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Robustness of Detrended Cross-correlation Analysis Method Under Outliers Observations

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The computation of the bivariate Hurst exponent constitutes an important technique to test the power-law cross-correlation of time series. For this objective, the detrended cross-correlation analysis method represents the most used one. In this article, we prove the robustness of the detrended cross-correlation analysis method, where the trend is estimated using the polynomial fitting, to estimate the bivariate Hurst exponent when time series are corrupted by outliers observations. On the other hand, we give the exact polynomial order and a regression region for computing a detrended cross-correlation function to obtain a least-square estimator of bivariate Hurst exponent. Our theoretical results are shown by a simulation study on a two-fractional Gaussian noise process corrupted by outliers observations. Additionally, our results are applied to financial time series. The empirical findings results are accompanied by interpretations.

Keywords: Power-law cross-correlation, detrended cross-correlation function, bivariate Hurst exponent, two-fractional Gaussian noise process.

1. Introduction

The trajectory and speed of motion of a particle in a fluid evolve randomly. The random process describing this evolution of the microscopic particles of a fluid under the effect of the molecules of the fluid is called Brownian motion. In 1968, Mandelbrot and Van Ness proposed an extension of the Brownian motion by introducing the fractional Brownian motion (fBm), this stochastic process was already defined in 1940 by Kolmogorov [1]. The fBm is one of the most popular stochastic fractional process due to its self-similarity properties and the possible fractals it can allow to build. The fBm is a centered Gaussian process with stationary increments and have a continuous trajectories almost surely, its covariance function depends on a parameter $H \in]0, 1[$ known as the Hurst parameter, the special case where $H = 1/2$ allows to find the ordinary Brownian motion. Recall that the fBm noted $B_H(t), t \in \mathbb{R}_+$ admits, for $H \in]0, 1[$ and $(s, t) \in \mathbb{R}_+^2$, the following covariance function:

$$\mathbb{E}[B_H(t)B_H(s)] = \frac{\text{var}(B_H(1))}{2} (|t|^{2H} + |s|^{2H} + |s - t|^{2H}).$$

Therefore, $B_H(t)$ is characterized by the regularity of its trajectory of order H , which means that $t \mapsto B_H(t)$ is Hölder continuous of order strictly less than H .

The fBm offers a suitable modeling framework for non-stationary self-similar stochastic processes with stationary increments. It has been extensively used to model stochastic phenomena in various fields of research such that queuing networks (see, for instance, [2, 3])

and mathematical finance [4]. On the other hand, the estimation of the Hurst exponent H constitutes an important goal in the analysis of the fBm. In this regard, numerous methods have been proposed. One of these methods is the detrended fluctuation analysis (DFA). The DFA method is a version, for time series with trend, of the aggregated variance method used for long-memory stationary process [5]. A straightforward generalization of the DFA method, in the case of fractal cross-correlated signals, is the Detrended Cross-Correlation Analysis (DCCA) [6]. However, this method can be used to investigate the power-law cross-correlation between two simultaneously recorded time series. The DCCA is the first method proposed to investigate the long-range cross-correlation between two stationary or non stationary time series [7]. In addition, the DCCA method can be generalized to exhibit the multifractal characteristics of two cross-correlated signals. The generalization of the DCCA method is a multifractal detrended cross-correlation analysis method (more details about DCCA method is given in section 2).

Outliers or observations which are not consistent with the rest of the time series can exist in different time series such as financial time series. The outliers observations can be due to measurement equipment error or data entry errors. These anomalies can affect the model parameter estimates, and then, distort the power of statistical tests based on biased estimates. Recall that the effect of outliers observations on the estimation of Hurst exponent of fBm is studied in [8], while the effect of outliers on the estimation of differencing parameter d ($H = d+1/2$) of ARFIMA process is studied in [9]. Although the DCCA method is frequently used in different research fields such as seismology [10], meteorology [11] and finance [12, 13], at the time of writing of this article we only found the article [14] that is interested in the DCCA method in the presence of outliers observations. Where, in [14], authors shows that the outliers observations lead to the spurious results of DCCA method.

The main goal of this article is to study the DCCA method, under presence of outlier observations, in order to test if outliers observations can affect the power-law cross-correlation coefficient. For this goal, the studied stochastic process is two-fractional Gaussian noise (2-fGn) process [15] contaminated by outliers. The 2-fGn process is the increment of size one of two-multivariate fractional Brownian motion process. Let $X(t)$ and $Y(t)$ the components of 2-fGn with Hurst index H_X and H_Y respectively and let $\gamma_{XY}(h) = \mathbb{E}[X(t)Y(t+h)]$, then

$$\gamma_{XY}(h) = \frac{\sigma_X \sigma_Y}{2} (w_{XY}(h-1) - 2w_{XY}(h) + w_{XY}(h+1)), \quad (1)$$

where $\sigma_X = \sqrt{\text{var}(X(1))}$, $\sigma_Y = \sqrt{\text{var}(Y(1))}$ and

$$w_{XY}(h) = \begin{cases} (\rho_{XY} - \eta_{XY} \text{sign}(h)) |h|^{H_X+H_Y}, & \text{if } H_X + H_Y \neq 1; \\ \tilde{\rho}_{XY} |h| + \tilde{\eta}_{XY} h \log(h), & \text{if } H_X + H_Y = 1. \end{cases} \quad (2)$$

If $H_X + H_Y \neq 1$ there exists $(\rho_{XY}, \eta_{XY}) \in [-1, 1] \times \mathbb{R}$ with $\rho_{XY} = \rho_{YX} = \text{corr}(X(1), Y(1))$ and $\eta_{XY} = -\eta_{YX}$. If $H_X + H_Y = 1$ there exists $(\tilde{\rho}_{XY}, \tilde{\eta}_{XY}) \in [-1, 1] \times \mathbb{R}$ with $\tilde{\rho}_{XY} = \tilde{\rho}_{YX} = \text{corr}(X(1), Y(1))$ and $\tilde{\eta}_{XY} = -\tilde{\eta}_{YX}$.

