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Optimal control of averaged state of a population dynamics model

Cyrille KENNE * Boniface NKEMZI †

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Abstract

In this article, we study the average control of a population dynamic model with age dependence and spatial structure in a bounded domain $\Omega \subset \mathbb{R}^3$. We assume that we can act on the system via a control in a sub-domain ω of Ω . We prove that we can bring the average of the state of our model at time $t = T$ to a desired state. By means of Euler-Lagrange first order optimality condition, we expressed the optimal control in terms of average of an appropriate adjoint state that we characterize by an optimality system.

Mathematics Subject Classification. 49J20, 92D25, 35Q93, 93C05

Key-words : Population dynamics, average control, Euler-Lagrange formula.

1 Introduction

The invasive species have a real impact on communities and ecosystems. They are viewed as a significant component of global change and they have severe negative consequences for biodiversity. We consider a model describing the dynamics of an invasive species with age dependence and spatial structure, the invasive species are diffusing in the habitat with a diffusion coefficient depending of the susceptibility of the habitat to invasion (ecological factors) and/or genetics factors of the species. We then consider a population with age dependence and spatial structure, and we assume that the population lives in a bounded domain $\Omega \subset \mathbb{R}^3$. We denote by Γ the boundary of the domain and we assume that it is of class C^2 . For the time $T > 0$, the life expectancy of an individual $A > 0$ and $\theta_{min}, \theta_{max} > 0$, we set $I = (\theta_{min}, \theta_{max})$, $U = (0, T) \times (0, A)$, $Q = U \times \Omega$, $\Sigma = U \times \Gamma$, $Q_A = (0, A) \times \Omega$, $Q_T = (0, T) \times \Omega$ and $Q_\omega = U \times \omega$, where ω is a non-empty open subset of Ω . for $\theta \in I$, the system reads as follows:

$$\left\{ \begin{array}{ll} \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - d(\theta)\Delta y + \mu y & = f + v\chi_\omega & \text{in } Q, \\ y & = 0 & \text{on } \Sigma, \\ y(0, \cdot, \cdot, \theta) & = y^0 & \text{in } Q_A, \\ y(\cdot, 0, \cdot, \theta) & = \int_0^A \beta(t, a, x)y(t, a, x, \theta) da & \text{in } Q_T, \end{array} \right. \quad (1)$$

where

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- $y = y(t, a, x, \theta)$ is the distribution of individuals of age $a \geq 0$, at time $t \geq 0$ and location $x \in \Omega$.
- The recruitment $f \in L^2(Q)$ is a positive periodic function.
- The control $v \in L^2(Q_\omega)$ which corresponds to the removal of the individuals in a sub-domain ω of Ω and χ_ω denote the characteristic function of the control set ω .
- The mortality rate $\mu = \mu(a) \geq 0$ is a known increasing positive function which is continuous on $[0, A]$, whereas the fertility rate $\beta = \beta(t, a, x) \in L^\infty(Q)$ is known and positive.
- $d(\theta) > 0$ is the diffusion coefficient of species dispersal in the environment and is assumed depending of susceptibility $\theta \in I$ and $d \in \mathcal{C}(I)$.

Model (1) is a system with varying parameter and our question is: Let z_d be a given age-dependent distribution of species, can the average of the solution to (1) at time $t = T$ be steered to z_d upon selecting a suitable control v corresponding to a removal (eradication) of species on the sub-domain ω ?

Remark 1 *Set*

$$W(T, A) = \left\{ \rho \in L^2(U; H_0^1(\Omega)); \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} \in L^2(U; H^{-1}(\Omega)) \right\}. \quad (2)$$

Then we have (see [4]) that

$$W(T, A) \subset \mathcal{C}([0, T], L^2(Q_A)) \text{ and } W(T, A) \subset \mathcal{C}([0, A], L^2(Q_T)). \quad (3)$$

Under the assumptions on the data, and for $\theta \in I$ fixed, (1) has a unique solution $y(\theta; v) = y(t, a, x, \theta; v)$ in $W(T, A)$. More precisely, we are concerned in this paper by the following optimization problem:

$$\inf_{v \in L^2(Q_\omega)} J(v), \quad (4)$$

where the cost function is given by

$$J(v) = \left\| \int_I y(\theta; v)(T) \, d\theta - z_d \right\|_{L^2(Q_A)}^2 + N \|v\|_{L^2(Q_\omega)}^2, \quad (5)$$

with $z_d \in L^2(Q_A)$ and $N > 0$ are given, $\int_I y(\theta; v)(T) \, d\theta \in L^2(Q_A)$, for all $v \in L^2(Q_\omega)$.

Optimal control for age-structured population was studied later by some authors like A. Ouedrogo and al. [9]. In this paper the authors considered a nonlinear age-structured population dynamics model and they study the existence of an optimal control making the density of the population as close as possible of some given density. In [1] B. Ainseba and al. investigated the optimal harvesting problem for a nonlinear age-dependent and spatially structured population dynamics model with a constant diffusion coefficient, where the birth process is described by a nonlocal and nonlinear boundary condition. The notion of averaged control was introduced by Zuazua [10] to analyse the problem of controlling parameter dependent systems. In this notion, the aim is to find a control, independent of the unknown parameter, so that the average of the state is controlled. For more literature on the topic, we refer for instance to Lohac and Zuazua [6], Lazar and Zuazua [5], Hafdallah and Ayadi [2] and LU and Zuazua [7], G. Mophou et al. [8] and the reference therein. In this paper, we are concerned with the control of a parameter dependent age structured population dynamics system.

