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MAGNETIC PERTURBATIONS OF THE ROBIN LAPLACIAN IN THE STRONG COUPLING LIMIT

RAYAN FAHS

ABSTRACT. This paper is devoted to the asymptotic analysis of the eigenvalues of the Laplace operator with a uniform magnetic field and Robin boundary condition on a smooth planar domain. We show how the singular limit when the Robin parameter tends to infinity is equivalent to a semi-classical limit involving a small positive parameter h (the semi-classical parameter). The main result is a comparison between the spectrum of the Robin Laplacian with an effective operator defined on the boundary of the domain via the Born-Oppenheimer approximation. More precisely, the n -th eigenvalue of the Robin Laplacian is approximated, modulo $O(h^2)$, by the n -th eigenvalue of the effective operator. When the curvature has a unique non-degenerate maximum, the eigenvalue asymptotics displays the contribution of the magnetic field explicitly.

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Key words and phrases. Magnetic Laplacian, Robin boundary condition, eigenvalues, Born-Oppenheimer approximation, projection de Feshbach, diamagnetic inequalities.

1. INTRODUCTION

1.1. Magnetic Robin Laplacian. Let Ω be a bounded open subset of \mathbb{R}^2 . It is assumed that the boundary $\Gamma = \partial\Omega$ is smooth C^∞ . In this paper, we study the eigenvalues of the magnetic Robin Laplacian in $L^2(\Omega)$ with a large negative parameter, in a continuation of the works initiated in [5, 15, 11]. The operator is

$$\mathcal{P}_\gamma^b = -(\nabla - ibA_0)^2,$$

with domain

$$\text{Dom}(\mathcal{P}_\gamma^b) = \{u \in H^2(\Omega) : \nu \cdot (\nabla - ibA_0)u + \gamma u = 0 \text{ on } \Gamma\},$$

where

- A_0 is the vector field,
- ν is the unit outward normal vector of Γ ,
- $\gamma < 0$ is the Robin parameter,
- $b = b_0|\gamma|^\eta$ is the intensity of the applied magnetic field, with b_0 and η are strictly positive constants,
- $\nu \cdot (\nabla - ibA_0)u + \gamma u = 0$ on Γ is the Robin boundary condition.

The magnetic potential A_0 generates a constant magnetic field B equal to 1 and it is defined by

$$A_0(x_1, x_2) = (A_{0_1}, A_{0_2}) = \frac{1}{2}(-x_2, x_1).$$

We have:

$$B := \nabla \times A_0 = \partial_{x_1}A_{0_2} - \partial_{x_2}A_{0_1} = 1.$$

To be more precise, the operator \mathcal{P}_γ^b is defined via the Lax-Milgram theorem, from the closed semi-bounded quadratic form [1, Ch. 4]

$$H^1(\Omega) \ni u \mapsto \mathcal{Q}_\gamma^b(u) := \int_\Omega |(\nabla - ibA_0)u(x)|^2 dx + \gamma \int_\Gamma |u(x)|^2 ds(x),$$

where ds is the standard surface measure on the boundary. Note that, by a classical trace theorem (see for instance [14] in the case of straight boundary), the trace of u is well defined as an element of $H^{\frac{1}{2}}(\Gamma)$, and the quadratic form \mathcal{Q}_γ^b is well defined and bounded from below.

Since Ω is bounded and regular, the operator \mathcal{P}_γ^b has a compact resolvent, by Sobolev embedding. Its spectrum is purely discrete, and since \mathcal{P}_γ^b is self-adjoint and bounded from below, it consists of an increasing sequence of eigenvalues. Our aim is to examine the magnetic effects on the principal eigenvalues $(\lambda_n(b, \gamma))_{n \in \mathbb{N}^*}$, when the Robin parameter γ tends to $-\infty$.

A part from its own interest, the study of the spectrum of the operator \mathcal{P}_γ^b arises in several contexts, where both situations subject to magnetic fields, $b > 0$, or without magnetic fields, $b = 0$, occur. Estimating the ground state energy of \mathcal{P}_γ^b , leads to information on the critical temperature/critical fields of certain superconductors surrounded by other materials [11, 18]. On the other hand, eigenvalue asymptotics in the singular limit $\gamma \rightarrow -\infty$, provide counter examples in the context of spectral geometry [19, 4], and has connections to the study of Steklov eigenvalues [20].

1.2. Main results. In the case without magnetic field, $b = 0$, the asymptotic expansion of the eigenvalues of the Robin Laplacian has been the subject of recent studies [2, 5, 3, 4, 9, 15, 17]. In particular, the derivation of an effective operator on the boundary, involving the Laplace-Beltrami operator as well as the mean curvature of the boundary. In the same spirit, we will study the negative spectrum of the Robin's Laplacian on a bounded domain with Robin condition at the boundary. Our aim is to improve/complement earlier estimates and also clarify the magnetic field's contribution in the spectral asymptotics.

The main contribution of this article is Theorem 1.1 below, which involves two effective operators that we introduce as follows. We denote by $|\Omega|$ the area of the domain Ω and by $|\partial\Omega|$ the arc-length

of its boundary. Let us parameterize the boundary, $\partial\Omega$, by the arc-length, which we denote by s . Let $\kappa(s)$ be the curvature of $\partial\Omega$ at the point defined by the arc-length s , with the convention that $\kappa(s) \geq 0$ when $\partial\Omega$ is convex in a neighborhood of s , and negative otherwise. The effective operators are self-adjoint operators in the Hilbert space $L^2(\partial\Omega)$, acting on periodic functions, and defined as follows

$$(1.1) \quad \begin{aligned} \mathcal{L}_\gamma^{\text{eff},\pm} &= -\gamma^{-2}(1 \pm c_\pm \gamma^{-1})(\partial_s - ib\beta_0)^2 - 1 + \gamma\kappa(s) - \frac{\kappa(s)^2}{2}\gamma^{-2}, \\ \text{Dom}(\mathcal{L}_\gamma^{\text{eff},\pm}) &= H^2(\mathbb{R}/|\partial\Omega|\mathbb{Z}) \end{aligned}$$

with c_-, c_+ are positive constants, and β_0 is the circulation of the magnetic field,

$$(1.2) \quad \beta_0 = \frac{|\Omega|}{|\partial\Omega|}.$$

We will describe the asymptotics of the negative eigenvalues of \mathcal{P}_γ^b , via those of the effective operators on the boundary. More precisely, the n -th eigenvalues for the Laplace operator \mathcal{P}_γ^b , is approximated modulo $o(1)$ by the n -th eigenvalue of the operators $\mathcal{L}_\gamma^{\text{eff},\pm}$.

Theorem 1.1. *Let $0 \leq \eta < \frac{1}{2}$ and $b_0 > 0$. Suppose that $b = b_0|\gamma|^\eta$. Then, as γ tends to $-\infty$, for any $n \in \mathbb{N}^*$, we have*

$$\gamma^2 \lambda_n(\mathcal{L}_\gamma^{\text{eff},-}) + o(1) \leq \lambda_n(b, \gamma) \leq \gamma^2 \lambda_n(\mathcal{L}_\gamma^{\text{eff},+}) + o(1),$$

where $\lambda_n(\mathcal{L}_\gamma^{\text{eff},\pm})$ is the n -eigenvalues of the effective operator introduced in (1.1), with c_\pm being constants dependent on η, b_0 and the domain Ω solely.

Remark 1.1. *Theorem 1.1 allows for the magnetic field to grow as the Robin parameter tends to $-\infty$, and at the same time, the spectral asymptotics remains valid. In the case of the unit disk (hence with constant curvature), and with large magnetic field ($0 < \eta < \frac{1}{2}$), Theorem 1.1 leads to an improvement of the work by Kachmar-Sundqvist [11, Thm. 1.1]. In fact, by well-chosen quasi modes, we find in Section 5.3, the following asymptotics of the lowest eigenvalue*

$$\lambda_1(b, \gamma) = -\gamma^2 + \gamma + \left(\inf_{m \in \mathbb{Z}} \left(m - \frac{b_0|\gamma|^\eta}{2} \right)^2 - \frac{1}{2} \right) + o(1),$$

which was derived in [11, Thm. 1.1] when b is fixed and $\gamma \rightarrow -\infty$.

In a generic situation where the curvature has a unique non-degenerate maximum, Theorem 1.1 allows us to determine the leading order term of the spectral gap between the successive eigenvalues. This is valid under the following assumption:

$$\textbf{Assumption A} \quad \begin{cases} \kappa \text{ attains its maximum } \kappa_{\max} \text{ at a unique point;} \\ \text{the maximum is non-degenerate, i.e. } \kappa''(0) < 0, \end{cases}$$

where $\kappa_{\max} = \max_{0 \leq s \leq |\partial\Omega|} \kappa(s)$ denotes the maximal curvature along the boundary $\partial\Omega$, and the arc-length parametrization of the boundary is selected so that $\kappa(0) = \kappa_{\max}$.

After the study of the spectrum of the effective operator, under **Assumption A** above, we show that, the contribution of the magnetic field is of lower order compared to that of the curvature. Note that, the dependence with respect to the labeling of the eigenvalues, n , appears in the third term of the expansion.

Corollary 1.1. *Let $0 \leq \eta \leq \frac{1}{8}$ and $b_0 > 0$. Suppose that $b = b_0|\gamma|^\eta$. Under the **Assumption A**, as γ tends to $-\infty$, for any $n \in \mathbb{N}^*$, we have*

$$\lambda_n(b, \gamma) = -\gamma^2 + \kappa_{\max}\gamma + (2n - 1)\sqrt{\frac{-\kappa''(0)}{2}}|\gamma|^{1/2} + \mathcal{O}(|\gamma|^{1/4}).$$

The first two terms in the asymptotics of $\lambda_n(b, \gamma)$ were known [15, Thm. 1.1] and [22, Prop. 5.1] while the third term, which displays the spectral gap, is new. Corollary 1.1 follows from the spectral asymptotics of the 1D effective operators, which is valid under a weaker assumption on the strength of the magnetic field ($b = \mathcal{O}(|\gamma|^{1/8})$) when compared to the one in Theorem 1.1 ($b = o(|\gamma|^{1/2})$).

In the absence of the magnetic field, $b = 0$, the full asymptotic expansion for the low-lying eigenvalues was determined in [5] and they read as follows. For a fixed label $n \geq 1$ and as $\gamma \rightarrow -\infty$, we have

$$(1.3) \quad \lambda_n(0, \gamma) \sim -\gamma^2 + \kappa_{\max}\gamma + (2n-1)\sqrt{\frac{-\kappa''(0)}{2}}|\gamma|^{1/2} + \beta_{0,n} + \sum_{j \geq 1} \beta_{j,n}|\gamma|^{-j/2},$$

where $(\beta_{j,n})_{j \geq 0} \subset \mathbb{R}$ is a sequence defined by recursion. Notice that the first three terms in Corollary 1.1 are consistent with those in (1.3).

In the presence of a magnetic field, knowing the spectral gap, we can improve the asymptotics in Corollary 1.2 by constructing accurate quasi-modes for the operator \mathcal{P}_γ^b (not for the effective operator).

Corollary 1.2. *Let $0 \leq \eta \leq \frac{1}{8}$ and $b_0 > 0$. Suppose that $b = b_0|\gamma|^\eta$. Under the **Assumption A**, as γ tends to $-\infty$, for any $n \in \mathbb{N}^*$, we have*

$$\lambda_n(b, \gamma) = -\gamma^2 + \kappa_{\max}\gamma + (2n-1)\sqrt{\frac{-\kappa''(0)}{2}}|\gamma|^{1/2} + \zeta_{1,n}(b) + \mathcal{O}(|\gamma|^{-1/4}),$$

where

$$\zeta_{1,n}(b) = \beta_{0,n} + \beta_0^2 b^2 (1 - 4I_{2,n}),$$

where $\beta_{0,n}$ is the constant in (1.3), β_0 is the constant in (1.2), and $I_{2,n}$ is a constant independent of b (see (1.6)).

Let us introduce the constant $I_{2,n}$ explicitly. Let f_n be the n -th normalized eigenfunction of the harmonic oscillator

$$(1.4) \quad H_{\text{harm}} = -\frac{d^2}{d\sigma^2} - \frac{\kappa''(0)}{2}\sigma^2.$$

Note that, the eigenfunctions f_n can be determined by recursion as follows,

$$(1.5) \quad f_1(\sigma) = c_1 \exp^{-\sqrt{\frac{-\kappa''(0)}{2}}\frac{\sigma^2}{2}}, \quad f_n(\sigma) = c_n (L^+)^{n-1} f_1(\sigma) \quad (n \geq 1),$$

where $L^+ = -\frac{d}{d\sigma} + \sqrt{\frac{-\kappa''(0)}{2}}\sigma$, and, for every $n \geq 1$, the constant c_n normalizes $f_n(\sigma)$ in $L^2(\mathbb{R})$. We introduce then the constant $I_{2,n}$ as follows

$$(1.6) \quad I_{2,n} := \langle i\partial_\sigma f_n, R_n(i\partial_\sigma f_n) \rangle_{L^2(\mathbb{R})} = \langle \partial_\sigma f_n, R_n(\partial_\sigma f_n) \rangle_{L^2(\mathbb{R})},$$

where R_n is the regularized resolvent, the self-adjoint operator which vanishes on the eigenspace $E_n = \text{span}(f_n)$, and equals to the inverse of $H_{\text{harm}}^n := H_{\text{harm}} - (2n-1)\sqrt{\frac{-\kappa''(0)}{2}}$ on the orthogonal complement of E_n , then extended by linearity to all of $L^2(\mathbb{R})$.

Remark 1.2. *When $n = 1$, $I_{2,n} = \frac{1}{4}$, hence the term that depends on b in the asymptotics of Corollary 1.2 vanishes. To see this, we observe that*

$$\partial_\sigma f_1(\sigma) = -\sqrt{\frac{-\kappa''(0)}{2}}\sigma f_1(\sigma) \perp f_1(\sigma).$$

Hence

$$I_{2,1} = \frac{-\kappa''(0)}{2} \langle \sigma f_1, R_1(\sigma f_1) \rangle.$$

Now, arguing as [21, Prop. A.3], we find that

$$2(1 - 4I_{2,1}) = E''(0),$$

where $E(z)$ is the ground state energy of the harmonic oscillator

$$H_{\text{harm}}(z) = -\frac{d^2}{d\sigma^2} - \frac{\kappa''(0)}{2}(\sigma + z)^2.$$

Since \mathbb{R} is invariant by translation, we observe that $E(z) = E(0) = \sqrt{\frac{-\kappa''(0)}{2}}$, hence $E''(0) = 0$.

1.3. Organization of the paper. The paper is organized as follows. In Section 3, we introduce an effective semiclassical parameter, and we recall why the eigenfunctions are localized, via Agmon estimates near the boundary. As a consequence, we replace the initial problem by a problem on a thin tubular neighborhood of the boundary. In Section 4, by using the Born-Oppenheimer approximation, we derive an effective operator whose eigenvalues simultaneously describe the eigenvalues of the magnetic Robin Laplacian. In Section 5, we estimate the eigenvalues for the effective operator with large magnetic field, thereby proving Corollary 1.1. Finally, in Section 6, we describe the term which determines the influence of the magnetic field on the spectrum of Robin Laplacian, thereby proving corollary 1.2. In Appendix A, we recall the known results related to some one dimensional auxiliary differential operators.

2. BOUNDARY COORDINATES

The key to proving Theorem 1.1 is a reduction to the boundary. Indeed, the eigenfunctions associated with eigenvalues lower than $-\epsilon h$ with $\epsilon > 0$ concentrate exponentially near the boundary (cf. Proposition 3.1). To single out the influence of the boundary curvature, we need a special coordinate system displaying the arc-length along the boundary and the normal distance to the boundary. Those are the tubular coordinates. In this section, we introduce the necessary notation to use these coordinates (cf. [7, 6]). Let

$$\mathbb{R}/(|\partial\Omega|\mathbb{Z}) \ni s \mapsto M(s) \in \partial\Omega$$

be the arc-length parametrization of $\partial\Omega$. We will always work with $|\partial\Omega|$ -periodic functions sometimes restricted to the interval

$$\left] -\frac{|\partial\Omega|}{2}, \frac{|\partial\Omega|}{2} \right] =] -L, L].$$

At the point $M(s) \in \partial\Omega$, $T(s) = M'(s)$ is the unit tangent vector and $v(s)$ is the unit normal vector such that

$$\forall s \in \mathbb{R}/(|\partial\Omega|\mathbb{Z}), \quad \det(T(s), v(s)) = 1.$$

The curvature $\kappa(s)$ at point $M(s)$ is then defined as follows

$$T'(s) = \kappa(s)v(s).$$

The smoothness of the boundary yields the existence of a constant $t_0 > 0$ such that, upon defining

$$V_{t_0} = \{x \in \Omega : \text{dist}(x, \partial\Omega) < t_0\}$$

the change of coordinates

$$\Phi : \mathbb{R}/(|\partial\Omega|\mathbb{Z}) \times (0, t_0) \ni (s, t) \mapsto x = M(s) - tv(s) \in V_{t_0}$$

becomes a diffeomorphism. Let us note that, for $x \in V_{t_0}$, one can write

$$\Phi^{-1}(x) := (s(x), t(x)) \in \mathbb{R}/(|\partial\Omega|\mathbb{Z}) \times (0, t_0)$$

where $t(x) = \text{dist}(x, \partial\Omega)$ and $s(x)$ is the coordinate of the point $M(s(x)) \in \mathbb{R}/(|\partial\Omega|\mathbb{Z})$ satisfying $\text{dist}(x, \partial\Omega) = |x - M(s(x))|$.

The inverse of Φ defines a coordinate system on a tubular neighborhood of $\partial\Omega$ in $\bar{\Omega}$ that can be used locally or semi-globally. Now we express various integrals in the new coordinates (s, t) .

