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| Rabah Souam. On stable capillary hypersurfaces with planar boundaries. 2021. hal-03410986v2

**HAL Id: hal-03410986**

**<https://hal.science/hal-03410986v2>**

Preprint submitted on 14 Nov 2021 (v2), last revised 7 Jan 2022 (v3)

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# ON STABLE CAPILLARY HYPERSURFACES WITH PLANAR BOUNDARIES

RABAH SOUAM

**ABSTRACT.** We study stable immersed capillary hypersurfaces  $\Sigma$  in domains  $\mathcal{B}$  of  $\mathbb{R}^{n+1}$  bounded by hyperplanes. When  $\mathcal{B}$  is a half-space, we show  $\Sigma$  is a spherical cap. When  $\mathcal{B}$  is a domain bounded by  $k$  hyperplanes  $P_1, \dots, P_k$ ,  $2 \leq k \leq n+1$ , having independent normals, and  $\Sigma$  has contact angle  $\theta_i$  with  $P_i$  and does not touch the edges of  $\mathcal{B}$ , we prove there exists  $\delta > 0$ , depending only on  $P_1, \dots, P_k$ , so that if  $\theta_i \in (\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta)$  for each  $i$ , then  $\Sigma$  has to be a piece of a sphere.

## 1. INTRODUCTION

A capillary hypersurface in a domain  $\mathcal{B}$  of a Riemannian manifold is a constant mean curvature (CMC) hypersurface in  $\mathcal{B}$  meeting  $\partial\mathcal{B}$  at a constant angle. Capillary hypersurfaces are critical points for an energy functional for variations which preserve the *enclosed volume*. More precisely, given an angle  $\theta \in (0, \pi)$ , for a compact hypersurface  $\Sigma$  inside  $\mathcal{B}$  such that  $\partial\Sigma \subset \partial\mathcal{B}$  and  $\partial\Sigma$  bounds a compact domain  $W$  in  $\partial\mathcal{B}$ , the energy of  $\Sigma$  is by definition the quantity

$$E(\Sigma) := |\Sigma| - \cos \theta |W|.$$

Here and throughout this paper, we use the notation  $|M|$  to denote the volume of a Riemannian manifold  $M$ . The energy functional  $E$  can be defined for general immersed hypersurfaces, not only those with boundary bounding a compact domain in  $\partial\mathcal{B}$ . Details can be found in [1]. The stationary hypersurfaces of  $E$  for variations preserving the *enclosed volume* are precisely the CMC hypersurfaces making a constant angle  $\theta$  with  $\partial\mathcal{B}$ .

Existence and uniqueness problems of capillary hypersurfaces are interesting on their own right. In the minimal case, when  $\mathcal{B}$  is the unit ball in  $\mathbb{R}^3$  and the angle of contact is  $\frac{\pi}{2}$ , these questions proved to be related to deep problems in Geometric Analysis, see for instance the work of Fraser-Schoen [4]. When  $\mathcal{B}$  is a domain in  $\mathbb{R}^3$ , capillary surfaces correspond to models of incompressible liquids inside a container  $\mathcal{B}$  in the absence of gravity. The free interface of the liquid (locally) minimizes the energy functional  $E$ . It is then a natural problem to study the stable ones, that is, those for which the second variation of the energy functional is non-negative for all volume preserving variations. Stability issues for capillary hypersurfaces have been recently actively addressed in different ambient

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2010 *Mathematics Subject Classification.* 53A10, 49Q10, 53C42, 76B45.

*Key words and phrases.* Capillary hypersurfaces, constant mean curvature hypersurfaces, stability.

manifolds and domains, see for instance [1, 3, 5, 7, 8, 9, 10]. One of the achievements in this direction is the proof by Wang-Xia [10] that in a unit ball in a space form, totally umbilical hypersurfaces are the only stable capillary hypersurfaces (Nunes [8] previously solved the free boundary case in a unit ball in  $\mathbb{R}^3$ ).

