\rho)(\xi_1,\xi_2,\xi_3,\xi_4)\\ =\ &2\rho(\xi_1,\xi_2,\xi_3,\xi_4)_1+\rho(\xi_1,\xi_2,\xi_4,\xi_3)_4+\rho(\xi_2,\xi_4,\xi_1,\xi_3)_5\\ +\ &2\rho(\xi_2,\xi_3,\xi_1,\xi_4)_1+\rho(\xi_2,\xi_3,\xi_4,\xi_1)_5+\rho(\xi_3,\xi_4,\xi_2,\xi_1)_6\\ +\ &2\rho(\xi_3,\xi_1,\xi_2,\xi_4)_1+\rho(\xi_3,\xi_1,\xi_4,\xi_2)_6+\rho(\xi_1,\xi_4,\xi_3,\xi_2)_4.\end{aligned}$$

Using the circular assumptions on ρ we infer

$$\begin{aligned}&(\operatorname{Circ}\operatorname{Alt}_2\operatorname{Sym}_{3,4}\rho)(\xi_1,\xi_2,\xi_3,\xi_4)\\ =\ &-\rho(\xi_1,\xi_3,\xi_2,\xi_4)-\rho(\xi_2,\xi_1,\xi_3,\xi_4)-\rho(\xi_3,\xi_2,\xi_1,\xi_4)\\ =\ &0.\end{aligned}$$

Then by the proof of lemma 11 in the case $p=3$, we infer that a 4-linear form S which is symmetric with respect to the last three entries satisfies the equation (6.6) if and only if

$$S = \operatorname{Sym}_{2,3,4}\operatorname{Alt}_2\operatorname{Sym}_{3,4}\rho+\sigma,$$

with σ any 4-linear form which is symmetric with respect to all its entries, satisfies (6.6). We write now

$$
\begin{aligned}
& (\mathrm{Sym}_{2,3,4}\,\mathrm{Alt}_2\,\mathrm{Sym}_{3,4}\,\rho)\,(\xi_1,\xi_2,\xi_3,\xi_4) \\
= \;& \rho\,(\xi_1,\xi_2,\xi_3,\xi_4)_{_1} + \rho\,(\xi_1,\xi_2,\xi_4,\xi_3)_{_2} - \rho\,(\xi_2,\xi_1,\xi_3,\xi_4)_{_3} - \rho\,(\xi_2,\xi_1,\xi_4,\xi_3)_{_4} \\
+ \;& \rho\,(\xi_1,\xi_2,\xi_4,\xi_3)_{_1} + \rho\,(\xi_1,\xi_2,\xi_3,\xi_4)_{_2} - \rho\,(\xi_2,\xi_1,\xi_4,\xi_3)_{_4} - \rho\,(\xi_2,\xi_1,\xi_3,\xi_4)_{_3} \\
+ \;& \rho\,(\xi_1,\xi_3,\xi_2,\xi_4)_{_1} + \rho\,(\xi_1,\xi_3,\xi_4,\xi_2)_{_2} - \rho\,(\xi_3,\xi_1,\xi_2,\xi_4)_{_5} - \rho\,(\xi_3,\xi_1,\xi_4,\xi_2)_{_6} \\
+ \;& \rho\,(\xi_1,\xi_3,\xi_4,\xi_2)_{_1} + \rho\,(\xi_1,\xi_3,\xi_2,\xi_4)_{_2} - \rho\,(\xi_3,\xi_1,\xi_4,\xi_2)_{_6} - \rho\,(\xi_3,\xi_1,\xi_2,\xi_4)_{_5} \\
+ \;& \rho\,(\xi_1,\xi_4,\xi_2,\xi_3)_{_1} + \rho\,(\xi_1,\xi_4,\xi_3,\xi_2)_{_2} - \rho\,(\xi_4,\xi_1,\xi_2,\xi_3)_{_7} - \rho\,(\xi_4,\xi_1,\xi_3,\xi_2)_{_8} \\
+ \;& \rho\,(\xi_1,\xi_4,\xi_3,\xi_2)_{_1} + \rho\,(\xi_1,\xi_4,\xi_2,\xi_3)_{_2} - \rho\,(\xi_4,\xi_1,\xi_3,\xi_2)_{_8} - \rho\,(\xi_4,\xi_1,\xi_2,\xi_3)_{_7}\,.
\end{aligned}
$$

The fact that ρ is skew-symmetric with respect to the second and third variables implies that $\mathrm{Sym}_{2,3,4}\,\rho = 0$. We infer

$$
\begin{aligned}
& (\mathrm{Sym}_{2,3,4}\,\mathrm{Alt}_2\,\mathrm{Sym}_{3,4}\,\rho)\,(\xi_1,\xi_2,\xi_3,\xi_4) \\
= \;& -2\rho\,(\xi_2,\xi_1,\xi_3,\xi_4) - 2\rho\,(\xi_2,\xi_1,\xi_4,\xi_3) \\
- \;& 2\rho\,(\xi_3,\xi_1,\xi_2,\xi_4) - 2\rho\,(\xi_3,\xi_1,\xi_4,\xi_2) \\
- \;& 2\rho\,(\xi_4,\xi_1,\xi_2,\xi_3) - 2\rho\,(\xi_4,\xi_1,\xi_3,\xi_2) \\
= \;& 2\rho\,(\xi_2,\xi_3,\xi_1,\xi_4) + 2\rho\,(\xi_2,\xi_4,\xi_1,\xi_3) \\
+ \;& 2\rho\,(\xi_3,\xi_2,\xi_1,\xi_4) + 2\rho\,(\xi_3,\xi_4,\xi_1,\xi_2) \\
+ \;& 2\rho\,(\xi_4,\xi_2,\xi_1,\xi_3) + 2\rho\,(\xi_4,\xi_3,\xi_1,\xi_2)\,,
\end{aligned}
$$

which shows the required expressions for S. □

By the equation (6.5) we can apply lemma 12 to the tensor $\rho := \nabla^{S_1} R^{\nabla^{S_1}}$. We infer the equation

$$
S_3 = \frac{1}{4!3}\,\mathrm{Sym}_{2,3,4}(\nabla^{S_1} R^{\nabla^{S_1}})_2 + \frac{i}{3}\nabla^{S_1}\sigma_2 + \sigma_3\,, \tag{6.7}
$$

We deduce that the equation (6.4) is equivalent to the equation (6.7). This concludes the proof of the proposition 2 thanks to lemma 10.

7 Third reduction of the integrability equations and proof of the main theorem

In this section we will prove the following result.