For all s , the Jacobian matrix J_Φ is written, as follows

$$J_\Phi(s, t) = (M'(s) - t\nu'(s), \nu(s)) .$$

As $\nu'(s) = \kappa(s)M'(s)$, the determinant of the Jacobian matrix of the transformation Φ^{-1} is given by:

$$\det J_\Phi(s, t) = 1 - t\kappa(s).$$

In the new coordinates, the components of the vector field \mathbf{A}_0 are given as follows,

$$\begin{aligned} \tilde{A}_1(s, t) &= (1 - t\kappa(s))\mathbf{A}_0(\Phi(s, t)) \cdot M'(s) , \\ \tilde{A}_2(s, t) &= \mathbf{A}_0(\Phi(s, t)) \cdot v(s) . \end{aligned}$$

For all $u \in L^2(V_{t_0})$, we assign the pull-back function \tilde{u} defined in the new coordinates as follows

$$\tilde{u} = u \circ \Phi .$$

Consequently, for all $u \in H^1(V_{t_0})$, we have

$$\begin{aligned} &\int_{V_{t_0}} |(\nabla - ibA_0)u(x)|^2 dx \\ &= \int_{\Phi^{-1}(V_{t_0})} \left[(1 - t\kappa(s))^{-2} |(\partial_s - ib\tilde{A}_1)\tilde{u}|^2 + |(\partial_t - ib\tilde{A}_2)\tilde{u}|^2 \right] (1 - t\kappa(s)) dt ds , \end{aligned}$$

$$\int_{V_{t_0}} |u(x)|^2 dx = \int_{\Phi^{-1}(V_{t_0})} |\tilde{u}|^2 (1 - t\kappa(s)) dt ds ,$$

and

$$\int_{V_{t_0} \cap \partial\Omega} |u(x)|^2 dx = \int |\tilde{u}(s, t = 0)|^2 ds .$$

The following lemma can be found in [8] .

Lemma 2.1. *For $t_0 > 0$ small enough, there exists a gauge transformation $\varphi(s, t)$ on $\mathbb{R}/(|\partial\Omega|\mathbb{Z}) \times (0, t_0)$ such that \bar{A}_0 defined by*

$$\bar{A}_0 = \tilde{A}_0 - \nabla_{(s,t)}\varphi ,$$

satisfies

$$\bar{A}_0(s, t) = \begin{pmatrix} \bar{A}_1(s, t) \\ \bar{A}_2(s, t) \end{pmatrix} = \begin{pmatrix} \beta_0 - t + \frac{t^2}{2}\kappa(s) \\ 0 \end{pmatrix} ,$$

with

$$\beta_0 = \frac{1}{|\partial\Omega|} \int_{\Omega} \operatorname{curl} A_0 dx = \frac{|\Omega|}{|\partial\Omega|} ,$$

and for all $u \in H^1(V_{t_0})$, we have

$$\int_{V_{t_0}} |(\nabla - ibA_0)u(x)|^2 dx = \int_{\Phi^{-1}(V_{t_0})} \left((1 - t\kappa(s))^{-2} |(\partial_s - ib\bar{A}_1)w|^2 + |\partial_t w|^2 \right) (1 - t\kappa(s)) ds dt$$

where we noted $w = e^{i\varphi}\tilde{u}$ and $\tilde{u} = u \circ \Phi$.

3. TRANSFORMATION INTO A SEMI-CLASSICAL PROBLEM

We will prove the results of Section 1.2 by a semiclassical framework. Let us consider the semi-classical parameter

$$h = \gamma^{-2}.$$

The limit $\gamma \rightarrow -\infty$ is now equivalent to the semi-classical limit $h \rightarrow 0^+$. The quadratic form can be written as

$$\mathcal{Q}_\gamma^b(u) = h^{-2} \left(\int_\Omega |(h\nabla - ibhA_0)u(x)|^2 dx - h^{\frac{3}{2}} \int_\Gamma |u(x)|^2 ds(x) \right) := h^{-2} \mathcal{Q}_h^b(u).$$

Consequently, we obtain the self-adjoint operator depending on h

$$\mathcal{L}_h^b = -(h\nabla - ibhA_0)^2,$$

with domain

$$\text{Dom}(\mathcal{L}_h^b) = \{ u \in H^2(\Omega) : \nu \cdot (\nabla - ibA_0)u - h^{\frac{1}{2}}u = 0 \text{ on } \Gamma \}.$$

Clearly,

$$\mathcal{P}_\gamma^b = h^{-2} \mathcal{L}_h^b,$$

and thus the relation between the spectra of the operators \mathcal{P}_γ^b and \mathcal{L}_h^b is displayed as follows :

$$(3.7) \quad \sigma(\mathcal{P}_\gamma^b) = h^{-2} \sigma(\mathcal{L}_h^b).$$

Let $(\mu_n(h, b))_{n \in \mathbb{N}^*}$ be the sequence of eigenvalues of the operator \mathcal{L}_h^b . Theorem 1.1 and Corollary 1.1 can be reformulated in semi-classical form as follows.

Theorem 3.1. *Let $0 \leq \eta < \frac{1}{2}$ and $b_0 > 0$. Suppose that $b = b_0 h^{\frac{-\eta}{2}}$. There exist constants $c_\pm > 0$ and $h_0 > 0$, for all $h \in (0, h_0)$, for any $n \in \mathbb{N}^*$, we have :*

$$h \lambda_n(\mathcal{L}_h^{\text{eff}, -}) + o(h^2) \leq \mu_n(h, b) \leq h \lambda_n(\mathcal{L}_h^{\text{eff}, +}) + o(h^2),$$

with

$$\mathcal{L}_h^{\text{eff}, \pm} = -h(1 \pm c_\pm h^{\frac{1}{2}})(\partial_s - ib\gamma_0)^2 - 1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h.$$

Corollary 3.1. *Let $\eta \leq \frac{1}{8}$ and $b_0 > 0$. Suppose that $b = b_0 h^{\frac{-\eta}{2}}$. Under the Assumption A, there exist constants c_\pm and $h_0 > 0$, for all $h \in (0, h_0)$, for any $n \in \mathbb{N}^*$, we have :*

$$\mu_n(h, b) = -h - \kappa_{\max} h^{3/2} + (2n - 1) \sqrt{\frac{-\kappa''(0)}{2}} h^{7/4} + \mathcal{O}(h^{15/8}).$$

Corollary 3.2. *Let $\eta \leq \frac{1}{8}$ and $b_0 > 0$. Suppose that $b = b_0 h^{\frac{-\eta}{2}}$. Under the Assumption A, there exist constants c_\pm and $h_0 > 0$, for all $h \in (0, h_0)$, for any $n \in \mathbb{N}^*$, we have :*

$$\mu_n(h, b) = -h - h^{3/2} \kappa_{\max} + h^{7/4} \sqrt{\frac{\kappa''(0)}{2}} (2n - 1) + h^2 \zeta_{1,n}(b) + \mathcal{O}(h^{17/8}),$$

where

$$\zeta_{1,n}(b) = \beta_{0,n} + \beta_0^2 b^2 (1 - 4I_{2,n}),$$

where $\beta_{0,n}$ is the constant in (1.3), β_0 is the constant in (1.2), and $I_{2,n}$ is a constant independent of b .

3.1. Reduction near the boundary via Agmon estimates. The goal of this section is to show that the eigenfunctions associated to eigenvalues less than $-\epsilon h$, with $0 < \epsilon < 1$ a fixed constant, are concentrated exponentially near the boundary at the scale $h^{1/2}$.

Proposition 3.1. *Let $M \in (-1, 0)$. For all $\alpha < \sqrt{-M}$, there exist constants $C > 0$ and $h_0 \in (0, 1)$ such that, for $h \in (0, h_0)$, if $u_{h,b}$ is a normalized ground state of \mathcal{L}_h^b with eigenvalue $\mu(h, b)$ such that $\mu(h, b) < Mh$, then*

$$(3.8) \quad \int_{\Omega} (|u_{h,b}(x)|^2 + h|(\nabla - ibA_0)u_{h,b}(x)|^2) \exp\left(\frac{2\alpha \text{dist}(x, \Gamma)}{h^{1/2}}\right) dx \leq C.$$

Proof. The proof is similar to the one of [5, Thm. 5.1], and only notice that it is a consequence modulo a partition of unity with balls of size $Rh^{1/2}$ with R large enough and the fact, if the support of u avoids the boundary, we have $\mathcal{Q}_h^b(u) \geq -Ch$. \square

We record the following simple corollary of Proposition 3.1.

Corollary 3.3. *Let $M \in (-1, 0)$, $\rho \in (0, \frac{1}{2})$ and $c > 0$. For all $0 < \alpha < \sqrt{-M}$, there exist $h_0 \in (0, 1)$ such that, if $u_{h,b}$ is a normalized ground state of \mathcal{L}_h^b with eigenvalue $\mu(h, b)$ such that $\mu(h, b) < Mh$, then for all $h \in (0, h_0)$,*

$$(3.9) \quad \int_{\text{dist}(x, \Gamma) \geq ch^{\frac{1}{2}-\rho}} (|u_{h,b}(x)|^2 + h|(\nabla - ibA_0)u_{h,b}(x)|^2) dx \leq \exp(-2\alpha ch^{-\rho}).$$

As a consequence, for small h , the ground states of the operator \mathcal{L}_h^b are concentrated near the boundary of Ω (cf. Corollary 3.3). This will allow us to work in a tubular neighborhood of $\partial\Omega$.

Let $\rho \in (0, 1/2)$, we introduce the δ -neighborhood of the boundary

$$\Omega_{\delta} = \{x \in \Omega : \text{dist}(x, \Gamma) < \delta\}, \quad \delta = h^{\frac{1}{2}-\rho}.$$

The quadratic form, defined on the variational space

$$W_{\delta} = \{u \in H^1(\Omega_{\delta}) : u(x) = 0, \forall x \in \Omega \text{ such that } \text{dist}(x, \Gamma) = \delta\},$$

is given by the formula

$$(3.10) \quad q_h^{b,\rho}(u) = h^2 \int_{\Omega_{\delta}} |(\nabla - ibA_0)u(x)|^2 dx - h^{\frac{3}{2}} \int_{\partial\Omega_{\delta}} |u(x)|^2 ds(x).$$

Note again that the trace is well-defined by a classical trace theorem. The associated self-adjoint operator is:

$$\mathcal{L}_h^{b,\rho} = -(h\nabla - ihbA_0)^2,$$

with domain

$$\text{Dom}(\mathcal{L}_h^{b,\rho}) = \{u \in H^2(\Omega_{\delta}) : \nu \cdot (\nabla - ibA_0)u - h^{\frac{1}{2}}u = 0 \text{ on } \Gamma \text{ and } u(x) = 0, \forall x \in \Omega \text{ such that } \text{dist}(x, \Gamma) = \delta\}.$$

That is, we consider the realization with mixed boundary conditions (Robin and Dirichlet conditions). Let $(\mu_n(h, b, \rho))_{n \in \mathbb{N}^*}$ be the sequence of eigenvalues of the operator $\mathcal{L}_h^{b,\rho}$. The following proposition allows the reduction of the analysis to the operator $\mathcal{L}_h^{b,\rho}$.

Proposition 3.2. *Let $\epsilon > 0$ and $\alpha \in (0, \sqrt{\epsilon})$, there exist constants $C > 0$, $h_0 \in (0, 1)$ such that, for all $h \in (0, h_0)$, $n \geq 1$ and $\mu_n(h, b) < -\epsilon h$,*

$$\mu_n(h, b, \rho) \leq \mu_n(h, b) + C \exp(-\alpha h^{-\rho}).$$

Moreover, we have, for all $n \geq 1$, $h > 0$

$$\mu_n(h, b) \leq \mu_n(h, b, \rho).$$

Proof. The inequality $\mu_n(h, b) \leq \mu_n(h, b, \rho)$ is not asymptotic. Let $(v_{k,h})_{1 \leq k \leq n}$ be an orthonormal family of eigenfunctions associated with eigenvalues $(\tilde{\mu}_k(h, b, \rho))_{1 \leq k \leq n}$. We define the function

$$u_{k,h} = \begin{cases} v_{k,h} & \text{if } x \in \Omega_\delta \\ 0 & \text{if } x \in \Omega \setminus \Omega_\delta. \end{cases}$$

Let E be the vector subspace generated by the family $(u_{k,h})_{1 \leq k \leq n}$ and $u \in E$ which is written as follows:

$$u = \sum_{k=1}^n \alpha_k u_{k,h}.$$

Inserting u into the quadratic form $\mathcal{Q}_h^b(u)$, we obtain

$$\begin{aligned} \mathcal{Q}_h^b(u) &= \left\langle \mathcal{L}_h^{b,\rho} \sum_{k=1}^n \alpha_k v_{k,h}, \sum_{k=1}^n \alpha_k v_{k,h} \right\rangle \\ &= \sum_{k=1}^n \alpha_k q_h^{b,\rho}(v_{k,h}) \leq \mu_n(h, b, \rho) \sum_{k=1}^n \alpha_k \|u_{k,h}\|^2 = \mu_n(h, b, \rho) \|u\|^2. \end{aligned}$$

By the min-max theorem, we get the non-asymptotic inequality $\mu_n(h, b) \leq \mu_n(h, b, \rho)$.

Now we determine an upper bound of $\mu_n(h, b, \rho)$ in terms of $\mu_n(h, b)$. Let $(u_{k,h})_{1 \leq k \leq n}$ be an orthonormal family of eigenvectors associated with the eigenvalues $(\mu_k(h, b))_{1 \leq k \leq n}$. Let us consider the cut off function define on \mathbb{R} as following:

$$0 \leq \chi \leq 1, \quad \chi = 1 \text{ on }]-\infty, 1/2] \text{ and } \chi = 0 \text{ on } [1, +\infty[.$$

We define the function

$$\chi_1(x) = \chi\left(\frac{t(x)}{\delta}\right).$$

For $k = 1, \dots, n$, we set

$$v_{k,h} = \chi_1 u_{k,h}.$$

Let F_h the vector subspace of $\text{Dom}(\mathcal{L}_h^{b,\rho})$ spanned by the family $(v_{k,h})_{1 \leq k \leq n}$ and $w_h \in F_h$ which is written as follows:

$$w_h = \sum_{k=1}^n \beta_k v_{k,h}.$$

The functions $(v_{k,h})$ are almost orthonormal (Proposition 3.1). We note that $\dim F_h = n$ and

$$\|w_h\|^2 = \sum_{k=1}^n |\beta_k|^2 + \mathcal{O}(\exp(-\alpha h^{-\rho})).$$

Inserting w_h into the quadratic form $q_h^{b,\rho}(w_h)$

$$q_h^{b,\rho}(w_h) = \langle \mathcal{L}_h^{b,\rho} w_h, w_h \rangle = \sum_{j,k=1}^n \beta_j \beta_k \langle \mathcal{L}_h^{b,\rho} v_{j,h}, v_{k,h} \rangle.$$

For j, k fixed and for h small enough, we have

$$\begin{aligned} \langle \mathcal{L}_h^{b,\rho} v_{j,h}, v_{k,h} \rangle &= \delta_{j,k} \mu_j(h, b) - h^2 \langle u_{j,h} \Delta \chi_1, \chi_1 u_{k,h} \rangle - 2h^2 \langle \nabla \chi_1 \cdot (\nabla - ibA_0) u_{j,h}, \chi_1 u_{k,h} \rangle \\ &\quad + \mathcal{O}(\exp(-\alpha h^{-\rho})). \end{aligned}$$

From Hölder's inequality and the Corollary 3.3, we obtain

$$h^2 \langle u_{j,h} \Delta \chi_1, \chi_1 u_{k,h} \rangle + 2h^2 \langle \nabla \chi_1 \cdot (\nabla - ibA_0) u_{j,h}, \chi_1 u_{k,h} \rangle = \mathcal{O}(\exp(-\alpha h^{-\rho})).$$

Therefore,

$$\begin{aligned} q_h^{b,\rho}(w_h) &= \sum_{j=1}^n \mu_j(h, b) |\beta_j|^2 + \mathcal{O}(\exp(-\alpha h^{-\rho})) \sum_{j,k=1}^n \beta_j \beta_k \\ &\leq \left(\mu_n(h, b) + C \exp(-\alpha h^{-\rho}) \right) \|w_h\|^2. \end{aligned}$$

By the min-max theorem, we have $\mu_n(h, b, \rho) \leq \mu_n(h, b) + C \exp(-\alpha h^{-\rho})$. \square

Proposition 3.2 leads us to replace the initial problem by a new Robin-Dirichlet leaving in a δ -neighborhood of the boundary Γ .