The rest of this article is organized as follows: the DCCA method is recalled in section 2. The theoretical results are given in section 3. Section 4 is devoted to Monte-Carlo's experiments on the estimation of power law cross correlation coefficient for outliers contam-

inated 2-fGn process. An application of the estimation procedure to a financial time series is presented in section 5. We conclude with section 6.

2. Detrended Cross-Correlation Analysis Method

Assume a processes X and Y with zero means and are long-range temporally autocorrelated with power-law auto-correlations as:

$$\mathbb{E}[X(i)X(i+s)] \sim s^{2H_X-2} \text{ and } \mathbb{E}[Y(i)Y(i+s)] \sim s^{2H_Y-2}, \quad (3)$$

where H_X and H_Y are the Hurst exponents of X and Y respectively and are assumed to lie in the range $[0.5, 1[$. The power-law cross-correlations are defined by the following relations:

$$\mathbb{E}[X(i)Y(i+s)] \sim As^{2\gamma-2} \text{ and } \mathbb{E}[Y(i)X(i+s)] \sim Bs^{2\gamma-2}, \text{ where } (A, B) \in \mathbb{R}_+^2. \quad (4)$$

The scaling exponent γ characterizes the cross-correlation properties of the processes X and Y . For estimating γ , the DCCA method [16] is proposed in the presence of non-stationary trends.

Considering two time series $\{X_i\}$ and $\{Y_i\}$ where $i = 1, \dots, N$. Each time series is covered with $n_s = [N/s]$ non overlapping boxes of size s where $[.]$ is a lower integer sign. The profiles within the v -th box $[\ell_v + 1, \ell_v + s]$, where $\ell_v = (v - 1)s$, are determined to be:

$$X_v(k) = \sum_{j=1}^k X(\ell_v + j) \text{ and } Y_v(k) = \sum_{j=1}^k Y(\ell_v + j), \text{ for } k = 1, \dots, s. \quad (5)$$

Assume that the local trends of $X_v(k)$ and $Y_v(k)$ are respectively $\widetilde{X}_v(k)$ and $\widetilde{Y}_v(k)$. We recall that there are many different methods for the determination of \widetilde{X}_v . For example, the trend functions could be polynomials [17], or the non parametric detrending procedure based on the empirical mode decomposition method [18]. The cross-correlation or the local detrended covariance function for each box [19] is calculated as follows:

$$f_v^{X,Y}(s) = \frac{1}{s} \sum_{k=1}^s (X_v(k) - \widetilde{X}_v(k))(Y_v(k) - \widetilde{Y}_v(k)), \text{ for } v = 1, \dots, n_s, \quad (6)$$

and the q -th order cross-correlation function is calculated as follows:

$$F_{X,Y}(q, s) = \begin{cases} \left(\frac{1}{n_s} \sum_{v=1}^{n_s} (f_v^{X,Y}(s))^{\frac{q}{2}} \right)^{\frac{1}{q}}, & \text{when } q \neq 0; \\ \exp\left(\frac{1}{2n_s} \sum_{v=1}^{n_s} \log(f_v^{X,Y}(s)) \right), & \text{when } q = 0. \end{cases} \quad (7)$$

Then, we expect the following scaling relation:

$$F_{X,Y}(q, s) \sim s^{H_{X,Y}(q)}, \quad (8)$$

where the Hurst exponent $H_{X,Y}(q)$ characterizes the long-range cross-correlation properties of processes X and Y . If $H_{X,Y}(2) = 0.5$ or $F_{X,Y}(2, s)$ oscillates around zero then there is no cross-correlation. If $H_{X,Y}(2) > 0.5$ then there is a positive power-law cross-correlation, which indicates that a large increment in one process is more likely to be followed by a large increment in the other process. If $H_{X,Y}(2) < 0.5$ then two processes are anti-correlated.

3. Main Results

In [16], authors use a stationary linear ARFIMA process to validate the DCCA method. In this article, the used outliers-contaminated process is given by:

$$\begin{cases} Z(t) = X(t) + \theta_X(1 - B^X(t))N^X(t); \\ W(t) = Y(t) + \theta_Y(1 - B^Y(t))N^Y(t), \end{cases} \quad (9)$$

where

- $X(t)$ and $Y(t)$ are the components of 2-fGn.
- $B^X(t)$ and $B^Y(t)$ are independent Bernoulli random variables with parameters p_X and p_Y and such that $\text{cov}[B^X(j), B^X(j')] = \text{cov}[B^Y(j), B^Y(j')] = 0$, $\forall j' \neq j$.
- $N^X(t)$ and $N^Y(t)$ are an independent Gaussian random variables with means respectively μ and ν and variances σ_X^2 and σ_Y^2 respectively, and such that $\text{cov}[N^X(j), N^X(j')] = \text{cov}[N^Y(j), N^Y(j')] = 0$, $\forall j' \neq j$.
- $\text{cov}[X(j), (1 - B^Y(j'))N^Y(j')] = \text{cov}[Y(j), (1 - B^X(j'))N^X(j')] = 0$, $\forall j$ and j' .
- θ_X and θ_Y are a positive real numbers.

Recall that the model given by Eq. (9) is used, for unidimensional process, in [20] and [21] to testing the robustness of a proposed estimator of correlation dimension for deterministic dynamics corrupted by outliers observations. The advantage of the model given by Eq.(9) is that the outliers observations are added to free process in randomly positions, which is logical since in reality we do not find successive outliers.

Property 3.1. If B is a Bernoulli random variable with parameter p which is independent of the normal random variable N with mean μ and variance σ^2 . Then $\mathbb{E}[(1 - B)N] = p\mu$ and, for $\ell \in \mathbb{N}$, we have $\mathbb{E}[\sum_{j=1}^k (1 - B(\ell + j))N(\ell + j)] = kp\mu$.

Lemma 3.1. Consider a matrix S such that $S_{i,j} = i^{j-1}$ with $i = 1, \dots, s$ and $j = 1, \dots, p+1$. Then $P = S(S^\top S)^{-1}S^\top$ is the projection matrix of the p -th order polynomial fitting for any time series of uniform intervals.

