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MIXED SPECTRA FOR STABLE SIGNALS FROM DISCRETE OBSERVATIONS

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ABSTRACT

This paper concerns the continuous-time stable alpha symmetric processes which are inivitable in the modeling of certain signals with indefinitely increasing variance. Particularly the case where the spectral measurement is mixed: sum of a continuous measurement and a discrete measurement. Our goal is to estimate the spectral density of the continuous part by observing the signal in a discrete way. For that, we propose a method which consists in sampling the signal at periodic instants. We use Jackson's polynomial kernel to build a periodogram which we then smooth by two spectral windows taking into account the width of the interval where the spectral density is non-zero. Thus, we bypass the phenomenon of aliasing often encountered in the case of estimation from discrete observations of a continuous time process.

KEYWORDS

Spectral density, stable processes, periodogram, smoothing estimate, aliasing.

1. INTRODUCTION

The multiple applications where the random signals whose variance increases indefinitely impose the interest of using stable alpha processes by several research authors in various fields. In particular, Stable symmetric harmonizable processes and their properties have been widely studied by many authors such as [1] - [10] to name a few.

Concrete applications of stable symmetric processes cover a wide spectrum of fields such as: physics, biology, electronics and electricity, hydrology, economics, communications and radar applications., ...ect. See: [11]-[22]. In this paper we consider a symmetric alpha stable harmonizable process $X = \{X(t) : t \in R\}$. Alternatively X has the integral representation:

$$X(t) = \int \exp\left[i(t\lambda)\right] d\xi(\lambda) \tag{1}$$

where $1 < \alpha < 2$ and ξ is a complex valued symmetric α-stable random measure on R with independent and isotropic increments. The control measure is defined by $m(A) = |\xi(A)|_\alpha^\alpha$ (see [4]) is called spectral measure. The estimation of the spectral density function was already studied in different cases: by E.Masry and S.Combanis [4] when the time of the process is continuous, by Sabre [23] when the time of the process is discrete and by R.Sabre [24]-[25]when the time of the process is p-adic.

In this paper we consider a general case where the spectral measure is the sum of an absolutely continuous measure with respect to Lebesgue measure and a discrete measure:

$$d\mu(\lambda) = \phi(x)dx + \sum_{i=1}^{q} c_i \delta_{w_i}$$

where δ is a Dirac measure, the specral density ϕ is nonnegative integrable and bounded function. c_i is an unknown positive real number and w_i is an unknown real number. Assume that $w_i \neq 0$. The discrete measurement comes from the repeated random energy jumps during the experimental measurements. Spectral density represents the distribution of the energy carried by the signal.

Our objective is to propose a nonparametric estimator of the spectral density after discrete sampling of the process $X(t)$. This work is motivated by the fact that in practice it is impossible to observe the process over a continuous time interval. However, we sampled the process at equidistant times, i.e., $t_n = n\tau$, $\tau > 0$. It is known that aliasing of ϕ occurs. For more details about aliasing phenomenon, see [26]. To avoid this difficulty, we suppose that the spectral density ϕ is vanishing for $|\lambda| > \Omega$ where Ω is a nonnegative real number. We create an estimate of the spectral density based on smoothing methods. We show that it is asymptotically unbiased and consistent.

Briefly, the organization of this paper will be as follows: in the second section two technical lemmas will be presented as well as a preiodogram and we will show that this periodogram is an asymptotically unbiased but inconsistent estimator. In the third section, the periodogram will be smoothed by two well chosen spectral windows to estimate the spectral density at the jump points. We show that the smoothing periodogram is a consistent estimator. The fourth section gives conclusions and working perspectives.

2. THE PERIODOGRAM AND ITS PROPRIETIES

In this section we give some basic notations and properties of the Jackson's polynomial kernel. Let N is the size of sample of X. Let k and n are the numbers satisfying: $N-1 = 2k(n-1)$ *with* $n \in N$ $k \in N \cup \left\{\frac{1}{2}\right\}$ if $k = \frac{1}{2}$ then $n = 2n_1 - 1,\ n_1 \in N$.

The Jackson's polynomial kernel is defined by: $\left|H_N(\lambda)\right|^{\alpha} = \left|A_N H^{(N)}(\lambda)\right|^{\alpha}$ where

$$H^{(N)}(\lambda) = \frac{1}{q_{k,n}} \left(\frac{\sin \frac{n\lambda}{2}}{\sin \frac{\lambda}{2}} \right)^{2k} \quad \textit{with} \quad q_{k,n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{\sin \frac{n\lambda}{2}}{\sin \frac{\lambda}{2}} \right)^{2k} d\lambda.$$

where $A_N = (B_{\alpha,N})^{\frac{-1}{\alpha}}$ with $B_{\alpha,N} = \int_{-\pi}^{\pi} \left|H^{(N)}(\lambda)\right|^{\alpha} d\lambda.$

We cite two lemmas which are used in this paper. Their proof are given in [23].

Lemma 2.1 *There is a non negative function* h_k *such as:*

$$H^{(N)}(\lambda) = \sum_{m=-k(n-1)}^{k(n-1)} h_k\left(\frac{m}{n}\right)\cos(m\lambda)$$

Lemma 2.2 *Let* $B'_{\alpha,N} = \int_{-\pi}^{\pi}\left|\frac{\sin\frac{n\lambda}{2}}{\sin\frac{\lambda}{2}}\right|^{2k\alpha} d\lambda$ *and* $J_{N,\alpha} = \int_{-\pi}^{\pi}|u|^{\gamma}|H_N(u)|^{\alpha}\,du$, *where* $\gamma \in]0,2]$, *then*

$$B'_{\alpha,N}\begin{cases} \geq 2\pi\left(\frac{2}{\pi}\right)^{2k\alpha} n^{2k\alpha-1} & if \quad 0<\alpha<2 \\ \leq \frac{4\pi k\alpha}{2k\alpha-1} n^{2k\alpha-1} & if \quad \frac{1}{2k}<\alpha<2 \end{cases}$$

and

$$J_{N,\alpha} \leq \begin{cases} \frac{\pi^{\gamma+2k\alpha}}{2^{2k\alpha}(\gamma-2k\alpha+1)}\frac{1}{n^{2k\alpha-1}} & if \quad \frac{1}{2k}<\alpha<\frac{\gamma+1}{2k} \\ \frac{2k\alpha\pi^{\gamma+2k\alpha}}{2^{2k\alpha}(\gamma+1)(2k\alpha-\gamma-1)}\frac{1}{n^{\gamma}} & if \quad \frac{\gamma+1}{2k}<\alpha<2 \end{cases}$$

This paragraph gives a periodogram and develops its proprieties. Consider the process $X(t)$, defined in (1), observed at instants $t_j = j\tau$, $j = 1,2,...N$ and $\tau = \frac{2\pi}{\omega}$, where ω is a real number strictly greater than 2Ω. Define the periodogram $\hat{I}_N$ on $]-\Omega,\Omega[$ as follows:

$$\hat{I}_N(\lambda) = C_{p,\alpha}\,|I_N(\lambda)|^p, \qquad 0<p<\frac{\alpha}{2}$$

Where

$$I_N(\lambda) = [\tau]^{\frac{1}{\alpha}} A_N \mathrm{Re}[\sum_{n'=-k(n-1)}^{n'=k(n-1)} h_k\left(\frac{n'}{n}\right)\exp\{-i(n'\tau\lambda)\}X(n'\tau+k(n-1)\tau)],$$

The normalisation constant $C_{p,\alpha}$ is given by $C_{p,\alpha} = \frac{D_p}{F_{p,\alpha}[C_\alpha]^{p/\alpha}}$, with

$$D_p = \int_{-\infty}^{\infty}\frac{1-\cos(u)}{|u|^{1+p}}du \quad and \quad F_{p,\alpha} = \int_{-\infty}^{\infty}\frac{1-e^{-|u|^{a}}}{|u|^{1+p}}du\ .$$

Lemma 2.3

The characteristic function of $I_N(\lambda)$, $E\exp\left[irI_N(\lambda)\right]$, *converges to* $\exp[-C_\alpha |r|^\alpha (\psi_{N,1}(\lambda)+\psi_{N,2}(\lambda)]$. *where*

$$\psi_{N,1}(\lambda)=\int_{-\pi}^{\pi}|H_N(y-\tau\lambda)|^\alpha\,\phi\left(\frac{y}{\tau}\right)dy \quad \text{and} \quad \psi_{N,2}(\lambda)=\sum_{i=1}^{q}c_i\left|H_N(w_i-\tau\lambda)\right|^\alpha$$

Proof

From (1) and the expression of I_N, we have:

$$I_N(\lambda)=[\tau]^{\frac{1}{\alpha}}A_N\mathrm{Re}\int_R\sum_{n'=-k(n-1)}^{n'=k(n-1)}h_k\left(\frac{n'}{n}\right)\exp\left\{i[n'\tau(\lambda-u)]\right\}\exp\left\{i[\tau uk(n-1)]\right\}d\xi(u).$$