3.2. The Robin Laplacian in boundary coordinates. By Lemma 2.1, the new quadratic form is expressed in tubular coordinates and is written as follows:

$$\begin{aligned} \tilde{q}_h^{b,\rho}(u) &= h^2 \int_{-L}^L \int_0^\delta \left| \left(\partial_s - ib\beta_0 - ib \left(-t + \frac{t^2}{2} \kappa(s) \right) \right) \tilde{u} \right|^2 (1 - t\kappa(s))^{-1} dt ds \\ &\quad + h^2 \int_{-L}^L \int_0^\delta |\partial_t \tilde{u}|^2 (1 - t\kappa(s)) dt ds - h^{3/2} \int_{-L}^L |\tilde{u}(s, t=0)|^2 ds, \end{aligned}$$

with $\tilde{u} = e^{i\varphi} u \circ \Phi$ and $L = \frac{|\partial\Omega|}{2}$. The operator $\mathcal{L}_h^{b,\rho}$ is expressed in tubular coordinates (s, t) as follows:

$$\begin{aligned} \tilde{\mathcal{L}}_h^{b,\rho} &= -h^2 a^{-1} \left(\partial_s - ib\beta_0 - ib \left(-t + \frac{t^2}{2} \kappa(s) \right) \right) a^{-1} \left(\partial_s - ib\beta_0 - ib \left(-t + \frac{t^2}{2} \kappa(s) \right) \right) \\ &\quad - h^2 a^{-1} \partial_t a \partial_t \quad (\text{in } (L^2(a dt ds))). \end{aligned}$$

With $a = 1 - t\kappa(s)$. In boundary coordinates, the Robin condition becomes

$$\partial_t u(s, t=0) = -h^{-1/2} u(s, t=0).$$

We introduce,

$$\begin{aligned} \tilde{\Omega}_\delta &= \{(s, t) : s \in [-L, L[\text{ and } 0 < t < \delta\}, \\ \text{Dom}(\tilde{q}_h^{b,\rho}) &= \{u \in H^1(\tilde{\Omega}_\delta) : u(s, \delta) = 0\}, \\ \text{Dom}(\tilde{\mathcal{L}}_h^{b,\rho}) &= \{u \in H^2(\tilde{\Omega}_\delta) : u(s, \delta) = 0 \text{ and } \partial_t u(s, t=0) = -h^{-1/2} u(s, t=0)\}. \end{aligned}$$

We know that the eigenfunctions are localized near the boundary, at the scale $h^{\frac{1}{2}}$ (cf. Prop 3.1) and in order to obtain a Robin condition independent of h , we get a partially semiclassical problem by changing the variable

$$(s, t) = (s, h^{\frac{1}{2}} \tau).$$

This change of variable transforms the above expression of $\tilde{\mathcal{L}}_h^{b,\rho}$ into an operator $\tilde{\mathcal{L}}_h^{b,\rho}$ as follows

$$\begin{aligned} h \left[-h \hat{a}^{-1} \left(\partial_s - ib\beta_0 - ib \left(-h^{\frac{1}{2}} \tau + h \frac{\tau^2}{2} \kappa(s) \right) \right) \hat{a}^{-1} \left(\partial_s - ib\beta_0 - ib \left(-h^{\frac{1}{2}} \tau + h \frac{\tau^2}{2} \kappa(s) \right) \right) \right. \\ \left. - \hat{a}^{-1} \partial_\tau \hat{a} \partial_\tau \right], \end{aligned}$$

where the new weight

$$\hat{a}(s, \tau) = 1 - h^{\frac{1}{2}} \tau \kappa(s).$$

The boundary condition becomes

$$\partial_\tau u(s, \tau=0) = -u(s, \tau=0).$$

We consider rather the operator

$$\widehat{\mathcal{L}}_h^{b,\rho} = h^{-1} \widetilde{\mathcal{L}}_h^{b,\rho},$$

and the domain of integration becomes

$$[-L, L] \times (0, \delta/h^{1/2}) = [-L, L] \times (0, h^{-\rho}).$$

The associated quadratic form is

$$\begin{aligned} \widehat{q}_h^{b,\rho}(\psi) &= \int_{-L}^L \int_0^{h^{-\rho}} \left| \left(h^{1/2} \partial_s - i b h^{1/2} \beta_0 - i b h \left(-\tau + h^{1/2} \frac{\tau^2}{2} \kappa(s) \right) \right) \psi \right|^2 \widehat{a}^{-1} \, d\tau \, ds \\ &\quad + \int_{-L}^L \int_0^{h^{-\rho}} |\partial_\tau \psi|^2 \widehat{a} \, d\tau \, ds - \int_{-L}^L |\psi(s, \tau = 0)|^2 \, ds. \end{aligned}$$

We let

$$\widehat{\Omega}_\delta = \{(s, \tau) : s \in [-L, L] \text{ and } 0 < \tau < h^{-\rho}\},$$

$$\text{Dom}(\widehat{q}_h^{b,\rho}) = \{u \in H^1(\widehat{\Omega}_\delta) : u(s, h^{-\rho}) = 0\},$$

$$\text{Dom}(\widehat{\mathcal{L}}_h^{b,\rho}) = \{u \in H^2(\widehat{\Omega}_\delta) : u(s, h^{-\rho}) = 0 \text{ and } \partial_\tau u(s, \tau = 0) = -u(s, \tau = 0)\}.$$

Let $\widehat{\mu}_n(h, b, \rho)$ the n -th eigenvalue of the self-adjoint operator $\widehat{\mathcal{L}}_h^{b,\rho}$. We have

$$(3.11) \quad \widehat{\mu}_n(h, b, \rho) = h^{-1} \mu_n(h, b, \rho).$$

4. ASYMPTOTIC OF THE EIGENVALUES OF THE ROBIN'S LAPLACIAN

The aim of this section is to prove the following result, which implies Theorem 1.1.

Theorem 4.1. *Let $0 \leq \eta < \frac{1}{2}$, $\rho \in (0, 1/4)$ and $b_0 > 0$. Suppose that $b = b_0 h^{-\frac{\eta}{2}}$. There exist constants $c_\pm > 0$ and $h_0 > 0$, for all $h \in (0, h_0)$, for any $n \in \mathbb{N}^*$, we have:*

$$h \lambda_n(\mathcal{L}_h^{\text{eff},-}) + o(h^2) \leq \widehat{\mu}_n(h, b, \rho) \leq h \lambda_n(\mathcal{L}_h^{\text{eff},+}) + o(h^2).$$

4.1. The Feshbach projection. Following the writing of the magnetic Robin Laplacian in tubular coordinates in Section 3.2, there appears a one-dimensional operator defined, for s fixed, by

$$\begin{cases} -\tilde{a}^{-1} \partial_\tau \tilde{a} \partial_\tau = -\partial_\tau^2 + h^{1/2} \kappa(s) \tilde{a}^{-1} \partial_\tau \\ \partial_\tau u(s, \tau = 0) = -u(s, \tau = 0) \text{ in } L^2((0, h^{-\rho}), (1 - \tau h^{1/2} \kappa(s)) \, d\tau). \end{cases}$$

This operator is denoted by

$$\mathcal{H}_{\kappa(s),h} = \mathcal{H}_B^{\{T\}},$$

with

$$T = h^{-\rho} \text{ and } B = h^{1/2} \kappa(s).$$

and where $\mathcal{H}_B^{\{T\}}$ is defined in A.2. Let $v_{\kappa(s),h}$ be a normalized groundstate of $\mathcal{H}_{\kappa(s),h}$ associated with $\lambda_1(\mathcal{H}_{\kappa(s),h})$. By [11, Lemma 2.5], we have

$$(4.12) \quad \lambda_1(\mathcal{H}_{\kappa(s),h}) = -1 - \kappa(s) h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2} h + o(h),$$

and, for $C > 0$

$$(4.13) \quad \lambda_2(\mathcal{H}_{\kappa(s),h}) \geq -C h^{1/2-\rho}.$$

According to Lemma A.1 and Proposition A.2, there exist constants C and C_k , such that

$$(4.14) \quad \|\partial_s v_{\kappa(s),h}\|_{L^2((0, h^{-\rho}), (1 - h^{1/2} \kappa(s) \tau) \, d\tau)} \leq C h^{1/2},$$

and

$$(4.15) \quad \int_0^{h^{-\rho}} \tau^k |v_{\kappa(s),h}|^2 (1 - h^{1/2} \kappa(s) \tau) \, d\tau \leq C_k.$$

We introduce for $s \in [-L, L]$, the Feshbach projection Π_s on the normalized ground state $v_{\kappa(s),h}$ of $\mathcal{H}_{\kappa(s),h}$

$$(4.16) \quad \Pi_s \psi = \langle \psi, v_{\kappa(s),h} \rangle_{L^2((0, h^{-\rho}), \hat{a} \, d\tau)} v_{\kappa(s),h}.$$

We also let

$$\Pi_s^\perp = \text{Id} - \Pi_s,$$

and

$$R_h(s) = \|\partial_s v_{\kappa(s),h}\|_{L^2((0, h^{-\rho}), \hat{a} \, d\tau)}^2.$$

The quantity R_h is sometimes called ‘‘Born-Oppenheimer correction’’.

Lemma 4.1. *There exist constants $C > 0$ and $h_0 > 0$ such that, for all $\psi \in \text{Dom}(\hat{q}_h^{b,\rho})$ and $h \in (0, h_0)$, we have:*

$$\|[\Pi_s, \partial_s] \psi\|_{L^2(\hat{a} \, d\tau \, ds)} = \|[\Pi_s^\perp, \partial_s] \psi\|_{L^2(\hat{a} \, d\tau \, ds)} \leq \int_{-L}^L \left[2R_h(s)^{\frac{1}{2}} + ch^{1/2} \right] \|\psi\|_{L^2(\hat{a} \, d\tau)} \, ds.$$

Proof. We estimate the commutator:

$$\begin{aligned} [\Pi_s, \partial_s] \psi &= -[\Pi_s^\perp, \partial_s] \psi \\ &= \langle \partial_s \psi, v_{\kappa(s),h} \rangle_{L^2(\hat{a} \, d\tau)} v_{\kappa(s),h} - \partial_s \left(\langle \psi, v_{\kappa(s),h} \rangle_{L^2(\hat{a} \, d\tau)} v_{\kappa(s),h} \right) \\ &= -\langle \psi, \partial_s v_{\kappa(s),h} \rangle_{L^2(\hat{a} \, d\tau)} v_{\kappa(s),h} - \langle \psi, v_{\kappa(s),h} \rangle_{L^2(\hat{a} \, d\tau)} \partial_s v_{\kappa(s),h} \\ &\quad + h^{1/2} \kappa'(s) \left(\int_0^{h^{-\rho}} \psi v_{\kappa(s),h} \tau \, d\tau \right) v_{\kappa(s),h}. \end{aligned}$$

We get, thanks to the Cauchy-Schwarz inequality and the inequality (4.15)

$$\|[\Pi_s, \partial_s] \psi\|_{L^2(\hat{a} \, d\tau \, ds)} \leq \int_{-L}^L \left[2R_h(s)^{\frac{1}{2}} + ch^{1/2} \right] \|\psi\|_{L^2(\hat{a} \, d\tau)} \, ds.$$

□

4.2. Approximation of the norm on the weighted space. In this section, we introduce an approximation of the weight.

Lemma 4.2. *Let $\psi \in L^2(\hat{a} \, d\tau \, ds)$. There exist constant $c > 0$ such that:*

$$(4.17) \quad \|\psi\|_{L^2(\hat{a}^{-1} \, d\tau \, ds)}^2 \leq (1 + ch^{1/2}) \|\Pi_s \psi\|_{L^2(\hat{a} \, d\tau \, ds)}^2 + (1 + ch^{1/4}) \|\Pi_s^\perp \psi\|_{L^2(\hat{a} \, d\tau \, ds)}^2,$$

and

$$(4.18) \quad \|\psi\|_{L^2(\hat{a}^{-1} \, d\tau \, ds)}^2 \geq (1 - ch^{1/2}) \|\Pi_s \psi\|_{L^2(\hat{a} \, d\tau \, ds)}^2 + (1 - ch^{1/4}) \|\Pi_s^\perp \psi\|_{L^2(\hat{a} \, d\tau \, ds)}^2.$$

Proof. We have

$$\left| \int_{-L}^L \int_0^{h^{-\rho}} |\psi|^2 \hat{a} \, d\tau \, ds - \int_{-L}^L \int_0^{h^{-\rho}} |\psi|^2 \hat{a}^{-1} \, d\tau \, ds \right| \leq C \int_{-L}^L \int_0^{h^{-\rho}} h^{1/2} \tau |\psi|^2 \hat{a} \, d\tau \, ds.$$

Then, we use an orthogonal decomposition to get

$$\begin{aligned} &\left| \int_{-L}^L \int_0^{h^{-\rho}} |\psi|^2 \hat{a} \, d\tau \, ds - \int_{-L}^L \int_0^{h^{-\rho}} |\psi|^2 \hat{a}^{-1} \, d\tau \, ds \right| \\ &\leq C \left[\int_{-L}^L \int_0^{h^{-\rho}} h^{1/2} \tau |\Pi_s \psi|^2 \hat{a} \, d\tau \, ds + \int_{-L}^L \int_0^{h^{-\rho}} h^{1/2} \tau |\Pi_s^\perp \psi|^2 \hat{a} \, d\tau \, ds \right]. \end{aligned}$$

Thanks to (4.15), we get

$$\int_{-L}^L \int_0^{h^{-\rho}} h^{1/2} \tau |\Pi_s \psi|^2 \hat{a} \, d\tau \, ds = h^{1/2} \int_{-L}^L |\varphi(s)|^2 \left[\int_0^{h^{-\rho}} \tau |v_{\kappa(s),h}|^2 \hat{a} \, d\tau \right] ds \leq ch^{1/2} \|\Pi_s \psi\|_{L^2(\hat{a} \, d\tau \, ds)}^2.$$

We use that $h^{1/2-\rho} \leq h^{1/4}$, we obtain:

$$\int_{-L}^L \int_0^{h^{-\rho}} h^{1/2} \tau |\Pi_s^\perp \psi|^2 \hat{a} \, d\tau ds \leq h^{1/2-\rho} \int_{-L}^L \int_0^{h^{-\rho}} |\Pi_s^\perp \psi|^2 \hat{a} \, d\tau ds \leq h^{1/4} \|\Pi_s^\perp \psi\|_{L^2(\hat{a} \, d\tau ds)}^2.$$

□

The following corollary is a direct consequence of (4.17).

Corollary 4.1. *Let $\psi \in L^2(\hat{a} \, d\tau ds)$. There exist $c > 0$ such that :*

$$\|\Pi_s \psi\|_{L^2(\hat{a}^{-1} \, d\tau ds)}^2 \leq (1 + ch^{1/2}) \int_{-L}^L |\varphi(s)|^2 \, ds,$$

with $\varphi = \langle \psi, v_{\kappa(s),h} \rangle_{L^2((0, h^{-\rho}), \hat{a} \, d\tau)}$.

4.3. Upper bound. To separate the variables, we consider the function of the form :

$$\psi(s, t) = \varphi(s) \cdot v_{\kappa(s),h}(\tau),$$

with $\varphi \in H^1(\mathbb{R}/2L\mathbb{Z})$.

The following proposition provides an upper bound of the quadratic form on a subspace.

Proposition 4.1. *Let $0 \leq \eta < \frac{1}{2}$, $\rho \in (0, 1/4)$ and $b_0 > 0$. Suppose that $b = b_0 h^{-\frac{\eta}{2}}$. There exist constant $c > 0$, $h_0 > 0$, and a quadratic form $q_h^{\text{eff},+}$ in $L^2(\mathbb{R}/2L\mathbb{Z})$*

$$q_h^{\text{eff},+}(\varphi) = \int_{-L}^L \left(-1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h \right) |\varphi(s)|^2 \, ds + h(1 + ch^{\frac{1}{2}}) \int_{-L}^L |(\partial_s - ib\beta_0)\varphi|^2 \, ds,$$

such that, for all $h \in (0, h_0)$, we have:

$$\hat{q}_h^{b,\rho}(\psi) \leq q_h^{\text{eff},+}(\varphi) + o(h)\|\varphi\|_{L^2(\mathbb{R}/2L\mathbb{Z})}^2.$$

Proof. The proof will be done in a five steps.

i. We write

$$\hat{q}_h^{b,\rho}(\psi) = q^{\text{tang}}(\psi) + q^{\text{trans}}(\psi),$$

where

$$q^{\text{trans}}(\psi) = \int_{-L}^L \int_0^{h^{-\rho}} \left| \left(h^{\frac{1}{2}} \partial_s - ibh^{\frac{1}{2}} \beta_0 - ibh \left(-\tau + h^{\frac{1}{2}} \frac{\tau^2}{2} \kappa(s) \right) \right) \psi \right|^2 \hat{a}^{-1} \, d\tau ds,$$

and

$$q^{\text{tang}}(\psi) = \int_{-L}^L \int_0^{h^{-\rho}} |\partial_\tau \psi|^2 \hat{a} \, d\tau ds - \int_{-L}^L |\psi(s, \tau = 0)|^2 \, ds.$$

ii. Upper bound of $q^{\text{tang}}(\psi)$. We get, by using the min-max principle

$$\begin{aligned} q^{\text{tang}}(\psi) &= \int_{-L}^L |\varphi(s)|^2 \left[\int_0^{h^{-\rho}} |\partial_\tau v_{\kappa(s),h}(\tau)|^2 \hat{a} \, d\tau - |v_{\kappa(s),h}(0)|^2 \right] ds \\ &= \int_{-L}^L q_{\kappa(s),h}(v_{\kappa(s),h}) |\varphi(s)|^2 \, ds \\ &= \int_{-L}^L \lambda_1(\mathcal{H}_{\kappa(s),h}) |\varphi(s)|^2 \, ds \\ &\leq \int_{-L}^L \left(-1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h \right) |\varphi(s)|^2 \, ds + o(h)\|\varphi\|^2. \end{aligned}$$

iii. Upper bound of $q^{trans}(\psi)$. Using that for any $a, b \in \mathbb{R}$ and $h > 0$

$$|a + b|^2 \leq (1 + h^{\frac{1}{2}}) |a|^2 + (1 + h^{-\frac{1}{2}}) |b|^2,$$

we have

$$q^{trans}(\psi) \leq (1 + h^{\frac{1}{2}}) q_1^{trans}(\psi) + (1 + h^{-\frac{1}{2}}) q_2^{trans}(\psi),$$

where

$$q_1^{trans}(\psi) = \int_{-L}^L \int_0^{h^{-\rho}} \left| \left(h^{\frac{1}{2}} \partial_s - i b h^{\frac{1}{2}} \beta_0 \right) \psi \right|^2 \hat{a}^{-1} d\tau ds,$$

and

$$q_2^{trans}(\psi) = b^2 h^2 \int_{-L}^L \int_0^{h^{-\rho}} \left| \left(-\tau + h^{\frac{1}{2}} \frac{\tau^2}{2} \kappa(s) \right) \psi \right|^2 \hat{a}^{-1} d\tau ds.$$

iv. We use Lemma 4.2, there exist constants $c > 0$:

$$q_2^{trans}(\psi) \leq b^2 h^2 (1 + c h^{1/4}) \int_{-L}^L |\varphi(s)|^2 \left[\int_0^{h^{-\rho}} \left| \left(-\tau + h^{\frac{1}{2}} \frac{\tau^2}{2} \kappa(s) \right) v_{\kappa, h} \right|^2 \hat{a} d\tau \right] ds.$$

Using the inequality (4.15), and since the curvature is bounded, there exists a constant $c > 0$, such that:

$$\int_0^{h^{-\rho}} \left| \left(-\tau + h^{\frac{1}{2}} \frac{\tau^2}{2} \kappa(s) \right) v_{\kappa, h} \right|^2 \hat{a} d\tau \leq c.$$

Therefore,

$$(1 + h^{-\frac{1}{2}}) q_2^{trans}(\psi) \leq c b^2 h^{\frac{3}{2}} \|\varphi\|^2.$$

For $\eta < \frac{1}{2}$, we have:

$$\frac{b^2 h^{\frac{3}{2}}}{h} = b_0 h^{-\eta + \frac{1}{2}} \xrightarrow{h \rightarrow 0^+} 0,$$

then,

$$(1 + h^{-\frac{1}{2}}) q_2^{trans}(\psi) = o(h) \|\varphi\|^2.$$

v. We have

$$\begin{aligned} q_1^{trans}(\psi) &\leq h (1 + h^{\frac{1}{2}}) \int_{-L}^L \int_0^{h^{-\rho}} |v_{\kappa(s), h} (\partial_s - i b \beta_0) \varphi|^2 \tilde{a}^{-1} d\tau ds \\ &\quad + h (1 + h^{-\frac{1}{2}}) \int_{-L}^L \int_0^{h^{-\rho}} |\partial_s v_{\kappa(s), h}|^2 |\varphi(s)|^2 \tilde{a}^{-1} d\tau ds. \end{aligned}$$

We use Lemma 4.2 and Corollary 4.1, there exists a constant $c > 0$, such that:

$$\begin{aligned} q_1^{trans}(\psi) &\leq h (1 + h^{\frac{1}{2}}) (1 + c h^{1/2}) \int_{-L}^L |(\partial_s - i b \beta_0) \varphi|^2 ds \\ &\quad + h (1 + h^{-\frac{1}{2}}) (1 + c h^{1/4}) \int_{-L}^L |\varphi(s)|^2 \left[\int_0^{h^{-\rho}} |\partial_s v_{\kappa(s), h}|^2 \tilde{a} d\tau \right] ds. \end{aligned}$$

From (4.14), we deduce that

$$(1 + h^{\frac{1}{2}}) q_1^{trans}(\psi) \leq h (1 + c h^{\frac{1}{2}}) \int_{-L}^L |(\partial_s - i b \beta_0) \varphi|^2 ds + o(h) \|\varphi\|^2.$$

Then the conclusion follows. \square

4.4. **Lower bound.** The following proposition provides a lower bound of the quadratic form.

