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► **To cite this version:**

Ayşe N Arslan, Boris Detienne. Robust Strategic Planning of Phytosanitary Treatments in Agriculture. 2021. hal-03410635

HAL Id: hal-03410635

<https://hal.science/hal-03410635>

Preprint submitted on 1 Nov 2021

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Robust Strategic Planning of Phytosanitary Treatments in Agriculture

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Abstract

This paper deals with robust planning and scheduling of activities in agriculture and in particular the application of phytosanitary treatments. The crops are subject to many diseases that may arise during different time windows of the planning horizon. In response, a phytosanitary treatment can be applied to protect against a subset of these diseases. However, the effective duration of some treatments is uncertain, it depends on the type of treatment applied as well as on the weather conditions. In this study we introduce a penalty function based approach to handle this uncertainty without being overly conservative akin to light robustness approach proposed in the literature. We discuss different forms for this penalty function and elaborate on solution methodologies for the resulting models. We test the effectiveness of our approach with realistically-sized instances, which correspond to a typical vineyard in Bordeaux area, and present a numerical analysis of different optimization models and solution methods.

Keywords: Robust optimization, Light robustness, Penalty function, Phytosanitary treatment planning.

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Nomenclature

Indices and sets

- S set of sites
 T set of time periods of the planning horizon
 R set of diseases (requests) against which the sites must be protected
 P set of treatment mixtures
 C set of components that make up treatment mixtures
 M set of machines that can be used to apply treatment (product) mixtures
 Ω precomputed set of site clusters
 Z_s precomputed set of meta-treatment-sequences for site s

Subsets

- $\Omega_m \subseteq \Omega$ set of clusters that can be treated by machine m
 $\Omega_s \subseteq \Omega$ set of clusters that include site s
 $M_\omega \subseteq M$ set of machines that can be used to treat ω
 $R_s \subseteq R$ set of diseases that affect site s
 $R_{st} \subseteq R_s$ set of diseases that must be covered at site s in time period t
 $R_{ps} \subseteq R_s$ set of diseases that can be treated by treatment mixture p at site s
 $P^{S/NS} \subseteq P$ set of systemic/non-systemic treatments
 $P_s \subseteq P$ set of treatment mixtures that can be used at site s
 $P_{rs} \subseteq P$ set of treatment mixtures that can treat disease r at site s
 $T_{ps} \subseteq T$ set of time periods in which treatment mixture p can be applied at site s
 $T_{rs} \subseteq T$ set of time periods in which disease r should be treated at site s
 $Z_{sp} \subseteq Z_s$ set of all sequences of site s that contain mixture p
 \mathcal{K}_{sp}^ζ set of all indices k in sequence $\zeta \in Z_{sp}$ that are associated with mixture p
 $Succ_p \subseteq P$ set of mixtures that can be applied directly following mixture p

Parameters

D_{srpt}	active duration of treatment mixture p at site s in time period t for request r
D_{pst}^{\max}	$\max_{r \in R_{ps}} D_{srpt}$
ς_{sp}	cost of applying mixture p at site s
f_m	fixed cost of utilizing machine m
tol_s	toxicity tolerance of site s
ς_ω	cost associated with treating cluster ω
tox_{sp}	toxicity induced by using mixture $p \in P_s$ at site s
tox_ζ	cumulative toxicity level induced by sequence $\zeta \in Z_s$
p_k^ζ	k^{th} mixture in sequence ζ
$[e_k^\zeta, l_k^\zeta]$	time window in which p_k^ζ can be applied using sequence ζ

Decision Variables

$x_m \in \{0, 1\}$	1 if machine m is leased
$\lambda_s^\zeta \in \{0, 1\}$	1 if meta-sequence $\zeta \in Z_s$ is selected for the planning of treatments at site s
$\delta_{spt} \in \{0, 1\}$	1 if treatment mixture p is to be spread at site s in period t
$\mu_{mpt}^\omega \in \{0, 1\}$	1 if machine m is allocated to spread treatment mixture p in period t at cluster ω

1 Introduction

In this paper, we consider the planning and scheduling of phytosanitary treatments in agriculture. Crops are subject to many diseases that may arise over different time windows during the season. Phytosanitary treatments are applied as a preventive measure, where each treatment is effective against a subset of these diseases, and has a nominal duration of “guaranteed protection” that can vary depending on the type of the treatment as well as the disease. But the effective duration of protection depends largely on the weather conditions. While treatments may appear to be helpful to the producers, in the recent years growing concern of the consumers as well as of public health organizations has led the governments to impose more strict regulation on their use that the production practices must adapt to (5). Apart from environmental and health considerations, the topic also plays an important role in international trade relations (2). Government regulations pose restrictions on the repetitive usage of certain toxic treatments and limit the total amount of toxicity of a treatment plan. Additionally, treatments are categorized as systemic or non-systemic. The latter are surface treatments that are much more sensitive to the effects of rainfall. On the other hand, non-systemic treatments are cost-effective and more environmentally friendly. An important issue for producers is therefore to establish a reasonable use of protection treatments by phytosanitary products that protect the crops as effectively as possible while respecting the environment and government regulations. The goal is to reduce the use level of phytosanitary products by means of a better selection of products and time spacing between treatments, as well as allocation of equipment to treatments, assuring that the crops are protected during the season, while minimizing the cost of products and equipment and taking the uncertainty caused by the usage of non-systemic treatments into account.

The rest of the paper is organized as follows: in Section 2 we introduce the deterministic mathematical model for the phytosanitary treatment planning problem, in Section 3 we discuss the uncertainty we consider and give details about the modeling of robustness along with a review of the related literature, introducing our penalty measure approach and ensuing solution methodologies, in Section 4 we discuss further improvements to the model to implicitly improve robustness of solutions, in Section 5 we present our computational results with instances from an industrial partner, finally we conclude in Section 6.

2 Deterministic Mathematical Model

Given a set of sites, a set of diseases to be treated at each site, a set of *site clusters* that correspond to a subset of sites that can be treated together on the same day by the same machine, as defined in Section 2.1, and, a set of *treatment sequences* for each site that correspond to a disease coverage plan for the site as defined in Section 2.2, the strategic phytosanitary treatment planning problem is to determine the timing of treatments at each site and to allocate machine resources to perform these treatments. The objective is to minimize the costs associated with the use of phytosanitary products as well as the use of the machines over the season.

2.1 Site clusters

One of the critical resources in the agricultural planning problem is the machines that are used to apply treatments. The costs associated with dispatching machines (along with their operators) to sites are significant. They include hourly charges for the time spent traveling to sites and spreading treatments to sites, as well as fixed leasing costs. Promoting simultaneous treatment of multiple sites is therefore imperative to reduce the number of trips and hence the number of machines required. The set of sites that can be treated in one trip are restricted by a combination of factors such as the site surface areas and distances, as well machine tank capacities. Our model assumes that a feasible groupings of sites, that we call *clusters*, have been pre-generated. When there are a huge number of possible clusters, a subset is heuristically selected prior to optimization. A site typically belongs to multiple clusters, and not all machines can serve a certain cluster. In our optimization model, we assume that a site can only be treated on a given day if a machine is assigned that day to one of the clusters containing it. The following definition formalizes the concept of site clusters.

Definition 2.1. *Given a machine m , a cluster $\omega \in \Omega_m$ is defined by a set of contiguous sites, $\omega \subseteq S$, that can be treated within a work shift by machine m , ensuring that the machine has enough tank capacity to carry the product quantity required for treating all the sites of ω .*

We consider only maximal clusters with respect to both work shift and tank capacity, *i.e.*, one cannot add a site $s \in S$ to $\omega \in \Omega$ without violating one of the two capacity constraints.

2.2 Treatment sequences

Another input to our model is the set of treatment sequences for each site. A treatment sequence is a selection of phytosanitary treatments to be applied consecutively over the agricultural season. Given

that each treatment can be used to protect against multiple diseases, that different treatments have different protection durations, that some treatment pairs cannot be applied consecutively, and that government regulations restrict the use of some of the treatments, this problem is highly complex. We assume that a set of sequences has been pre-generated for each site. As there is possibly a huge number of such sequences, a subset is selected by experts of the application prior to optimization. These sequences are generated by taking the toxicity, precedence and application time-window restrictions of each treatment into account. Therefore, each treatment sequence is constructed to yield a feasible treatment solution, which means that there exists a planning of the treatments prescribed by the sequence that allows to cover all the diseases of the site over the season in the absence of weather perturbations. The feasibility of the treatment sequence translates into a time window of application for each treatment of the sequence. These time windows do not overlap to guarantee that the ordering of the treatments in the sequence is obeyed. A treatment sequence yields a feasible treatment planning for any timing that obeys the associated time window restrictions. As a given treatment sequence can yield many feasible treatment plans, we call it a *meta-treatment-sequence*. The following two definitions formalize these concepts.