The rest of this paper is structured as follows. In Section 2, we give some regularity results. In Section 3, we prove the existence and the uniqueness of the control and we characterize with an optimality system. A conclusion is given in Section 4.

2 Preliminary results

In order to solve the optimization problem (4), we need some preliminary results.

In what follows, we will sometime adopt the following notation

$$\begin{cases} L &= \frac{\partial}{\partial t} + \frac{\partial}{\partial a} - d(\theta)\Delta + \mu I, \\ L^* &= -\frac{\partial}{\partial t} - \frac{\partial}{\partial a} - d(\theta)\Delta + \mu I, \end{cases} \quad (6)$$

where I is the identity operator.

Remark 2 From now on, we use $C(X)$ to denote a positive constant whose value varies from a line to another but depends on X ; the positive constant $d_0 = \inf_{\theta \in I} d(\theta)$ and we will denote by $(\cdot, \cdot)_H$ the scalar product in H .

Lemma 2.1 Let $v \in L^2(Q_\omega)$ and $y \in L^2(U; H_0^1(\Omega))$ be solution of (1), then we have the following estimations:

$$\begin{aligned} \|y\|_{L^2(U; H_0^1(\Omega))} &\leq C(T, \|\beta\|_{L^2(Q)}) (\|y^0\|_{L^2(Q_A)} + \|f\|_{L^2(Q)} + \|v\|_{L^2(Q_\omega)}), \\ \|y(T, \cdot, \cdot, \theta)\|_{L^2(Q_A)} &\leq C(T, \|\beta\|_{L^2(Q)}) (\|y^0\|_{L^2(Q_A)} + \|f\|_{L^2(Q)} + \|v\|_{L^2(Q_\omega)}), \\ \|y(\cdot, A, \cdot, \theta)\|_{L^2(Q_T)} &\leq C(T, \|\beta\|_{L^2(Q)}) (\|y^0\|_{L^2(Q_A)} + \|f\|_{L^2(Q)} + \|v\|_{L^2(Q_\omega)}). \end{aligned}$$

Proof. We proceed as in [3]. We recall that $y = y(t, a, x, \theta; v)$ is solution of the problem

$$\begin{cases} \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - d(\theta)\Delta y + \mu y &= f + v\chi_{Q_\omega} && \text{in } Q, \\ y &= 0 && \text{on } \Sigma, \\ y(0, \cdot, \cdot, \theta) &= y^0 && \text{in } Q_A, \\ y(\cdot, 0, \cdot, \theta) &= \int_0^A \beta(t, a, x)y(t, a, x, \theta) da && \text{in } Q_T. \end{cases}$$

By defining $z = e^{-rt}y$ with $r > 0$, we obtain that z is solution of the problem

$$\begin{cases} \frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} - d(\theta)\Delta z + (\mu + r)z &= f + v\chi_{Q_\omega} && \text{in } Q, \\ z &= 0 && \text{on } \Sigma, \\ z(0, \cdot, \cdot, \theta) &= y^0 && \text{in } Q_A, \\ z(\cdot, 0, \cdot, \theta) &= \int_0^A \beta(t, a, x)z(t, a, x, \theta) da && \text{in } Q_T. \end{cases} \quad (7)$$

Multiplying the first equation of system (7) by z and integrating by parts over Q , we get:

$$\begin{aligned} &\frac{1}{2} \|z(T, \cdot, \cdot, \theta)\|_{L^2(Q_A)}^2 - \frac{1}{2} \|z(0, \cdot, \cdot, \theta)\|_{L^2(Q_A)}^2 + \frac{1}{2} \|z(\cdot, A, \cdot, \theta)\|_{L^2(Q_T)}^2 \\ &\quad - \frac{1}{2} \|z(\cdot, 0, \cdot, \theta)\|_{L^2(Q_T)}^2 + d(\theta) \|\nabla z\|_{L^2(Q)}^2 + \int_Q (r + \mu) z^2 dx dt da \\ &\quad = \int_Q (f + v\chi_{Q_\omega}) z dx dt da. \end{aligned}$$

Then using the fact that $\mu \geq 0$, it follows the inequality

$$\begin{aligned} & \frac{1}{2} \|z(T, \cdot, \cdot, \theta)\|_{L^2(Q_A)}^2 + \frac{1}{2} \|z(\cdot, A, \cdot, \theta)\|_{L^2(Q_T)}^2 + d_0 \|\nabla z\|_{L^2(Q)}^2 + r \|z\|_{L^2(Q)}^2 \\ & \leq \frac{1}{2} \|y^0\|_{L^2(Q_A)}^2 + \frac{1}{2} \|z(\cdot, 0, \cdot, \theta)\|_{L^2(Q_T)}^2 + \frac{1}{2} \|f + v\chi_{Q_\omega}\|_{L^2(Q)}^2 \\ & \quad + \frac{1}{2} \|z\|_{L^2(Q)}^2. \end{aligned} \tag{8}$$