Proposition 3.1. Let $f_v^{Z,W}(s)$ the cross-correlation function of the outliers-contaminated process, given by Eq. (9), where the trend is computed by \mathcal{P} -order polynomial fitting. Then we have

$$\mathbb{E}[f_v^{Z,W}(s)] = \frac{1}{s} (\text{Tr}(A \text{cov}[X_v, Y_v]) + \theta_X \theta_Y \eta_Y^\top A \eta_X), \quad (10)$$

where $A = I_s - S(S^\top S)^{-1}S^\top$, I_s is a s -identity matrix, S is a matrix such that $S_{i,j} = i^{j-1}$ with $i = 1, \dots, s$ and $j = 1, \dots, \mathcal{P} + 1$, X_v is the the vector of profiles within the v -th box, $\eta_Y = (p_Y \nu, 2p_Y \nu, \dots, sp_Y \nu)$ and $\eta_X = (p_X \mu, 2p_X \mu, \dots, sp_X \mu)$.

Corollary 3.1. Let $F_{Z,W}(q, s)$ the q -th order cross-correlation function of the outliers-contaminated process, given by Eq. (9), where the trend is computed by \mathcal{P} -order polynomial fitting. If $\mathcal{P} = 2$, $H_X + H_Y \neq 1$ and s is large enough, then

$$\mathbb{E}[F_{Z,W}(2, s)^2] = \frac{\sigma_X \sigma_Y \rho_{XY}}{2} s^{H_X + H_Y} \mathcal{O}(1) + \frac{\sigma_X \sigma_Y \rho_{XY}}{2} s. \quad (11)$$

Remark 3.1. The dimension of the matrix $P = S(S^\top S)^{-1}S^\top$, where S is given in lemma 3.1, is $s \times s$ for all values of \mathcal{P} , so the calculation of $P_{i,j}$, if s is large enough, is very difficult. However, using numerical computations, we notice that for $s > 75$ and $\mathcal{P} > 4$ and for $s > 350$ and $\mathcal{P} > 3$ the matrix $S^\top S$ is not invertible, then $A = I_s - S(S^\top S)^{-1}S^\top$ is not defined. On the other hand, for $s > 2$ and $\mathcal{P} = 3$, $\eta_Y^\top A \eta_X$ is very near to zero and is independent of s .

4. Estimation of Power-law Cross-correlation Exponent

According to corollary 3.1 and Eq. (8), for outliers-contaminated process given by Eq. (9), if s is large enough, we assume that

$$F_{Z,W}(2, s)^2 = \frac{\sigma_X \sigma_Y \rho_{XY}}{2} s^{H_X + H_Y} \mathcal{O}(1) + \frac{\sigma_X \sigma_Y \rho_{XY}}{2} s. \quad (12)$$

Then

$$\begin{aligned} \frac{F_{Z,W}(2, s)^2}{s} &= \frac{\sigma_X \sigma_Y \rho_{XY}}{2} s^{H_X + H_Y - 1} \mathcal{O}(1) + \frac{\sigma_X \sigma_Y \rho_{XY}}{2} \\ &= s^{H_X + H_Y - 1} \left(\mathcal{O}(1) + \frac{\sigma_X \sigma_Y \rho_{XY}}{2} s^{1 - H_X - H_Y} \right), \end{aligned}$$

so $\log\left(\frac{F_{Z,W}(2, s)^2}{s}\right) = (H_X + H_Y - 1) \log(s) + \log\left(\mathcal{O}(1) + \frac{\sigma_X \sigma_Y \rho_{XY}}{2} s^{1 - H_X - H_Y}\right)$. Thus,

if s is large enough and $H_X + H_Y > 1$, we have $\log\left(\frac{F_{Z,W}(2, s)^2}{s}\right) = (H_X + H_Y - 1) \log(s) + \log(\mathcal{O}(1))$. (13)

From Eq. (13), we have $F_{Z,W}(2, s) = s^{\frac{H_Z + H_W}{2}} \mathcal{O}(1)$, while from Eq. (8), with $q = 2$, we have $F_{Z,W}(2, s) \sim s^{H_{Z,W}(2)}$, then we can conclude that the bivariate Hurst exponent $H_{Z,W}(2) \sim \frac{H_Z + H_W}{2}$. On the other hand, according to Eq. (13), the slope obtained by least square regression method of $\log(F_{Z,W}(2, s)^2)$ over $\log(s)$ can be given an estimator of $H_X + H_Y$.

In order to validate the theoretical results, the used outliers-contaminated process is given by:

$$\begin{cases} Z(i) = X(i) + \theta_X(1 - B^X(i))N^X(i); \\ W(i) = Y(i) + \theta_Y(1 - B^Y(i))N^Y(i), \end{cases} \quad (14)$$

where $N^X(i)$ and $N^Y(i)$ are i.i.d. standard Gaussian random variables and $B^X(i)$ and $B^Y(i)$ are independent Bernoulli random variables with parameters $p_X = p_Y = 0.2$. This means that the expectation of the number of contaminated observations, in a signal of length N , is $N \times 20\%$. θ_X and θ_Y are chosen such that the contaminated observation achieves a given Signal Noise Rate (SNR), that is such that $SNR = 10 \log_{10}\left(\frac{1}{\theta_X}\right) = 10 \log_{10}\left(\frac{1}{\theta_Y}\right)$ so $\theta_X^2 = \theta_Y^2 = 10^{-\frac{SNR}{10}}$. $X(i)$ and $Y(i)$ are the components of 2-fGn process with Hurst parameters H_X and H_Y respectively. The simulation of 2-fGn process is done for $\rho_{XY} = \rho_{YX} = 0.8$ and $\eta_{XY} = \eta_{YX} = 0$ using the R software functions of the articles [15, 22], this corresponds to the long-range interdependent case (see, for instance, [15]). To obtain an estimator of $H_X + H_Y$, we use the model given by the Eq. (13) for s in the interval $[\exp(2), \exp(4)]$ with equidistance step in logarithmic scale equal to 10^{-1} . Figure 1 illustrates the evolution of $\log(F_{Z,W}(2, s)^2/s)$ onto $\log(s)$ for free (blue curve) and outliers-contaminated (red curve)

2-fGn process of length $N = 80000$ where $SNR = 5$, $H_X = 0.6$ and $H_Y = 0.7$, where we remark the similarity behavior of evolutions of $\log(F_{Z,W}(2, s)^2/s)$. On the other hand, the behavior of the error between evolutions of $\log(F_{Z,W}(2, s)^2/s)$ wish tends to zero is illustrated by Figure 2. The results of estimation for 1000 independent replications using SNR equal to 5 and 10 are detailed in tables 1 and 2. The estimation results reveal the convergence of the least square estimator to the true value $H_X + H_Y$ for all tested values N . As well, the decline, from $N = 1000$, of the standard deviation (std) and the mean square error (mse) to zero.

