Recursive Algorithms for Identification of Dual Youla-Kucera Parametrized Plant Models in Closed Loop Operation

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Abstract

The growing interest in using dual Youla Kucera plant parametrization for modeling plant uncertainties raises the need for recursive identification algorithms dedicated to the identification of these structures in closed loop operation in view of developing appropriate iterative tuning and adaptive control strategies. The paper presents recursive algorithms for identification in closed loop operation of dual Youla-Kucera parametrized plant models. These algorithms assure global asymptotic stability in the deterministic environment and allow to obtain unbiased parameter estimation in the presence of measurement noise when the plant model is in the model set. The paper also re-visit the Hansen scheme which allows to associate open loop type recursive identification algorithms for the identification of these structures in closed loop operation. When the plant model is not in the model set, comparison of the various algorithms is done in terms of the bias distribution . Further comparisons and performance evaluation is provided by simulations on some relevant examples.

Keywords: System identification, Youla-Kucera parametrization, Recursive algorithms, Identification in closed loop

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1. Introduction

There is a growing interest for modeling plant uncertainties using the dual Youla- Kucera plant parametrization [1], [2] when operating in closed loop. Dual Youla-Kucera parametrization appears to be an efficient way for evaluating robustness of adaptive regulation schemes ([3]). Certainly this representation can be used as well for evaluating the robustness of adaptive feedforward disturbance compensation schemes in the presence of plant uncertainties. Application in the field of active noise control have been reported ([3]).

Moving towards handling large plant model uncertainties in adaptive regulation and adaptive feedforward disturbance compensation[1] will require identification in closed loop of dual Youla Kucera parametrized plant models in order to implement iterative tuning and adaptive control strategies. As far as the authors' knowledge goes, the standard way of recursively identifying a dual parametrized Youla-Kucera plant model in closed loop operation is to use the so called "Hansen scheme" [1],[4] which transforms this operation in an open loop type identification scheme. This approach however does not take advantage of using dedicated methods for recursive identification in closed loop operation where the aim is to find a plant model estimate which allows to obtain the best closed loop predictor [5],[6]. The objective is also to get a better approximation in the frequency region close to the Nyquist point when the plant model is not in the model set. There are number of off-line identification procedure for identification of dual Youla Kucera parametrized models in closed loop([7]).

The objective of this paper is to introduce novel recursive identification algorithms dedicated to the identification in closed loop operation of dual Youla Kucera parametrized models which try to minimize the closed loop output prediction error. The algorithms will be developed under the assumptions that the controller is constant and known and that the plant model is in the model set.

[1] These schemes are adaptive with respect to variations of dynamic characteristics of the disturbances but the plant model is assumed to be known and almost invariant

The algorithms will assure:

- global asymptotic stability for any initial parameter estimate and closed loop error prediction,
- asymptotically optimal prediction for different types of measurement noise,
- unbiased parameter estimation in the presence of measurement noise under appropriate richness conditions on the excitation signal

Then the behavior of the algorithms in the case when the plant model is not in the model set will be analyzed by evaluating the bias in the frequency domain. The "Hansen scheme" will be re-visited with the objective of enhancing the qualities of the estimated models from the perspective of the bias distribution in the frequency domain when the plant model is not in the model set. A comparison of the various algorithms in terms of bias distribution will be provided. Simulations on relevant examples will further illustrate the performance of the various algorithms.

The paper is organized as follows: Basic equation will be presented in Section 2. The algorithms will be developed and analyzed in Sections 3 through 6 under the hypothesis that the plant model is in the model set. The bias distribution in the frequency domain when the plant is not in the model set will be discussed in Section 7. Section 8 will present simulation results .

2. Basic Equations

The closed loop output error configuration for the identification of a dual Youla-Kucera parametrized plant models is shown in Figure 1

The initial available plant model (nominal model) is described by the transfer operator:

$$G_o(q^{-1}) = \frac{q^{-d_o} B'_o(q^{-1})}{A_o(q^{-1})} = \frac{B_o(q^{-1})}{A_o(q^{-1})} \tag{1}$$

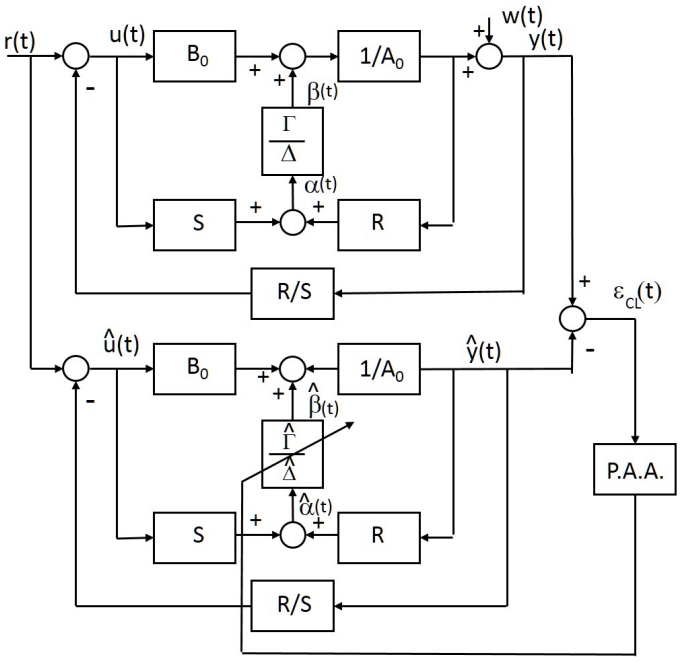

Figure 1: Closed loop ouput error identification of dual Youla Kucera parametrized model

with:

d_0 = the plant pure time delay in

number of sampling periods

$$A_o = 1 + a_1 q^{-1} + \cdots + a_{n_A} q^{-n_A} = 1 + q^{-1} A_0^* \ ;$$

$$B_o = b_1 q^{-1} + \cdots + b_{n_B} q^{-n_B} = q^{-1} B_0^* \ ;$$

$$B_o^* = b_1 + \cdots + b_{n_B} q^{-n_B+1} \ ;$$

where $A_0(q^{-1})$, $B_0(q^{-1})$, $B_0^*(q^{-1})$ are polynomials in the delay operator q^{-1} and n_{A_0}, n_{B_0} and $n_B - 1$ represent their orders[2]. The feedback fixed polynomial controller K is given by:

$$K(q^{-1}) = \frac{R(q^{-1})}{S(q^{-1})} \tag{2}$$

with:

$$S = 1 + s_1 q^{-1} + \cdots + s_{n_S} q^{-n_S} = 1 + q^{-1} S^* \ ;$$

$$R = r_0 + \cdots + r_{n_R} q^{-n_R} \ ;$$

It will be assumed that S is an asymptotically stable polynomial

The unknown part of the plant model is described by the input output block Π:

$$\Pi(q^{-1}) = \frac{\Delta(q^{-1})}{\Gamma(q^{-1})} \tag{3}$$

where:

$$\Gamma = 1 + \gamma_1 q^{-1} + \cdots + \gamma_{n_\Gamma} q^{-n_\Gamma} = 1 + q^{-1} \Gamma^* \ ;$$

$$\Delta = \delta_1 q^{-1} + \cdots + \delta_{n_\Delta} q^{-n_\Delta} = q^{-1} \Delta^* \ ; \tag{4}$$

The unknown plant model is described by:

$$G(q^{-1}) = \frac{B(q^{-1})}{A(q^{-1})} \tag{5}$$

[2]The complex variable z^{-1} will be used for characterizing the system's behavior in the frequency domain and the delay operator q^{-1} will be used for describing the system's behavior in the time domain.

and can be expressed as[3]

$$G(q^{-1}) = \frac{\Gamma B_0 + \Delta S}{\Gamma A_0 - \Delta R} \tag{6}$$

The closed loop output predictor can be described similarly except that one uses an estimation of Π:

$$\hat{\Pi}(q^{-1}) = \frac{\hat{\Delta}(q^{-1})}{\hat{\Gamma}(q^{-1})} \tag{7}$$

where:

$$\hat{\Gamma} = 1 + \hat{\gamma}_1 q^{-1} + \cdots + \hat{\gamma}_{n_\Gamma} q^{-n_\Gamma} = 1 + q^{-1}\hat{\Gamma}^* \ ;$$
$$\hat{\Delta} = \hat{\delta}_1 q^{-1} + \cdots + \hat{\delta}_{n_\Delta} q^{-n_\Delta} = q^{-1}\hat{\Delta}^* \ ; \tag{8}$$

The input and output of Π will be denoted $\alpha(t)$ and $\beta(t)$ respectively and the input and output of $\hat{\Pi}$ will be denoted $\hat{\alpha}(t)$ and $\hat{\beta}(t)$ respectively.