Lemma 13 *Under the assumptions of the theorem 2 the M-totally real almost complex structure J_A is integrable over U if and only if*

$$\begin{aligned}
S_1 &\in C^\infty\left(M, S^2T^*_M \otimes_{\mathbb{R}} \mathbb{C}T_M\right), \quad \textit{i.e. } \nabla^{S_1} \textit{ is torsion free,}\\
S_2 &= S^0_2 + \sigma_2,\\
S^0_2\left(\xi_1,\xi_2,\xi_3\right) &:= \frac{i}{6}\left[R^{\nabla^{S_1}}\left(\xi_1,\xi_2\right)\xi_3 + R^{\nabla^{S_1}}\left(\xi_1,\xi_3\right)\xi_2\right],\\
\sigma_2 &\in C^\infty\left(M, S^3T^*_M \otimes_{\mathbb{R}} \mathbb{C}T_M\right),
\end{aligned}$$

and for all $k \geqslant 3$,

$$\begin{aligned}
S_k &= \frac{i}{k}\nabla^{S_1}\sigma_{k-1} + \frac{i}{(k+1)!}\operatorname{Sym}_{2,\dots,k+1}\beta_{k-1} + \sigma_k,\\
\sigma_k &\in C^\infty\left(M, S^{k+1}T^*_M \otimes_{\mathbb{R}} \mathbb{C}T_M\right),\\
\beta_k &:= \frac{i}{k}d_1^{\nabla^{S_1}}\nabla^{S_1}\sigma_{k-1} + \frac{i}{(k+1)!}d_1^{\nabla^{S_1}}\operatorname{Sym}_{2,\dots,k+1}\beta_{k-1}\\
&+ \frac{1}{k!}\operatorname{Sym}_{3,\dots,k+2}\left(\sum_{p=2}^{k-1} pS_p \wedge_1 S_{k-p+1}\right),\\
\beta_2 &:= -\frac{i}{3}(\nabla^{S_1}R^{\nabla^{S_1}})_2,\\
\operatorname{Circ}\beta_k &= 0.
\end{aligned}$$

Proof We show that the statement of proposition 2 is equivalent to the statement of lemma 13. We show indeed by induction on $k \geqslant 3$ the following statement.

Statement 1 *The tensors S_h, $h = 3,\dots,k+1$, satisfy the equations*

$$\left[d_1^{\nabla^{S_1}}S_h + \sum_{p=2}^{h-1} pS_p \wedge_1 S_{h-p+1} + i\,(h+1)\operatorname{Alt}_2 S_{h+1}\right]\left(\xi_1,\xi_2,\eta^h\right) = 0, \quad (7.1)$$

for all $h = 3,\dots,k$, for all $\xi_1,\xi_2,\eta \in T_{M,\pi(\eta)}$ and

$$S_3 = \frac{i}{3}\nabla^{S_1}\sigma_2 + \frac{1}{4!3}\operatorname{Sym}_{2,3,4}(\nabla^{S_1}R^{\nabla^{S_1}})_2 + \sigma_3,$$

with $\sigma_3 \in C^\infty\left(M, S^4T^*_M \otimes_{\mathbb{R}} \mathbb{C}T_M\right)$*, if and only if the tensors* S_h *satisfy for all* $h = 3, \ldots, k+1$*, the identities*

$$S_h = \frac{i}{h}\nabla^{S_1}\sigma_{h-1} + \frac{i}{(h+1)!}\operatorname{Sym}_{2,\ldots,h+1}\beta_{h-1} + \sigma_h\,, \tag{7.2}$$

with $\sigma_h \in C^\infty\left(M, S^{h+1}T^*_M \otimes_{\mathbb{R}} \mathbb{C}T_M\right)$ *and where for all* $r = 3, \ldots, k$*,*

$$\begin{aligned}\beta_r &:= \frac{i}{r}d_1^{\nabla^{S_1}}\nabla^{S_1}\sigma_{r-1} + \frac{i}{(r+1)!}d_1^{\nabla^{S_1}}\operatorname{Sym}_{2,\ldots,r+1}\beta_{r-1}\\ &+ \frac{1}{r!}\operatorname{Sym}_{3,\ldots,r+2}\left(\sum_{p=2}^{r-1} pS_p \wedge_1 S_{r-p+1}\right),\end{aligned}$$

with $\beta_2 := -\frac{i}{3}(\nabla^{S_1}R^{\nabla^{S_1}})_2$ *satisfies the equation* $\operatorname{Circ}\beta_r = 0$*.*

The statement 1 follows directly from the following fact.

Fact 1 *Let* S_h*, for some* $h = 3, \ldots, k$*, be the tensor given by (7.2). Then the tensor* S_{h+1} *satisfies the equation (7.1) if and only if* S_{h+1} *satisfies the identity (7.2), with* h *replaced by* $h+1$ *and* β_h *satisfies the equation* $\operatorname{Circ}\beta_h = 0$*.*

In order to show the fact 1 we observe first that (7.1) rewrites as

$$d_1^{\nabla^{S_1}}S_h + \frac{1}{h!}\operatorname{Sym}_{3,\ldots,h+2}\left(\sum_{p=2}^{h-1} pS_p \wedge_1 S_{h-p+1}\right) + i\,(h+1)\operatorname{Alt}_2 S_{h+1} = 0\,.$$

Using the expression (7.2) for S_h and the definition of β_h, we can rewrite the previous identity as

$$\beta_h = -\operatorname{Alt}_2\left[\nabla^{S_1}\sigma_h + i\,(h+1)\;S_{h+1}\right]. \tag{7.3}$$

By the proof of lemma 11 we deduce $\operatorname{Circ}\beta_h = 0$ and

$$-\nabla^{S_1}\sigma_h - i\,(h+1)\;S_{h+1} = C_{h+1}\operatorname{Sym}_{2,\ldots,h+2}\beta_h - i\,(h+1)\;\sigma_{h+1}\,.$$

Therefore the identity (7.3) is equivalent to; $\operatorname{Circ}\beta_h = 0$ and S_{h+1} satisfies (7.2), with h replaced by $h+1$. This concludes the proof fact 1. We infer the required conclusion of lemma 13. □