Proposition 4.2. *Let $0 \leq \eta < \frac{1}{2}$, $\rho \in (0, 1/4)$ and $b_0 > 0$. Suppose that $b = b_0 h^{-\frac{\eta}{2}}$. There exist constants $c_- > 0$, $c > 0$, $h_0 > 0$, and a quadratic form $q_h^{\text{eff},-}$ on $L^2(\mathbb{R}/2L\mathbb{Z})$*

$$q_h^{\text{eff},-}(\varphi) = \int_{-L}^L \left(-1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h \right) |\varphi(s)|^2 ds + h(1 - c_- h^{\frac{1}{2}}) \int_{-L}^L |(\partial_s - ib\beta_0)\varphi|^2 ds,$$

such that, for all $h \in (0, h_0)$, $\psi \in \text{Dom}(\widehat{q}_h^{b,\rho})$, we have:

$$\widehat{q}_h^{b,\rho}(\psi) \geq q_h^{\text{eff},-}(\varphi) + o(h)\|\varphi\|_{L^2(\mathbb{R}/2L\mathbb{Z})}^2 - ch^{1/4}\|\Pi_s^\perp \psi\|_{L^2(\hat{a} d\tau ds)}^2,$$

and with

$$\varphi(s) := \langle \psi(s, \cdot), v_{\kappa(s),h} \rangle_{L^2((0, h^{-\rho}), \hat{a} d\tau)}.$$

Proof. We write

$$\widehat{q}_h^{b,\rho}(\psi) = q^{\text{trans}}(\psi) + q^{\text{tang}}(\psi),$$

with

$$q^{\text{trans}}(\psi) = \int_{-L}^L \int_0^{h^{-\rho}} \left| \left(h^{\frac{1}{2}} \partial_s - ibh^{\frac{1}{2}} \beta_0 - ibh \left(-\tau + h^{\frac{1}{2}} \frac{\tau^2}{2} \kappa(s) \right) \right) \psi \right|^2 \hat{a}^{-1} d\tau ds,$$

and

$$q^{\text{tang}}(\psi) = \int_{-L}^L \int_0^{h^{-\rho}} |\partial_\tau \psi|^2 \hat{a} d\tau ds - \int_{-L}^L |\psi(s, \tau = 0)|^2 ds.$$

i. We get, by using an orthogonal decomposition,

$$q_{\kappa(s),h}(\psi) = q_{\kappa(s),h}(\Pi_s \psi) + q_{\kappa(s),h}(\Pi_s^\perp \psi),$$

Then, by using the min-max principle, we get

$$\begin{aligned} q^{\text{tang}}(\psi) &= \int_{-L}^L q_{\kappa(s),h}(\psi) ds \\ &= \int_{-L}^L q_{\kappa(s),h}(\varphi v_{\kappa(s),h}) ds + \int_{-L}^L q_{\kappa(s),h}(\Pi_s^\perp \psi) ds \\ &\geq \int_{-L}^L \lambda_1(\mathcal{H}_{\kappa(s),h}) \|\varphi v_{\kappa(s),h}\|_{L^2((0, h^{-\rho}), \hat{a} d\tau)}^2 ds \\ &\quad + \int_{-L}^L \lambda_2(\mathcal{H}_{\kappa(s),h}) \|\Pi_s^\perp \psi\|_{L^2((0, h^{-\rho}), \hat{a} d\tau)}^2 ds, \end{aligned}$$

From (4.12) and (4.13), we have

$$q^{\text{tang}}(\psi) \geq \int_{-L}^L \left(-1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h \right) |\varphi(s)|^2 ds + o(h)\|\varphi\|^2 - Ch^{1/2-\rho}\|\Pi_s^\perp \psi\|_{L^2(\hat{a} d\tau ds)}^2.$$

ii. We use Lemma 4.2, we have

$$\begin{aligned} q^{\text{trans}}(\psi) &\geq (1 - ch^{1/2}) \int_{-L}^L \int_0^{h^{-\rho}} \left| h^{\frac{1}{2}} \Pi_s \partial_s \psi - ibh^{\frac{1}{2}} \beta_0 \Pi_s \psi - ibh \Pi_s(\alpha_s \psi) \right|^2 \hat{a} d\tau ds \\ &\quad + (1 - ch^{1/4}) \int_{-L}^L \int_0^{h^{-\rho}} \left| h^{\frac{1}{2}} \Pi_s^\perp \partial_s \psi - ibh^{\frac{1}{2}} \beta_0 \Pi_s^\perp \psi - ibh \Pi_s^\perp(\alpha_s \psi) \right|^2 \hat{a} d\tau ds. \end{aligned}$$

with

$$\alpha_s(\tau) = -\tau + h^{1/2} \frac{\tau^2}{2} \kappa(s).$$

We write

$$\Pi_s \partial_s \psi = \partial_s \Pi_s \psi + [\Pi_s, \partial_s] \psi,$$

and use

$$|a + b|^2 \geq (1 - h^{\frac{1}{2}})|a|^2 - h^{-\frac{1}{2}}|b|^2.$$

We obtain

$$\begin{aligned} q^{trans}(\psi) &\geq (1 - ch^{1/2})(1 - h^{\frac{1}{2}})q_1^{trans}(\psi) - (1 - ch^{1/2})h^{-\frac{1}{2}}q_2^{trans}(\psi) \\ &\quad + (1 - ch^{1/4})(1 - h^\alpha)q_3^{trans}(\psi) - (1 - ch^{1/4})h^{-\alpha}q_2^{trans}(\psi), \end{aligned}$$

with

- $q_1^{trans}(\psi) = \int_{-L}^L \int_0^{h^{-\rho}} \left| h^{\frac{1}{2}} \partial_s \Pi_s \psi - ibh^{\frac{1}{2}} \beta_0 \Pi_s \psi - ibh \Pi_s(\alpha_s \psi) \right|^2 \hat{a} \, d\tau ds,$
- $q_2^{trans}(\psi) = h \int_{-L}^L \int_0^{h^{-\rho}} \left| [\Pi_s, \partial_s] \psi \right|^2 \hat{a} \, d\tau ds = h \int_{-L}^L \int_0^{h^{-\rho}} \left| [\Pi_s^\perp, \partial_s] \psi \right|^2 \hat{a} \, d\tau ds,$
- $q_3^{trans}(\psi) = \int_{-L}^L \int_0^{h^{-\rho}} \left| h^{\frac{1}{2}} \partial_s \Pi_s^\perp \psi - ibh^{\frac{1}{2}} \beta_0 \Pi_s^\perp \psi - ibh \Pi_s^\perp(\alpha_s \psi) \right|^2 \hat{a} \, d\tau ds.$

iii. We have

$$\begin{aligned} q_1^{trans}(\psi) &\geq (1 - h^{\frac{1}{2}}) \int_{-L}^L \int_0^{h^{-\rho}} \left| h^{\frac{1}{2}} \partial_s \Pi_s \psi - ibh^{\frac{1}{2}} \beta_0 \Pi_s \psi \right|^2 \hat{a} \, d\tau ds \\ &\quad - h^{-\frac{1}{2}} \int_{-L}^L \int_0^{h^{-\rho}} \left| bh \Pi_s(\alpha_s \psi) \right|^2 \hat{a} \, d\tau ds. \end{aligned}$$

Clearly,

$$\begin{aligned} \int_{-L}^L \int_0^{h^{-\rho}} \left| \Pi_s(\alpha_s \psi) \right|^2 \hat{a} \, d\tau ds &\leq \int_{-L}^L \int_0^{h^{-\rho}} \left| \alpha_s \psi \right|^2 \hat{a} \, d\tau ds \\ &\leq \int_{-L}^L \int_0^{h^{-\rho}} \left| \alpha_s \Pi_s \psi \right|^2 \hat{a} \, d\tau ds + \int_{-L}^L \int_0^{h^{-\rho}} \left| \alpha_s \Pi_s^\perp \psi \right|^2 \hat{a} \, d\tau ds. \end{aligned}$$

With the same type of reasoning as for the upper bound, there exists a constant $c > 0$ such that

$$q_1^{trans}(\psi) \geq h(1 - ch^{\frac{1}{2}}) \int_{-L}^L |(\partial_s - ib\beta_0)\varphi|^2 \, ds - ch^{1/2} \|\Pi_s^\perp \psi\|^2 + o(h) \|\varphi\|^2.$$

vi. We use Lemma 4.1 and the inequality (4.14), we obtain

$$q_2^{trans}(\psi) \leq ch^2 \|\psi\|_{L^2(\hat{a} \, d\tau ds)}^2 \leq ch^2 \|\varphi\|^2 + ch^2 \|\Pi_s^\perp \psi\|_{L^2(\hat{a} \, d\tau ds)}^2.$$

v. As $q_3^{trans}(\psi) \geq 0$, we obtain:

$$q^{trans}(\psi) \geq h(1 - ch^{\frac{1}{2}}) \int_{-L}^L |(\partial_s - ib\gamma_0)\varphi|^2 \, ds - ch^{1/2} \|\Pi_s^\perp \psi\|_{L^2(\hat{a} \, d\tau ds)}^2 + o(h) \|\varphi\|^2.$$

Then the conclusion follows. \square

4.5. End of the proof. We now have everything we need to finish the proof of Theorem 4.1. The self-adjoint operator associated with the quadratic form $q_h^{\text{eff}, \pm}$ is :

$$\mathcal{L}_h^{\text{eff}, \pm} = -h(1 \pm c_\pm h^{\frac{1}{2}})(\partial_s - ib\beta_0)^2 - 1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h$$

in $L^2(\mathbb{R}/2L\mathbb{Z})$, where c_\pm is a constant independent of h . The operator $\mathcal{L}_h^{\text{eff}, \pm}$ is with compact resolvent and it is bounded from below. Its spectrum is purely discrete and it is formed by an increasing sequence of eigenvalues $\lambda_n(\mathcal{L}_h^{\text{eff}, \pm})$.

Corollary 4.2. *Let $0 \leq \eta < \frac{1}{2}$, $\rho \in (0, 1/4)$ and $b_0 > 0$. Suppose that $b = b_0 h^{\frac{-\eta}{2}}$. For $h \rightarrow 0_+$, we have:*

$$\widehat{\mu}_n(h, b, \rho) \leq \lambda_n(\mathcal{L}_h^{\text{eff},+}) + o(h).$$

Proof. Let $(\varphi_j)_{1 \leq j \leq n}$ an orthonormal family of eigenvectors associated with eigenvalues $(\lambda_j)_{1 \leq j \leq n}$. Let E the subspace of $\text{Dom}(q_h^{\text{eff},+})$ generated by the family $(\varphi_j)_{1 \leq j \leq n}$. We use Proposition 4.1, we have:

$$\widehat{q}_h^{b,\rho}(v_{k(s),h} \varphi) \leq q_h^{\text{eff},+}(\varphi) + o(h) \|\varphi\|^2, \quad \forall \varphi \in E.$$

We use the min-max theorem, we have:

$$q_h^{\text{eff},+}(\varphi) \leq \lambda_n(\mathcal{L}_h^{\text{eff},+}) \|\varphi\|^2.$$

Consequently,

$$\begin{aligned} \widehat{q}_h^{b,\rho}(v_{k(s),h} \varphi) &\leq \lambda_n(\mathcal{L}_h^{\text{eff},+}) \|\varphi\|^2 + o(h) \|\varphi\|^2 \\ &\leq \left(\lambda_n(\mathcal{L}_h^{\text{eff},+}) + o(h) \right) \|v_{k(s),h} \varphi\|^2. \end{aligned}$$

For all $g \in v_{k(s),h} E$, we have:

$$\widehat{q}_h^{b,\rho}(g) \leq \left(\lambda_n(\mathcal{L}_h^{\text{eff},+}) + o(h) \right) \|g\|^2.$$

Since

$$\int_0^{h^{-\rho}} |v_{k(s),h}(\tau)|^2 \widehat{a} \, d\tau = 1,$$

then the functions of the family $(v_{k(s),h} \varphi_j)_{1 \leq j \leq n}$ are linearly independent, so $\dim(f_{k(s),h} E) = n$. By application of the min-max principle, we obtain

$$\widehat{\mu}_n(h, b, \rho) \leq \lambda_n(\mathcal{L}_h^{\text{eff},+}) + o(h).$$

□

Corollary 4.3. *Let $0 \leq \eta < \frac{1}{2}$, $\rho \in (0, 1/4)$ and $b_0 > 0$. Suppose that $b = b_0 h^{\frac{-\eta}{2}}$. For $h \rightarrow 0_+$, we have:*

$$\widehat{\mu}_n(h, b, \rho) \geq \lambda_n(\mathcal{L}_h^{\text{eff},-}) + o(h).$$

Proof. According to Proposition 4.2, there exists $c > 0$, and $\epsilon_0 \in (0, 1)$ such that, for all $\psi \in \text{Dom}(\widehat{q}_h^{b,\rho})$ and h small enough, we have:

$$(4.19) \quad \widehat{q}_h^{b,\rho}(\psi) \geq Q_h^{\text{tens}}(\langle \psi, v_{k(s),h} \rangle, \Pi^\perp \psi),$$

where, $\forall (\varphi, f) \in \text{Dom}(q_h^{\text{eff},-}) \times \text{Dom}(\widehat{q}_h^{b,\rho})$

$$Q_h^{\text{tens}}(\varphi, f) = Q_h^{\text{eff}}(\varphi) - \frac{\epsilon_0}{2} \|f\|_{L^2(\widehat{a} \, d\tau ds)}^2,$$

with

$$Q_h^{\text{eff}}(\varphi) = q_h^{\text{eff},-}(\varphi) + o(h) \|\varphi\|_{L^2(\mathbb{R}/2L\mathbb{Z})}^2.$$

By application of min-max principle (see [12, Chapter 13]), we have the comparison of the Rayleigh quotients:

$$(4.20) \quad \widehat{\mu}_n(h, b, \rho) \geq \widehat{\mu}_n^{\text{tens}}(h).$$

We note that the self-adjoint operators associated with the quadratic forms Q_h^{eff} and Q_h^{tens} are respectively $\mathcal{L}_h^{\text{eff}}$ and $\mathcal{L}_h^{\text{tens}}$. It is easy to see that the spectrum of $\mathcal{L}_h^{\text{tens}}$ lying below $-\epsilon_0$ is coincides with the spectrum of $\mathcal{L}_h^{\text{eff}}$. Then, for all $n \in \mathbb{N}^*$, $\widehat{\mu}_n^{\text{tens}}(h)$ and $\widehat{\mu}_n^{\text{eff}}(h)$ are respectively the n -th eigenvalues of $\mathcal{L}_h^{\text{tens}}$ and $\mathcal{L}_h^{\text{eff}}$ and satisfy

$$(4.21) \quad \widehat{\mu}_n^{\text{tens}}(h) = \widehat{\mu}_n^{\text{eff}}(h) = \lambda_n(\mathcal{L}_h^{\text{eff},-}) + o(h).$$

□

5. SPECTRUM OF EFFECTIVE OPERATOR

Let $\lambda_n(\mathcal{L})$ be the n -th eigenvalue of an operator \mathcal{L} . Let $c \in \mathbb{R}$, $b_0 > 0$, $\eta < \frac{1}{2}$ and $b = b_0 h^{\frac{-\eta}{2}}$. The aim of this section is to study the spectrum of the effective operator of the form:

$$\mathcal{L}_{h,0}^{\text{eff}} = -h(1 - c h^{\frac{1}{2}})(\partial_s - ib\beta_0)^2 - 1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h,$$

and the associate quadratic form on $L^2(\mathbb{R}/2L\mathbb{Z})$ is

$$q_{h,0}^{\text{eff}}(\varphi) = \int_{-L}^L \left(-1 - \kappa(s)h^{\frac{1}{2}} - \frac{\kappa(s)^2}{2}h \right) |\varphi(s)|^2 ds + h(1 - c h^{\frac{1}{2}}) \int_{-L}^L |(\partial_s - ib\beta_0)\varphi|^2 ds.$$

The calculations in this section are classical. The operator that we are going to study its spectrum is a perturbation of the harmonic oscillator. Then, the calculations are based on the context of the harmonic approximation (cf. [12, Section 4.3]), with a small adjustment, since the operator is defined on a periodic space.