In the time domain the following input/output relationships will be used:

$$y(t) = \frac{\Gamma B_0 + \Delta S}{\Gamma A_0 - \Delta R} u(t) \tag{9}$$

$$u(t) = r(t) - \frac{R}{S} y(t) \tag{10}$$

where $r(t)$ is the external excitation.

$$\hat{y}(t) = \frac{\hat{\Gamma} B_0 + \hat{\Delta} S}{\hat{\Gamma} A_0 - \hat{\Delta} R} \hat{u}(t) \tag{11}$$

$$\hat{u}(t) = r(t) - \frac{R}{S} \hat{y}(t) \tag{12}$$

Note that:

$$\alpha(t) = Su(t) + Ry(t) \tag{13}$$

$$\beta(t) = A_0 y(t) - B_0 u(t) = \frac{\Delta}{\Gamma} \alpha(t) \tag{14}$$

[3] In order to simplify the writing the argument q^{-1} has been dropped out in many equations

$$\hat{\alpha}(t) = S\hat{u}(t) + R\hat{y}(t) \tag{15}$$

$$\hat{\beta}(t) = A_0\hat{y}(t) - B_0\hat{u}(t) = \frac{\hat{\Delta}}{\hat{\Gamma}}\hat{\alpha(t)} \tag{16}$$

The closed loop output error is defined as:

$$\varepsilon_{CL} = y(t) - \hat{y}(t) \tag{17}$$

and the filtered closed loop output error is defined as:

$$\nu_{CL} = \beta(t) - \hat{\beta}(t) \tag{18}$$

Using Eqs. (10), (12),(14) and (16) one effectively gets:

$$\nu_{CL}(t) = \frac{P_0}{S}\varepsilon_{CL}(t) \tag{19}$$

where:

$$P_0 = A_0S + B_0R \tag{20}$$

is a Hurwitz polynomial. Define:

$$\theta^T = [\gamma_1, \gamma_2 \ldots \delta_1, \delta_2, \ldots] \; ; \tag{21}$$

$$\hat{\theta}^T = [\hat{\gamma}1, \hat{\gamma}2 \ldots \hat{\delta}1, \hat{\delta}2...], \tag{22}$$

$$\varphi^T(t) = [-\hat{\beta}(t), -\hat{\beta}(t-1), \ldots \hat{\alpha}(t), \hat{\alpha}(t-1), \ldots] \tag{23}$$

For developing the identification algorithm it will be assumed that the plant model is in the model set. The algorithms will be analyzed first in this context and then in the case when the plant model is not in the model set.

3. Dual-CLOE Algorithms

In a deterministic context ($w = 0$ in Fig. 1), for the case of a constant estimated parameter vector $\hat{\theta}$ using the predictor given in Eq (11) one has the

following result:

Lemma 1 For a constant estimated vector $\hat{\theta}$ in a deterministic environment $(w(t) = 0)$ the closed loop output error is given by:

$$\varepsilon_{CL}(t+1) = \frac{S}{\Gamma P_0}(\theta - \hat{\theta})^T \varphi(t) \tag{24}$$

and the filtered closed loop output error is given by:

$$\nu_{CL}(t+1) = \frac{1}{\Gamma}(\theta - \hat{\theta})^T \varphi(t) \tag{25}$$

The proof of this lemma is given in appendix A.

In the presence of time varying parameter estimates one can define an *a priori* and and *a posteriori* filtered closed loop output error, i.e.

$$\varepsilon^0_{CL}(t+1) = \frac{S}{\Gamma P_0}[\theta - \hat{\theta}(t)]^T \varphi(t) \tag{26}$$

$$\varepsilon_{CL}(t+1) = \frac{S}{\Gamma P_0}[\theta - \hat{\theta}(t+1)]^T \varphi(t) \tag{27}$$

Similarly one can define an *a priori* and and *a posteriori* filtered closed loop output error, i.e.

$$\nu^0_{CL}(t+1) = \frac{1}{\Gamma}[\theta - \hat{\theta}(t)]^T \varphi(t) \tag{28}$$

$$\nu_{CL}(t+1) = \frac{1}{\Gamma}[\theta - \hat{\theta}(t+1)]^T \varphi(t) \tag{29}$$

Eqs. (27)and (29) have the standard form for using parameter adaptation algorithms (PAA) of the form: [6, 8]

$$\hat{\Theta}(t+1) = \hat{\Theta}(t) + F(t)\phi(t)\nu(t+1) \tag{30}$$

$$\nu(t+1) = \frac{\nu^o(t+1)}{1 + \phi^T F(t)\phi(t)} \tag{31}$$

$$F(t+1)^{-1} = \lambda_1 F(t)^{-1} + \lambda_2 \phi(t)\phi^T(t) \tag{32}$$

$$0 < \lambda_1 < 1 \quad 0 \leq \lambda_2 < 2, F_0 > 0^4 \tag{33}$$

In the case of using the closed loop output error one has:

$Dual - CLOE$ algorithm:

$\hat{\Theta}(t) = \hat{\theta}(t)$, $\phi(t) = \varphi(t)$ and $\nu(t) = \varepsilon_{CL}(t)$.

In the case of using the filtered closed loop output error one has:

$FDual - CLOE$ algorithm:

$\hat{\Theta}(t) = \hat{\theta}(t)$, $\phi(t) = \varphi(t)$ and $\nu(t) = \nu_{CL}(t)$.

3.1. Stability Analysis

Using the Theorem 4.1 [6] or Theorem 3.3 [8] one can straightforwardly conclude that:

Lemma 2: Using the PAA given in Eqs. (30) through (33) with $\hat{\Theta}(t) = \hat{\theta}(t)$ and $\phi(t) = \varphi(t)$ and assuming that S is stable and the external excitation r(t) is bounded, one has :

for the case $\nu(t) = \varepsilon_{CL}(t)$:

$$\lim_{t\to\infty} \varepsilon_{CL}(t+1) = \lim_{t\to\infty} \varepsilon^0_{CL}(t+1) = 0 \tag{34}$$

together with the boundedness of $\varepsilon_{CL}(t)$, and $\varphi(t)$ for any initial conditions, provided that:

$$\frac{S(z^{-1})}{\Gamma(z^{-1})P_0(z^{-1})} - \frac{\lambda}{2}; \quad max\lambda_2(t) \leq \lambda_2 \leq 2 \tag{35}$$

is a *strictly positive real* (SPR) transfer function.

and for the case $\nu(t) = \nu_{CL}(t)$:

$$\lim_{t\to\infty} \nu_{CL}(t+1) = \lim_{t\to\infty} \nu^0_{CL}(t+1) = 0 \tag{36}$$

[4] $\lambda_1(t)$ and $\lambda_2(t)$ allow to obtain various profiles for the evolution of the adaptation gain $F(t)$

together with the boundedness of $\varepsilon_{CL}(t)$, $\nu_{CL}(t)$ and $\varphi(t)$ for any initial conditions, provided that:

$$\frac{1}{\Gamma(z^{-1})} - \frac{\lambda}{2}; \quad max\lambda_2(t) \leq \lambda_2 \leq 2 \tag{37}$$

is a *strictly positive real* (SPR) transfer function.

Remark 1:For $\Gamma = 1$, which corresponds to uncertainties modeled as a FIR filter, the SPR condition is suppressed for the algorithm using the filtered closed loop output error

Remark 2: If an a priori estimation of Γ denoted Γ_0 is available one can relax the SPR condition by filtering the adaptation error through Γ_0 or filtering the observation vector $\varphi(t)$ through $1/\Gamma_0$ (see next section).

4. The Dual - FCLOE algorithms

Consider now Eq. (24), for the case with time varying estimated parameters. Neglecting the non-commutativity of the time-varying operators (but an exact algorithms can be derived - see [8]) one can consider to filter the observation vector as follows:

$$\varphi_f(t) = \frac{S}{\Gamma_0 P_0}\varphi(t) \tag{38}$$

where Γ_0 is an *a priori* estimation of Γ (Γ_0 =1 in the absence of *a priori* information or in the case of FIR models). In this case one gets:

$$\varepsilon_{CL}(t+1) = \frac{\Gamma_0}{\Gamma}[\theta - \hat{\theta}(t+1)]^T\varphi_f(t) \tag{39}$$

This equation has the standard form for using the PAA given in Eqs. (30) through (33) with:

$Dual - FCLOE$ algorithm:

$\hat{\Theta}(t) = \hat{\theta}(t)$, $\phi(t) = \varphi_f(t)$ and $\nu(t) = \varepsilon_{CL}(t)$.

The stability condition in this case will be that the transfer function:

$$\frac{\Gamma_0(z^{-1})}{\Gamma(z^{-1})} - \frac{\lambda}{2}; \quad max\ \lambda_2(t) \leq \lambda_2 \leq 2 \tag{40}$$

be *strictly positive real*

For the case of using the filtered closed loop output error one can define the regressor vector as:

$$\varphi_{ff}(t) = \frac{1}{\Gamma_0}\varphi(t) \tag{41}$$

Eq. (25) becomes:

$$\nu_{CL}(t+1) = \frac{\Gamma_0}{\Gamma}(\theta - \hat{\theta}(t+1))^T \varphi_{ff}(t) \tag{42}$$

In this case, the algorithm will take the form:

FDual -FCLOE algorithm:

$\hat{\Theta}(t) = \hat{\theta}(t)$, $\phi(t) = \varphi_{ff}(t), \nu(t) = \nu_{CL}(t)$

and the stability condition will be that the transfer function given in Eq. (40) is a *strictly positive real* (SPR) transfer function.

