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A MEAN FIELD GAME MODEL OF FIRM-LEVEL INNOVATION

MATT BARKER*, PIERRE DEGOND†, RALF MARTIN‡, AND MIRABELLE MUÛLS

Abstract. Knowledge spillovers occur when a firm researches a new technology and that technology is adapted or adopted by another firm, resulting in a social value of the technology that is larger than the initially predicted private value. As a result, firms systematically under-invest in research compared with the socially optimal investment strategy. Understanding the level of under-investment, as well as policies to correct it, is an area of active economic research. In this paper, we develop a new model of spillovers, taking inspiration from the available microeconomic data. We prove existence and uniqueness of solutions to the model, and we conduct some initial simulations to understand how indirect spillovers contribute to the productivity of a sector.

AMS subject classifications. 35Q89, 91B69, 91A16, 49N80

Key words. mean field games, knowledge spillovers, innovation model

1. Introduction. When a business invests in research and development (R&D), such strategy only takes into account how a potential innovation may increase the investing company's private value. However, other businesses may utilise innovations made by the original investing company to increase their own profits. This is known in economic literature as the knowledge spillover effect. By only considering its private return, businesses systematically undervalue their own innovations and hence under-invest in R&D, compared with the socially optimal investment level. To counteract the under-investment, governments introduce R&D subsidy policies for certain sectors of the economy. In order to effectively allocate such subsidies, it is therefore important to understand the extent of under-investment and how it varies between sectors.

To understand the spillover effect we develop a mean field game (MFG) model of firms distributed heterogeneously between sectors and according to their productivity level, taking into account their microscopic behaviour. From a microeconomic perspective, the size of knowledge spillovers can be inferred from the network of patent citations [15]. When an industrial technology is developed, it often gets patented. As part of the patent any previous technology that has been used must be cited. This results in a network of patent citations, where each citation can be used as a proxy for a spillover from one technology to another, so spillover sizes can be evaluated [11]. In the model we develop, sectors are connected by a graph that is informed by and can be calibrated to the microeconomic network of patent citation data.

A first model of knowledge spillovers, by Cohen and Levinthal [10], considered the stock of knowledge of a firm to depend on the amount of investment in R&D of that firm and the total amount of investment by all other firms, through a mean field-type interaction. Only an initial analysis of the model was conducted in [10]. A later model, acknowledged in Section 13.2 of [1], started from a macroscopic perspec-

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38 tive, hence only the aggregate knowledge of the entire economy was considered and
39 spillovers were assumed to increase the aggregate uniformly. This does not explain
40 how spillovers heterogeneously affect firms. Similar models have also been used to
41 study entrepreneurship and intellectual property rights, such as in [2]. There has
42 been particularly extensive research of knowledge spillovers in cross-country models.
43 In such models, a country's own output is aggregated and the knowledge level increases
44 at a rate that depends on the leading country's knowledge level. However, simplifying
45 assumptions are made that may affect their accuracy, such as in [12], where interac-
46 tions take place in a discrete time setting, or [3, 16] where the interactions between
47 firms was described only through the evolution of aggregate quantities. In this paper,
48 we use an MFG model to both increase the complexity of the description of firms,
49 and to link firm-level evolution directly to microeconomic data for spillover sizes.
50 There have been several other papers focussing on MFG-type models of knowledge
51 spillovers, see [7, 21]. The Boltzmann model studied in both papers does not con-
52 sider how innovation among firms evolves, nor did it incorporate the microeconomic
53 data related to patent citations in its formulation. As a result, the model studied in
54 this paper can give greater insight into firm-level dynamics.

55 In this paper, we analyse a stationary MFG model describing the spillover effect.
56 The MFG model describes the long-term behaviour of firms with full anticipation
57 of the future. MFGs were described mathematically by Lasry and Lions [18, 19],
58 and simultaneously by Huang, Caines and Malhamé [17] and they build on the work
59 of Aumann and related authors on anonymous games [5, 22]. The novelties of the
60 system we develop are, first, that the distribution dependence enters into the drift
61 term rather than in the cost functional and, second, that we are considering more
62 than one population of agents. Therefore our MFG model can be classed as a multi-
63 population MFG with a non-separable Hamiltonian. There has been some work in
64 both multi-population MFGs (see [9]) and MFG models with non-separable Hamil-
65 tonians (see [4, 14]). However, we are aware of no literature for models that display
66 both characteristics, so although our model is one dimensional, its interest reaches
67 beyond this setting. As a result of the novelty of our model, the techniques we use
68 to prove existence and uniqueness are also novel. However, they rely heavily on the
69 ability to write a stationary Fokker-Planck equation in the form of an exponential.
70 This characteristic has previously been used in [6] to prove existence and uniqueness
71 in MFG and BRS models in a slightly different framework.