Let

$$\mathcal{L}_{h,1}^{\text{eff}} = \mathcal{L}_{h,0}^{\text{eff}} + 1.$$

By definition of the spectrum, we have

$$(5.22) \quad \lambda_n(\mathcal{L}_{h,0}^{\text{eff}}) = -1 + \lambda_n(\mathcal{L}_{h,1}^{\text{eff}}).$$

Either the operator

$$\mathcal{L}_{h,2}^{\text{eff}} = h^{-1/2} \mathcal{L}_{h,1}^{\text{eff}},$$

then,

$$(5.23) \quad \lambda_n(\mathcal{L}_{h,1}^{\text{eff}}) = h^{1/2} \lambda_n(\mathcal{L}_{h,2}^{\text{eff}}).$$

We introduce the semiclassical parameter

$$\hbar = h^{1/4}.$$

Lemma 5.1. *Let us consider:*

$$\mathcal{L}_{h,3}^{\text{eff}} = -\hbar^2(1 - c \hbar^2)(\partial_s - ib\beta_0)^2 - \kappa(s),$$

and the associated quadratic form on $L^2(\mathbb{R}/2L\mathbb{Z})$ is:

$$q_{h,3}^{\text{eff}}(\varphi) = - \int_{-L}^L \kappa(s) |\varphi(s)|^2 ds + \hbar^2(1 - c \hbar^2) \int_{-L}^L |(\partial_s - ib\beta_0)\varphi|^2 ds.$$

We have,

$$(5.24) \quad \lambda_n(\mathcal{L}_{h,2}^{\text{eff}}) = \lambda_n(\mathcal{L}_{h,3}^{\text{eff}}) + \mathcal{O}(\hbar^2).$$

Proof. Let $\varphi \in H^1(\mathbb{R}/2L\mathbb{Z})$, we have

$$\left| q_{h,2}^{\text{eff}}(\varphi) - q_{h,3}^{\text{eff}}(\varphi) \right| = \hbar^2 \left| \int_{-L}^L \frac{\kappa(s)^2}{2} |\varphi(s)|^2 ds \right|,$$

we use that Ω is a bounded open, then the curvature κ is also bounded, hence there exists $c > 0$ independent of s , such that

$$\left| q_{h,2}^{\text{eff}}(\varphi) - q_{h,3}^{\text{eff}}(\varphi) \right| \leq c \hbar^2 \|\varphi\|^2.$$

By application of min-max theorem, this ends the proof. \square

From (5.22), (5.23) and (5.26), we obtain

$$(5.25) \quad \lambda_n(\mathcal{L}_{h,0}^{\text{eff}}) = -1 + \hbar^2 \lambda_n(\mathcal{L}_{h,3}^{\text{eff}}) + \mathcal{O}(\hbar^4).$$

Therefore, to study the spectrum of the effective operator $\mathcal{L}_{h,0}^{\text{eff}}$, it suffices to study the spectrum of the operator $\mathcal{L}_{h,3}^{\text{eff}}$, introduced in Lemma 5.1.

5.1. Localization near the point of maximum curvature. Let κ_{\max} be the maximum curvature along the boundary Γ . Consider the effective operator of the form:

$$\mathcal{L}_{h,4}^{\text{eff}} = -\hbar^2(1 - c\hbar^2)(\partial_s - ib\beta_0)^2 + \kappa_{\max} - \kappa(s),$$

and the quadratic form on $L^2(\mathbb{R}/2L\mathbb{Z})$ associated to it is:

$$q_{h,4}^{\text{eff}}(\varphi) = \int_{-L}^L (\kappa_{\max} - \kappa(s)) |\varphi(s)|^2 ds + \hbar^2(1 - c\hbar^2) \int_{-L}^L |(\partial_s - ib\beta_0)\varphi|^2 ds.$$

Then,

$$(5.26) \quad \lambda_n(\mathcal{L}_{h,3}^{\text{eff}}) = \lambda_n(\mathcal{L}_{h,4}^{\text{eff}}) - \kappa_{\max}.$$

The goal is to show that the eigenfunctions of $\mathcal{L}_{h,4}^{\text{eff}}$, associated with the eigenvalues of orders \hbar concentrate exponentially near the point of maximum curvature.

Proposition 5.1. *For $\epsilon \in (0, 1)$, there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$ and for all eigenfunctions ψ of $\mathcal{L}_{h,4}^{\text{eff}}$, associated with the eigenvalues of orders \hbar , we have:*

$$\|e^{\epsilon\phi_0/\hbar} \psi\|^2 \leq C\|\psi\|^2, \quad q_{h,\alpha,4}^{\text{eff}}(e^{\epsilon\phi_0/\hbar} \psi) \leq C\hbar\|\psi\|^2,$$

where

$$\phi_0(s) = \min_{k \in \mathbb{Z}} \left| \int_0^{s+2kL} \sqrt{\kappa_{\max} - \kappa(s)} ds \right|.$$

Proof. The proof consists of three steps. Let $\epsilon \in (0, 1)$. Consider an eigenvalue $\lambda = \mathcal{O}(\hbar)$ and an associated eigenfunction ψ .

1. First step.

Let ϕ be a regular function, $2L$ periodic, with real value. We have:

$$\mathcal{L}_{h,4}^{\text{eff}}(e^\phi \psi) = e^\phi \mathcal{L}_{h,4}^{\text{eff}}(\psi) - \hbar^2(1 - c\hbar^2) \left(\psi \partial_s^2 e^\phi + 2\partial_s e^\phi \partial_s \psi - 2ib\beta_0 \psi \partial_s e^\phi \right).$$

Then

$$\begin{aligned} q_{h,4}^{\text{eff}}(e^\phi \psi) &= \langle \mathcal{L}_{h,4}^{\text{eff}}(e^\phi \psi), e^\phi \psi \rangle \\ &= \langle e^\phi \mathcal{L}_{h,4}^{\text{eff}}(\psi), e^\phi \psi \rangle \\ &\quad - \hbar^2(1 - c\hbar^2) \int_{-L}^L \left[\psi \partial_s^2 e^\phi + 2\partial_s e^\phi \partial_s \psi - 2ib\beta_0 \psi \partial_s e^\phi \right] \bar{\psi} e^\phi ds \\ &= \lambda \|e^\phi \psi\|^2 - \hbar^2(1 - c\hbar^2) \int_{-L}^L \left[\psi \partial_s^2 e^\phi + 2\partial_s e^\phi \partial_s \psi - 2ib\beta_0 \psi \partial_s e^\phi \right] \bar{\psi} e^\phi ds. \end{aligned}$$

By the formula of integration by parts and as ϕ is $2L$ periodic, we have

$$\int_{-L}^L \left[\psi \partial_s^2 e^\phi + 2\partial_s e^\phi \partial_s \psi - 2ib\beta_0 \psi \partial_s e^\phi \right] \bar{\psi} e^\phi ds = -\|\partial_s e^\phi \psi\|^2.$$

Then, we get

$$(5.27) \quad q_{h,4}^{\text{eff}}(e^\phi \psi) = \lambda \|e^\phi \psi\|^2 + \hbar^2(1 - c\hbar^2) \|\partial_s e^\phi \psi\|^2.$$

2. Second step.

For $\phi = \frac{\epsilon\phi_0}{\hbar}$, we notice that:

$$(5.28) \quad |\phi'|^2 = \frac{\epsilon^2}{\hbar^2} |\phi_0'|^2 = \frac{\epsilon^2}{\hbar^2} (\kappa_{\max} - \kappa(s)).$$

We have

$$q_{h,4}^{\text{eff}}(\varphi) = \int_{-L}^L (\kappa_{\max} - \kappa(s)) |\varphi(s)|^2 ds + \hbar^2(1 - c\hbar^2) \int_{-L}^L |(\partial_s - ib\beta_0)\varphi|^2 ds.$$

Consequently

$$(5.29) \quad q_{h,4}^{\text{eff}}(\varphi) \geq \int_{-L}^L (\kappa_{\max} - \kappa(s)) |\varphi(s)|^2 ds.$$

From (5.27) and (5.29), we have

$$\int_{-L}^L \left[(\kappa_{\max} - \kappa(s)) - \lambda - \hbar^2(1 - c\hbar^2)|\phi'|^2 \right] e^{2\phi} |\psi|^2 ds \leq 0.$$

From (5.28), we get

$$\int_{-L}^L \left[1 - \epsilon^2(1 - c\hbar^2) \right] (\kappa_{\max} - \kappa(s)) e^{2\phi} |\psi|^2 ds \leq \lambda \|e^\phi \psi\|^2 \leq C\hbar \|e^\phi \psi\|^2.$$

We notice that for any $\epsilon \in (0, 1)$ and for \hbar small enough, there exists a constant $C > 0$ independent of \hbar such that

$$1 - \epsilon^2(1 - c\hbar^2) \geq C.$$

We deduce the inequality

$$(5.30) \quad \int_{-L}^L (\kappa_{\max} - \kappa(s)) e^{2\phi} |\psi|^2 ds \leq C\hbar \|e^\phi \psi\|^2.$$

From (5.30) and (5.28), we have

$$(5.31) \quad q_{h,4}^{\text{eff}}(e^\phi \psi) = \lambda \|e^\phi \psi\|^2 + \hbar^2(1 - c\hbar^2) \|\partial_s e^\phi \psi\|^2 \leq C\hbar \|e^\phi \psi\|^2.$$

3. Third step.

Show that:

$$(5.32) \quad \int_{-L}^L e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds \leq C \|\psi\|^2,$$

$$(5.33) \quad q_{h,4}^{\text{eff}}(e^{\epsilon\phi_0/\hbar} \psi) \leq C\hbar \|\psi\|^2.$$

We use that $u = \kappa_{\max} - \kappa$ admits a unique non-degenerate minimum, we deduce that there exists $c > 0$ such that for any $C_0 > 0$, there exist constants C, \hbar_0 such that, for all $\hbar \in (0, \hbar_0)$,

$$(5.34) \quad \int_{|s| \leq C_0 \hbar^{1/2}} (\kappa_{\max} - \kappa(s)) e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds \leq C\hbar \|\psi\|^2,$$

$$(5.35) \quad \int_{|s| \leq C_0 \hbar^{1/2}} e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds \leq C \|\psi\|^2,$$

$$(5.36) \quad \int_{|s| \geq C_0 \hbar^{1/2}} (\kappa_{\max} - \kappa(s)) e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds \geq cC_0^2 \hbar \int_{|s| \geq C_0 \hbar^{1/2}} e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds.$$

By the inequality (5.35), to show (5.32), it suffices to show the following inequality:

$$(5.37) \quad \int_{|s| \geq C_0 \hbar^{1/2}} e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds \leq C \|\psi\|^2.$$

Taking C_0 large enough, we deduce that

$$\begin{aligned} \int_{|s| \geq C_0 \hbar^{1/2}} e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds &\leq \frac{1}{c C_0^2 \hbar} \int_{|s| \geq C_0 \hbar^{1/2}} u(s) e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds \\ &\leq \frac{1}{c C_0^2 \hbar} \int_{-L}^L u(s) e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds. \end{aligned}$$

By inequality (5.30), we get

$$\begin{aligned} \int_{|s| \geq C_0 \hbar^{1/2}} e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds &\leq \frac{C}{c C_0^2} \int_{-L}^L e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds \\ &\leq \frac{C}{c C_0^2} \int_{|s| \leq C_0 \hbar^{1/2}} e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds + \frac{C}{c C_0^2} \int_{|s| \geq C_0 \hbar^{1/2}} e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds. \end{aligned}$$

Thus,

$$\left(1 - \frac{C}{c C_0^2}\right) \int_{|s| \geq C_0 \hbar^{1/2}} e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds \leq \frac{C}{c C_0^2} \int_{|s| \leq C_0 \hbar^{1/2}} e^{2\epsilon\phi_0/\hbar} |\psi|^2 ds.$$

From the inequality (5.35), we get (5.37) and hence we show the point (5.32).

From (5.31), we have

$$q_{\hbar,4}^{\text{eff}}(e^{\epsilon\phi_0/\hbar} \psi) \leq C \hbar \|e^{\epsilon\phi_0/\hbar} \psi\|^2,$$

and from the inequality (5.32), we get the inequality (5.33). \square

5.2. Construction of quasimodes. In this section we will construct trial states (Φ_n) in the domain of the effective operator $\mathcal{L}_{\hbar,4}^{\text{eff}}$, such that, for every fixed $n \in \mathbb{N}^*$ and $\eta < \frac{1}{2}$, we have

$$(5.38) \quad \left\langle \mathcal{L}_{\hbar,4}^{\text{eff}} \Phi_n - (2n-1)\hbar \sqrt{\frac{-\kappa''(0)}{2}} \Phi_n, \Phi_n \right\rangle_{L^2(\mathbb{R}/2L\mathbb{Z})} \leq C \hbar^{\frac{3}{2}}.$$

We know that, the eigenfunctions are localized near the boundary at the scale $\hbar^{\frac{1}{2}}$ (cf. Prop 5.1). It suggests to introduce the rescaling

$$(5.39) \quad s = \hbar^{1/2} \sigma.$$

The effective operator $\mathcal{L}_{\hbar,4}^{\text{eff}}$ becomes

$$\mathcal{L}_{\hbar,4}^{\text{eff}} = -\hbar^2 (1 - c\hbar^2) \left(\hbar^{-\frac{1}{2}} \partial_\sigma - ib\beta_0 \right)^2 + \kappa_{\max} - \kappa(\hbar^{\frac{1}{2}} \sigma).$$

Recall that the value of the maximum curvature is $\kappa_{\max} = \kappa(0)$. We can use a Taylor expansion of κ near 0, as follows

$$\kappa(\hbar^{\frac{1}{2}} \sigma) = \kappa_{\max} + \frac{\hbar \sigma^2}{2} \kappa''(0) + \sum_{j \geq 3} \frac{\hbar^{\frac{j}{2}}}{j!} \sigma^j \kappa^{(j)}(0).$$

We consider the formal expansion of the operator $\mathcal{L}_{h,4}^{\text{eff}}$ (recall that $b = b_0 \hbar^{-2\eta}$)

$$\begin{aligned}
& -\hbar^2 (1 - c\hbar^2) \left(\hbar^{-\frac{1}{2}} \partial_\sigma - ib\beta_0 \right)^2 + \kappa_{\max} - \kappa(\hbar^{\frac{1}{2}} \sigma) \\
&= -\hbar (1 - c\hbar^2) \left(\partial_\sigma - ib\hbar^{\frac{1}{2}} \beta_0 \right)^2 - \frac{\hbar \sigma^2}{2} \kappa''(0) - \sum_{j \geq 3} \frac{\hbar^{\frac{j}{2}}}{j!} \sigma^j \kappa^{(j)}(0) \\
&= -\hbar (1 - c\hbar^2) \left(\partial_\sigma^2 - 2ib\hbar^{\frac{1}{2}} \beta_0 \partial_\sigma - b^2 \hbar \beta_0^2 \right) - \frac{\hbar \sigma^2}{2} \kappa''(0) - \sum_{j \geq 3} \frac{\hbar^{\frac{j}{2}}}{j!} \sigma^j \kappa^{(j)}(0) \\
&= \hbar \left(-\partial_\sigma^2 - \frac{\sigma^2}{2} \kappa''(0) \right) + 2i\beta_0 b_0 \hbar^{\frac{3}{2}-2\eta} \partial_\sigma + \hbar^{2-4\eta} b_0^2 \beta_0^2 + c\hbar^3 \partial_\sigma^2 \\
&\quad + 2icb_0 \beta_0 \hbar^{\frac{7}{2}-2\eta} \partial_\sigma - c\hbar^{4-4\eta} b_0^2 \beta_0^2 - \hbar^{3/2} \sum_{j \geq 3} \frac{\hbar^{\frac{j-3}{2}}}{j!} \sigma^j \kappa^{(j)}(0).
\end{aligned}$$

Consequently, we have the formal expansion of the effective operator $\mathcal{L}_{h,4}^{\text{eff}}$,

$$(5.40) \quad \hbar^{-1} \mathcal{L}_{h,4}^{\text{eff}} = P_0 + R_h + \hbar^{1/2} Q_h,$$

where

$$\begin{aligned}
P_0 &= -\partial_\sigma^2 - \frac{\sigma^2}{2} \kappa''(0), \\
R_h &= 2i\beta_0 b_0 \hbar^{\frac{1}{2}-2\eta} \partial_\sigma + \hbar^{1-4\eta} b_0^2 \beta_0^2 + c\hbar^2 \partial_\sigma^2 + 2icb_0 \beta_0 \hbar^{\frac{3}{2}-2\eta} \partial_\sigma - c\hbar^{3-4\eta} b_0^2 \beta_0^2, \\
Q_h &= -\sum_{j \geq 3} \frac{\hbar^{\frac{j-3}{2}}}{j!} \sigma^j \kappa^{(j)}(0).
\end{aligned}$$

Since $\kappa(0)$ is the non-degenerate maximum of κ , we have $\kappa''(0) < 0$. The eigenvalues of the harmonic oscillator

$$-\partial_\sigma^2 + \frac{-\kappa''(0)}{2} \sigma^2 \quad \text{in } L^2(\mathbb{R})$$

are $(2n-1) \sqrt{\frac{-\kappa''(0)}{2}}$ with $n \in \mathbb{N}^*$. The corresponding normalized eigenfunctions are denoted by $f_n(\sigma)$. They have the form

$$(5.41) \quad f_n(\sigma) = h_n(\sigma) \exp\left(-\sqrt{\frac{-\kappa''(0)}{2}} \frac{\sigma^2}{2}\right),$$

where the $h_n(\sigma)$ are the rescaled Hermite polynomials.

Define the trial state Φ_n in $L^2(\mathbb{R}/2L\mathbb{Z})$ as follows:

$$(5.42) \quad \Phi_n(s) = \hbar^{-1/4} \chi\left(\frac{s}{2L}\right) f_n(\hbar^{-1/2} s),$$

where χ is a cut-off function defined on \mathbb{R} as follows:

$$0 \leq \chi \leq 1, \quad \chi = 1 \quad \text{in }]-1/2, 1/2] \quad \text{and} \quad \chi = 0 \quad \text{in } \mathbb{R} \setminus]-1, 1[.$$

Using the exponential decay of f_n to infinity, it is easy to verify

$$(5.43) \quad \|\Phi_n\|_{L^2(\mathbb{R}/2L\mathbb{Z})} = 1 + o(1).$$

We have

$$\begin{aligned}
\langle R_h \Phi_n, \Phi_n \rangle_{L^2(\hbar^{\frac{1}{2}} d\sigma)} &= 2i\beta_0 b_0 \hbar^{\frac{1}{2}-2\eta} \langle \partial_\sigma \Phi_n, \Phi_n \rangle + 2icb_0 \beta_0 \hbar^{\frac{3}{2}-2\eta} \langle \partial_\sigma \Phi_n, \Phi_n \rangle + c\hbar^2 \langle \partial_\sigma^2 \Phi_n, \Phi_n \rangle \\
&\quad + \hbar^{1-4\eta} b_0^2 \beta_0^2 \|\Phi_n\|_{L^2(\mathbb{R}/2L\mathbb{Z})}^2 - c\hbar^{3-4\eta} b_0^2 \beta_0^2 \|\Phi_n\|_{L^2(\mathbb{R}/2L\mathbb{Z})}^2.
\end{aligned}$$

We observe that, the function $\partial_\sigma f_n f_n$ is odd. Using the exponential decay of f_n at infinity, we have

$$\langle \partial_\sigma \Phi_n, \Phi_n \rangle_{L^2(\hbar^{\frac{1}{2}} d\sigma)} = \mathcal{O}(\hbar^{\frac{1}{2}}).$$

For $\eta \leq \frac{1}{8}$, using the expression of $\langle R_\hbar \Phi_n, \Phi_n \rangle_{L^2(\hbar^{\frac{1}{2}} d\sigma)}$, it is easy to check that

$$\langle R_\hbar \Phi_n, \Phi_n \rangle_{L^2(\hbar^{\frac{1}{2}} d\sigma)} = \mathcal{O}(\hbar^{\frac{1}{2}}).$$

Then, we observe that after a change of variables and by construction of Φ_n (5.40) yields

$$(5.44) \quad \begin{aligned} \hbar^{-1} \mathcal{L}_{h,4}^{\text{eff}} \Phi_n(s) &= (2n-1) \sqrt{\frac{-\kappa''(0)}{2}} \Phi_n(\sigma) + \hbar^{-\frac{1}{4}} \hbar \chi'' \left(\frac{\hbar^{\frac{1}{2}} \sigma}{2L} \right) f_n(\sigma) \\ &+ 2\hbar^{-\frac{1}{4}} \hbar^{\frac{1}{2}} \chi' \left(\frac{\hbar^{\frac{1}{2}} \sigma}{2L} \right) f_n'(\sigma) + R_\hbar \Phi_n(\sigma) + \hbar^{3/2} Q_\hbar \Phi_n(\sigma), \end{aligned}$$

where $\Phi_n(\sigma) = \Phi_n(\hbar^{1/2} s)$.