Remark: One can also consider to filter the regressor vector through the current estimation of $\hat{\Gamma}(t)$ but including a stability test (Dual-AFCLOE algorithm). .

4.1. The Noisy Case

In the presence of measurement noise $w(t)$, the output of the plant will be given by:

$$y((t+1) = -Ay(t) + Bu(t) + Aw(t+1) \tag{43}$$

One assumes that the measurement noise $w(t)$ is bounded and independent with respect to the external excitation $r(t)$ and that one uses a decreasing adaptation gain ($\lambda_1(t) \equiv 1, \lambda_2(t) > 0$). One can use for analysis, the ODE approach [9]. Following a similar path as for the proof of *Lemma 1* it can be shown that the closed loop output prediction error and the filtered closed loop output error for a fixed value of the estimated parameter vector will be given by:

$$\varepsilon_{CL}(t+1) = \frac{S}{\Gamma P_0}(\theta - \hat{\theta})^T \varphi(t) + \frac{AS}{\Gamma P_0} w(t+1) \tag{44}$$

$$\nu_{CL}(t+1) = \frac{1}{\Gamma}(\theta - \hat{\theta})^T \varphi(t) + \frac{A}{\Gamma} w(t+1) \tag{45}$$

For the Dual-FCLOE algorithm the expression of the adaptation error for a constant estimated paramter vector will take the form:

$$\varepsilon_{CL}(t+1) = \frac{\Gamma_0}{\Gamma}(\theta - \hat{\theta})^T \varphi_f(t) + \frac{AS}{\Gamma P_0} w(t+1) \tag{46}$$

$$\nu_{CL}(t+1) = \frac{\Gamma_0}{\Gamma}(\theta - \hat{\theta})^T \varphi_{ff}(t) + \frac{A}{\Gamma} w(t+1) \tag{47}$$

One observes that $\varphi(t)$, $\varphi_f(t)$ and $\varphi_{ff}(t)$ are independent with respect to $w(t+1)$ (by the way that they are generated) and one can straightforwardly use Theorem 4.1 [8] leading to the following conclusions:

- Assume that the stationary process $\phi(t, \hat{\theta})$ and $\nu(t+1, \hat{\theta})$ can be defined for $\hat{\theta}(t) \equiv \hat{\theta}$.
- Assume that $\hat{\theta}(t)$ generated by the algorithm belongs infinitely often to the domain (D_s) for which the stationary processes $\phi(t, \hat{\theta})$ and $\nu(t+1, \hat{\theta})$ can be defined.
- Assume that either $w(t+1, \hat{\theta})$ is a sequence of independent equally distributed normal random variables $(0, s)$, or $E\{\phi(t, \hat{\theta}), w(t+1, \hat{\theta})\} = 0$

Then, if the strictly positive real condition of Eq. (35) for Dual-CLOE, of Eq. (37) for FDual-CLOE, of Eq. (40) Dual-FCLOE and FDual-FCLOE are satisfied, one has:

$$Prob\{\lim_{t \to \infty} \hat{\theta}(t) \in D_c\} = 1 \tag{48}$$

where the convergence domain D_c is defined as :

$$D_c : \hat{\theta} : \phi^T(t, \hat{\theta})[\theta - \hat{\theta}] = 0 \tag{49}$$

Under richness conditions the domain D_c reduces to $\hat{\theta} = \theta$ and one gets unbiased parameter estimates i.e.

$$Prob\{\lim_{t \to \infty} \hat{\theta}(t) = \theta\} = 1 \tag{50}$$

5. Dual - XCLOE Algorithm

In order to remove the SPR condition (37) depending upon the unknown polynomial $\Gamma(q^{-1})$ one can consider to use an extended estimation model. However, the independence between the measurement noise and the regressor vector will be lost in the noisy case even for constant estimated parameters. So in order to remove the SPR condition (37) and to assure unbiased estimates in the presence of noise, we will consider an ARMAX model representation for the plant + noise:

$$y((t+1) = -Ay(t) + Bu(t) + Ce(t+1) \tag{51}$$

where e(t) is a gaussian discrete time white noise (0,σ) and $C = 1 + q^{-1}C^*$ is an asymptotically stable polynomial. The objective will be first to construct an optimal predictor for the case of known parameters which will assure that the prediction error will become asymptotically white and then an adaptation algorithm will be developed for the case of unknown parameters. One has the following result:

Lemma 3 Under the hypotheses that $A_0, B_0, R, S, \Delta, \Gamma$ and C are perfectly known, the asymptotically optimal one step ahead closed loop output predictor is given by:

$$\hat{\Gamma}\hat{\beta}(t+1) = \hat{\Delta}^*\hat{\alpha}(t) + \hat{H}^*\frac{\varepsilon_{CL}}{S} \tag{52}$$

$$\hat{y}(t+1) = -A_0^*\hat{y}(t) + \hat{\beta}(t+1) + B_0^*\hat{u}(t) \tag{53}$$

where:

$$H^* = C^*S - (\Gamma A_0)^*S - \Gamma^* B_0^* R: \quad H = 1 + q^{-1}H^*$$

$$\hat{H}^* = \hat{h}_1 + \hat{h}_2 q^{-1} + ... + \hat{h}_{n_H} q^{-n_H+1};$$

$$\hat{H} = 1 + q^{-1}\hat{H}^*$$

$$n_H = max(n_C + n_S, n_\Gamma + n_{A_0} + n_S, n_\Gamma + n_{B_0} + n_R); \tag{54}$$

leading to:

$$\lim_{t\to\infty}[\epsilon_{CL}(t+1) - e(t+1)] = 0 \tag{55}$$

for $\hat{\theta} = \theta$ and $\hat{H} = H$. The proof of this lemma is given in Appendix B.
Note that Eq. (52) can be rewritten as:

$$\hat{\beta}(t+1) = \hat{\theta}_e^T \varphi_e(t) \tag{56}$$

where:

$$\hat{\theta}_e^T = [\hat{\theta}^T, \hat{h}_1, \hat{h}_2, ..., \hat{h}_{nH}] \tag{57}$$

$$\varphi_e^T(t) = [\varphi^T, \varepsilon_{CLf}(t), ...\varepsilon_{CLf}(t - n_H + 1)] \tag{58}$$

$$\varepsilon_{CLf}(t) = \frac{1}{S}\varepsilon_{CL}(t) \tag{59}$$

To identify the vector of parameter, one replaces $\hat{\theta}_e$ in the predictor by a time-varying vector $\hat{\theta}_e(t)$ which will be estimated.
In a deterministic context $e(t+1) = 0$ and $C = 1$. From Eqs. (99) through (102) it results that the *a posteriori* output of the $\hat{\Pi}$ predictor will be given by:

$$\hat{\beta}(t+1) = \hat{\theta}_e^T(t+1)\varphi_e(t) \tag{60}$$

and the *a posteriori* closed loop prediction error will be given by:

$$\varepsilon_{CL}(t+1) = [\theta_e - \hat{\theta}_e(t+1)]^T \varphi_e(t) \tag{61}$$

This equation has the standard form leading to the use of the parameter adaptation algorithm given in Eqs. (30) through (33) with;
Dual-XCLOE algorithm:
$\hat{\Theta}(t) = \hat{\theta}_e, \phi(t) = \varphi_e(t)$, $\nu(t) = \varepsilon_{CL}(t)$.
Lemma 3bis Under the hypotheses that $A_0, B_0, R, S, \Delta, \Gamma$ and C are perfectly known, the asymptotically optimal one step ahead filtered closed loop output predictor is given by:

$$\hat{\Gamma}\hat{\beta}(t+1) = \hat{\Delta}^*\hat{\alpha}(t) + \hat{H}^*\nu_{CL} \tag{62}$$

where:

$$H^* = C^* - \Gamma^*; \quad \hat{H}^* = C * -\hat{\Gamma}^*; \quad \hat{H} = 1 + q^{-1}\hat{H}^* \\ = \hat{h}_1 + \hat{h}_2 q^{-1} + ... + \hat{h}_{n_H} q^{-n_H+1}; \\ n_H = max(n_C, n_\Gamma); \tag{63}$$

leading to:

$$\lim_{t\to\infty}[\nu_{CL}(t+1) - e(t+1)] = 0 \tag{64}$$

for $\hat{\theta} = \theta$ and $\hat{H} = H$. The proof of this lemma is given in Appendix B. Note that Eq. (62) can be rewritten as:

$$\hat{\beta}(t+1) = \hat{\theta}_{ef}^T \varphi_{ef}(t) \tag{65}$$

where:

$$\hat{\theta}_{ef}^T = [\hat{\theta}^T, \hat{h}_1, \hat{h}_2, ..., \hat{h}_{n_H}] \tag{66}$$

$$\varphi_{ef}^T(t) = [\varphi^T, \nu_{CL}(t), ...\nu_{CL}(t - n_H + 1)] \tag{67}$$

To identify the vector of parameters, one replaces $\hat{\theta}_{ef}$ in the predictor by a time-varying vector $\hat{\theta}_{ef}(t)$ which will be estimated.