Definition 2.2. *Given site $s \in S$ and request r , a request treatment sequence ζ_s^r for a request r is an ordered list of mixtures $\langle p_1, p_2, \dots, p_{|\zeta_s^r|} \rangle$ that treat request r at site s for which there exists a feasible planning. That is, the time windows of application and efficacy duration of treatments $\langle p_1, p_2, \dots, p_{|\zeta_s^r|} \rangle$ permit treating request r at site s for $t \in T_{rs}$.*

Definition 2.3. *Given site $s \in S$, a meta-treatment-sequence $\zeta \in Z_s$ is a partially ordered list of mixtures such that for all requests $r \in R_s$, the mixtures that treat r form a valid request-treatment-sequence ζ_s^r and there exists a feasible planning for the mutual realization of treatments in ζ_s .*

2.3 Formulation

Finally, using the above inputs and the notations introduced in the Nomenclature, the treatment planning problem can be formulated as:

$$\min \sum_{m \in M} f_m x_m + \sum_{s \in S, p \in P_s, t \in T_{ps}} \varsigma_{sp} \delta_{spt} + \sum_{\omega \in \Omega} \varsigma_{\omega} \sum_{m \in M_{\Omega}, p \in P, t \in T} \mu_{mpt}^{\omega} \quad (1)$$

$$\text{s.t.} \quad \sum_{p \in P, \omega \in \Omega_m} \mu_{mpt}^{\omega} \leq x_m \quad \forall t \in T, m \in M \quad (2)$$

$$\sum_{\zeta \in Z_s} \lambda_{\zeta} = 1 \quad \forall s \in S \quad (3)$$

$$\sum_{p \in P_{rs}, \tau \in T_{ps} \mid \tau \in [t - D_{srp\tau} + 1, t]} \delta_{sp\tau} \geq 1 \quad \forall s \in S, r \in R_s, t \in T_{rs} \quad (4)$$

$$\delta_{spt} \leq \sum_{\omega \in \Omega_s, m \in M_\omega} \mu_{mpt}^\omega \quad \forall s \in S, p \in P_s, t \in T_{ps} \quad (5)$$

$$\delta_{spt} \leq \sum_{\zeta \in Z_s \mid p=p_k^\zeta, t \in [e_k^\zeta, l_k^\zeta]} \lambda_\zeta \quad \forall s \in S, p \in P_s, t \in T_{ps} \quad (6)$$

$$\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\delta} \in \{0, 1\}. \quad (7)$$

The objective function (1) minimizes the global cost of planning, including machine usage cost and treatment cost. Constraints (2) enforce that machines can only be used to apply treatments if they are leased for the treatment campaign. Constraints (3) impose that a single treatment sequence should be chosen for each site s . Constraints (4) ensure that each request $r \in R_{st}$ is covered at each site s in each time period t . Constraints (5)-(6) impose the relationship between treatment and cluster and sequence selection variables, respectively. Accordingly, a treatment of mixture p can only be applied at site s in time period t if a machine is allocated to apply treatment p at cluster $\omega \in \Omega_s$ in time period t , and a compatible treatment sequence is selected. A treatment sequence is considered compatible with applying treatment δ_{spt} if mixture p is in the sequence ($p = p_k^\zeta$) and time period t is within the time window in which this mixture can be applied ($t \in [e_k^\zeta, l_k^\zeta]$).

We next introduce a set of valid inequalities that generalize and strengthen constraints (6) and significantly reduce the computational time required to solve the above model. To describe them, for each $p \in P_s$, let us sort in increasing order the combined list of e_k^ζ and $(l_k^\zeta + 1)$ values for all $\zeta \in Z_{sp}$ and all $k \in \mathcal{K}_{sp}^\zeta$. Let this sorted list be denoted by $\{d_1, \dots, d_n\}$. We then create the time windows $a_i = [d_i, d_{i+1} - 1]$ for $i = 1, \dots, n-1$. Let \mathcal{A}_{sp} denote the collection of application time windows of mixture p at site s over all treatment sequences $\zeta \in Z_{sp}$ constructed in this manner, *i.e.*, $\mathcal{A}_{sp} = \bigcup_{i=1}^{n-1} a_i$. Moreover, for $a \in \mathcal{A}_{sp}$, let $Z_{sp}(a)$ be the set of sequences that allow for the application of mixture p during time window a , *i.e.*, $\zeta \in Z_{sp}(a)$ satisfies $a \subseteq [e_k^\zeta, l_k^\zeta]$ for some $k \in \mathcal{K}_{sp}^\zeta$. Using this notation, we write

$$\sum_{t \in a} \delta_{spt} \leq \sum_{\zeta \in Z_{sp}(a)} \lambda_s^\zeta \quad \forall s \in S, p \in P_s, a \in \mathcal{A}_{sp}, \quad (8)$$

where the sum on the right-hand-side is assumed to be zero if $Z_{sp}(a) = \emptyset$. These inequalities are valid and generalize the special case of constraints (6) where a is a singleton. Therefore, in the sequel we replace constraints (6) with constraints (8). We illustrate these constraints on a small example.

Example 2.1. Consider the treatment sequences given in the table below. The following inequalities

$\zeta = 1$	e^1	l^1	$\zeta = 2$	e^2	l^2	$\zeta = 3$	e^3	l^3
p_1	9	12	p_2	9	12	p_2	7	12
p_1	17	20	p_2	17	20	p_2	13	16
p_1	25	28	p_2	25	28	p_2	17	28

are valid for the treatment planning problem where we eliminate the site index for ease of lecture:

$$\begin{aligned}
\delta_{p_19} + \delta_{p_110} + \delta_{p_111} + \delta_{p_112} &\leq \lambda^1 & \delta_{p_29} + \delta_{p_210} + \delta_{p_211} + \delta_{p_212} &\leq \lambda^2 + \lambda^3 \\
\delta_{p_117} + \delta_{p_118} + \delta_{p_119} + \delta_{p_120} &\leq \lambda^1 & \delta_{p_213} + \delta_{p_214} + \delta_{p_215} + \delta_{p_216} &\leq \lambda^3 \\
\delta_{p_125} + \delta_{p_126} + \delta_{p_127} + \delta_{p_128} &\leq \lambda^1 & \delta_{p_217} + \delta_{p_218} + \delta_{p_219} + \delta_{p_220} &\leq \lambda^2 + \lambda^3 \\
\delta_{p_27} + \delta_{p_28} &\leq \lambda^3 & \delta_{p_221} + \delta_{p_222} + \delta_{p_223} + \delta_{p_224} &\leq \lambda^3 \\
\delta_{p_225} + \delta_{p_226} + \delta_{p_227} + \delta_{p_228} &\leq \lambda^2 + \lambda^3.
\end{aligned}$$

3 Robust Mathematical Model

We next describe the nature of uncertainty present in the phytosanitary treatment planning problem. Based on this description, we introduce a naive robust model with no recourse action (i.e., static) and outline the challenges posed by column-wise uncertainty in the context of robust optimization. We then present a review of relevant work in the literature, and present our approach based on a penalty function for static robust optimization.

3.1 Characterizing the uncertainty

The uncertainty we consider affects the active duration of non-systemic treatments and is caused by weather conditions. A non-systemic treatment will lose its effectiveness if the total amount of rainfall in a day exceeds a certain threshold, which practitioners estimate to be around 20 centimeter cubes per meter square. The event of losing all non-systemic treatments as a result of such rainfall will be referred to as a “wash-out” in the sequel.

The robust optimization approach in the context of phytosanitary treatment planning problem is appropriate for several reasons. The first is the risk-averse behavior expressed by agricultural experts who stand to lose their crops in the case of a wash-out. The second is the difficulty associated with describing a distribution that governs meteorological conditions. We further remark that, risk-averse stochastic optimization based on chance-constrained models are highly non-convex and are very challenging to solve.

Our naive robust approach models the uncertainty associated to the wash-out events with the help of discrete uncertain parameters. Let ξ_t be equal to 1 if a wash-out occurs in period t and 0 otherwise, and let Ξ be the associated uncertainty set. The impact of a wash-out is to reduce the effective duration of a treatment. We write

$$D_{srp\tau}(\boldsymbol{\xi}) = \min\{D_{srp\tau}, t' - \tau\} \quad \boldsymbol{\xi} \in \Xi,$$

where $t' = \min\{t \in [\tau, \tau + D_{srp\tau} - 1] \mid \xi_t = 1\}$ is the first wash-out period during the effective duration of a treatment and Ξ is the considered uncertainty set. We remark that t' is set to $\tau + D_{srp\tau}$ when $\{t \in [\tau, \tau + D_{srp\tau} - 1] \mid \xi_t = 1\} = \emptyset$. A (strictly) robust solution should be feasible for all realizations of the uncertain parameters in the uncertainty set Ξ .

