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The center value : a sharing rule for cooperative games on acyclic graphs

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Abstract

A cooperative game with transferable utility is a situation in which players can generate worth by cooperating and obtain a certain payoff via a sharing rule. In this paper we assume that the cooperation between players is restricted by an acyclic graph. We introduce a new sharing rule for this type of game. We also offer an axiomatization of this sharing rule, based mainly on linearity and a consistency axiom. For unanimity games, this sharing rule identifies the center of the graph.

Keywords game theory, cooperative games, graphs, associated consistency, centrality measures

1 Introduction

Cooperation restricted games structured by a graph have first been introduced by Myerson (1977, [18]). In his article, he extended the Shapley Value (1953, [20]) to situations in which communication was incomplete. In this framework, direct communication between two players is feasible only if an edge is linking these two players. Coalitions can only form between connected players. This work set the path for numerous different extensions of the Shapley value on communication structures. We can cite, among others, the Position value (Borm *et al.*, 1992 [5]), its generalization (Ghintran, 2010 [7]) and two sharing rules introduced by Hamiache (1999, [9] ; 2004, [11]). Another popular sharing rule defined on the same framework is the Average Tree solution proposed by Herings *et al.* (2008, [15]). The restriction of communication was also studied using other frameworks. Coalition structures (Aumann and Dreze, 1974, [2]) partition players into pre-determined coalitions that can cooperate. Conference structures (Myerson, 1980 [19])

allow for a more flexible framework in which players can belong to several conferences, and can be modeled as an hypergraph. Permission structures (van den Brink and Gilles, 1996 [4]) restrict cooperation by the positions of the players in a hierarchical framework. Games on antimatroids were also developed (Algaba *et al.*, 2003 [1]). In the present work we will focus solely on communication structures as introduced in Myerson (1977, [18]). Similarly to the work of Herings *et al.* (2008, [15]), we will focus on acyclic graphs.

As surveyed by Thomson (2006, [21]), the concept of consistency has played a fundamental role in the axiomatic approach in game theory, economics and political science. This concept is built upon the idea that the sharing rule considered must give the same payoff when applied to a game and some modification of that game. In one of his works (2001, [10]), Hamiache builds in this fashion a game associated to the original game and shows that the associated game converges to an inessential game, which is then solved by the use of the axioms of inessentiality and continuity. This way, he characterized the Shapley value by relying mainly on consistency and inessential games. Hamiache and Navarro (2019, [13]) recently extended the result of 2001 to games with restricted communication and offered a new sharing rules characterized using the same axioms used in 2001 to characterize the Shapley value.

In a previous work, Hamiache (1999, [9]) already relied on consistency, used in a very specific manner. He introduced an associated game that depends on the sharing rule itself. In addition to the use of other axioms, finding the fixed point in the resulting equation gave the formula for a new sharing rule without relying on convergence and continuity¹. The present work builds upon the work of 1999. We use the axioms of linearity, independence of irrelevant players² and associated consistency already used by Hamiache. In addition, we use an axiom of initial conditions (similar in spirit to the Standardness axiom used by Hart and Mas-Colell (1989, [14])) and efficiency. Moreover, we define a different associated game which allows us to characterize a (widely) different sharing rule than the ones existing in the literature.

Since its introduction by Hamiache in 1999 and 2001, the axiom of associated consistency has been proven a powerful tool for characterizing different sharing rules of the literature. Hwang (2006, [16]) used it to characterized

¹For a discussion on the uniqueness of this sharing rule, please refer to Bilbao *et al.* (2006, [3]).

²We use the version of the axiom used in Hamiache 2011, [12].

the equal allocation of non-separable costs. Driessen (2010, [6]) proposed a characterization of the family of symmetric, efficient and linear sharing rules and Xu *et al.* (2013, [22]) characterized the Equal Surplus Division value. The ability to tailor the associated game in order to offer characterization of different sharing rules gives to this principle its flexibility and usefulness.

The problematic of sharing rules on games with communication structures is closely related to the concept of centrality in graphs. In their article, Gomez *et al.* (2003, [8]) investigate centrality with a game theoretical approach. Their approach is similar to ours, however, whereas the present work uses game theoretical axioms to characterize a known centrality concept (the center), their article explores properties of a centrality measure build upon the Shapley value that are linked to centrality issues.

In a unanimity game over the grand coalition, the only asymmetry present is induced by the graph, since all players are equal in the game. Hence, sharing the generated worth of 1 between the players is then similar to measuring the position of the players in the graph, *i.e.* measuring their centrality. This issue of centrality measures is all the more obvious if we consider (as we do in this article) of linear sharing rules since finding a solution on unanimity games leads to a solution for all games (Hamiache, 1999 [9]). The simplest method used to measure centrality is to identify the center of the graph. In this work we introduce a sharing rule for games on acyclic graph which, when applied to unanimity games, gives the same payoff to all players in the center of the graph and gives nothing to players outside the center. Hence we are able to offer a definition of the center of a graph in a cooperative game theoretic approach relying mainly on linearity and consistency. We also prove that the axioms used in the characterization are independent. The results we obtain are widely different from those provided by other sharing rules proposed in the literature such as the Average tree value (Herings *et al.* 2008, [15]) and the Myerson value (1977, [18]) in that they give more importance to the information from the graph over information from the game.

2 The formal framework

Let U be a non-empty and finite set of players. A coalition is a non-empty subset of U . We write (N, v) to represent a coalitional game with transferable utility (TU-game) where $N \subseteq U$ is the set of players and $v : 2^N \rightarrow \mathbb{R}$ (with $v(\emptyset) = 0$) is the characteristic function of the game. Let Γ be the set of all such games. For any $S \subsetneq N$ we will write $(S, v|_S)$ the sub-game

restricted to coalition S with $v|_S(T) = v(T)$, $\forall T \subseteq S$.

Given a coalition $R \subseteq N$, $R \neq \emptyset$, we call a unanimity game, a game (N, u_R) with $u_R(S) = 1$ if $R \subseteq S$ and $u_R(S) = 0$ otherwise. We call null game, a game (N, v) where $v(S) = 0$, $\forall S \subseteq N$. We write the null game over coalition N as $(N, \mathbf{0})$.

Let $g_N = \{\{i, j\} | i \neq j, i, j \in N\}$ be the set of all possible links between two players in N . We denote as $g \subseteq g_N$ a subset of those links. We call a graph a pair (N, g) where the vertices of the graph represent the players in N and the edges represent the ability for two players to communicate. For all $S \subseteq N$ and all $g \subseteq g_N$, we define $g(S) = \{\{i, j\} | i \in S, j \in S, \{i, j\} \in g\}$ the links of the subgraph $(S, g(S))$ induced by coalition S . We write $g_i = \{\{i, j\} \in g | j \in N\}$, the set of links of player i . Given a set $S \subseteq N$, we define by $S^* = \{i \in N | \exists j \in S \text{ such that } \{i, j\} \in g\} \cup S$ the set of all players connected to at least one player of S . We call a path between two vertices i and $j \in N$, a string of vertices $i = i_1, i_2, \dots, i_k = j$ such that, $\forall q$ with $1 \leq q \leq k - 1$, we have $\{i_q, i_{q+1}\} \in g$. If there exists a path between two vertices of N , we say they are connected by the graph (N, g) . We symbolize the connectedness of two players $i, j \in N$ by the graph (N, g) with the following notation : $i \xrightarrow{(N, g)} j$. For all graphs (N, g) , we denote

$N/g = \left\{ \{i | i \xrightarrow{(N, g)} j, i \in N\} \cup \{j\} | j \in N \right\}$ as the set of the components of N . A graph is connected if $|N/g| = 1$. A subgraph $(S, g(S))$ is connected if $|S/(g(S))| = 1$.

We say that a graph g is an acyclic graph if two distincts vertices are connected by exactly one path. In this article we will only consider connected acyclic graphs. We say that a vertex $i \in N$ is an extremity in an acyclic graph g if $|g_i| = 1$. We write $X(g) = \{i \in N | |g_i| = 1\}$ the set of extremities of graph g . When there is no ambiguity we write this set simply X . We also note x the cardinal of X .

We define the eccentricity of a given vertex i as $\xi_i^g = \max_j \{d(i, j) | j \in N\}$, with $d(i, j)$ the number of edges between i and j . The radius of the graph (N, g) is written $r(g) = \min_{i \in N} \xi_i^g$. We note $C(g) = \{i \in N | \xi_i^g = r(g)\}$, the set of vertices in the center of the graph, *i.e.* the set of vertices of minimal eccentricity.

A game with communication structure is a triplet (N, v, g) , where (N, v) is a TU-game and (N, g) is a graph. When the graph is acyclic (which will be the case in this paper) we will call them cooperative games on acyclic graphs. We shall write as Γ_G the set of those games. We define the quotient

game v/g as

$$(v/g)(S) = \sum_{R \in S/(g(S))} v(R), \quad \forall S \subseteq N. \quad (1)$$

The worth of a coalition in the quotient game is the sum of the worth of its components in the original game.

We call sharing rule on Γ a function ψ that associates to each game $(N, v) \in \Gamma$ a vector $\psi(N, v) \in \mathbb{R}^N$. In a similar way, we define a sharing rule for cooperative games on acyclic graphs as a function ϕ which associates with each game $(N, v, g) \in \Gamma_G$ a vector $\phi(N, v, g) \in \mathbb{R}^N$.

3 Center of acyclic subgraphs

Let (N, g) be an acyclic graph, we know that $1 \leq |C(g)| \leq 2$ (Jordan, 1869 [17]). In this section we are interested in the center of specific subgraphs of (N, g) .

Lemma 1

Let (N, g) be an acyclic graph. For any player $i \in N$, if a vertex $k \in N$ is such that $k \in \underset{j}{\operatorname{argmax}}\{d(i, j) \mid j \in N\}$ then $k \in X(g)$.

Proposition 1

Let (N, g) be an acyclic graph. The center of the subgraph obtained by deleting all the graph extremities is the same as the center of the original graph and the radius is equal to the radius of the original graph minus one. Formally, $C(g(N \setminus X(g))) = C(g)$ and $r(g(N \setminus X)) = r(g) - 1$.

Proof : The eccentricity of player i is defined by

$$\xi_i^g = \underset{j}{\max}\{d(i, j) \mid j \in N\}.$$

By lemma 1 we know that, $\forall i \in N$, the vertices $k \in N$ such that

$$k \in \underset{j}{\operatorname{argmax}}\{d(i, j) \mid j \in N\}$$

are extremities of (N, g) . Consequently, the eccentricity of player i can be defined as

$$\xi_i^g = \underset{x_0}{\max}\{d(i, x_0) \mid x_0 \in X(g)\}.$$

Therefore, by deleting the extremities $X(g)$ from graph g , the eccentricity in graph $g(N \setminus X(g))$ of every player $i \in N \setminus X(g)$ will be

$$\xi_i^g = \max_{x_0} \{d(i, x_0) \mid x_0 \in X(g(N \setminus X(g)))\}.$$

Every player's eccentricity will then be reduced by 1. Thus,

$$C(g) = C(g(N \setminus X(g))) \text{ and } r(g(N \setminus X)) = r(g) - 1.$$

□

Proposition 2

Let (N, g) be an acyclic graph. Let $(R, g(R))$ be a subgraph of g with $N \setminus X \subseteq R \subseteq N$. There exists coalitions R such that $C(g(R)) \neq C(g)$ and

- if $|C(g)| = |C(g(R))|$ then $C(g) = C(g(R))$,
- if $|C(g)| = 1$ and $|C(g(R))| = 2$ then $C(g) \subsetneq C(g(R))$,
- if $|C(g)| = 2$ and $|C(g(R))| = 1$ then $|C(g) \cap C(g(R))| = 1$.