Proof of the main theorem

Proof We show that the recursive definition of β_k in the statement of lemma 13 yields the formula

$$\beta_k = \frac{i}{k}d_1^{\nabla^{S_1}}\nabla^{S_1}\sigma_{k-1} + \frac{1}{(k+1)!k!}\operatorname{Sym}_{3,\ldots,k+2}\theta_k\,, \tag{7.4}$$

$$\begin{aligned}\theta_k &:= \sum_{r=2}^{k-2}\frac{(r+2)!}{r+1}(i\,d_1^{\nabla^{S_1}})^{k-r}\nabla^{S_1}\sigma_r + 3!\left(i\,d_1^{\nabla^{S_1}}\right)^{k-2}\beta_2\\ &+ \sum_{r=3}^{k}(r+1)!\sum_{p=2}^{r-1}(i\,d_1^{\nabla^{S_1}})^{k-r}\left(pS_p \wedge_1 S_{r-p+1}\right),\end{aligned}$$

for all $k \geqslant 3$. We show (7.4) by induction on k. We notice first that the recursive definition of β_k rewrites as

$$\beta_k = \frac{i}{k} d_1^{\nabla^{S_1}} \nabla^{S_1} \sigma_{k-1} + \mathrm{Sym}_{3,\dots,k+2} \left[\frac{i}{(k+1)!} d_1^{\nabla^{S_1}} \beta_{k-1} + \frac{1}{k!} \sum_{p=2}^{k-1} p S_p \wedge_1 S_{k-p+1} \right],$$

and we write

$$\begin{aligned} \beta_{k+1} &= \frac{i}{k+1} d_1^{\nabla^{S_1}} \nabla^{S_1} \sigma_k + \mathrm{Sym}_{3,\dots,k+3} \left[\frac{i}{(k+2)!} d_1^{\nabla^{S_1}} \beta_k \right], \\ &+ \mathrm{Sym}_{3,\dots,k+3} \left[\frac{1}{(k+1)!} \sum_{p=2}^{k} p S_p \wedge_1 S_{k-p+2} \right]. \end{aligned}$$

Using the inductive assumption (7.4) we infer the expressions

$$\begin{aligned} \frac{i}{(k+2)!} d_1^{\nabla^{S_1}} \beta_k &= \frac{1}{(k+2)!k} (i\, d_1^{\nabla^{S_1}})^2 \nabla^{S_1} \sigma_{k-1} \\ &+ \frac{1}{(k+2)!\,(k+1)!k!} \mathrm{Sym}_{4,\dots,k+3}\, i\, d_1^{\nabla^{S_1}} \theta_k, \\ i\, d_1^{\nabla^{S_1}} \theta_k &= \sum_{r=2}^{k-2} \frac{(r+2)!}{r+1} (i\, d_1^{\nabla^{S_1}})^{k+1-r} \nabla^{S_1} \sigma_r + 3! (i\, d_1^{\nabla^{S_1}})^{k-1} \beta_2 \\ &+ \sum_{r=3}^{k} (r+1)! \sum_{p=2}^{r-1} (i\, d_1^{\nabla^{S_1}})^{k+1-r} \left(p S_p \wedge_1 S_{r-p+1} \right). \end{aligned}$$

This combined with the identity $\mathrm{Sym}_{3,\dots,k+3} \mathrm{Sym}_{4,\dots,k+3} = k!\, \mathrm{Sym}_{3,\dots,k+3}$, yields

$$\begin{aligned} &\beta_{k+1} \\ &= \frac{i}{k+1} d_1^{\nabla^{S_1}} \nabla^{S_1} \sigma_k + \frac{1}{(k+2)!k} \mathrm{Sym}_{3,\dots,k+3} (i\, d_1^{\nabla^{S_1}})^2 \nabla^{S_1} \sigma_{k-1} \\ &+ \frac{1}{(k+2)!\,(k+1)!} \mathrm{Sym}_{3,\dots,k+3} \sum_{r=2}^{k-2} \frac{(r+2)!}{r+1} (i\, d_1^{\nabla^{S_1}})^{k+1-r} \nabla^{S_1} \sigma_r \\ &+ \frac{3!}{(k+2)!\,(k+1)!} \mathrm{Sym}_{3,\dots,k+3} (i\, d_1^{\nabla^{S_1}})^{k-1} \beta_2 \\ &+ \frac{1}{(k+2)!\,(k+1)!} \mathrm{Sym}_{3,\dots,k+3} \sum_{r=3}^{k} (r+1)! \sum_{p=2}^{r-1} (i\, d_1^{\nabla^{S_1}})^{k+1-r} \left(p S_p \wedge_1 S_{r-p+1} \right) \\ &+ \frac{1}{(k+1)!} \mathrm{Sym}_{3,\dots,k+3} \sum_{p=2}^{k} p S_p \wedge_1 S_{k-p+2}. \end{aligned}$$

Putting the terms together we obtain (7.4) for β_{k+1}. Then the obvious identity $d_1^{\nabla} \nabla = \mathrm{Alt}_2 \nabla^2$ combined with the formula (3.16) allows to conclude the required expression of $\beta_k \equiv \beta_k(\sigma_{k-1})$ in the statement of the main theorem. This concludes the proof of the main theorem. □

8 Appendix

In this appendix we provide some well known basic facts about the geometric theory of linear connections needed for the reading of the paper. (See also [Gau]).

8.1 The horizontal distribution associated to a linear connection

We start with the following fact.

Lemma 14 *Let ∇ be a linear connection acting on sections of a vector bundle E over a manifold M. Then the linear map*

$$T_{M,p} \ni \xi \longmapsto H_\eta(\xi) := d_p\sigma(\xi) - T_\eta \nabla_\xi \sigma \in T_{E,\eta},$$

is independent of the sections σ such that $\sigma(p) = \eta$.

Proof Let $e = (e_k)_{k=1}^r$ be a local frame of E over an open set $U \subset M$. We consider the local expression $\sigma = e \cdot f$ with $f \in C^1(U, \mathbb{R}^r)$. Let $A \in C^\infty(U, T^*_M \otimes \mathrm{Matrix}_{r\times r}(\mathbb{R}))$ be the connection form of ∇ with respect to the local frame e, i.e $\nabla e = e \cdot A$. Then $\nabla\sigma = e \otimes (df + A \cdot f)$. If we denote by $\theta_e : U \times \mathbb{R}^r \longrightarrow E_{|U}$ then the differential of this map at the point $(p, f(p))$ provides an isomorphism

$$d_{p,f(p)}\theta_e : T_{U,p} \oplus \mathbb{R}^r \quad \longrightarrow \quad T_{E,\sigma(p)}.$$

With respect to it, the equality hold

$$d_{p,f(p)}\theta_e\left[\xi \oplus d_pf(\xi)\right] \quad = \quad d_p\sigma(\xi).$$