72 The paper is organised as follows. In Section 2, we develop the spillover model by
73 describing firm behaviour at a microscopic level and formally deriving the mean field
74 limit. In section 3, we describe the MFG problem and prove existence of solutions to
75 it. We also show uniqueness of such solutions holds, provided the coupling strength
76 between sectors is small enough. In Section 4, we provide some deeper insights into
77 the effects of the modelling parameters, through numerical simulations. The first sim-
78 ulations show how parameters describing effects unrelated to spillovers (for example
79 the discount factor, the noise level and the labour efficiency) change the MFG model.
80 Our second group of simulations demonstrate the effect of the spillover network on
81 the model. The spillover network is a sector-level network that aggregates the patent
82 citation network. We show that the effect of a spillover on any sector is a result of all
83 paths to that sector in the associated network, and not just the immediate connections
84 between sectors, which is contrary to the current economic state of the art. Finally,
85 in Section 5, we briefly discuss future research prospects for the model, including how
86 we intend to apply the model to economic questions relevant to R&D subsidy policy.

87 **2. Model development.**

88 **2.1. The microscopic model.**

89 **Firms.** Assume there are L sectors within the economy, and in sector ℓ there are
 90 N_ℓ firms. We assume firm i in group ℓ has $s_{i,j}^{\ell,\ell'}$ links with firm j in group ℓ' , where
 91 $s_{i,j}^{\ell,\ell'}$ is a random variable, taking a value $s \in \mathbb{N}$ with probability $\frac{1}{N_{\ell'}} p(\ell, \ell', s)$. The
 92 i^{th} firm in sector ℓ has a productivity level $Z_{\ell,i} \in \Omega = (0, \bar{z})$, which increases as a
 93 result of employing labour $h_{\ell,i}$ or due to knowledge spillovers from firms that they
 94 are linked with. The productivity dynamics are also affected by noise with strength
 95 $\sigma \in (0, \infty)$. As a result, $Z_{\ell,i}$ evolves according to the following SDE

96 (2.1a)
$$dZ_{\ell,i}(t) = \left((h_{\ell,i}(t))^\gamma + \frac{1}{N} \sum_{\ell'=1}^L \sum_{j=1}^{N_{\ell'}} s_{i,j}^{\ell,\ell'} Z_{\ell',j}(t) \right) dt + \sigma dB_{\ell,i}(t)$$

97 (2.1b)
$$\mathcal{L}(Z_{\ell,i}(0)) = m_\ell^0,$$

99 where $N = \sum_{\ell=1}^L N_\ell$, $B_{\ell,i}$ is an independent Brownian motion with reflection at
 100 boundaries 0 and \bar{z} , and $\gamma \in (0, 1)$ represents the inefficiency in converting one unit
 101 of labour to one unit of knowledge. In the initial condition (2.1b), $\mathcal{L}(Z_{\ell,i}(0))$ denotes
 102 the law of the random variable $Z_{\ell,i}(0)$ and m_ℓ^0 is an initial distribution, which may be
 103 different for each sector. We assume firms produce a quantity of differentiated good
 104 at a rate $q_{\ell,i}$ according to the production function

105 (2.2)
$$q_{\ell,i} = Z_{\ell,i}.$$

106 Each firm sells their product at a market-determined price $r_{\ell,i}$ and maximises their
 107 profit subject to the other firms' decisions. Each agent's profit functional is given by

108 (2.3)
$$J_{\ell,i}(h) = \mathbb{E} \left[\int_0^\infty (r_{\ell,i}(t)q_{\ell,i}(t) - wh_{\ell,i}(t)) e^{-\rho t} dt \right],$$

109 where $h = \left((h_{\ell,i})_{i=1}^{N_\ell} \right)_{\ell=1}^L$. The wage, w , and the discount rate, ρ , are given constants.