On the other hand, one can write for $(t, x) \in (0, T) \times \Omega$

$$z(t, 0, x, \theta) = \int_0^A \beta(t, a, x) z(t, a, x, \theta) da,$$

then

$$\|z(\cdot, 0, \cdot, \theta)\|_{L^2(Q_T)}^2 \leq \|\beta\|_{L^2(Q)}^2 \|z\|_{L^2(Q)}^2.$$

Thus (8) gives

$$\begin{aligned} & \frac{1}{2} \|z(T, \cdot, \cdot, \theta)\|_{L^2(Q_A)}^2 + \frac{1}{2} \|z(\cdot, A, \cdot, \theta)\|_{L^2(Q_T)}^2 + d_0 \|\nabla z\|_{L^2(Q)}^2 \\ & + \left(r - \frac{1}{2} \|\beta\|_{L^2(Q)}^2 - \frac{1}{2} \right) \|z\|_{L^2(Q)}^2 \leq \frac{1}{2} \|f + v\chi_{Q_\omega}\|_{L^2(Q)}^2 \\ & \quad + \frac{1}{2} \|y^0\|_{L^2(Q_A)}^2. \end{aligned}$$

By choosing r such that $r = \frac{1}{2} \|\beta\|_{L^2(Q)}^2 + \frac{1}{2}$, one has

$$\begin{aligned} & \|z(T, \cdot, \cdot, \theta)\|_{L^2(Q_A)}^2 + \|z(\cdot, A, \cdot, \theta)\|_{L^2(Q_T)}^2 + \|z\|_{L^2(U; H_0^1(\Omega))}^2 \\ & \leq C \left(\|y^0\|_{L^2(Q_A)}^2 + \|f\|_{L^2(Q)}^2 + \|v\|_{L^2(Q_\omega)}^2 \right). \end{aligned}$$

This implies

$$\begin{aligned} & \|y(T, \cdot, \cdot, \theta)\|_{L^2(Q_A)}^2 + \|y(\cdot, A, \cdot, \theta)\|_{L^2(Q_T)}^2 + \|y\|_{L^2(U; H_0^1(\Omega))}^2 \\ & \leq C(T, \|\beta\|_{L^2(Q)}) \left(\|y^0\|_{L^2(Q_A)}^2 + \|f\|_{L^2(Q)}^2 + \|v\|_{L^2(Q_\omega)}^2 \right). \end{aligned}$$

So that,

$$\begin{aligned} & \|y\|_{L^2(U; H_0^1(\Omega))} \leq C(T, \|\beta\|_{L^2(Q)}) \left(\|y^0\|_{L^2(Q_A)} + \|f\|_{L^2(Q)} + \|v\|_{L^2(Q_\omega)} \right), \\ & \|y(T, \cdot, \cdot, \theta)\|_{L^2(Q_A)} \leq C(T, \|\beta\|_{L^2(Q)}) \left(\|y^0\|_{L^2(Q_A)} + \|f\|_{L^2(Q)} + \|v\|_{L^2(Q_\omega)} \right), \\ & \|y(\cdot, A, \cdot, \theta)\|_{L^2(Q_T)} \leq C(T, \|\beta\|_{L^2(Q)}) \left(\|y^0\|_{L^2(Q_A)} + \|f\|_{L^2(Q)} + \|v\|_{L^2(Q_\omega)} \right). \end{aligned}$$

■

Proposition 2.1 *Let $\theta \in I$, then the map $v \mapsto y(\theta; v)$ is a continuous function from $L^2(Q_\omega)$ onto $L^2(U, H_0^1(\Omega))$.*

Proof. Let $\theta \in I$ and $v_0 \in L^2(Q_\omega)$. We show that $v \mapsto y(\theta; v)$ is continuous at v_0 . Set $\bar{y} = y(\theta; v) - y(\theta; v_0)$, then \bar{y} is solution to the problem

$$\begin{cases} L\bar{y} = v\chi_{Q_\omega} - v_0\chi_{Q_\omega} & \text{in } Q, \\ \bar{y} = 0 & \text{on } \Sigma, \\ \bar{y}(0, \cdot, \cdot, \theta) = 0 & \text{in } Q_A, \\ \bar{y}(\cdot, 0, \cdot, \theta) = \int_0^A \beta(t, a, x)\bar{y} da & \text{in } Q_T, \end{cases} \quad (9)$$

In view of the Lemma 2.1, we have that

$$\|\bar{y}\|_{L^2(U; H_0^1(\Omega))} \leq C(T, \|\beta\|_{L^2(Q)}) \|v - v_0\|_{L^2(Q_\omega)}.$$

As $v \rightarrow v_0$, we have $\bar{y} \rightarrow 0$ strongly in $L^2(U; H_0^1(\Omega))$. Hence $y(\theta; v) \rightarrow y(\theta; v_0)$ strongly in $L^2(U; H_0^1(\Omega))$ as $v \rightarrow v_0$. ■