3.2 A naive robust model

The robust counterpart of the planning problem presented in Section 2.3 is obtained by replacing constraints (4) with their robust counterpart:

$$\sum_{p \in P_{rs}^S, \tau \in T_{ps} \mid \tau \in [t - D_{srp\tau} + 1, t]} \delta_{sp\tau} + \sum_{p \in P_{rs}^N, \tau \in T_{ps} \mid \tau \in [t - D_{srp\tau}(\boldsymbol{\xi}) + 1, t]} \delta_{sp\tau} \geq 1 \quad \forall s \in S, r \in R_s, t \in T_{rs}, \boldsymbol{\xi} \in \Xi,$$

As the uncertainty is column-wise, *i.e.*, row-wise dependent, each uncertain parameter will assume its worst-case realization (see for instance (4),(28)). As a result, assuming that, for each time period $t \in T$, there is a wash-out event in at least one scenario $\boldsymbol{\xi} \in \Xi$, this constraint is equivalent to imposing

$$\sum_{p \in P_{rs}^S, \tau \in T_{ps} \mid \tau \in [t - D_{srp\tau} + 1, t]} \delta_{ps\tau} \geq 1 \quad \forall s \in S, r \in R_s, t \in T_{rs}.$$

This model is unrealistically conservative as it immunizes against the possibility that all non-systemic treatments may be washed-out as soon as they are applied, and imposes therefore that only systemic treatments should be used. Alternatively, we may consider an adaptive robust model where treatment and cluster selection decisions are taken after the realization of uncertainty. However, as the recourse problem is a mixed integer program and the recourse matrix is uncertain, the resulting model is extremely difficult to solve. Indeed, even in the simple case of two-stage adjustable robust optimization with linear programming problems in both stages, and a polyhedral uncertainty set, the problem is shown to be NP-Hard (7).

In the following, we present a review of the existing literature on two-stage adjustable robust opti-

mization, and methodologies proposed to mitigate the conservativeness of static robust optimization, as well as cutting plane methods in robust optimization.

3.3 Literature review

Robust optimization is an approach to handling uncertainty in optimization where the probability distributions are replaced with uncertainty sets. In robust optimization, constraints are imposed for all realizations whereas the objective function is evaluated for the worst-case realization within the uncertainty set. As such, in applications where the effects of uncertainty can be catastrophic, robust optimization presents itself as a viable modeling approach. The idea behind robust optimization was first proposed by (28), who described linear programming models with column-wise uncertainty that should be feasible for all realizations of columns within a convex compact set, and proved that these models are equivalent to each uncertain parameter assuming its worst-case value, and are therefore too conservative. The interest in robust optimization was later revived by the seminal works of (8; 15) that proposed row-wise uncertainty with smaller uncertainty sets that would still guarantee satisfaction of uncertain constraints with high probability, and developed complexity-preserving deterministic equivalent formulations. Advances in static robust optimization are presented in (6),(9) and (21).

On the other hand, as outlined above, robust optimization can sometimes be “over-conservative”, especially when uncertainty does not have a row-independent structure. To remedy this problem, one might consider introducing recourse (or adjustability/adaptability) after the realization of uncertainty. Unfortunately, there are no exact solution methods for two-stage robust optimization with integer recourse that can be effective with large scale formulations. In (31), the constraint-and-column generation idea presented in (32) is extended to the mixed-integer recourse case where the authors propose a nested constraint-and-column generation scheme. In this framework, violated uncertainty realizations are added to a restricted master as needed along with the second stage constraints that are concerned. However, finding a violated realization is equivalent to solving a bilevel problem with a mixed-integer second level, and is therefore extremely difficult. Handling the addition of columns and constraints at the same time further undermines the efficiency of this method (in a typical implementation the optimization is restarted after adding the violated realizations).

Given the evident difficulty of solving two-stage robust optimization problems with mixed-integer recourse, most studies in the literature focus on obtaining approximate solutions. In (30), authors develop a conservative approximation for multistage robust MILPs presented in the context of in-

formation discovery in multistage stochastic programming. They partition the uncertainty set into hyperrectangles and restrict the continuous and binary recourse decisions to piecewise affine and constant functions of the uncertain parameter over each hyperrectangle, respectively. The resulting conservative approximation can be formulated as an MILP; see also (23). In (13), authors propose a piecewise constant decision rule for binary recourse variables used in conjunction with a scenario generation scheme resulting in an endogenous design of the decision rule. In (14), piecewise constant functions are again used to describe linearly parametrized binary decision rules. Another line of recent research in the approximate solution methods literature is “ K -adaptability” or “finite adaptability”. The K -adaptability problem consists of selecting a first-stage solution along with K recourse solutions at the first stage. In the second stage, after the realization of uncertainty, the recourse problem reduces to selecting the best solution among these K solutions. The authors of (24) extend the idea of finite adaptability introduced in (10), for two-stage robust optimization with pure binary recourse under objective function and right-hand-side uncertainty. In (29), the concept of K -adaptability is extended to problems with mixed-integer first- and second-stage feasible regions as well as uncertainty affected technology and recourse matrices. While in these papers, a partition of the uncertainty set and an assignment of K recourse policies to subsets defined by this partition is sought concurrently, it is also possible to define the finite adaptability problem for a given partition of the uncertainty set. This partition is then iteratively improved using the information from the solution obtained. This idea is explored in (11) and (26) in the context of multi-stage adjustable robust mixed-integer optimization. Recently, exact solution methods have been proposed for two-stage robust optimization problems with binary recourse in the case of only objective function uncertainty and some structural properties ((3),(25)).

Trying to bridge the gap between the conservativeness of robust optimization and the difficulty of solving adaptable robust optimization problems, (18) proposed the concept of *light robustness*, of which the methods are utilized in the context of railway timetable optimization in (20). The idea behind light robustness is to introduce slacks to constraints while at the same time imposing feasibility for all realizations of uncertain parameters in the Bertsimas-Sim budgeted uncertainty set expressed through the linear programming dual. The slacks are then minimized while keeping the cost within a tolerance of the nominal cost. A heuristic version is also proposed associating the robustness of a given solution with the amount of slack on the left-hand-side of each constraint. The minimum slack is then maximized and distributed among constraints in as much of an equivalent way as possible. In (27), authors generalize the concept of light robustness by introducing a norm in the objective function. They additionally show that, light robustness can be used to find Pareto optimal solutions

to the bi-objective optimization problem with cost and robustness as the objectives. In (22), authors explore the price of robustness, *i.e.*, the increase in travel time, for strict versus light robustness in the context of train timetable information.

Finally, the use of cutting plane methods for solving static robust optimization problems was explored in (19) and (12), where the authors numerically compared this method to the widely utilized reformulation method. In (19), the authors additionally present an uncertain covering problem where the columns may disappear with a certain probability, which they model as a robust optimization problem with a binary uncertainty set. Although this problem is similar to phytosanitary treatment planning problem, we remark that in our version columns do not disappear completely but starting from a certain time index.

3.4 Penalty measure approach

To tackle the conservativeness of the naive robust model, we propose an alternative model, implementing a heuristic measure of robustness, assuming that the treatment coverage is not completely lost, but that the rainfall exposes the crops to a risk of developing diseases. We associate an uncertain cost, that we call a “penalty measure” in the sequel, to non-systemic treatments that depends on the amount and timing of the rainfall during the activity of each treatment.

To allow for a linear expression of this cost, we introduce the binary variables

$$\delta_{sptt'} = 1 \text{ if site } s \text{ is treated with mixture } p \text{ in time period } t \text{ until the end of time period } t',$$

for $s \in S$, $p \in P_s$, $t \in T_{ps}$ and $t' \in [t, t + D_{pst}^{\max} - 1]$. We additionally define the function $a_{prs}(t, t', \tau)$ that computes the number of days of coverage affected by rainfall in period $\tau \in [t, t']$ if $\delta_{sptt'} = 1$, and request $r \in R_s$ is being covered by treatment $p \in P^{NS}$ at site $s \in S$. This latter condition is satisfied when $p \in P_{rs}$ and $T_{rs} \cap [t, t'] \neq \emptyset$. We let for $s \in S$, $r \in R_s$, $p \in P_{rs}^{NS}$, $t \in T_{ps}$ and $t' \in [t, t + D_{pst}^{\max} - 1]$

$$a_{prs}(t, t', \tau) = \begin{cases} 0 & \text{if } T_{rs} \cap [t, t'] = \emptyset, \\ \max\{0, \min\{t', t + D_{srpt} - 1, \max_{\kappa \in T_{rs} \cap [t, t']} \kappa\} - \max\{\tau, \min_{\kappa \in T_{rs} \cap [t, t']} \kappa\} + 1\}. \end{cases}$$

Here the expression $\min\{t', t + D_{srpt} - 1, \max_{\kappa \in T_{rs} \cap [t, t']} \kappa\}$ determines the earliest of three events: end of the decided coverage of treatment p , end of the effective duration of treatment p , end of the coverage requirement for disease r , respectively. Whereas the expression $\max\{\tau, \min_{\kappa \in T_{rs} \cap [t, t']} \kappa\}$ determines

the latest of two events: the rainfall event and the beginning of the coverage requirement for disease r , respectively. Therefore the difference between these two expressions (plus one) gives the total number of days of coverage lost for disease r . We remark that if this quantity is negative, meaning that the rainfall event comes after the end of coverage, then the lost coverage is set to 0.