In this paper, we are interested in domains  $\mathcal{B}$  in  $\mathbb{R}^{n+1}$  bounded by a finite family of hyperplanes. The first case we consider is when  $\mathcal{B}$  is a half-space in  $\mathbb{R}^{n+1}$ . In this case, it was shown by Ainouz-Souam [1] that a stable capillary hypersurface with contact angle  $\theta < \frac{\pi}{2}$  is a spherical cap provided each of its boundary components is embedded. Choe-Koiso [3] obtained the same result when  $\theta > \frac{\pi}{2}$  under the assumption that the boundary is convex and Marinov [7] treated the case  $n = 2$  assuming the boundary is embedded. In our main result (Theorem 3.2) we remove the boundary embeddedness assumption thus characterizing the spherical caps as the only stable immersed capillary hypersurfaces in half-spaces in  $\mathbb{R}^{n+1}$ ,  $n \geq 2$ . The second case we study is when  $\mathcal{B}$  is a domain in  $\mathbb{R}^{n+1}$  bounded by a finite number of hyperplanes  $P_1, \dots, P_k$  with linearly independent normals. Li-Xiong [5] showed that, in this situation, a stable capillary hypersurface meeting each  $P_i$  with a constant angle  $\theta_i \in [\pi/2, \pi]$  and not touching the edges of  $\mathcal{B}$  is a piece of a sphere under the assumption that its boundary is embedded for  $n = 2$  and that each of its boundary components is convex for  $n \geq 3$ . The case  $k = 2$ , that is, when  $\mathcal{B}$  is a wedge, was previously obtained by Choe-Koiso [3]. We here prove (Theorem 3.3) the existence of a positive number  $\delta > 0$  such that if  $\Sigma$  is a stable immersed capillary hypersurface in  $\mathcal{B}$  not touching the edges of  $\mathcal{B}$  and making an angle  $\theta_i \in (\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta)$  with each  $P_i$ , then  $\Sigma$  is a piece of a sphere. We emphasize that in our results we do not assume the boundary of the hypersurface is embedded.

## 2. PRELIMINARIES

We here recall briefly some basic facts about capillary hypersurfaces and refer to [1] for more details.

Let  $n \geq 2$  be an integer and  $\mathcal{B}$  a domain in  $\mathbb{R}^{n+1}$  bounded by a finite number  $P_1, \dots, P_k$  of hyperplanes,  $k \geq 1$ . A capillary hypersurface in  $\mathcal{B}$  is a compact and constant mean curvature (CMC) hypersurface contained in  $\mathcal{B}$  with boundary on  $\partial\mathcal{B}$  and meeting each  $P_i$  with a constant angle  $\theta_i \in (0, \pi)$ . Consider a capillary hypersurface defined by a smooth immersion  $\psi : \Sigma \rightarrow \mathcal{B}$  and let  $N$  be a global unit normal to  $\Sigma$  along  $\psi$  chosen so that its (constant) mean curvature satisfies  $H \geq 0$ . Write  $\partial\Sigma = \cup_{i=1}^k \Gamma_i$ , with  $\psi(\Gamma_i) \subset P_i$ . By an edge of  $\mathcal{B}$  we mean a non empty intersection  $P_i \cap P_j$  with  $i \neq j$ . In the sequel we will suppose that the hypersurface does not touch the edges of  $\mathcal{B}$  so that  $\psi(\partial\Sigma)$  lies on the smooth parts of  $\partial\mathcal{B}$  where the unit normal is well defined. This condition can be weakened slightly but we will restrict ourselves to this case for simplicity.