We observe now the linear identity $d\tau_{\sigma(p)} \cdot d_{p,0}\theta_{e|0\oplus\mathbb{R}^r} = d_{p,f(p)}\theta_{e|0\oplus\mathbb{R}^r}$. We infer

$$T_{\sigma(p)} \cdot \theta_{e|\{p\}\times\mathbb{R}^r} = d_{p,f(p)}\theta_{e|0\oplus\mathbb{R}^r}, \tag{8.1}$$

and

$$\begin{aligned} & T_{\sigma(p)}\left[e(p)\cdot\Big(d_pf(\xi) + A(\xi)\cdot f(p)\Big)\right] \\ = \quad & d_{p,f(p)}\theta_e\left[0\oplus\Big(d_pf(\xi) + A(\xi)\cdot f(p)\Big)\right], \\ T_{\sigma(p)}\nabla_\xi\sigma \quad = \quad & d_{p,f(p)}\theta_e\left[0\oplus\Big(d_pf(\xi) + A(\xi)\cdot f(p)\Big)\right]. \end{aligned}$$

Thus

$$H_{\sigma(p)}(\xi) \quad = \quad d_{p,f(p)}\theta_e\left[\xi\oplus\Big(-A(\xi)\cdot f(p)\Big)\right],$$

i.e. if $\eta = e\cdot h$, then

$$H_\eta(\xi) \quad = \quad d_{p,h}\theta_e\left[\xi\oplus\Big(-A(\xi)\cdot h\Big)\right],$$

which shows the required conclusion. □

Let $\pi_E : E \longrightarrow M$ be the projection map and notice the equality $\mathrm{Ker}\, d_\eta \pi_E = T_{E_p,\eta}$, for any $\eta \in E_p$. The identity $\pi_E \circ \sigma = \mathrm{id}_M$ implies

$$d_{\sigma(p)}\pi_E \circ d_p\sigma\,(\xi) \quad = \quad \xi\,.$$

We deduce the identity $d_\eta \pi_E \circ H_\eta\,(\xi) = \xi$. We define the horizontal distribution $\mathcal{H} \subset T_E$ associated to ∇ as

$$\mathcal{H}_\eta \quad := \quad H_\eta\left(T_{M,\pi_E(\eta)}\right) \subset T_{E,\eta}\,.$$

We notice now that the tangent bundle of the vector bundle $E \oplus E$ is given by the fibers

$$T_{E\oplus E,(\eta_1,\eta_2)} \quad = \quad \left\{(v_1, v_2) \in T_{E,\eta_1} \oplus T_{E,\eta_2} \mid d_{\eta_1}\pi_E\,(v_1) = d_{\eta_2}\pi_E\,(v_2)\right\},$$

and that the differential of the sum bundle map $sm_{_E} : E \oplus E \longrightarrow E$ satisfies

$$d_{(\eta_1,\eta_2)}\,(sm_{_E})\,(v_1, v_2) \quad = \quad T_{\eta_1+\eta_2}\left(T_{\eta_1}^{-1}v_1 + T_{\eta_2}^{-1}v_2\right),$$

for any $(v_1, v_2) \in T_{E,\eta_1} \oplus T_{E,\eta_2}$ such that $d_{\eta_1}\pi_E\,(v_1) = d_{\eta_2}\pi_E\,(v_2) = 0$. We infer that for any sections σ_j of E such that $\sigma_j\,(p) = \eta_j$, $j = 1, 2$, hold the equalities

$$\begin{aligned}
H_{\eta_1+\eta_2}\,(\xi) \quad &= \quad d_p\,(\sigma_1 + \sigma_2)\,(\xi) - T_{\eta_1+\eta_2}\nabla_\xi\,(\sigma_1 + \sigma_2) \\
&= \quad d_{(\eta_1,\eta_2)}\,(sm_{_E})\left(d_p\sigma_1\,(\xi)\,, d_p\sigma_2\,(\xi)\right) - T_{\eta_1+\eta_2}\nabla_\xi\sigma_1 - T_{\eta_1+\eta_2}\nabla_\xi\sigma_2 \\
&= \quad d_{(\eta_1,\eta_2)}\,(sm_{_E})\left(d_p\sigma_1\,(\xi) - T_{\eta_1}\nabla_\xi\sigma_1, d_p\sigma_2\,(\xi)\ - T_{\eta_2}\nabla_\xi\sigma_2\right).
\end{aligned}$$

We conclude the property

$$H_{\eta_1+\eta_2}\,(\xi) = d_{(\eta_1,\eta_2)}\,(sm_{_E})\,(H_{\eta_1}\,(\xi)\,, H_{\eta_2}\,(\xi))\,. \tag{8.2}$$

Lemma 15 *For any section $\sigma \in C^1\,(M, E)$ and for any function $u \in C^1\,(M, \mathbb{R})$ the identity holds*

$$d_p\,(u\sigma) \quad = \quad d_p u \otimes T_{u\sigma(p)}\sigma\,(p) + d_{\sigma(p)}[u\,(p)\,\mathbb{I}_E] \cdot d_p\sigma\,,$$

for any point $p \in M$.

Proof With the notations in the proof of lemma 14

$$\begin{aligned}
d_p\,(u\sigma)\,(\xi) \quad &= \quad d_{p,uf(p)}\theta_e\left[\xi \oplus d_p\,(uf)\,(\xi)\right] \\
&= \quad d_{p,uf(p)}\theta_e\left\{\xi \oplus \left[d_p u\,(\xi)\,f\,(p) + u\,(p)\,d_p f\,(\xi)\right]\right\} \\
&= \quad d_{p,uf(p)}\theta_e\left[0 \oplus d_p u\,(\xi)\,f\,(p)\right] \\
&+ \quad d_{p,uf(p)}\theta_e\left[\xi \oplus u\,(p)\,d_p f\,(\xi)\right] \\
&= \quad T_{u\sigma(p)}\theta_e\left(p, d_p u\,(\xi)\,f\,(p)\right) + d_p\,(u\,(p)\,\sigma)\,(\xi)\,,
\end{aligned}$$

thanks to (8.1). Using the identity

$$d_p(\lambda\sigma) = d_{\sigma(p)}(\lambda \mathbb{I}_E) \cdot d_p\sigma\,, \tag{8.3}$$

for any $\lambda \in \mathbb{R}$, we conclude

$$d_p(u\sigma)(\xi) \quad = \quad d_p u(\xi)\, T_{u\sigma(p)}\sigma(p) + d_{\sigma(p)}[u(p)\,\mathbb{I}_E] \cdot d_p\sigma(\xi)\,.$$

□

We observe also the elementary identity

$$d_\eta(\lambda \mathbb{I}_E) \cdot T_\eta = \lambda T_{\lambda\eta}\,, \tag{8.4}$$

for all $\eta \in E$. We show now the identity

$$H_{\lambda\eta} = d_\eta(\lambda \mathbb{I}_E) \cdot H_\eta\,, \tag{8.5}$$

for all $\eta \in E$. Indeed let σ be a section such that $\sigma(p) = \eta$. Using (8.3) and (8.4) we obtain the equalities

$$\begin{aligned} H_{\lambda\eta} &= d_p(\lambda\sigma) - T_{\lambda\eta}\nabla(\lambda\sigma) \\ &= d_{\sigma(p)}(\lambda \mathbb{I}_E) \cdot d_p\sigma - \lambda T_{\lambda\eta}\nabla\sigma \\ &= d_{\sigma(p)}(\lambda \mathbb{I}_E) \cdot [d_p\sigma - T_\eta\nabla\sigma] \\ &= d_\eta(\lambda \mathbb{I}_E) \cdot H_\eta\,. \end{aligned}$$

The property (8.5) implies in particular $H_{0_p} = d_p 0_M$, where 0_M is the zero section of T_M.