Using [1] and the definition of the Jackson polynomial kernel we obtain that the characteristic function is the form:

$$E\exp\left[irI_N(\lambda)\right]=\exp\left[-C_a|r|^a\,\psi_N(\lambda)\right]. \tag{2}$$

where $\psi_N(\lambda)=\psi_{N,1}(\lambda)+\psi_{N,2}(\lambda)$ with

$$\psi_{N,1}(\lambda)=\int_R|H_N(v-\tau\lambda)|^a\,\phi\left(\frac{v}{\tau}\right)dv \quad \text{and} \quad \psi_{N,2}(\lambda)=\sum_{i=1}^{q}c_i\left|H_N(w_i-\tau\lambda)\right|^\alpha$$

$$\psi_{N,1}(\lambda)=\int_R|H_N(v-\tau\lambda)|^a\,\phi\left(\frac{v}{\tau}\right)dv=\sum_{j\in Z}\int_{(2j-1)\pi}^{(2j+1)\pi}|H_N(v-\tau\lambda)|^a\,\phi\left(\frac{v}{\tau}\right)dv.$$

Putting $v=y-2\pi j$ and using the fact that H_N is 2π-periodic, we get $\psi_N(\lambda)=\sum_{j\in Z}\int_{-\pi}^{\pi}|H_N(y-\tau\lambda)|^a\,\phi_j(y)dy$, where $\phi_j(y)=\phi\left(\frac{y}{\tau}-\frac{2\pi}{\tau}j\right)$. Let j be an integer such that $-\Omega<\frac{y-2\pi j}{\tau}<\Omega$. Since $\tau\Omega<\pi$ and $|y|<\pi$, we get $|j|<\frac{\tau\Omega}{2\pi}+\frac{1}{2}<1$ and then $j=0$. Therefore

$$\psi_{N,1}(\lambda)=\int_{-\pi}^{\pi}|H_N(y-\tau\lambda)|^\alpha\,\phi\left(\frac{y}{\tau}\right)dy. \tag{3}$$

Theorem 2.4 *Let* $-\Omega<\lambda<\Omega$ *then* $E\left[\hat{I}_N(\lambda)\right]=\left[\psi_N(\lambda)\right]^{\frac{p}{\alpha}}$,

Proof

From the following equality used in [4], for all real x and $0<p<2$,

$$|x|^p=D_p^{-1}\int_{-\infty}^{\infty}\frac{1-\cos(xu)}{|u|^{1+p}}du=D_p^{-1}\mathrm{Re}\int_{-\infty}^{\infty}\frac{1-e^{ixu}}{|u|^{1+p}}du, \tag{4}$$

and replacing x by I_N, we obtain

$$\hat{I}_N(\lambda) = \frac{1}{F_{p,\alpha}[C_a]^{p/\alpha}} Re\int_{-\infty}^{\infty} \frac{1-\exp\{iuI_N(\lambda)\}}{|u|^{1+p}} du, \tag{5}$$

The equation (5) and the definition of the $F_{p,\alpha}$, give

$$\begin{aligned} E\hat{I}_N(\lambda) &= \frac{1}{F_{p,\alpha}[C_a]^{p/\alpha}} \int_R \frac{1-\exp\left\{-C_a |u|^{\alpha} \psi_N(\lambda)\right\}}{|u|^{1+p}} du. \\ &= \left[\psi_N(\lambda)\right]^{p/\alpha}. \end{aligned}$$

3. SMOOTHING PERIODOGRAM

In order to obtain a consistent estimate of $[\phi(\lambda)]^{\frac{p}{\alpha}}$, we smooth the periodogram via spectral windows depending on whether $\tau\lambda$ is a jump point or not $(\tau\lambda \neq w_i)$.

$$f_N(\lambda) = \begin{cases} f_N^{(1)}(\lambda) & if \;\; \tau\lambda \notin \{w_1, w_2, \cdots, w_q\} \\ \dfrac{f_N^{(2)}(\lambda) - c f_N^{(1)}(\lambda)}{1-c} & else \end{cases}$$

where $f_N^{(1)}(\lambda) = \int_{-\pi}^{\pi} W_N^{(1)}(\lambda - u)\hat{I}_N(u)du$ and $f_N^{(2)}(\lambda) = \int_{-\pi}^{\pi} W_N^{(2)}(\lambda - u)\hat{I}_N(u)du.$

The spectral windows $W_N^{(1)}$ and $W_N^{(2)}$ are defined by: $W_N^{(1)}(x) = M_N^{(1)} W(M_N^{(1)} x)$ and $W_N^{(2)}(x) = M_N^{(2))} W(M_N^{(2)} x)$ with W is an even nonnegative, continuous function , vanishing for $|\lambda| > 1$ such that $\int_{-1}^{1} W(u)du = 1$. The bandwidths $M_N^{(1)})$ and $M_N^{(2)})$ satisfying: $c = \dfrac{M_N^{(2)}}{M_N^{(1)}}$ $\lim_{N\to\infty} M_N^{(i)} = +\infty$, $\lim_{N\to\infty} \dfrac{M_N^{(i)}}{N} = 0$ for $i = 1,2$, $\lim_{N\to+\infty} \dfrac{M_N^{(2)}}{M_N^{(1)}} = 0$ and such that $W(M_N^{(2)}\theta) = W(M_N^{(1)}\theta) \;\forall\, \theta \in \left]-\dfrac{1}{M_N^{(1)}}, \dfrac{1}{M_N^{(1)}}\right[$.

Th following theorem shows that $f_N(\lambda)$ is an asymptotically unbiased estimator of $[\phi(\lambda)]^{\frac{p}{\alpha}}$ for $-\Omega < \lambda < \Omega$ and $\tau\lambda \notin \{w_1, w_2, \cdots, w_q\}$.

Theorem 3.1

Let $-\Omega < \lambda < \Omega$, such that $\tau\lambda \notin \{w_1, w_2, \cdots, w_q\}$. Then, $E\left[f_N(\lambda)\right] - \left[\phi(\lambda)\right]^{\frac{p}{\alpha}} = o(1)$.

If ϕ satisfies the hypothesis $\left|\phi(x)-\phi(y)\right| \le cste\left|(x-y)\right|^{-\gamma}$, with $\gamma < 2k\alpha - 1$, then,

$$E\left[f_N(\lambda)\right]-[\phi(\lambda)]^{\frac{p}{\alpha}} = \begin{cases} O\left(\dfrac{1}{n^{(2k\alpha-1)}}+\dfrac{1}{M_N^{(1)\gamma}}\right) & if \quad \lambda \neq 0 \\ O\left(\dfrac{1}{M_N^{(1)}n^{(2k\alpha-1)}}+\dfrac{1}{M_N^{(1)\gamma}}+\dfrac{1}{n^{2k\alpha-1}}\right) & if \quad \lambda = 0 \end{cases}.$$

Proof

It is easy to see that:

$$E\left[f_N(\lambda)\right] = \int_R M_N^{(1)} W\left[M_N^{(1)}(\lambda-u)\right] E\left[\hat{I}_N(u)\right] du.$$

Let $M_N^{(1)}(\lambda-u) = v$, we obtain:

$$E[f_N(\lambda)] = \int_{-1}^{1} W(v)\left[\psi_N\left(\lambda-\frac{v}{M_N^{(1)}}\right)\right]^{\frac{p}{\alpha}} dv. \tag{7}$$

Since $\int_{-1}^{1} W(u)du = 1$ and the inequality (3), we get:

$$\left|E\left[f_N(\lambda)\right]-\left[\phi(\lambda)\right]^{\frac{p}{\alpha}}\right| \le \int_{-1}^{1} W(v)\left|\psi_N\left(\lambda-\frac{v}{M_N^{(1)}}\right)-\phi(\lambda)\right|^{\frac{p}{\alpha}} dv.$$

As $\dfrac{p}{\alpha} < 1$, we obtain

$$\left|\psi_N\left(\lambda-\frac{v}{M_N^{(1)}}\right)-\phi(\lambda)\right|^{\frac{p}{\alpha}} \le \left|\psi_{N,1}\left(\lambda-\frac{v}{M_N^{(1)}}\right)-\phi(\lambda)\right|^{\frac{p}{\alpha}} + \left|\psi_{N,2}\left(\lambda-\frac{v}{M_N^{(1)}}\right)\right|^{\frac{p}{\alpha}}$$

We now examine the limit of $\psi_{N,1}\left(\lambda-\dfrac{v}{M_N^{(1)}}\right)$. From (3) we get:

$$\psi_{N,1}\left(\lambda-\frac{v}{M_N^{(1)}}\right) = \int_{-\infty}^{\infty}\left|H_N\left(u-\tau\left(\lambda-\frac{v}{M_N^{(1)}}\right)\right)\right|^{\alpha} \phi\left(\frac{u}{\tau}\right) du.$$