Proof : We consider any graph (N, g) . Let c be a player in $C(g)$ and R be a coalition such that $N \setminus X \subseteq R \subseteq N$. We will prove that there exists no player $c' \in R \setminus C(g)$ such that $\xi_{c'}^{g(R)} < \xi_c^{g(R)}$. Let us engage in *reductio ad absurdum* and consider a player c' such that $\xi_{c'}^{g(R)} < \xi_c^{g(R)}$. From proposition 1 we know that $\xi_c^{g(N \setminus X)} = r(g) - 1$ and that $c \in C(g(N \setminus X))$, therefore we would obtain

$$r(g) - 1 = \xi_c^{g(N \setminus X)} < \xi_{c'}^g - 1 = \xi_{c'}^{g(N \setminus X)} \leq \xi_{c'}^{g(R)} < \xi_c^{g(R)} \leq \xi_c^g = r(g),$$

from which we could conclude

$$r(g) - 1 < \xi_{c'}^{g(R)} < r(g).$$

Since the radius of a graph and the eccentricity of a player are integers, this relation is not possible. Therefore there exists no player $c' \in R \setminus C(g)$ such that $\xi_{c'}^{g(R)} < \xi_c^{g(R)}$ for $R, N \setminus X \subseteq R \subseteq N$. Consequently, if $c \in C(g)$ then $\xi_c^{g(R)} \leq \xi_i^{g(R)}$, $\forall i \in R \setminus C(g)$, $\forall R, N \setminus X \subseteq R \subseteq N$ and thus we can make the following conclusions :

- if $|C(g)| = |C(g(R))|$ then $C(g) = C(g(R))$,
- if $|C(g)| = 1$ and $|C(g(R))| = 2$ then $C(g) \subsetneq C(g(R))$,
- if $|C(g)| = 2$ and $|C(g(R))| = 1$ then $|C(g) \cap C(g(R))| = 1$.

which concludes the proof of proposition 2. □

4 The main value

Given a sharing rule ϕ , for all cooperative games on acyclic graphs (N, v, g) we define its associated game (N, v_ϕ^*, g) so that, for all $S \subseteq N$, we have,

$$v_\phi^*(S) = \begin{cases} v(S^*) - \sum_{j \in S^* \setminus S} \phi_j(S^* \cap \{j\}^*, v_{|_{S^* \cap \{j\}^*}}, g(S^* \cap \{j\}^*)) & \text{if } |S/g| = 1, \\ (v_\phi^*/g)(S) & \text{if } |S/g| > 1. \end{cases} \quad (2)$$

When there is no ambiguity we will write v_ϕ^* as simply v^* .

We offer the following interpretation for this associated game. Let S be connected. The world seen by coalition S is restricted to S^* . They aim to acquire the worth $v(S^*)$ but need to interact with all players in $j \in S^* \setminus S$. The world of j is limited to $\{j\}^*$. When interacting together, player j and coalition S are careful not to share any information that the other might not already have. Therefore they settle on a common view of the world represented by $S^* \cap \{j\}^*$, which allows to reach an agreement on a share of a potentially smaller pie than what is really at stake. For instance, if S is a large coalition with only one player connected to a certain player $j \notin S$ they will offer him the payoff he would get in a two-player game in exchange for his cooperation, since it is the only subgraph on which they can both agree. This allows well connected players or coalitions to hide part of their knowledge of the graph and avoid having to share a part of a bigger pie than necessary.

We now formulate our system of axioms.

Axiom 1 (*Associated consistency*) :

For all cooperative games on acyclic graphs (N, v, g) ,

$$\phi(N, v, g) = \phi(N, v_\phi^*, g).$$

Axiom 2 (*Efficiency*) :

For all cooperative games on acyclic graphs (N, v, g) ,

$$\sum_{i \in N} \phi_i(N, v, g) = v(N).$$

Axiom 3 (*Linearity with respect to games*) :

For all parameters $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ and all cooperative games on acyclic graphs (N, v, g) and (N, w, g) ,

$$\phi(N, \alpha v + \beta w, g) = \alpha \phi(N, v, g) + \beta \phi(N, w, g).$$

Axiom 4 (*Initial conditions*) :

Given the complete graph (N, g) where $|N| = 2$ and a game (N, v) in which the two players are equal,

$$\phi_1(N, v, g) = \phi_2(N, v, g).$$

In the game (N, v, g) , when $|N| = 2$, the two players are interchangeable in the graph and equal in the game. Hence we require their payoffs to be equal, which is in line with axioms commonly found in the literature, such as the equal treatment of equals.³

Axiom 5 (*Independence of Irrelevant Players*) :

Given a graph (N, g) , for all connected coalitions R ,

$$\phi_i(N, u_R, g) = \begin{cases} \phi_i(R, u_R, g(R)) & \text{if } i \in R, \\ 0 & \text{if } i \notin R. \end{cases}$$

³Note that we impose this requirements only in this specific case (as Hart and Mas-Colell did with their Standardness axiom ([14], 1989)), since it is strong enough for our main theorem.

The axiom of independence of irrelevant players has been used in two different forms by Hamiache (1999, [9] ; 2011, [12]). Players who are not members of coalition R receive a null payoff, as is usual in the literature for null players. In order to share the sum generated by players in R we also do not rely on the graph $(N \setminus R, g(N \setminus R))$.

We define the set $\Gamma_G/g = \{(N, v/g, g) \mid (N, v, g) \in \Gamma_G\}$. We know from Hamiache (1999, [9]) that the set of functions $\{u_R \mid R \subseteq N, |R/g| = 1\}$ is a basis of the set of functions $\{v \mid (N, v, g) \in \Gamma_G/g\}$.

Also from Hamiache (1999, [9]), using axioms 3 and 5, we can write, for all games $(N, v/g, g) \in \Gamma_G/g$,

$$\phi_i(N, v/g, g) = \sum_{\substack{S \subseteq N \\ |S/g|=1 \\ i \in S^*}} \sum_{\substack{R \\ S \subseteq R \subseteq S^*}} (-1)^{|R|-|S|} \phi_i(R, u_R, g(R)) v(S). \quad (3)$$

By definition of v^* we have that $(v/g)^* = v^*$ for all v . Using axiom 1, we obtain that

$$\phi(N, v, g) = \phi(N, v^*, g) = \phi(N, (v/g)^*, g) = \phi(N, v/g, g).$$

Therefore, for all games $(N, v, g) \in \Gamma_G$, we have

$$\phi_i(N, v, g) = \sum_{\substack{S \subseteq N \\ |S/g|=1 \\ i \in S^*}} \sum_{\substack{R \\ S \subseteq R \subseteq S^*}} (-1)^{|R|-|S|} \phi_i(R, u_R, g(R)) v(S). \quad (4)$$

Lemma 2

Let R and K be two coalitions such that $R, K \subseteq N$ with $\emptyset \neq K \subseteq R$,

$$\sum_{\substack{S \\ K \subseteq S \subseteq R}} (-1)^{|R|-|S|} = \begin{cases} 0 & \text{if } K \neq R, \\ 1 & \text{if } K = R. \end{cases}$$

Proof :

$$\sum_{\substack{S \\ K \subseteq S \subseteq R}} (-1)^{|R|-|S|} = \sum_{t=0}^{r-k} \binom{r-k}{t} (-1)^{r-k-t}.$$

Let us use Newton's binomial formula. If $K \neq R$ then

$$\sum_{t=0}^{r-k} \binom{r-k}{t} (-1)^{r-k-t} = (-1+1)^{r-k} = 0.$$

If $K = R$ then

$$\sum_{t=0}^0 \binom{0}{t} (-1)^{0-t} = (-1)^0 = 1.$$

□

Theorem 1

The center value for cooperative games with acyclic graphs defined as

$$CV_i(N, v, g) = \sum_{\substack{S \subseteq N \\ |S/g|=1 \\ i \in S^*}} \sum_{\substack{R \\ S \subseteq R \subseteq S^* \\ i \in C(g(R))}} (-1)^{|R|-|S|} \frac{1}{|C(g(R))|} v(S),$$

satisfies axioms 1 to 5.

Proof : First, it is obvious that the center value satisfies linearity with respect to games. Hence, we can infer from linearity that

$$CV_i(N, u_R, g) = \begin{cases} \frac{1}{|C(g(R))|} & \text{if } i \in R, \\ 0 & \text{else,} \end{cases}$$

for any $R \subseteq N$, $|R/g| = 1$.

From there simple computations show that $CV_i(N, u_R, g) = CV_i(R, u_R, g(R))$ if $i \in R$ and 0 if $i \notin R$ which proves that CV satisfies the independence of irrelevant players axiom. Direct computations show that CV also satisfies the initial conditions axiom. Let us show that CV is efficient. From linearity

with respect to games we can write

$$\begin{aligned}
\sum_{i \in N} CV_i(N, v, g) &= \sum_{i \in N} \sum_{\substack{S \subseteq N \\ |S/g|=1 \\ i \in S^*}} \sum_{\substack{R \\ S \subseteq R \subseteq S^*}} (-1)^{|R|-|S|} CV_i(N, u_R, g) v(S) \\
&= \sum_{\substack{S \subseteq N \\ |S/g|=1}} \sum_{i \in S^*} \sum_{\substack{R \\ S \subseteq R \subseteq S^*}} (-1)^{|R|-|S|} CV_i(N, u_R, g) v(S) \\
&= \sum_{\substack{S \subseteq N \\ |S/g|=1}} \sum_{\substack{R \\ S \subseteq R \subseteq S^*}} (-1)^{|R|-|S|} \sum_{i \in R} CV_i(R, u_R, g(R)) v(S)
\end{aligned}$$

The second equality is obtained by inverting the sums, the last equality stems from the independence of irrelevant players axiom. We know that the sums of the payoff of players in $(R, u_R, g(R))$ will be equal to one. Thus, using Newton's binomial formula,

$$\begin{aligned}
\sum_{i \in N} CV_i(N, v, g) &= \sum_{\substack{S \subseteq N \\ |S/g|=1}} \sum_{\substack{R \\ S \subseteq R \subseteq S^*}} (-1)^{|R|-|S|} v(S) \\
&= \sum_{\substack{S \subseteq N \\ |S/g|=1}} \sum_{r=|S|}^{|S^*|} \binom{|S^*|-|S|}{r-|S|} (-1)^{r-|S|} v(S) \\
&= \sum_{\substack{S \subseteq N \\ |S/g|=1}} \sum_{t=0}^{|S^*|-|S|} \binom{|S^*|-|S|}{t} (-1)^t v(S) \\
&= v(N).
\end{aligned}$$

which proves that CV is efficient.

The proof that CV satisfies the associated consistency is lengthy and can be found in the appendix. □

Theorem 2

If a sharing rule ϕ for cooperative games on acyclic graphs satisfies axioms 1 to 5 then

$$\phi_i(N, v, g) = CV_i(N, v, g), \quad \forall i \in N.$$

Proof :

We will prove this theorem by induction. By axiom 2, the theorem is true for one player games.