110 **Consumers.** Assume there is a representative consumer with preferences given
 111 by $Q = \frac{1}{N} \sum_{\ell=1}^L \sum_{i=1}^{N_\ell} q_{\ell,i}^\alpha$, the Dixit–Stiglitz constant elasticity of substitution (CES)
 112 form, and with average income Y . The value $\alpha \in (0, 1)$ is related to the elasticity
 113 of substitution. The demand for each variety can be found by maximising Q under
 114 the budget constraint that average expenditure is equal to average income, i.e.
 115 $\frac{1}{N} \sum_{\ell=1}^L \sum_{i=1}^{N_\ell} r_{\ell,i} q_{\ell,i} = Y$. This gives

116 (2.4)
$$q_{\ell,i} = Br_{\ell,i}^{\frac{1}{\alpha-1}}, \quad B = YR^{\frac{\alpha}{1-\alpha}}, \quad R = \left(\frac{1}{N} \sum_{\ell=1}^L \sum_{i=1}^{N_\ell} r_{\ell,i}^{\frac{\alpha}{\alpha-1}} \right)^{\frac{\alpha-1}{\alpha}}.$$

117 For the purposes of the firm-level optimisation problem, we assume R is fixed, in
 118 that it can't be changed by any individual firm — this becomes true as $N_\ell \rightarrow \infty$ for
 119 each ℓ . For the mathematical analysis we assume B to be a fixed constant, which
 120 simplifies matters and enhances the model's relevant features. Later in the numerical
 121 simulations B will be determined as the solution to a fixed point problem, which
 122 endogenises the price formation through the interaction between firms and consumers.

123 **Firm profits revisited.** Now, the profit functional (2.3) can be rewritten as
 124 $J_{\ell,i}(h) = \mathbb{E} \left[\int_0^\infty \left(\frac{(Z_{\ell,i}(t))^\alpha}{B^{\alpha-1}} - wh_{\ell,i}(t) \right) e^{-\rho t} dt \right]$, using the consumer behaviour (2.4)
 125 and the production function (2.2).

126 **2.2. Mean field limit.** When there are large numbers of firms in each sector,
 127 the microscopic model developed in Section 2.1 can become intractable. Instead, we
 128 assume the number of firms in each sector, N_ℓ , goes to infinity while $\frac{N_\ell}{N} \rightarrow A_\ell$ for
 129 some $A_\ell \in (0, 1)$, which represents the proportion of firms in sector ℓ . In order to
 130 derive the limiting mean field model, we first define the empirical distributions for
 131 each sector $\ell = 1, \dots, L$ by $m_\ell^{N_\ell} = \frac{1}{N_\ell} \sum_{i=1}^{N_\ell} \delta_{Z_{\ell,i}}$, where $\delta_{Z_{\ell,i}}$ is a Dirac delta at the
 132 point $Z_{\ell,i}$. We can then rewrite the dynamics (2.1) using $m_\ell^{N_\ell}$ as

$$133 \quad dZ_{\ell,i}(t) = \left((h_{\ell,i}(t))^\gamma + \sum_{\ell'=1}^L \frac{(N_{\ell'})^2}{N} \int_{\Omega} z' dm_{\ell'}^{N_{\ell'}}(z', t) \frac{1}{N_{\ell'}} \sum_{j=1}^{N_{\ell'}} s_{i,j}^{\ell,\ell'} \right) dt + \sigma dB_{\ell,i}(t)$$

$$134 \quad \mathcal{L}(Z_{\ell,i}(0)) = m_\ell^0,$$

134 Assuming $m_\ell^{N_\ell}$ has a limit, m_ℓ , as $N_\ell \rightarrow \infty$ then, in the limiting model, a representa-
 135 tive firm in sector ℓ evolves according to the SDE

$$136 \quad (2.5a) \quad dZ_\ell^{h,m}(t) = \left((h_\ell(t))^\gamma + \sum_{\ell'=1}^L A_{\ell'} p(\ell, \ell') \int_{\Omega} z' dm_{\ell'}(z', t) \right) dt + \sigma dB_\ell(t)$$

$$137 \quad (2.5b) \quad \mathcal{L}(Z_\ell^{h,m}(0)) = m_\ell^0,$$

139 where, by the law of large numbers, $p(\ell, \ell') = \sum_{s=0}^\infty p(\ell, \ell', s)$. The corresponding
 140 profit functional is

$$141 \quad (2.6) \quad J_\ell(h; m) = \mathbb{E} \left[\int_0^\infty \left(\frac{(Z_\ell^{h,m}(t))^\alpha}{B^{\alpha-1}} - wh_\ell(t) \right) e^{-\rho t} dt \right].$$