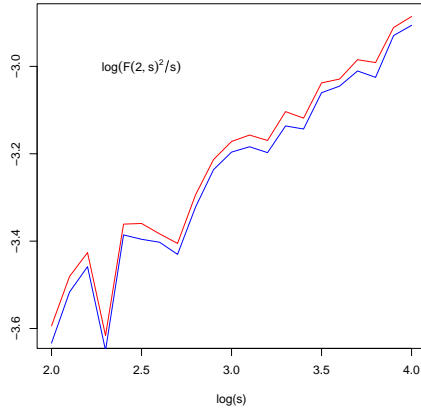


Fig. 1. Evolution of $\log(F_{Z,W}(2, s)^2/s)$ versus $\log(s)$ of free (blue curve) and contaminated (red curve) 2-fGn process.

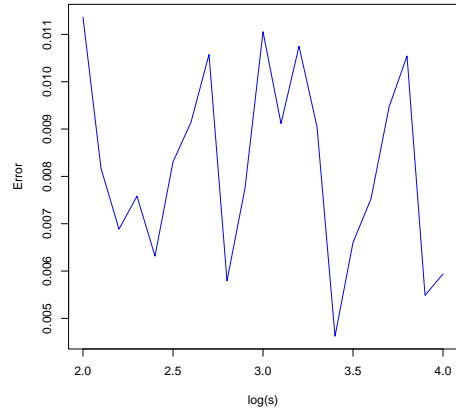


Fig. 2. Error between evolutions of $\log(F_{Z,W}(2, s)^2/s)$.

5. Application to Financial Time Series

Recall that the interbank rates are the interest rates traded in the interbank market by agents and financial institutions. In this section, the analyzed data was the conditional stochastic volatilities time series of interbank rates of Russia and China. These interbank rates are named TIRUSSIA and SHIBOR respectively. TIRUSSIA and SHIBOR time series are collected in the period time from January, 11, 2016 until November, 31, 2020 as a 1204 observations and are downloaded from the web sites http://www.cbr.ru/eng/hd_base/mkr/mkr_base/ and <http://www.shibor.org/> respectively. The interbank rates time series comprising the sample are all annual daily overnight rates (for 252 working days) and were transformed into daily rates per working day (used as the daily return), calculated as follows:

$$y_t = (1 + x_t)^{\frac{1}{252}} - 1,$$

where y_t represents the interbank rate of a working day and x_t the overnight interbank rate. We focused on this study on the analysis of the conditional stochastic volatility of

Table 1. Estimation results of $H_X + H_Y$ for different N .

	SNR = 5				SNR = 10		
	N	mean	std	mse	mean	std	mse
$H_X = 0.55, H_Y = 0.6$ $H_X + H_Y = 1.15$	$N = 500$	1.148	0.130	0.130	1.146	0.123	0.123
	$N = 1000$	1.153	0.088	0.088	1.154	0.087	0.088
	$N = 2000$	1.159	0.063	0.064	1.160	0.060	0.061
$H_X = 0.55, H_Y = 0.65$ $H_X + H_Y = 1.2$	$N = 500$	1.195	0.127	0.127	1.191	0.127	0.128
	$N = 1000$	1.207	0.089	0.089	1.204	0.087	0.087
	$N = 2000$	1.206	0.065	0.065	1.207	0.060	0.060
$H_X = 0.55, H_Y = 0.7$ $H_X + H_Y = 1.25$	$N = 500$	1.244	0.128	0.128	1.249	0.134	0.134
	$N = 1000$	1.252	0.094	0.094	1.259	0.092	0.093
	$N = 2000$	1.257	0.065	0.066	1.257	0.067	0.067
$H_X = 0.55, H_Y = 0.75$ $H_X + H_Y = 1.3$	$N = 500$	1.296	0.132	0.132	1.295	0.127	0.128
	$N = 1000$	1.299	0.090	0.090	1.304	0.092	0.092
	$N = 2000$	1.306	0.066	0.066	1.307	0.064	0.065
$H_X = 0.55, H_Y = 0.8$ $H_X + H_Y = 1.35$	$N = 500$	1.349	0.128	0.128	1.345	0.125	0.125
	$N = 1000$	1.355	0.091	0.091	1.357	0.089	0.089
	$N = 2000$	1.356	0.063	0.064	1.359	0.066	0.067
$H_X = 0.55, H_Y = 0.85$ $H_X + H_Y = 1.4$	$N = 500$	1.396	0.125	0.125	1.399	0.123	0.123
	$N = 1000$	1.402	0.091	0.091	1.408	0.089	0.089
	$N = 2000$	1.406	0.060	0.060	1.409	0.063	0.064
$H_X = 0.55, H_Y = 0.9$ $H_X + H_Y = 1.45$	$N = 500$	1.409	0.156	0.161	1.427	0.158	0.160
	$N = 1000$	1.424	0.111	0.114	1.434	0.109	0.110
	$N = 2000$	1.432	0.080	0.082	1.439	0.079	0.080
$H_X = 0.55, H_Y = 0.95$ $H_X + H_Y = 1.5$	$N = 500$	1.488	0.157	0.158	1.476	0.151	0.153
	$N = 1000$	1.480	0.108	0.110	1.492	0.108	0.108
	$N = 2000$	1.489	0.073	0.073	1.496	0.076	0.076

such time series y_t . To estimate the stochastic volatility, we use the stochastic volatility model with moving average innovations (SVM-MA) [23], using that there are different studies that reject the use of ARCH or GARCH models to estimate volatility time series (see, for instance, [24, 23, 25]). We remind that the SVM-MA model is given by:

$$\begin{cases} y_t = \mu + \varepsilon_t^y; \\ \varepsilon_t^y = u_t + \psi u_{t-1}, \end{cases} \quad (15)$$

where u_t is a Gaussian random variable with zero mean and having $\exp(h_t)$ as a variance, $u_0 = 0$ and $|\psi| < 1$. On the other hand, the log-volatility h_t is assumed to follow the autoregressive process as:

$$h_t = \mu_h(1 - \phi_h) + \phi_h h_{t-1} + \varepsilon_t^h, \quad (16)$$

where the error term ε_t^h is a Gaussian random variable with zero mean and variance w_h^2 , and also independent of the error term ε_t^y . The estimation method for fitting model given by Eq. (15) and Eq. (16) is detailed in [23]. Estimated volatilities time series are illustrated in figures 3 and 4, and some of their descriptive statistics are given in Table 3.

The augmented Dickey-Fuller (ADF) unit root test [26] and Phillips-Perron (PP) test [27] are used to explore the stationarity characteristics of conditional stochastic volatility time series. We used both tests in order to check the robustness of the results. One advantage of the

Table 2. Estimation results of $H_X + H_Y$ for different N , in the case where $H_X = H_Y$.

	SNR=5				SNR=10		
	N	mean	std	mse	mean	std	mse
$H_X = H_Y = 0.55,$ $H_X + H_Y = 1.1$	$N = 500$	1.100	0.128	0.128	1.103	0.123	0.123
	$N = 1000$	1.102	0.088	0.088	1.111	0.085	0.086
	$N = 2000$	1.107	0.062	0.063	1.110	0.060	0.061
$H_X = H_Y = 0.6,$ $H_X + H_Y = 1.2$	$N = 500$	1.194	0.133	0.133	1.196	0.133	0.133
	$N = 1000$	1.209	0.089	0.090	1.208	0.088	0.088
	$N = 2000$	1.207	0.063	0.063	1.206	0.064	0.064
$H_X = H_Y = 0.65,$ $H_X + H_Y = 1.3$	$N = 500$	1.292	0.137	0.137	1.300	0.130	0.130
	$N = 1000$	1.305	0.093	0.093	1.304	0.093	0.093
	$N = 2000$	1.306	0.068	0.068	1.305	0.068	0.068
$H_X = H_Y = 0.7,$ $H_X + H_Y = 1.4$	$N = 500$	1.396	0.139	0.139	1.390	0.139	0.139
	$N = 1000$	1.390	0.102	0.102	1.403	0.096	0.096
	$N = 2000$	1.407	0.070	0.071	1.406	0.069	0.069
$H_X = H_Y = 0.75,$ $H_X + H_Y = 1.5$	$N = 500$	1.490	0.146	0.146	1.498	0.140	0.140
	$N = 1000$	1.496	0.097	0.097	1.505	0.105	0.105
	$N = 2000$	1.501	0.072	0.072	1.505	0.071	0.071
$H_X = H_Y = 0.8,$ $H_X + H_Y = 1.6$	$N = 500$	1.588	0.146	0.147	1.597	0.147	0.147
	$N = 1000$	1.601	0.104	0.104	1.599	0.102	0.102
	$N = 2000$	1.606	0.071	0.071	1.607	0.070	0.070
$H_X = H_Y = 0.85,$ $H_X + H_Y = 1.7$	$N = 500$	1.686	0.151	0.151	1.684	0.147	0.148
	$N = 1000$	1.701	0.104	0.104	1.707	0.104	0.104
	$N = 2000$	1.708	0.075	0.076	1.707	0.076	0.077
$H_X = H_Y = 0.9,$ $H_X + H_Y = 1.8$	$N = 500$	1.785	0.147	0.147	1.794	0.159	0.159
	$N = 1000$	1.796	0.108	0.108	1.802	0.106	0.106
	$N = 2000$	1.802	0.077	0.077	1.807	0.076	0.077
$H_X = H_Y = 0.95,$ $H_X + H_Y = 1.9$	$N = 500$	1.878	0.158	0.160	1.888	0.160	0.160
	$N = 1000$	1.899	0.106	0.106	1.907	0.113	0.113
	$N = 2000$	1.896	0.079	0.079	1.910	0.077	0.077

Table 3. Descriptive statistics of studied time series.

Time series	Mean	Standard deviation	Skewness	Kurtosis
Volatility of SHIBOR	0.0144	1.122e-05	-0.025	2.85
Volatility of TIRUSSIA	0.014	1.113e-05	0.065	2.918

PP test over the ADF test is that the former is robust to general forms of heteroscedasticity in the error term. On the other hand, Akaike information criterion was used to select the lag length in the ADF test, while Newey-West Bartlett kernel is used to select the bandwidth for the PP test. Instead, a robust version of Jarque Bera (JB) test [28] and Shapiro-Wilk (SW) test [29] are applied to test the normality of stochastic volatilities time series. The results of stationarity and normality tests are illustrated in Table 4. According to Table 4, the stochastic volatilities time series are stationary and distributed according to the normal distribution.

Recall that many methods for detecting outliers have been proposed. These methods

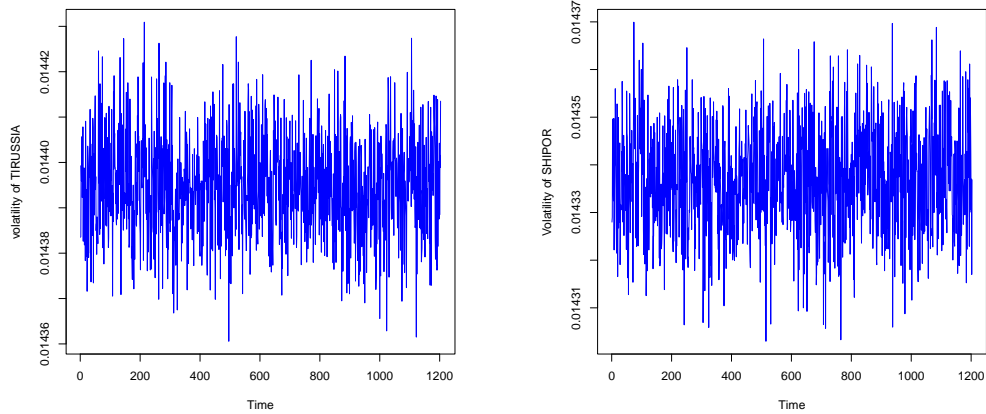


Fig. 3. Volatility of TIRUSSIA interbank interest rate. Fig. 4. Volatility of SHIPOR interbank interest rate.