Capillary immersions in  $\mathcal{B}$  are critical points of an energy functional for deformations through immersions in  $\mathcal{B}$ , mapping  $\partial\Sigma$  into  $\partial\mathcal{B}$  and preserving the *enclosed volume*. When  $\psi$  is an embedding with each  $\psi(\Gamma_i)$  bounding a domain  $W_i \subset P_i$ , the energy writes:

$$E(\psi) = |\psi(\Sigma)| - \sum_{i=1}^k \cos \theta_i |W_i|$$

and the enclosed volume is the volume of the domain in  $\mathbb{R}^{n+1}$  bounded by  $\psi(\Sigma)$  and  $\cup_{i=1}^k W_i$ . This energy functional and the notion of enclosed volume can be extended in a suitable way to include general immersions, details can be found in [1].

We denote by  $\mathbf{n}_i$  the exterior unit normal to  $P_i$ . The angle of contact along  $\Gamma_i$  is the angle  $\theta_i \in (0, \pi)$  between  $N$  and  $\mathbf{n}_i$ . We let  $\nu_i$  be the exterior unit normal to  $\Gamma_i$  in  $\Sigma$  and  $\bar{\nu}_i$  be the unit normal to  $\Gamma_i$  in  $\partial\mathcal{B}$  chosen so that  $\{N, \nu_i\}$  and  $\{\mathbf{n}_i, \bar{\nu}_i\}$  have the same orientation in  $(T\partial\Sigma)^\perp$ . The angle between  $\nu_i$  and  $\bar{\nu}_i$  is also equal to  $\theta_i$ .

The index form  $\mathcal{I}$  of  $\psi$  is the symmetric bilinear form defined on  $\mathcal{C}^\infty(\Sigma)$  by

$$\mathcal{I}(f, g) = - \int_{\Sigma} f (\Delta g + |\sigma|^2 g) dA + \sum_{i=1}^k \int_{\Gamma_i} f \left( \frac{\partial g}{\partial \nu_i} - q_i g \right) ds,$$

where  $\Delta$  stands for the Laplacian of the metric induced by  $\psi$ ,  $\sigma$  is the second fundamental form of  $\psi$  and

$$q_i = \cot \theta_i \sigma(\nu_i, \nu_i).$$

A capillary immersion is said to be stable if the second variation of the energy  $E$  is nonnegative for all volume preserving deformations. This condition is equivalent to

$$\mathcal{I}(f, f) \geq 0 \quad \text{for all } f \in \mathcal{C}^\infty(\Sigma) \quad \text{with} \quad \int_{\Sigma} f dA = 0,$$

see [1].

Let  $D$  denote the standard covariant differentiation on  $\mathbb{R}^{n+1}$ . A basic well-known fact we will use is the following (see for instance [1]):

**Proposition 2.1.** *Let  $\mathcal{B}$  be as above and  $\psi : \Sigma \rightarrow \mathcal{B}$  a capillary immersion into  $\mathcal{B}$ . Then, the unit conormal  $\nu_i$  along  $\Gamma_i$  is a principal direction of  $\psi$ , that is,  $D_{\nu_i} N = -\sigma(\nu_i, \nu_i) \nu_i$ .*

### 3. THE RESULTS

A key ingredient we will use is the following fact obtained in [1], we include the proof for completeness.

**Proposition 3.1.** *Let  $\psi : \Sigma \rightarrow \mathbb{R}^{n+1}$  be a smooth immersion in the Euclidean space  $\mathbb{R}^{n+1}$  of a smooth compact orientable  $n$ -dimensional manifold  $\Sigma$  with or without boundary. Let  $N : \Sigma \rightarrow \mathbb{S}^n \subset \mathbb{R}^{n+1}$  be a global unit normal of  $\psi$  and  $\nu$  the exterior unit conormal to  $\partial\Sigma$  in  $\Sigma$ . Then*

$$(3.1) \quad n \int_{\Sigma} N dA = \int_{\partial\Sigma} \{ \langle \psi, \nu \rangle N - \langle \psi, N \rangle \nu \} ds$$

where  $dA$  and  $ds$  denote the volume elements of  $\Sigma$  and  $\partial\Sigma$ , respectively.