Proposition 2.2 *Let $\lambda > 0$. Let $v, w \in L^2(Q_\omega)$. Let also $y = y(\theta; v)$ be a solution of system (1). Set $\bar{y}_\lambda = \frac{y(\theta; v + \lambda w) - y(\theta; v)}{\lambda}$, then (\bar{y}_λ) converges strongly in $L^2(U; H_0^1(\Omega))$ as $\lambda \rightarrow 0$ to a function \bar{y} which is solution of*

$$\begin{cases} L\bar{y} = w\chi_{Q_\omega} & \text{in } Q, \\ \bar{y} = 0 & \text{on } \Sigma, \\ \bar{y}(0, \cdot, \cdot, \theta) = 0 & \text{in } Q_A, \\ \bar{y}(\cdot, 0, \cdot, \theta) = \int_0^A \beta(t, a, x)\bar{y} da & \text{in } Q_T. \end{cases} \quad (10)$$

Proof. \bar{y}_λ is a solution to the problem

$$\begin{cases} L\bar{y}_\lambda = w\chi_{Q_\omega} & \text{in } Q, \\ \bar{y}_\lambda = 0 & \text{on } \Sigma, \\ \bar{y}_\lambda(0, \cdot, \cdot, \theta) = 0 & \text{in } Q_A, \\ \bar{y}_\lambda(\cdot, 0, \cdot, \theta) = \int_0^A \beta(t, a, x)\bar{y}_\lambda da & \text{in } Q_T. \end{cases}$$

Define $y_\lambda = \bar{y}_\lambda - \bar{y}$, where \bar{y} is a solution to (10). Then y_λ is a solution to

$$\begin{cases} Ly_\lambda = 0 & \text{in } Q, \\ y_\lambda = 0 & \text{on } \Sigma, \\ y_\lambda(0, \cdot, \cdot, \theta) = 0 & \text{in } Q_A, \\ y_\lambda(\cdot, 0, \cdot, \theta) = \int_0^A \beta(t, a, x)y_\lambda da & \text{in } Q_T. \end{cases} \quad (11)$$

From the Lemma 2.1, we obtain that

$$\|y_\lambda\|_{L^2(U; H_0^1(\Omega))} \leq 0. \quad (12)$$

Passing to the limit in this latter identity when $\lambda \rightarrow 0$, it follows that $y_\lambda \rightarrow 0$ strongly in $L^2(U; H_0^1(\Omega))$. This means that (\bar{y}_λ) converges to \bar{y} strongly in $L^2(U; H_0^1(\Omega))$ as $\lambda \rightarrow 0$. ■

3 Existence and characterization of the control

In this section, we will show that the optimization problem (4) has a unique solution. Moreover, we will give the equations that characterize the control.

Proposition 3.1 *There exists a unique control $u \in L^2(Q_\omega)$ solution of (4).*

Proof. Observing that we have $J(0) \geq 0$, we have that the set $\{J(v) : J(v) \geq 0, v \in L^2(Q_\omega)\}$ is a non-empty lower bounded subset of \mathbb{R} , consequently $\alpha = \inf_{v \in L^2(Q_\omega)} J(v)$ exists. Let $(v_n)_n$ be a minimizing sequence such that

$$J(v_n) \rightarrow \alpha, \text{ when } n \rightarrow +\infty. \quad (13)$$

Then we have that there exists $C > 0$ independent of n such that for all $n \in \mathbb{N}$, $J(v_n) \leq C$. i.e.

$$\left\| \int_I y(\theta; v_n)(T) d\theta - z_d \right\|_{L^2(Q_A)}^2 + N \|v_n\|_{L^2(Q_\omega)}^2 \leq C,$$

so

$$\|v_n\|_{L^2(Q_\omega)} \leq C, \quad (14)$$

$$\left\| \int_I y(\theta; v_n)(T) d\theta \right\|_{L^2(Q_A)} \leq C. \quad (15)$$

Now $y_n = y(t, a, x, \theta; v_n)$ is solution of the problem

$$\begin{cases} \frac{\partial y_n}{\partial t} + \frac{\partial y_n}{\partial a} - d(\theta)\Delta y_n + \mu y_n = f + v_n \chi_{Q_\omega} & \text{in } Q, \\ y_n = 0 & \text{on } \Sigma, \\ y_n(0, \cdot, \cdot, \theta) = y^0 & \text{in } Q_A, \\ y_n(\cdot, 0, \cdot, \theta) = \int_0^A \beta(t, a, x) y_n(t, a, x, \theta) da & \text{in } Q_T. \end{cases} \quad (16)$$

In view of (14), we obtain from Lemma 2.1 that

$$\|y_n\|_{L^2(U; H_0^1(\Omega))} \leq C(T, \|\beta\|_{L^2(Q)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}), \quad (17)$$

$$\|y_n(T, \cdot, \cdot, \theta)\|_{L^2(Q_A)} \leq C(T, \|\beta\|_{L^2(Q)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}), \quad (18)$$

$$\|y_n(\cdot, A, \cdot, \theta)\|_{L^2(Q_T)} \leq C(T, \|\beta\|_{L^2(Q)}, \|y^0\|_{L^2(Q_A)}, \|f\|_{L^2(Q)}). \quad (19)$$