For two-players games, using equation (4) we have

$$\begin{aligned}\phi_1(N, v, g) &= v(1) + \phi_1(N, u_N, g) [v(12) - v(2) - v(1)], \\ \phi_2(N, v, g) &= v(1) + \phi_2(N, u_N, g) [v(12) - v(1) - v(2)].\end{aligned}$$

We also have the following associated game :

$$\begin{aligned}v^*(1) &= v(12) - \phi_2(N, v, g), \\ v^*(2) &= v(12) - \phi_1(N, v, g), \\ v^*(12) &= v(12).\end{aligned}$$

Using axiom 4 we know $\phi_1(N, u_N, g) = \phi_2(N, u_N, g) = a$. Therefore

$$\begin{aligned}\phi_1(N, v^*, g) &= v^*(1) + a [v^*(12) - v^*(2) - v^*(1)], \\ \phi_1(N, v^*, g) &= v(12) - v(2) - a(v(12) - v(2) - v(1)) \\ &\quad + a [-v(12) + v(1) + 2a(v(12) - v(2) - v(1)) + v(2)]\end{aligned}$$

Consequently

$$\begin{aligned}\phi_1(N, v^*, g) - \phi_1(N, v, g) &= 0 \\ &= [2a^2 - 3a + 1] v(12) + [-2a^2 + 3a - 1] v(2) + [-2a^2 + 3a - 1] v(1)\end{aligned}$$

Since this previous equation must be true for every game (N, v) we need to find a such that

$$1 - 3a + 2a^2 = 0.$$

Which yields two solutions : $a = 1$ and $a = \frac{1}{2}$. Since $a = 1$ is violating efficiency, the only solution is $a = \frac{1}{2}$. Therefore $\phi_1(N, u_N, g) = \phi_2(N, u_N, g) = \frac{1}{2}$.

For two-players game, the theorem is then true.

For three players games, the only acyclic graph is a path of three players. We consider then (N, g) with $N = \{1, 2, 3\}$ and $g = \{(1, 2), (2, 3)\}$. Using equation (4) and the payoffs for the two-players graph we have

$$\begin{aligned}\phi_1(N, v, g) &= v(1) + \left(\frac{1}{2} - \phi_1(N, u_N, g)\right) [v(12) - v(1) - v(2)] \\ &\quad + \phi_1(N, u_N, g) [v(123) - v(1) - v(23)].\end{aligned}$$

$$\begin{aligned}
\phi_2(N, v, g) &= v(2) + \left(\frac{1}{2} - \phi_2(N, u_N, g) \right) [v(12) - v(1) - v(2)] \\
&\quad + \left(\frac{1}{2} - \phi_2(N, u_N, g) \right) [v(23) - v(2) - v(3)] \\
&\quad + \phi_2(N, u_N, g) [v(123) - v(1) - v(2) - v(3)].
\end{aligned}$$

$$\begin{aligned}
\phi_3(N, v, g) &= v(3) + \left(\frac{1}{2} - \phi_3(N, u_N, g) \right) [v(23) - v(2) - v(3)] \\
&\quad + \phi_3(N, u_N, g) [v(123) - v(12) - v(3)].
\end{aligned}$$

We also have the following associated game :

$$\begin{aligned}
v^*(1) &= v(12) - \left[v(2) - \frac{1}{2}(v(12) - v(1) - v(2)) \right], \\
v^*(2) &= v(123) - \left[v(1) - \frac{1}{2}(v(12) - v(1) - v(2)) \right] - \left[v(3) - \frac{1}{2}(v(23) - v(2) - v(3)) \right], \\
v^*(3) &= v(23) - \left[v(2) - \frac{1}{2}(v(23) - v(2) - v(3)) \right], \\
v^*(12) &= v(123) - \left[v(3) - \frac{1}{2}(v(23) - v(2) - v(3)) \right], \\
v^*(13) &= v^*(1) + v^*(3), \\
v^*(23) &= v(123) - \left[v(1) - \frac{1}{2}(v(12) - v(1) - v(2)) \right], \\
v^*(123) &= v(123).
\end{aligned}$$

We have then

$$\begin{aligned}\phi_1(N, v^*, g) &= v^*(1) + \left(\frac{1}{2} - \phi_1(N, u_N, g)\right) [v^*(12) - v^*(1) - v^*(2)] \\ &\quad + \phi_1(N, u_N, g) [v^*(123) - v^*(1) - v^*(23)],\end{aligned}$$

$$\begin{aligned}\phi_2(N, v^*, g) &= v^*(2) + \left(\frac{1}{2} - \phi_2(N, u_N, g)\right) [v^*(12) - v^*(1) - v^*(2)] \\ &\quad + \left(\frac{1}{2} - \phi_2(N, u_N, g)\right) [v^*(23) - v^*(2) - v^*(3)] \\ &\quad + \phi_2(N, u_N, g) [v^*(123) - v^*(1) - v^*(2) - v^*(3)],\end{aligned}$$

$$\begin{aligned}\phi_3(N, v^*, g) &= v^*(3) + \left(\frac{1}{2} - \phi_3(N, u_N, g)\right) [v^*(23) - v^*(2) - v^*(3)] \\ &\quad + \phi_3(N, u_N, g) [v^*(123) - v^*(12) - v^*(3)].\end{aligned}$$

Hence the following system

$$\begin{cases} \phi_1(N, v^*, g) - \phi_1(N, v, g) = 0, \\ \phi_2(N, v^*, g) - \phi_2(N, v, g) = 0, \\ \phi_3(N, v^*, g) - \phi_3(N, v, g) = 0. \end{cases}$$

$$\Leftrightarrow \begin{cases} \phi_1(N, u_N, g)v(12) - \phi_1(N, u_N, g)v(2) - \phi_1(N, u_N, g)v(123) + \phi_1(N, u_N, g)v(23) = 0, \\ (1 - \phi_2(N, u_N, g))v(123) + (\phi_2(N, u_N, g) - 1)v(23) \\ + (\phi_2(N, u_N, g) - 1)v(12) + (1 - \phi_2(N, u_N, g))v(2) = 0, \\ \phi_3(N, u_N, g)v(23) - \phi_3(N, u_N, g)v(2) - \phi_3(N, u_N, g)v(123) + \phi_3(N, u_N, g)v(12) = 0. \end{cases}$$

This must be true for all v . Hence,

$$\Leftrightarrow \begin{cases} \phi_1(N, u_N, g) = 0, \\ \phi_2(N, u_N, g) = 1, \\ \phi_3(N, u_N, g) = 0. \end{cases}$$

We showed that the theorem is true for $n = 3$ players.

For $n = 4$ we have two possible graphs, a star graph and a line graph. For space and ease of reading, we will skip the proofs for $n = 4$ as they follow the same pattern as above.

We can therefore assume that the theorem is true for $n \leq 4$.

Let us assume now that the theorem is true for any graphs with at most $n - 1$ players, we will prove it is true for n -player games. We make the following hypothesis :

- $\phi_i(R, u_R, g(R)) = 1$ if $i \in C(g(R))$, $|C(g(R))| = 1$ and $R \subsetneq N$,
- $\phi_i(R, u_R, g(R)) = \frac{1}{2}$ if $i \in C(g(R))$, $|C(g(R))| = 2$ and $R \subsetneq N$,
- $\phi_i(R, u_R, g(R)) = 0$ if $i \notin C(g(R))$, $R \subsetneq N$.

By axiom 1 we have $\phi(N, v, g) = \phi(N, v^*, g)$ and therefore $\phi(R, u_R, g(R)) = \phi(R, (u_R)^*, g(R))$ for all connected coalitions $R \subseteq N$. We obtain

$$\begin{aligned} \phi_i(N, v^*, g) &= \phi_i(N, v, g) \\ &= \sum_{\substack{S \subseteq N \\ |S/g|=1 \\ i \in S^*}} \sum_{\substack{R \\ S \subseteq R \subseteq S^*}} (-1)^{|R|-|S|} \phi_i(R, (u_R)^*, g(R)) v(S). \end{aligned}$$

Since $\phi(N, v, g) = \phi(N, v^*, g)$ for all games, the coefficients of $v(S)$ must be equal in the two expressions $\forall S \subseteq N$. Using our induction hypothesis, the value of $\phi(R, (u_R)^*, g(R)) = \phi(R, u_R, g(R))$ are already known. So in order to satisfy axiom 1 we only need to find the value such that $\phi(N, (u_N)^*, g) = \phi(N, u_N, g)$.

By definition, $(u_N)^*(S)$, for a connected coalition S is given by,

$$(u_N)^*(S) = u_N(S^*) - \sum_{j \in S^* \setminus S} \phi_j(S^* \cap \{j\}^*, (u_N)_{|S^* \cap \{j\}^*}, g(S^* \cap \{j\}^*)),$$

From the definition of a unanimity game, $(u_N)_{|S^* \cap \{j\}^*}$ is a null game except when $S^* \cap \{j\}^* = N$. This is possible only if $\{j\}^* = N$ and $S^* = N$. Since g is acyclic, the only possible case where $\{j\}^* = N$ is when g is a star graph with j as its center. However in this case $|S/g| \neq 1$, for all $S \subseteq N \setminus \{j\}$. Therefore we obtain

$$(u_N)^*(S) = u_N(S^*).$$

We can write

$$\begin{aligned}
\phi_i(N, (u_N)^*, g) &= \sum_{\substack{S \subseteq N \\ |S/g|=1 \\ i \in S^*}} \sum_{\substack{R \\ S \subseteq R \subseteq S^*}} (-1)^{|R|-|S|} \phi_i(R, u_R, g(R)) u_N(S^*) \\
&= \sum_{\substack{S \subseteq N \\ |S/g|=1 \\ i \in S^* \\ S^*=N}} \sum_{\substack{R \\ S \subseteq R \subseteq N}} (-1)^{|R|-|S|} \phi_i(R, u_R, g(R)) = \phi_i(N, u_N, g)
\end{aligned} \tag{5}$$

We will split the rest of the proof in three different cases.

First case :

We know that in an acyclic connected graph, $1 \leq |C(g)| \leq 2$ (Jordan, 1869 [17]). We will now consider the case where $|C(g)| = 1$ and $i \in C(g)$.

The coalitions S such that $S \subseteq N$, $|S/g| = 1$, $i \in S^*$ and $S^* = N$ can be written as $N \setminus X \subseteq S \subseteq N$. Therefore, coalition S is always connected and i always belongs to S . From equation (5) we can write

$$\begin{aligned}
\phi_i(N, u_N, g) &= \sum_{\substack{S \\ N \setminus X \subseteq S \subseteq N}} \sum_{\substack{R \\ S \subseteq R \subseteq N}} (-1)^{|R|-|S|} \phi_i(R, u_R, g(R)) \\
&\quad + \sum_{\substack{S \\ N \setminus X \subseteq S \subseteq N}} \phi_i(N, u_N, g) (-1)^{|N|-|S|} \\
&= \sum_{\substack{S \\ N \setminus X \subseteq S \subseteq N}} \sum_{\substack{R \\ S \subseteq R \subseteq N}} (-1)^{|R|-|S|} \phi_i(R, u_R, g(R)) \\
&\quad + \sum_{s=n-x}^n \binom{x}{s-(n-x)} (-1)^{|N|-|S|} \phi_i(N, u_N, g).
\end{aligned}$$

By changing the running index, the second term becomes

$$\sum_{t=0}^x \binom{x}{t} (-1)^{x-t}.$$

Using Newton's binomial formula we know that this term is equal to 0. Consequently,

$$\phi_i(N, u_N, g) = \sum_{\substack{S \\ N \setminus X \subseteq S \subseteq N}} \sum_{\substack{R \\ S \subseteq R \subseteq N}} (-1)^{|R|-|S|} \phi_i(R, u_R, g(R)).$$