Now, we infer from (5.44) that

$$\left\langle \mathcal{L}_{h,4}^{\text{eff}} \Phi_n - (2n-1) \hbar \sqrt{\frac{-\kappa''(0)}{2}} \Phi_n, \Phi_n \right\rangle_{L^2(\mathbb{R}/2L\mathbb{Z})} \leq C \hbar^{\frac{3}{2}}.$$

As a consequence of the spectral theorem, (5.38) yields that, for every fixed $n \in \mathbb{N}^*$, there exists an eigenvalue $\lambda_n(\mathcal{L}_{h,4}^{\text{eff}})$ of the operator $\mathcal{L}_{h,4}^{\text{eff}}$ such that

$$(5.45) \quad \tilde{\lambda}_n(\mathcal{L}_{h,4}^{\text{eff}}) = (2n-1) \hbar \sqrt{\frac{-\kappa''(0)}{2}} + \mathcal{O}(\hbar^{3/2}).$$

Proposition 5.2. *For all $n \geq 1$ and $\eta \leq \frac{1}{8}$, there exist $h_0 > 0$ such that, for all $h \in (0, h_0)$*

$$\lambda_n(\mathcal{L}_{h,4}^{\text{eff}}) = (2n-1) \hbar \sqrt{\frac{-\kappa''(0)}{2}} + \mathcal{O}(\hbar^{3/2}).$$

Proof. For all $N \geq 1$, we can take a family of eigenvalues and eigenfunctions

$(\lambda_n(\mathcal{L}_{h,4}^{\text{eff}}), \psi_{n,h})_{n=1,\dots,N}$ such that $(\psi_{n,h})_{n=1,\dots,N}$ is an orthonormal family. Let $\mathcal{E}_N(\hbar)$ the vector subspace of $\text{Dom}(\mathcal{L}_{h,4}^{\text{eff}})$ spanned by the family $(\psi_{n,h})_{n=1,\dots,N}$.

For all $\psi \in \mathcal{E}_N(\hbar)$, ψ also satisfies the Agmon estimates (5.1). By using the proposition 5.1 which gives that the eigenfunctions of $\mathcal{L}_{h,4}^{\text{eff}}$, associated with the eigenvalues of orders \hbar concentrate exponentially near the point of maximum curvature and using the change of variable introduced in (5.39), we easily obtain that, for all $\psi \in \mathcal{E}_N(\hbar)$, there exists $C > 0$ such that

$$\begin{aligned} q_{h,4}^{\text{eff}}(\psi) &\geq \int_{-\hbar^{-1/2}L}^{\hbar^{-1/2}L} \hbar \frac{-\kappa''(0)}{2} \sigma^2 |\psi(\hbar^{1/2}\sigma)|^2 d(\hbar^{1/2}\sigma) \\ &\quad + \hbar(1-c\hbar^2) \int_{-\hbar^{-1/2}L}^{\hbar^{-1/2}L} |(\partial_\sigma - ib\hbar^{\frac{1}{2}}\beta_0)\psi|^2 d(\hbar^{1/2}\sigma) \\ &\geq \hbar \left[\int_{-\hbar^{-1/2}L}^{\hbar^{-1/2}L} \left(|\partial_\sigma \psi(\hbar^{1/2}\sigma)|^2 + \frac{-\kappa''(0)}{2} \sigma^2 |\psi(\hbar^{1/2}\sigma)|^2 \right) d(\hbar^{1/2}\sigma) \right] \\ &\quad - C \hbar^{3/2} \|\psi\|_{L^2(\mathbb{R}/2L\mathbb{Z})}^2 \\ &\geq (2N-1) \hbar \sqrt{\frac{-\kappa''(0)}{2}} \|\psi\|_{L^2(\mathbb{R}/2L\mathbb{Z})}^2 - C \hbar^{3/2} \|\psi\|_{L^2(\mathbb{R}/2L\mathbb{Z})}^2. \end{aligned}$$

Then, by the min-max principle, we have

$$\lambda_N(\mathcal{L}_{h,4}^{\text{eff}}) \geq (2N-1) \hbar \sqrt{\frac{-\kappa''(0)}{2}} - C \hbar^{3/2}.$$

Finally, from (5.45), the asymptotic expansion of the n -th eigenvalue is:

$$\lambda_n(\mathcal{L}_{h,4}^{\text{eff}}) = (2n-1)\hbar\sqrt{\frac{-\kappa''(0)}{2}} + \mathcal{O}(\hbar^{3/2}).$$

□

Now, by inequalities (5.27) and (5.26), we deduce the asymptotic expansion of the n -th eigenvalue of the effective operator $\mathcal{L}_{h,0}^{\text{eff}}$

$$(5.46) \quad \lambda_n(\mathcal{L}_{h,0}^{\text{eff}}) = -1 - \kappa_{\max}\hbar^2 + (2n-1)\sqrt{\frac{-\kappa''(0)}{2}}\hbar^3 + \mathcal{O}(\hbar^{7/2}).$$

From Theorem 4.1 and Proposition 3.2, we have shown Corollary 3.1, which was a reformulation of Corollary 1.1.

5.3. Unit disk case. This section is devoted to the proof of Remark 1.1. We will study the spectrum of the effective operator $\mathcal{L}_{h,0}^{\text{eff}}$, where $\Omega = D(0, 1)$ and $\eta < \frac{1}{2}$. We have

$$L = \pi, \quad \beta_0 = \frac{1}{2}, \quad b = b_0 h^{\frac{-\eta}{2}} \quad \text{and} \quad \kappa = 1.$$

Thus

$$\mathcal{L}_{h,0}^{\text{eff}} = -h \left(1 - ch^{\frac{1}{2}}\right) \left(\partial_s - i\frac{b}{2}\right)^2 - 1 - h^{\frac{1}{2}} - \frac{h}{2},$$

the associate quadratic form on $H^1(\mathbb{R}/2\pi\mathbb{Z})$ is

$$q_{h,0}^{\text{eff}}(\varphi) = \left(-1 - h^{\frac{1}{2}} - \frac{h}{2}\right) \int_{-\pi}^{\pi} |\varphi(s)|^2 ds + h \left(1 - ch^{\frac{1}{2}}\right) \int_{-\pi}^{\pi} \left| \left(\partial_s - i\frac{b}{2}\right) \varphi \right|^2 ds.$$

We use the completeness of the orthonormal family $\{e^{ims}/\sqrt{2\pi}\}_{m \in \mathbb{Z}}$ in $L^2(\mathbb{R}/2\pi\mathbb{Z})$, then, for all $\varphi \in H^1(\mathbb{R}/2\pi\mathbb{Z})$, we have

$$\varphi(s) = \sum_{m \in \mathbb{Z}} \widehat{\varphi}(m) \frac{e^{ims}}{\sqrt{2\pi}},$$

where

$$\widehat{\varphi}(m) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \varphi(s) e^{-ims} ds.$$

For all $\varphi \in H^1(\mathbb{R}/2\pi\mathbb{Z})$, we have

$$\begin{aligned} q_{h,0}^{\text{eff}}(\varphi) &= \left(-1 - h^{\frac{1}{2}} - \frac{h}{2}\right) \int_{-\pi}^{\pi} |\varphi(s)|^2 ds + h \left(1 - ch^{\frac{1}{2}}\right) \int_{-\pi}^{\pi} \left| \left(\partial_s - i\frac{b}{2}\right) \varphi \right|^2 ds \\ &= \sum_{m \in \mathbb{Z}} \left(-1 - h^{\frac{1}{2}} - \frac{h}{2} + h \left(1 - ch^{\frac{1}{2}}\right) \left(m - \frac{b}{2}\right)^2\right) |\widehat{\varphi}(m)|^2 \end{aligned}$$

Let $k \in \mathbb{Z}$, we set

$$\varphi_k(s) = \frac{e^{iks}}{\sqrt{2\pi}} := e_k(s).$$

Inserting φ_k into the quadratic form, we have for all $k \in \mathbb{Z}$

$$q_{h,0}^{\text{eff}}(\varphi_k) = -1 - h^{\frac{1}{2}} - \frac{h}{2} + h \left(1 - ch^{\frac{1}{2}}\right) \left(k - \frac{b}{2}\right)^2.$$

By application of min-max principle, we have

$$\forall k \in \mathbb{Z}, \quad \lambda_1(\mathcal{L}_{h,0}^{\text{eff}}) \leq \frac{q_{h,0}^{\text{eff}}(e_k)}{\|e_k\|^2} \leq -1 - h^{\frac{1}{2}} - \frac{h}{2} + h \left(1 - ch^{\frac{1}{2}}\right) \left(k - \frac{b}{2}\right)^2,$$

then

$$\lambda_1(\mathcal{L}_{h,0}^{\text{eff}}) \leq -1 - h^{\frac{1}{2}} - \frac{h}{2} + h \left(1 - ch^{\frac{1}{2}}\right) \inf_{k \in \mathbb{Z}} \left(k - \frac{b}{2}\right)^2.$$

For $\eta < \frac{1}{2}$, we get as $h \rightarrow 0$

$$h^{\frac{1}{2}} \inf_{k \in \mathbb{Z}} \left(k - \frac{b_0 h^{-\frac{\eta}{2}}}{2}\right)^2 = o(1),$$

therefore, we have the following upper bound of the first eigenvalue

$$(5.47) \quad \lambda_1(\mathcal{L}_{h,0}^{\text{eff}}) \leq -1 - h^{\frac{1}{2}} + \left(\inf_{m \in \mathbb{Z}} \left(m - \frac{b}{2}\right)^2 - \frac{1}{2}\right) h + o(h).$$

Now, for lower bound. For all $\varphi \in H^1(\mathbb{R}/2\pi\mathbb{Z})$, we have

$$\begin{aligned} q_{h,0}^{\text{eff}}(\varphi) &= \sum_{m \in \mathbb{Z}} \left(-1 - h^{\frac{1}{2}} - \frac{h}{2} + h \left(1 - ch^{\frac{1}{2}}\right) \left(m - \frac{b}{2}\right)^2\right) |\hat{\varphi}(m)|^2 \\ &\geq \left(-1 - h^{\frac{1}{2}} - \frac{h}{2} + h \left(\inf_{m \in \mathbb{Z}} \left(m - \frac{b}{2}\right)^2\right)\right) \sum_{m \in \mathbb{Z}} |\hat{\varphi}(m)|^2 + o(h) \sum_{m \in \mathbb{Z}} |\hat{\varphi}(m)|^2 \\ &= \left(-1 - h^{\frac{1}{2}} - \frac{h}{2} + h \left(\inf_{m \in \mathbb{Z}} \left(m - \frac{b}{2}\right)^2\right)\right) \|\varphi\|_{L^2(\mathbb{R}/2L\mathbb{Z})}^2 + o(h) \|\varphi\|_{L^2(\mathbb{R}/2L\mathbb{Z})}^2. \end{aligned}$$

Then, by the min-max principle, we have

$$(5.48) \quad \lambda_1(\mathcal{L}_{h,0}^{\text{eff}}) \geq -1 - h^{\frac{1}{2}} + \left(\inf_{m \in \mathbb{Z}} \left(m - \frac{b}{2}\right)^2 - \frac{1}{2}\right) h + o(h).$$

From (5.47) and (5.48), we deduce the asymptotic expansion of the first eigenvalue of the effective operator $\mathcal{L}_{h,0}^{\text{eff}}$

$$(5.49) \quad \lambda_1(\mathcal{L}_{h,0}^{\text{eff}}) = -1 - h^{\frac{1}{2}} + \left(\inf_{m \in \mathbb{Z}} \left(m - \frac{b}{2}\right)^2 - \frac{1}{2}\right) h + o(h).$$

According to Theorem 3.1, we get

$$\mu_1(h, b) = -h - h^{\frac{3}{2}} + \left(\inf_{m \in \mathbb{Z}} \left(m - \frac{b}{2}\right)^2 - \frac{1}{2}\right) h^2 + o(h^2).$$

To obtain the proof of Remark 1.1, it suffices to use the relation (3.7) between the spectrum of the departure operator and the semi-classical operator.

Then, in the case with constant magnetic field ($\eta = 0$), we found by a very simple proof, Theorem 1.1 of the article [11].

6. THE INFLUENCE OF THE MAGNETIC FIELD ON THE EIGENVALUES

This section is devoted to the proof of Corollary 3.2. The analysis in this section is inspired by the article of B. Helffer and A. Kachmar [5] and its aim is to reveal the influence of the magnetic field, which is not seen by the effective operator. The goal is to construct good trial states and use the spectral theorem to establish existence of n -eigenvalues of the operator $\mathcal{L}_h^{b,\rho}$ satisfying the refined asymptotics

$$\bar{\mu}_n = -h - h^{3/2} \kappa_{\max} + (2n - 1) h^{7/4} \sqrt{\frac{\kappa''(0)}{2}} + h^2 \zeta_{1,n} + \mathcal{O}(h^{17/8}),$$

which in light of three-term asymptotics for $\mu_n(h, b, \rho)$ yields the equality

$$\mu_n(h, b, \rho) = \bar{\mu}_n,$$

for $h > 0$ sufficiently small.

6.1. Construction of approximate eigenfunctions. In this section, we construct trial states (ϕ_n) in the domain of the operator which determines the term in accurate expansion of the eigenvalues which depends of the magnetic field b . The trial states (ϕ_n) will be supported near the boundary of Ω , so that the use of the boundary coordinates (s, t) recalled in Section 2 is valid. By Section 3.2, we recall that the operator $\mathcal{L}_h^{b,\rho}$ is expressed in tubular coordinates (s, t) as follows:

$$\begin{aligned} \mathcal{L}_h^{b,\rho} = & -h^2 a^{-1} \left(\partial_s - ib\beta_0 - ib \left(-t + \frac{t^2}{2} \kappa(s) \right) \right) a^{-1} \left(\partial_s - ib\beta_0 - ib \left(-t + \frac{t^2}{2} \kappa(s) \right) \right) \\ & - h^2 a^{-1} \partial_t a \partial_t \quad (\text{in } (L^2(a dt ds)), \end{aligned}$$

where

$$a(s, t) = 1 - t\kappa(s),$$

and with the Robin and Dirichlet conditions

$$\partial_t u(s, t=0) = -h^{-1/2} u(s, t=0) \quad \text{and} \quad u(s, h^{1/2-\rho}) = 0.$$

We know that, the eigenfunctions are localized near the boundary at the scale $h^{\frac{1}{2}}$ in the t direction (cf. Prop 3.1), and $h^{\frac{1}{8}}$ in the s direction (cf. [5, Thm. 5.5]), then it suggests to introduce this change of variable

$$(s, t) \mapsto (h^{1/8}\sigma, h^{1/2}\tau).$$

This change of variable transforms the above expression of $\mathcal{L}_h^{b,\rho}$ into

$$\begin{aligned} \mathcal{L}_h^{b,\rho} = & h \left[-\tilde{a}^{-1} \left(h^{3/8} \partial_\sigma - ibh^{1/2} \beta_0 - ibh\alpha(\sigma, \tau) \right) \tilde{a}^{-1} \left(h^{3/8} \partial_\sigma - ibh^{1/2} \beta_0 - ibh\alpha(\sigma, \tau) \right) \right. \\ & \left. - \tilde{a}^{-1} \partial_\tau (\tilde{a} \partial_\tau) \right], \end{aligned}$$

with

$$\alpha(\sigma, \tau) = -\tau + \frac{h^{1/2}}{2} \tau^2 \kappa(h^{1/8}\sigma) \quad \text{and} \quad \tilde{a}(\sigma, \tau) = 1 - h^{1/2} \tau \kappa(h^{1/8}\sigma).$$

The boundary condition is now transformed into

$$\partial_\tau \tilde{u} = -\tilde{u} \text{ on } \tau = 0.$$

Recall that the value of the maximal curvature is $\kappa_{\max} = \kappa(0)$. We have the following asymptotic expansions:

$$(6.50) \quad \tilde{a}(\sigma, \tau) = 1 - h^{1/2} \tau \kappa(0) - h^{3/4} \tau \frac{\kappa''(0)}{2!} \sigma^2 + h^{7/8} \tau \epsilon_1(\sigma),$$

$$(6.51) \quad \partial_\tau \tilde{a}(\sigma, \tau) = -h^{1/2} \kappa(0) - h^{3/4} \frac{\kappa''(0)}{2!} \sigma^2 - h^{7/8} \frac{\kappa^{(3)}(0)}{3!} \sigma^3 - h \frac{\kappa^{(4)}(0)}{4!} \sigma^4 + h^{9/8} \epsilon_2(\sigma),$$

$$(6.52) \quad \tilde{a}(\sigma, \tau)^{-1} = 1 + h^{1/2} \epsilon_3(\sigma, \tau),$$

$$(6.53) \quad \tilde{a}(\sigma, \tau)^{-2} = 1 + h^{1/2} \epsilon_4(\sigma, \tau)$$

$$(6.54) \quad \partial_\sigma \{ \tilde{a}(\sigma, \tau)^{-1} \} = -h^{5/8} \epsilon_5(\sigma, \tau),$$

where, for $0 < h < \frac{1}{2}$, $0 \leq \tau \leq h^{-\rho}$ and $\sigma = \mathcal{O}(h^{-1/8})$, the functions ϵ_j , $j = 1, \dots, 5$, satisfy

$$|\epsilon_1(\sigma)| \leq C|\sigma|^3, \quad |\epsilon_2(\sigma)| \leq C|\sigma|^5 \quad \text{and} \quad |\epsilon_3(\sigma, \tau)| + |\epsilon_4(\sigma, \tau)| + |\epsilon_5(\sigma, \tau)| \leq C\tau.$$