Using (14), (17), (18) and (19) there exist $u \in L^2(Q_\omega)$, $y \in L^2(U; H_0^1(\Omega))$, $y_T \in L^2(Q_A)$, $y_A \in L^2(Q_T)$ and extracted sequences from the sequences $(v_n)_n$, $(y_n)_n$, $(y_n(T, \cdot, \cdot, \theta))_n$, $(y_n(\cdot, A, \cdot, \theta))_n$ still denoted $(v_n)_n$, $(y_n)_n$, $(y_n(T, \cdot, \cdot, \theta))_n$, $(y_n(\cdot, A, \cdot, \theta))_n$ such that the following convergences hold

$$v_n \rightharpoonup u \text{ weakly in } L^2(Q_\omega), \quad (20)$$

$$y_n \rightharpoonup y \text{ weakly in } L^2(U; H_0^1(\Omega)), \quad (21)$$

$$y_n(T, \cdot, \cdot, \theta) \rightharpoonup y_T \text{ weakly in } L^2(Q_A), \quad (22)$$

$$y_n(\cdot, A, \cdot, \theta) \rightharpoonup y_A \text{ weakly in } L^2(Q_T). \quad (23)$$

Now let us prove that (u, y) satisfies (1). Let $\phi \in \mathcal{D}(Q)$ a test function. Multiplying the first equation in (16) by ϕ and integrating by parts over Q , we obtain

$$\langle y_n, L^* \phi \rangle = \langle f, \phi \rangle + \langle v_n \chi_\omega, \phi \rangle, \quad \forall \phi \in \mathcal{D}(Q). \quad (24)$$

Taking the limit as $n \rightarrow +\infty$ in (24) and using (20) and (21) yields

$$\langle y, L^* \phi \rangle = \langle f, \phi \rangle + \langle u \chi_\omega, \phi \rangle, \quad \forall \phi \in \mathcal{D}(Q).$$

That is

$$\langle Ly, \phi \rangle = \langle f, \phi \rangle + \langle u \chi_\omega, \phi \rangle, \quad \forall \phi \in \mathcal{D}(Q).$$

Thus,

$$Ly = f + u \chi_\omega \quad \text{in } Q. \quad (25)$$

Now on one hand as $y \in L^2(U; H_0^1(\Omega))$, using (25), we have that $\frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} \in L^2(U; H^{-1}(\Omega))$. This implies that $y \in W(T, A)$, by (3) of Remark 1, $y(T, \cdot, \cdot, \theta)$, $y(0, \cdot, \cdot, \theta)$ exist and belong to $L^2(Q_A)$ and $y(\cdot, A, \cdot)$, $y(\cdot, 0, \cdot, \theta)$ exist and belong to $L^2(Q_T)$. On the other hand $y \in L^2(Q)$ and $d(\theta)\Delta y \in H^{-1}(U; L^2(\Omega))$; consequently $y|_\Sigma$ and $\frac{\partial y}{\partial \nu}|_\Sigma$ exist and belong respectively to $H^{-1}(U; H^{-\frac{1}{2}}(\Gamma))$ and $H^{-1}(U; H^{-\frac{3}{2}}(\Gamma))$. Multiplying the first equation in (16) by $\phi \in \mathcal{C}^\infty(\bar{Q})$ such that $\phi = 0$ on Σ , $\phi(\cdot, A, \cdot) = 0$ in Q_T and integrating by parts over Q and using initials and boundary conditions, we obtain

$$\begin{aligned} & (y_n(T, \cdot, \cdot, \theta), \phi(0, \cdot, \cdot))_{L^2(Q_A)} - (y^0, \phi(0, \cdot, \cdot))_{L^2(Q_A)} \\ & - (y_n(\cdot, 0, \cdot, \theta), \phi(\cdot, 0, \cdot))_{L^2(Q_T)} + (y_n, L^* \phi)_{L^2(Q)} \\ & = (f, \phi)_{L^2(Q)} + (v_n, \phi)_{L^2(Q_\omega)} \\ & \forall \phi \in \mathcal{C}^\infty(\bar{Q}), \phi = 0 \text{ on } \Sigma, \phi(\cdot, A, \cdot) = 0 \text{ in } Q_T. \end{aligned} \quad (26)$$

Note that

$$y_n(\cdot, 0, \cdot, \theta) \rightharpoonup y^1 = \int_0^A \beta(t, a, x) y(t, a, x, \theta) da \quad \text{weakly in } L^2(Q_T). \quad (27)$$

Indeed let $\phi \in L^2(Q_T)$, then

$$\begin{aligned} \int_{Q_T} y_n(t, 0, x, \theta) \phi(x, t) dx dt & = \int_{Q_T} \left(\int_A \beta(t, a, x) y_n(t, a, x, \theta) da \right) \phi(x, t) dx dt \\ & = \int_Q y_n(t, a, x, \theta) \psi(t, a, x) dx dt da, \end{aligned} \quad (28)$$

where $\psi(t, a, x) = \beta(t, a, x) \phi(x, t) \in L^2(Q)$. By letting $n \rightarrow +\infty$ in (28) while using (21), we obtain

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{Q_T} y_n(t, 0, x, \theta) \phi(x, t) dx dt & = \int_Q y(t, a, x, \theta) \psi(t, a, x) dx dt da \\ & = \int_Q \beta(t, a, x) y(t, a, x, \theta) \phi(x, t) dx dt da \\ & = \int_{Q_T} \left(\int_A \beta(t, a, x) y(t, a, x, \theta) da \right) \phi(x, t) dx dt. \end{aligned}$$

