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Segre classes of tautological bundles on Hilbert schemes of surfaces

Claire Voisin*

Abstract

We first give an alternative proof, based on a simple geometric argument, of a result of Marian, Oprea and Pandharipande on top Segre classes of the tautological bundles on Hilbert schemes of $K3$ surfaces equipped with a line bundle. We then turn to the blow-up of $K3$ surface at one point and establish vanishing results for the corresponding top Segre classes in a certain range. This determines, at least theoretically, all top Segre classes of tautological bundles for any pair (Σ, H), $H \in \operatorname{Pic}\Sigma$.

Classification. 14N10 (primary), 14J99 (secondary).
Keywords: Punctual Hilbert scheme, Segre classes, tautological bundles.

1 Introduction

Let S be a smooth projective (or compact complex) surface. The Hilbert scheme $S^{[k]}$ is smooth projective (or compact complex) of dimension $2k$. For any line bundle H on S, we get an associated vector bundle $\mathcal{H}_{[k]}$ on $S^{[k]}$, whose fiber at a point $[Z] \in S^{[k]}$ is the vector space $H^0(H_{|Z})$. If S is a $K3$ surface and $c_1(H)^2 = 2g-2$, we denote

$$s_{k,g} := \int_{S^{[k]}} s_{2k}(\mathcal{H}_{[k]}).$$

This is indeed a number which depends only on k and g (see Theorem 2). The following result is proved in [4]:

Theorem 1. *One has* $s_{k,g} = 2^k\binom{g-2k+1}{k}$.

Here the binomial coefficient is defined for $k \geq 0$. It is always 1 for $k = 0$ and the formula for $\binom{n}{k}$ for any n is

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}.$$

In particular, we have $\binom{n}{k} = 0$ if $n \geq 0$ and $n < k$. The theorem above thus gives in particular the vanishing

$$s_{k,g} = 0 \text{ when } g-2k+1 \geq 0 \text{ and } k > g-2k+1. \tag{1}$$

The proof of this vanishing statement in [4] is rather involved and we are going to give in Section 2 a direct geometric proof of (1), based on a small improvement of Lazarsfeld's arguments in [2].

We will then show how the vanishing (1), even only in the smaller range $g = 2k-1$, $g = 2k$, implies Theorem 1. We simply use for this the following result which is due to Tikhomirov [7] (see also Ellingsrud-Göttsche-Lehn [1] and Lehn [3] for related statements) :

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Theorem 2. *The Segre numbers $\int_{S^{[k]}} s_{2k}(H_{[k]})$ for a projective surface S equipped with a line bundle H depend only on the four numbers*

$$\pi = H \cdot K_S,\ d = H^2,\ \kappa = K_S^2,\ e = c_2(S).$$

We will denote these Segre numbers $s_{k,d,\pi,\kappa,e}$. It follows from Theorem 2 that the numbers $s_{k,g}$ can be computed as well by considering a surface Σ which is the disjoint union of a $K3$ surface S', equiped with a line bundle H' of self intersection $2(g-1)-2$, and an abelian surface A equiped with a line bundle θ with $\theta^2 = 2$. We will show in Section 3 that the formula obtained by this observation (this is a particular case of (3) below), combined with the vanishing result (1), uniquely determine the numbers $b_k := \int_{A^{[k]}} s_{2k}(\theta_{[2k]})$ and finally the numbers $s_{k,g}$ for all k, g, knowing that $s_{1,g} = 2g-2$, $b_0 = 1$, $b_1 = 2$.

In Section 2, we will establish similar vanishing results for a $K3$ surface S blown-up at one point. Let $\widetilde{S}$ be such a surface and let $H = \tau^* L(-lE)$ with $2g-2 = L^2$, where L generates $\operatorname{Pic} S$.

Theorem 3. *For $k \geq 2$, one has the following vanishing for the Segre numbers $\tilde{s}_{k,g,l} := \int_{\widetilde{S}^{[k]}} s_{2k}(\mathcal{H}_{[k]})$:*

$$\tilde{s}_{k,g,l} = 0 \text{ for } k = l,\ l+1 \text{ and } g - \frac{l(l+1)}{2} = 3k-2. \tag{2}$$

We will also prove that these vanishing statements together with Theorem 1 determine all Segre numbers $s_k(d,\pi,\kappa,e)$. We use for this the following complement to Theorem 2, (see [3], [1],) obtained by observing that the Hilbert scheme $S^{[k]}$ of a disjoint union $S_1 \sqcup S_2$ is the disjoint union for $l = 0,\dots,k$, of $S_1^{[l]} \times S_2^{[k-l]}$, while all the data d, π, κ, c_2 for the pairs (Σ, H) are additive under disjoint unions $(S,L) = (S_1,L_1) \sqcup (S_2,L_2)$:

Lemma 4. *With the notation $s_{d,\pi,\kappa,e}(z) = \sum_k s_{k,d,\pi,\kappa,e} z^k$*

$$s_{d,\pi,\kappa,e}(z) = s_{d_1,\pi_1,\kappa_1,e_1}(z) s_{d_2,\pi_2,\kappa_2,e_2}(z) \tag{3}$$

with $d = d_1 + d_2$, $\pi = \pi_1 + \pi_2$ etc.