Let $u-\tau\left(\lambda-\dfrac{v}{M_N^{(1)}}\right) = y$, we obtain:

$$\psi_{N,1}\left(\lambda-\frac{v}{M_N^{(1)}}\right) = \int_R |H_N(y)|^\alpha \phi\left(\lambda-\frac{v}{M_N^{(1)}}+\frac{y}{\tau}\right)dy$$

$$= \sum_{j\in Z}\int_{(2j-1)\pi}^{(2j+1)\pi} |H_N(y)|^\alpha \phi\left(\lambda-\frac{v}{M_N^{(1)}}+\frac{y}{\tau}\right)dy. \quad (8)$$

Let $y-2j\pi = s$. Since $|H_N(.)|^\alpha$ is $2\pi-$periodic function, we get

$$\psi_{N,1}\left(\lambda-\frac{v}{M_N^{(1)}}\right)=\sum_{j\in Z}\int_{-\pi}^{+\pi}|H_N(s)|^\alpha \phi\left(\lambda-\frac{v}{M_N^{(1)}}+\frac{s}{\tau}+\frac{2\pi}{\tau}j\right)ds.$$

The function ϕ is uniformly continuous on $[-\Omega,\Omega]$ and since $|H_N|^\alpha$ is a kernel, the right hand side of the last equality converges to $\sum_{j\in Z}\phi\left(\lambda+\frac{2\pi j}{\tau}\right)$. Let j be an integer such that $-\Omega<\frac{\tau\lambda+2\pi j}{\tau}<\Omega$. The definition of τ implies that $|\tau\lambda|<|\tau\Omega|<\pi$. It is easy to see that $|j|<1$ and then $j=0$. Since $H_N^{(1)}$ is a kernel, we obtain that $\psi_{N,1}\left(\lambda-\frac{v}{M_N^{(1)}}\right)$ converges to $\phi(\lambda)$. On the other hand, $\psi_{N,2}\left(\lambda-\frac{v}{M_N^{(1)}}\right)=\sum_{i=1}^{q}\left|H_N^{(1)}\left(w_i-\tau\left(\lambda-\frac{v}{M_N^{(1)}}\right)\right)\right|^\alpha$

Since w_i is different from $\tau\lambda$ and from the lemma 2.2, we get $\psi_{N,2}\left(\lambda-\frac{v}{M_N^{(1)}}\right)\le\left[2\pi\left(\frac{2}{\pi}\right)^{2k\alpha}n^{2k\alpha-1}\right]^{-1}\frac{1}{cte}\sum_{i=1}^{q}c_i$, where $\text{cte}=\inf\left|\sin\left(\frac{w_i-\tau\left(\lambda-\frac{v}{M_N^{(1)}}\right)}{2}\right)\right|^\alpha$.

Therefore, $\psi_{N,2}\left(\lambda-\frac{v}{M_N^{(1)}}\right)=O\left(\frac{1}{n^{2k\alpha-1}}\right)$. Thus, we have $E\left[f_N(\lambda)\right]-\left[\phi(\lambda)\right]^{\frac{p}{\alpha}}=o(1)$.

The rate of convergence:

Assume that the spectral density ϕ satisfies the hypothesis H. Let $F=|Bias(f_N(\lambda))|=|E[f_N(\lambda)]-[\phi(\lambda)]^{p/\alpha}|$. It follows that

$$F\le\frac{p}{2\alpha}\int_{-1}^{1}W(v)\left[\left[\psi_N\left(\lambda-\frac{v}{M_N^{(1)}}\right)\right]^{\frac{p}{\alpha}-1}+[\phi(\lambda)]^{\frac{p}{\alpha}-1}\right]\left|\psi_N\left(\lambda-\frac{v}{M_N^{(1)}}\right)-\phi(\lambda)\right|dv.$$

Since $\psi_N\left(\lambda-\frac{v}{M_N}\right)$ converges to $\phi(\lambda)$, in order to study the rate of the convergence for F we examine the rate of convergence of $\int_{-1}^{1}W(v)\left|\psi_N\left(\lambda-\frac{v}{M_N^{(1)}}\right)-\phi(\lambda)\right|dv$. Indeed, from (3), we obtain $\psi_N\left(\lambda-\frac{v}{M_N^{(1)}}\right)=\int_{-\pi}^{\pi}|H_N\left(y-\tau\lambda+\frac{\tau v}{M_N^{(1)}}\right)|^{\alpha}\,\phi\left(\frac{y}{\tau}\right)dy.$

Let $\Delta\left(\psi_N,\phi\right)=\psi_N\left(\lambda-\frac{v}{M_N^{(1)}}\right)-\phi(\lambda)$. Putting $t=-\left(y-\tau\lambda+\frac{\tau v}{M_N^{(1)}}\right)$ and using the condition H, we get $\left|\Delta\left(\psi_N,\phi\right)\right|\le C_1\int_{\tau\lambda-\frac{\tau v}{M_N^{(1)}}-\pi}^{\tau\lambda-\frac{\tau v}{M_N^{(1)}}+\pi}\left|H_N\left(t\right)\right|^{\alpha}\left|\frac{v}{M_N^{(1)}}+\frac{t}{\tau}\right|^{\gamma}dt.$

We can maximize as follows:

$$\begin{aligned}\int_{-1}^{1}W(v)\left|\Delta(\psi_N,\phi)\right|dv &\le 2^{\gamma}C_1\left|\frac{1}{M_N^{(1)}}\right|^{\gamma}\int_{-1}^{1}W(v)\,|v|^{\gamma}\,dv\\ &+ 2^{\gamma}\frac{C_1}{\tau^{\gamma}}\int_{-1}^{1}W(v)\int_{\tau_1\lambda-\frac{\tau v}{M_N^{(1)}}-\pi}^{\tau\lambda-\frac{\tau_1 v}{M_N^{(1)}}+\pi}\left|H_N\left(t\right)\right|^{\alpha}\left|t\right|^{\gamma}dtdv\end{aligned}$$

The second integral of the right hand side is bounded as follows:

$$\begin{aligned}\int_{\tau\lambda-\frac{\tau v}{M_N^{(1)}}-\pi}^{\tau\lambda-\frac{\tau v}{M_N^{(1)}}+\pi}\left|H_N\left(t\right)\right|^{\alpha}\left|t\right|^{\gamma}dt &\le \int_{-|\tau\lambda|-|\frac{\tau v}{M_N^{(1)}}|-\pi}^{-\pi}\left|H_N\left(t\right)\right|^{\alpha}\left|t\right|^{\gamma}dt\\ &+ \int_{-\pi}^{\pi}\left|H_N\left(t\right)\right|^{\alpha}\left|t\right|^{\gamma}dt \qquad (9)\\ &+ \int_{\pi}^{|\tau\lambda|+|\frac{\tau v}{M_N^{(1)}}|+\pi}\left|H_N\left(t\right)\right|^{\alpha}\left|t\right|^{\gamma}dt.\end{aligned}$$

Since $|H_N(.)|$ is even, the first and the last integrals in the right hand side of (9) are equal. As $\frac{\tau v}{M_N}$ converges to zero and $\tau\lambda<\tau\Omega<\pi$, for a large N we have:

$$\int_{\pi}^{|\tau\lambda|+|\frac{\tau v}{M_N^{(1)}}|+\pi} \left|H_N(t)\right|^{\alpha} |t|^{\gamma}\, dt \;\leq\; (2\pi)^{\gamma} \int_{\pi}^{|\tau\lambda|+|\frac{\tau v}{M_N^{(1)}}|+\pi} \left|H_N(t)\right|^{\alpha} dt$$

$$\leq \frac{(2\pi)^{\gamma}}{B'_{\alpha,N}} \frac{|\tau\lambda| + \frac{\tau}{M_N^{(1)}}}{\left|\sin\left(\pi + |\tau\lambda| + \frac{\tau}{M_N^{(1)}}\right)\right|^{2k\alpha}}$$

The lemma 2.1 gives:

$$\int_{\pi}^{|\tau\lambda|+|\frac{\tau v}{M_N^{(1)}}|+\pi} \left|H_N(t)\right|^{\alpha} |t|^{\gamma}\, dt = \begin{cases} \mathrm{O}\left(\dfrac{1}{n^{2k\alpha-1}}\right) & if\ \lambda \neq 0, \\ \mathrm{O}\left(\dfrac{1}{M_N^{(1)} n^{2k\alpha-1}}\right) & if\ \lambda = 0, \end{cases}$$

Thus, we obtain the result of the theorem.