Let ξ_t represent the volume of rainfall in period t and let Ξ be the associated uncertainty set. We then define for $\xi \in \Xi$, the penalty measure

$$\theta(\xi, \delta) = \sum_{\tau \in T} \sum_{\substack{s \in S, r \in R_s, p \in P_{rs}^{NS} \\ t \in T_{ps} \mid t \in [\tau - D_{srpt} + 1, \tau], t' \in [\tau, t + D_{pst}^{\max} - 1]}} a_{prs}(t, t', \tau) \delta_{sptt'} \varrho(\xi_\tau), \quad (9)$$

where $\varrho(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is a function that associates a certain cost representing the risk of developing a disease to a given volume of rainfall (we do not yet pose any assumptions on the form of $\varrho(\cdot)$). Using the variables $\delta_{sptt'}$, we may then rewrite the robust treatment planning problem as

$$\min \sum_{m \in M} f_m x_m + \sum_{s \in S, p \in P_s} \varsigma_{sp} \sum_{\substack{t \in T_{ps} \\ t' \in [t, t + D_{pst}^{\max} - 1]}} \delta_{sptt'} + \sum_{\omega \in \Omega} \varsigma_\omega \sum_{m \in M_\Omega, p \in P, t \in T} \mu_{mpt}^\omega + \max_{\xi \in \Xi} \theta(\xi, \delta) \quad (10)$$

s.t. (2) – (3)

$$\sum_{p \in P_{rs}, \tau \in T_{ps} \mid \tau \in [t - D_{srp\tau} + 1, t], \tau' \in [t, \tau + D_{ps\tau}^{\max} - 1]} \delta_{spt\tau\tau'} \geq 1 \quad \forall s \in S, r \in R_s, t \in T_{rs} \quad (11)$$

$$\sum_{t' \in [t, t + D_{pst}^{\max} - 1]} \delta_{sptt'} \leq \sum_{\omega \in \Omega_s, m \in M_\omega} \mu_{mpt}^\omega \quad \forall s \in S, p \in P_s, t \in T_{ps} \quad (12)$$

$$\sum_{t \in a, t' \in [t, t + D_{pst}^{\max} - 1]} \delta_{sptt'} \leq \sum_{\zeta \in Z_{sp}(a)} \lambda_s^\zeta \quad \forall s \in S, p \in P_s, a \in \mathcal{A}_{sp} \quad (13)$$

$$\sum_{\substack{t \in [e_k^\zeta, l_k^\zeta] \\ t' \in [t, t + D_{pst}^{\max} - 1]}} \delta_{sptt'} \leq \sum_{\zeta' \in Z_{sp}^\zeta} N_{sp_k^\zeta}^{\zeta'} \lambda_s^{\zeta'} \quad \forall s \in S, \zeta \in Z_s, k = 1, \dots, |\zeta| \quad (14)$$

$$\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\delta} \in \{0, 1\}, \quad (15)$$

where $N_{sp}^\zeta[e, l] = \left| \{k \in \mathcal{K}_{sp}^\zeta \mid [e_k^\zeta, l_k^\zeta] \cap [e, l] \neq \emptyset\} \right|$ is the number of times mixture p can be applied at site s with treatment sequence ζ in the time window $[e, l]$.

Additional constraints (14) are required to impose that a mixture can only be applied as many times as the cardinality of the set \mathcal{K}_{sp}^ζ using sequence $\zeta \in Z_{sp}$. This set of constraints are not necessary for the deterministic model since there is no economic incentive to apply additional treatments. However, in the case of the robust model it is profitable to introduce additional treatments as long as the cost of the treatment is less than the reduction obtained in the penalty measure as a result of its application.

In this case, without the addition of constraints (14), it is possible to apply more treatments than allowed by the selected sequence.

We remark here that, once the model is written in this form, bi-objective optimization methods may also be used, just like in light robustness, to find the most robust solution corresponding to a certain threshold deviation from the nominal cost. Similarly, one may find the minimum cost solution corresponding to a certain threshold deviation from the minimum robustness (value $\theta(\boldsymbol{\xi}, \boldsymbol{\delta})$ obtained without taking the cost into account).

We illustrate next the reason why this approach is less conservative than the naive model presented in Section 3.2, as well as the light robustness approach.

Example 3.1. *Consider the following example where we drop the site and treatment indices, we consider a single disease to treat over 5 periods, with a single treatment mixture of effective duration 3 applied twice over the 5 periods. Assume that the first treatment can only be applied in period 1, whereas the second treatment can be applied anytime in the time window [3,4]. Further assume that the uncertainty set $\Xi = \{\boldsymbol{\xi} \in \{0, 1\}^5 \mid \sum_{i=1}^5 \xi_t \leq 1\}$ is given. The deterministic coverage constraints can be written as:*

$$\delta_{11} + \delta_{12} + \delta_{13} \geq 1 \quad (16)$$

$$\delta_{12} + \delta_{13} \geq 1 \quad (17)$$

$$\delta_{13} + \delta_{33} + \delta_{34} + \delta_{35} \geq 1 \quad (18)$$

$$\delta_{34} + \delta_{35} + \delta_{44} + \delta_{45} + \delta_{46} \geq 1 \quad (19)$$

$$\delta_{35} + \delta_{45} + \delta_{46} \geq 1 \quad (20)$$

where the variables $\delta_{tt'}$ represent a treatment from time period t to t' . When naive robustness is applied to these constraints the problem becomes infeasible as vectors $(1, 0, 0, 0, 0)$, $(0, 0, 1, 0, 0)$ and $(0, 0, 0, 1, 0)$ belong to the uncertainty set Ξ . On the other hand, a light robust version can be obtained by introducing slacks to each constraint with an optimal solution $\mathbf{s} \geq (1, 1, 1, 1, 1)$ when the sum of the slack variables is minimized. Assuming that $\varrho(\xi) = \xi$, the penalty measure approach we propose is expressed as (omitting the cost from the objective function for ease of exposition):

$$\begin{aligned} \min \max_{\boldsymbol{\xi} \in \Xi} & \xi_1 \delta_{11} + 2\xi_1 \delta_{12} + 3\xi_1 \delta_{13} + \xi_2 \delta_{12} + 2\xi_2 \delta_{13} + \xi_3 \delta_{13} + \xi_3 \delta_{33} + 2\xi_3 \delta_{34} + 3\xi_3 \delta_{35} \\ & + \xi_4 \delta_{34} + 2\xi_4 \delta_{35} + \xi_4 \delta_{44} + 2\xi_4 \delta_{45} + 2\xi_4 \delta_{46} + \xi_5 \delta_{35} + \xi_5 \delta_{45} + \xi_5 \delta_{46} \end{aligned} \quad (21)$$

(16) – (20)

We remark that the uncertainty is now concentrated in one row, and becomes therefore row-wise independent. As a result, only one parameter ξ_t for $t = 1, \dots, 5$ can take the value 1. Therefore, a feasible solution with a penalty score of 3 can be obtained by taking $\delta_{13} = 1$ and $\delta_{45} = 1$.

We also highlight the parallels with the heuristic light robustness approach presented in (18). They proposed that the robustness of a solution is associated with the amount of slack in the uncertain rows, and tried to maximize this slack first, then distributing it among the constraints as equivalently as possible. We remark that, the function a we introduce can be seen as a calculation of the total slack that needs to be introduced to constraints to render them feasible again, where we use our knowledge of the problem structure in its closed form calculation without introducing slack variables. In this sense, our methodology poses the question of what makes a solution susceptible rather than what makes a solution robust. We then try to minimize this susceptibility as much as possible also scaling it with a function of the amount of rainfall.

We believe that this type of methodology can be carried over to problems that present a network structure to mitigate conservativeness stemming from the row-wise independence assumption, see for example (4).