To conclude this introduction, we mention Lehn's conjecture [3, Conjecture 4.9]:

Conjecture 5. *One has*

$$s_{d,\pi,\kappa,e}(z) = \frac{(1-w)^a (1-2w)^b}{(1-6w+6w^2)^c}, \tag{4}$$

where $a = \pi - 2\kappa$, $b = d - 2\pi + \kappa + 3\chi$, $c = \frac{d-\pi}{2} + \chi$, $\chi = \frac{\kappa+e}{12}$ and the variable w is related to z by

$$z = \frac{w(1-w)(1-2w)^4}{(1-6w+6w^2)^3}.$$

This conjecture is proved in [4] for $K3$ and more generally K-trivial surfaces, that is for $\kappa = \pi = 0$. Although we were not able to prove it in general, our results imply the following:

Corollary 6. *Lehn's conjecture is equivalent to the fact that the development in power series of z of the Lehn function $f_{d,\pi,\kappa,e}(z)$ defined as the right hand side in (4) has vanishing Taylor coefficient of order k for $e = 25$, $\kappa = -1$ and $(d,\pi) = (7(k-1), k-1)$ or $(d,\pi) = (7(k-1)+1, k)$*

Shortly after this paper was written, Marian-Oprea-Pandharipande (see [5]) and Szenes-Vergne independently were able to check that the Lehn function satisfies the vanishing properties stated in Corollary 6, thus completing the proof of Lehn's conjecture.

Let us mention the following intriguing question: Lehn's conjecture (now a theorem) singles out the class of pairs (S, H) with the following numerical properties:

$$H^2 = 0,\ H \cdot K_S = 2K^2 = 2\chi(\mathcal{O}_S). \tag{5}$$

These conditions are indeed equivalent to the vanishing of the exponents a, b and c above, so that for these pairs, one has the vanishing $s_{2k}(\mathcal{H}_{[k]}) = 0$. It would be nice to have a geometric proof of this.

Thanks. I thank Rahul Pandharipande for discussions and in particular for suggesting, after I had given a geometric proof of the vanishings (1) on $K3$ surfaces, to look at surfaces other than $K3$'s. This work has been done during my stay at ETH-ITS. I acknowledge the support of Dr. Max Rössler, the Walter Haefner Foundation and the ETH Zurich Foundation.

2 Geometric vanishing

Let S be a $K3$ surface with $\operatorname{Pic} S = \mathbb{Z}H$, where H is an ample line bundle of self-intersection $2g-2$. We give in this section a geometric proof of the vanishing result (1) proved in [4].

Proposition 7. *The Segre classes $s_{2k}(H_{[k]})$ vanish in the range*

$$3k - 1 > g > 2k - 2. \tag{6}$$

In particular, $s_{k,2k} = 0$ and $s_{k,2k-1} = 0$ when $k \geq 2$.

Proof. Sections of H provide sections of $\mathcal{H}_{[k]}$, or equivalently of the line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{H}^*_{[k]})}(1)$. In fact, all sections of $\mathcal{H}_{[k]}$ come from $H^0(S, H)$. As we are on a $K3$ surface, $H^0(S, H)$ has dimension $g+1$. We thus have a rational map $\phi : \mathbf{P}(\mathcal{H}^*_{[k]}) \dashrightarrow \mathbf{P}^g$ such that $\phi^*\mathcal{O}_{\mathbf{P}^g}(1) = \mathcal{O}_{\mathbf{P}(\mathcal{H}^*_{[k]})}(1)$. The top Segre class of $\mathcal{H}^*_{[k]}$ (or $\mathcal{H}_{[k]}$) is the top self-intersection of $c_1(\mathcal{O}_{\mathbf{P}(\mathcal{H}^*_{[k]})}(1))$ on $\mathbf{P}(\mathcal{H}^*_{[k]})$. We observe that the first inequality in (6) says that $\dim \mathbf{P}(\mathcal{H}^*_{[k]}) > \dim \mathbf{P}^g$, so the proposition is a consequence of the following lemma which is a mild generalization of Lazarsfeld's result in [2], saying that smooth curves in $\mid H \mid$ are Brill-Noether generic:

Lemma 8. *If $g > 2k - 2$, the vector bundle $\mathcal{H}_{[k]}$ is generated by the sections coming from $H^0(S, H)$.*

Indeed, this last statement says that the rational map ϕ is actually a morphism so that the top self-intersection of a line bundle pulled-back via ϕ is 0. □

Proof of Lemma 8. The proof is by contradiction. It is obtained by applying Lazarsfeld's arguments in [2]. For convenience of the reader and because Lazarsfeld considers only subschemes supported on smooth curves, we give the complete argument: If $z \in S^{[k]}$ is a point such that $H^0(S, H) \to \mathcal{H}_{[k],z}$ is not surjective, z corresponds to a length k subscheme $Z \subset S$ such that the restriction map $H^0(S, H) \to H^0(H_{|Z})$ is not surjective, hence $H^1(S, \mathcal{I}_Z(H)) \neq 0$. By Serre duality, we thus have a nonzero class $e \in \operatorname{Ext}^1(\mathcal{I}_Z, H^{-1})$, which provides a torsion free rank 2 sheaf $\mathcal{E}$ fitting into an exact sequence

$$0 \to H^{-1} \to \mathcal{E} \to \mathcal{I}_Z \to 0. \tag{7}$$

Note that the original Lazarsfeld argument deals with all subschemes which are locally complete intersection, for which $\mathcal{E}$ is locally free (assuming k is minimal). We have $c_1(\mathcal{E}) = H^{-1}$ and $c_2(\mathcal{E}) = k$. It thus follows that

$$\chi(\mathcal{E}, \mathcal{E}) := h^0(End(\mathcal{E})) - \dim Ext^1(\mathcal{E}, \mathcal{E}) + \dim Ext^2(\mathcal{E}, \mathcal{E}) = 4\chi(\mathcal{O}_S) + c_1(\mathcal{E})^2 - 4c_2(\mathcal{E})$$

$$= 8 + 2g - 2 - 4k.$$

The second inequality in (6) thus gives

$$\chi(\mathcal{E}, \mathcal{E}) > 2.$$

We thus conclude (applying Serre duality showing that $\dim Ext^2(\mathcal{E}, \mathcal{E}) = h^0(End(\mathcal{E}))$) that $\mathcal{E}$ has an endomorphism $f : \mathcal{E} \to \mathcal{E}$ which is not proportional to the identity, hence can be assumed to be of generic rank 1. Let B be the line bundle defined as $\mathcal{F}^{**}$ where $\mathcal{F}$ is the saturation of $\operatorname{Im} f$ in $\mathcal{E}$. The line bundle B must be a power of H. The non-split exact sequence (7) shows that $\operatorname{Hom}(\mathcal{E}, H^{-1}) = 0$ since the exact sequence (7) is not split, so B must be trivial or a positive power of H. It follows that $\mathcal{F}$ is equal to $H^{\otimes k} \otimes \mathcal{I}_W$ for some $k \geq 0$ and for some 0-dimensional subscheme $W \subset Z$ (which can appear only where $\mathcal{E}$ is not locally free). As $H^{\otimes k} \otimes \mathcal{I}_W$ is not contained in H^{-1}, it must map nontrivially to $\mathcal{I}_Z$ via $f : \mathcal{E} \to \mathcal{I}_Z$, so that finally $k = 0$ and $\mathcal{I}_W \subset \mathcal{I}_Z$. As $\mathcal{I}_Z \subset \mathcal{I}_W$ and $\operatorname{End}(\mathcal{I}_Z) = \mathbb{C}Id$, we conclude that in fact f induces an isomorphism $\mathcal{I}_W \cong \mathcal{I}_Z$ and the sequence (7) is split, which is a contradiction. □

We note for later reference the following simple fact on which the proof of Proposition 7 rests. We will say that H is k-ample if $\mathcal{H}_{[k]}$ is generated by its global sections. 1-ample means that H is generated by sections, and 2-ample means that H is very ample.

Lemma 9. *Let Σ be a surface, H a line bundle on Σ. Assume that H is k-ample and $h^0(\Sigma, H) < 3k$. Then $s_{2k}(\mathcal{H}_{[k]}) = 0$.*