This gives us the following identities:

$$\begin{aligned} -\tilde{a}^{-1}\partial_\tau(\tilde{a}\partial_\tau) &= -\partial_\tau^2 - (\tilde{a}^{-1}\partial_\tau\tilde{a})\partial_\tau \\ &= -\partial_\tau^2 + h^{1/2}\kappa(0)\partial_\tau + h^{3/4}\frac{\kappa''(0)}{2!}\sigma^2\partial_\tau + h^{7/8}\frac{\kappa^{(3)}(0)}{3!}\sigma^3\partial_\tau + h\frac{\kappa^{(4)}(0)}{4!}\sigma^4\partial_\tau \\ &\quad + h\tau\kappa^2(0)\partial_\tau + h^{9/8}q_1(\sigma, \tau)\partial_\tau, \end{aligned}$$

and

$$\begin{aligned} &-\tilde{a}^{-1}\left(h^{3/8}\partial_\sigma - ibh^{1/2}\beta_0 - ibh\alpha(\sigma, \tau)\right)\tilde{a}^{-1}\left(h^{3/8}\partial_\sigma - ibh^{1/2}\beta_0 - ibh\alpha(\sigma, \tau)\right) \\ &= -\tilde{a}^{-1}h^{3/8}\partial_\sigma\left[\tilde{a}^{-1}\left(h^{3/8}\partial_\sigma - ibh^{1/2}\beta_0 - ibh\alpha(\sigma, \tau)\right)\right] \\ &\quad + ibh^{1/2}\beta_0\tilde{a}^{-2}\left(h^{3/8}\partial_\sigma - ibh^{1/2}\beta_0 - ibh\alpha(\sigma, \tau)\right) \\ &\quad + ibh\alpha(\sigma, \tau)\tilde{a}^{-2}\left(h^{3/8}\partial_\sigma - ibh^{1/2}\beta_0 - ibh\alpha(\sigma, \tau)\right) \\ &= -h^{3/8}\tilde{a}^{-1}\partial_\sigma\tilde{a}^{-1}\left(h^{3/8}\partial_\sigma - ibh^{1/2}\beta_0 - ibh\alpha(\sigma, \tau)\right) \\ &\quad - h^{3/8}\tilde{a}^{-2}\left(h^{3/8}\partial_\sigma^2 - ibh^{1/2}\beta_0\partial_\sigma - ibh\partial_\sigma\alpha(\sigma, \tau) - ibh\alpha(\sigma, \tau)\partial_\sigma\right) \\ &\quad + ibh^{1/2}\beta_0\tilde{a}^{-2}\left(h^{3/8}\partial_\sigma - ibh^{1/2}\beta_0 - ibh\alpha(\sigma, \tau)\right) \\ &\quad + ibh\alpha(\sigma, \tau)\tilde{a}^{-2}\left(h^{3/8}\partial_\sigma - ibh^{1/2}\beta_0 - ibh\alpha(\sigma, \tau)\right). \end{aligned}$$

We have the following identities:

$$\begin{aligned} &-h^{3/8}\tilde{a}^{-1}\partial_\sigma\tilde{a}^{-1}\left(h^{3/8}\partial_\sigma - ibh^{1/2}\beta_0 - ibh\alpha(\sigma, \tau)\right) = h^{11/8}q_2(\sigma, \tau)\partial_\sigma + h^{\frac{3-\eta}{2}}q_3(\sigma, \tau). \\ &-h^{3/8}\tilde{a}^{-2}\left(h^{3/8}\partial_\sigma^2 - ibh^{1/2}\beta_0\partial_\sigma - ibh\partial_\sigma\alpha(\sigma, \tau) - ibh\alpha(\sigma, \tau)\partial_\sigma\right) \\ &\quad = -h^{6/8}\partial_\sigma^2 + ibh^{7/8}\beta_0\partial_\sigma + h^{\frac{11-4\eta}{8}}q_4(\sigma, \tau)\partial_\sigma + h^{\frac{3-\eta}{2}}q_5(\sigma, \tau) + h^{10/8}q_6(\sigma, \tau)\partial_\sigma^2. \\ &ibh^{1/2}\beta_0\tilde{a}^{-2}\left(h^{3/8}\partial_\sigma - ibh^{1/2}\beta_0 - ibh\alpha(\sigma, \tau)\right) \\ &\quad = ibh^{7/8}\beta_0\partial_\sigma + b^2\beta_0^2h + h^{\frac{11-4\eta}{8}}q_7(\sigma, \tau)\partial_\sigma + h^{\frac{3-2\eta}{2}}q_8(\sigma, \tau). \\ &ibh\alpha(\sigma, \tau)\tilde{a}^{-2}\left(h^{3/8}\partial_\sigma - ibh^{1/2}\beta_0 - ibh\alpha(\sigma, \tau)\right) \\ &\quad = h^{\frac{11-4\eta}{8}}q_9(\sigma, \tau)\partial_\sigma + h^{\frac{3-2\eta}{2}}q_{10}(\sigma, \tau). \end{aligned}$$

where the functions q_i , for $1 \leq i \leq 10$ satisfy for $0 < h < \frac{1}{2}$, $0 \leq \tau \leq h^{-\rho}$ and $|\sigma| = \mathcal{O}(h^{-1/8})$

$$|q_1(\sigma, \tau)| \leq C(|\sigma|^3 + h^{1/8}\tau) \text{ and } |q_i(\sigma, \tau)| \leq C\tau \text{ for } 1 \leq i \leq 10.$$

This gives us the following identities:

$$\begin{aligned} & -\tilde{a}^{-1} \left(h^{3/8} \partial_\sigma - ibh^{1/2} \beta_0 - ibh\alpha(\sigma, \tau) \right) \tilde{a}^{-1} \left(h^{3/8} \partial_\sigma - ibh^{1/2} \beta_0 - ibh\alpha(\sigma, \tau) \right) \\ & = -h^{3/4} \left[\partial_\sigma^2 - 2ibh^{1/8} \beta_0 \partial_\sigma - h^{1/4} b^2 \beta_0^2 \right. \\ & \quad \left. + h^{3/8} \left(h^{1/8} Q_1(\sigma, \tau) \partial_\sigma^2 + h^{\frac{2-4\eta}{8}} Q_2(\sigma, \tau) \partial_\sigma + h^{\frac{3-8\eta}{8}} Q_3(\sigma, \tau) \right) \right], \end{aligned}$$

where

$$\begin{aligned} Q_1 &= q_6 \\ Q_2 &= h^{\eta/2} q_2 + q_4 + q_7 + q_9, \\ Q_3 &= q_8 + q_{10} + h^{\eta/2} (q_3 + q_5). \end{aligned}$$

Consequently, we have the formal expansion of the operator $\mathcal{L}_h^{b,\rho}$,

$$h^{-1} \mathcal{L}_h^{b,\rho} = \mathcal{L}_h = P_0 + h^{1/2} P_1 + h^{3/4} P_2 + h^{7/8} P_3 + h P_4 + h^{9/8} Q_h$$

where

$$\begin{aligned} P_0 &= -\partial_\tau^2 \\ P_1 &= \kappa(0) \partial_\tau = \kappa_{\max} \partial_\tau \\ P_2 &= -\partial_\sigma^2 + \frac{\kappa''(0)}{2} \sigma^2 \partial_\tau \\ P_3 &= \frac{\kappa^{(3)}(0)}{3!} \sigma^3 \partial_\tau + 2ib\beta_0 \partial_\sigma \\ P_4 &= \frac{\kappa^{(4)}(0)}{4!} \sigma^4 \partial_\tau + \tau \kappa^2(0) \partial_\tau + b^2 \beta_0^2 \\ Q_h &= q_1(\sigma, \tau) \partial_\tau + h^{1/8} Q_1(\sigma, \tau) \partial_\sigma^2 + h^{\frac{2-4\eta}{8}} Q_2(\sigma, \tau) \partial_\sigma + h^{\frac{3-8\eta}{8}} Q_3(\sigma, \tau). \end{aligned}$$

Remember from (5.41), $f_n(\sigma)$ is the n -th normalized eigenfunction of the harmonic oscillator

$$-\partial_\sigma^2 + \frac{-\kappa''(0)}{2} \sigma^2.$$

The function u_0 should be interpreted as the ground state of the model operator $\mathcal{H}_{0,0}$ introduced in Section A.1.

The construction of the trial state ϕ_n is based on the simple observation that the function $g(\sigma, \tau) = u_0(\tau) f_n(\sigma)$ satisfies

$$(6.55) \quad P_0 g = -g, \quad P_1 g = -\kappa_{\max} g, \quad P_2 g = (2n-1)g.$$

Let $n \in \mathbb{N}^*$. We will construct a function

$$\Psi_n(\sigma, \tau) = u_0(\tau) f_n(\sigma) + h^{1/8} u_0(\tau) v_{n,1}(\sigma) + h^{2/8} u_0(\tau) v_{n,2}(\sigma) + h^{7/8} g_{n,0}(\sigma, \tau) + h g_{n,0}(\sigma, \tau)$$

in the Schwartz space $\mathcal{S}(\mathbb{R} \times \mathbb{R}_+)$ such that the real-valued Schwartz functions $(v_{n,j})_{j=1,2} \subset \mathcal{S}(\mathbb{R})$, $(g_{n,j})_{j=0,1} \subset \mathcal{S}(\mathbb{R} \times \overline{\mathbb{R}_+})$ satisfies

$$\forall j, \quad \partial_\tau g_{n,j}|_{\tau=0} = -g_{n,j}|_{\tau=0},$$

and as $h \rightarrow 0_+$ and $\eta \leq \frac{1}{8}$, there exist constant $C_n > 0$, such that

$$(6.56) \quad \|(\mathcal{L}_h - \mu_n) \Psi_n\|_{L^2(\mathbb{R} \times \mathbb{R}_+)} \leq C_n h^{\frac{9}{8}}$$

where

$$\mu_n = -1 - h^{1/2} \kappa_{\max} + h^{3/4} \sqrt{-\frac{\kappa''(0)}{2}} (2n-1) + h^{7/8} \zeta_{0,n} + h \zeta_{1,n}.$$

Before we proceed in the construction of the aforementioned sequences, we show how they give us refined expansions of the eigenvalues of the operator $\mathcal{L}_h^{b,\rho}$. Let us consider the cut off function define on \mathbb{R} as following:

$$0 \leq \chi \leq 1 \quad \text{and} \quad \chi = 0 \quad \text{on} \quad]-\infty, -1[\cup]1, +\infty[.$$

Define a function Φ_n in $L^2(\Omega)$ by means of the boundary coordinates as follows:

$$\Phi_n(s, t) = \chi\left(\frac{2s}{|\partial\Omega|}\right) \chi\left(\frac{t}{h^{1/2-\rho}}\right) \Psi_n(h^{-1/8}s, h^{-1/2}t).$$

The function Φ_n is in the domain of the operator $\mathcal{L}_h^{b,\rho}$ because the functions $g_{n,j}$ and u_0 satisfy the Robin condition at $\tau = 0$. Since all involved functions in the expression of Ψ are in the Schwartz space, multiplication by cut-off functions will produce small errors in the various calculations—all the error terms are $\mathcal{O}(h^\infty)$. Also, since the function $u_0(\tau) f_n(\sigma)$ is normalized in $L^2(\mathbb{R} \times \mathbb{R}^+)$, the norm of Φ_n in $L^2(\Omega)$ is equal to $1 + o(1)$, as $h \rightarrow 0^+$.

Thanks to (6.56), we get that

$$\|\mathcal{L}_h^{b,\rho} \Phi_n - h \mu_n \Phi_n\|_{L^2(\Omega)} = \mathcal{O}(h^{17/8}) \|\Phi_n\|_{L^2(\Omega)}.$$

As a consequence of the spectral theorem, for every fixed $n \in \mathbb{N}^*$, there is an eigenvalue $\bar{\mu}_n$ of the operator $\mathcal{L}_h^{b,\rho}$ such that

$$(6.57) \quad \bar{\mu}_n = -h - h^{3/2} \kappa_{\max} + h^{7/4} \sqrt{-\frac{\kappa''(0)}{2}} (2n-1) + h^{15/8} \zeta_{0,n} + h^2 \zeta_{1,n} + \mathcal{O}(h^{17/8}),$$

with $\zeta_{0,n}, \zeta_{1,n} \in \mathbb{R}$.

Thanks to Corollary 3.1 and Proposition 3.2, we observe that there exists h_0 such that, for all $h \in (0, h_0)$,

$$\mu_n(h, b) = \bar{\mu}_n.$$

Thus, $\mu_n(h, b)$ satisfies the asymptotic expansion in Corollary 3.2.

6.2. The sequence $(\zeta_{j,n})_{j=0,1}$. We want to find a real number ζ_0 and two functions $v_1(\sigma)$ and $g_0(\sigma, \tau)$. According to the equalities in (6.55), for every Schwartz function $v(\sigma)$, we have the two simple identities

$$(P_0 + h^{1/2} P_1) u_0(\tau) v(\sigma) = (-1 - h^{1/2} \kappa_{\max}) u_0(\tau) v(\sigma) \quad \text{and}$$

$$P_2 u_0(\tau) v(\sigma) = u_0(\tau) H_{\text{harm}} v(\sigma)$$

where H_{harm} is the operator introduce in (1.4). A straightforward computation yields

$$\begin{aligned} & (\mathcal{L}_h - \mu_n) \Psi_n \\ &= h^{7/8} \left[(P_0 + 1) g_0(\sigma, \tau) + u_0(\tau) \left(H_{\text{harm}} - \sqrt{-\frac{\kappa''(0)}{2}} (2n-1) \right) v_1(\sigma) + (P_3 - \zeta_{0,n}) u_0(\tau) f_n(\sigma) \right] \\ &+ h \left[(P_0 + 1) g_1(\sigma, \tau) + u_0(\tau) \left(H_{\text{harm}} - \sqrt{-\frac{\kappa''(0)}{2}} (2n-1) \right) v_2(\sigma) + (P_3 - \zeta_{0,n}) u_0(\tau) v_1(\sigma) \right. \\ &\left. + (P_4 - \zeta_{1,n}) u_0(\tau) f_n(\sigma) \right] + \mathcal{O}(h^{9/8}). \end{aligned}$$

In order that (6.56) is satisfied, it is sufficient to select ζ_0 , $g_0(\sigma, \tau)$ and $v_1(\sigma)$ as follows:

(6.58)

$$(P_0 + 1)g_0(\sigma, \tau) + u_0(\tau) \left(H_{\text{harm}} - \sqrt{\frac{-\kappa''(0)}{2}}(2n-1) \right) v_1(\sigma) + (P_3 - \zeta_{0,n})u_0(\tau)f_n(\sigma) = 0.$$

We can select ζ_0 such that

$$(P_3 - \zeta_{0,n})u_0(\tau)f_n(\sigma) \perp u_0(\tau)f_n(\sigma) \quad \text{in } L^2(\mathbb{R} \times \mathbb{R}_+).$$

The number $\zeta_{0,n}$ is given by:

$$\begin{aligned} \zeta_{0,n} &= \int_{\mathbb{R} \times \mathbb{R}_+} u_0(\tau)f_n(\sigma) \overline{P_3(u_0(\tau)f_n(\sigma))} \, d\tau d\sigma \\ &= \int_{\mathbb{R}} -\frac{\kappa^{(3)}(0)}{3!} \sigma^3 |f_n(\sigma)|^2 \, d\sigma - ib\gamma_0 \int_{\mathbb{R}} f_n(\sigma) \partial_\sigma f_n(\sigma) \, d\sigma, \end{aligned}$$

we have the integral of the odd functions on \mathbb{R} , then

$$(6.59) \quad \zeta_{0,n} = 0.$$

Consequently,

$$h_1(\sigma) := - \int_0^\infty u_0(\tau) P_3(u_0(\tau)f_n(\sigma)) \, d\tau \perp f_n(\sigma) \quad \text{in } L^2(\mathbb{R}).$$

Since $H_{\text{harm}}^n := H_{\text{harm}} - (2n-1)\sqrt{\frac{-\kappa''(0)}{2}}$ can be inverted in the orthogonal of the n -th eigenfunction $f_n(\sigma)$, we may select $v_1(\sigma)$ to be

$$v_1(\sigma) = \left(H_{\text{harm}} - \sqrt{\frac{-\kappa''(0)}{2}}(2n-1) \right)^{-1} (h_1(\sigma)) = R_n(h_1(\sigma)).$$

where R_n is the regularized resolvent, the self-adjoint operator which vanishes on the eigenspace $E_n = \text{span}(f_n)$, and equals to the inverse of H_{harm}^n on the orthogonal complement of E_n , then extended by linearity to all of $L^2(\mathbb{R})$.