Definition 5 *A distribution $\mathcal{H} \subset T_E$, is called horizontal if the map*

$$d_\eta \pi_{E|\mathcal{H}_\eta} : \mathcal{H}_\eta \longrightarrow T_{M,\pi_E(\eta)}\,,$$

is an isomorphism for all $\eta \in E$.

Lemma 16 *Any horizontal distribution $\mathcal{H} \subset T_E$, which satisfies the conditions (8.2) and (8.5) with $H_\eta := \left(d_\eta \pi_{E|\mathcal{H}_\eta}\right)^{-1}$, determines a connection ∇ over E with associated horizontal distribution $\mathcal{H}$.*

Proof The connection ∇ is defined by the formula

$$\nabla_\xi \sigma \quad = \quad T^{-1}_{\sigma(p)} \cdot \left[d_p\sigma - H_{\sigma(p)}\right](\xi)\,,$$

for any $\xi \in T_{M,p}$. The definition is well posed because

$$\left[d_p\sigma - H_{\sigma(p)}\right](\xi) \quad \in \quad T_{E_p,\sigma(p)}\,,$$

which follows from the identity

$$d_{\sigma(p)}\pi_E \cdot \left[d_p\sigma - H_{\sigma(p)}\right](\xi) \quad = \quad 0\,.$$

It is obvious that the additive property of ∇ is equivalent to the condition (8.2). We observe now that with the previous definition, the covariant Leibniz property

$$\nabla_{\xi}(u\sigma) \quad = \quad d_p u(\xi)\,\sigma(p) + u(p)\,\nabla_{\xi}\sigma\,,$$

is equivalent to the identity

$$\begin{aligned} & d_p(u\sigma)(\xi) - H_{u\sigma(p)}(\xi) \\ = \quad & T_{u\sigma(p)}\left\{ d_p u(\xi)\,\sigma(p) \; + u(p)\,T^{-1}_{\sigma(p)} \cdot \left[d_p\sigma(\xi) \; - H_{\sigma(p)}(\xi) \right] \right\}. \end{aligned}$$

We develop the right hand side using (8.4). We infer that the previous identity is equivalent to the following one

$$\begin{aligned} d_p(u\sigma)(\xi) - H_{u\sigma(p)}(\xi) \quad = \quad & d_p u(\xi)\,T_{u\sigma(p)}\sigma(p) \\ + \quad & d_{\sigma(p)}\left[u(p) \; \mathbb{I}_E \right] \cdot \left[d_p\sigma(\xi) \; - H_{\sigma(p)}(\xi) \right]. \end{aligned}$$

The later hold true thanks to lemma 15 and the assumption (8.5). □

The data of a smooth horizontal distribution over E coincides with the one of section

$$H \in C^{\infty}(E, \pi_E^* T_M^* \otimes T_E)$$

such that $d\pi_E \cdot H = \mathbb{I}_{\pi_E^* T_E}$. (We notice that $d\pi_E \in C^{\infty}(E, T_E^* \otimes \pi_E^* T_M)$). Such type of section determines a connection if and only if it satisfies the identity (8.5).

For any vector $\Xi \in T_{E,\eta}$ we denote by

$$\gamma_{\eta}^{\mathcal{H}}(\Xi) := \Xi - H_{\eta} \circ d_{\eta}\pi_E(\Xi)\,,$$

its vertical component with respect to the horizontal distribution $\mathcal{H}$. In particular

$$\gamma_{\sigma(p)}^{\mathcal{H}} \cdot d_p\sigma(\xi) = T_{\sigma(p)}[\nabla_{\xi}\sigma(p)]\,.$$

8.2 The induced connection

Let $\psi : N \longrightarrow M$ be a smooth map. We define the vector bundle $\psi^* E := N \times_{\psi} E$ over N. In explicit terms

$$\psi^* E \quad = \quad \left\{ (y,\eta) \in N \times E \mid \psi(y) = \pi_E(\eta) \right\},$$

and the projection over N is given by the restriction of the projection to the first factor. We will denote by $\Psi : \psi^* E \longrightarrow E$ the restriction of the projection to the second factor. The sections of $\psi^* E$ are identified with the maps $\sigma : N \longrightarrow E$ such that $\pi_E \circ \sigma = \psi$. In this way, if s is a section of E then the section $\psi^* s := s \circ \psi$ is a section of $\psi^* E$. More in general if α is a section of $\Lambda^p T_M^* \otimes E$, we define the section $\psi^*\alpha \in \Lambda^p T_N^* \otimes \psi^* E$ as

$$(\psi^*\alpha)(y) \quad := \quad (\alpha \circ \psi)(y) \cdot \Lambda^p(d_y\psi)\,.$$

We provide a generalization of lemma (15).

Lemma 17 *For any section $\sigma \in C^1(N, \psi^* E)$ and for any function $u \in C^1(N, \mathbb{R})$ the identity holds*

$$d_p(u\sigma) = d_p u \otimes T_{u\sigma(p)}\sigma(p) + d_{\sigma(p)}[u(p)\mathbb{I}_E] \cdot d_p\sigma,$$

for any point $p \in N$.