3 Proof of Theorem 1

We are going to prove here Theorem 1 for $2g - 2 \geq 0$, i.e. $g \geq 1$, by induction on g. The case where g is nonpositive works similarly, by induction on $-g$. Let S' be a $K3$ surface equiped with a line bundle H' such that $c_1(H')^2 = 2(g-1) - 2$. Let A be an abelian surface with a principal polarization θ, so that $c_1(\theta)^2 = 2$. The surface $\Sigma = S' \sqcup A$ equiped with the line bundle H_Σ which is equal to H' on S' and θ on A, has the same characteristic numbers as our original pair (S, H) where S is a $K3$ surface, and H is a polarization with self-intersection $2g - 2$. On the other hand, $\Sigma^{[k]}$ is the disjoint union

$$\Sigma^{[k]} = \sqcup_{l=0}^{l=k} {S'}^{[k-l]} \times A^{[l]},$$

and on each summand ${S'}^{[k-l]} \times A^{[l]}$, the vector bundle $H_{\Sigma,[k]}$ equals $pr_1^* H'_{[k-l]} \oplus pr_2^* \theta_{[l]}$. We thus conclude that we have the following formula, where $b_l := \int_{A^{[l]}} s_{2l}(\theta_{[2l]})$ (this is a particular case of (3)):

$$s_{k,g} = \sum_{l=0}^{l=k} b_l s_{k-l,g-1}. \tag{8}$$

Corollary 10. *The numbers $s_{k,g}$ for $g \geq 1$ are fully determined by the numbers b_l, $0 \leq l \leq k$ and the numbers $s_{l,1}$, $l \leq k$, $s_{1,g}$, $g \geq 1$ (or $s_{0,g}$).*

Remark 11. We have $b_0 = 1$, $b_1 = 2$, and similarly $s_{0,g} = 1$, $s_{1,g} = 2g - 2$.

Lemma 12. *Suppose that the numbers b_l, $0 \leq l \leq k-1$ and the numbers $s_{l,1}$, $0 \leq l \leq k-1$ are given, with $b_0 = 1$, $b_1 = 2$. Then the numbers $s_{k,1}$ and b_k are determined by the condition $b_0 = 1$, $b_1 = 2$, equation (8), and the vanishing equations*

$$s_{k,2k} = 0,\ s_{k,2k-1} = 0 \tag{9}$$

for $k \geq 2$ proved in Proposition 7.

Proof. Indeed, by Corollary 10, all the numbers $s_{l,g'}$ for $g' \leq g-1$ and $l \leq k-1$ are determined by b_l, $0 \leq l \leq k-1$ and $s_{l,1}$, $0 \leq l \leq k-1$. We thus can write (8) as

$$s_{k,g} = s_{k,g-1} + (\dots) + b_k,$$

$$s_{k,g-1} = s_{k,g-2} + (\dots) + b_k,$$

$$\dots$$

where the expressions $(\dots)$ in the middle are determined by b_l, $0 \leq l \leq k-1$ and $s_{l,1}$, $0 \leq l \leq k-1$. Combining these equations, we get

$$\begin{aligned} s_{k,2k} &= s_{k,1} + (\dots) + (2k-1)b_k \\ s_{k,2k-1} &= s_{k,1} + (\dots) + (2k-2)b_k, \end{aligned} \tag{10}$$

hence we can see the equations $s_{k,2k} = 0$, $s_{k,2k-1} = 0$ as a system of two affine equations in the two variables $s_{k,1}$ and b_k, whose linear part is invertible and the constants are determined by b_l, $0 \leq l \leq k-1$ and $s_{l,1}$, $0 \leq l \leq k-1$. The numbers $s_{k,1}$ and b_k are thus uniquely determined by these equations and the numbers b_l, $0 \leq l \leq k-1$ and $s_{l,1}$, $0 \leq l \leq k-1$. □

Corollary 13. *There exist unique sequences of numbers $s_{k,g}$, $k \geq 0$, $g \geq 1$ and b_l, $l \geq 0$ satisfying:*

1. $b_0 = 1$, $b_1 = 2$,
2. $s_{0,g} = 1$, $s_{1,g} = 2g-2$,
3. $s_{k,2k} = 0$, $s_{k,2k-1} = 0$ *for* $k \geq 2$.
4. $s_{k,g} = \sum_{l=0}^{l=k} b_l s_{k-l,g-1}$.

Proof of Theorem 1. The numbers $s'_{k,g} := 2^k \binom{g-2k+1}{k}$ satisfy the vanishings $s'_{k,2k} = 0$, $s'_{k,2k-1} = 0$ for $k \geq 2$, that is, condition 3 of Corollary 13. They also satisfy the condition $s'_{1,g} = 2g-2$, that is, condition 2 of Corollary 13. In order to show that $s_{k,g} = s'_{k,g}$, it suffices by Corollary 13 to show that they also satisfy condition 4 for adequate numbers b'_l, which is proved in the following Lemma 14. □