142 If all firms act in the same way as the representative firm, then the distribution of firms
 143 with respect to productivity level is given by a system of L Fokker–Planck equations

$$144 \quad (2.7a) \quad \partial_t m_\ell = -\partial_z \left[\left((h_\ell)^\gamma + \sum_{\ell'=1}^L A_{\ell'} p(\ell, \ell') \int_{\Omega} z' dm_{\ell'}(z', t) \right) m_\ell \right] + \frac{\sigma^2}{2} \partial_{zz}^2 m_\ell$$

$$145 \quad (2.7b) \quad - \left((h_\ell)^\gamma + \sum_{\ell'=1}^L A_{\ell'} p(\ell, \ell') \int_{\Omega} z' dm_{\ell'}(z', t) \right) m_\ell + \frac{\sigma^2}{2} \partial_z m_\ell \Big|_{z=0, \bar{z}} = 0$$

$$146 \quad (2.7c) \quad m_\ell(z, 0) = m_\ell^0(z).$$

148 3. The MFG model.

149 **3.1. Problem formulation.** The MFG problem is related to the search for
 150 Nash equilibria in the optimisation of the profit functional (2.6) while agents evolve
 151 according to the dynamics (2.5).

152 **DEFINITION 3.1.** *The MFG problem is to find a pair (h^*, m^*) , where $h^* = (h_\ell^*)_{\ell=1}^L$
 153 is a sequence of controls and $m^* = (m_\ell^*)_{\ell=1}^L$ is a sequence of probability distributions*

154 on $\bar{\Omega}$, such that for any other sequence of controls h and every ℓ

$$155 \quad (3.1a) \quad J_\ell(h_\ell^*, m^*) \geq J_\ell(h_\ell, m^*)$$

$$156 \quad (3.1b) \quad \text{and } m_\ell^* = \mathcal{L} \left(Z_\ell^{h^*, m^*} \right). \\ 157$$

158 Such a distribution is called an MFG equilibrium.

159 To find an MFG equilibrium we first describe the Hamilton–Jacobi–Bellman
160 (HJB) PDE related to the optimisation part of the problem (3.1a). Then we couple
161 the HJB PDE to the Fokker–Planck PDE (2.7) to solve the consistency part (3.1b).
162 We start by defining L Hamiltonians $H_\ell : \Omega \times (H^1(\Omega))^L \times \mathbb{R} \rightarrow \mathbb{R}$, for $\ell = 1, \dots, L$ as

$$163 \quad (3.2) \quad H_\ell(z, m, \lambda) = \sup_{h \geq 0} \left(h^\gamma + \sum_{\ell'=1}^L A_{\ell'} p(\ell, \ell') \int_{\Omega} z' m_{\ell'}(z') dz' \right) \lambda + \frac{z^\alpha}{B^{\alpha-1}} - wh \\ = (1 - \gamma) \left(\frac{\gamma}{w} \right)^{\frac{\gamma}{1-\gamma}} \max(0, \lambda)^{\frac{1}{1-\gamma}} + \lambda \sum_{\ell'=1}^L A_{\ell'} p(\ell, \ell') \int_{\Omega} z' m_{\ell'}(z') dz' + \frac{z^\alpha}{B^{\alpha-1}},$$

164 where $z \in \Omega$ is productivity, $m = (m_\ell)_{\ell=1}^L$ is a distribution of firms in each sector and
165 λ is an adjoint variable. The optimal control is given by $h_\ell^* = \left(\frac{\gamma}{w} \max(0, \lambda) \right)^{\frac{1}{1-\gamma}}$, for
166 $\ell = 1, \dots, L$. Then we define the running profit $V_\ell(z, t)$, for $\ell = 1, \dots, L$, by

$$167 \quad (3.3) \quad V_\ell(z, t) = \sup_{h_\ell} \mathbb{E} \left[\int_t^\infty \left(\frac{(Z_\ell(s))^\alpha}{B^{\alpha-1}} - wh_\ell(z) \right) e^{-\rho(s-t)} ds \middle| Z_\ell(t) = z \right],$$