3.4.1 Using a linear penalty function

One variant of the above static robust model is to define the function $\varrho(\cdot)$ as a linear function. We assume that

$$\varrho(\xi_t) = 0.05\xi_t \text{ for } t \in T.$$

Here, the number 0.05 is chosen so that exactly $a_{prs}(\tau, \tau', t)$ days of coverage is lost when ξ_t is equal to twenty centimeter cubes. Consider the uncertainty set

$$\Xi^L = \{\boldsymbol{\xi} \in \mathbb{R}^T \mid \xi_t \leq u_t \quad \forall t \in T, \sum_{t \in T_i} \xi_t \leq \Gamma_i \quad \forall i = 1, \dots, K\}. \quad (22)$$

In this setting, u_t represents an upper bound on the amount of rainfall per day, and T_i for $i = 1, \dots, K$ represent a set of time-windows. The parameter Γ_i represents a budget of uncertainty over time-

window T_i for $i = 1, \dots, K$. We may then write the inner maximization problem in (10) as

$$\max_{\xi \in \Xi^L} \sum_{\tau \in T} \sum_{\substack{s \in S, r \in R_s, p \in P_{rs}^{NS} \\ t \in T_{ps} \mid t \in [\tau - D_{srpt} + 1, \tau], t' \in [\tau, t + D_{pst}^{\max} - 1]}} a_{prs}(t, t', \tau) \delta_{sptt'} 0.05 \xi_\tau \quad (23)$$

Here the choice to define $\varrho(\cdot)$ as a linear function is not superfluous, as (23) is a linear program over $\xi \in \Xi^L$, and therefore reformulation through the linear programming dual is possible. We associate the dual variables χ with bound constraints and π with budget constraints of (22) and obtain (10)-(15) equivalently as

$$\begin{aligned} \min \quad & \sum_{m \in M} f_m x_m + \sum_{s \in S, p \in P_s} \varsigma_{sp} \sum_{\substack{t \in T_{ps} \\ t' \in [t, t + D_{pst}^{\max} - 1]}} \delta_{sptt'} + \sum_{\omega \in \Omega} \varsigma_\omega \sum_{m \in M, p \in P, t \in T} \mu_{mpt}^\omega + \sum_{i=1}^K \Gamma_i \chi_i + \sum_{t \in T} \pi_t u_t \\ \text{s.t.} \quad & \sum_{i \in \mathcal{N}_t} \chi_i + \pi_t \geq \sum_{\substack{s \in S, r \in R_s, p \in P_{rs}^{NS} \\ \tau \in T_{ps} \mid \tau \in [t - D_{srp\tau} + 1, t], \tau' \in [t, \tau + D_{pst}^{\max} - 1]}} 0.05 a_{prs}(\tau, \tau', t) \delta_{sp\tau\tau'} \quad \forall t \in T \\ & (2) - (3), (11) - (15). \end{aligned}$$

where $\mathcal{N}_t = \{i = 1, \dots, K \mid t \in T_i\}$ is the set of budget constraints that apply to ξ_t for $t \in T$. This problem is of the same theoretical difficulty as its nominal counterpart. Although $\varrho(\xi_t) = 0.05 \xi_t$ is an oversimplification and overestimation of the true penalty function, it allows for reformulation and solution with off-the-shelf optimization software. One can solve directly the deterministic equivalent formulation, or employ a row-generation approach. However, our preliminary results showed that reformulation was a much more effective solution method for this model. In our numerical tests in Section 5, we therefore solve this model directly using its reformulation.

3.4.2 Using a piecewise linear penalty function

The main shortcoming of the linear penalty measure is that it does not reflect the disproportionate difference between higher volumes of rain that cause wash-outs and lower volumes of rain that are assumed not to affect the coverage of treatments. Hence, it overestimates the true risk, therefore potentially leading to over-conservative decisions. To overcome this drawback, we introduce a piecewise linear function of the volume of rainfall with I pieces defined as follows:

$$\varrho(\xi_t) = \begin{cases} a_i \xi_t + b_i & \text{for } l_i \leq \xi_t < l_{i+1}, i = 1, \dots, I-1 \\ 1 & \text{for } l_I = 20 \leq \xi_t < l_{I+1} = \xi^{\max} \end{cases} \quad (24)$$

with $l_1 = 0$, where ξ^{\max} is the maximum value ξ_t for $t \in T$ can take.

To model this function (and obtain a linear robust equivalent formulation), we represent ξ_t for $t \in T$ as a convex combination of the breakpoints l_1, \dots, l_{I+1} . We introduce the uncertain parameters $\xi_{t,i} \geq 0$ for $t \in T$ and $i = 1, \dots, I+1$, and write $\xi_t = \sum_{i=1}^{I+1} l_i \xi_{t,i}$ and $\varrho(\xi_t) = \sum_{i=1}^{I+1} \varrho(l_i) \xi_{t,i}$ such that $\sum_{i=1}^{I+1} \xi_{t,i} = 1$. In order for the function $\varrho(\xi_t)$ to be evaluated correctly, we need to impose additional constraints to ensure that at most two variables $\xi_{t,i}, \xi_{t,i+1}$ are positive. To this end, we introduce the binary uncertain parameters $w_{t,i}$ for $t \in T$ and $i = 1, \dots, I+1$, and construct the uncertainty set

$$\Xi^{\text{PW},I} = \left\{ \xi \in \mathbb{R}^{T \times I+1}, \mathbf{w} \in \mathbb{Z}^{T \times I+1} \left| \begin{array}{ll} \sum_{i=1}^{I+1} l_i \xi_{t,i} \leq u_t & \forall t \in T \\ \sum_{t \in T_k} \sum_{i=1}^{I+1} l_i \xi_{t,i} \leq \Gamma_k & \forall k = 1, \dots, K \\ \xi_{t,i} \leq w_{t,i} & \forall t \in T, i = 1, \dots, I+1 \\ \sum_{i=1}^{I+1} w_{t,i} \leq 2 & \forall t \in T \\ w_{t,i} + w_{t,j} \leq 1 & \forall t \in T, i = 1, \dots, I \\ & j = i+2, \dots, I+1 \end{array} \right. \right\}.$$

The set of constraints that are imposed on the binary uncertain variables w are known as SOS2 constraints which can be efficiently handled by many commercial mixed integer programming solvers. As $\Xi^{\text{PW},I}$ involves binary parameters, we resort to a row-generation approach to solve (10), writing it in the form

$$\min \sum_{m \in M} f_m x_m + \sum_{s \in S, p \in P_s} \varsigma_{sp} \sum_{t \in T_{ps}, t' \in [t, t + D_{pst}^{\max} - 1]} \delta_{sptt'} + \sum_{\omega \in \Omega} \varsigma_{\omega} \sum_{m \in M_{\omega}, p \in P, t \in T} \mu_{mpt}^{\omega} + \Theta \quad (25)$$

$$\text{s.t. } \Theta \geq \theta((\xi, \mathbf{w})^i, \delta) \quad \forall i = 1, \dots, k \quad (26)$$

$$(2) - (3), (11) - (15),$$

where cuts (26) are generated by solving the mixed integer linear separation problem

$$\max_{(\xi, \mathbf{w}) \in \Xi^{\text{PW},I}} \sum_{t \in T} \sum_{\substack{s \in S, r \in R_s, p \in P_{rs}^{NS} \\ \tau \in T_{ps} \mid \tau \in [t - D_{srp\tau} + 1, t], \tau' \in [t, \tau + D_{ps\tau}^{\max} - 1]}} a_{prs}(\tau, \tau', t) \delta_{ps\tau\tau'} \sum_{i=1}^5 \varrho(l_i) \xi_{t,i}.$$

Thus, the piecewise linear representation of the penalty function comes at the additional cost of introducing more variables in the uncertainty set, including the binary variables to model the SOS2 constraints. These binary variables in turn necessitate using a cut-generation approach (as a deter-

ministic equivalent reformulation through the linear programming dual is not possible) which induces an increase in the solution time and slows down the convergence to an optimal solution. In our computational section we evaluate whether or not this additional computational burden is justified by comparing the performance and cost of solutions obtained with this model to those of the linear penalty function model.

4 Improving the robustness of treatment sequences

In this section, we consider enlarging the input set of treatment sequences to generate further sequences that take the robustness objective of the problem into account. This process is implemented in our model by considering the option of exchanging each non-systemic treatment in the sequence with a “compatible” treatment that is preferable in terms of robustness (akin to local search heuristics). This new treatment can be either systemic or non-systemic.

To this end, we introduce the set $P_k^{\zeta,s}$, the set of compatible mixtures that can replace the k^{th} treatment p_k^ζ of sequence $\zeta \in Z_s$. Let k' and k'' be the indices preceding and succeeding index k in ζ , respectively. We assume that the replacing treatment p will be applied within the same time window as p_k^ζ , that is $[e_k^\zeta, l_k^\zeta]$, so as not to disturb the precedence relations satisfied by sequence $\zeta \in Z_s$. Thus, $p \in P_k^{\zeta,s}$ if and only if:

- (i) $p \in Succ_{p_{k'}}^\zeta$ and $p_{k''}^\zeta \in Succ_p$, *i.e.*, mixture p satisfies the precedence relationships with $p_{k'}^\zeta$ and $p_{k''}^\zeta$,
- (ii) $\text{tox}_{sp} - \text{tox}_{sp_k^\zeta} \leq \text{tol}_s - \text{tox}_\zeta$, *i.e.*, the level of toxicity induced by replacing p_k^ζ with p does not violate the toxicity tolerance constraint of site s ,
- (iii) $R_p = R_{p_k^\zeta}$, *i.e.*, p treats all of the requests that p_k^ζ treats,
- (iv) $D_{srpt} \geq D_{srp_k^\zeta t}$ for $r \in R_s$ and $t \in [e_k^\zeta, l_k^\zeta]$, *i.e.*, the activity duration of p is long enough to allow for a feasible treatment solution using the current application time windows,
- (v) $T_{ps} \cap [e_k^\zeta, l_k^\zeta] \neq \emptyset$, *i.e.*, p can be applied at site s during the time window $[e_k^\zeta, l_k^\zeta]$.