Lemma 14. *There exist numbers b'_l, $l \geq 0$ with $b'_0 = 1$, $b'_1 = 2$ such that for any $g \geq 1$*

$$s'_{k,g} = \sum_{l=0}^{k} b'_l s'_{k-l,g-1}. \tag{11}$$

Proof. We observe that $s'_{k,g}$ is, as a function of g, a polynomial of degree exactly k, with leading coefficient 2^k. Hence the $s'_{l,g}$ for $0 \leq l \leq k-1$ form a basis of the space of polynomials of degree $k-1$, and for k fixed, there exist uniquely defined numbers $b'_{l,k}$, $l = 0, \dots, k$, with $b'_{0,k} = 1$, such that for any g:

$$s'_{k,g} = \sum_{l=0}^{k} b'_{l,k} s'_{k-l,g-1}. \tag{12}$$

Let us prove that $b'_{l,k} = b'_{l,k-1}$ for $l \leq k-1$. We have

$$\binom{g-2k+1}{k} = \binom{g-2k}{k} + \binom{g-2k}{k-1},$$

that is,

$$2s'_{k-1,g-3} = s'_{k,g} - s'_{k,g-1}, \tag{13}$$

with the convention that $s'_{k,g} = 0$ for $k < 0$. It follows by definition of $b'_{l,k}$ that

$$2s'_{k-1,g-3} = \sum_{l=0}^{k} b'_{l,k} s'_{k-l,g-1} - \sum_{l=0}^{k} b'_{l,k} s'_{k-l,g-2} = \sum_{l=0}^{k} b'_{l,k}(s'_{k-l,g-1} - s'_{k-l,g-2}),$$

which gives, by applying (13) again to each term in the right hand side:

$$2s'_{k-1,g-3} = 2\sum_{l=0}^{k} b'_{l,k} s'_{k-l-1,g-4} = 2\sum_{l=0}^{k-1} b'_{l,k} s'_{k-l-1,g-4}.$$

By definition of $b'_{l,k-1}$, this provides $b'_{l,k} = b'_{l,k-1}$. □

4 Further geometric vanishing

We discuss in this section similar geometric vanishing results for the Segre classes on the blow-up of a $K3$ surface at one point. The setting is thus the following: S is a $K3$ surface with $\operatorname{Pic} S = \mathbb{Z}L$, $L^2 = 2g-2$, and $x \in S$ is a point. The surface $\tau : \widetilde{S} \to S$ is the blow-up of S at x with exceptional curve E, and $H := \tau^* L(-lE) \in \operatorname{Pic} \widetilde{S}$ for some positive integer l. Our main goal is to discuss the analogue of Lemma 8 in this context. Note that, when H is very ample, the curve E has degree l in the embedding given by $|H|$, so that the vector bundle $\mathcal{H}_{[k]}$ can be generated by sections only when $k \leq l+1$.

To start with, we have:

Proposition 15. *Let S be a $K3$ surface with Picard group generated by L, $L^2 = 2g-2$. Let $\tau : \widetilde{S} \to S$ be the blow-up at a point $x \in S$. Then, denoting $H = \tau^* L(-lE)$, if*

$$4 + 2g > (l+1)^2, \tag{14}$$

one has $H^1(\widetilde{S}, H) = 0$. It follows that $h^0(\widetilde{S}, H) = g + 1 - \frac{l(l+1)}{2}$.

Proof. We argue by contradiction. The proof follows Reider's [6] and Lazarsfeld's [2] methods. Assume $H^1(\widetilde{S}, H) \neq 0$. Then, by Serre duality, $\operatorname{Ext}^1(H, \mathcal{O}_{\widetilde{S}}(E)) \neq 0$, which provides a rank 2 vector bundle $\mathcal{E}$ on $\widetilde{S}$ which fits in an exact sequence

$$0 \to \tau^* L^{-1}((l+1)E) \to \mathcal{E} \to \mathcal{O}_{\widetilde{S}} \to 0. \tag{15}$$

The fact that the extension class of (15) is not trivial translates into $h^0(\widetilde{S}, \mathcal{E}) = 0$. We have $c_2(\mathcal{E}) = 0$ and $c_1(\mathcal{E})^2 = 2g - 2 - (l+1)^2$, so that (14) gives the inequality

$$\chi(End\,\mathcal{E}) = 8 + c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) > 2.$$