Theorem 3.2. Let λ a real nuber belonging to $]-\Omega,\Omega[$, and $\tau\lambda = w_i$. Choose k such that $\lim_{N\to\infty} \dfrac{\left(M_N^{(1)}\right)^{2k\alpha}}{n^{2k\alpha-1}} = 0$. Then,

i) $E[f_N(\lambda)] - [\phi(\lambda)]^{\frac{p}{\alpha}} = o(1)$

ii) If ϕ satisfiies the hypothesis $\left|\varphi(x) - \varphi(y)\right| \leq cste\left|(x-y)\right|^{-\gamma}$, with $\frac{\gamma+1}{2k} < \alpha < 2$, then

$$\left|E[f_N(\lambda)] - [\phi(\lambda)]^{\frac{p}{\alpha}}\right| = \begin{cases} O\left(\dfrac{1}{\left(M_N^{(2)}\right)^{\gamma}} + \dfrac{\left(M_N^{(1)}\right)^{2k\alpha}}{n^{2k\alpha-1}}\right) & if \quad 0 < \gamma \leq 1 \\ O\left(\dfrac{1}{M_N^{(2)}} + \dfrac{\left(M_N^{(1)}\right)^{2k\alpha}}{n^{2k\alpha-1}}\right) & if \quad 1 < \gamma \leq 2 \end{cases}$$

Proof :

The form of estimator gives:

$$E[f_N(\lambda)] = \int_{-\pi}^{\pi} \frac{W_N^{(2)}(\lambda-u) - \frac{M_N^{(2)}}{M_N^{(1)}} W_N^{(1)}(\lambda-u)}{1-\frac{M_N^{(2)}}{M_N^{(1)}}} [\psi_N(u)]^{\frac{p}{\alpha}} du$$

$$E[f_N(\lambda)] = \int_{-\pi}^{\lambda-\frac{1}{M_N^{(1)}}} + \int_{\lambda-\frac{1}{M_N^{(1)}}}^{\lambda+\frac{1}{M_N^{(1)}}} + \int_{\lambda+\frac{1}{M_N^{(1)}}}^{\pi} = E_1 + E_2 + E_3$$

Put $\lambda - u = v$, $E_2 = \int_{-\frac{1}{M_N^{(1)}}}^{\frac{1}{M_N^{(1)}}} \frac{M_N^{(2)}\left[W[M_n^{(2)}v] - W[M_N^{(1)}v]\right]}{1-\frac{M_N^{(2)}}{M_N^{(1)}}} [\psi_N(\lambda-v)]^{\frac{p}{\alpha}} dv.$ Therefore $E_2 = 0$ for a large N.

$$E_1 = \frac{M_N^{(2)}}{1-\frac{M_N^{(2)}}{M_N^{(1)}}} \int_{-\pi}^{\lambda-\frac{1}{M_N^{(1)}}} W[M_N^{(2)}(\lambda-u)][\psi_N(u)]^{\frac{p}{\alpha}} du -$$

$$\frac{M_N^{(2)}}{1-\frac{M_N^{(2)}}{M_N^{(1)}}} \int_{-\pi}^{\lambda-\frac{1}{M_N^{(1)}}} W[M_N^{(1)}(\lambda-u)][\psi_N(u)]^{\frac{p}{\alpha}} du$$

Put $M_N^{(2)}(\lambda-u) = v$ in the first integral and put $M_N^{(1)}(\lambda-u) = w,$ in the second integral and for a large *N*, we have $M_N^{(1)}(\lambda+\pi) > 1 \quad et \quad M_N^{(2)}(\lambda+\pi) > 1.$ As W is null outside of $[-1,1]$, for large *N*, the second integral of E_1 is zero.Therefore,

$$E_1 = \frac{1}{1-\frac{M_N^{(2)}}{M_N^{(1)}}} \int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^{1} W(v) \left[\psi_N\left(\lambda - \frac{v}{M_N^{(2)}}\right)\right]^{\frac{p}{\alpha}} dv. \tag{10}$$

$$E_3 = \frac{M_N^{(2)}}{1-\frac{M_N^{(2)}}{M_N^{(1)}}} \int_{\lambda+\frac{1}{M_N^{(1)}}}^{\pi} W[M_N^{(2)}(\lambda-u)][\psi_N(u)]^{\frac{p}{\alpha}} du - \frac{M_N^{(2)}}{1-\frac{M_N^{(2)}}{M_N^{(1)}}} \int_{\lambda+\frac{1}{M_N^{(1)}}}^{\pi} W[M_N^{(1)}(\lambda-u)][\psi_N(u)]^{\frac{p}{\alpha}} du$$

Putting $M_N^{(2)}(\lambda-u) = v$ in the first integral and $M_N^{(1)}(\lambda-u) = w$ in the second integral, we obtain

$$E_3 = \frac{1}{1-\frac{M_N^{(2)}}{M_N^{(1)}}} \int_{M_N^{(2)}(\lambda-\pi)}^{-\frac{M_N^{(2)}}{M_N^{(1)}}} W(v) \left[\psi_N\left(\lambda - \frac{v}{M_N^{(2)}}\right)\right]^{\frac{p}{\alpha}} dv - \frac{\frac{M_N^{(2)}}{M_N^{(1)}}}{1-\frac{M_N^{(2)}}{M_N^{(1)}}} \int_{M_N^{(1)}(\lambda-\pi)}^{-1} W(w) \left[\psi_N\left(\lambda - \frac{w}{M_N^{(1)}}\right)\right]^{\frac{p}{\alpha}} dw.$$

For large N we have, $M_N^{(2)}(\lambda-\pi)<-1$ *and* $M_N^{(1)}(\lambda-\pi)<-1.$

$$E_3=\frac{1}{1-\frac{M_N^{(2)}}{M_N^{(1)}}}\int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^{1}W(v)\left[\psi_N\left(\lambda+\frac{v}{M_N^{(2)}}\right)\right]^{\frac{p}{\alpha}}dv. \tag{11}$$

It is easy to show that for a large N

$$\frac{1}{1-\frac{M_N^{(2)}}{M_N^{(1)}}}\int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^{1}W(v)dv=\frac{1}{2}. \tag{12}$$

$$\left|E[f_N(\lambda)]-[\phi(\lambda)]^{\frac{p}{\alpha}}\right|=\left|E_1+E_3-[\phi(\lambda)]^{\frac{p}{\alpha}}\right|\le\left|E_1-\frac{1}{2}[\phi(\lambda)]^{\frac{p}{\alpha}}\right|+\left|E_3-\frac{1}{2}[\phi(\lambda)]^{\frac{p}{\alpha}}\right|$$

From (10) and (12), for a large N, we have

$$\left|E_1-\frac{1}{2}[\phi(\lambda)]^{\frac{p}{\alpha}}\right|\le\frac{1}{1-\frac{M_N^{(2)}}{M_N^{(1)}}}\int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^{1}W(v)\left|\psi_{N,1}\left(\lambda-\frac{v}{M_N^{(2)}}\right)+\psi_{N_2}\left(\lambda-\frac{v}{M_N^{(2)}}\right)-\phi(\lambda)\right|^{\frac{p}{\alpha}}dv$$

As $\frac{p}{\alpha}<1$, we obtain

$$\left|E_1-\frac{1}{2}[\phi(\lambda)]^{\frac{p}{\alpha}}\right|\le\frac{1}{1-\frac{M_N^{(2)}}{M_N^{(1)}}}\int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^{1}W(v)\left|\psi_{N,1}\left(\lambda-\frac{v}{M_N^{(2)}}\right)-\phi(\lambda)\right|^{\frac{p}{\alpha}}dv$$

$$+\frac{1}{1-\frac{M_N^{(2)}}{M_N^{(1)}}}\int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^{1}W(v)\left|\psi_{N,2}\left(\lambda-\frac{v}{M_N^{(2)}}\right)\right|^{\frac{p}{\alpha}}dv.$$

On the other hand,

$$\int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^{1}W(v)\left|\psi_{N,1}\left(\lambda\pm\frac{v}{M_N^{(2)}}\right)-\phi(\lambda)\right|^{\frac{p}{\alpha}}dv\le\int_0^1W(v)\left|\psi_{N,1}\left(\lambda\pm\frac{v}{M_N^{(2)}}\right)-\phi(\lambda)\right|^{\frac{p}{\alpha}}dv$$

For all λ belonging to $]-\pi,\pi[$ $\psi_{N,1}\left(\lambda\pm\frac{v}{M_N^{(2)}}\right)$ converges to $\phi(\lambda)$, uniformly in $v\in[-1,1]$. Therefore, $\frac{1}{1-\frac{M_N^{(2)}}{M_N^{(1)}}}\int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^{1}W(v)\left|\psi_{N,1}\left(\lambda\pm\frac{v}{M_N^{(2)}}\right)-\phi(\lambda)\right|^{\frac{p}{\alpha}}dv$ converge to zero.