As a consequence of the choice of ζ_0 and $v_1(\sigma)$, we get that, for all σ ,

$$w_1(\sigma, \cdot) := -u_0(\cdot)H_{\text{harm}}^n v_1(\sigma) - (P_3 - \zeta_{0,n})u_0(\cdot)f_n(\sigma)$$

is in the orthogonal of $\{u_0(\tau)\}$ in $L^2(\mathbb{R}_+)$, i.e.

$$\int_0^\infty w_1(\sigma, \tau)u_0(\tau) \, d\tau = 0.$$

The operator $P_0 + 1$ can be inverted in the orthogonal complement of $\{u_0(\tau)\}$, and the inverse is an operator in $L^2(\mathbb{R}_+)$ which sends $\mathcal{S}(\mathbb{R}_+)$ into itself. The proof of this is standard and follows the same argument as in [21, Lemma A.5]. In that way, we may select $g_0(\sigma, \tau)$ to be

$$g_0(\sigma, \cdot) = (P_0 + 1)^{-1}(w_1(\sigma, \cdot)).$$

Now, we want to find a real number ζ_1 and two functions $v_2(\sigma)$, $g_1(\sigma, \tau)$. It is sufficient to select ζ_1 , $g_1(\sigma, \tau)$ and $v_2(\sigma)$ as follows:

$$(P_0 + 1)g_1(\sigma, \tau) + u_0(\tau)H_{\text{harm}}^n v_2(\sigma) + (P_3 - \zeta_{0,n})u_0(\tau)v_1(\sigma) + (P_4 - \zeta_{1,n})u_0(\tau)f_n(\sigma) = 0.$$

We can select ζ_1 such that

$$(P_3 - \zeta_{0,n})u_0(\tau)v_1(\sigma) + (P_4 - \zeta_{1,n})u_0(\tau)f_n(\sigma) \perp u_0(\tau)f_n(\sigma) \quad \text{in } L^2(\mathbb{R} \times \mathbb{R}_+).$$

i.e.

$$\zeta_{1,n} = \int_{\mathbb{R} \times \mathbb{R}_+} u_0(\tau) f_n(\sigma) \left(\overline{P_4(u_0(\tau) f_n(\sigma))} + (P_3 - \zeta_{0,n}) u_0(\tau) v_1(\sigma) \right) d\tau d\sigma.$$

Then,
(6.60)

$$\zeta_{1,n} = -\frac{\kappa^{(4)}(0)}{4!} \int_{\mathbb{R}} \sigma^4 |f_n(\sigma)|^2 d\sigma + b^2 \beta_0^2 - \frac{\kappa^2(0)}{2} + \int_{\mathbb{R} \times \mathbb{R}_+} u_0(\tau) f_n(\sigma) \overline{P_3 u_0(\tau) v_1(\sigma)} d\tau d\sigma.$$

Now, the goal is to separate the terms which depend on b and which do not depend on b . In fact, the terms which are independent of b are already calculated in the article [5] which are denoted by $\beta_{0,n}$. We have

$$(6.61) \quad P_3 u_0(\tau) v_1(\sigma) = -\frac{\kappa^{(3)}(0)}{3!} \sigma^3 u_0(\tau) v_1(\sigma) + 2ib\beta_0 u_0(\tau) \partial_\sigma v_1(\sigma).$$

We recall that

$$(6.62) \quad h_1(\sigma) = -\int_0^\infty u_0(\tau) P_3(u_0(\tau) f_n(\sigma)) d\tau = \frac{\kappa^{(3)}(0)}{3!} \sigma^3 f_n(\sigma) - 2ib\beta_0 \partial_\sigma f_n(\sigma).$$

$$(6.63) \quad v_1(\sigma) = R_n(h_1(\sigma)) = \frac{\kappa^{(3)}(0)}{3!} R_n(\sigma^3 f_n(\sigma)) - 2ib\beta_0 R_n(\partial_\sigma f_n(\sigma)).$$

Then, (6.61), (6.62) and (6.63), imply

$$\begin{aligned} P_3 u_0(\tau) v_1(\sigma) &= -\frac{\kappa^{(3)}(0)}{3!} \sigma^3 u_0(\tau) \left(\frac{\kappa^{(3)}(0)}{3!} R_n(\sigma^3 f_n(\sigma)) - 2ib\beta_0 R_n(\partial_\sigma f_n(\sigma)) \right) \\ &\quad + 2ib\beta_0 u_0(\tau) \partial_\sigma \left(\frac{\kappa^{(3)}(0)}{3!} R_n(\sigma^3 f_n(\sigma)) - 2ib\beta_0 R_n(\partial_\sigma f_n(\sigma)) \right). \end{aligned}$$

We have

$$\begin{aligned} &\int_{\mathbb{R} \times \mathbb{R}_+} u_0(\tau) f_n(\sigma) \overline{P_3 u_0(\tau) v_1(\sigma)} d\tau d\sigma \\ &= \int_{\mathbb{R} \times \mathbb{R}_+} u_0(\tau) f_n(\sigma) \left[-\frac{\kappa^{(3)}(0)}{3!} \sigma^3 u_0(\tau) \left(\frac{\kappa^{(3)}(0)}{3!} R_n(\sigma^3 f_n(\sigma)) + 2ib\beta_0 R_n(\partial_\sigma f_n(\sigma)) \right) \right. \\ &\quad \left. - 2ib\beta_0 u_0(\tau) \partial_\sigma \left(\frac{\kappa^{(3)}(0)}{3!} R_n(\sigma^3 f_n(\sigma)) + 2ib\beta_0 R_n(\partial_\sigma f_n(\sigma)) \right) \right] d\tau d\sigma \\ &= \int_{\mathbb{R}} f_n(\sigma) \left[-\frac{\kappa^{(3)}(0)}{3!} \sigma^3 \left(\frac{\kappa^{(3)}(0)}{3!} R_n(\sigma^3 f_n(\sigma)) + 2ib\beta_0 R_n(\partial_\sigma f_n(\sigma)) \right) \right. \\ &\quad \left. - 2ib\beta_0 \partial_\sigma \left(\frac{\kappa^{(3)}(0)}{3!} R_n(\sigma^3 f_n(\sigma)) + 2ib\beta_0 R_n(\partial_\sigma f_n(\sigma)) \right) \right] d\sigma \\ &= -\left(\frac{\kappa^{(3)}(0)}{3!} \right)^2 \int_{\mathbb{R}} \sigma^3 f_n(\sigma) R_n(\sigma^3 f_n(\sigma)) d\sigma - 2ib\beta_0 \frac{\kappa^{(3)}(0)}{3!} \langle \sigma^3 f_n, R_n(\partial_\sigma f_n) \rangle \\ &\quad - 2ib\beta_0 \frac{\kappa^{(3)}(0)}{3!} \langle f_n, \partial_\sigma R_n(\sigma^3 f_n(\sigma)) \rangle + 4b^2 \beta_0^2 \langle f_n, \partial_\sigma R_n(\partial_\sigma f_n) \rangle. \end{aligned}$$

Using the exponential decay of f_n at infinity, and since R_n is self-adjoint operator, then

$$\langle f_n, \partial_\sigma R_n(\sigma^3 f_n(\sigma)) \rangle = -\langle \sigma^3 f_n, R_n(\partial_\sigma f_n) \rangle,$$

and

$$\langle f_n, \partial_\sigma R_n(\partial_\sigma f_n) \rangle = \langle \partial_\sigma f_n, R_n(\partial_\sigma f_n) \rangle := I_{2,n}.$$

Then, we obtain

$$\int_{\mathbb{R} \times \mathbb{R}_+} u_0(\tau) f_n(\sigma) \overline{P_3 u_0(\tau) v_1(\sigma)} \, d\tau d\sigma = - \left(\frac{\kappa^{(3)}(0)}{3!} \right)^2 \int_{\mathbb{R}} \sigma^3 f_n(\sigma) R_n(\sigma^3 f_n(\sigma)) \, d\sigma - 4b^2 \beta_0^2 I_{2,n}.$$

Consequently, (6.60) gives

$$\begin{aligned} \zeta_{1,n} = & - \frac{\kappa^{(4)}(0)}{4!} \int_{\mathbb{R}} \sigma^4 |f_n(\sigma)|^2 \, d\sigma - \frac{\kappa^2(0)}{2} - \left(\frac{\kappa^{(3)}(0)}{3!} \right)^2 \int_{\mathbb{R}} \sigma^3 f_n(\sigma) R_n(\sigma^3 f_n(\sigma)) \, d\sigma \\ & + b^2 \beta_0^2 (1 - 4I_{2,n}). \end{aligned}$$

The terms which are independent of b in the asymptotic estimate of $\zeta_{1,n}$ are the same which can be found in the article [5, Thm. 1.1] and denoted by $\beta_{0,n}$.

Finally, we have

$$(6.64) \quad \zeta_{1,n} = \beta_{0,n} + b^2 \beta_0^2 (1 - 4I_{2,n}).$$

Consequently,

$$h_2(\sigma) := - \int_0^\infty u_0(\tau) \left[(P_3 - \zeta_{0,n}) u_0(\tau) v_1(\sigma) + (P_4 - \zeta_{1,n}) u_0(\tau) f_n(\sigma) \right] \, d\tau \perp f_n(\sigma) \quad \text{in } L^2(\mathbb{R}).$$

We take:

$$v_2(\sigma) = (H_{\text{harm}}^n)^{-1} h_2(\sigma).$$

As a consequence of the choices of $\zeta_{1,n}$ and $v_2(\sigma)$, we get that, for all σ ,

$$w_2(\sigma, \cdot) := -u_0(\cdot) H_{\text{harm}}^n v_2(\sigma) - (P_3 - \zeta_{0,n}) u_0(\tau) v_1(\sigma) - (P_4 - \zeta_{1,n}) u_0(\tau) f_n(\sigma)$$

is in the orthogonal of $\{u_0(\tau)\}$ in $L^2(\mathbb{R}_+)$.

Finally, we select $g_1(\sigma, \tau)$ as follows

$$g_1(\sigma, \cdot) = (P_0 + 1)^{-1} (w_2(\sigma, \cdot)).$$

APPENDIX A. AUXILIARY OPERATORS

The aim of this section is to recall some spectral properties related to the Robin Laplacian in dimension one. This models naturally arise in our strategy of dimensional reduction and already appears in [12], [5], [11], [9], [16] and [13].

A.1. 1D Laplacian on the half line. We recall here some spectral properties of Robin's Laplacian on \mathbb{R}_+ . Let the operator

$$\mathcal{H}_{0,0} := - \frac{d^2}{d\tau^2} \quad \text{dans } L^2(\mathbb{R}_+),$$

with domain $\text{Dom}(\mathcal{H}_{0,0}) = \{u \in H^2(\mathbb{R}_+) : u'(0) = -u(0)\}$.

This operator is self-adjoint but is not a compact resolvent, then the spectrum of this operator is $\sigma(\mathcal{H}_0) = \text{sp}_{\text{dis}}(\mathcal{H}_{0,0}) \cup \text{sp}_{\text{ess}}(\mathcal{H}_{0,0}) = \{-1\} \cup [0, +\infty[$. The subspace associated with the unique eigenvalue -1 is generated by the normalized eigenfunction in L^2

$$(1.65) \quad u_0(\tau) = \sqrt{2} \exp(-\tau).$$

A.2. A weighted 1D Laplacian. Let $B \in \mathbb{R}$, $T > 0$ such that $|B|T < \frac{1}{3}$. In weighted space $L^2((0, T), (1 - B\tau) d\tau)$, we introduce the operator

$$\mathcal{H}_B^{\{T\}} = -\frac{d^2}{d\tau^2} + \frac{B}{1 - B\tau} \frac{d}{d\tau},$$

with domain

$$\text{Dom}(\mathcal{H}_B^{\{T\}}) = \{u \in H^2(0, T) : u'(0) = -u(0) \text{ and } u(T) = 0\}.$$

This weight will come to measure the effect of the curvature. The operator $\mathcal{H}_B^{\{T\}}$ is the self-adjoint operator in $L^2((0, T), (1 - B\tau) d\tau)$ associated with the following quadratic form:

$$q_B^{\{T\}}(u) = \int_0^T |u'(\tau)|^2 (1 - B\tau) d\tau - |u(0)|^2.$$

with domain

$$\text{Dom}(\mathcal{H}_B^{\{T\}}) = \{u \in H^1(0, T) : u(T) = 0\}.$$

The operator $\mathcal{H}_B^{\{T\}}$ is a compact resolvent. Hence the spectrum $\sigma(\mathcal{H}_B^{\{T\}}) \subseteq \mathbb{R}$, purely discrete and it is formed by a strictly increasing sequence of eigenvalues $(\lambda_n(\mathcal{H}_B^{\{T\}}))_{n \in \mathbb{N}^*}$.

The following proposition gives an asymptotic two-term expansion of the eigenvalue $\lambda_1(\mathcal{H}_B^{\{T\}})$.

Proposition A.1. (Asymptotic of $\lambda_1(\mathcal{H}_B^{\{T\}})$) *there exist constant $C > 0$ and $T_0 > 0$, such that for all $T \geq T_0$, $B \in (-1/(3T), 1/(3T))$, we have:*

$$|\lambda_1(\mathcal{H}_B^{\{T\}}) - (-1 - B)| \leq CB^2.$$

Either the change of function $u = (1 - B\tau)^{-\frac{1}{2}} \tilde{u}$, the new Hilbert space becomes $L^2(0, T), d\tau$, the form domain is always independent of the parameter and the expression of the operator depends on B :

$$\tilde{\mathcal{H}}_B^{\{T\}} = -\frac{d^2}{d\tau^2} - \frac{B^2}{4(1 - B\tau)^2},$$

with the new Robin condition at 0 donate by $\tilde{u}'(0) = \left(-1 - \frac{B}{2}\right) \tilde{u}(0)$ et $\tilde{u}(T) = 0$.

Note that the associated quadratic form is defined on

$$\tilde{q}_B^{\{T\}}(\psi) = \int_0^T |\partial_\tau \psi|^2 d\tau - \int_0^T \frac{B^2}{4(1 - B\tau)^2} |\psi|^2 d\tau - \left(1 + \frac{B}{2}\right) |\psi(0)|^2.$$

For further use, we would like to estimate $\|\tau \tilde{u}_B^{\{T\}}\|_{L^2(0, T), d\tau}$ and $\|\partial_B \tilde{u}_B^{\{T\}}\|_{L^2(0, T), d\tau}$ uniformly with regard to B and T .

Proposition A.2. *There exist constants $C > 0$, $\alpha > 0$ and $T_0 > 0$ such that, for all $T \geq T_0$, $B \in (-1/(3T), 1/(3T))$, we have:*

$$\|e^{\alpha\tau} \tilde{u}_B^{\{T\}}\|_{L^2(0, T), d\tau} \leq C,$$

with $\tilde{u}_B^{\{T\}}$ is the normalized eigenfunction associated with $\lambda_1(\mathcal{H}_B^{\{T\}})$.

Proof. Let ϕ a regular function. By the formula of integration by parts, we get:

$$\begin{aligned} \left\langle \mathcal{H}_B^{\{T\}} \tilde{u}_B^{\{T\}}, e^{2\phi} \tilde{u}_B^{\{T\}} \right\rangle &= \int_0^T \left| \partial_\tau \left(e^{2\phi} \tilde{u}_B^{\{T\}} \right) \right|^2 d\tau - \int_0^T \frac{B^2}{4(1 - B\tau)^2} e^{2\phi} |\tilde{u}_B^{\{T\}}|^2 d\tau \\ &\quad - \left(1 + \frac{B}{2}\right) e^{2\phi(0)} |\tilde{u}_B^{\{T\}}(0)|^2 - \|\phi' e^\phi \tilde{u}_B^{\{T\}}\|^2 \\ &= \lambda_1(\mathcal{H}_B^{\{T\}}) \|e^\phi \tilde{u}_B^{\{T\}}\|^2. \end{aligned}$$

We have

$$|\tilde{u}_B^{\{T\}}(0)|^2 = -2 \int_0^T \partial_\tau \tilde{u}_B^{\{T\}}(\tau) \tilde{u}_B^{\{T\}}(\tau) \, d\tau,$$

according to a trace theory, there are constants $C > 0$ such that, for all $\epsilon > 0$, we have:

$$(1.66) \quad |\tilde{u}_B^{\{T\}}(0)|^2 \leq C\epsilon \|\partial_\tau \tilde{u}_B^{\{T\}}\|_{L^2(0,T)}^2 + C\epsilon^{-1} \|\tilde{u}_B^{\{T\}}\|_{L^2(0,T)}^2$$

We replace (1.66) in the form of $\tilde{q}_B^{\{T\}}$ and we use Proposition A.2, there exist constants $C > 0$ such that:

$$\left(1 - C\epsilon - \frac{C\epsilon B}{2}\right) \|\partial_\tau \tilde{u}_B^{\{T\}}\|_{L^2(0,T)}^2 \leq \left(\lambda_1(\mathcal{H}_B^{\{T\}}) + C\epsilon^{-1}\left(1 + \frac{B}{2}\right) + C\right) \|\tilde{u}_B^{\{T\}}\|_{L^2(0,T)}^2$$

for $\epsilon = \frac{1}{3C}$ and $B \rightarrow 0$, we obtain:

$$\|\tilde{u}_B^{\{T\}}\|_{H^1(0,T)} \leq C,$$

then,

$$|\tilde{u}_B^{\{T\}}(0)|^2 \leq C.$$

Which implies the simple estimate

$$-\int_0^T \frac{B^2}{4(1-B\tau)^2} e^{2\phi} |\tilde{u}_B^{\{T\}}|^2 \, d\tau - \left(1 + \frac{B}{2}\right) C - \int_0^T |\phi'|^2 e^{2\phi} |\tilde{u}_B^{\{T\}}|^2 \, d\tau \leq \lambda_1(\mathcal{H}_B^{\{T\}}) \|e^\phi \tilde{u}_B^{\{T\}}\|^2,$$

and thus,

$$\int_0^T \left[-\frac{B^2}{4(1-B\tau)^2} - \lambda_1(\mathcal{H}_B^{\{T\}}) - |\phi'|^2 \right] e^{2\phi} |\tilde{u}_B^{\{T\}}|^2 \, d\tau \leq C.$$

For $T \rightarrow +\infty$, we have $-1 \leq B \leq 1$, whence

- $\frac{B^2}{4(1-B\tau)^2} \leq \frac{9}{16} B^2$,
- $\lambda_1(\mathcal{H}_B^{\{T\}}) \leq -1 - B$.

We ask that $\phi = \alpha\tau$ with α a real positive constant. For $\alpha < \frac{\sqrt{5}}{4}$, we obtain:

$$-\frac{9}{16} B^2 + 1 + B - \alpha^2 \geq \frac{1}{16},$$

then,

$$\int_0^T e^{2\alpha\tau} |\tilde{u}_B^{\{T\}}|^2 \, d\tau \leq C.$$

□

Lemma A.1. *There exist constants $C > 0$ and $T_0 > 0$, such that, for all $T \geq T_0$ and $B \in (-1/(3T), 1/(3T))$, we have:*

$$\left| \lambda_1(\mathcal{H}_B^{\{T\}}) \right| \leq C,$$

$$\|\partial_B \tilde{u}_B^{\{T\}}\|_{L^2(0,T), d\tau} \leq C.$$

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