We then introduce binary decision variables y_{skp}^ζ for $s \in S, \zeta \in Z_s, k = 1, \dots, |\zeta|$ and $p \in P_k^{\zeta,s}$ to represent the decision of replacing treatment mixture p_k^ζ in sequence $\zeta \in Z_s$ with treatment mixture $p \in P_k^{\zeta,s}$. The following set of constraints are added to the model:

$$\sum_{p \in P_k^{\zeta,s}} y_{skp}^\zeta \leq \lambda_s^\zeta \quad \forall s \in S, \zeta \in Z_s, k = 1, \dots, |\zeta| \quad (27)$$

$$\sum_{k=1, \dots, |\zeta|, p \in P_k^{\zeta, s}} (\text{tox}_{sp} - \text{tox}_{sp_k^\zeta}) y_{skp}^\zeta \leq (\text{tol}_s - \text{tox}_{|\zeta|}) \lambda_s^\zeta \quad \forall s \in S, \zeta \in Z_s \quad (28)$$

$$y_{skp}^\zeta + y_{sk'p'}^\zeta \leq \lambda_s^\zeta \quad \forall p, p' \mid p \in P_k^{\zeta, s}, p' \in P_{k'}^{\zeta, s}, p' \notin \text{Succ}_p \quad (29)$$

Constraints (27) impose that we can replace a treatment in sequence $\zeta \in Z_s$ only if the sequence is selected to treat site s , while limiting the choice of replacement treatment to one of $p \in P_k^{\zeta, s}$. Constraints (28) make sure that the toxicity introduced by new treatments does not exceed the toxicity tolerance on site s after accounting for the toxicity of removed treatments. Finally, constraints (29) ensure that the added treatment mixtures satisfy the precedence constraints.

We next generalize constraints (13) and (14) in order to incorporate the mixture exchange decisions. To do so, we first introduce relevant notation, we repeat some of the notation from Section 2.3 for convenience.

For $p \in P_s$, let \bar{Z}_{sp} be the set of all sequences of site s such that $p \in P_k^{\zeta, s}$ for some $k = 1, \dots, |\zeta|$. Further, for $\zeta \in \bar{Z}_{sp}$, let $\bar{\mathcal{K}}_{sp}^\zeta$ be the set of all indices such that $p \in P_k^{\zeta, s}$ for $k \in \bar{\mathcal{K}}_{sp}^\zeta$. For $p \in P_s$, we first sort the combined list of e_k^ζ and $(l_k^\zeta + 1)$ values for $\zeta \in Z_{sp} \cup \bar{Z}_{sp}$, and $k \in \mathcal{K}_{sp}^\zeta \cup \bar{\mathcal{K}}_{sp}^\zeta$ in increasing order, let this list be denoted by $\{d_1, \dots, d_n\}$. We then create the time windows $a_i = [d_i, d_{i+1} - 1]$ for $i = 1, \dots, n - 1$. Let \mathcal{A}_{sp} denote the collection of application time windows of mixture p at site s over all treatment sequences $\zeta \in Z_{sp} \cup \bar{Z}_{sp}$ constructed in this manner, *i.e.*, $\mathcal{A}_{sp} = \bigcup_{i=1}^{n-1} a_i$. Moreover, for $a \in \mathcal{A}_{sp}$, let $Z_{sp}(a)$ be the set of sequences that allow for the application of mixture p during time window a and let $\bar{Z}_{sp}(a)$ be the set of pairs (ζ, k) that allow for the application of mixture p during time window a when mixture p is substituted for p_k^ζ , *i.e.*, $a \subseteq [e_k^\zeta, l_k^\zeta]$. We remark that each of these sets maybe empty for a given time window a . Finally, we let $C_{[e, l]} = \{(\zeta, k) \mid [e_k^\zeta, l_k^\zeta] \cap [e, l] \neq \emptyset\}$, and write

$$\sum_{\substack{t \in a \\ t' \in [t, t + D_{pst}^{\max} - 1]}} \delta_{sp tt'} \leq \sum_{\zeta \in Z_{sp}(a)} \lambda_s^\zeta - \sum_{k \in \mathcal{K}_{sp}^\zeta \mid a \subseteq [e_k^\zeta, l_k^\zeta], p' \in P_k^{\zeta, s}} y_{skp'}^\zeta + \sum_{(\zeta, k) \in \bar{Z}_{sp}(a)} y_{skp}^\zeta \quad \forall s \in S, p \in P_s, a \in \mathcal{A}_{sp} \quad (30)$$

$$\begin{aligned} \sum_{\substack{t \in [e_k^\zeta, l_k^\zeta] \\ t' \in [t, t + D_{pst}^{\max} - 1]}} \delta_{sp_k^\zeta tt'} &\leq \sum_{\zeta' \in Z_{sp_k^\zeta}} N_{sp_k^\zeta [e_k^\zeta, l_k^\zeta]}^{\zeta'} \lambda_s^{\zeta'} - \sum_{(\zeta', k') \in C_{[e_k^\zeta, l_k^\zeta]} \mid \zeta' \in Z_{sp_k^\zeta}, p \in P_{k'}^{\zeta', s}} y_{sk'p}^{\zeta'} \\ &+ \sum_{(\zeta', k') \in C_{[e_k^\zeta, l_k^\zeta]} \mid \zeta' \in \bar{Z}_{sp_k^\zeta}, p_k^\zeta \in P_{k'}^{\zeta', s}} y_{sk'p_k^\zeta}^{\zeta'} \quad \forall s \in S, \zeta \in Z_s, k = 1, \dots, |\zeta| \end{aligned} \quad (31)$$

We write the robust sequence generation model as:

$$\begin{aligned}
\min \quad & \sum_{m \in M} f_m x_m + \sum_{s \in S, p \in P_s} \varsigma_{sp} \sum_{t \in T_{ps}, t' \in [t, t + D_{pst}^{\max} - 1]} \delta_{sptt'} + \sum_{\omega \in \Omega} \varsigma_{\omega} \sum_{m \in M_{\omega}, p \in P, t \in T} \mu_{mpt}^{\omega} + \max_{\xi \in \Xi} \theta(\xi, \delta) \\
\text{s.t.} \quad & (2) - (3), (11) - (12), (27) - (31) \\
& \mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\delta} \in \{0, 1\}.
\end{aligned}$$

This model can be used with both of the penalty measures presented in Section 3.4.1 and Section 3.4.2. In the next section, we do so to compare the solutions obtained using pregenerated sequences with those obtained using the robust sequence improvement model.

5 Computational Results

In this section, we present our computational experience on realistically-sized instances, which correspond to a typical vineyard in Bordeaux area. We are interested in comparing the time required to solve each model, the cost of operations using each model, and the effectiveness of the two different penalty measures introduced in Section 3.4.1 and Section 3.4.2, as well as comparing the performance with pre-generated treatment sequences to that of the treatment sequence improvement model of Section 4. In particular, we would like to evaluate whether the more sophisticated calculation of the penalty measure and treatment sequences leads to more effective decision-making.

In our computations, we use the uncertainty set Ξ^L for the robust model of Section 3.4.1, and the piecewise linear function

$$\varrho(\xi_t) = \begin{cases} 0 & \text{for } l_1 = 0 \leq \xi_t < l_2 = 10 \\ 0.03\xi_t - 0.3 & \text{for } l_2 = 10 \leq \xi_t < l_3 = 15 \\ 0.17\xi_t - 2.4 & \text{for } l_3 = 15 \leq \xi_t < l_4 = 20 \\ 1 & \text{for } l_4 = 20 \leq \xi_t \leq l_5 = \xi^{\max}, \end{cases} \quad (32)$$

with the uncertainty set $\Xi^{\text{PW},4}$ for the robust model of Section 3.4.2. We let $T_i = \{7(i-1)+1, \dots, 14+7(i-1)\}$, therefore each T_i represents a two-week time window, with T_i and T_{i+1} overlapping over a week. In order to determine the parameters u and Γ of the uncertainty set, we utilize part of the historical data available for the past ten years for Bordeaux, France ([Met](#)). Let $H = \{1, \dots, 10\}$ and $H' \subset H$ denote the set of years selected to construct the uncertainty set. Further let ξ_t^h denote the

amount of rainfall in period $t \in T$ in year $h \in H'$, and define $\xi_t^{\max} = \max_{h \in H'} \xi_t^h$ for $t \in T$. In constructing the uncertainty set, we let $u_t = \max_{t' \in [t-2, t+2]} \xi_{t'}^{\max}$, estimating that the weather conditions on a given day of the year can be expected to be similar within a five-day interval over different years. Further, we let $\Gamma_i = \max_{t' \in [7(i-2), 28+7(i-2)]} \xi_{t'}^{\max}$, estimating that the weather conditions over two given weeks of the year can be expected to be similar within a four-week interval over different years.

We let

$$\text{cost} = \sum_{m \in M} f_m x_m + \sum_{s \in S, p \in P_s} \varsigma_{sp} \sum_{t \in T_{ps}, t' \in [t, t+D_{pst}^{\max}-1]} \delta_{sptt'} + \sum_{\omega \in \Omega} \varsigma_{\omega} \sum_{m \in M_{\omega}, p \in P, t \in T} \mu_{mpt}^{\omega}$$

and

$$\text{penalty measure} = \max_{\xi \in \Xi} \theta(\xi, \delta)$$

for each model and introduce parameter α to balance these two components of the objective function. We test each robust model with the objective function $\alpha \times \text{cost} + (1 - \alpha) \times \text{penalty measure}$ where $\alpha \in \{0.1, 0.05, 0.01\}$.