Proof A local frame e of E induces a local frame $\psi^* e$ of $\psi^* E$ over the open set $\psi^{-1}(U)$. Then $\sigma = \psi^* e \cdot f$ with $f \in C^1\left(\psi^{-1}(U), \mathbb{R}^r\right)$. We denote by $\theta_e : U \times \mathbb{R}^r \longrightarrow E_{|U}$ the trivialization map induced by the local frame e of E. Then the differential of this map at the point $(\psi(p), f(p))$ provides an isomorphism

$$d_{\psi(p),f(p)}\theta_e : T_{U,\psi(p)} \oplus \mathbb{R}^r \longrightarrow T_{E,\sigma(p)},$$

and

$$d_p\sigma(\xi) = d_{\psi(p),uf(p)}\theta_e\left[d_p\psi(\xi) \oplus d_p f(\xi)\right].$$

for any $\xi \in T_{N,p}$ we have

$$\begin{aligned}
d_p(u\sigma)(\xi) &= d_{\psi(p),uf(p)}\theta_e\left[d_p\psi(\xi) \oplus d_p(uf)(\xi)\right] \\
&= d_{\psi(p),uf(p)}\theta_e\left\{d_p\psi(\xi) \oplus \left[d_p u(\xi) f(p) + u(p) d_p f(\xi)\right]\right\} \\
&= d_{\psi(p),uf(p)}\theta_e\left[0 \oplus d_p u(\xi) f(p)\right] \\
&+ d_{\psi(p),uf(p)}\theta_e\left[d_p\psi(\xi) \oplus u(p) d_p f(\xi)\right] \\
&= T_{u\sigma(p)} \cdot \theta_e\left(\psi(\xi), d_p u(\xi) f(p)\right) + d_p(u(p)\sigma)(\xi),
\end{aligned}$$

thanks to (8.1). Using the equality

$$d_p(\lambda\sigma) = d_{\sigma(p)}(\lambda\mathbb{I}_E) \cdot d_p\sigma,$$

for any $\lambda \in \mathbb{R}$, we conclude the required identity

$$d_p(u\sigma)(\xi) = d_p u(\xi) T_{u\sigma(p)}\sigma(p) + d_{\sigma(p)}[u(p)\mathbb{I}_E] \cdot d_p\sigma(\xi).$$

□

The induced connection ∇^ψ over $\psi^* E$ is defined by the formula

$$\begin{aligned}
\nabla^\psi_\xi \sigma &:= T^{-1}_{\sigma(p)} \gamma^{\mathcal{H}}_{\sigma(p)} d_p\sigma(\xi) \\
&= T^{-1}_{\sigma(p)}\left[d_p\sigma(\xi) - H_{\sigma(p)} d_p\psi(\xi)\right],
\end{aligned}$$

for any $\xi \in T_{N,p}$. It is obvious that the additive property of ∇^ψ follows from the condition (8.2). We show now that ∇^ψ satisfies the Leibniz property

$$\nabla^\psi_\xi(u\sigma) = d_p u(\xi)\sigma(p) + u(p)\nabla^\psi_\xi\sigma.$$

Indeed using lemma 17 and the identity (8.5) we have

$$\begin{aligned}
\nabla^{\psi}_{\xi}(u\sigma) &= T^{-1}_{u\sigma(p)}\gamma^{\mathcal{H}}_{u\sigma(p)}d_p(u\sigma)(\xi)\\
&= T^{-1}_{u\sigma(p)}\gamma^{\mathcal{H}}_{u\sigma(p)}\left[d_pu(\xi)\ T_{u\sigma(p)}\sigma(p)+d_{\sigma(p)}[u(p)\,\mathbb{I}_E]d_p\sigma(\xi)\right]\\
&= d_pu(\xi)\,\sigma(p)+T^{-1}_{u\sigma(p)}\left[d_{\sigma(p)}[u(p)\,\mathbb{I}_E]d_p\sigma(\xi)\ -H_{u\sigma(p)}d_p\psi(\xi)\right]\\
&= d_pu(\xi)\,\sigma(p)\\
&+ T^{-1}_{u\sigma(p)}\left[d_{\sigma(p)}[u(p)\,\mathbb{I}_E]d_p\sigma(\xi)\ -d_{\sigma(p)}[u(p)\,\mathbb{I}_E]H_{\sigma(p)}d_p\psi(\xi)\right]\\
&= d_pu(\xi)\,\sigma(p)+T^{-1}_{u\sigma(p)}\left[d_{\sigma(p)}[u(p)\,\mathbb{I}_E]\gamma^{\mathcal{H}}_{\sigma(p)}d_p\sigma(\xi)\right]\\
&= d_pu(\xi)\,\sigma(p)+T^{-1}_{u\sigma(p)}\left[u(p)\,\gamma^{\mathcal{H}}_{\sigma(p)}d_p\sigma(\xi)\right]\\
&= d_pu(\xi)\,\sigma(p)+u(p)\,\nabla^{\psi}_{\xi}\sigma\,.
\end{aligned}$$

We observe also that for any $s\in C^{\infty}(M,E)$ and $\xi\in T_{N,p}$ we have the equalities

$$\begin{aligned}
\nabla^{\psi}_{\xi}(\psi^*s) &= T^{-1}_{s\circ\psi(p)}\gamma^{\mathcal{H}}_{s\circ\psi(p)}d_{\psi(p)}s\cdot d_p\psi(\xi)\\
&= \nabla s(\psi(p))\cdot d_p\psi(\xi),
\end{aligned}$$

in other terms the functorial formula

$$\nabla^{\psi}(\psi^*s)=\psi^*(\nabla s), \tag{8.6}$$

holds.

8.2.1 The induced connection (second approach)

We observe that the tangent space of ψ^*E at the point (y,η) is given by the equality

$$T_{\psi^*E,(y,\eta)} = \left\{(\xi,\theta)\in T_{N,y}\oplus T_{E,\eta}\mid d_y\psi(\xi)=d_\eta\pi_E(\theta)\right\}.$$

Given any horizontal distribution $H\in C^{\infty}(E,\pi_E^*T^*_M\otimes T_E)$ over E, we define the horizontal distribution

$$H^{\psi}:=\Psi^*H \in C^{\infty}\left(\psi^*E,\pi^*_{\psi^*E}T^*_N\otimes T_{\psi^*E}\right).$$

In explicit terms

$$H^{\psi}_{(y,\eta)} = \mathbb{I}_{T_{N,y}}\oplus H_\eta\cdot d_y\psi.$$

If H satisfies the identities (8.2) and (8.5) then so does H^{ψ}. This follows indeed from the identities

$$\begin{aligned} d_{(y,\eta_1,\eta_2)}\left(sm_{_{\psi^*E}}\right) &= \mathbb{I}_{T_{N,y}} \oplus d_{(\eta_1,\eta_2)}\left(sm_{_E}\right), \\ d_{(y,\eta)}(\lambda \mathbb{I}_{\psi^*E}) &= \mathbb{I}_{T_{N,y}} \oplus d_{\eta}\left(\lambda \mathbb{I}_E\right). \end{aligned}$$

By definition of H^{ψ} we infer that the induced connection ∇^{ψ} over ψ^*E satisfies the formula

$$\nabla^{\psi}_{\xi}\sigma = T^{-1}_{\sigma(y)} \cdot \left[d_y\sigma\left(\xi\right) - H_{\sigma(y)} \cdot d_y\psi\left(\xi\right) \right],$$

for any $\xi \in T_{N,y}$.