So that (27) holds. Moreover by taking the limit as $n \rightarrow +\infty$ in (26) and by using (20)-(22) and (27) we are lead to

$$\begin{aligned} & (y_T, \phi(0, \cdot, \cdot))_{L^2(Q_A)} - (y^0, \phi(0, \cdot, \cdot))_{L^2(Q_A)} \\ & - (y^1, \phi(\cdot, 0, \cdot))_{L^2(Q_T)} + (y, L^* \phi)_{L^2(Q)} = (f, \phi)_{L^2(Q)} + (u, \phi)_{L^2(Q_\omega)}, \\ & \forall \phi \in \mathcal{C}^\infty(\bar{Q}), \phi = 0 \text{ on } \Sigma, \phi(\cdot, A, \cdot) = 0 \text{ in } Q_T. \end{aligned}$$

an integration by parts gives

$$\begin{aligned}
& (y_T, \phi(T, \cdot, \cdot))_{L^2(Q_A)} - (y^0, \phi(0, \cdot, \cdot))_{L^2(Q_A)} - (y^1, \phi(\cdot, 0, \cdot))_{L^2(Q_T)} \\
& - (y(T, \cdot, \cdot, \theta), \phi(T, \cdot, \cdot))_{L^2(Q_A)} + (y(0, \cdot, \cdot, \theta), \phi(0, \cdot, \cdot))_{L^2(Q_A)} \\
& - (y(\cdot, 0, \cdot, \theta), \phi(\cdot, 0, \cdot))_{L^2(Q_T)} + (Ly, \phi)_{L^2(Q)} \\
& - \left\langle y, \frac{\partial \phi}{\partial \nu} \right\rangle_{H^{-1}(U; H^{-\frac{1}{2}}), H_0^1(U; H^{\frac{1}{2}})} = (f, \phi)_{L^2(Q)} + (u, \phi)_{L^2(Q_\omega)}, \\
& \forall \phi \in \mathcal{C}^\infty(\overline{Q}), \phi = 0 \text{ on } \Sigma, \phi(\cdot, A, \cdot) = 0 \text{ in } Q_T.
\end{aligned}$$

That is in view of (25),

$$\begin{aligned}
& (y_T, \phi(T, \cdot, \cdot))_{L^2(Q_A)} - (y^0, \phi(0, \cdot, \cdot))_{L^2(Q_A)} - (y^1, \phi(\cdot, 0, \cdot))_{L^2(Q_T)} \\
& - (y(T, \cdot, \cdot, \theta), \phi(T, \cdot, \cdot))_{L^2(Q_A)} + (y(0, \cdot, \cdot, \theta), \phi(0, \cdot, \cdot))_{L^2(Q_A)} \\
& - (y(\cdot, 0, \cdot, \theta), \phi(\cdot, 0, \cdot))_{L^2(Q_T)} - \left\langle y, \frac{\partial \phi}{\partial \nu} \right\rangle_{H^{-1}(U; H^{-\frac{1}{2}}), H_0^1(U; H^{\frac{1}{2}})} = 0, \\
& \forall \phi \in \mathcal{C}^\infty(\overline{Q}), \phi = 0 \text{ on } \Sigma, \phi(\cdot, A, \cdot) = 0 \text{ in } Q_T. \tag{29}
\end{aligned}$$

Choosing respectively in (29), ϕ such that $\phi(T, \cdot, \cdot) = 0$, $\phi(\cdot, 0, \cdot) = 0$, $\frac{\partial \phi}{\partial \nu} = 0$ and $\phi(\cdot, 0, \cdot) = 0$, $\frac{\partial \phi}{\partial \nu} = 0$ and $\frac{\partial \phi}{\partial \nu} = 0$; we successively obtain

$$y(0, \cdot, \cdot, \theta) = y^0 \quad \text{in } Q_A. \tag{30}$$

$$y_T = y(T, \cdot, \cdot, \theta) \quad \text{in } Q_A. \tag{31}$$

$$y(\cdot, 0, \cdot, \theta) = \int_0^A \beta(t, a, x) y(t, a, x, \theta) da \quad \text{in } Q_T. \tag{32}$$

and finally from (29)

$$y = 0 \quad \text{on } \Sigma. \tag{33}$$

By (25), (30)-(33), it follows that (u, y) solves (1). Moreover if we set

$$V_n = \int_I y_n(T, \cdot, \cdot, \theta) d\theta,$$

then in view of (15), there exist a subsequence of the sequence $(V_n)_n$ still denoted $(V_n)_n$ and $V \in L^2(Q_A)$ such that as $n \rightarrow +\infty$, $\forall \phi \in L^2(Q_A)$,

$$\begin{aligned}
\int_{Q_A} V_n(a, x) \phi(a, x) dadx &= \int_I \left(\int_{Q_A} y_n(T, \cdot, \cdot, \theta) \phi(a, x) dadx \right) d\theta \\
&\rightarrow \int_{Q_A} V(a, x) \phi(a, x) dadx. \tag{34}
\end{aligned}$$