We know that, depending on elements in R , the center $C(g(R))$ can be different from $C(g)$. By separating the sums according to proposition 2 and using our induction hypothesis,

$$\phi_i(N, u_N, g) = \sum_{\substack{S \\ N \setminus X \subseteq S \subsetneq N}} \left[\sum_{\substack{R \\ S \subseteq R \subseteq N \\ \{i\} = C(g(R))}} (-1)^{|R|-|S|} + \sum_{\substack{R \\ S \subseteq R \subseteq N \\ |C(g(R))|=2}} \frac{1}{2} (-1)^{|R|-|S|} \right].$$

Since $\frac{1}{2} = 1 - \frac{1}{2}$,

$$\phi_i(N, u_N, g) = \sum_{\substack{S \\ N \setminus X \subseteq S \subsetneq N}} \left[\sum_{\substack{R \\ S \subseteq R \subsetneq N}} (-1)^{|R|-|S|} - \sum_{\substack{R \\ S \subseteq R \subseteq N \\ |C(g(R))|=2}} \frac{1}{2} (-1)^{|R|-|S|} \right],$$

$$\phi_i(N, u_N, g) = \sum_{\substack{S \\ N \setminus X \subseteq S \subsetneq N}} \sum_{\substack{R \\ S \subseteq R \subsetneq N}} (-1)^{|R|-|S|} - \sum_{\substack{S \\ N \setminus X \subseteq S \subsetneq N}} \sum_{\substack{R \\ S \subseteq R \subseteq N \\ |C(g(R))|=2}} \frac{1}{2} (-1)^{|R|-|S|}. \quad (6)$$

Let us focus for a bit on the first term. Inverting the order of sums, we obtain

$$\sum_{\substack{R \\ N \setminus X \subseteq R \subsetneq N}} \sum_{\substack{S \\ N \setminus X \subseteq S \subseteq R}} (-1)^{|R|-|S|}.$$

By using lemma 2 we know that $\sum_{\substack{S \\ N \setminus X \subseteq S \subseteq R}} (-1)^{|R|-|S|}$ is equal to 0 except

when $R = N \setminus X$. We then obtain

$$\sum_{\substack{R \\ N \setminus X \subseteq R \subsetneq N}} \sum_{\substack{S \\ N \setminus X \subseteq S \subseteq R}} (-1)^{|R|-|S|} = (-1)^{(n-x)-(n-x)} = 1.$$

Now we will prove that the second term of equation (6) is equal to 0. Again, let us invert the sums and the expression

$$\sum_{\substack{S \\ N \setminus X \subseteq S \subseteq N \\ |S/g|=1 \\ i \in S}} \sum_{\substack{R \\ S \subseteq R \subseteq N \\ |C(g(R))|=2}} \frac{1}{2} (-1)^{|R|-|S|}$$

can be written as

$$\sum_{\substack{R \\ N \setminus X \subseteq R \subseteq N \\ |C(g(R))|=2}} \sum_{\substack{S \\ N \setminus X \subseteq S \subseteq R}} \frac{1}{2}(-1)^{|R|-|S|}.$$

From lemma 2, we know that $\sum_{\substack{S \\ N \setminus X \subseteq S \subseteq R}} \frac{1}{2}(-1)^{|R|-|S|}$ is not equal to 0 only

if $R = N \setminus X$. However, we know that $C(g) = C(g(N \setminus X))$ and therefore $|C(g)| = |C(g(N \setminus X))| = 1$. Consequently, there exists no coalition R such that $R = N \setminus X$ and $|C(g(R))| = 2$. Therefore the second term of equation (6) is null and we obtain

$$\phi_i(N, u_N, g) = 1.$$

This concludes our first case.

Second case :

We assume that $|C(g)| = 2$ and $i \in C(g)$. Coalitions $S \subseteq N$, $|S/g| = 1$ such that $i \in S^*$ and $S^* = N$ can be written as $N \setminus X \subseteq S \subseteq N$. Therefore, coalition S is always connected and i always belongs to S . From equation (5) we can write

$$\begin{aligned} \phi_i(N, u_N, g) = & \sum_{\substack{S \\ N \setminus X \subseteq S \subsetneq N}} \sum_{\substack{R \\ S \subseteq R \subsetneq N}} (-1)^{|R|-|S|} \phi_i(R, u_R, g(R)) \\ & + \sum_{\substack{S \\ N \setminus X \subseteq S \subseteq N}} \phi_i(N, u_N, g) (-1)^{|N|-|S|}. \end{aligned}$$

Using the same steps as the previous case we can write

$$\phi_i(N, u_N, g) = \sum_{\substack{S \\ N \setminus X \subseteq S \subseteq N \\ |S/g|=1 \\ i \in S}} \sum_{\substack{R \\ S \subseteq R \subsetneq N}} (-1)^{|R|-|S|} \phi_i(R, u_R, g(R)).$$

From proposition 2 we know that there are subgraphs $g(R)$ in which $C(g(R)) = C(g)$, others in which $\{i\} = C(g(R))$ and finally some in which $i \notin C(g(R))$. Using our inductions hypothesis,

$$\phi_i(N, u_N, g) = \sum_{\substack{S \\ N \setminus X \subseteq S \subsetneq N}} \left[\sum_{\substack{S \subseteq R \subseteq N \\ |C(g(R))|=2 \\ i \in C(g(R))}} \frac{1}{2}(-1)^{|R|-|S|} + \sum_{\substack{S \subseteq R \subseteq N \\ \{i\} = C(g(R))}} (-1)^{|R|-|S|} \right].$$

Since $0 = \frac{1}{2} - \frac{1}{2}$ and $1 = \frac{1}{2} + \frac{1}{2}$ we write

$$\phi_i(N, u_N, g) = \sum_{\substack{S \\ N \setminus X \subseteq S \subsetneq N}} \left[\sum_{\substack{R \\ S \subseteq R \subsetneq N}} \frac{1}{2} (-1)^{|R|-|S|} + \sum_{\substack{R \\ S \subseteq R \subsetneq N \\ \{i\} = C(g(R))}} \frac{1}{2} (-1)^{|R|-|S|} - \sum_{\substack{R \\ S \subseteq R \subsetneq N \\ i \notin C(g(R))}} \frac{1}{2} (-1)^{|R|-|S|} \right].$$

From what we computed in the previous case we know that

$$\sum_{\substack{S \\ N \setminus X \subseteq S \subsetneq N \\ |S/g|=1 \\ i \in S}} \sum_{\substack{R \\ S \subseteq R \subsetneq N}} \frac{1}{2} (-1)^{|R|-|S|} = \frac{1}{2}.$$

So we are left with the coalitions R where $C(g(R))$ is composed of a single player. We will first focus on the second term,

$$\sum_{\substack{S \\ N \setminus X \subseteq S \subsetneq N}} \sum_{\substack{R \\ S \subseteq R \subsetneq N \\ \{i\} = C(g(R))}} \frac{1}{2} (-1)^{|R|-|S|}.$$

Inverting the order of sums, we obtain

$$\sum_{\substack{R \\ N \setminus X \subseteq R \subsetneq N \\ \{i\} = C(g(R))}} \sum_{\substack{S \\ N \setminus X \subseteq S \subsetneq R}} \frac{1}{2} (-1)^{|R|-|S|}.$$

From lemma 2, we know that $\sum_{\substack{S \\ N \setminus X \subseteq S \subsetneq R}} \frac{1}{2} (-1)^{|R|-|S|}$ is not equal to 0 only if

$R = N \setminus X$. However, if $R = N \setminus X$, then $|C(g(R))| = 2 \neq 1$ therefore this term cancels.

The same reasoning can be applied to the third term:

$$- \sum_{\substack{S \\ N \setminus X \subseteq S \subsetneq N}} \sum_{\substack{R \\ S \subseteq R \subsetneq N \\ i \notin C(g(R))}} \frac{1}{2} (-1)^{|R|-|S|} = 0.$$

Therefore we obtain $\phi_i(N, u_N, g) = \frac{1}{2}$ which concludes our second case.

Third case :

Let us consider now that $i \notin C(g)$. Coalitions $S \subseteq N$, $|S/g| = 1$ such that $i \in S^*$ and $S^* = N$ can be written as $N \setminus X \subseteq S \subseteq N$. Therefore, coalition S is always connected and i always belongs to S^* . From equation (5) we obtain

$$\begin{aligned} \phi_i(N, u_N, g) = & \sum_{\substack{S \\ N \setminus X \subseteq S \subseteq N}} \sum_{\substack{R \\ S \subseteq R \subseteq N}} (-1)^{|R|-|S|} \phi_i(R, u_R, g(R)) \\ & + \sum_{\substack{S \\ N \setminus X \subseteq S \subseteq N}} \phi_i(N, u_N, g) (-1)^{|N|-|S|}. \end{aligned}$$

Using the same steps as the previous cases we can write

$$\phi_i(N, u_N, g) = \sum_{\substack{S \\ N \setminus X \subseteq S \subseteq N}} \sum_{\substack{R \\ S \subseteq R \subseteq N}} (-1)^{|R|-|S|} \phi_i(R, u_R, g(R)).$$

If there exists no coalition R such that $i \in C(g(R))$ then from our hypothesis we conclude $\phi_i(N, u_N, g) = 0$. From proposition 2 we know there exist players $i \in N$ and coalitions R , $N \setminus X \subseteq R \subsetneq N$ such that $i \in C(g(R))$. We also know that this is only possible if $|C(g)| = 1$. In those cases then $\phi_i(R, u_R, g(R)) = \frac{1}{2}$ and $|C(g(R))| = 2$. Consequently,

$$\phi_i(N, u_N, g) = \sum_{\substack{S \\ N \setminus X \subseteq S \subseteq N}} \sum_{\substack{R \\ S \subseteq R \subseteq N \\ i \in C(g(R))}} \frac{1}{2} (-1)^{|R|-|S|}.$$

Inverting the order of sums,

$$\sum_{\substack{R \\ N \setminus X \subseteq R \subseteq N \\ i \in C(g(R))}} \sum_{\substack{S \\ N \setminus X \subseteq S \subseteq R}} (-1)^{|R|-|S|} \frac{1}{2}.$$

From lemma 2 we know that $\sum_{\substack{S \\ N \setminus X \subseteq S \subseteq R}} \frac{1}{2} (-1)^{|R|-|S|}$ is not equal to 0 only if

$R = N \setminus X$. However if $R = N \setminus X$ then $C(g(R)) = C(g)$ and therefore $i \notin C(g(R))$. Consequently, when $i \notin C(g)$,

$$\phi_i(N, u_N, g) = 0,$$

which concludes our last case and our proof. \square

5 Independence of axioms

In this section we prove that the five axioms used in our characterizations are independent.

Efficiency

Let us consider the sharing rule f cooperative games on acyclic graphs defined such that

$$f(N, v, g) = \frac{1}{2}CV(N, v, g),$$

Obviously this sharing rule satisfies all axioms but efficiency.