168 where $Z_\ell(s)$ follows (2.5). If we let the equilibrium distribution be given by m_ℓ (for
169 $\ell = 1, \dots, L$), then the MFG PDE system is stationary and given by

$$170 \quad (3.4a) \quad V_\ell \in H^1(\Omega)$$

$$171 \quad (3.4b) \quad m_\ell \in H^1(\Omega)$$

$$172 \quad (3.4c) \quad -\frac{\sigma^2}{2} V_\ell'' + \rho V_\ell - H_\ell(z, m, V_\ell') = 0$$

$$173 \quad (3.4d) \quad -\frac{\sigma^2}{2} m_\ell'' + (\partial_\lambda H_\ell(z, m, V_\ell') m_\ell)' = 0$$

$$174 \quad (3.4e) \quad V_\ell'|_{z=0, \bar{z}} = 0$$

$$175 \quad (3.4f) \quad -\frac{\sigma^2}{2} m_\ell' + \partial_\lambda H_\ell(z, m, 0) m_\ell \Big|_{z=0, \bar{z}} = 0$$

$$176 \quad (3.4g) \quad \int_{\Omega} m_\ell(z) dz = 1. \\ 177$$

178 It can be shown, using either the dynamic programming principle (c.f [25]) or the
179 stochastic maximum principle (c.f. [8]), that $V_\ell(z)$, as defined by (3.3), satisfies the
180 HJB equation (3.4c), (3.4e). The Fokker–Planck system (3.4d), (3.4f), (3.4g) comes
181 from the distribution in the previous section (2.7) and the consistency condition (3.1b).

182 3.2. Existence and uniqueness of solutions to the MFG.

183 **DEFINITION 3.2.** A solution to the innovation MFG model (3.4) is defined to be
184 a tuple $(m, V) = (m_1, \dots, m_L, V_1, \dots, V_L)$ such that $m_\ell : \Omega \rightarrow (0, \infty)$, $V_\ell : \Omega \rightarrow \mathbb{R}$
185 satisfy (3.4) in the weak sense for each $\ell = 1, \dots, L$.

186 **THEOREM 3.3.** *There exists a solution $(m, V) \in [C^2(\Omega) \cap C^1(\bar{\Omega})]^{2L}$ to (3.4).*
 187 *Furthermore, if $\sum_{\ell'=1}^L A_{\ell'} p(\ell, \ell')$ is small enough for every $\ell = 1, \dots, L$, then the solu-*
 188 *tion is unique.*

189 *Proof outline.* As noted in the introduction, this proof is based on the proof of
 190 existence and uniqueness in [6]. The proof presented here has some technical differ-
 191 ences compared with the one in [6], hence it is reproduced in full. However, it follows
 192 a similar framework and so we do not claim the proof to be new. First, for $k \in [0, \infty)$
 193 we introduce an auxiliary system of PDEs defined by

$$194 \quad (3.5a) \quad V^k \in H^1(\Omega)$$

$$195 \quad (3.5b) \quad -\frac{\sigma^2}{2} (V^k)'' + \rho V^k - H^k(z, (V^k)') = 0$$

$$196 \quad (3.5c) \quad (V^k)' \Big|_{z=0, \bar{z}} = 0,$$

198

$$199 \quad (3.6a) \quad m^k \in H^1(\Omega)$$

$$200 \quad (3.6b) \quad -\frac{\sigma^2}{2} (m^k)'' + \left(\left[\left(\frac{\gamma}{w} \max(0, (V^k)') \right)^{\frac{\gamma}{1-\gamma}} + k \right] m^k \right)' = 0$$

$$201 \quad (3.6c) \quad -\frac{\sigma^2}{2} (m^k)' + km^k \Big|_{z=0, \bar{z}} = 0$$

$$202 \quad (3.6d) \quad \int_{\Omega} m^k(z) dz = 1,$$

203

204 where $H^k(z, \lambda) = (1 - \gamma) \left(\frac{\gamma}{w} \right)^{\frac{\gamma}{1-\gamma}} (\max(0, \lambda))^{\frac{1}{1-\gamma}} + k\lambda + \frac{z^\alpha}{B^{\alpha-1}}$. We use a modified
 205 version of upper and lower solutions (c.f. [23]) to prove existence and uniqueness
 206 of a weak solution V^k to (3.5) for any $k \in [0, \infty)$, and use elliptic regularity the-
 207 ory to show $V^k \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Next we define $m^k = \frac{1}{\|\bar{m}^k\|_1} \bar{m}^k$, where $\bar{m}^k =$
 208 $e^{\frac{\sigma^2}{2} \left(kz + \int_0^z \left(\frac{\gamma}{w} \max(0, (V^k)') \right)^{\frac{\gamma}{1-\gamma}} dy \right)}$ and $\|\bar{m}^k\|_1 = \int_{\Omega} \bar{m}^k dz$, for $k \in [0, \infty)$. We prove
 209 that $m^k \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and that m^k is the unique solution of (3.6). Finally, we de-
 210 fine a map $\Phi : [0, \infty)^L \rightarrow [0, \infty)^L$ by $\Phi_\ell(k) = \sum_{\ell'=1}^L A_{\ell'} p(\ell, \ell') \int_{\Omega} z m^{k_{\ell'}}(z) dz$ for $\ell =$
 211 $1, \dots, L$, and using the Brouwer fixed point theorem we prove there exists $\bar{k} \in [0, \infty)^L$
 212 such that $\Phi(\bar{k}) = \bar{k}$. We use the contraction mapping theorem to prove uniqueness
 213 under certain smallness assumptions for the data. Then it follows, by replacing \bar{k}_ℓ
 214 with $\Phi(\bar{k}_\ell)$ in (3.5) and (3.6), that $(m^{\bar{k}}, V^{\bar{k}}) = (m^{\bar{k}_1}, \dots, m^{\bar{k}_L}, V^{\bar{k}_1}, \dots, V^{\bar{k}_L})$ is a
 215 (unique) solution to (3.4) with the required regularity. \square