In particular, if  $\Sigma$  has no boundary, then

$$\int_{\Sigma} N dA = 0.$$

*Proof.* Let  $a$  be a constant vector field on  $\mathbb{R}^{n+1}$ . Consider the following vector field on  $\Sigma$

$$X = \langle a, N \rangle \psi^T - \langle \psi, N \rangle a^T,$$

where,  $\psi^T = \psi - \langle \psi, N \rangle N$  (resp.  $a^T = a - \langle a, N \rangle N$ ) is the projection of  $\psi$  (resp. of  $a$ ) on  $T\Sigma$ . Using the following well known formulas that can be thoroughly checked:

$$\operatorname{div} \psi^T = n + nH \langle \psi, N \rangle, \quad \operatorname{div} a^T = nH \langle a, N \rangle,$$

we compute the divergence of  $X$ :

$$\begin{aligned} \operatorname{div} X &= \langle a, N \rangle \operatorname{div} \psi^T + \langle a, D_{\psi^T} N \rangle - \langle \psi, N \rangle \operatorname{div} a^T - \langle a^T, N \rangle - \langle \psi, D_{a^T} N \rangle \\ &= n \langle a, N \rangle + nH \langle \psi, N \rangle \langle a, N \rangle + \langle a^T, D_{\psi^T} N \rangle - nH \langle a, N \rangle \langle \psi, N \rangle - \langle \psi^T, D_{a^T} N \rangle \\ &= n \langle a, N \rangle, \end{aligned}$$

where we used the symmetry of the Weingarten map, that is,  $\langle a^T, D_{\psi^T} N \rangle = \langle \psi^T, D_{a^T} N \rangle$ . Integrating on  $\Sigma$  and using the divergence theorem we get

$$n \int_{\Sigma} \langle a, N \rangle dA = \int_{\partial\Sigma} \{ \langle a, N \rangle \langle \psi, \nu \rangle - \langle \psi, N \rangle \langle a, \nu \rangle \} ds$$

Since this is true for any  $a$ , (3.1) follows.  $\square$

We now prove our first result characterizing the spherical caps as the only stable immersed capillary hypersurfaces in half-spaces in  $\mathbb{R}^{n+1}$ . This was proved under additional assumptions on the boundary and the angle of contact in [1], [3] and [7]. As it will be clear from the proof, it is not necessary to assume the hypersurface is contained in a half-space, we only use the fact that it meets a hyperplane with a constant angle.

**Theorem 3.2.** *Let  $\psi : \Sigma \rightarrow \mathbb{R}^{n+1}$ , be a capillary immersion of a compact and connected orientable manifold  $\Sigma$  of dimension  $n$  in a half-space in  $\mathbb{R}^{n+1}$ ,  $n \geq 2$ . If the immersion is stable then  $\psi(\Sigma)$  is a spherical cap.*

*Proof.* Without loss of generality, we may suppose the half-space is the upper half-space  $x_{n+1} \geq 0$ . The starting point is to derive a suitable test function inspired by the work of Barbosa-do Carmo [2] proving that round spheres are the only compact and closed stable CMC hypersurfaces in  $\mathbb{R}^{n+1}$ .

Integrating the equation  $\operatorname{div} \psi^T = n + nH \langle \psi, N \rangle$ , we get

$$(3.2) \quad \int_{\partial\Sigma} \langle \psi, \nu \rangle ds = n \int_{\Sigma} \{ 1 + H \langle \psi, N \rangle \} dA.$$

We denote, as usual, by  $e_1, \dots, e_{n+1}$  the vectors of the canonical basis of  $\mathbb{R}^{n+1}$ . On  $\partial\Sigma$ , we have  $\cos \theta N + \sin \theta \nu = -e_{n+1}$ ,  $\langle \psi, \nu \rangle = \cos \theta \langle \psi, \bar{\nu} \rangle$  and  $\langle \psi, N \rangle = -\sin \theta \langle \psi, \bar{\nu} \rangle$ . Proposition 3.1 gives in this case

$$(3.3) \quad n \cos \theta \int_{\Sigma} N dA = - \left( \int_{\partial \Sigma} \langle \psi, \nu \rangle ds \right) e_{n+1}.$$