Linearity with respect to games

Let us consider the sharing rule f for cooperative games on acyclic graphs defined such that

$$f_i(N, v, g) = \begin{cases} 0 & \forall i \in N \text{ if } v = \delta_T, |T| > 2, T \subsetneq N, \\ CV_i(N, v, g) & \text{otherwise,} \end{cases}$$

where $\delta_T(S) = 1$ if $S = T$ and 0 otherwise (also known as a dirac game over coalition T). This sharing rule is not linear but is efficient and satisfies the initial conditions and the independence of irrelevant players axioms. We need to show it satisfies the associated consistency axiom. Obviously, for any $v \neq \delta_T$, $|T| > 2$, $T \subsetneq N$, this is true. Let us then consider the associated game $(\delta_T)_f^*$, with $|T| > 2$, $T \subsetneq N$. We consider, without loss of generality, any connected coalition $S \subseteq N$.

$$(\delta_T)_f^*(S) = \delta_T(S^*) - \sum_{j \in S^* \setminus S} f_j(S^* \cap \{j\}^*, (\delta_T)_{|S^* \cap \{j\}^*}, g(S^* \cap \{j\}^*)).$$

Since the graph g is acyclic, $S^* \cap \{j\}^*$ can only be a coalition of size 2 at most. Therefore we necessarily have $S^* \cap \{j\}^* \neq T$ thus $\delta_T(S^*) = 0$. That leaves us two cases. When $T \not\subseteq S^* \cap \{j\}^*$, $(\delta_T)_{|S^* \cap \{j\}^*}$ is a null game and therefore $f_i(N, \delta_T, g) = f_i(N, (\delta_T)_f^*, g) = CV_i(N, \mathbf{0}, g) = 0$, $\forall i \in N$. When $T \subsetneq S^* \cap \{j\}^*$, $(\delta_T)_{|S^* \cap \{j\}^*}$ is a dirac game over coalition T and therefore $f_i(N, \delta_T, g) = f_i(N, (\delta_T)_f^*, g) = 0$, $\forall i \in N$, which concludes our proof.

Initial conditions

Let us consider the sharing rule f for cooperative games on acyclic graphs

defined as

$$f_i(N, v, g) = \sum_{\substack{S \subseteq N \\ |S/g|=1 \\ i \in S^*}} \sum_{\substack{R \subseteq S \\ i \in C(g(R))}} (-1)^{|R|-|S|} \frac{1}{\alpha_R} v(S), \quad \forall i \in N,$$

with

$$\alpha_R = \begin{cases} 1 & \text{if } |C(g(R))| = 1, \\ 4 & \text{if } |C(g(R))| = 2 \text{ and } i = \min(C(g(R))), \\ \frac{4}{3} & \text{if } |C(g(R))| = 2 \text{ and } i = \max(C(g(R))). \end{cases}$$

Following the proof of theorem 1, we can see that the sharing rule f satisfies all our axioms except for the initial conditions.

Independence of Irrelevant Players⁴

Let us consider the sharing rule f for cooperative games on acyclic graphs such that

$$f(N, v, g) = ED(N, v),$$

where $ED_i(N, v) = \frac{v(N)}{|N|}$, $\forall i \in N$. Obviously this sharing rule satisfies all axioms except for axiom 5.

Associated consistency⁵

Let us consider the sharing rule f for cooperative games on acyclic graphs such that

$$f(N, v, g) = Sh(N, v),$$

where $Sh(N, v)$ is the Shapley value. It is well known that the Shapley value satisfies axioms 2 to 5, however it does not satisfies axiom 1.

⁴We thank an anonymous referee for providing this example

⁵We thank an anonymous referee for providing this example.

6 Conclusion

In this paper, we introduced the center value for cooperative games on acyclic graphs. This sharing rule CV is defined such that

$$CV_i(N, u_N, g) = \begin{cases} \frac{1}{|C(g)|} & \text{if } i \in C(g), \\ 0 & \text{if } i \notin C(g). \end{cases}$$

The general form is given by

$$CV_i(N, v, g) = \sum_{\substack{S \subseteq N \\ |S/g|=1 \\ i \in S^*}} \sum_{\substack{R \subseteq S^* \\ S \subseteq R \subseteq S^* \\ i \in C(g(R))}} (-1)^{|R|-|S|} \frac{1}{|C(g(R))|} v(S).$$

We can also write this sharing rule in the following form :

$$CV_i(N, v, g) = \sum_{\substack{R \subseteq N \\ |R/g|=1 \\ i \in C(g(R))}} \frac{1}{|C(g(R))|} \Delta_{v/g}(R),$$

with $\Delta_{v/g}(R) = \sum_{\substack{S \\ |S/g|=1 \\ S \subseteq R \subseteq S^*}} (-1)^{|R|-|S|} v(S)$. This form is a weighted sum of

Harsanyi dividends of the graph-restricted game and echoes the forms given by Herings *et al.* (2008, [15]) for the Average tree solution and the Myerson value.

The way the center value considers the graph is radically different from the Myerson value (1977, [18]) and the Average tree solution (Herings *et al.* 2008, [15]). Indeed, in a connected graph, the Myerson value will consider all players as equal in term of usefulness in the graph. In this sense, the Myerson value gives no weight to centrality, checks only for connectedness and measures the marginal contribution. However, this new sharing rule offers the opposite stance. Even though in a unanimity games over the grand coalition all players have positive marginal contribution, this sharing rule mainly considers centrality, giving the utmost importance to players in the center of the graph as opposed to the Myerson value and the Average tree solution which would consider all players equal in every aspects. Moreover, the center value offers results that blatantly go against marginality although its characterization relies on two axioms that can be linked to this kind of approach. Indeed, the independence of irrelevant players echoes the null

player axiom and the initial conditions axiom shares some similarity with the equal treatment of equals. Still, this sharing rule offers payoffs that are widely different to the one usually seen in the literature.

Thus, we obtain effectively a spectrum of sharing rules in terms of the measure of centrality, going from the Myerson value to the center value. In addition, this sharing rule opens up yet another new door between the cooperative game theory framework and the problematic of centrality measures. In future works we hope to generalize this result to all graphs.

A Appendix

Proof that CV satisfies the associated consistency axiom :

We know that

$$v^*(S) = v(S^*) - \sum_{j \in S^* \setminus S} CV_j(S^* \cap \{j\}^*, v_{|S^* \cap \{j\}^*}, g(S^* \cap \{j\}^*))$$

for any $S \subseteq N$ such that $|S/g| = 1$. Since g is an acyclic graph, $S^* \cap \{j\}^*$ can only be a coalition of size two. Direct computations show that

$$\begin{aligned} CV_i(N, v^*, g) &= \sum_{\substack{S \subseteq N \\ |S/g|=1 \\ i \in S^*}} \sum_{\substack{R \subseteq S^* \\ S \subseteq R \subseteq S^* \\ i \in C(g(R))}} (-1)^{|R|-|S|} \frac{1}{|C(g(R))|} \\ &\quad \times \left[v(S^*) - \frac{1}{2} \sum_{\substack{j \\ j \in S^* \setminus S}} (v(\{j\}) - v(\{j, k^{S_j}\}) - v(\{k^{S_j}\})) \right], \end{aligned}$$

where k^{S_j} is the player $k \in S$ linked to player j . Since we need to prove $CV_i(N, v, g) = CV_i(N, v^*, g)$ for all v , it is sufficient to prove that the coefficients associated with every $v(S)$ are equal in $CV_i(N, v, g)$ and in $CV_i(N, v^*, g)$. Let us regroup the coefficients of $v(S)$, $\forall S \subseteq N$. We will separate the $S \subseteq N$ into different cases.

Case 1 :

We assume $|S| \geq 3$. Let us first compute the coefficients of $v(S)$ in $CV_i(N, v, g)$. The coefficients are

$$\sum_{\substack{R \\ S \subseteq R \subseteq S^* \\ i \in C(g(R))}} (-1)^{|R|-|S|} \frac{1}{|C(g(R))|}. \quad (7)$$

Since we consider coalitions R such that $S \subseteq R \subseteq S^*$, the center of R can only be moved by one player in comparison to the center of S . Therefore, since $|S| \geq 3$ if $i \notin S$ the above expression is null.

If $i \notin (C(g(S)))^*$ the above expression is also null.

If $i \in (C(g(S)))^* \setminus C(g(S))$ and $|C(g(S))| = 2$, the above expression is also null.

For the following cases let us consider the players $j \in (C(g(S)))^*$ who can belong to the center of at least one R . For everyone of these players we will divide the set $S^* \setminus S$ into three subsets. We will regroup in E_j the players that, when added, are moving the center towards j . We will regroup in E_{-j} the players that, when added, are moving the center away from j . Finally, in E_0 we will regroup the players that, when added, are not moving the center.

If $i \in (C(g(S)))^* \setminus C(g(S))$ and $|C(g(S))| = 1$, since $0 = \frac{1}{2} - \frac{1}{2}$, the coefficients in this case are

$$\sum_{\substack{R \\ S \subsetneq R \subseteq S^* \setminus E_{-i}}} (-1)^{|R|-|S|} \frac{1}{2} - \sum_{\substack{R \\ S \subsetneq R \subseteq S^* \setminus E_{-i} \setminus E_i}} (-1)^{|R|-|S|} \frac{1}{2}.$$

From lemma 2 we can deduce that $\sum_{\substack{R \\ S \subsetneq R \subseteq S^* \setminus E_{-i}}} (-1)^{|R|-|S|} = (-1)$ if $S \subsetneq S^* \setminus E_{-i}$.

We obtain an analog result for the second term. Considering the different possible case of emptiness and non-emptiness of E_i , E_{-i} and E_0 we find that above expression will cancel in every cases except the case where $E_{-i} \neq \emptyset$ and $E_i \neq \emptyset$ and $E_0 = \emptyset$. In this case, the above expression will equal $-\frac{1}{2}$.

If $i \in C(g(S))$ and $|C(g(S))| = 2$ we have

$$\begin{aligned} & \sum_{\substack{R \\ S \subseteq R \subseteq S^*}} (-1)^{|R|-|S|} \frac{1}{2} \\ & + \left[\sum_{\substack{R \\ S \subsetneq R \subseteq S^* \setminus E_{-i}}} (-1)^{|R|-|S|} \frac{1}{2} - \sum_{\substack{R \\ S \subsetneq R \subseteq S^* \setminus E_{-i} \setminus E_i}} (-1)^{|R|-|S|} \frac{1}{2} \right] \\ & - \left[\sum_{\substack{R \\ S \subsetneq R \subseteq S^* \setminus E_i}} (-1)^{|R|-|S|} \frac{1}{2} - \sum_{\substack{R \\ S \subsetneq R \subseteq S^* \setminus E_i \setminus E_{-i}}} (-1)^{|R|-|S|} \frac{1}{2} \right]. \end{aligned}$$

The above expression will cancel in every case except when $E_{-i} = \emptyset$ and $E_i \neq \emptyset$ and $E_0 = \emptyset$ in which case it is equal to $-\frac{1}{2}$.

If $i \in C(g(S))$ and $|C(g(S))| = 1$ we have

$$\sum_{\substack{R \\ S \subseteq R \subseteq S^*}} (-1)^{|R|-|S|} - \sum_{j \in \{i\}^* \setminus \{i\}} \left[\sum_{\substack{R \\ S \subseteq R \subseteq S^* \setminus E_{-j}}} (-1)^{|R|-|S|} \frac{1}{2} - \sum_{\substack{R \\ S \subseteq R \subseteq S^* \setminus E_{-j} \setminus E_j}} (-1)^{|R|-|S|} \frac{1}{2} \right].$$

There are three cases where the above expression does not cancel.

When $E_0 = \emptyset$ and for every players $j \in (C(g(S)))^*$, $E_j \neq \emptyset$. In this case⁶, the above expression is equal to $\frac{|\{i\}^* \setminus \{i\}|}{2}$.

When $E_0 = \emptyset$ and $\exists j \in (C(g(S)))^*$ such that $E_j = \emptyset$, $E_{-j} \neq \emptyset$. In this case, the expression is equal to

$$\sum_{\substack{j \in \{i\}^* \setminus \{i\} \\ E_j \neq \emptyset}} (-\frac{1}{2}).$$

When $E_0 = \emptyset$ and $\exists! j \in S$ such that $E_{-j} = \emptyset$ ⁷. In this case the expression is equal to $\frac{1}{2}$.