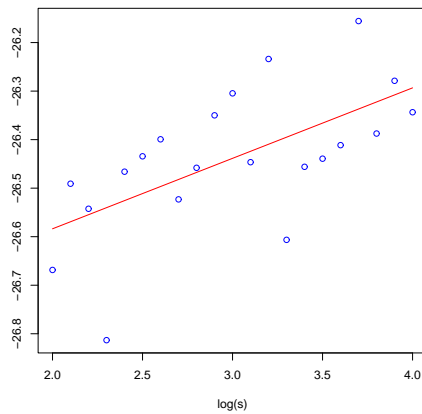


Fig. 5. $\log(F_{Z,W}(2, s)^2/s)$ onto $\log(s)$ and polynomial fitting.

include, for example, Z-score method, box-plot method, statistical measures, and asymmetric winsorized mean method. In this study, in order to detect outliers in volatilities time series, we applied Tietjen-Moore (TM) test [30] for all windows of length 30, using that TM test is recommended for a small time series [30]. The results of TM test is given in the last column of Table 4 where the value between parenthesis indicates the 5%-quantile value of 10000 values of TM statistical test of simulated normal random vector having the same length as the studied stochastic volatility time series. The results of TM test reveals that the volatility of SHIBOR rate have 7 outlier observations, whereas for the volatility of TIRUSSIA rate there are 5 outlier observations.

Table 4. Results of stationarity, normality and outliers tests for studying time series.

Time series	Stationarity tests		Normality tests		TM test
	ADF test	PP test	JB test	SW test	
Volatility of SHIBOR	-3.435(1)***	-3.435(2)***	1.194***	0.998***	0.573(0.588)**
Volatility of TIRUSSIA	-3.435(1)***	-3.435(9)***	1.161***	0.999***	0.237(0.242)**

Note: (***) and (**) denotes the p -value statistical significance at 1% and 5% respectively.

For studied volatilities time series, the evolution of $\log(F_{Z,W}(2,s)^2/s)$ onto $\log(s)$ for $s \in [\exp(2), \exp(4)]$ is illustrated in Figure 5 with blue points, whereas the least square line is presented in red curve where the estimator of $H_X + H_Y$ is equal to 1.145. This result indicates a positive power-law cross-correlation of studied time series, which indicates that a large increment in one volatility time series is more likely to be followed by a large increment in the other volatility, which implies some dependence between volatility of interbank rates. This dependence can be interpreted as a consequence of general monetary policy of BRICS group and some individual interbank relationship between China and Russia.

There is a general effect that could explain the interbank dependence between BRICS countries. Recall that the Brazilian development bank, the state bank for development and foreign economic affairs of Russia, the export and import bank of India, the development bank of China and development bank of Southern Africa established an interbank cooperation mechanism in April 2010. This interbank cooperation mechanism would serve as a useful instrument for cutting trading costs and enhancing intra-BRICS trade and investment. In addition, at the Durban summit in 2013, the BRICS countries signed two agreements in the framework of interbank cooperation. One of these agreements is for the co-financing of infrastructure in Africa and the second one for financing the green economy and for controlling climate change. Also, in 2014, the BRICS group announced the creation of the new development bank (NDB) and the Contingent Reserve Arrangement (CRA). The BRICS governments promoted the NDB and the CRA respectively as alternatives to the World Bank and the International Monetary Fund.

On the other hand, China and Russia enhance inter-bank cooperation. China development bank and sberbank bank of Russia have signed an agreement that will advance financial cooperation between both countries. Recall that sberbank is the largest bank in Russia and eastern Europe, and the third largest in Europe. In addition, China development bank is one of China's three policy banks, and is primarily responsible for raising funding for large infrastructure projects. Under the five-year accord, China development bank and sberbank will cooperate in financing Russia's large-scale government projects and infrastructure projects, international settlement and trade financing, correspondent banking, custody services, personnel training and experience exchanges that have a total value of some 2 billion dollars. The agreement gives priority to Chinese-funded projects in oil and gas, electricity, infrastructure, metal mining, telecommunications, agriculture, and forestry.

According to the empirical finding results about the behavior of volatilities time series, a vectorial ARFIMA process [31, 32] can be applied to modeling the dynamics of such volatil-

ities. Additionally, we can propose a vectorial version of SVM-MA model given by Eq. (15) under hypothesis that log-volatilities time series have a vectorial ARFIMA structure as a new model for interbank interest rates. These proposed econometrics models can be applied to confirm the detected long-range dependence between stochastic volatilities, and can be applied to other stochastic volatilities of interbank interest rates for other countries of BRICS group.

6. Conclusion

This paper is interested in the problem of power-law cross-correlation under outlier observations. Where we prove the robustness of a detrended cross-correlation analysis method, where the trend is estimated using polynomial fitting, on the estimation of the coefficient characterizing the power-law cross-correlation. We give a polynomial order and a regression region permits to obtain a good least square estimator of the power-law cross-correlation coefficient. We confirm our theoretical results by a simulation study on a 2-fGn process corrupted by a outlier observations. An application of our theoretical results, to financial time series, is also conducted.