From (3.2) and (3.3), we conclude that:

$$\int_{\Sigma} \{1 + H \langle \psi, N \rangle + \cos \theta \langle N, e_{n+1} \rangle\} dA = 0.$$

So we may use  $\phi := 1 + H \langle \psi, N \rangle + \cos \theta \langle N, e_{n+1} \rangle$  as a test function in the stability inequality. Set  $u = \langle \psi, N \rangle$  and  $v = \langle N, e_{n+1} \rangle$ . These functions satisfy the following well-known formulas

$$(3.4) \quad \Delta u + |\sigma|^2 u = -nH,$$

and

$$(3.5) \quad \Delta v + |\sigma|^2 v = 0.$$

Using these equations, we compute:

$$\begin{aligned} \Delta \phi &= H(-nH - |\sigma|^2 u) - \cos \theta |\sigma|^2 v \\ &= -nH^2 - |\sigma|^2 (Hu + \cos \theta v) \\ &= -nH^2 - |\sigma|^2 (\phi - 1). \end{aligned}$$

Therefore

$$\phi \Delta \phi + |\sigma|^2 \phi^2 = (|\sigma|^2 - nH^2) \phi.$$

Moreover:

$$\frac{\partial u}{\partial \nu} = \langle \nu, N \rangle + \langle \psi, D_{\nu} N \rangle = -\sigma(\nu, \nu) \langle \psi, \nu \rangle$$

and

$$\frac{\partial v}{\partial \nu} = \langle D_{\nu} N, e_{n+1} \rangle = -\sigma(\nu, \nu) \langle \nu, e_{n+1} \rangle = \sigma(\nu, \nu) \sin \theta,$$

Now, taking into account that, along  $\partial \Sigma$  :  $\langle \psi, \nu \rangle = \cos \theta \langle \psi, \bar{\nu} \rangle$  and  $\langle \psi, N \rangle = -\sin \theta \langle \psi, \bar{\nu} \rangle$ , one can check after direct computations that:

$$(3.6) \quad \frac{\partial \phi}{\partial \nu} = \cot \theta \sigma(\nu, \nu) \phi.$$

It follows that

$$(3.7) \quad \mathcal{I}(\phi, \phi) = - \int_{\Sigma} (|\sigma|^2 - nH^2) \phi dA = - \int_{\Sigma} (|\sigma|^2 - nH^2) (1 + H \langle \psi, N \rangle + \cos \theta \langle N, e_{n+1} \rangle) dA.$$

We now introduce the following function:

$$F = |H\psi + N|^2 = H^2 |\psi|^2 + 2H \langle \psi, N \rangle + 1.$$

One has  $\Delta |\psi|^2 = 2n(1 + H \langle \psi, N \rangle)$  and  $\Delta \langle \psi, N \rangle = -|\sigma|^2 \langle \psi, N \rangle - nH$ , so that

$$\begin{aligned} \Delta F &= 2nH^2(1 + H \langle \psi, N \rangle) + 2H(-|\sigma|^2 \langle \psi, N \rangle - nH) \\ &= 2H(nH^2 - |\sigma|^2) \langle \psi, N \rangle \end{aligned}$$

Integrating we get

$$2H \int_{\Sigma} (nH^2 - |\sigma|^2) \langle \psi, N \rangle dA = \int_{\Sigma} \Delta F dA = \int_{\partial \Sigma} \frac{\partial F}{\partial \nu} ds,$$

that is,

$$(3.8) \quad H \int_{\Sigma} (nH^2 - |\sigma|^2) \langle \psi, N \rangle dA = H \int_{\partial\Sigma} (H - \sigma(v, v)) \langle \psi, v \rangle ds.$$