It follows that $h^0(\widetilde{S}, End\,\mathcal{E}) + h^0(\widetilde{S}, End\,\mathcal{E}(E)) > 2$, hence $h^0(\widetilde{S}, End\,\mathcal{E}(E)) > 1$. Thus there exists a $\phi \in \operatorname{Hom}(\mathcal{E}, \mathcal{E}(E))$ which is not proportional to the identity. The characteristic polynomial of ϕ has its trace in $H^0(\widetilde{S}, \mathcal{O}_{\widetilde{S}}(E)) = H^0(\widetilde{S}, \mathcal{O}_{\widetilde{S}})$ and determinant in $H^0(\widetilde{S}, \mathcal{O}_{\widetilde{S}}(2E)) = H^0(\widetilde{S}, \mathcal{O}_{\widetilde{S}})$. It is thus a polynomial with coefficients in $\mathbb{C}$ and has a rood λ. Replacing ϕ by $\phi - \lambda Id_{\mathcal{E}}$ (where we see $Id_{\mathcal{E}}$ as an element of $\operatorname{Hom}(\mathcal{E}, \mathcal{E}(E))$), we can in fact assume that ϕ is generically of rank 1. Let $A = \operatorname{Ker}\phi \subset \mathcal{E}$. We have $A = \tau^* L^{\alpha}(\beta E)$ and $\mathcal{E}$ fits in an exact sequence

$$0 \to A \to \mathcal{E} \to B \otimes \mathcal{I}_W \to 0, \tag{16}$$

where B is the line bundle $\tau^* L^{-1-\alpha}((l+1-\beta)E)$. As $B = \operatorname{Im}\phi$, we have $B \hookrightarrow \mathcal{E}(E)$. From the exact sequence (15), we immediately conclude that $\alpha \leq 0$ and $(-1-\alpha) \leq 0$, so that $\alpha = 0$ or $\alpha = -1$.

Assume first $\alpha = 0$. Then as $h^0(\widetilde{S}, \mathcal{E}) = 0$, we conclude that $\beta < 0$, hence $l + 1 - \beta > 0$. Then (16) gives

$$c_2(\mathcal{E}) = A \cdot B + \deg W \geq -\beta(l + 1 - \beta) > 0,$$

which is a contradiction.

In the remaining case $\alpha = -1$, we conclude that $B = \mathcal{O}_{\widetilde{S}}((l + 1 - \beta)E)$, so that we have a nonzero morphism $\mathcal{O}_{\widetilde{S}}((l - \beta)E) \to \mathcal{E}$. This provides a line bundle $A' \subset \mathcal{E}$ defined as the saturation of the image of this morphims, and we know that $A' = \tau^* L^{\alpha'}(\beta' E)$ with $\alpha' \geq 0$. We can then apply the previous argument with A replaced by A', getting a contradiction. $\square$

Pushing forward the arguments above, we now prove the following result:

Theorem 16. *Let S be a general $K3$ surface with Picard group generated by L, and $x \in S$ a general point. Then for $k \geq 2$, $H = \tau^* L(-lE)$ is k-ample for $k = l$ or $k = l + 1$, and $g - \frac{l(l+1)}{2} = 3k - 2$.*

Remark 17. *When $g - \frac{l(l+1)}{2} = 3k - 2$, with $k = l$ or $k = l + 1$, one has for $l > 0$*

$$4 + 2g = l(l + 1) + 6k \geq (l + 7)l > (l + 1)^2$$

so that Proposition 15 applies, which gives $H^1(\widetilde{S}, H) = 0$ and $h^0(\widetilde{S}, H) = g + 1 - \frac{l(l+1)}{2} = 3k - 1$.

Proof of Theorem 16. With the assumptions of Theorem 16, assume H is not k-ample. Therefore there exists a 0-dimensional subscheme $Z \subset \widetilde{S}$ of length k such that $H^1(\widetilde{S}, H \otimes \mathcal{I}_Z) \neq 0$. Using the duality $H^1(\widetilde{S}, H \otimes \mathcal{I}_Z)^* = \mathrm{Ext}^1(\mathcal{I}_Z, -H + E)$, this provides us with a rank 2 torsion free sheaf $\mathcal{E}$ on $\widetilde{S}$ fitting in an exact sequence

$$0 \to \tau^* L^{-1}((l + 1)E) \to \mathcal{E} \to \mathcal{I}_Z \to 0. \tag{17}$$

The numerical invariants of $\mathcal{E}$ are given by

$$c_2(\mathcal{E}) = k,\ c_1(\mathcal{E})^2 = 2g - 2 - (l + 1)^2,$$

from which we conclude that

$$\chi(\mathcal{E}, \mathcal{E}) = 8 + 2g - 2 - (l + 1)^2 - 4k,$$

hence

$$h^0(End\,\mathcal{E}) + h^0(End\,\mathcal{E}(E)) \geq 8 + 2g - 2 - (l + 1)^2 - 4k. \tag{18}$$