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Appendix A. Appendix

Proof of Property 2.1. Let B is a Bernoulli random variable with parameter p and N a normal random variable with mean μ and variance σ^2 . If two random variables X and Y are

independent, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$. Since B and N are independent, then

$$\mathbb{E}[(1 - B)N] = \mathbb{E}[(1 - B)]\mathbb{E}[N] = p\mu, \quad (\text{A.1})$$

and

$$\begin{aligned} \mathbb{E}\left[\sum_{j=1}^k (1 - B(\ell + j))N(\ell + j)\right] &= \sum_{j=1}^k \mathbb{E}[(1 - B(\ell + j))N(\ell + j)] \\ &= \sum_{j=1}^k \mathbb{E}[(1 - B(\ell + j))]\mathbb{E}[N(\ell + j)] \\ &= \sum_{j=1}^k p\mu \\ &= kp\mu. \end{aligned}$$

□

Proof of lemma 2.1. Let

$$\begin{aligned} \tilde{y}_i &= a_0 + a_1x_i + a_2x_i^2 + \cdots + a_px_i^p \text{ and} \\ y_i &= a_0 + a_1x_i + a_2x_i^2 + \cdots + a_px_i^p + e_i. \end{aligned}$$

Now, for $\vec{y} = (y_1, y_2, \dots, y_s)$, $X = (x_i^{j-1})$ with $i = 1, \dots, s$ and $j = 1, \dots, p + 1$ and $\vec{a} = (a_0, a_1, \dots, a_p)$, we have

$$\begin{aligned} \tilde{\vec{y}} &= X\vec{a} \text{ and} \\ \vec{y} &= X\vec{a} + \vec{e}. \end{aligned}$$

Then, using that $\text{span}\{X\} \perp \text{span}\{\vec{e}\}$ so $X^\top \vec{e} = 0$, we have

$$X^\top \vec{y} = X^\top X\vec{a} + X^\top \vec{e} = X^\top X\vec{a}. \quad (\text{A.2})$$

So

$$(X^\top X)^{-1} X^\top \vec{y} = \vec{a} \quad (\text{A.3})$$

and

$$X(X^\top X)^{-1} X^\top \vec{y} = X\vec{a} = \tilde{\vec{y}}. \quad (\text{A.4})$$

Thus $X(X^\top X)^{-1} X^\top$ is the projection matrix of \vec{y} on X . If \vec{y} is a time series of uniform intervals, we can set $x_i = 1, 2, 3, \dots, s$ without loss of generality. That is $P = S(S^\top S)^{-1} S^\top$ is the projection matrix of the p -th order polynomial fitting for any time series of uniform intervals where $S_{i,j} = i^{j-1}$ with $i = 1, \dots, s$ and $j = 1, \dots, p + 1$. □

Proof of Proposition 2.1. Let $Z(t)$ and $W(t)$ the stochastic processes generating according to Eq. 9. The trend, obtained by \mathcal{P} -order polynomial fitting, of the vector $Z_v = (Z_v(1), \dots, Z_v(s))^\top$ is given by $\tilde{Z}_v = PZ_v$ where P is given by lemma 2.1 with $p = \mathcal{P}$. Then, the detrended covariance function of such outliers-contaminated processes can be written as:

$$f_v^{Z,W}(s) = \frac{1}{s} \mathcal{Z}^\top \mathcal{W} \text{ where } \mathcal{Z} = Z_v - PZ_v \text{ and } \mathcal{W} = W_v - PW_v. \quad (\text{A.5})$$

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Using that $Z_v = X_v + \theta_X \mu_v^X$ and $W_v = Y_v + \theta_Y \mu_v^Y$ where $\mu_v^X = (\mu_v^X(1), \dots, \mu_v^X(s))^\top$, $\mu_v^Y = (\mu_v^Y(1), \dots, \mu_v^Y(s))^\top$, $\mu_v^X(k) = \sum_{j=1}^k (1 - B^X(\ell_v + j)) N^X(\ell_v + j)$ and $\mu_v^Y(k) = \sum_{j=1}^k (1 - B^Y(\ell_v + j)) N^Y(\ell_v + j)$. So $\mathcal{Z} = AX_v + \theta_X A \mu_v^X$ and $\mathcal{W} = AY_v + \theta_Y A \mu_v^Y$ where $A = I_s - P$ and I_s is a s -identity matrix. Then

$$\begin{aligned}
\mathbb{E}[f_v^{Z,W}(s)] &= \frac{1}{s} \mathbb{E}[\mathcal{Z}^\top \mathcal{W}] \\
&= \frac{1}{s} \mathbb{E}[(AX_v + \theta_X A \mu_v^X)^\top (AY_v + \theta_Y A \mu_v^Y)] \\
&= \frac{1}{s} \mathbb{E}[(X_v^\top A^\top + \theta_X \mu_v^{X^\top} A^\top) (AY_v + \theta_Y A \mu_v^Y)] \\
&= \frac{1}{s} \mathbb{E}[X_v^\top A^\top AY_v + \theta_Y X_v^\top A^\top A \mu_v^Y + \theta_X \mu_v^{X^\top} A^\top AY_v + \theta_X \theta_Y \mu_v^{X^\top} A^\top A \mu_v^Y]; \quad A^\top = A \text{ and } A^2 = A \\
&= \frac{1}{s} \mathbb{E}[X_v^\top AY_v + \theta_Y X_v^\top A \mu_v^Y + \theta_X \mu_v^{X^\top} AY_v + \theta_X \theta_Y \mu_v^{X^\top} A \mu_v^Y] \\
&= \frac{1}{s} \mathbb{E}[X_v^\top AY_v] + \frac{1}{s} \theta_X \theta_Y \mathbb{E}[\mu_v^{X^\top} A \mu_v^Y]; \quad \text{cov}(X_v, \mu_v^X) = \text{cov}(Y_v, \mu_v^Y) = 0 \text{ and } \mathbb{E}[X_v] = \mathbb{E}[Y_v] = 0 \\
&= \frac{1}{s} \text{Tr}[A \text{cov}(X_v, Y_v)] + \frac{1}{s} \mathbb{E}[X_v]^\top A \mathbb{E}[Y_v] + \frac{1}{s} \theta_X \theta_Y \text{Tr}[A \text{cov}(\mu_v^X, \mu_v^Y)] + \frac{1}{s} \theta_X \theta_Y \mathbb{E}[\mu_v^{X^\top}]^\top A \mathbb{E}[\mu_v^Y] \\
&= \frac{1}{s} (\text{Tr}(A \text{cov}[X_v, Y_v]) + \theta_X \theta_Y \eta_Y^\top A \eta_X); \quad \text{cov}(\mu_v^X, \mu_v^Y) = 0.
\end{aligned}$$