216 Solutions to the auxiliary HJB PDE.

217 **THEOREM 3.4.** *There exists a unique solution $V^k \in C^{2,\tau}(\bar{\Omega})$ to the auxiliary*
 218 *HJB PDE (3.5) for any $k \in [0, \infty)$ and some $\tau \in (0, 1)$, where $C^{2,\tau}(\bar{\Omega})$ is the set*
 219 *of C^2 functions on $\bar{\Omega}$ whose second derivative is Hölder continuous with exponent τ .*
 220 *Furthermore, $0 \leq V^k \leq \frac{z^\alpha}{\rho B^{\alpha-1}}$*

221 *Proof.* The existence part of the proof uses the theory of upper and lower solu-
 222 tions, specifically Theorem 4.3. in [20], and follows along similar lines to the proof
 223 of Proposition 3.12 in [6]. This shows that a solution $V^k \in W^{1,p}(\Omega)$ to the auxiliary
 224 HJB PDE exists, for some $p \geq 1$, provided the following hold true:

- 225 1. There exist constants $V \leq \bar{V}$ such that $\rho V - \frac{z^\alpha}{B^{\alpha-1}} \leq 0 \leq \rho \bar{V} - \frac{z^\alpha}{B^{\alpha-1}}$, for
 226 every $z \in \bar{\Omega}$.
 227 2. There exist constants $a_k \in \mathbb{R}$ and $b_k > 0$ such that

$$228 \quad \left| \rho u - (1 - \gamma) \left(\frac{\gamma}{w} \right)^{\frac{\gamma}{1-\gamma}} (\max(0, \lambda))^{\frac{1}{1-\gamma}} - k\lambda - \frac{z^\alpha}{B^{\alpha-1}} \right| \leq a_k + b_k |\lambda|^p,$$

229 for every $z \in \Omega$, $u \in [V, \bar{V}]$ and every $\lambda \in \mathbb{R}$.
 230 If these two properties hold, then $V \leq V^k \leq \bar{V}$. The first assertion is true by taking
 231 $V = 0$ and $\bar{V} = \frac{\bar{z}^\alpha}{\rho B^{\alpha-1}}$, which also gives the required bounds for V^k . The second
 232 assertion is true with $b_k = k + (1 - \gamma) \left(\frac{\gamma}{w} \right)^{\frac{\gamma}{1-\gamma}}$, $a_k = \frac{2\bar{z}^\alpha}{B^{\alpha-1}} + b_k$, and $p = \frac{2}{1-\gamma}$, as then

$$233 \quad \left| \rho u - (1 - \gamma) \left(\frac{\gamma}{w} \right)^{\frac{\gamma}{1-\gamma}} (\max(0, \lambda))^{\frac{1}{1-\gamma}} - k\lambda - \frac{z^\alpha}{B^{\alpha-1}} \right| \\ \leq \rho |u| + \left(k + (1 - \gamma) \left(\frac{\gamma}{w} \right)^{\frac{\gamma}{1-\gamma}} \right) \max \left(1, |\lambda|^{\frac{1}{1-\gamma}} \right) + \frac{\bar{z}^\alpha}{B^{\alpha-1}} \leq a_k + b_k |\lambda|^p.$$