By assumption, $g - \frac{l(l+1)}{2} = 3k - 2$, so $2g - 2 - (l + 1)^2 = 6k - 6 - (l + 1)$ and (18) gives

$$2h^0(End\,\mathcal{E}(E)) \geq 2 + 2k - (l + 1),$$

hence $2h^0(End\,\mathcal{E}(E)) > 2$ because $k \geq 2$ and $k = l$ or $k = l + 1$. Thus there exists a morphism

$$\phi : \mathcal{E} \to \mathcal{E}(E)$$

which is not proportional to the identity. As before, we can even assume that ϕ is generically of rank 1. One difference with the previous situation is the fact that $\mathcal{E}$ is not necessarily locally free, and furthermore $c_2(\mathcal{E}) \neq 0$. The kernel of ϕ and its image are torsion free of rank 1, hence are of the form $A \otimes \mathcal{I}_W$, $B \otimes \mathcal{I}_{W'}$ for some line bundles A, B on $\widetilde{S}$ which are of the form

$$A = \tau^* L^{\alpha}(\beta E),\ B = \tau^* L^{-1-\alpha}((l + 1 - \beta)E).$$

As before, we must have $\alpha \leq 0$ and $-1 - \alpha \leq 0$ because B injects into $\mathcal{E}(E)$. Hence we conclude that $\alpha = 0$ or $\alpha = -1$.

(i) If $\alpha = 0$, then we have a nonzero morphism $\mathcal{O}(\beta E) \otimes \mathcal{I}_W \to \mathcal{I}_Z$. It follows that $\beta \leq 0$. If $\beta = 0$, this says that $\mathcal{I}_W \subset \mathcal{I}_Z$ and that the extension class of (17) vanishes in $\mathrm{Ext}^1(\mathcal{I}_W, \tau^* L^{-1}((l+1)E))$. But the restriction map

$$\mathrm{Ext}^1(\mathcal{I}_Z, \tau^* L^{-1}((l+1)E)) \to \mathrm{Ext}^1(\mathcal{I}_W, \tau^* L^{-1}((l+1)E))$$

is injective as it is dual to the map $H^1(\widetilde{S}, \mathcal{I}_W(H)) \to H^1(\widetilde{S}, \mathcal{I}_Z(H))$ which is surjective. Indeed, the spaces are respective quotients of $H^0(H_{|W})$, $H^0(H_{|Z})$ by Proposition 15 which applies in our case as noted in Remark 17. So we conclude that $\beta < 0$. We now compute $c_2(\mathcal{E})$ using the exact sequence

$$0 \to A \otimes \mathcal{I}_W \to \mathcal{E} \to B \otimes \mathcal{I}_{W'} \to 0,$$

with $A = \mathcal{O}(\beta E)$, $B = \tau^* L^{-1}((l+1-\beta)E)$. This gives

$$c_2(\mathcal{E}) = \deg W + \deg W' - \beta(l+1-\beta) \geq -\beta(l+1-\beta) \geq l+2.$$

This contradicts $c_2(\mathcal{E}) = k \leq l+1$.

(ii) If $\alpha = -1$, then we use instead the inclusion $B \otimes \mathcal{I}_{W'} \subset \mathcal{E}(E)$, with $B = \mathcal{O}((l+1-\beta)E)$ and argue exactly as before. □

We deduce the following Corollary 18 concerning the numbers $s_k(d, \pi, \kappa, e)$ (we adopt here Lehn's notation [3]) defined as the top Segre class of $\mathcal{H}_{[k]}$ for a pair (Σ, H) where Σ is a smooth compact surface, and

$$d = H^2,\ \pi = H \cdot c_1(K_\Sigma),\ \kappa = c_1(\Sigma)^2,\ e = c_2(\Sigma).$$

Corollary 18. *(Cf. Theorem 3.) One has the following vanishing for $s_k(d, \pi, -1, 25)$*

$$s_k(7(k-1), k-1, -1, 25) = 0,\ \ s_k(7(k-1)+1, k, -1, 25) = 0 \tag{19}$$

for $k \geq 2$.

Proof. Take for Σ the blow-up of a $K3$ surface at a point so

$$\kappa = -1,\ e = 25.$$

Furthermore, assuming $\mathrm{Pic}\, S = \mathbb{Z}L$ with $L^2 = 2g-2$, and letting $H = \tau^* L(-lE)$ as above, we have

$$d = H^2 = 2g - 2 - l^2,\ \pi = H \cdot c_1(K_\Sigma) = l. \tag{20}$$

We consider the cases where

$$g - \frac{l(l+1)}{2} = 3k - 2 \tag{21}$$

with (i) $k = l+1$ or (ii) $k = l$.