□

Proof of Corollary 2.1. Let $\mathcal{P} = 2$, $a = \sum_{k=1}^s k$ and $b = \sum_{k=1}^s k^2$ then $(S^\top S)^{-1} = (sb - a^2)^{-1} \begin{pmatrix} b & -a \\ -a & s \end{pmatrix}$ and $\eta_Y^\top I_s \eta_X = \mu \nu p_Y p_X b$. After computation we obtain $\eta_Y^\top S (S^\top S)^{-1} S^\top \eta_X = \mu \nu p_Y p_X b$. Then, according to proposition 2.1 and using that $A = I_s - S (S^\top S)^{-1} S^\top$, we have

$$\mathbb{E}[f_v^{Z,W}(s)] = \frac{\text{Tr}(A \text{cov}[X_v, Y_v])}{s}. \quad (\text{A.6})$$

Let $\lambda(\star)$ denotes the eigenvalues \star , by theorem 2 in [33], we have

$$-\frac{1}{2} \lambda_{\max}(-2A) \text{Tr}(\text{cov}[X_v, Y_v]) \leq \text{Tr}(A \text{cov}[X_v, Y_v]) \leq \frac{1}{2} \lambda_{\max}(2A) \text{Tr}(\text{cov}[X_v, Y_v]), \quad (\text{A.7})$$

where λ_{\max} is the largest eigenvalue. Using that $-\frac{1}{2} \lambda(-2A) = \frac{1}{2} \lambda(2A) = \lambda(A)$ and that all eigenvalues of A are either 0 or 1, so $\text{Tr}(A \text{cov}[X_v, Y_v]) = \text{Tr}(\text{cov}[X_v, Y_v])$. Then

$$\mathbb{E}[f_v^{Z,W}(s)] = \frac{\text{Tr}(\text{cov}[X_v, Y_v])}{s}. \quad (\text{A.8})$$

On the other hand, we have

$$\begin{aligned}\text{cov}[X_v(\ell), Y_v(\ell)] &= \text{cov}\left[\sum_{j=1}^{\ell} X(\ell_v + j), \sum_{j'=1}^{\ell} Y(\ell_v + j')\right] \\ &= \sum_{j=1}^{\ell} \sum_{j'=1}^{\ell} \gamma_{XY}(j' - j) \\ &= \sum_{k=1}^{\ell-1} (\ell - k)(\gamma_{XY}(k) + \gamma_{XY}(-k)) + \gamma_{XY}(0)\ell,\end{aligned}$$

then

$$\text{Tr}(\text{cov}[X_v, Y_v]) = \sum_{\ell=1}^s \sum_{k=1}^{\ell-1} (\ell - k)(\gamma_{XY}(k) + \gamma_{XY}(-k)) + \gamma_{XY}(0) \sum_{\ell=1}^s \ell. \quad (\text{A.9})$$

By Eq. A.8 and Eq. A.9, we have

$$\mathbb{E}[f_v^{Z,W}(s)] = \frac{1}{s} \sum_{\ell=1}^s \sum_{k=1}^{\ell-1} (\ell - k)(\gamma_{XY}(k) + \gamma_{XY}(-k)) + \frac{\sigma_X \sigma_Y \rho_{XY}}{s} \sum_{\ell=1}^s \ell. \quad (\text{A.10})$$

Let $\alpha = H_X + H_Y \neq 1$. For $k \geq 1$, we have $\text{sign}(k - 1) = \text{sign}(k) = \text{sign}(k + 1) = 1$, then

$$\gamma_{XY}(k) = \frac{\sigma_X \sigma_Y}{2} k^\alpha (\rho_{XY} - \eta_{XY}) B(k) \text{ where } B(k) = (1 - 1/k)^\alpha - 2 + (1 + 1/k)^\alpha. \quad (\text{A.11})$$

On the other hand, $\text{sign}(-k - 1) = \text{sign}(-k) = \text{sign}(-k + 1) = -1$, then $\gamma_{XY}(-k) = \frac{\sigma_X \sigma_Y}{2} k^\alpha (\rho_{XY} + \eta_{XY}) B(-k)$. Using that $B(-k) = B(k)$, we have $\gamma_{XY}(k) + \gamma_{XY}(-k) = \frac{\sigma_X \sigma_Y \rho_{XY}}{2} k^\alpha B(k)$, so

$$\mathbb{E}[F_{Z,W}(2, s)^2] = \frac{\sigma_X \sigma_Y \rho_{XY}}{2s} \sum_{\ell=1}^s \sum_{k=1}^{\ell-1} (\ell - k) k^\alpha B(k) + \frac{\sigma_X \sigma_Y \rho_{XY}}{s} \sum_{\ell=1}^s \ell. \quad (\text{A.12})$$

If s is large enough, the principal term in s of $\mathbb{E}[F_{Z,W}(2, s)^2]$ is given by $\frac{\sigma_X \sigma_Y \rho_{XY}}{2s} \sum_{k=1}^s (s - k) k^\alpha B(k) + \frac{\sigma_X \sigma_Y \rho_{XY}}{2} s$. Using that $k^\alpha B(k) = (k - 1)^\alpha - 2k^\alpha + (k + 1)^\alpha$, then

$$\frac{1}{s} \sum_{k=1}^s (s - k) k^\alpha B(k) = s^{\alpha+1} \left(\int_0^1 (1 - x) \left((x - \frac{1}{s})^\alpha - 2x^\alpha + (x + \frac{1}{s})^\alpha \right) dx + \mathcal{O}(s^{-1}) \right). \quad (\text{A.13})$$

If s is large enough, the integral $\int_0^1 (1 - x) \left((x - \frac{1}{s})^\alpha - 2x^\alpha + (x + \frac{1}{s})^\alpha \right) dx$ tends to zero. Thus

$$\mathbb{E}[F_{Z,W}(2, s)^2] = \frac{\sigma_X \sigma_Y \rho_{XY}}{2} s^\alpha \mathcal{O}(1) + \frac{\sigma_X \sigma_Y \rho_{XY}}{2} s. \quad (\text{A.14})$$

□