In a deterministic context $e(t+1) = 0$ and $C = 1$, the a posteriori output of the predictor will be given by:

$$\hat{\beta}(t+1) = \hat{\theta}_e^T(t+1)\varphi_{ef}(t) \tag{68}$$

and the a posteriori prediction error will be given by:

$$\nu_{CL}(t+1) = [\theta_{ef} - \hat{\theta_{ef}}(t+1)]^T \varphi_{ef}(t) \tag{69}$$

90 This equation has the standard form leading to the use of the parameter adaptation algorithm given in Eqs. (30) through (33) with:

FDual-XCLOE algorithm:

$\hat{\Theta}(t) = \hat{\theta}_{ef}, \quad \phi(t) = \varphi_{ef}(t), \quad \nu(t) = \nu_{CL}(t).$

Remark: The algorithm using the adjustable predictor for the filtered closed loop output error will need less parameters to estimate than the algorithm using the adjustable predictor for the closed loop output error.

5.1. Stability analysis

Using the Theorem 4.1 [6] or Theorem 3.3 [8] one can straightforwardly conclude that:

Lemma 4: Using the predictor given in Eqs. (52) and (53) with adjustable parameters and the PAA given in Eqs. (30) through (33) with $\hat{\Theta}(t) = \hat{\theta}_e, \phi(t) = \varphi_e(t)$ and $\nu(t) = \varepsilon_{CL}$, one has :

$$\lim_{t\to\infty} \varepsilon_{CL}(t+1) = \lim_{t\to\infty} \varepsilon^0_{CL}(t+1) = 0 \tag{70}$$

together with the boundedness of ε_{CL} and $\varphi_e(t)$ for any initial conditions.

Lemma 4bis: Using the predictor given in Eq. (62) with adjustable parameters and the PAA given in Eqs. (30) through (32) with $\hat{\Theta}(t) = \hat{\theta}_{ef}, \phi(t) = \varphi_{ef}(t)$ and $\nu(t) = \nu_{CL}$. one has :

$$\lim_{t\to\infty} \nu_{CL}(t+1) = \lim_{t\to\infty} \nu^0_{CL}(t+1) = 0 \tag{71}$$

together with the boundedness of ν_{CL} and $\varphi_e(t)$ for any initial conditions.

5.2. The noisy case

In the noisy case, using Eqs. (99) and (100)from Appendix B, the prediction error equation takes the form:

$$\varepsilon_{CL}(t+1) = [\theta_e - \hat{\theta}_e(t+1)]^T \varphi_e(t) + Ce(t+1) - C^*\varepsilon_{CL}(t) \tag{72}$$

which leads to:

$$\varepsilon_{CL}(t+1) = \frac{1}{C}[\theta_e - \hat{\theta}_e(t+1)]^T \varphi_e(t) + e(t+1) \tag{73}$$

One can use for analysis the ODE method [9]. Since e(t+1) is a white noise sequence, $\varphi_e(\hat{\theta}_e, t)$ will be independent with respect to $e(t+1)$ and applying Theorem 4.1 from [8] one concludes that asymptotic unbiased parameter estimates

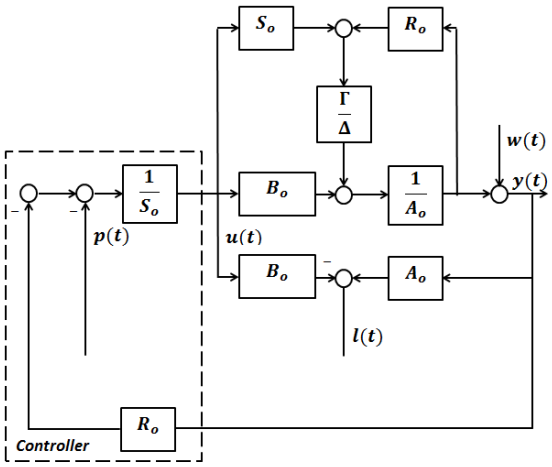

Figure 2: Dual YK identification from a YK structure (without Q-filter)

are obtained under richness conditions provided that

$$\frac{1}{C} - \frac{\lambda_2}{2}, \quad \lambda_2 = max\lambda_2(t) \tag{74}$$

is a *strictly positive real* transfer function. The same convergence result is obtained when using the predictor of Eq. (62) (filtered closed loop output predictor) with adjustable parameters and the corresponding parameter and regressor vector.

A summary of the algorithms presented above is given in Table 1.

6. Modified Hansen approach for recursive identification of Youla Kucera structure in closed loop operation

Adding a Youla-Kucera observer whose output is $\{l(t)\}$ (it is the input to a Youla-Kucera control filter), a direct identification of the Dual-Youla Kucera structure is possible by introducing an excitation sequence $\{p(t)\}$ and collecting the sequence $\{l(t)\}$ as shown in Fig. 2. The identification can be performed with any open-loop type algorithm. One has indeed

$$p(t) = Su(t) + Ry(t) \tag{75a}$$

$$l(t) = -B_0 u(t) + A_0 y(t) \tag{75b}$$

and on the other hand

$$\alpha(t) = Su(t) + R(y(t) - w(t)) \tag{76a}$$

$$\beta(t) = -B_0 u(t) + A_0 (y(t) - w(t)) \tag{76b}$$

But since

$$\beta(t) = \frac{\Delta}{\Gamma}\alpha(t) \tag{77}$$

and combining with (5) and (6), one obtains:

$$l(t) = \frac{\Delta}{\Gamma}p(t) + \frac{A}{\Gamma}w(t) \tag{78}$$

If one uses F-OLOE algorithm [8] in order to identify the Dual-YK filter with this scheme, the parameters vectors remain the one given by (21) and (22), the observation vector is $\varphi^T(t) = [-\hat{l}(t),\ -\hat{l}(t-1)\cdots p(t),\ p(t-1)\cdots]$, the predicted output is $\hat{l}(t+1) = \hat{\theta}^T(t)\varphi(t)$, the regressor vector is $\varphi(t)$ filtered by $\frac{1}{\Gamma_0}$, and the adaptation error is given by $\nu(t+1) = \varepsilon(t+1) = l(t+1) - \hat{l}(t+1)$ (MH-FOLOE algorithm). Another choice consists in using an ARMAX predictor model, and one can use the recursive extended least-squares (RELS- see [8]), where the regressor and the observation vector are in this case: $\varphi_e^T(t) = [\varphi^T(t),\ \varepsilon(t),\ \varepsilon(t-1)\cdots]$, and the parameter vector is $\hat{\theta}_e^T = [\hat{\theta}^T, \hat{c}_1, \hat{c}_2 \cdots]$.
This structure can be compared to the well-known Hansen scheme where an open-loop identification is performed with input-output sequences $\{x(t)\}$ and $\{z(t)\}$ defined as

$$x(t) = \frac{S^2}{P_0}r(t) \tag{79a}$$

$$z(t) = y(t) - \frac{B_0 S}{P_0}r(t) \tag{79b}$$

One can verify that

$$z(t) = \frac{\Delta}{\Gamma}x(t) + \frac{AS}{\Gamma P_0}w(t) \tag{80}$$

Consequently the Hansen structure induces an implicit frequency weighting by $\left|\frac{S^2(e^{i\omega})}{P_0(e^{i\omega})}\right|$ with respect to the excitation sequence $\{r(t)\}$ which in general has not a specific interest, whereas the scheme proposed in this section is a truly open-loop procedure (without any weighting) for the identification of the the Dual Youla-Kucera model in closed loop operation.