Since $\tau\lambda = w_i$, $\quad \psi_{N,2}\left(\lambda_i \pm \frac{v}{M_N^{(2)}}\right) \le \sum_{\substack{m=1 \\ m\neq i}}^{q} \frac{a_m}{B'_{\alpha,N}} \frac{1}{\left|\sin\left[\frac{1}{2}\left(w_i \pm \frac{\tau v}{M_N^{(2)}} - w_m\right)\right]\right|^{2k\alpha}}$

$$+\frac{a_i}{B'_{\alpha,N}} \frac{1}{\left|\sin\left[\frac{1}{2}\left(\pm\frac{\tau v}{M_N^{(2)}}\right)\right]\right|^{2k\alpha}}.$$

For all $\quad m \neq i$, $\lim_{N\to+\infty} \sup_{v\in[-1,1]} \frac{1}{\left|\sin\left[\frac{1}{2}\left(w_i \pm \frac{\tau v}{M_N^{(2)}} - w_m\right)\right]\right|^{2k\alpha}} = \frac{1}{\left|\sin\left[\frac{1}{2}\left(w_i - w_m\right)\right]\right|^{2k\alpha}}.$

Thus, for large N, we get

$$\sum_{\substack{m=1 \\ m\neq i}}^{q} \frac{a_m}{B'_{\alpha,N}} \frac{1}{\left|\sin\left[\frac{1}{2}\left(w_i \pm \frac{\tau v}{M_N^{(2)}} - w_m\right)\right]\right|^{2k\alpha}} \le \left(\varepsilon + \frac{1}{\inf_{m\in\{1,2,\cdots,q\}-\{i\}} \left|\sin\left[\frac{1}{2}\left(w_i - w_m\right)\right]\right|^{2k\alpha}}\right) \sum_{\substack{m=1 \\ m\neq i}}^{q} \frac{a_m}{B'_{\alpha,N}}.$$

The lemma 2.1 gives $\quad \sum_{\substack{m=1 \\ m\neq i}}^{q} \frac{a_m}{B'_{\alpha,N}} \frac{1}{\left|\sin\left[\frac{1}{2}\left(w_i \pm \frac{\tau v}{M_N^{(2)}} - w_m\right)\right]\right|^{2k\alpha}} = O\left(\frac{1}{n^{2k\alpha-1}}\right).$ (13)

For large N, we have $\sup_{v\in]0,1]} \left|\frac{\tau v}{M_N^{(2)}}\right| = \frac{\tau}{M_N^{(2)}} < \pi$. Consequently

$$\frac{a_i}{B'_{\alpha,N}} \frac{1}{\left|\sin\left[\frac{1}{2}\left(\pm\frac{\tau v}{M_N^{(2)}}\right)\right]\right|^{2k\alpha}} \le \frac{a_i}{B'_{\alpha,N}} \frac{\pi^{2k\alpha}}{\left|\frac{\tau v}{M_N^{(2)}}\right|^{2k\alpha}}.$$

As $\quad \frac{\tau}{M_N^{(1)}} \le \frac{\tau v}{M_N^{(2)}} \le \frac{\tau}{M_N^{(2)}}$, we obtain $\frac{a_i}{B'_{\alpha,N}} \frac{1}{\left|\sin\left[\frac{1}{2}\left(\pm\frac{\tau v}{M_N^{(2)}}\right)\right]\right|^{2k\alpha}} \le \frac{a_i}{B'_{\alpha,N}} \frac{\pi^{2k\alpha}}{\left|\frac{\tau}{M_N^{(1)}}\right|^{2k\alpha}}.$

Frome the lemma 2.2 , we obtain

$$\frac{a_i}{B'_{\alpha,N}} \frac{1}{\left|\sin\left[\frac{1}{2}\left(+-\frac{\tau v}{M_N^{(2)}}\right)\right]\right|^{2k\alpha}} = O\left(\frac{\left(M_N^{(1)}\right)^{2k\alpha}}{n^{2k\alpha-1}}\right). \tag{14}$$

Therefore, we get

$$\frac{2^{\frac{p}{\alpha}}}{1-\frac{M_N^{(2)}}{M_N^{(1)}}} \int_{\frac{M_N^{(2)}}{M_N^{(1)}}}^{1} W(v) \left|\psi_{N,2}\left(\tau\lambda \pm \frac{\tau v}{M_N^{(2)}}\right)\right|^{\frac{p}{\alpha}} dv = O\left[\frac{1}{n^{(2k\alpha-1)\frac{p}{\alpha}}} + \left(\frac{\left(M_N^{(1)}\right)^{2k\alpha}}{n^{2k\alpha-1}}\right)^{\frac{p}{\alpha}}\right] \tag{15}$$

Choosing $M_N^{(1)}$ such that $\frac{\left(M_N^{(1)}\right)^{2k\alpha}}{n^{2k\alpha-1}}$ converges to 0. For example $M_N^{(1)} = n^b$ with $0 < b < 1 - \frac{1}{2k\alpha}$. Thus, $\lim_{N\to+\infty} E[f_N(\lambda)] - [\phi(\lambda)]^{\frac{p}{\alpha}} = 0.$

Theorem 4.2 *Let* $-\Omega < \lambda < \Omega$ *such that* $\phi(\lambda) > 0$. *Then,* $var\left[f_N(\lambda))\right]$ *converges to zero. If* $M_N^{(1)} = n^c$ *with* $\frac{1}{2k^2\alpha^2} < c < \frac{1}{2}$, *then* $var\left[f_N(\lambda))\right] = \mathrm{O}\left(\frac{1}{n^{(1-2c)}}\right)$.

Proof

Fisrt suppose that $\tau\lambda \notin \{w_1, w_2, \cdots, w_q\}$. It is clear that the variance of f_N can be written as follows:

$$var[f_N(\lambda)] = \int_{R^2} W_N^{(1)}(\lambda - u) W_N^{(1)}(\lambda_1 - u') cov\left[\hat{I}_N(u), \hat{I}_N(u')\right] du du'.$$

Since W is zero for $|\lambda| > 1$, for large N, we have

$$var[f_N(\lambda)] = \int_{-1}^{1} cov\left[\hat{I}_N\left(\lambda - \frac{x_1}{M_N}\right), \hat{I}_N\left(\lambda - \frac{x_1'}{M_N}\right)\right] W(x_1) W(x_1') dx_1 dx_1'.$$

Define two subsets of the $[-1,1]^2$ by:

- $L_1 = \left\{(x_1, x_1') \in [-1,1]^2; \quad |x_1 - x_1'| > \sigma_N\right\}$,
- $L_2 = \left\{(x_1, x_1') \in [-1,1]^2; \quad |x_1 - x_1'| \le \sigma_N\right\}$,

where σ_N is a nonnegative real, converging to 0. We split the integral into an integral over the subregion L_2 and an integral over L_1: $\text{var}[f_N(\lambda)] = \int_{L_2} + \int_{L_1} \overset{\Delta}{=} J_1 + J_2$.

Cauchy-Schwartz inequality and theorem 3.1, give

$J_1 \le C\int_{|x_1 - x_1^{'}| \le \sigma_N} W(x_1)W(x_1^{'})dx_1 dx_1^{'}$. where C is constant. Thus, we obtain

$$J_1 = O(\sigma_N) \tag{16}$$

It remains to show that J_2 converges to zero. For simplicity, we define

$$\lambda_1 = \lambda - \frac{x_1}{M_N^{(1)}}; \quad \lambda_2 = \lambda - \frac{x_1^{'}}{M_N^{(1)}}, \text{ and } C(\lambda) = cov\left[\hat{I}_N\left(\lambda - \frac{x_1}{M_N^{(1)}}\right), \hat{I}_N\left(\lambda - \frac{x_1^{'}}{M_N^{(1)}}\right)\right].$$

We first show that $C(\lambda)$ converges to zero uniformly in $x_1, x_1^{'} \in [-1,1]$. Indeed, from lemma 2.3, we have

$$E\hat{I}_N(v) - \hat{I}_N(v) = F_{p,\alpha}^{-1}[C_\alpha]^{-p/\alpha}\int_{-\infty}^{\infty}\frac{\text{Re}\left(e^{iuI_N(v)}\right) - e^{-C_\alpha|u|^\alpha \psi_N(v)}}{|u|^{1+p}}du.$$

Thus, the expression of the covariance becomes

$$\begin{aligned} C(\lambda) \quad = \quad & F_{p,\alpha}^{-2}C_\alpha^{-\frac{2p}{\alpha}}\int_{R^2}E\left[\prod_{k=1}^{2}\cos\left(u_k I_N(\lambda_k)\right)\right] \\ & -\exp\left\{-C_\alpha\sum_{k=1}^{2}|u_k|^\alpha \psi_N(\lambda_k)\right\}\frac{du_1 du_2}{|u_1 u_2|^{1+p}} \end{aligned}$$

Using the following equality: $2\cos x \cos y = \cos(x+y) + \cos(x-y)$, we have

$$\begin{aligned} E\left[\prod_{k=1}^{2}\left(\cos u_k I_N(\lambda_k)\right)\right] = & \frac{1}{2}\exp\left[-C_\alpha\int\left|(\tau)^{\frac{1}{\alpha}}\sum_{k=1}^{2}u_k H_N(\tau\lambda_k - \tau v)\right|^\alpha d\mu(v)\right] \\ & + \frac{1}{2}\exp\left[-C_\alpha\int\left|(\tau)^{\frac{1}{\alpha}}\sum_{k=1}^{2}(-1)^{k+1}u_k H_N(\tau\lambda_k - \tau v)\right|^\alpha d\mu(v)\right]. \end{aligned}$$