The local frame e induces a local frame $\eta := e \circ \psi$ of ψ^*E over $\psi^{-1}\left(U\right)$. We compute the local connection A^{ψ} form of ∇^{ψ} with respect to such frame. We notice that $\nabla^{\psi}\eta = \psi^*\left(e \cdot A\right) = \eta \cdot \psi^*A$ by the previous remark. We infer the equality $A^{\psi} = \psi^*A$.

8.2.2 Parallel transport

We consider a smooth curve $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ and a section $\sigma \in C^1\left((-\varepsilon,\varepsilon), \gamma^*E\right)$ which satisfies the equation

$$\nabla^{\gamma}_{\frac{d}{dt}}\sigma = 0,$$

over $(-\varepsilon, \varepsilon)$ with $\sigma\left(0\right) = \eta \in E_{\gamma(0)}$. If we write $\sigma\left(t\right) = e\left(\gamma\left(t\right)\right) \cdot f\left(t\right)$ then

$$\nabla^{\gamma}_{\frac{d}{dt}}\sigma = e\left(\gamma\left(t\right)\right) \cdot \left[\dot{f}\left(t\right) + A\left(\dot{\gamma}\left(t\right)\right) \cdot f\left(t\right) \right].$$

We infer that the parallel transport map $\tau_{\gamma,t} : E_{\gamma(0)} \longrightarrow E_{\gamma(t)}$, $t \in (-\varepsilon, \varepsilon)$ given by $\tau_{\gamma,t}\left(\eta\right) = \sigma\left(t\right)$, is linear. We show the following fact.

Lemma 18 *For any smooth curve $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ and for any section $\sigma \in C^1\left((-\varepsilon,\varepsilon), \gamma^*E\right)$, holds the identity*

$$\nabla^{\gamma}_{\frac{d}{dt}}\sigma\left(0\right) = \frac{d}{dt}_{|_{t=0}} \left[\tau^{-1}_{\gamma,t} \cdot \sigma\left(t\right) \right]. \tag{8.7}$$

Proof We notice first that the term $\tau^{-1}_{\gamma,t} \cdot \sigma\left(t\right)$ is given by the intrinsic identities

$$\begin{aligned} \frac{du_t}{ds} + A\left(\dot{\gamma}\left(s\right)\right) \cdot u_t\left(s\right) &= 0, \\ u_t\left(t\right) &= f\left(t\right), \\ e\left(\gamma\left(0\right)\right) \cdot u_t\left(0\right) &= \tau^{-1}_{\gamma,t} \cdot \sigma\left(t\right). \end{aligned}$$

Integrating the first equation we infer

$$u_t\left(t\right) - u_t\left(0\right) = -\int_0^t A\left(\dot{\gamma}\left(s\right)\right) \cdot u_t\left(s\right) ds.$$

Using the second equation we obtain

$$f(t) - u_t(0) = -\int_0^t A(\dot{\gamma}(s)) \cdot u_t(s)\, ds.$$

Deriving with respect to the variable t we obtain

$$\begin{aligned} \frac{d}{dt} u_t(0) &= \dot{f}(t) + A(\dot{\gamma}(t)) \cdot u_t(t) \\ &= \dot{f}(t) + A(\dot{\gamma}(t)) \cdot f(t). \end{aligned}$$

Evaluating at $t = 0$ and multiplying both sides with $e(\gamma(0))$ we infer the required conclusion. □

We consider now a C^1-vector field ξ over M and let $\varphi_{\xi,t}$ be the associated 1-parameter subgroup of transformations of M. Let $\Phi_{\xi,t} : E \longrightarrow E$ be the parallel transport map along the flow lines of $\varphi_{\xi,t}$. It is obvious by definition, that the map $\Phi_{\xi,t}$ satisfies $\pi_E \circ \Phi_{\xi,t} = \varphi_{\xi,t} \circ \pi_E$.

The vector field $\Xi := \dot{\Phi}_{\xi,0}$ over E satisfies the equality $\Xi(\eta) = H_\eta(\xi)$, for any $\eta \in E$. This is a direct consequence of the definition of the induced connection along the flow lines of ξ.

To any section $\sigma \in C^1(M, E)$ we can associate a C^1-vector field Σ over E defined as $\Sigma(\eta) := T_\eta[\sigma \circ \pi_E(\eta)]$. Let $\Phi_{\Sigma,t}$ be the associated 1-parameter subgroup of transformations of E. In explicit terms it satisfies

$$\Phi_{\Sigma,t}(\eta) = \eta + t\sigma \circ \pi_E(\eta)\ .$$

Then

$$[\Xi, \Sigma] = \frac{d}{dt}_{|_{t=0}} \frac{d}{ds}_{|_{s=0}} (\Phi_{\xi,-t} \circ \Phi_{\Sigma,s} \circ \Phi_{\xi,t}).$$

The fact that the map $\Phi_{\xi,-t}$ is linear on the fibers implies

$$\begin{aligned} \Phi_{\xi,-t} \circ \Phi_{\Sigma,s} \circ \Phi_{\xi,t} &= \Phi_{\xi,-t}[\Phi_{\xi,t} + s\sigma \circ \pi_E \circ \Phi_{\xi,t}] \\ &= \mathbb{I}_E + s\Phi_{\xi,-t} \cdot \sigma \circ \pi_E \circ \Phi_{\xi,t} \\ &= \mathbb{I}_E + s\Phi_{\xi,-t} \cdot \sigma \circ \varphi_{\xi,t} \circ \pi_E. \end{aligned}$$

Thus for any $\eta \in E_p$ holds

$$\Phi_{\xi,-t} \circ \Phi_{\Sigma,s} \circ \Phi_{\xi,t}(\eta) = \eta + s\Phi_{\xi,-t} \cdot \sigma \circ \varphi_{\xi,t}(p) \in E_p.$$

We conclude

$$\begin{aligned} [\Xi, \Sigma](\eta) &= \frac{d}{dt}_{|_{t=0}} T_\eta \left[\Phi_{\xi,-t} \cdot \sigma \circ \varphi_{\xi,t}(p)\right] \\ &= T_\eta[\nabla_\xi \sigma(p)], \end{aligned}$$

i.e for any $\eta \in E$ the equality holds

$$[\Xi, \Sigma](\eta) = T_\eta\left[(\nabla_\xi \sigma) \circ \pi_E(\eta)\right]. \tag{8.8}$$