We use three instances for different characteristics to present our results. For each of these instances number of requests is 5, number of machines is 6, number of mixture sequences is 32, and the number of periods is 152. Instance dependent characteristics, number of sites, average number of treatment sequences generated per size, and number of site clusters, are summarized in Table 1. These instances are inspired by a collaboration with an agricultural partner.

	Instance 1	Instance 2	Instance 3
$ S $	21	23	10
$ Z_s^{\text{avg}} $	7	11	15
$ \Omega $	7	10	3

Table 1: Instance summary

All our implementations are done using the Julia (16) language along with the mathematical programming package JuMP (17). All our experiments are conducted using a 2 Dodeca-core Haswell Intel Xeon E5-2680 v3 2.5 GHz machine with 128Go RAM running Linux OS. The resources of this machine are strictly partitioned using Slurm Workload Manager¹ to run several tests in parallel. The resources available for each run (algorithm-instance) are set to two threads and a 20 Go RAM limit. This virtually creates 12 independent machines, each running one single instance at a time.

In our computations, we use the CPLEX solver in its default settings, imposing a four hour time limit

¹<https://slurm.schedmd.com/> (accessed June 2019)

(14400 seconds) and a 0.5% optimality gap. We implement cutting plane algorithms with the lazy cut callback functionalities of the CPLEX solver, therefore searching for violated cuts at each node where an integer feasible solution is found. In comparing different methods, we use $H \setminus H'$ (set of available realizations that have not been used to construct the uncertainty set) to evaluate the performance of solutions. As such, we report our results as a summary of 100 replications, each time constructing the uncertainty set based on five years of available data and then evaluating the resulting solution with each of the realizations of the five years that remain.

5.1 Comparing deterministic and robust approaches

In this section, we compare the deterministic optimal solution (D) obtained by solving the extended model of Section 3 to those obtained by solving the linear robust model (L) of Section 3.4.1 and the piecewise linear robust model (PW) of Section 3.4.2 using the three instances of different characteristics introduced in the previous section.

In Table 2, we present the solution time, the percentage optimality gap, the cost component of the objective function, and the penalty measure component of the objective function for each model, each instance and each value of α . For each of these parameters we report the mean, median, and standard deviation over one hundred replications.

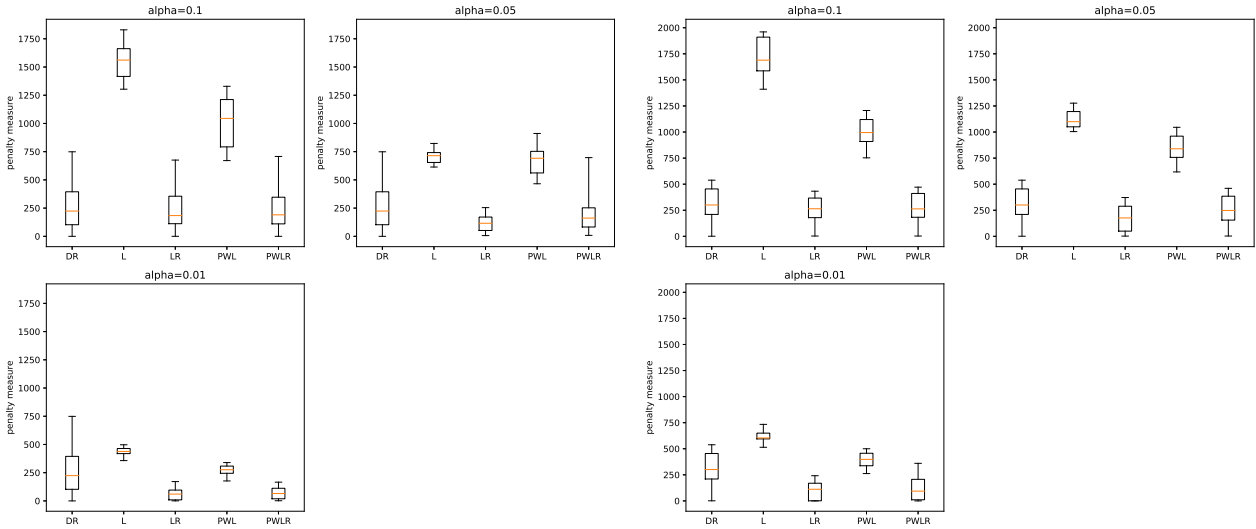
Results presented under the header “Solution Time” clearly indicate the additional computational burden for the piecewise linear penalty robust model both compared to its deterministic counterpart and the reformulation solution of the linear robust model. We remark also that all instances are solved within seconds using the deterministic model, whereas both robust models may reach the time limit for some instances. Results presented under the header “Cost” confirm our observation from Section 3.4.2 that the linear robust model will tend to be more conservative. The cost of treatment decisions using this model is higher on average and can be as much as %25 more expensive compared to the piecewise linear model. The results presented under the header “Measure” give further insight to this comparison, we observe that the penalty measure for the linear robust model can be twice as large as its piecewise linear counterpart. As such, the linear robust model justifies further investment in expensive systemic treatments in order to minimize the overestimated penalty measure.

We present, in Figure 1, an analysis of the performance of solutions obtained by each model using the historical realizations of data. We plot the penalty measure of both robust models along with what we will refer to as the “realized measure” for both the deterministic and robust models. This is computed as $\sum_{t \in T} \sum_{s \in S, r \in R_s, p \in P_{rs}^{NS}, \tau \in T_{ps} \mid \tau \in [t - D_{srp\tau} + 1, t], \tau' \in [t, \tau + D_{ps\tau}^{\max} - 1]} \varrho(\xi_t^h) a_{prs}(\tau, \tau', t) \bar{\delta}_{spt\tau'}$ for year

		Solution Time			Gap		Cost			Measure	
Instance 1		D	L	PW	L	PW	D	L	PW	L	PW
$\alpha = 0.1$	mean	1.1	5.2	109.3	0.22	0.40	27647	27962	27857	1551	1014
	med	1.1	4.6	91.0	0.20	0.41	27647	27960	27938	1561	1045
	std	0.0	2.5	72.8	0.13	0.09	0	29	114	178	235
$\alpha = 0.05$	mean	1.1	19.2	313.5	0.48	0.48	27647	40449	32451	712	675
	med	1.1	15.9	251.0	0.49	0.49	27647	40829	34774	715	692
	std	0.0	11.9	259.4	0.04	0.03	0	2621	4700	84	159
$\alpha = 0.01$	mean	1.1	2.5	331.3	0.19	0.46	27647	48427	46368	435	272
	med	1.1	2.5	134.9	0.14	0.48	27647	48331	47714	438	276
	std	0.0	0.6	585.5	0.16	0.08	0	189	2445	48	55
Instance 2		D	L	PW	L	PW	D	L	PW	L	PW
$\alpha = 0.1$	mean	19.6	127.1	956.1	0.29	0.32	39638	40758	40018	1714	985
	med	19.6	121.6	582.8	0.37	0.39	39638	40847	40070	1689	995
	std	0.7	77.9	1311.8	0.18	0.17	0	368	186	200	160
$\alpha = 0.05$	mean	19.6	947.3	4579.6	0.50	0.74	39638	49215	41982	1121	841
	med	19.6	996.7	3462.4	0.50	0.49	39638	50394	41235	1099	839
	std	0.7	490.4	3904.1	0.00	1.21	0	3417	1431	99	141
$\alpha = 0.01$	mean	19.6	4732.0	13205.8	0.51	3.96	39638	65878	58669	616	399
	med	19.6	2033.1	14400.1	0.50	2.77	39638	65930	58239	604	398
	std	0.7	5428.4	2697.8	0.04	3.71	0	270	3869	78	86
Instance 3		D	L	PW	L	PW	D	L	PW	L	PW
$\alpha = 0.1$	mean	2.0	1783.9	8116.4	0.50	0.61	27602	28101	27997	520	322
	med	1.9	1157.4	8425.5	0.50	0.50	27602	28099	27996	529	340
	std	0.2	2172.5	6240.3	0.00	0.19	0	31	48	57	68
$\alpha = 0.05$	mean	2.0	4979.9	10455.1	0.53	0.86	27602	28210	28198	511	307
	med	1.9	2117.1	14400.0	0.50	0.76	27602	28204	28088	519	283
	std	0.2	5384.0	5628.4	0.06	0.38	0	35	244	57	67
$\alpha = 0.01$	mean	2.0	7038.9	8609.2	0.54	1.99	27602	34120	33635	409	193
	med	1.9	6467.4	14400.0	0.50	1.13	27602	33909	33907	428	182
	std	0.2	5186.7	6579.0	0.10	1.80	0	1934	2953	76	62

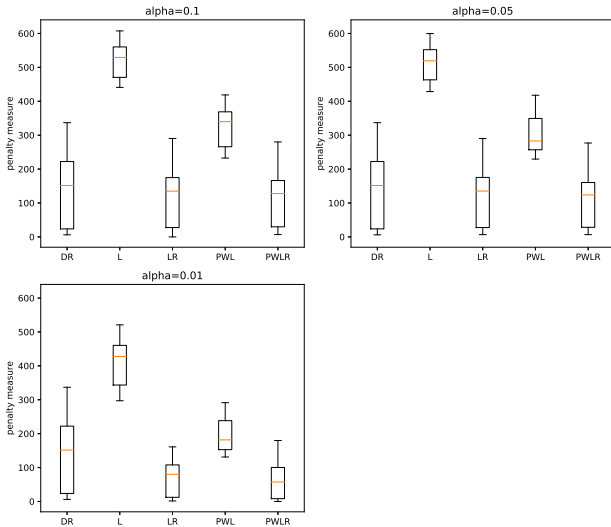
Table 2: Comparison of the deterministic, linear robust and piecewise linear robust models with a priori generation of sequences.