Let us now look at $CV_i(N, v^*, g)$. When $|S| \geq 3$, the worth of coalition S can only appear in $CV_i(N, v^*, g)$ for coalition T such that $T = S^*$. The coefficients of $v(S)$ in $CV_i(N, v^*, g)$ are therefore equal to

$$\sum_{\substack{T \subseteq N \\ |T/g|=1 \\ T^*=S \\ i \in T^*}} \sum_{\substack{R \\ T \subseteq R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|}. \quad (8)$$

First we can observe that if $i \notin S$, there is no T such that $i \in T^*$, $T^* = S$, $i \notin S$. The expression thus cancels and matches the result for $CV_i(N, v, g)$.

Second, we can observe that if $(S \setminus X(g(S)))^* \neq S$ there exists no coalition $T \subseteq N$ such that $T^* = S$. This means that the coefficients associated with

⁶Note that if a single j has a non-empty E_j then every other players $k \in S \setminus j$ has a non-empty E_{-k} .

⁷Note that in this case j is the only player in S such that $E_k \neq \emptyset$.

such S will be null. If a coalition S is such that $(S \setminus X(g(S)))^* \neq S$ then there exists a player in $S^* \setminus S$ which is also in $(S \setminus X(g(S)))^* \setminus (S \setminus X(g(S)))$. Therefore, if a coalition S is such that $(S \setminus X(g(S)))^* \neq S$ then $\exists j \in S^* \setminus S$ such that

$$\max_{c \in C(g(S))} d(c, j) \leq \max_{k \in S} \left[\max_{c \in C(g(S))} d(c, k) \right].^8$$

This is equivalent to saying that some players in $S^* \setminus S$ do not move the center when added to S .

We showed that the coefficients of $v(S)$ in $CV_i(N, v, g)$ these coefficients were null anytime the set E_0 was not empty. Therefore, for coalitions $S \subseteq N$ with $|S| \geq 3$ and $(S \setminus X(g(S)))^* \neq S$, the coefficients associated to $v(S)$ in $CV_i(N, v, g)$ and in $CV_i(N, v^*, g)$ are equal, $\forall i \in N$.

We will now focus on coalition S with $|S| \geq 3$, $(S \setminus X(g(S)))^* = S$ and $i \in S$. If $i \notin (C(g(S)))^*$, since $T^* = S$ then $i \notin C(g(R))$ for any R such that $T \subseteq R \subseteq T^*$ hence expression (8) cancels as does expression (7) in this case. Similarly, if $i \in (C(g(S)))^* \setminus C(g(S))$ with $|C(g(S))| = 2$ then $i \notin C(g(R))$ for any R such that $T \subseteq R \subseteq T^*$ hence expression (8) cancels. Now, we will consider the remaining cases : if $i \in (C(g(S)))^* \setminus C(g(S))$ with $|C(g(S))| = 1$ and if $i \in C(g(S))$.

The coalitions $T \subseteq N$ such that $T^* = S$ range from $T = S \setminus X(g(S))$ to a certain coalition $Q \subseteq N$ with $Q^* = S$ and $\forall k \in Q^* \setminus Q$, $(Q \cup k)^* \neq S$. We can write expression (8) as

$$\sum_{\substack{T \\ S \setminus X(g(S)) \subseteq T \subseteq Q}} \sum_{\substack{R \\ T \subseteq R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|}.$$

Inverting the sums with have

$$= \sum_{\substack{R \\ S \setminus X(g(S)) \subseteq R \subseteq S \\ i \in C(g(R))}} \sum_{\substack{T \\ S \setminus X(g(S)) \subseteq T \subseteq R \cap Q}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|}. \quad (9)$$

The latter equation cancels except when $S \setminus X(g(S)) = R \cap Q$. From here we can distinguish between two cases : $S \cap X(g) = \emptyset$ and $S \cap X(g) \neq \emptyset$.

Case 1.1 :

We assume $S \cap X(g) = \emptyset$. Since $S \cap X(g) = \emptyset$, every players in $X(g(S))$ has

⁸Note that the converse is not true.

neighbours outside of S . Therefore, there exists a unique $T \subseteq N$ such that $T^* = S$, which is $T = S \setminus X(g(S))$. In this case expression (9) reduces to

$$\sum_{\substack{R \\ S \setminus X(g(S)) \subseteq R \subseteq S \\ i \in C(g(R))}} (-1)^{|R| - |S \setminus X(g(S))|} \frac{1}{|C(g(R))|}. \quad (10)$$

Since $S \cap X(g) = \emptyset$, for every player $j \in (C(g(S)))^*$ who is a potential center we have $E_j \neq \emptyset$ and therefore $E_{-j} \neq \emptyset$ for S , otherwise the graph wouldn't be connected.

In addition, for a given player $j \in (C(g(S)))^*$ who is a potential center, $|E_j|$ with respect to S is equal or greater than $|E_j|$ with respect to $S \setminus X(g(S))$.

However, E_j with respect to $S \setminus X(g(S))$ cannot be empty. Moreover, if E_0 with respect to S is empty then E_0 with respect to $S \setminus X(g(S))$ is empty as well.

If E_0 with respect to S is not empty then E_0 with respect to $S \setminus X(g(S))$ is not empty.

Therefore we can conclude that the expression (10) will always be equal to expression (7), since the sets E_j , E_{-j} and E_0 will have the same characteristics for any potential center j with respect to the set S or with respect to the set $S \setminus X(g(S))$.

Case 1.2 :

We assume $S \cap X(g) \neq \emptyset$. In this case $Q = (S \setminus X(g(S))) \cup (X(g) \cap S)$, hence expression (9) becomes

$$\sum_{\substack{R \\ S \setminus X(g(S)) \subseteq R \subseteq S \setminus X(g) \\ i \in C(g(R))}} (-1)^{|R| - |S \setminus X(g(S))|} \frac{1}{|C(g(R))|}. \quad (11)$$

Since $S \cap X(g) \neq \emptyset$, there exists at least a player $j \in (C(g(S)))^*$ who is a potential center such that E_j with respect to S is empty and E_j with respect to $S \setminus X(g(S))$ is non-empty. However, since R is such that $S \setminus X(g(S)) \subseteq R \subseteq S \setminus X(g)$ the coalition including players from E_j with respect to $S \setminus X(g(S))$ will not appear.

Similarly if E_0 with respect to S is empty but E_0 with respect to $S \setminus X(g(S))$ is not this means that the players in E_0 with respect to $S \setminus X(g(S))$ are all in $X(g)$. If E_0 with respect to S is not empty, then as in the previous case,

E_0 with respect to $S \setminus X(g(S))$ is not empty either.

In summary, we can conclude that expression (11) is equal to (7).

We proved that, for any $S \subseteq N$ such that $|S| \geq 3$, the coefficients associated with $v(S)$ in $CV_i(N, v, g)$ are equal to the coefficients associated with $v(S)$ in $CV_i(N, v^*, g)$, for any $i \in N$.

Case 2 :

We assume $|S| = 2$. We will note $S = \{s_1, s_2\}$. Let us first compute the coefficients of $v(S)$ in $CV_i(N, v, g)$. The coefficients are

$$\sum_{\substack{R \\ S \subseteq R \subseteq S^* \\ i \in C(g(R))}} (-1)^{|R|-|S|} \frac{1}{|C(g(R))|}.$$

If $i \notin S$, since $|S| = 2$ there exists no coalition R , $S \subseteq R \subseteq S^*$ such that $i \in C(g(R))$.

If $i \in S$ then player i is in the center of S and, depending on coalition R , can either be a single center of R , outside the center of R or in the double center of R . Since $1 = \frac{1}{2} + \frac{1}{2}$ and $0 = \frac{1}{2} - \frac{1}{2}$ the above expression is thus

$$= \sum_{\substack{R \\ S \subseteq R \subseteq S^*}} (-1)^{|R|-|S|} \frac{1}{2} + \sum_{\substack{R \\ S \subseteq R \subseteq S^* \setminus E_{-i}}} (-1)^{|R|-|S|} \frac{1}{2} - \sum_{\substack{R \\ S \subsetneq R \subseteq S^* \setminus E_i}} (-1)^{|R|-|S|} \frac{1}{2}.$$

The first term is equal to zero. If $S \cap X(g) = \emptyset$ then neither E_i nor E_{-i} are emptysets and we obtain

$$= (-1) \times \frac{1}{2} - (-1) \times \frac{1}{2} = 0.$$

If $S \cap X(g) \neq \emptyset$ then, either $E_i = \emptyset$, $E_{-i} \neq \emptyset$ or $E_i \neq \emptyset$, $E_{-i} = \emptyset$. In the first case we obtain a coefficient of $\frac{1}{2}$, in the second case the coefficient of $v(S)$ is $-\frac{1}{2}$.

In summary, the coefficients associated with $v(S)$ in $CV_i(N, v, g)$ for any S

such that $|S| = 2$ are

$$\rightarrow \text{Case 2.1 : } 0 \quad (12)$$

$$\rightarrow \text{Case 2.2}$$

$$\hookrightarrow \text{Case 2.2.1 : } 0 \quad (13)$$

$$\hookrightarrow \text{Case 2.2.2}$$

$$\hookrightarrow \text{Case 2.2.2.1 : } \frac{1}{2} \quad (14)$$

$$\hookrightarrow \text{Case 2.2.2.2 : } -\frac{1}{2} \quad (15)$$

When $|S| = 2$, the worth of coalition S appear in $CV_i(N, v^*, g)$ only when :

- only one of its members belongs to a given coalition,
- as the neighbourhood of a single extremity of g .

The coefficients of $v(S)$ in $CV_i(N, v^*, g)$ are then

$$\begin{aligned} & \sum_{\substack{T \subseteq N \\ |T/g|=1 \\ i \in T^* \\ s_1 \in T \\ s_2 \notin T}} \sum_{\substack{R \\ T \subseteq R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{1}{2} \\ & + \sum_{\substack{T \subseteq N \\ |T/g|=1 \\ i \in T^* \\ s_1 \notin T \\ s_2 \in T}} \sum_{\substack{R \\ T \subseteq R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{1}{2} \\ & + \sum_{\substack{T \subseteq N \\ |T/g|=1 \\ i \in T^* \\ T^*=S}} \sum_{\substack{R \\ T \subseteq R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|}. \end{aligned} \quad (16)$$

Case 2.1 :

We assume that $i \notin S$. This contradicts $i \in T^*$, hence the last term does not appear. In addition, the first and second terms of the previous equation cannot exist simultaneously, since i will necessarily be on one “side” of S . Without loss of generality we will keep the first term. Furthermore, there exists a certain minimal coalition T such that $i \in T^*$, $s_1 \in T$, $s_2 \notin T$. We

will note this coalition Q_1 . Note that Q_1 can be as small as $Q_1 = \{s_1\}$ if $i \in S^* \setminus S$. There exists also a maximal coalition T that satisfies these conditions. We will note it Q_2 . Q_2 is composed of every players $j \in N$ such the path between player j and s_2 includes s_1 . Note that $Q_2^* = Q_2 \cup \{s_2\}$. Expression (16) is therefore

$$= \sum_{\substack{T \\ Q_1 \subseteq T \subseteq Q_2 \\ |T/g|=1}} \sum_{\substack{R \\ T \subseteq R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{1}{2}.$$

By inverting the sums we obtain

$$= \sum_{\substack{R \\ Q_1 \subseteq R \subseteq Q_2^* \\ |R/g|=1 \\ i \in C(g(R))}} \sum_{\substack{T \\ Q_1 \subseteq T \subseteq R \cap Q_2 \\ R \subseteq T^*}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{1}{2}.$$

In order for i to be in the center of R we need for i to be in R . In addition, there exists a minimal coalition T for every R such that $Q_1 \subseteq T \subseteq R \cap Q_2$ with $R \subseteq T^*$. We will note this coalition Q_R . We obtain

$$= \sum_{\substack{R \\ Q_1 \cup \{i\} \subseteq R \subseteq Q_2^* \\ |R/g|=1 \\ i \in C(g(R))}} \sum_{\substack{T \\ Q_R \subseteq T \subseteq R \cap Q_2}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{1}{2}.$$

The above expression cancels except for R such that $Q_R = R \cap Q_2$. However, when $Q_R = Q_1$, since $i \in Q_2$ we necessarily have $Q_R \subsetneq R \cap Q_2$. When $Q_R \supsetneq Q_1$, since Q_R is minimal we always have $Q_R \subsetneq R$ and therefore $Q_R \subsetneq R \cap Q_2$. In conclusion, the above expression is null and therefore equal to (12).