234 Now, since Ω is bounded and $p > 2$, $V^k \in H^1(\Omega)$. To show $V^k \in C^{2,\tau}(\bar{\Omega})$, take any
 235 solution V^k to (3.5) and define

$$236 \quad f = \frac{2}{\sigma^2} \left(\left(\frac{\sigma^2}{2} - \rho \right) V^k + k (V^k)' + (1 - \gamma) \left(\frac{\gamma}{w} \right)^{\frac{\gamma}{1-\gamma}} \left(\max(0, (V^k)') \right)^{\frac{1}{1-\gamma}} + \frac{z^\alpha}{B^{\alpha-1}} \right).$$

237 Then V^k is a solution of $-u'' + u = f$, where $f \in L^2(\Omega)$. So, from the elliptic regularity
 238 result of Proposition 7.2. p.404 in [24], $V^k \in H^2(\Omega)$. Therefore $(V^k)' \in H^1(\Omega)$, and
 239 so $f \in H^1(\Omega)$ because $\alpha \in (0, 1)$. So, from the elliptic regularity result of Proposition
 240 7.4. p.407 in [24], $V^k \in H^3(\Omega)$. Then, by the Sobolev inequality (c.f. Theorem 6
 241 p.270 in [13]) $V^k \in C^{2,\tau}(\bar{\Omega})$.

242 To prove uniqueness we use the strong maximum principle and Hopf's lemma, as
 243 stated in [13] Section 6.4.2. pp. 330–333. Suppose, for some $k \in [0, \infty)$, there are
 244 two solutions $V_1, V_2 \in C^2(\Omega) \cap C^1(\bar{\Omega})$ to (3.5) and $V_1 \neq V_2$. If we define $u = V_1 - V_2$,
 245 then u must attain its maximum at some point $z^* \in \bar{\Omega}$. Suppose at this point $u > 0$.
 246 Note that if this were not the case, we could consider the minimum, as either its
 247 maximum or its minimum must be non-zero. The argument for the minimum is the
 248 same as the one for the maximum, so it is omitted. First suppose $z^* \in \Omega$. Since this
 249 is the maximal point, $u'(z^*) = 0$, so $V_1'(z^*) = V_2'(z^*)$. Hence, there exists an open,
 250 connected and bounded region U such that $U \subset \Omega$, $z^* \in U$ and

$$251 \quad -\frac{\sigma^2}{2} u'' = -\rho u + k u' + (1 - \gamma) \left(\frac{\gamma}{w} \right)^{\frac{\gamma}{1-\gamma}} \left[\max(0, V_1')^{\frac{1}{1-\gamma}} - \max(0, V_2')^{\frac{1}{1-\gamma}} \right] \leq 0,$$

252 for every $z \in U$. So, by the strong maximum principle, u is constant in U . In
 253 particular, using (3.5b), $u(z^*) = 0$. But this is a contradiction. The only other case
 254 is $z^* \in \partial\Omega$ and $u(z) < u(z^*)$ for every $z \in \Omega$. Then, $\frac{\partial u}{\partial \nu} \Big|_{z^*} > 0$ by Hopf's Lemma,
 255 but by (3.5c), $\frac{\partial u}{\partial \nu} = \frac{\partial V_1}{\partial \nu} - \frac{\partial V_2}{\partial \nu} = 0$. This again leads to a contradiction. Therefore
 256 $V_1 = V_2$ and solutions to (3.5) are unique for every $k \in [0, \infty)$. \square

257 PROPOSITION 3.5. Fix $k, k_1, k_2 \in [0, \infty)$. Then, the unique classical solution to
 258 the auxiliary HJB PDE (3.5), as found in Theorem 3.4, satisfies the following prop-
 259 erties:

- 260 1. V^k is an increasing function on $\bar{\Omega}$ i.e. $(V^k)' \geq 0$

- 261 2. $(V^k)' > 0$ for all $z \in \Omega$
 262 3. $\|(V^k)'\|_\infty = \sup_{z \in \Omega} (V^k)'(z) \leq \left[\frac{\bar{z}^\alpha}{(1-\gamma)B^{\alpha-1}} \right]^{1-\gamma} \left(\frac{w}{\gamma} \right)^\gamma$
 263 4. $\|V^{k_1} - V^{k_2}\|_\infty \leq \frac{1}{\rho} \left[\frac{\bar{z}^\alpha}{(1-\gamma)B^{\alpha-1}} \right]^{1-\gamma} \left(\frac{w}{\gamma} \right)^\gamma |k_1 - k_2|$
 264 5. V^k is strictly increasing with respect to k
 265 6. $\|(V^{k_1})' - (V^{k_2})'\|_\infty \leq \frac{4\bar{z}}{\sigma^2} \left[\frac{\bar{z}^\alpha}{(1-\gamma)B^{\alpha-1}} \right]^{1-\gamma} \left(\frac{w}{\gamma} \right)^\gamma |k_1 - k_2|$
 266 7. $(V^k)''(0) > 0 > (V^k)''(\bar{z})$.