Alg.	Dual-CLOE	Dual-FCLOE	Dual-XCLOE	FDual-CLOE	FDual-FCLOE	FDual-XCLOE
Plant+Noise	$y = \frac{B}{A} + w$	$y = \frac{B}{A} + w$	$y = \frac{B}{A} + \frac{C}{A}e$	$y = \frac{B}{A} + w$	$y = \frac{B}{A} + w$	$y = \frac{B}{A} + \frac{C}{A}e$
Adj. Par. Vect. $\hat{\Theta}$	$\hat{\theta}^T = [\hat{\gamma}_1.\hat{\delta}_1.]$	$\hat{\theta}^T == [\hat{\gamma}_1.\hat{\delta}_1.]$	$\hat{\theta}_e^T = [\hat{\gamma}_1.\hat{\delta}_1.\hat{h}_1.]$	$\hat{\theta}^T = [\hat{\gamma}_1.\hat{\delta}_1.]$	$\hat{\theta}^T = [\hat{\gamma}_1.\hat{\delta}_1.]$	$\hat{\theta}_e^T = [\hat{\gamma}_1.\hat{\delta}_1.\hat{h}_1.]$
Obs. vect. φ	$[-\beta(i), \alpha(i)]$	$[-\beta(i), \alpha(i)]$	$[-\beta(i), \alpha(i), \varepsilon_f(i)]$	$[-\beta(i), \alpha(i)]$	$[-\beta(i), \alpha(i)]$	$[-\beta(i), \alpha(i), \nu_f(i)]$
Pred.out.$\hat{y}^0(t+1)$	$-A_0^* y + B_0^* \hat{u} + \hat{\beta}^0$	$-A_0^* y + B_0^* \hat{u} + \hat{\beta}^0$	$-A_0^* y + B_0^* \hat{u} + \hat{\beta}^0$	$\hat{\beta} = \hat{\theta}^T(t)\varphi(t)$	$\hat{\beta} = \hat{\theta}^T(t)\varphi(t)$	$\hat{\beta} = \hat{\theta}^T(t)\varphi(t)$
Adapt err.	$y - \hat{y}$	$y - \hat{y}$	$y - \hat{y}$	$\beta - \hat{\beta}$	$\beta - \hat{\beta}$	$\beta - \hat{\beta}$
Regr.vect. ϕ	φ	$\frac{S}{\Gamma_0 P_0}\varphi$	φ	φ	$\frac{S}{\Gamma_0 P_0}\varphi$	φ
Stab.cond. (deterministic)	$\frac{S}{\Gamma P_0} - \frac{\lambda_2}{2}$ SPR	$\frac{\Gamma_0}{\Gamma} - \frac{\lambda_2}{2}$ SPR	None	$\frac{1}{\Gamma} - \frac{\lambda_2}{2}$ SPR	$\frac{\Gamma_0}{\Gamma} - \frac{\lambda_2}{2}$ SPR	None
Conv. cond.) (stochastic)	$\frac{S}{\Gamma P_0} - \frac{\lambda_2}{2}$ SPR	$\frac{\Gamma_0}{\Gamma} - \frac{\lambda_2}{2}$ SPR	$\frac{1}{C} - \frac{\lambda_2}{2}$ SPR	$\frac{1}{\Gamma} - \frac{\lambda_2}{2}$ SPR	$\frac{\Gamma_0}{\Gamma} - \frac{\lambda_2}{2}$ SPR	$\frac{1}{C} - \frac{\lambda_2}{2}$ SPR

Table 1: Summary of closed loop identification algorithms for dual Youla Kucera plant models

7. Asymptotic bias distribution in the frequency domain

It is important to assess the bias distribution in the frequency domain of the estimated model when the plant model is not in the model set. Contrary to prediction error methods which aim at minimizing the prediction error variance asymptotically, it has been shown in [10], that in general this is not the case for the algorithms belonging to the Pseudo-Linear Regression (PLR) class, like Dual-CLOE. The non-measurable signal whose variance is asymptotically minimized by algorithms of the PLR class, is called the equivalent prediction error denoted as ε_E, and from the expression of this latter signal it is possible to infer

a limit model in the frequency domain, useful to analyze the bias distribution of the algorithm.

7.1. Limit model for the Dual-CLOE algorithm

One has the following result

Lemma 4 The limit model for $Dual-CLOE$ algorithm ($\nu(t)=\varepsilon_{CL}(t)$) or for the $FDual-CLOE$ algorithm ($\nu(t)=\nu_{CL}(t)$) is given by

$$\hat{\theta}^{*}=Argmin\int_{-\pi}^{+\pi}\left|\hat{A}(e^{i\omega})\right|^{2}\left|G(e^{i\omega})-\hat{G}(e^{i\omega})\right|^{2}\left|S_{yw}(e^{i\omega})\right|^{2}\Psi_{r}(\omega)d\omega \quad (81)$$

where $S_{yw}=\frac{AS}{\Gamma P0}$ is the output sensitivity function and $\Psi_r(\omega)$ is the PSD (Power Spectral Density) of the external excitation signal.
The proof is given in Appendix C (section13).

7.2. Limit model for the Dual-FCLOE algorithm

Lemma 5 The limit model of Dual-FCLOE algorithm is given by

$$\hat{\theta}^{*}=Argmin\int_{-\pi}^{+\pi}\left|\frac{\hat{A}(e^{i\omega})S(e^{i\omega})}{\Gamma_{0}(e^{i\omega})P_{0}(e^{i\omega})}\right|^{2}\left|G(e^{i\omega})-\hat{G}(e^{i\omega})\right|^{2}\left|S_{yw}(e^{i\omega})\right|^{2}\Psi_{r}(\omega)d\omega \quad (82)$$

The proof is given in Appendix D (section 14).

7.3. Limit model for the FDual-FCLOE algorithm

Lemma 6 The limit model of $FDual$-FCLOE algorithm is given by

$$\hat{\theta}^{*}=Argmin\int_{-\pi}^{+\pi}\left|\frac{\hat{A}(e^{i\omega})}{\Gamma_{0}(e^{i\omega})}\right|^{2}\left|G(e^{i\omega})-\hat{G}(e^{i\omega})\right|^{2}\left|S_{yw}(e^{i\omega})\right|^{2}\Psi_{r}(\omega)d\omega \quad (83)$$

The proof is obtained by combining results of Appendices C and D (sections 13 and 14).

7.4. Limit model for the Dual-XCLOE algorithm

Lemma 7 The limit model of Dual-XCLOE algorithm is given by

$$\hat{\theta}^* = Argmin \int_{-\pi}^{+\pi} \left|\hat{A}(e^{i\omega})\right|^2 \left|G(e^{i\omega}) - \hat{G}(e^{i\omega})\right|^2 \left|S_{yw}(e^{i\omega})\right|^2 \Psi_r(\omega) + \left|\hat{A}(e^{i\omega})\right|^2 \left|W(e^{i\omega}) \frac{A(e^{i\omega})\hat{\Gamma}(e^{i\omega})}{\hat{A}(e^{i\omega})\Gamma(e^{i\omega})} - \frac{\hat{C}(e^{i\omega})}{\hat{A}(e^{i\omega})}\right|^2 \Psi_e(\omega) d\omega \quad (84)$$

where $\Psi_e(\omega)$ is the PSD of the noise The proof is given in Appendix E (section 15)

7.5. Limit model for the MH-FOLOE

Lemma 8 The limit model of this algorithm is given by

$$\hat{\theta}^* = Argmin \int_{-\pi}^{+\pi} \left|\frac{\hat{A}(e^{i\omega})}{\Gamma_0(e^{i\omega})}\right|^2 \left|G(e^{i\omega}) - \hat{G}(e^{i\omega})\right|^2 \left|\frac{A(e^{i\omega})}{P_0(e^{i\omega})\Gamma(e^{i\omega})}\right|^2 \Psi_p(\omega) d\omega \quad (85)$$

where $\Psi_p(\omega)$ is the PSD of the excitation p, when the identification of the Dual-YK filter is performed with F-OLOE [8] (where the filter of the observation vector is $1/\Gamma_0$), or with H-OLOE [11] (where the basis poles are the roots of Γ_0). The proof is given in Appendix F (section 16).

By comparison, the limit model of the Hansen scheme, when identified with F-OLOE where the observation vector is the regressor filtered through $\frac{1}{\Gamma_0}$, or with H-OLOE where the basis poles are the roots of Γ_0 (it can be inferred from the one of PEM [9]), is given by:

$$\hat{\theta}^* = Argmin \int_{-\pi}^{+\pi} \left|\frac{\hat{A}(e^{i\omega})S(e^{i\omega})}{\Gamma_0(e^{i\omega})P_0(e^{i\omega})}\right|^2 \left|G(e^{i\omega}) - \hat{G}(e^{i\omega})\right|^2 \left|S_{yw}(e^{i\omega})\right|^2 \Psi_r(\omega) d\omega \quad (86)$$

7.6. Comparison of the various algorithms in terms of bias

The Dual-FCLOE algorithm seems to offer the best approximation properties since the model errors in the frequency domain are weighted by the true and the estimated sensitivity function. This means that the best approximation is obtained in the the frequency zone close the Nyquist point (where the modulus of the output sensitivity function has its maximum). The other algorithms from the CLOE family induce a larger penalty in high frequencies.

8. Simulations

One considers the following configuration:

$B_0(q^{-1}) = q^{-1} + 0.5q^{-2}$ $A_0(q^{-1}) = 1 - 1.5q^{-1} + 0.7q^{-2}$

$R(q^{-1}) = 0.2419 - 0.1491q^{-1} - 0.2144q^{-2} + 0.1766q^{-3}$

$S(q^{-1}) = 1 - 0.5043q^{-1} - 0.3696q^{-2} - 0.1262q^{-3}$

$\Delta(q^{-1}) = 2.8q^{-1} - 0.7q^{-2}$

$\Gamma(q^{-1}) = 1 - 1.1q^{-1} + 0.3q^{-2}$

The simulations are performed under the following conditions:

The excitation is a PRBS (2048 samples, decimation 1). The sequence {w(t)} is a centered gaussian white noise, and in closed-loop the noise/noise ratio is about 32% (-10dB) (output variance/noise variance) for all simulations [5]. In this context we are far from the asymptotic behaviour of the algorithms but closer to real situations. Fig. 3 displays the Bode Diagrams of the nominal system (B_0, A_0), and the current one (note that this latter is unstable in open-loop, but is stabilized by the controller), and Fig. 4 compares the output sensitivity functions of the loops including these two systems. Note that the maximum of the sensitivity function including the true system is roughly equal to 10 dB, which implies a modulus margin lower than 0.32.