Denote by  $H_{\partial\Sigma}$  the mean curvature of the immersion  $\psi|_{\partial\Sigma} : \partial\Sigma \longrightarrow \mathbb{R}^n \times \{0\}$  computed with respect to the unit normal  $\bar{v}$ . We claim that, on  $\partial\Sigma$  :

$$(3.9) \quad H - \sigma(v, v) = -(n-1)(H + \sin \theta H_{\partial\Sigma}).$$

Indeed, let  $i \in \{1, \dots, k\}$  and  $\{v_1, \dots, v_{n-1}\}$  be a local orthonormal frame on  $\partial\Sigma$ . Then:

$$\sigma(v, v) = nH - \sum_{j=1}^{n-1} \sigma(v_j, v_j).$$

Now, considering the unit normal  $\bar{v}$  to  $\partial\Sigma$  in  $\mathbb{R}^n \times \{0\}$  along  $\psi$ , as chosen in Sec. 2, we have  $N = -\sin \theta \bar{v} - \cos \theta e_{n+1}$ . We can thus write

$$\sigma(v_j, v_j) = -\langle \nabla_{v_j} N, v_j \rangle = \sin \theta \langle \nabla_{v_j} \bar{v}, v_j \rangle.$$

Therefore the following relation holds on  $\partial\Sigma$

$$\sigma(v, v) = nH + (n-1) \sin \theta H_{\partial\Sigma}.$$

This proves (3.9). Using (3.9) and the fact that  $\langle \psi, v \rangle = \cos \theta \langle \psi, \bar{v} \rangle$ , we can write (3.8) as follows:

$$(3.10) \quad H \int_{\Sigma} (nH^2 - |\sigma|^2) \langle \psi, N \rangle dA = -(n-1)H \cos \theta \int_{\partial\Sigma} (H + \sin \theta H_{\partial\Sigma}) \langle \psi, \bar{v} \rangle ds.$$

We claim that

$$(3.11) \quad \int_{\partial\Sigma} (H + \sin \theta H_{\partial\Sigma}) \langle \psi, \bar{v} \rangle ds = 0.$$

To see this, we first apply Minkowski formula to the immersion  $\psi|_{\partial\Sigma} : \partial\Sigma \longrightarrow \mathbb{R}^n \times \{0\}$ , obtaining:  $\int_{\partial\Sigma} H_{\partial\Sigma} \langle \psi, \bar{v} \rangle ds = -|\partial\Sigma|$ . Thus

$$\int_{\partial\Sigma} (H + \sin \theta H_{\partial\Sigma}) \langle \psi, \bar{v} \rangle ds = H \int_{\partial\Sigma} \langle \psi, \bar{v} \rangle ds - \sin \theta |\partial\Sigma|.$$

Integrating the relation  $\Delta\psi = nHN$ , we obtain

$$\int_{\partial\Sigma} v ds = nH \int_{\Sigma} N dA.$$

Therefore

$$\int_{\partial\Sigma} \langle v, e_{n+1} \rangle ds = nH \int_{\Sigma} \langle N, e_{n+1} \rangle dA,$$

that is,

$$(3.12) \quad -\sin \theta |\partial\Sigma| = nH \int_{\Sigma} \langle N, e_{n+1} \rangle dA.$$

Along  $\partial\Sigma$ , we have  $\langle \psi, v \rangle = \cos \theta \langle \psi, \bar{v} \rangle$  and  $\langle \psi, N \rangle = -\sin \theta \langle \psi, \bar{v} \rangle$ . So (3.1) gives

$$(3.13) \quad n \int_{\Sigma} \langle N, e_{n+1} \rangle dA = - \int_{\partial\Sigma} \langle \psi, \bar{v} \rangle ds.$$

(3.12) and (3.13) show that

$$H \int_{\partial\Sigma} \langle \psi, \bar{\nu} \rangle ds - \sin \theta |\partial\Sigma| = 0.$$

This proves (3.11). So (3.10) gives

$$H \int_{\Sigma} (nH^2 - |\sigma|^2) \langle \psi, N \rangle = 0.$$

Going back to (3.7), we get

$$\mathcal{I}(\phi, \phi) = - \int_{\Sigma} (|\sigma|^2 - nH^2) (1 + \cos \theta \langle N, e_{n+1} \rangle).$$