By substituting the expression for $C(\lambda)$ and changing the variable u_2 to $(-u_2)$ in the second term, we obtain

$$C(\lambda) = F_{p,\alpha}^{-2}C_\alpha^{-\frac{2p}{\alpha}}\int_{R^2}\left(e^{-K} - e^{-K^{'}}\right)\frac{du_1 du_2}{|u_1 u_2|^{1+p}}, \tag{17}$$

where $K = C_\alpha \int_R \left| (\tau)^{\frac{1}{\alpha}} \sum_{k=1}^{2} u_k H_N(\tau\lambda_k - \tau v) \right|^\alpha d\xi(v)$ and

$K' = C_\alpha \tau \sum_{k=1}^{2} |u_k|^\alpha \int_R \left| H_N(\tau\lambda_k - \tau v) \right|^\alpha \xi(v) dv$

Since $K, K' > 0$, $\left| e^{-K} - e^{-K'} \right| \le \left| K - K' \right| \exp\{\left| K - K' \right| - K'\}$, we obtain:

$\left| K - K' \right| \le 2C_\alpha \tau |u|^{\frac{\alpha}{2}} Q_N(\lambda_1; \lambda_2),$ where

$Q_N(\lambda_1; \lambda_2) = \int_{-\Omega}^{\Omega} |H_N(\tau\lambda_1 - \tau u)|^{\frac{\alpha}{2}} |H_N(\tau\lambda_2 - \tau u)|^{\frac{\alpha}{2}} d\xi(u)$

Now, let us show that $Q_N(\lambda_1; \lambda_2)$ converges to zero. Indeed, since ϕ is bounded on $[-\Omega, \Omega]$, we have

$$\begin{aligned} Q_N(\lambda_1; \lambda_2) \le \sup(\phi) \int_{-\Omega}^{\Omega} |H_N(\tau\lambda_1 - \tau u) H_N(\tau\lambda_2 - \tau u)|^{\frac{\alpha}{2}} du \\ + \sum_{i=1}^{q} c_i \left| H_N(\tau\lambda_1 - \tau w_i) H_N(\tau\lambda_2 - \tau w_i) \right|^{\frac{\alpha}{2}} \end{aligned} \tag{18}$$

From the definition of H_N , we can write

$$\int_{-\Omega}^{\Omega} |H_N(\tau\lambda_1 - \tau v) H_N(\tau\lambda_2 - \tau v)|^{\frac{\alpha}{2}} dv = \int_{-\Omega}^{\Omega} \frac{1}{B'_{\alpha,N}} \left| \frac{\sin\left[\frac{n}{2}(\tau\lambda_1 - \tau v)\right]}{\sin\left[\frac{1}{2}(\tau\lambda_1 - \tau v)\right]} \right|^{k\alpha} \left| \frac{\sin\left[\frac{n}{2}(\tau\lambda_2 - \tau v)\right]}{\sin\left[\frac{1}{2}(\tau\lambda_2 - \tau v)\right]} \right|^{k\alpha} dv.$$

a) *First step:*

We show that the denominators of the first and second terms under the last integral do not vanish for the same v, so we suppose that v exists, belonging to $[-\Omega, \Omega]$ and $z, z' \in Z$ such as: $\tau\lambda_1 - \tau v = 2z\pi$ *and* $\tau\lambda_2 - \tau v = 2z'\pi$. Since $\lambda_1 \neq \lambda_2$, then z and z' are different.

Therefore, $z - z' = \frac{\tau}{2\pi}(\lambda_1 - \lambda_2)$. Hence, $|z - z'| = \frac{1}{w}|\lambda_1 - \lambda_2|$. As $\lim_{N\to\infty} |\lambda_1 - \lambda_2| = 0$, consequently, for a large N we get: $|z - z'| < \frac{1}{2}$.

Thus, we obtain a contradiction with the fact that z and z' are different integers.

b) *Second step:*

Asuume there exist q points, $V_1, V_2, \cdots, V_q \in [-\Omega, \Omega]$ such as:

for $j = 1, 2, \cdots, q$ $\tau\lambda_1 - \tau V_j \in 2\pi Z$ then $\frac{\lambda_1}{w} - \frac{V_j}{w} \in Z$, and we assume that there exist q' points $V_1', V_2', \cdots, V_{q'}' \in [-\Omega, \Omega]$ such that for $i = 1, 2, \cdots, q'$ $\frac{\lambda_2}{w} - \frac{V_i'}{w} \in Z$. Showing that, $|V_j| \not\asymp \Omega$, $|V_i'| \not\asymp \Omega$ for $1 \le j \le q$ and $1 \le i \le q'$. Indeed, $-1 < \frac{\lambda - \Omega}{w} < 0$ and $0 < \frac{\lambda + \Omega}{w} < 1$ because $w > 2\Omega$. Hence $\frac{\lambda - \Omega}{w} \notin Z$ and $\frac{\lambda + \Omega}{w} \notin Z$. For a large N, we get that $\left[\frac{\lambda - \Omega}{w}\right]_E < \frac{\lambda_1}{w} - \frac{\Omega}{w} < 1 + \left[\frac{\lambda - \Omega}{w}\right]_E$, where $[x]_E$ is the integer part of x. Hence, $\frac{\lambda_1}{w} - \frac{\Omega}{w} \notin Z$. In the same manner, we show that $\frac{\lambda_1}{w} + \frac{\Omega}{w} \notin Z$. Similarly, it can be shown that $\frac{\lambda_2 + \Omega}{w} \notin Z$. Thus, $|V_j| \not\asymp \Omega$ and $|V_i'| \not\asymp \Omega$.

c) *Third step:*

We classify V_j and V_i' by increasing order: $-\Omega < V_{j_1} < V_{j_2} < \cdots < V_{j_{q+q'}} < \Omega$, and we write the integral in the following manner:

$$\int_{-\Omega}^{\Omega} \left| \frac{\sin\left[\frac{n}{2}(\tau\lambda_1 - \tau v)\right]}{\sin\left[\frac{1}{2}(\tau\lambda_1 - \tau v)\right]} \right|^{k\alpha} \left| \frac{\sin\left[\frac{n}{2}(\tau\lambda_2 - \tau v)\right]}{\sin\left[\frac{1}{2}(\tau\lambda_2 - \tau_1 v)\right]} \right|^{k\alpha} dv = I_1 + \sum_{i=1}^{q+q'} I_{2,i} + \sum_{i=1}^{q+q'-1} I_{3,i} + I_4$$

where

$$I_1 = \int_{-\Omega_1}^{V_{j_1}-\delta(N)} \left| \frac{\sin\left[\frac{n}{2}(\tau\lambda_1 - \tau v)\right]}{\sin\left[\frac{1}{2}(\tau\lambda_1 - \tau v)\right]} \right|^{k\alpha} \left| \frac{\sin\left[\frac{n}{2}(\tau\lambda_2 - \tau v)\right]}{\sin\left[\frac{1}{2}(\tau\lambda_2 - \tau v)\right]} \right|^{k\alpha} dv$$

$$I_{2,i} = \int_{V_{j_i}-\delta(N)}^{V_{j_i}+\delta(N)} \left| \frac{\sin\left[\frac{n}{2}(\tau\lambda_1 - \tau v)\right]}{\sin\left[\frac{1}{2}(\tau\lambda_1 - \tau v)\right]} \right|^{k\alpha} \left| \frac{\sin\left[\frac{n}{2}(\tau\lambda_2 - \tau v)\right]}{\sin\left[\frac{1}{2}(\tau\lambda_2 - \tau v)\right]} \right|^{k\alpha} dv$$

$$I_{3,i} = \int_{V_{j_i}+\delta(N)}^{V_{j_{i+1}}-\delta(N)} \left| \frac{\sin\left[\frac{n}{2}(\tau\lambda_1 - \tau v)\right]}{\sin\left[\frac{1}{2}(\tau\lambda_1 - \tau v)\right]} \right|^{k\alpha} \left| \frac{\sin\left[\frac{n}{2}(\tau\lambda_2 - \tau v)\right]}{\sin\left[\frac{1}{2}(\tau\lambda_2 - \tau v)\right]} \right|^{k\alpha} dv$$

$$I_4 = \int_{V_{j_{q+q'}}+\delta(N)}^{\Omega} \left| \frac{\sin\left[\frac{n}{2}(\tau\lambda_1 - \tau v)\right]}{\sin\left[\frac{1}{2}(\tau\lambda_1 - \tau v)\right]} \right|^{k\alpha} \left| \frac{\sin\left[\frac{n}{2}(\tau\lambda_2 - \tau v)\right]}{\sin\left[\frac{1}{2}(\tau_1\lambda_2 - \tau v)\right]} \right|^{k\alpha} dv$$

where $\delta(N)$ is a nonnegative real number converging to zero and satisfying:

$-\Omega < V_{j_1} - \delta(N) < V_{j_1} + \delta(N) < V_{j_2} - \delta(N) < V_{j_2} + \delta(N) < \cdots\cdots < V_{j_{q+q'}} - \delta(N) < V_{j_{q+q'}} + \delta(N) < \Omega$, and $\delta(N) < \left|\frac{\lambda_1 - \lambda_2}{2}\right|$.