[5] It corresponds to a high level of the measurement noise compared to the usual ratio used in the literature which is around 10% (-20 dB)

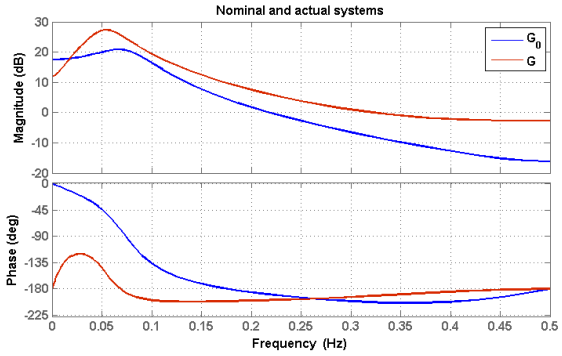

Figure 3: Bode diagrams of the nominal system and the current one

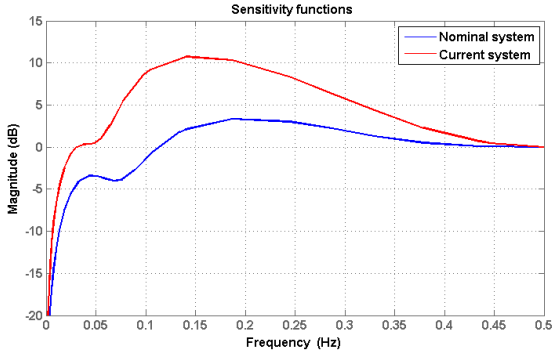

Figure 4: Sensitivity functions for the closed-loop including the nominal system and the current one

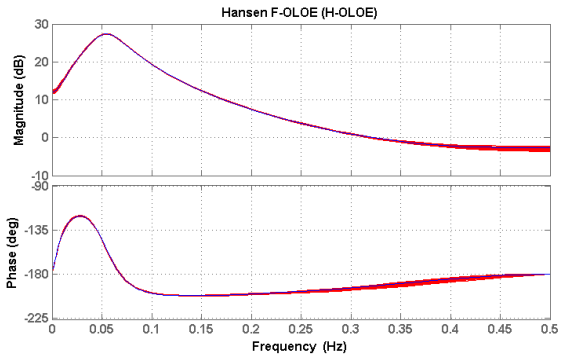

Figure 5: Identification result (50 realizations) for the Hansen scheme, estimation performed with F-OLOE

The first set of simulations compares 50 realizations in the case "plant model is in the model set" for the following algorithms:

a) The Hansen scheme where the identification routine is F-OLOE (or equivalently H-OLOE see [11]), where the zeros of the filter (or the basis poles of H-OLOE) are equal to 0.5 (Fig. 5),

b) The modified Hansen scheme, where the parameter estimation is performed with F-OLOE or H-OLOE, and where the zeros of the filter (or the basis poles of H-OLOE) are equal to 0.5 (Fig. 6),

c) Dual-FCLOE, where the roots of Γ_0 are set to 0.5 (Fig. 7),

d) FDual-FCLOE, where the roots of Γ_0 are set to 0.5 (Fig 8).

These figures reveal that in this highly noisy context the Hansen scheme provides the lowest variance in the low and high frequency ranges and the modified Hansen Scheme is the one yielding the larger variance in the low and high frequency range. However these frequencies domains are not essential for control. In the critical frequency range (around the maximum of the sensitivity function and of the system resonance) all algorithms show a very little difference

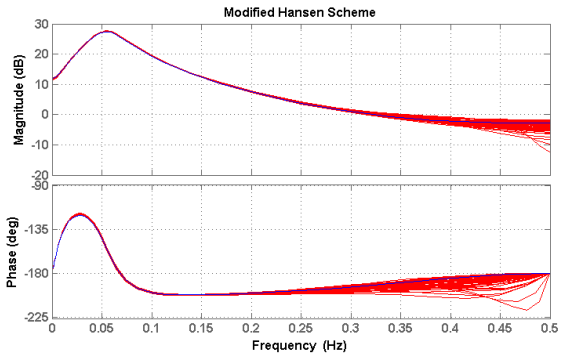

Figure 6: Identification result (50 realizations)for the modified Hansen scheme, estimation performed with F-OLOE (MH-OLOE)

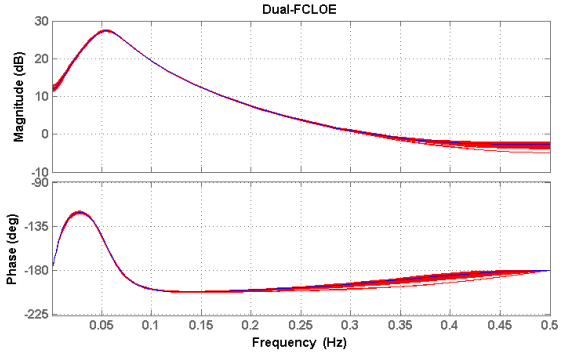

Figure 7: Identification result (50 realizations) for Dual-FCLOE

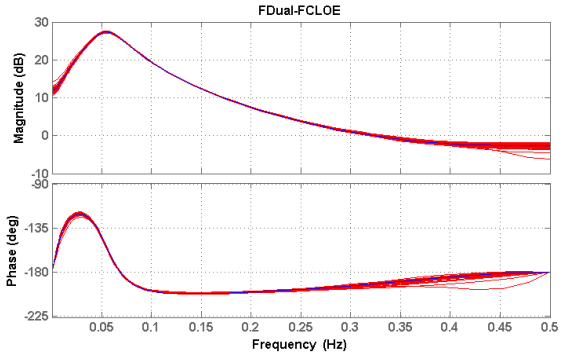

Figure 8: Identification result (50 realizations) for FDual-FCLOE

in performance in terms of variance. These observations are confirmed if one considers the standard deviation of the estimated parameters around their true values, as shown in Tab. 2, and the additional indicators:

$\bar{\tilde{\theta}} = \frac{1}{n_\theta} \sum_{j=1}^{n_\theta} \left| \bar{\tilde{\theta}}(j) - \theta(j) \right|$

$\bar{\sigma} = \frac{1}{n_\theta} \sum_{j=1}^{n_\theta} \sigma(j)$

$\bar{v} = \frac{1}{n_\theta} \sum_{j=1}^{n_\theta} \sigma^2(j)$

displayed in Tab. 3 (the bar corresponds to the mean value, n_θ is the lenght of θ, $\sigma(j)$ the standard deviation of the ith parameter)

True Parameters	MH-OLOE	Dual-FCLOE	FDual-FCLOE	Hansen Scheme/F-OLOE
$\gamma_1 = -1$	0.1939	0.1121	0.1981	0.1137
$\gamma_2 = 0.3$	0.1393	0.0783	0.1443	0.0820
$\delta_1 = 2.8$	0.1569	0.0880	0.0965	0.0580
$\delta_2 = -0.7$	0.5938	0.4096	0.6602	0.3894

Table 2: Standard deviation of the estimated parameters

The second set of simulations consists in performing identification in the case where the true Dual-Youla parameter is not in the model set (here $\hat{\Delta}$ and $\hat{\Gamma}$

Indicators	MH-OLOE	Dual-FCLOE	FDual-FCLOE	Hansen Scheme/F-OLOE
$\bar{\tilde{\theta}}$	0.3502	0.0687	0.0597	0.0566
$\bar{\sigma}$	0.2631	0.1790	0.2471	0.1476
$\bar{v}$	0.1050	0.0523	0.0992	0.0364

Table 3: Standard indicators of the parameters' dispersion

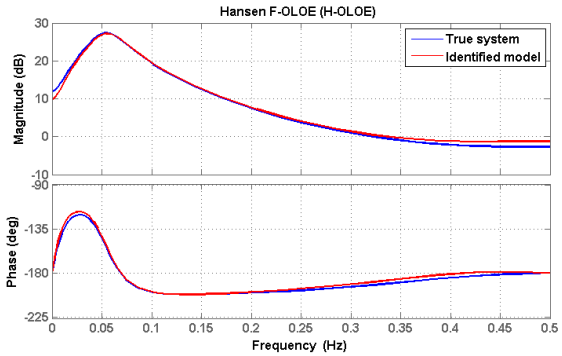

Figure 9: Resulting model for a Dual-YK Filter of order one, Hansen scheme

have a degree equal to 1), thus the identified models are necessarily biased. One can observe that the asymptotic models are identical for the schemes a), c), (see Fig. 9 and Fig. 10) and this is explained by the expressions of the bias in the frequency domain given in Eqs. (82) and (86) which are the same. One can notice that the misfit is strongly penalized for the frequencies around the maximum of the sensitivity function. As expected from expression (83) , $FDual-CLOE$ penalizes more significantly the fit in high frequency (Fig. 11), whereas the modified Hansen scheme, in accordance with Eq. (85) spreads the bias over the all spectrum, as shown in Fig. 12.