As  $\theta \in (0, \pi)$ , we have  $1 + \cos \theta \langle N, e_{n+1} \rangle > 0$ . Also  $|\sigma|^2 - nH^2 \geq 0$  with equality only at umbilics. Hence  $\mathcal{I}(\phi, \phi) \leq 0$ . However, by the stability assumption,  $\mathcal{I}(\phi, \phi) \geq 0$ . We conclude that necessarily  $|\sigma|^2 - nH^2 \equiv 0$  and so  $\psi(\Sigma)$  is totally umbilical, that is, a spherical cap.  $\square$

In the above proof we strongly used in our computations that the origin lies on the bounding hyperplane. This explains the hypothesis in our second result dealing with a domain bounded by a finite family of hyperplanes. It was inspired by the work of Li-Xiong [5].

**Theorem 3.3.** *Let  $\mathcal{B}$  be a domain in  $\mathbb{R}^{n+1}$  bounded by  $k$  hyperplanes  $P_1, \dots, P_k$ ,  $2 \leq k \leq n+1$ , having linearly independent normals. Then there exists a constant  $\delta > 0$  so that if  $\psi : \Sigma \rightarrow \mathcal{B}$  is a stable compact and connected immersed capillary hypersurface in  $\mathcal{B}$  not touching the edges of  $\mathcal{B}$  and having a contact angle  $\theta_i \in (\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta)$  with  $P_i$ , then  $\psi(\Sigma)$  is a part of a sphere.*

*Proof.* Let  $\mathbf{n}_i$  denote, as above, the unit normal to  $P_i$  that points outward  $\mathcal{B}$ . The independence of  $\mathbf{n}_1, \dots, \mathbf{n}_k$  ensures that  $\cap_{i=1}^k P_i \neq \emptyset$  and so we may assume, without loss of generality, that the origin lies in  $\cap_{i=1}^k P_i$ . Following [5], we consider a linear combination  $a := \sum_{i=1}^k c_i \mathbf{n}_i$  verifying:  $\langle a, \mathbf{n}_i \rangle = -\cos \theta_i$ ,  $i = 1, \dots, k$ . The vector  $a$  exists and is unique by the independence of  $\mathbf{n}_1, \dots, \mathbf{n}_k$ . We consider the function

$$\phi = 1 + H \langle \psi, N \rangle + \langle a, N \rangle.$$

Proceeding as in the proof of Theorem 3.2, one shows that  $\int_{\Sigma} \phi dA = 0$ , so that  $\phi$  can be used as a test function, and that

$$\mathcal{I}(\phi, \phi) = - \int_{\Sigma} (|\sigma|^2 - nH^2) \phi dA = - \int_{\Sigma} (|\sigma|^2 - nH^2) (1 + H \langle \psi, N \rangle + \langle a, N \rangle) dA.$$

We claim that

$$(3.14) \quad \int_{\Sigma} (|\sigma|^2 - nH^2) H \langle \psi, N \rangle dA = 0.$$

To prove this, write  $\partial\Sigma = \cup_{i=1}^k \Gamma_i$ , with  $\psi(\Gamma_i) \subset P_i$ . As in the proof of Theorem 3.2, considering the function  $F = |H\psi + N|^2$ , one shows that

$$(3.15) \quad H \int_{\Sigma} (nH^2 - |\sigma|^2) \langle \psi, N \rangle dA = -(n-1)H \sum_{i=1}^k \cos \theta_i \int_{\Gamma_i} (H + \sin \theta_i) H_{\Gamma_i} \langle \psi, \bar{\nu}_i \rangle ds,$$



where  $H_{\Gamma_i}$  is the mean curvature of the immersion  $\psi|_{\Gamma_i} : \Gamma_i \longrightarrow P_i$  with respect to the unit normal  $\bar{v}_i$ . As before, Proposition 3.1 together with the relation  $\langle \psi, v_i \rangle = \cos \theta \langle \psi, \bar{v}_i \rangle$  on  $\Gamma_i$ , give

$$(3.16) \quad n \int_{\Sigma} N dA = \sum_{i=1}^k \left( \int_{\Gamma_i} \langle \psi, \bar{v}_i \rangle dA \right) \mathbf{n}_i.$$