Case 2.2 :

We assume that $i \in S$. Without loss of generality we consider $S = \{i, s_2\}$. We consider the first two terms of expression (16) :

$$\begin{aligned} &= \sum_{\substack{T \subseteq N \\ |T/g|=1 \\ i \in T \\ s_2 \notin T}} \sum_{\substack{R \\ T \subseteq R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{1}{2} \\ &+ \sum_{\substack{T \subseteq N \\ |T/g|=1 \\ i \in T^* \setminus T \\ s_2 \in T}} \sum_{\substack{R \\ T \subseteq R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{1}{2}. \end{aligned}$$

For the first term, there exists a minimal coalition T such that $i \in T$ and $s_2 \notin T$. We will write it Q_1 . We have $Q_1 = \{i\}$. There exists also a maximal coalition T such that $i \in T$ and $s_2 \notin T$. We will write it Q_2 . For the second term, since $i \in T^* \setminus T$, the only T for which there exists a R , $T \subseteq R \subseteq T^*$ such that $i \in C(g(R))$ is when $T = \{s_2\}$ and consequently $R = \{i, s_2\}$. Thus we obtain

$$= \sum_{\substack{T \subseteq T \subseteq Q_2 \\ |T/g|=1}} \sum_{\substack{R \\ T \subseteq R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{1}{2} \\ + (-1)^1 \frac{1}{2} \times \frac{1}{2}.$$

Inverting the sums as we did in the previous case, we have

$$= \sum_{\substack{R \\ Q_1 \subseteq R \subseteq Q_2^* \\ |R/g|=1 \\ i \in C(g(R))}} \sum_{\substack{T \\ Q_R \subseteq T \subseteq R \cap Q_2}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{1}{2} - \frac{1}{4}.$$

Since $Q_1 = \{i\}$, for R such that $Q_1 \subseteq R \subseteq Q_1 \cup \{s_2\}$, we have $Q_R = R \cap Q_2$ and the first terms does not cancel. We obtain

$$= (-1)^0 \times 1 \times \frac{1}{2} + (-1)^1 \times \frac{1}{2} \times \frac{1}{2} - \frac{1}{4} = 0.$$

Therefore, expression (16) is reduced to

$$\sum_{\substack{T \subseteq N \\ |T/g|=1 \\ i \in T^* \\ T^*=S}} \sum_{\substack{R \\ T \subseteq R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|}. \quad (17)$$

Case 2.2.1 :

When $S \cap X(g) = \emptyset$, since $|S| = 2$, there are no coalition $T \subseteq N$ such that $T^* = S$ and this term disappears. Hence, expression (16) equals expression (13).

Case 2.2.2.1 :

When $S \cap X(g) \neq \emptyset$ and $i \in X(g)$, the only coalition $T \subseteq N$ such that $T^* = S$ is when $T = \{i\}$. Expression (17) is thus

$$(-1)^0 \times 1 + (-1)^1 \times \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2},$$

which is equal to the expression (14).

Case 2.2.2.2 :

When $S \cap X(g) \neq \emptyset$ and $i \notin X(g)$, the only coalition $T \subseteq N$ such that $T^* = S$ is when $T = \{i\}$. Expression (17) equals

$$(-1)^1 \times \frac{1}{2} = -\frac{1}{2},$$

which is equal to the expression (15).

We proved that for any $S \subseteq N$ such that $|S| = 2$ the coefficients associated with $v(S)$ in $CV_i(N, v, g)$ are equal to the coefficients associated with $v(S)$ in $CV_i(N, v^*, g)$, $\forall i \in N$.

Case 3 :

We assume that $|S| = 1$. We will note $S = \{s\}$. Let us compute the coefficients of $v(S)$ in $CV_i(N, v, g)$. The coefficients are

$$\sum_{\substack{R \\ S \subseteq R \subseteq S^* \\ i \in C(g(R))}} (-1)^{|R|-|S|} \frac{1}{|C(g(R))|}.$$

If $S = \{i\}$, we have

$$\begin{aligned} &= 1 + \sum_{\substack{R \\ S \subsetneq R \subseteq S^*}} (-1)^{|R|-|S|} - \sum_{\substack{R \\ S \subsetneq R \subseteq S^* \\ |C(g(R))|=2}} (-1)^{|R|-|S|} \frac{1}{2} \\ &= 1 - 1 - \sum_{\substack{j \\ j \in \{i\}^* \setminus \{i\}}} \sum_{\substack{R \\ S \subsetneq R \subseteq S^* \setminus E_{-j}}} (-1)^{|R|-|S|} \frac{1}{2} \\ &= \frac{|\{i\}^* \setminus \{i\}|}{2}. \end{aligned}$$

If $i \in S^* \setminus S$ we have only one R , $S \subseteq R \subseteq S^*$ such that $i \in C(g(R))$ and it is $R = \{s, i\}$. We obtain

$$\sum_{\substack{R \\ S \subseteq R \subseteq S^* \\ i \in C(g(R))}} (-1)^{|R|-|S|} \frac{1}{|C(g(R))|} = -\frac{1}{2}.$$

If $i \notin S^*$, then the coefficients of $v(S)$ are null. In summary, we need to prove that the coefficients of $v(S)$ in $CV_i(N, v^*, g)$ are

$$\rightarrow \text{Case 3.1 : } \frac{|\{i\}^* \setminus \{i\}|}{2} \quad (18)$$

$$\rightarrow \text{Case 3.2 : } -\frac{1}{2} \quad (19)$$

$$\rightarrow \text{Case 3.3 : } 0 \quad (20)$$

When $S = |1|$, coalition S only appears when it is either a neighbour of a coalition or if it is an extremity of a coalition and linked to a player outside the coalition. The coefficients of $v(S)$ in $CV_i(N, v^*, g)$ are then

$$\begin{aligned} & \sum_{\substack{T \subseteq N \\ |T/g|=1 \\ i \in T^* \\ s \in T^* \setminus T}} \sum_{\substack{R \\ T \subseteq R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \left(-\frac{1}{2}\right) \\ & + \sum_{\substack{T \subseteq N \\ |T/g|=1 \\ i \in T^* \\ s \in T \\ s \in X(g(T)) \setminus X(g)}} \sum_{\substack{R \\ T \subseteq R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{|S^* \setminus T|}{2}. \end{aligned} \quad (21)$$

Case 3.1 :

We assume that $S = \{i\}$. Expression (21) becomes

$$\begin{aligned} & \sum_{\substack{T \subseteq N \\ |T/g|=1 \\ i \in T^* \setminus T}} \sum_{\substack{R \\ T \subseteq R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \left(-\frac{1}{2}\right) \\ & + \sum_{\substack{T \subseteq N \\ |T/g|=1 \\ i \in T \\ i \in X(g(T)) \setminus X(g)}} \sum_{\substack{R \\ T \subseteq R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{|\{i\}^* \setminus T|}{2}. \end{aligned}$$

For the first term, since $i \in T^* \setminus T$, the only coalition T such that there exists coalitions $R, T \subseteq R \subseteq T^*$ such that $i \in C(g(R))$ are singletons of

neighbours of i . We obtain

$$\begin{aligned}
&= \sum_{\substack{j \in \{i\}^* \setminus \{i\} \\ T \subseteq N \\ |T/g|=1 \\ i \in T}} \sum_{\substack{R \subseteq \{j\}^* \\ i \in C(g(R))}} (-1)^{|R|-1} \frac{1}{|C(g(R))|} \left(-\frac{1}{2}\right) \\
&+ \sum_{\substack{T \subseteq N \\ |T/g|=1 \\ i \in T}} \sum_{\substack{R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{|\{i\}^* \setminus T|}{2} \\
&= \frac{|\{i\}^* \setminus \{i\}|}{4} + \sum_{\substack{T \subseteq N \\ |T/g|=1 \\ i \in T}} \sum_{\substack{R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{|\{i\}^* \setminus T|}{2}.
\end{aligned}$$

On the second term, when $i \notin (X(g(T)) \setminus X(g))$, $|\{i\}^* \setminus T| = 0$ hence we can consider all coalitions $T \subseteq N$, $|T/g| = 1$, $i \in T$. Inverting the sums we obtain

$$= \frac{|\{i\}^* \setminus \{i\}|}{4} + \sum_{\substack{R \subseteq N \\ |R/g|=1 \\ i \in C(g(R))}} \sum_{\substack{T \subseteq R \\ R \subseteq T^*}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{|\{i\}^* \setminus T|}{2}.$$

There exists a certain coalition Q_R with $\{i\} \subseteq Q_R \subseteq R$ for every R defined as above which is the minimal coalition such that for T , $Q_R \subseteq T \subseteq R$ we have $R \subseteq T^*$. Thus,

$$= \frac{|\{i\}^* \setminus \{i\}|}{4} + \sum_{\substack{R \subseteq N \\ |R/g|=1 \\ i \in C(g(R))}} \sum_{\substack{T \subseteq R \\ Q_R \subseteq T \subseteq R}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{|\{i\}^* \setminus T|}{2}.$$

The players in $T \setminus Q_R$ can be neighbours of i or not. Noting, for ease of reading, $\alpha = |(R \setminus Q_R) \cap \{i\}^*|$ and $\beta = |R \setminus \{i\}^*|$, we can rewrite the second term of the above expression as

$$\begin{aligned}
&= \frac{|\{i\}^* \setminus \{i\}|}{4} \\
&+ \sum_{\substack{R \subseteq N \\ |R/g|=1 \\ i \in C(g(R))}} \frac{1}{|C(g(R))|} \sum_{v=0}^{\alpha} \left[\binom{\alpha}{v} \times \sum_{u=0}^{\beta} \binom{\beta}{u} (-1)^{|R \setminus Q_R|+v+u} \times \frac{|\{i\}^* \setminus Q_R| - v}{2} \right].
\end{aligned}$$

The second term of the above expression cancels except when

- $\alpha = |(R \setminus Q_R) \cap \{i\}^*| = 0$ and $\beta = |R \setminus \{i\}^*| = 0$
- $\alpha = |(R \setminus Q_R) \cap \{i\}^*| = 1$ and $\beta = |R \setminus \{i\}^*| = 0$

By definition of Q_R , these cases only appear, respectively, when $R = \{i\}$ and $R = \{i, j\}$, $\forall j \in \{i\}^* \setminus \{i\}$. Therefore we obtain

$$\begin{aligned}
&= \frac{|\{i\}^* \setminus \{i\}|}{4} + 1 \times \left[\binom{0}{0} \left(\binom{0}{0} \right) (-1)^0 \frac{|\{i\}^* \setminus \{i\}|}{2} \right] \\
&+ \frac{1}{2} \sum_{j \in \{i\}^* \setminus \{i\}} \left[\binom{1}{0} \left(\binom{0}{0} (-1)^{-1} \frac{|\{i\}^* \setminus \{i\}|}{2} \right) + \binom{1}{1} \left(\binom{0}{0} (-1)^0 \frac{|\{i\}^* \setminus \{i\}| - 1}{2} \right) \right] \\
&= \frac{|\{i\}^* \setminus \{i\}|}{4} + \frac{|\{i\}^* \setminus \{i\}|}{2} + \frac{1}{2} \sum_{j \in \{i\}^* \setminus \{i\}} \left[\left(-\frac{|\{i\}^* \setminus \{i\}|}{2} \right) + \frac{|\{i\}^* \setminus \{i\}|}{2} - \frac{1}{2} \right] \\
&= \frac{|\{i\}^* \setminus \{i\}|}{4} + \frac{|\{i\}^* \setminus \{i\}|}{4} \\
&= \frac{|\{i\}^* \setminus \{i\}|}{2},
\end{aligned}$$

which equals expression (18) and concludes our case.