Iterating twice we deduce the identity

$$\Big[\Xi_1,[\Xi_2,\Sigma]\Big](\eta)=T_\eta\Big[(\nabla_{\xi_1}\nabla_{\xi_2}\sigma)\circ\pi_E(\eta)\Big]. \tag{8.9}$$

Moreover the fact that by (8.8) the vector fields $[\Xi_j,\Sigma]$, $j=1,2$ are tangent to the fibers of E and constant along them implies

$$\Big[[\Xi_1,\Sigma],[\Xi_2,\Sigma]\Big]=0. \tag{8.10}$$

8.3 The geometric meaning of the curvature tensor

Lemma 19 *Let $R:=\nabla^2$ be the curvature tensor of the connection ∇. Then for any vector fields ξ_1,ξ_2 over M and for any $\eta\in E$ the identity holds*

$$\gamma_\eta^\nabla\Big([\Xi_1,\Xi_2](\eta)\Big)=T_\eta[R(\xi_2,\xi_1)\eta].$$

Proof Let σ be a local section of E such that $\sigma(p)=\eta$. By definition of horizontal lift Ξ of a vector field ξ we have

$$\Xi(\eta) \;=\; [d\sigma(\xi)]\circ\pi_E(\eta)-T_\eta\Big[(\nabla_\xi\sigma)\circ\pi_E(\eta)\Big].$$

We infer by (8.8) the identity

$$[d\sigma(\xi)]\circ\pi_E \;=\; \Xi+[\Xi,\Sigma].$$

We infer $\sigma_*\xi=\Xi+[\Xi,\Sigma]$ over $\operatorname{Im}\sigma$. Thus

$$\begin{aligned}\sigma_*[\xi_1,\xi_2] &= [\sigma_*\xi_1,\sigma_*\xi_2]\\ &= [\Xi_1,\Xi_2]+\Big[\Xi_1,[\Xi_2,\Sigma]\Big]+\Big[[\Xi_1,\Sigma],\Xi_2\Big],\end{aligned}$$

thanks to (8.10). We rewrite the previous equality as

$$[\Xi_1,\Xi_2] \;=\; \Big[\Xi_2,[\Xi_1,\Sigma]\Big]-\Big[\Xi_1,[\Xi_2,\Sigma]\Big]-\sigma_*[\xi_2,\xi_1].$$

Using (8.9) we deduce

$$\begin{aligned}[\Xi_1,\Xi_2](\eta) &= T_\eta\Big[(\nabla_{\xi_2}\nabla_{\xi_1}\sigma-\nabla_{\xi_1}\nabla_{\xi_2}\sigma)(p)\Big]-d_p\sigma([\xi_2,\xi_1])\\ &= T_\eta\Big[(\nabla_{\xi_2}\nabla_{\xi_1}\sigma-\nabla_{\xi_1}\nabla_{\xi_2}\sigma-\nabla_{[\xi_2,\xi_1]}\sigma)(p)\Big]-H_\eta([\xi_2,\xi_1])\\ &= T_\eta[R(\xi_2,\xi_1)\sigma(p)]+H_\eta([\xi_1,\xi_2]).\end{aligned}$$

We infer the required conclusion. □

References

[ABI] Aslam, V., Burns, D. M., Irvine, D. *Left-invariant Grauert tubes on SU(2)*, Q. J. Math. 69 (2018), no. 3, 871-885.

[Bu1] Burns, D. M. *On the uniqueness and characterization of Grauert tubes*, Complex analysis and geometry (Trento 1993), 119-133, Lecture Notes in Pure and Appl. Math., 173, Dekker, New York, (1996).

[Bu2] Burns, D. M. *Symplectic geometry and the uniqueness of Grauert tubes*, Geom. Funct. Anal. 11, (2001), no.1,1-10.

[BHH] Burns, D. M., Halverscheid, S., Hind, R. *The geometry of Grauert tubes and complexification of symmetric spaces*, Duke Math. J. 118 (2003), no. 3, 465-491.

[Bie] Bielawski, R. *Complexification and hypercomplexification of manifolds with a linear connection*, Internat. J. Math. 14, (2003), 813-824.

[Br-Wh] Bruhat, F., Whitney, H. *Quelque propriétés fondamentales des ensembles analytiques-réels*, comment. Math. Helv. 33, (1959), 132-160.

[Dom] Dombrowsky, P. *On the geometry of the tangent bundle*, J.Reine Angew. Math. 210, (1962), 73-88.

[Gau] Gauduchon, P. *Connecxions linéaires, classes de Chern, théorème de Riemann-Roch*, Holomorphic curves in symplectic geometry, 113-162, Progr. Math., 117, Birkhäuser, Basel, (1994).

[Gra] Grauert, H. *On Levi's problem and the imbedding of real analytic manifolds*, Ann. of Math. (2), 68, (1958), 460-472.

[Gu-St] Guillemin, V., Stenzel, M. *Grauert tubes and the homogeneous Monge-Ampère equation*, J. Differential Geom. 34, (1991), 561-570.

[Ha-Ki] Hall B. C, Kirwin W. D. *Adapted complex structures and the geodesic flow*, Math. Ann. 350, (2011), 455-474.

[Hir] Hirsch, M. W. *Differential topology*, GTM 33. Springer Verlag, (1991).

[Kli] Klingenberg, W. *Riemannian Geometry*, Walter de Gruyter, Berlin, (1982).

[Lem] Lempert, L. *Complex structures on the tangent bundle of Riemannian manifolds*, Complex analysis and Geometry , Ancona V., Silva A. (eds), The university Series in Mathematics. Springer, boston, MA, (1993), 235-251.

[Le-Sz1] Lempert, L., Szöke, R. *Global solutions of the homogeneous complex Monge-Ampère equation and complex structures on the tangent bundle of Riemannian manifolds*, Math. Ann. 290, (1991), 689-712.

[Le-Sz2] Lempert, L., Szöke, R. *A new look at adapted complex structures*, Bulletin of the London Math. Soc. 44, (2012), 367-374.

[Mo-Nu] Mourao, J. M., Nunes, J. P. *On Complexified Analytic Hamiltonian Flows and Geodesics on the space of Kähler Metrics*, International Mathematics Research notices, 20, (2015), 10624-10656.

[Szo1] Szöke, R. *Complex structures on tangent bundles of Riemannian manifolds*, Math. Ann. 291, (1991), 409-928.

[Szo2] Szöke, R. *Canonical complex structures associated to connections and complexifications of Lie groups*, Math. Ann. 328, (2004), 553-591.

[Zel] Zelditch, S. *Pluri-potential theory on Grauert tubes of real analytic Riemannian manifolds, I*, Spectral geometry, 299-339, Proc. Sympos. Pure Math., 84, AMS (2012).

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