$h \in H \setminus H'$ where $\bar{\delta}$ is the optimal treatment solution of the respective model. The realized measure is the computation of the penalty observed in year $h \in H \setminus H'$ assuming that the cost function $\rho(w)$ takes the form given in (32). In Figure 1, we present each measure using a box-and-whiskers plot. On each box, the central mark is the median, the edges of the box are the 25th and 75th percentiles, the whiskers extend to the 5th and 95th percentiles.



(a) Penalty measure analysis for Instance 1.

(b) Penalty measure analysis for Instance 2.



(c) Penalty measure analysis for Instance 3.

Figure 1: Penalty measure analysis for deterministic, linear and piecewise linear robust models with a priori generation of sequences. Values (L) and (PWL) correspond, respectively, to the risk measure obtained from the objective function of the linear and piecewise linear robust models. Values (DR), (LR) and (PWLR) correspond to the simulated penalty measure obtained, respectively, using the solutions of deterministic, linear robust and piecewise linear robust models.

Results presented in Figure 1 clearly show the improvement in the realized penalty measure compared to the deterministic model (for parameter settings $\alpha = 0.05$ and $\alpha = 0.01$). They further provide

confirmation that the linear robust model overestimates the true penalty measure. This is evident from the significant gap between the distribution of the measure coming from the objective function of this model versus its realized measure. Whereas when comparing the piecewise linear robust measure to its realized counterpart we see that the mean value of the measure coming from the objective function is close to the worst-case realized measure. We also remark that although solutions provided by the piecewise linear robust model are less costly they can perform as well as the more expensive solutions provided by the linear robust model. We conclude that a sophisticated estimation of the penalty measure provides high quality solutions with reduced investment therefore justifying the more significant solution time for the piecewise linear robust model.

5.2 Comparing treatment sequences

We next present the results obtained using the sequence improvement model of Section 4 with both the linear and piecewise linear penalty measures. In Table 3, we present the results for the linear robust model (L) and the piecewise linear robust model (PW) using the same three instances as in the previous section. Results presented under the headers “Solution Time” and “Gap” clearly indicate that the piecewise linear model becomes more and more difficult to solve in this case with the optimality gap becoming as high as 29% on average for Instance 2. On the other hand, the linear penalty approach allows us to solve most instances to optimality while its average optimality gap is below 3%. As in the previous section, the cost difference between the two models remains significant while the difference between the penalty measures is less pronounced compared to those obtained using pregenerated sequences. We remark however that there is bias coming from the suboptimality of the piecewise linear model.

We conclude that although the piecewise linear model with sequence improvement may be the most realistic representation of the problem our capacity to solve it to optimality is limited. In this case the linear model with the sequence improvement turns out to be a good alternative to obtain robust solutions in an acceptable amount of solution time.

5.3 Measuring the price of robustness

We present in Table 4, the mean percentage increase in cost and the mean percentage decrease in the realized penalty measure compared to the deterministic model for both the linear and piecewise linear robust models with a priori generation of sequences versus the sequence improvement model. Our results show the importance of considering the selection of treatments within the optimization

		Solution Time		Gap		Cost		Penalty	
Instance 1		L	PW	L	PW	L	PW	L	PW
$\alpha = 0.1$	mean	44.8	554.7	0.45	0.46	30730	28674	1119	853
	med	29.4	381.6	0.49	0.49	30759	28318	1128	838
	std	46.9	787.7	0.07	0.06	920	829	100	166
$\alpha = 0.05$	mean	85.0	2354.5	0.49	0.48	35907	33654	747	497
	med	59.5	1827.5	0.50	0.50	35878	31473	727	471
	std	57.3	1963.8	0.02	0.03	1603	3836	116	110
$\alpha = 0.01$	mean	13.9	2495.6	0.49	0.50	51053	45089	226	148
	med	17.7	4358.0	0.05	0.02	923	2584	29	22
	std	686.8	7375.7	0.39	2.62	41821	41430	1570	940
Instance 2		L	PW	L	PW	L	PW	L	PW
$\alpha = 0.1$	mean	686.8	7375.7	0.39	2.62	41821	41430	1570	940
	med	725.1	6687.5	0.47	0.48	41646	40227	1580	966
	std	327.0	5108.2	0.16	6.30	1126	3569	153	167
$\alpha = 0.05$	mean	11853.8	13873.7	1.22	10.22	51651	44706	820	711
	med	14404.9	14400.1	0.89	10.65	51432	42746	815	711
	std	3980.4	1978.5	0.92	6.28	3054	5478	90	259
$\alpha = 0.01$	mean	14355.7	14400.3	2.30	29.17	69340	58600	274	365
	med	14411.0	14400.1	1.71	27.39	69490	59602	284	339
	std	389.9	1.0	1.75	9.00	638	9716	33	214
Instance 3		L	PW	L	PW	L	PW	L	PW
$\alpha = 0.1$	mean	14413.1	13989.1	1.08	1.25	28532	28139	456	305
	med	14414.6	14400.1	1.03	1.22	28500	28141	462	323
	std	10.5	1735.6	0.33	0.50	236	121	57	62
$\alpha = 0.05$	mean	14415.7	14250.1	1.45	2.90	30612	29167	298	237
	med	14416.2	14400.1	1.25	2.91	30695	29128	296	238
	std	6.3	1043.6	0.56	1.23	375	491	55	54
$\alpha = 0.01$	mean	12092.4	13050.7	2.58	5.48	36683	34947	153	88
	med	14410.0	14400.1	1.18	3.73	36754	33991	149	78
	std	4292.5	4001.2	3.40	5.22	960	2230	24	40

Table 3: Comparison of the linear robust and piecewise linear robust sequence improvement models.

framework when robustness is a concern. The sequence improvement model is able to reduce the penalty measure significantly while extending the same amount of investment as the a priori selection of treatments. Consider Instance 2 with $\alpha = 0.05$, here the percentage increase from the nominal cost is %24.16 for a priori selection, and %30.31 for the sequence improvement model whereas the reduction in the realized penalty measure is %39.53 percent for the former and %60.70 for the latter.

	% Cost increase				% Penalty decrease			
	A priori		Seq Imp		A priori		Seq Imp	
Instance 1	L	PW	L	PW	L	PW	L	PW
$\alpha = 0.1$	1.14	0.76	11.15	3.72	13.58	12.05	34.28	19.84
$\alpha = 0.05$	46.31	17.38	29.88	21.73	59.79	27.57	60.24	40.08
$\alpha = 0.01$	75.16	67.72	84.14	64.28	78.54	74.99	87.86	76.91
Instance 2	L	PW	L	PW	L	PW	L	PW
$\alpha = 0.1$	2.82	0.96	5.51	4.52	14.52	10.38	16.02	13.57
$\alpha = 0.05$	24.16	5.91	30.31	12.79	39.53	16.58	60.70	31.20
$\alpha = 0.01$	66.20	48.01	74.93	47.84	68.55	59.50	85.56	61.14
Instance 3	L	PW	L	PW	L	PW	L	PW
$\alpha = 0.1$	1.81	1.43	3.37	1.95	18.19	20.21	27.66	26.01
$\alpha = 0.05$	2.21	2.16	10.91	5.67	18.49	22.61	55.80	38.57
$\alpha = 0.01$	23.62	21.86	32.90	26.61	50.41	55.83	86.43	74.21

Table 4: Comparison of the deterministic, linear robust and piecewise linear robust models.

6 Conclusions

In this paper, we present an application of robust optimization to the strategic planning of phytosanitary treatments in agriculture. We discuss how to model the uncertainty as well as how to incorporate it into a meaningful robust model. We show that a naive application of robust optimization is too conservative as the uncertainty does not have a row-wise independent structure. We therefore propose an approach to handle robustness without being overly conservative using a penalty measure approach. We propose to use this approach with different penalty functions that require potentially adapting the solution method. We provide numerical insights in our computational section with realistic instances.

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