Case 3.2 :

We assume that $i \in S^* \setminus S$. Working from expression (21), there exists a minimal coalition Q_1 such that $i \in T^*$, $s \in T^* \setminus T$ and a maximal coalition Q_2 . Since $i \in S^* \setminus S$, $Q_1 = \{i\}$. The coalition Q_2 is composed of player i and of every players j such that there exists a path in g $j = a_1, a_2, \dots, a_{k-1} =$

i , $a_k = s$. We can thus write expression (21) as

$$\begin{aligned}
&= \sum_{\substack{T \subseteq Q_2 \\ |T/g|=1}} \sum_{\substack{R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \left(-\frac{1}{2}\right) \\
&+ \sum_{\substack{T \subseteq N \\ |T/g|=1 \\ i \in T^* \\ s \in X(g(T)) \setminus X(g)}} \sum_{\substack{R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{|S^* \setminus T|}{2}.
\end{aligned}$$

Inverting the sums

$$\begin{aligned}
&= \sum_{\substack{R \subseteq Q_2^* \\ |R/g|=1 \\ i \in C(g(R))}} \sum_{\substack{T \subseteq R \cap Q_2 \\ R \subseteq T^*}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \left(-\frac{1}{2}\right) \\
&+ \sum_{\substack{R \subseteq N \\ |R/g|=1 \\ i \in C(g(R))}} \sum_{\substack{T \subseteq R \\ R \subseteq T^*}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{|S^* \setminus T|}{2}.
\end{aligned}$$

For the first term and the second term, as in case 3.1, we can note Q_R the minimal coalition satisfying the corresponding conditions such that $R \subseteq T^*$. For the second term, we know that for i to be in the center of R , i needs to be in R . Thus,

$$\begin{aligned}
&= \sum_{\substack{R \subseteq Q_2^* \\ |R/g|=1 \\ i \in C(g(R))}} \sum_{\substack{T \subseteq R \cap Q_2 \\ Q_R \subseteq T \subseteq R \cap Q_2}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \left(-\frac{1}{2}\right) \\
&+ \sum_{\substack{R \subseteq N \\ |R/g|=1 \\ i \in C(g(R))}} \sum_{\substack{T \subseteq R \\ Q_R \subseteq T \subseteq R}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{|S^* \setminus T|}{2}.
\end{aligned}$$

The first term cancels except when $Q_R = R \cap Q_2$.

If $Q_R = Q_1 = \{i\}$ then only when $R = \{i\}$ or $R = \{s, i\}$ we have $Q_R = R \cap Q_2$.

If $Q_R \supsetneq \{i\}$ then by definition of Q_R we have $Q_R \subsetneq R$ and using the fact that $Q_R \subset Q_2$ therefore $Q_R \neq R \cap Q_2$.

Using the same reasoning as in case 3.1, we can conclude that the second

term cancels except when $R = \{s, i\}$. Therefore we obtain

$$\begin{aligned}
&= \sum_{\substack{R \\ \{i\} \subseteq R \subseteq \{s, i\}}} (-1)^{|R|-1} \frac{1}{|C(g(R))|} \left(-\frac{1}{2} \right) - \frac{1}{4} \\
&= \left(-\frac{1}{2} \right) + \frac{1}{4} - \frac{1}{4} = -\frac{1}{2},
\end{aligned}$$

which is equal to expression (19).

Case 3.3 :

We assume $i \notin S^*$. Working from equation (21), there exists a minimal coalition Q_1 such that $i \in T^*$, $s \in T^* \setminus T$ and a maximal coalition Q_2 . Since $i \notin S^*$, Q_1 is such that it contains every player on the path between i and s (excluding i and s). The coalition Q_2 is composed of player i and of every players $j \in N$ such that i is on the path from j to s . Note that $Q_2^* = Q_2 \cup \{s\}$. Expression (21) is thus

$$\begin{aligned}
&= \sum_{\substack{T \\ Q_1 \subseteq T \subseteq Q_2 \\ |T/g|=1}} \sum_{\substack{R \\ T \subseteq R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \left(-\frac{1}{2} \right) \\
&+ \sum_{\substack{T \\ T \subseteq N \\ |T/g|=1 \\ i \in T^* \\ s \in T \\ s \in X(g(T)) \setminus X(g)}} \sum_{\substack{R \\ T \subseteq R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{|S^* \setminus T|}{2}.
\end{aligned}$$

For the second term, since $i \notin S^*$, we need $i \in T$ in order for i to be in the center of at least one coalition R , $T \subseteq R \subseteq T^*$. We will also note Q the minimal connected coalition such that $i \in Q$, $s \in Q$. We obtain

$$\begin{aligned}
&= \sum_{\substack{T \\ Q_1 \subseteq T \subseteq Q_2 \\ |T/g|=1}} \sum_{\substack{R \\ T \subseteq R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \left(-\frac{1}{2} \right) \\
&+ \sum_{\substack{T \\ Q \subseteq T \subseteq N \\ |T/g|=1}} \sum_{\substack{R \\ T \subseteq R \subseteq T^* \\ i \in C(g(R))}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{|S^* \setminus T|}{2}.
\end{aligned}$$

Inverting the sums

$$\begin{aligned}
&= \sum_{\substack{R \\ Q_1 \cup \{i\} \subseteq R \subseteq Q_2^* \\ |R/g|=1 \\ i \in C(g(R))}} \sum_{\substack{T \\ Q_R \subseteq T \subseteq R \cap Q_2}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \left(-\frac{1}{2}\right) \\
&+ \sum_{\substack{R \\ Q \subseteq R \subseteq N \\ |R/g|=1 \\ i \in C(g(R))}} \sum_{\substack{T \\ Q_R \subseteq T \subseteq R}} (-1)^{|R|-|T|} \frac{1}{|C(g(R))|} \frac{|S^* \setminus T|}{2}.
\end{aligned}$$

We know the first term cancels except for coalitions R such that $Q_R = R \cap Q_2$. Since R is such that $Q_1 \cup \{i\} \subseteq R$, $R \cap Q_2$ is at the very least equal to $Q_1 \cup \{i\}$.

If $Q_R = Q_1$ then $Q_R \subsetneq R \cap Q_2$.

If $Q_R \supsetneq Q_1$, since $|Q_1| \geq 2$, we can always delete at least one extremity of R that belong to Q_2 to obtain Q_R . Therefore the first term cancels.

For the second term, using the result of the previous case, since we necessarily have $\{s, i\} \subsetneq Q$, we can conclude that the term is null. Therefore, the above expression is null and equal to (20).

We can therefore conclude that for all $S \subseteq N$, the coefficient associated in $CV_i(N, v, g)$ and $CV_i(N, v^*, g)$ are equal, $\forall i \in N$ which means that

$$CV_i(N, v, g) = CV_i(N, v^*, g), \quad \forall i \in N.$$

References

- [1] E. Algaba, J.M. Bilbao, R. van den Brink, A. Jiménez-Losada. Axiomatizations of the Shapley and value for cooperative and games on anti-matroids. *Mathematical Methods of Operations Research*, **57**, pp. 49–65, 2003.
- [2] R. J. Aumann, J. H. Dreze. Cooperative Games with Coalition Structures. *International Journal of Game Theory*, **3**, pp. 217–237, 1974.
- [3] J.M. Bilbao, N. Jiménez, J.J López. A note on a value with incomplete communication. *Games and Economic Behavior*, **54**, pp. 419–429, 2006.
- [4] R. van den Brink, R. P. Gilles. Axiomatizations of the conjunctive permission value for games with permission structures. *Games and Economic Behaviour*, **12**, pp. 113–126, 1996.

- [5] P. Borm, G. Owen, S. Tijs. On the position value for communication situation. *SIAM Journal on Discrete Mathematics*, **5**, pp. 305–320, 1992.
- [6] T. S. H. Driessn. Associated Consistency and Values for TU-games. *International Journal of Game Theory*, **39**, pp. 467–482, 2010.
- [7] A. Ghintran. A weighted position value. *Mathematical Social Sciences*, **65**, pp. 157–163, 2010.
- [8] D. Gomez, E. Gonzalez-Aranguena, C. Manuel, G. Owen, M. del Pozo, J. Tejada. Centrality and power in social networks: a game theoretic approach. *Mathematical Social Sciences*, **46**, pp. 27–54, 2003.
- [9] G. Hamiache. A value with incomplete communication. *Games and Economic Behaviour*, **26**, pp. 59–78, 1999.
- [10] G. Hamiache. Associated consistency and the Shapley value. *International Journal of Game Theory*, **30**, pp. 279–289, 2001.
- [11] G. Hamiache. A mean value for games with communication structures. *International Journal of Game Theory*, **32**, pp. 533–544, 2004.
- [12] G. Hamiache. Graph monotonic values. *Social Choice and Welfare*, **37**, pp. 287–307, 2011.
- [13] G. Hamiache, F. Navarro. Associated Consistency, Value and Graphs. *International Journal of Game Theory*, DOI: 10.1007/s00182-019-00688-y, 2019
- [14] S. Hart and A. Mas-Colell. Potential, Value and Consistency. *Econometrica*, **57**, pp. 589–614, 1989.
- [15] P. J.-J. Herings, G. van der Laan, D. Talman. The average tree solution for cycle-free graph games. *Games and Economic Behaviour*, **62**, pp. 77–92, 2008.
- [16] Y-H. Hwang. Associated concistency and equal allocation of nonseparable costs. *Economic Theory*, **28**, pp. 709–719, 2006.
- [17] C. Jordan. Sur les assemblages de lignes. *Journal für die reine und angewandte Mathematik*, **70**, pp. 185–190, 1869.
- [18] R. B. Myerson. Graph and cooperation in games. *Math. Oper. Res.*, **2**, pp. 225–229, 1977.

- [19] R. B. Myerson. Conference Structures and Fair Allocation Rules. International Journal of Game Theory, **9**, pp 169–182, 1980.
- [20] L.S. Shapley. A value for n -person games. Contributions to the Theory of Games II, Annals of Mathematics Studies, pp. 307–317, Princeton University Press, Princeton 1953.
- [21] W. Thomson. Consistent Allocation Rules, 2006. RCER Working Papers 418, University of Rochester - Center for Economic Research (RCER).
- [22] G. Xu, W. Wang, H. Dong. Axiomatization for the center-of-gravity of imputation set value. Linear Algebra and its Applications, **439**, pp. 2205–2215, 2013.