Showing that the first integral converges to zero. For a large N, it easy to see that $\lambda_1 < \Omega$. Without loss of generality, we assume that $\tau\lambda_1 - \tau V_{j_i} = 2k_i\pi$ for all i, with $k_i \in Z$. The fact that there is no v between $-\Omega$ and $V_{j_1} - \delta(N)$ on which the denominators are vanishing, gives

$$I_1 \le \frac{V_{j_1} - \delta(N) + \Omega}{\inf\left[\left|\sin\frac{\tau\delta(N)}{2}\right|^{k\alpha}, \left|\sin\frac{\tau(\lambda_1+\Omega)}{2}\right|^{k\alpha}\right]} \frac{1}{\inf\left[\left|\sin\frac{\tau(\lambda_2 - V_{j_1} + \delta(N))}{2}\right|^{k\alpha}, \left|\sin\frac{\tau(\lambda_2+\Omega)}{2}\right|^{k\alpha}\right]}.$$

Substituting V_{j_1} in the last inequality, we get

$\left|\sin\frac{\tau(\lambda_2 - V_{j_1} + \delta(N))}{2}\right|^{k\alpha} = \left|\sin\frac{\tau\,|\,\lambda_2 - \lambda_1 + \delta(N)\,|}{2}\right|^{k\alpha}$. For a large N ,we obtain

$$\frac{\tau\,|\,\lambda_2-\lambda_1+\delta(N)\,|}{2}\le \tau\Omega+\frac{\tau\delta(N)}{2}<\pi-\frac{\tau\delta(N)}{2}.$$

On the other hand, two cases are possible:

1. $\lambda_2-\lambda_1>0$, then we have $|\,\lambda_2-\lambda_1+\delta(N)\,|=\lambda_2-\lambda_1+\delta(N)>\delta(N)$
2. $\lambda_2-\lambda_1<0$, since $|\,\lambda_2-\lambda_1\,|>2\delta(N)$, we have

$|\,\lambda_2-\lambda_1+\delta(N)\,|=\lambda_1-\lambda_2-\delta(N)>\delta(N)$.

Therefore, $\frac{\tau\delta(N)}{2}<\frac{\tau\,|\,\lambda_2-\lambda_1+\delta(N)\,|}{2}<\pi-\frac{\tau\delta(N)}{2}$. For a large N, we have $\tau\delta(N)<2\pi-\tau(\lambda_1+\Omega)$ and $\tau\delta(N)<2\pi-\tau(\lambda_2+\Omega)$. Then,

$$\frac{\tau\delta(N)}{2}<\frac{\tau(\lambda_1+\Omega)}{2}<\pi-\frac{\tau\delta(N)}{2}\qquad\text{and}\qquad\frac{\tau\delta(N)}{2}<\frac{\tau(\lambda_2+\Omega)}{2}<\pi-\frac{\tau\delta(N)}{2}.$$

Therefore,

$$I_1\le\frac{V_{j_1}-\delta(N)+\Omega}{\left|\sin\frac{\tau\delta(N)}{2}\right|^{2k\alpha}}.$$

For the integral $I_{2,i}$, we bound the first fraction under integral by $n^{k\alpha}$:

$I_{2,i}\le n^{k\alpha}\int_{V_{j_i}-\delta(N)}^{V_{j_i}+\delta(N)}\frac{1}{\left|\sin\left[\frac{1}{2}\left(\tau\lambda_2-\tau v\right)\right]\right|^{k\alpha}}dv$. By substituting for V_{j_i} in the last inequality and putting $v=u-\frac{2k\pi}{\tau}$, we get

$I_{2,i}\le n^{k\alpha}\int_{\lambda_1-\delta(N)}^{\lambda_1+\delta(N)}\frac{1}{\left|\sin\left[\frac{1}{2}\left(\tau\lambda_2-\tau u\right)\right]\right|^{k\alpha}}du$. Since $|\,\lambda_1-u\,|<\delta(N)$, it is easy to see that :

$$|\,\lambda_2-u\,|\ge|\,\lambda_2-\lambda_1\,|-|\,\lambda_1-u\,|\ge|\,\lambda_2-\lambda_1\,|-\delta(N)>\frac{|\,\lambda_2-\lambda_1\,|}{2}$$

Since $\delta(N)$ converges to zero, for a large N we have $\delta(N)<\frac{2}{\tau}\left(\pi-\frac{\tau}{2}|\,\lambda_2-\lambda_1\,|\right)$, therefore $0<\tau\frac{|\,\lambda_2-\lambda_1\,|}{4}<\tau\frac{|\,\lambda_2-u\,|}{2}<\tau\frac{|\,\lambda_2-\lambda_1\,|+\delta(N)}{2}<\pi$. Consequently:

$$I_{2,i} \leq \frac{2\delta(N)n^{k\alpha}}{\inf\left[\sin\frac{|\lambda_1-\lambda_2|\tau}{4};\sin\frac{|\lambda_1-\lambda_2|\tau+\delta(N)\tau}{2}\right]},$$

where $|\lambda_2-\lambda_1|=\left|\frac{x_2-x_1}{M_N}\right|$. Then, for a large N, we have $\frac{3\tau}{2}|\lambda_2-\lambda_1|<2\pi$. Therefore, $\tau\frac{|\lambda_2-\lambda_1|}{4}<\frac{\tau|\lambda_2-\lambda_1|+\tau\delta(N)}{2}<\pi-\tau\frac{|\lambda_2-\lambda_1|}{4}$. Thus, we bound the integral as follows:

$$I_{2,i} \leq \frac{2\delta(N)n^{k\alpha}}{\left|\sin\frac{\tau|\lambda_2-\lambda_1|)}{4}\right|^{k\alpha}}$$

Since there is no v between $V_{j_i}+\delta(N)$ and $V_{j_{i+1}}-\delta(N)$ on which the denominators are vanishing, we get:

$$I_{3,i} \leq \frac{V_{j_{i+1}}-V_{j_i}-2\delta(N)}{A\times B},$$

where

$$A = \inf\left(\left|\sin\frac{\tau\lambda_1-\tau V_{j_i}-\tau\delta(N)}{2}\right|^{k\alpha},\left|\sin\frac{\tau\lambda_1-\tau V_{j_{i+1}}+\tau\delta(N)}{2}\right|^{k\alpha}\right)$$

$$B = \inf\left(\left|\sin\frac{\tau\lambda_2-\tau V_{j_i}-\tau\delta(N)}{2}\right|^{k\alpha},\left|\sin\frac{\tau\lambda_2-\tau V_{j_{i+1}}+\tau\delta(N)}{2}\right|^{k\alpha}\right)$$

It follows from the hypothesis on $\delta(N)$ that

$$\frac{\tau\delta(N)}{2}<\frac{\tau|\lambda_1-\lambda_2|}{2}-\frac{\tau|\delta(N)|}{2}<\frac{\tau|\lambda_1-\lambda_2-\delta(N)|}{2}<\frac{\tau|\lambda_1-\lambda_2|}{2}+\frac{\tau|\delta(N)|}{2}<\pi-\frac{\tau\delta(N)}{2}.$$

The definition of V_{j_i} gives

$\dfrac{1}{\left|\sin\dfrac{\tau\lambda_2-\tau V_{j_i}-\tau\delta(N)}{2}\right|^{k\alpha}}<\dfrac{1}{\left|\sin\dfrac{\tau\delta(N)}{2}\right|^{k\alpha}}$. Similarly we bound the other terms:

$$I_{3,i}\le\frac{V_{j_{i+1}}-V_{j_i}-2\delta(N)}{\left|\sin\dfrac{\tau\delta(N)}{2}\right|^{2k\alpha}}.$$

We can show by same majoration that :

$I_4\le\dfrac{V_{j_{i+1}}-V_{j_i}-2\delta(N)}{E\times F}$, where

$$E = \inf\left(\left|\sin\frac{\tau\lambda_1-\tau\Omega}{2}\right|^{k\alpha},\left|\sin\frac{\tau\lambda_1-\tau V_{j_{q+q'}}-\tau\delta(N)}{2}\right|^{k\alpha}\right)$$

$$F = \inf\left(\left|\sin\frac{\tau\lambda_2-\tau\Omega}{2}\right|^{k\alpha},\left|\sin\frac{\tau\lambda_2-\tau V_{j_{q+q'}}-\tau\delta(N)}{2}\right|^{k\alpha}\right)$$