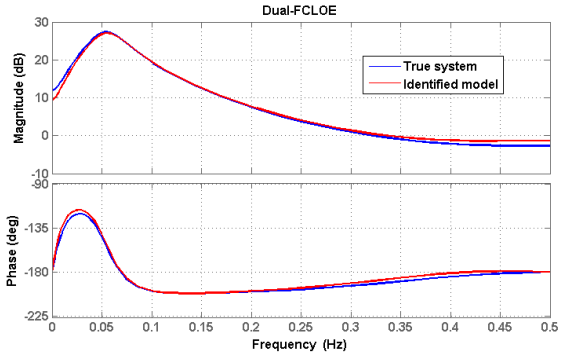

Figure 10: Resulting model for a Dual-YK Filter of order one, Dual-FCLOE

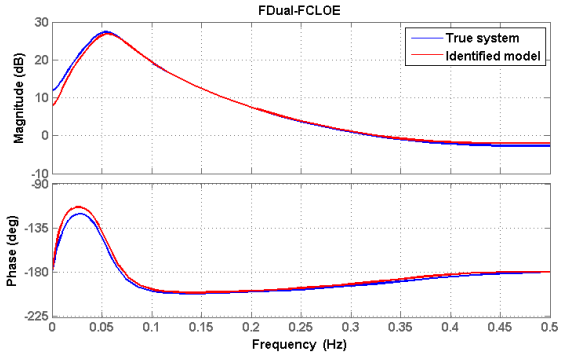

Figure 11: Resulting model for a Dual-YK Filter of order one, FDual-FCLOE

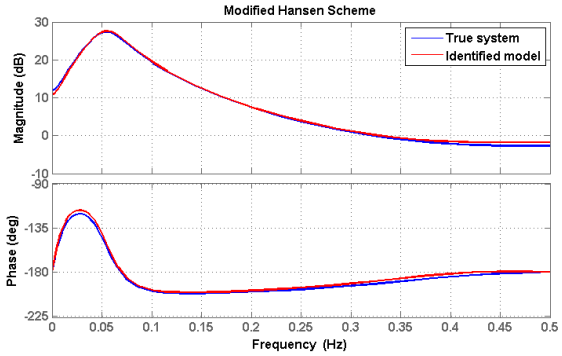

Figure 12: Resulting model for a Dual-YK Filter of order one, scheme b) with the modified Hansen Scheme (MH-OLOE)

9. Validation of the identified models

The techniques presented in [6] and [8] can be used also in this context.

10. Conclusions

The paper has presented a set of recursive algorithms for identification in closed loop operation of plant models represented in dual Youla Kucera form. The algorithms have been analyzed, compared and evaluated by simulations. In this context it appeared that there are three basic options for the choice of the prediction error:1) the closed loop output error 2) the closed loop error between the outputs of the true and estimated dual YK block which corresponds to a filtered closed loop output error) and 3) The Hansen scheme (and its modification) combined with open loop recursive identification algorithms. Some of these algorithms fully exploit the fact that one operates in closed loop allowing to get identified models which provide a better approximation in the crucial frequency regions close to the Nyquist point since the bias is heavily weighted by the sensitivity functions. Some of the algorithms allow to get models without any positive real condition related to a priori estimates.

11. Appendix A: Proof of Lemma 1

Proof: From Eq. (14) one has

$$(1 + q^{-1}\Gamma^*)\beta(t+1) = \Delta^*\alpha(t) \tag{87}$$

which can be alternatively written as:

$$\beta(t+1) = -\Gamma^*\beta(t) + \Delta^*\alpha(t) \tag{88}$$

Adding and subtracting the term $-\Gamma^*\hat{\beta}(t) + \Delta^*\hat{\alpha}(t)$ one gets:

$$\begin{aligned}\beta(t+1) &= -\Gamma^*\beta(t) + \Delta^*\alpha(t) \pm [-\Gamma^*\hat{\beta}(t) + \Delta^*\hat{\alpha}(t)] \\ &= -\Gamma^*[\beta(t) - \hat{\beta}(t)] + \Delta^*[\alpha(t) - \hat{\alpha}(t)] - \Gamma^*\hat{\beta}(t) + \Delta^*\hat{\alpha}(t)\end{aligned} \tag{89}$$

From Eq. (16) one gets

$$\hat{\beta}(t+1) = -\hat{\Gamma}^*\hat{\beta}(t) + \hat{\Delta}^*\hat{\alpha}(t) \tag{90}$$

Subtracting Eq. (90) from Eq. (89) and using Eq. (18) one gets:

$$\begin{aligned}\nu_{CL}(t+1) = -\Gamma^*\nu_{CL}(t) &+ \Delta^*[\alpha(t) - \hat{\alpha}(t)] \\ &-(\Gamma^* - \hat{\Gamma}^*)\hat{\beta}(t) + (\Delta^* - \hat{\Delta}^*)\hat{\alpha}(t)\end{aligned} \tag{91}$$

By using Eqs.(10),(12),(13),(15) one concludes that:

$$\alpha(t) - \hat{\alpha}(t) = 0 \tag{92}$$

and using now Eqs. (21) through (23), it results that Eq. (91) takes the form:

$$\Gamma\nu_{CL}(t+1) = (\theta - \hat{\theta})^T\varphi(t) \tag{93}$$

from which Eq. (25) results. Using now Eqs. (18),(14),(16), (10),(12),(17), one gets

$$\nu_{CL}(t) = A_0[y(t) - \hat{y}(t)] + \frac{B_0 R}{S}[y(t) - \hat{y}(t)] = \frac{P_0}{S}\varepsilon_{CL}(t). \tag{94}$$

Combining Eqs. (93) and (94) one gets Eq. (24)

End of the proof.

12. Appendix B: Proof of Lemmas 3 and 3bis

From Eq. (51) using the expression of A and B given in Eq. (6) one gets:

$$\Gamma[A_0 y(t+1 - B_0 u(t+1] = \Delta[Su(t+1) + Ry(t+1] + Ce(t+1) \tag{95}$$

which can be alternatively written as:

$$\begin{aligned} y(t+1) = -(\Gamma A_0)^* y(t) + \Gamma B_0 u(t+1) \\ +\Delta^*[Su(t) + Ry(t)] + Ce(t+1) \end{aligned} \tag{96}$$

where:

$$\Gamma A_0 = 1 + q^{-1}(\Gamma A_0)^* \tag{97}$$

Adding and subtracting $-(\Gamma A_0)^* \hat{y}(t) + \Gamma B_0 \hat{u}(t+1) + \Delta^*[S\hat{u}(t) + R\hat{y}(t)]$ one gets:

$$\begin{aligned} y(t+1) = -(\Gamma A_0)^* \varepsilon_{CL}(t) - \Gamma B_0^* \frac{R}{S} \varepsilon_{CL}(t) \\ -(\Gamma A_0)^* \hat{y}(t) + \Gamma B_0 \hat{u}(t+1) + \Delta^*[S\hat{u}(t) + R\hat{y}(t)] + Ce(t+1) \\ +\Delta^*[R\varepsilon_{CL}(t) - S\frac{R}{S}\varepsilon_{CL}(t)] \end{aligned} \tag{98}$$

where the last parenthesis is null. Adding and subtracting the term $C^*\varepsilon_{CL}(t)$, Eq. (98) will take the form:

$$\begin{aligned} y(t+1) = [C^*S - (\Gamma A_0)^* S - \Gamma^* B_0^* R]\frac{\varepsilon_{CL}(t)}{S} \\ -(\Gamma A_0)^* \hat{y}(t) + \Gamma B_0^* \hat{u}(t) + \Delta^*[S\hat{u}(t) + R\hat{y}(t)] \\ -C^*[\varepsilon_{CL}(t) - e(t)] + e(t+1) \end{aligned} \tag{99}$$

From Eq. (52) one gets:

$$\begin{aligned} \hat{y}(t+1) = -(\hat{\Gamma} A_0)^* \hat{y}(t) + \hat{\Gamma} B_0^* \hat{u}(t) \\ +\hat{\Delta}^*[S\hat{u}(t) + R\hat{y}(t)] + \hat{H}^* \frac{\varepsilon_{CL}}{S} \end{aligned} \tag{100}$$

Subtracting now Eq. (100) from Eq.(99) for $\hat{\theta} = \theta$ and taking into account that:

$$(\Gamma A_0)^* - (\hat{\Gamma} A_0)^* = (\Gamma^* - \hat{\Gamma}) A_0 \tag{101}$$

$$(\Gamma - \hat{\Gamma}) = q^{-1}(\Gamma^* - \hat{\Gamma}^*) \tag{102}$$

yields:

$$\varepsilon_{CL}(t+1) = -C^*(\varepsilon_{CL}(t) - e(t) + e(t+1) \tag{103}$$

which implies that (55) holds since $C(z^{-1})$ has all his roots inside the unit circle. Proof of lemma 3bis

Eq.(95) can be rewritten using Eqs.(13) and (14):

$$\begin{aligned} \beta(t+1) &= -\Gamma^* \beta(t) + \Delta^* \alpha(t) + Ce(t+1) \\ &= -\Gamma^* \nu_{CL}(t) - \Gamma^* \hat{\beta}(t) + \Delta^* \hat{\alpha}(t) + Ce(t+1) \end{aligned} \tag{104}$$

by adding and subtracting the term $-\Gamma^* \hat{\beta}(t) + \Delta^* \hat{\alpha}(t)$. Eq. (62) can be rewritten as:

$$\hat{\beta}(t+1) = -\hat{\Gamma}^* \hat{\beta}(t) + \hat{\Delta}^* \hat{\alpha}(t) + [-\hat{\Gamma}^* + C^*] \nu_{CL}(t) \tag{105}$$

Subtracting Eq. (105) from Eq. (104) one gets for the known parameter case:

$$\nu_{CL}(t+1) = -C^*(\nu_{CL}(t) - e(t) + e(t+1) \tag{106}$$

which implies that Eq. (64) holds.