Using (20), (21) gives in case (i), $d = 7(k-1)$, $\pi = k-1$ and in case (ii), $d = 7(k-1)+1, \pi = k$, so we are exactly computing $s_k(7(k-1), k-1, -1, 25) = 0$ in case (i) and $s_k(7(k-1)+1, k, -1, 25)$ in case (ii). Remark 17 says that assuming (21),

$$H^1(\widetilde{S}, H) = 0,\ h^0(\widetilde{S}, H) = 3k-1$$

in cases (i) and (ii). Theorem 16 says that under the same assumption, H is k-ample on $\widetilde{S}$. Lemma 9 thus applies and gives $s_{2k}(\mathcal{H}_{[k]}) = 0$ in both cases, which is exactly (19). □

Remark 19. Lehn gives in [3, Section 4] the explicit polynomial formulas for $2!s_2, \dots, 5!s_5$ as polynomial functions of d, π, κ, e with huge integral coefficients. For example

$$\begin{aligned} 5!s_5 = d^5 - 100d^4 + d^3(3740 + 10e - 50\pi - 10\kappa) \\ -d^2(62000 - 3420\pi + 700e - 860\kappa) + d(384384 + 15e^2 \\ +15960e - 30e\kappa - 150\pi e + 15\kappa^2 + 150\kappa\pi - 75610\pi - 24340\kappa + 375\pi^2) \\ -400e^2 - 117120e + 3920\pi e + 960\kappa e + 226560\kappa - 4720\kappa\pi \\ -560\kappa^2 + 530880\pi - 9600\pi^2 \end{aligned} \tag{22}$$

It is pleasant to check the vanishing statements (19) for $k = 2, \dots, 5$ using these formulas. For $k = 5$, one just has to plug-in the values $e = 25$, $\kappa = -1$, $d = 28$ and $\pi = 4$, or $e = 25$, $\kappa = -1$, $d = 29$ and $\pi = 5$ in (22).

We conclude this note by showing that all the Segre numbers are formally determined by the above results and formula (3).

Proposition 20. *The vanishings (19) together with the data of the numbers $s_k(d, 0, 0, 24)$ and $s_k(d, 0, 0, 0)$ determine all numbers $s_k(d, \pi, \kappa, e)$.*

Note that $s_k(d, 0, 0, 24)$ is for $d = 2g - 2$ the number $s_{k,g}$ of the introduction, and these numbers are given by Marian-Oprea-Pandharipande's Theorem 1. The numbers $s_k(d, 0, 0, 0)$ correspond for d even to the Segre classes of tautological sheaves on Hilbert schemes of abelian surfaces equipped with a line bundle of self-intersection d. They are fully determined, by multiplicativity, by the case of self-intersection 2, where one gets the numbers b'_k appearing in our proof of Theorem 1.

Proof of Proposition 20. According to [3], [1], and as follows from (3), the generating series

$$s(z) = \sum_k s_k(d, \pi, \kappa, e) z^k$$

is of the form

$$s(z) = A(z)^d B(z)^e C(z)^\pi D(z)^\kappa, \tag{23}$$

for power series A, B, C, D with 0-th order coefficient equal to 1. Theorem 1 determines the series $A(z)$ and $B(z)$. We thus only have to determine $C(z)$ and $D(z)$. The degree 1 coefficients of the power series $C(z)$, $D(z)$ are immediate to compute as $s_1 = d$. We now assume that the coefficients of the power series $C(z)$ and $D(z)$ are computed up to degree $k-1$. The degree k coefficient of $s(z) = A(z)^d B(z)^e C(z)^\pi D(z)^\kappa$ is of the form $\pi C_k + \kappa D_k + \nu$ where ν is determined by d, e, π, κ, the coefficients of A and B, and the coefficients of order $\leq k - 1$ of C and D. The vanishings (19) thus give the equations

$$0 = (k-1)C_k - D_k + \nu, \ 0 = kC_k - D_k + \nu',$$

which obviously determines C_k and D_k as functions of ν and ν'. □

We finally prove Corollary 6 of the introduction.

Proof of Corollary 6. Let $f_{d,\pi,\kappa,e}(z)$ be the Lehn function introduced in Conjecture 5. As Lehn's conjecture is proved by [4] for $\pi = \kappa = 0$ (the K-trivial case), the coefficients $f_{k,d,\pi,\kappa,e}$ of the Taylor expansion of $f_{d,\pi,\kappa,e}$ in z (not w) are the Segre numbers $s_{k,d,0,0,e}$ when $\pi = 0$, $\kappa = 0$. If furthermore they satisfy the vanishings $f_{k,d,\pi,\kappa,e} = 0$ for $e = 25$, $\kappa = -1$ and $d = 7(k-1)$, $\pi = k - 1$ or $d = 7(k-1) + 1$, $\pi = k$, the proof of Proposition 20 shows that $f_{k,d,\pi,\kappa,e} = s_{k,d,\pi,\kappa,e}$ for all k, d, π, κ, e as, by definition, f has the same multiplicative form (23) as s. □

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