Since $\delta(N)$ converges to zero, for a large N, we have $\delta(N)<\dfrac{2}{\tau}\left(\pi-\dfrac{\tau}{2}|\lambda_1-\Omega|\right)$, and $\delta(N)<\dfrac{2}{\tau}\left(\pi-\dfrac{\tau}{2}|\lambda_1-\lambda_2|\right)$. It follows that: $I_4\le\dfrac{\Omega-V_{j_{q+q'}}-\delta(N)}{\left|\sin\dfrac{\tau\delta(N)}{2}\right|^{2k\alpha}}$. We recapitulate, from the previous increases, we obtain

$$\int_{-\Omega}^{\Omega}\left|\frac{\sin\left[\frac{n}{2}(\tau\lambda_1-\tau v)\right]}{\sin\left[\frac{1}{2}(\tau\lambda_1-\tau v)\right]}\right|^{k\alpha}\left|\frac{\sin\left[\frac{n}{2}(\tau\lambda_2-\tau v)\right]}{\sin\left[\frac{1}{2}(\tau\lambda_2-\tau v)\right]}\right|^{k\alpha}dv\le\frac{\Omega+V_{j_1}-\delta(N)}{\left|\sin\frac{\tau\delta(N)}{2}\right|^{2k\alpha}}+\sum_{i=1}^{q+q'}\frac{2n^{k\alpha}\delta(N)}{\left|\sin\frac{\tau|\lambda_2-\lambda_1|}{4}\right|^{k\alpha}}$$
$$+\sum_{i=1}^{q+q'-1}\frac{V_{j_{i+1}}-V_{j_i}-\delta(N)}{\left|\sin\frac{\tau\delta(N)}{2}\right|^{2k\alpha}}+\frac{\Omega-V_{j_{q+q'}}-\delta(N)}{\left|\sin\frac{\tau\delta(N)}{2}\right|^{2k\alpha}}$$

After simplification, we have

$$\int_{-\Omega}^{\Omega}\left|\frac{\sin\left[\frac{n}{2}\left(\tau\lambda_1-\tau v\right)\right]}{\sin\left[\frac{1}{2}\left(\tau\lambda_1-\tau v\right)\right]}\right|^{k\alpha}\left|\frac{\sin\left[\frac{n}{2}\left(\tau\lambda_2-\tau v\right)\right]}{\sin\left[\frac{1}{2}\left(\tau\lambda_2-\tau v\right)\right]}\right|^{k\alpha}dv\le\frac{2\Omega-2(q+q'+1)\delta(N)}{\left|\sin\frac{\tau\delta(N)}{2}\right|^{2k\alpha}}+\frac{2n^{k\alpha}\delta(N)(q+q')}{\left|\sin\frac{\tau\,|\,\lambda_2-\lambda_1\,|}{4}\right|^{k\alpha}}.$$

Using the following inequality $\left|\sin\frac{x}{2}\right|\ge\frac{|\,x\,|}{\pi}$, we get

$$\int_{-\Omega}^{\Omega}\left|\frac{\sin\left[\frac{n}{2}\left(\tau\lambda_1-\tau v\right)\right]}{\sin\left[\frac{1}{2}\left(\tau\lambda_1-\tau v\right)\right]}\right|^{k\alpha}\left|\frac{\sin\left[\frac{n}{2}\left(\tau\lambda_2-\tau v\right)\right]}{\sin\left[\frac{1}{2}\left(\tau\lambda_2-\tau v\right)\right]}\right|^{k\alpha}dv\le\frac{2\Omega\pi^{2k\alpha}}{\left(\tau\delta(N)\right)^{2k\alpha}}+\frac{2n^{k\alpha}\delta(N)(q+q')(2\pi)^{k\alpha}}{\left(\frac{\tau\,|\,x_2-x_1\,|}{M_N}\right)^{k\alpha}}.$$

The lemma 2.1 gives

$$\frac{1}{B'_{\alpha,N}}\int_{-\Omega}^{\Omega}\left|\frac{\sin\left[\frac{n}{2}\left(\tau\lambda_1-\tau v\right)\right]}{\sin\left[\frac{1}{2}\left(\tau\lambda_1-\tau v\right)\right]}\right|^{k\alpha}\left|\frac{\sin\left[\frac{n}{2}\left(\tau\lambda_2-\tau v\right)\right]}{\sin\left[\frac{1}{2}\left(\tau\lambda_2-\tau v\right)\right]}\right|^{k\alpha}dv\le$$

$$\frac{1}{\pi}\left(\frac{\pi}{2}\right)^{2k\alpha}\left(\frac{2\Omega\pi^{2k\alpha}}{n^{2k\alpha-1}\left(\tau\delta(N)\right)^{2k\alpha}}+\frac{2\delta(N)(q+q')(2\pi)^{k\alpha}}{n^{k\alpha-1}\left(\frac{\tau\sigma_N}{M_N}\right)^{k\alpha}}\right).\tag{19}$$

In order to obtain the convergence of the last expression to zero, we choose $\delta(N)=n^{-\beta},\quad \beta>0$, such as

$$\lim_{n\to\infty}\frac{n^{2k\alpha\beta}}{n^{2k\alpha-1}}=0\qquad and\qquad \lim_{n\to\infty}\frac{1}{n^{\beta+k\alpha-1}\left(\frac{\sigma_N}{M_N}\right)^{k\alpha}}=0.\tag{20}$$

Thus, from (18) Q_N converges to zero. On the other hand,

$$C(\lambda)\le F_{p,\alpha}^{-2}C_\alpha^{-\frac{2p}{\alpha}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\,K-K'\,|\,e^{|K-K'|-K'}\frac{du_1du_2}{|\,u_1u_2\,|^{1+p}},$$

where $|\,K-K'\,|-K'\le -C_\alpha\sum_{k'=1}^{2}|\,u_{k'}\,|^{\alpha}\left|\psi_N\left(\lambda_{k'}\right)-\tau Q_N(\lambda_1;\lambda_2)\right|$.

We denote by: $\delta_{(N,k')}=\psi_N\left(\lambda_{k'}\right)-\tau Q_N(\lambda_1;\lambda_2)$. It follows from (18) and (20) that $\delta_{(N,k')}$ converges to $\phi(\lambda)$. Hence,

$$C(\lambda) \le F_{p,a}^{-2} C_\alpha^{-\frac{2p}{\alpha}} 2\tau C_\alpha Q_N(\lambda_1;\lambda_2) 4 \prod_{k'=1}^{2} \int_0^\infty \exp\left[-C_\alpha (u_{k'})^\alpha \mid \delta_{(N,k')} \mid\right] \frac{du_{k'}}{(u_{k'})^{1+p-\frac{\alpha}{2}}}.$$

Putting $u_{k'}\left(\delta_{N,k'}\right)^{\frac{1}{\alpha}} = v,$ we obtain

$$C(\lambda) \le F_{p,\alpha}^{-2} C_\alpha^{-\frac{2p}{\alpha}} 2\tau C_\alpha \frac{Q_N(\lambda_1;\lambda_2)}{\left[\delta_{(N,1)}\delta_{(N,2)}\right]^{\frac{1}{2}-\frac{p}{\alpha}}} \left(\int_{-\infty}^{\infty} \frac{e^{-C_\alpha |v|^\alpha}}{|v|^{1+p-\frac{\alpha}{2}}} dv \right)^2. \tag{20}$$

Since $\phi(\lambda) > 0$, $C(\lambda)$ converges uniformly in $x_1, x_1' \in [-1,1]$ to zero. From (20), we obtain $J_2 = O\left(\frac{1}{n^{2k\alpha(1-\beta)-1}} + \frac{1}{n^{\beta+k\alpha-1}\left(\frac{\sigma_N}{M_N}\right)^{k\alpha}} \right)$. Thus, $var[f_N(\lambda)]$ converges to zero and then, $f_N(\lambda)$ is an asymptotically unbiased and consistent estimator.

4. CONCLUSIONS

This paper gives an estimate of the spectral density of a mixed continuous-time stable process from observations at discrete instants. To avoid the phenomenon of aliasing we have assumed that the spectral density is a compact support. The applications of these processes are found in various fields. For example:

- The study of soil cracking where the observed signal is the resistance of the soil. This signal encounters random jumps due to the encounter of certain stones in the ground. The spectral measurement will therefore be composed of two parts, one continuous and the other discrete. This last corresponds to the resistance jumps encountered during the measurement.

- The growth of fruits on a tree can be seen as a continuous distribution, and when there is a fall of a fruit, the other fruits remaining on the tree absorb more energy and their growth will have a jump in value.

As perspective of this work is to find the optimum smoothing parameters to have a better rate of convergence. For this purpose, the cross-validation method will be the most appropriate tool.

We intend to complete this work by studying the case where the process is observed with random errors. For this, we will use the deconvolution methods known for their efficiency in the presence of random errors.

We think to give an estimator of the mode of the spectral density representing the frequency where the spectral density reaches the maximum of energy.

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