13. Appendix C: Proof of Lemma 4

Case 1: $\nu(t) = \varepsilon_{CL}(t)$. As in [10], let us define $Q(q^{-1}, \hat{\theta})$ such that $Q(q^{-1}, \hat{\theta}) \frac{\partial \varepsilon_{CL}}{\partial \hat{\theta}} = -\varphi(t, \hat{\theta})$. Notice that we have $\hat{\beta}(t+1) = \frac{P_0}{S} \hat{y}(t+1, \hat{\theta}) - B_0 r(t)$, and $\hat{\beta}(t+1) = \hat{\theta}^T \varphi(t)$. Therefore $\frac{\partial \hat{y}(t+1, \partial \hat{\theta})}{\hat{\theta}} = \frac{S}{P_0} \left[\varphi(t) + \hat{\theta}^T \frac{\partial \varphi(t)}{\partial q \varepsilon} \frac{\partial q \varepsilon}{\partial \hat{\theta}} \right]$, and $\left[\frac{P_0}{S} + \hat{\theta}^T \frac{\partial \varphi}{\partial q \varepsilon} \right] \frac{\partial \varepsilon(t+1, \hat{\theta})}{\partial \hat{\theta}} = -\varphi(t)$. Therefore $Q(q^{-1}, \hat{\theta}) = \frac{P_0}{S} + \hat{\theta}^T \frac{\partial \varphi}{q \varepsilon} = \frac{P_0 \hat{\Gamma}}{S}$. Now, Theorem 1 of [10] can be used with the present expression of $Q(q^{-1}, \hat{\theta}) = \frac{P_0 \hat{\Gamma}}{S}$. Since from [8], p.308, $\varepsilon_{CL}(t) = \frac{\hat{A} S}{P_0 \hat{\Gamma}} \left[G - \hat{G} \right] \frac{AS}{\Gamma P_0} r(t) + \frac{AS}{\Gamma P_0} w(t)$. One obtains, the equivalent prediction error $\varepsilon_E(t) = \hat{A} \left[G - \hat{G} \right] \frac{AS}{\Gamma P_0} r(t) + \frac{AS}{\Gamma P_0} w(t)$, and one has $\hat{\theta}^* =$

$Argmin \int_{-\pi}^{+\pi} \Psi_{\varepsilon_E}(\omega)d\omega$, where $\Psi_{\varepsilon_E}(\omega)$ is the PSD of ε_E, hence the limit model in the frequency domain.

Case 2: $\nu(t) = \nu_{CL}(t)$. The expression of $Q(q^{-1},\hat{\theta})$ such that $Q(q^{-1},\hat{\theta})\frac{\partial \nu_{CL}}{\partial \hat{\theta}} = -\varphi(t,\hat{\theta})$ leads to $Q(q^{-1},\hat{\theta}) = \hat{\Gamma}$, since $\hat{\beta}(t+1) = \hat{\theta}^T\varphi(t,\hat{\theta})$. One has $\nu_{CL}(t) = \frac{P_0}{S}\varepsilon_{CL}(t)$, and the equivalent prediction error $\varepsilon_E(t)$ is given by $\varepsilon_E(t) = Q(q^{-1},\hat{\theta})\nu_{CL}(t) + (1-Q(q^{-1},\theta))\frac{AS}{\Gamma P_0}w(t+1)$, and one gets $\varepsilon_E(t) = \hat{A}\left[G-\hat{G}\right]\frac{AS}{\Gamma P_0}r(t) + \frac{AS}{\Gamma P_0}w(t)$, leading to the same limit model as in case 1.

14. Appendix D: Proof of Lemma 5

Like in case 1 of $Dual-CLOE$ (where $\nu(t) = \varepsilon_{CL}(t)$), one has $Q(q^{-1},\hat{\theta}) = \frac{P_0\hat{\Gamma}}{S}$. On the other hand, one has $\varphi_f(t) = \frac{S}{P_0\Gamma_0}\varphi(t) = \frac{1}{Q_f(q^{-1})}\varphi(t)$, where $Q_f(q^{-1}) = \frac{P_0\Gamma_0}{S}$. Therefore one can apply Theorem 2 in [10], with the present expressions of $Q(q^{-1},\hat{\theta})$ and $Q_f(q^{-1})$, and the equivalent prediction error is $\varepsilon_E(t) = Q(q^{-1},\hat{\theta})\nu_{CL}(t) + (1-Q(q^{-1},\theta))\frac{AS}{\Gamma_0 P_0}w(t+1)$, and one gets $\varepsilon_E(t) = \frac{\hat{A}S}{\Gamma_0 P_0}\left[G-\hat{G}\right]\frac{AS}{\Gamma P_0}r(t) + \frac{AS}{\Gamma P_0}w(t)$, hence the limit model in the frequency domain.

15. Appendix E: Proof of Lemma 7

Let us determine $Q(q^{-1},\hat{\theta})$ such that $Q(q^{-1},\hat{\theta})\frac{\partial \varepsilon_{CL}}{\partial \hat{\theta}} = -\varphi(t,\hat{\theta})$. One has $\hat{\beta}(t+1) = \hat{\theta}_e^T\phi_e(t)$, and $\hat{y}(t+1) = \frac{S}{P_0}\hat{\theta}_e^T\phi_e(t)$. Therefore $\frac{\partial \hat{y}(t+1)}{\hat{\theta}} = \frac{S}{P_0}\varphi_e(t) + \frac{S}{P_0}\frac{\partial \varphi_e(t)}{\partial q\varepsilon}\frac{q\varepsilon}{\partial \hat{\theta}}$, and $Q(q^{-1},\hat{\theta}) = \frac{P_0}{S} + \hat{\theta}\frac{\partial \varphi_e}{\partial q\varepsilon} = \frac{P_0}{S} + \frac{\hat{\Gamma}^* P_0}{S} + \frac{\hat{H}^*}{S} = \hat{C}$. Therefore $\varepsilon_E(t) = \hat{A}\left[(G-\hat{G})\frac{AS}{P_0\Gamma}r(t) + \left(W\frac{A\hat{\Gamma}}{\hat{A}\Gamma} - \frac{\hat{C}}{\hat{A}}\right)e(t)\right]$, like in the classical X-CLOE algorithm.

16. Appendix F: Proof of Lemma 8

One has $l(t) = \frac{\Delta}{\Gamma}p(t) + \frac{A}{\Gamma}w(t)$. Let us define $\hat{l}(t)$ as the predicted output of F-OLOE (or H-OLOE), one has $\hat{l}(t) = \frac{\hat{\Delta}}{\hat{\Gamma}}p(t)$. Set $\nu(t)$ the prediction error of F-OLOE (or H-OLOE) such that $\nu(t) = l(t) - \hat{l}(t)$. On the other hand $l(t) = -B_0u(t) + A_0y(t)$ and $u(t) = -\frac{R}{S}y(t) - \frac{1}{S}p(t)$. Therefore one obtains $\nu(t) = \frac{P_0}{S}\varepsilon(t)$, where $\varepsilon(t) = y(t) - \hat{y}(t)$. Moreover, one has $r(t) = -\frac{1}{S}p(t)$. Thus

one has $\nu(t) = \frac{\hat{A}}{\hat{\Gamma}}\left(G - \hat{G}\right)\frac{A}{P_0\Gamma}p(t) + \frac{A}{\Gamma}w(t)$. Now the equivalent prediction error $\nu_E(t)$ of F-OLOE or H-OLOE is given by $\nu_E(t) = \frac{\hat{\Gamma}}{\Gamma 0}\nu(t) + (1 - \frac{\hat{\Gamma}}{)}\frac{A}{\Gamma}w(t)$. And finally $\nu_E(t) = \frac{\hat{A}}{\gamma_0}(G - \hat{G})\frac{A}{P_0\Gamma}p(t) + \frac{A}{\Gamma}w(t)$, hence the limit model in the frequency domain, and this ends the proof.

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