Now using (18), we deduce that the sequence $(y_n(T, \cdot, \cdot, \theta))_n$ is bounded independently of θ . Moreover using (22) and (31) it follows that

$$\lim_{n \rightarrow +\infty} \int_{Q_A} y_n(T, a, x, \theta) \phi(a, x) \, dadx = \int_{Q_A} y(T, a, x, \theta) \phi(a, x) \, dadx,$$

for all $\phi \in L^2(Q_A)$. If we set $z_n = \int_{Q_A} y_n(T, a, x, \theta) \phi(a, x) \, dadx$, then using (18) we get for all $n \in \mathbb{N}$,

$$|z_n| \leq C \|\phi\|_{L^2(Q_A)}.$$

It follows from the Lebesgue dominated convergence theorem that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_I z_n \, d\theta &= \int_I \lim_{n \rightarrow +\infty} z_n \, d\theta \\ &= \int_I \left(\int_{Q_A} y(T, a, x, \theta) \phi(a, x) \, dadx \right) \, d\theta. \end{aligned}$$

So $\int_{Q_A} V_n(a, x) \phi(a, x) \, dadx$ converges towards

$$\int_I \left(\int_{Q_A} y_T(a, x, \theta) \phi(a, x) \, dadx \right) \, d\theta = \int_{Q_A} \left(\int_I y(T, a, x, \theta) \, d\theta \right) \phi(a, x) \, dadx, \quad \forall \phi \in L^2(Q_A).$$

Using (34) and the uniqueness of the limit, we have that for $(a, x) \in Q_A$

$$V(a, x) = \int_I y(T, a, x, \theta) \, d\theta,$$

so we can write

$$\int_I y_n(T, \cdot, \cdot, \theta) \, d\theta \rightharpoonup \int_I y(T, \cdot, \cdot, \theta) \, d\theta \quad \text{weakly in } Q_A. \quad (35)$$

According to (13), from the weak lower semi-continuity of the function $v \mapsto J(v)$, (20) and (35), we obtain $J(u) \leq \liminf_{n \rightarrow +\infty} J(v_n)$. Which implies that $J(u) \leq \alpha$. But since α is the lower bound, we then have $\alpha = J(u)$. In addition, the function J is strictly convex. Therefore u is unique. ■
We can now characterize the control u .

Proposition 3.2 *Let u be the solution of (1). Then there exists $q \in L^2(U; H_0^1(\Omega))$ such that $\int_I q(\theta) \, d\theta \in L^2(Q)$ and $\{y, q\}$ is solution to*

$$\left\{ \begin{array}{ll} \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - d(\theta) \Delta y + \mu y &= f + u \chi_{Q_\omega} & \text{in } Q, \\ y &= 0 & \text{on } \Sigma, \\ y(0, \cdot, \cdot, \theta) &= y^0 & \text{in } Q_A, \\ y(\cdot, 0, \cdot, \theta) &= \int_0^A \beta(t, a, x) y(t, a, x, \theta) \, da & \text{in } Q_T, \end{array} \right. \quad (36)$$

$$\left\{ \begin{array}{ll} -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - d(\theta)\Delta q + \mu q = \beta(t, a, x)q(\cdot, 0, \cdot, \theta) & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(T, \cdot, \cdot, \theta) = \int_I y(\theta, u)(T) d\theta - z_d & \text{in } Q_A, \\ q(\cdot, A, \cdot, \theta) = 0 & \text{in } Q_T, \end{array} \right. \quad (37)$$

and

$$u = -\frac{1}{N} \int_I q(\theta) d\theta \quad \text{in } Q_\omega. \quad (38)$$

Proof. We write the Euler-Lagrange first order optimality condition which characterize the optimal control u .

$$\lim_{\lambda \rightarrow 0} \frac{J(u + \lambda v) - J(u)}{\lambda} = 0, \quad \forall v \in L^2(Q_\omega).$$

Let $v \in L^2(Q_\omega)$ and $\lambda > 0$. We have

$$\begin{aligned} J(u + \lambda v) - J(u) &= \left\| \int_I [y(\theta; u + \lambda v)(T) - y(\theta; u)(T)] d\theta \right\|_{L^2(Q_A)}^2 \\ &\quad + N^2 \lambda^2 \|v\|_{L^2(Q_\omega)}^2 + 2N\lambda \int_{L^2(Q_\omega)} uv \, dadxdt \\ + 2 \left(\int_I [y(\theta; u + \lambda v)(T) - y(\theta; u)(T)] d\theta; \int_I y(\theta; u)(T) d\theta - z_d \right)_{L^2(Q_A)}. \end{aligned}$$

Then

$$\begin{aligned} \frac{J(u + \lambda v) - J(u)}{\lambda} &= \lambda \left\| \int_I \left[\frac{y(\theta; u + \lambda v)(T) - y(\theta; u)(T)}{\lambda} \right] d\theta \right\|_{L^2(Q_A)}^2 \\ &\quad + N^2 \lambda \|v\|_{L^2(Q_\omega)}^2 + 2N \int_{L^2(Q_\omega)} uv \, dadxdt \\ + 2 \left(\int_I \left[\frac{y(\theta; u + \lambda v)(T) - y(\theta; u)(T)}{\lambda} \right] d\theta; \int_I y(\theta; u)(T) d\theta - z_d \right)_{L^2(Q_A)}. \end{aligned} \quad (39)$$