Integrating the relation  $\Delta \psi = nHN$ , we obtain

$$nH \int_{\Sigma} N dA = \sum_{i=1}^k \int_{\Gamma_i} v_i ds = \sum_{i=1}^k \int_{\Gamma_i} (\cos \theta_i \bar{v}_i + \sin \theta_i \mathbf{n}_i) ds.$$

By Proposition 3.1 applied to the immersion  $\psi|_{\Gamma_i} : \Gamma_i \longrightarrow P_i$ , one has  $\int_{\Gamma_i} \bar{v}_i ds = 0$ . So,

$$(3.17) \quad nH \int_{\Sigma} N dA = \sum_{i=1}^k \sin \theta_i |\Gamma_i| \mathbf{n}_i.$$

From (3.16), (3.17) and the independence of the  $\mathbf{n}_i$ 's, we infer that

$$(3.18) \quad \int_{\Gamma_i} H \langle \psi, \bar{v}_i \rangle ds = \sin \theta_i |\Gamma_i|.$$

Applying Minkowski formula to the immersion  $\psi|_{\Gamma_i} : \Gamma_i \longrightarrow P_i$ , we get

$$(3.19) \quad |\Gamma_i| = - \int_{\Gamma_i} H_{\Gamma_i} \langle \psi, \bar{v}_i \rangle ds.$$

(3.18) and (3.19) give

$$\int_{\Gamma_i} (H + \sin \theta_i H_{\Gamma_i}) \langle \psi, \bar{v}_i \rangle ds = 0.$$

Together with (3.15) this proves (3.14). Summarizing, we have proved that

$$\mathcal{I}(\phi, \phi) = - \int_{\Sigma} (|\sigma|^2 - nH^2)(1 + \langle a, N \rangle) dA.$$

As in the proof of Theorem 3.2, we can conclude that  $\psi(\Sigma)$  is totally umbilical provided  $|a| < 1$ . Now, recall that  $a$  is the unique solution, in the linear space spanned by  $\mathbf{n}_1, \dots, \mathbf{n}_k$ , to the system of linear equations  $\langle a, \mathbf{n}_i \rangle = -\cos \theta_i$ ,  $i = 1, \dots, k$ , and depends continuously on the  $\theta_i$ 's. When  $\theta_i = \frac{\pi}{2}$ , for  $i = 1, \dots, k$ , the solution to the system is  $a = 0$ . Therefore there exists  $\delta > 0$  such that if  $\theta_i \in (\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta)$ ,  $i = 1, \dots, k$ , then  $|a| < 1$  and  $\psi(\Sigma)$  is totally umbilical. Note that  $\psi(\Sigma)$  cannot be planar because otherwise it would not be disjoint from the edges of  $\mathcal{B}$ . Therefore  $\psi(\Sigma)$  is a piece of a sphere.  $\square$

Conversely, a part of a sphere as above is stable and even a minimizer of the energy functional  $E$  among all embedded hypersurfaces in  $\mathcal{B}$  enclosing the same volume, see [11].

The free boundary case, that is, when  $\theta_i = \frac{\pi}{2}$ , for each  $i = 1, \dots, k$ , was treated by Lopez [6]. The only possibility for a piece of a sphere to meet orthogonally the hyperplanes  $P_i$  is to be centered at a point in the intersection  $\cap_{i=1}^k P_i$ . The conclusion of Theorem 3.3 is therefore that a

stable hypersurface with free boundary has to touch some of the edges of  $\mathcal{B}$ .

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