Let us set $z_\lambda := z_\lambda(\theta; v) = \frac{y(\theta; u + \lambda v) - y(\theta; u)}{\lambda}$, then using Proposition 2.2, we obtain that z_λ converges strongly in $L^2(U; H_0^1(\Omega))$ as $\lambda \rightarrow 0$ to $z = z(\theta; v)$ solution of

$$\left\{ \begin{array}{ll} \frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} - d(\theta)\Delta z + \mu z = v\chi_{Q_\omega} & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(0, \cdot, \cdot, \theta) = 0 & \text{in } Q_A, \\ z(\cdot, 0, \cdot, \theta) = \int_0^A \beta(t, a, x)z(t, a, x, \theta) da & \text{in } Q_T. \end{array} \right. \quad (40)$$

Moreover since $\int_I y(\theta; u)(T) d\theta \in L^2(Q_A)$, then we also have $\int_I z(\theta; v)(T) d\theta \in L^2(Q_A)$. Taking (40) in account, passing to the limit as $\lambda \rightarrow 0$ in (39) and using the Lebesgue dominated convergence theorem,

we are lead to

$$\left(\int_I z(T, \cdot, \cdot, \theta) d\theta; \int_I y(\theta; u)(T) d\theta - z_d \right)_{L^2(Q_A)} + N \int_{L^2(Q_\omega)} uv dadxdt = 0, \quad (41)$$

$$\forall v \in L^2(Q_\omega).$$

To interpret (41), we consider the following adjoint system

$$\begin{cases} L^*q = \beta(t, a, x)q(\cdot, 0, \cdot, \theta) & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(T, \cdot, \cdot, \theta) = \int_I y(\theta; u)(T) d\theta - z_d & \text{in } Q_A, \\ q(\cdot, A, \cdot, \theta) = 0 & \text{in } Q_T, \end{cases} \quad (42)$$

where $q = q(\theta, u)$ is such that $\int_I q(\theta) d\theta \in L^2(Q)$. Since $\beta(t, a, x) \in L^\infty(0, A)$ and $q(\cdot, 0, \cdot, \theta) \in L^2(Q_T)$, then $\beta(t, a, x)q(\cdot, 0, \cdot, \theta) \in L^2(Q)$ and using that $\int_I y(\theta; u)(T) d\theta - z_d \in L^2(Q_A)$, it follows that $q \in L^2(U; H_0^1(\Omega))$ and $\frac{\partial q}{\partial t} + \frac{\partial q}{\partial a} \in L^2(U; H^{-1}(\Omega))$. So if multiply the first equation in (40) by q solution of (42) and integrate by parts over Q , we obtain

$$\begin{aligned} & (z(T, \cdot, \cdot, \theta), q(T, \cdot, \cdot, \theta))_{L^2(Q_A)} + (z(\cdot, A, \cdot, \theta), q(\cdot, A, \cdot, \theta))_{L^2(Q_T)} \\ & - (z(\cdot, 0, \cdot, \theta), q(\cdot, 0, \cdot, \theta))_{L^2(Q_T)} - \int_\Sigma q \frac{\partial z}{\partial \nu} d\sigma dxdt + (z, L^*q)_{L^2(Q)} = (v, q)_{L^2(Q_\omega)}. \end{aligned}$$

That is

$$\begin{aligned} & (z(T, \cdot, \cdot, \theta), q(T, \cdot, \cdot, \theta))_{L^2(Q_A)} + (z(\cdot, A, \cdot, \theta), q(\cdot, A, \cdot, \theta))_{L^2(Q_T)} \\ & - (z, \beta(t, a, x)q(\cdot, 0, \cdot, \theta))_{L^2(Q)} - \int_\Sigma q \frac{\partial z}{\partial \nu} d\sigma dxdt + (z, L^*q)_{L^2(Q)} = (v, q)_{L^2(Q_\omega)}. \end{aligned} \quad (43)$$

Using (42), (43) rewrites as

$$\left(z(T, \cdot, \cdot, \theta); \int_I y(\theta; u)(T) d\theta - z_d \right)_{L^2(Q_A)} = \int_{Q_\omega} vq dxdt, \quad (44)$$

then an integration by parts with respect to θ on J lead us to

$$\left(\int_I z(T, \cdot, \cdot, \theta) d\theta; \int_I y(\theta; u)(T) d\theta - z_d \right)_{L^2(Q_A)} = \left(v, \int_I q(\theta) d\theta \right)_{L^2(Q_\omega)}. \quad (45)$$

Combining (41) and (45), we obtain

$$\int_{L^2(Q_\omega)} v \left(\int_I q(\theta) d\theta \right) dadxdt + N \int_{L^2(Q_\omega)} uv dadxdt = 0 \quad \forall v \in L^2(Q_\omega).$$

That is

$$\int_{L^2(Q_\omega)} \left(\int_I q(\theta) d\theta + Nu \right) v dadxdt = 0 \quad \forall v \in L^2(Q_\omega),$$

which implies that

$$u = -\frac{1}{N} \int_I q(\theta) d\theta \quad \text{in } Q_\omega. \quad (46)$$

■

4 Conclusion

In this paper, we proved that after averaging the cost function related to our model, the system is still controllable and gives an optimal control which does not depend on the unknown parameter.

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