267 *Proof.* Property (1): Suppose, for a contradiction, there exists $z \in \bar{\Omega}$ such that
 268 $(V^k)'(z) < 0$. First, by the boundary condition (3.5c), $z \in \Omega$. So, by the boundary
 269 conditions and continuity of $(V^k)'$, there exists $z_0, z_1 \in \bar{\Omega}$ with $z_0 < z_1$, $(V^k)'(z_0) =$
 270 $(V^k)'(z_1) = 0$ and $(V^k)'(z) \leq 0$ for all $z \in (z_0, z_1)$. Suppose that $z_0, z_1 \in \Omega$. Then
 271 $(V^k)''(z_0) \leq 0 \leq (V^k)''(z_1)$ by construction of z_0, z_1 and differentiability of $(V^k)'$.
 272 Furthermore, $V^k(z_0) > V^k(z_1)$ because $(V^k)' < 0$ in (z_0, z_1) . So, using (3.5b)

$$273 \quad 0 = -\frac{\sigma^2}{2} \left((V^k)''(z_1) - (V^k)''(z_0) \right) + \rho (V^k(z_1) - V^k(z_0)) - \frac{1}{B^{\alpha-1}} (z_1^\alpha - z_0^\alpha) < 0.$$

274 This is a contradiction, so $z_0 = 0$ or $z_1 = \bar{z}$. Assume $z_0 = 0$, we will again prove
 275 a contradiction (the other two cases of $z_1 = \bar{z}$ and both $z_0 = 0, z_1 = \bar{z}$ follow along
 276 similar arguments so their proofs are omitted). Since $(V^k)'(0) = (V^k)'(z_1) = 0$ and
 277 $(V^k)'(z) < 0$ for all $z \in (0, z_1)$ then, by continuity of $(V^k)''$, we can find $\epsilon_1, \delta_1 \in$
 278 $(0, \frac{z_1}{2})$ such that $(V^k)''(z) \leq 0$ for all $z \in (0, \epsilon_1]$ and $(V^k)''(z) \geq 0$ for all $z \in$
 279 $[z_1 - \delta_1, z_1)$. Furthermore, V^k is strictly decreasing on (z_0, z_1) . So, using these two
 280 facts and continuity of $(V^k)'$ there exists $\delta \in (0, \delta_1]$ and $\epsilon \in (0, \epsilon_1]$ such that

- 281 1. $(V^k)'(\epsilon) = (V^k)'(z_1 - \delta) = \min \left((V^k)'(\epsilon_1), (V^k)'(z_1 - \delta_1) \right)$
 282 2. $V^k(\epsilon) > V^k(z_1 - \delta)$
 283 3. $(V^k)''(\epsilon) \leq 0 \leq (V^k)''(z_1 - \delta)$.

284 Then $-\frac{\sigma^2}{2} \left((V^k)''(z_1 - \delta) - (V^k)''(\epsilon) \right) + \rho (V^k(z_1 - \delta) - V^k(\epsilon)) - \frac{(z_1 - \delta)^\alpha - \epsilon^\alpha}{B^{\alpha-1}} < 0$,

285 which contradicts the fact that V^k is a solution to (3.5). Therefore, $(V^k)' \geq 0$ in $\bar{\Omega}$.

286 Property (2): From Property (1), we know $(V^k)' \geq 0$. Now suppose, for a
 287 contradiction, there exists $z^* \in \Omega$ such that $(V^k)'(z^*) = 0$. Then $(V^k)''(z^*) = 0$,
 288 since it is a minimum of $(V^k)'$. So, by (3.5b), $V^k(z^*) = \frac{z^{\alpha}}{\rho B^{\alpha-1}}$ and, since $(V^k)'(z^*) <$
 289 $\frac{d}{dz} \left(\frac{z^\alpha}{\rho B^{\alpha-1}} \right)$, there exists $z_0, z_1 \in \Omega$ with $z_0 < z^* < z_1$ such that

- 290 1. $(V^k)'(z_0) = (V^k)'(z_1)$
 291 2. $V^k(z_0) > \frac{z_0^\alpha}{\rho B^{\alpha-1}}$ and $V^k(z_1) < \frac{z_1^\alpha}{\rho B^{\alpha-1}}$
 292 3. $(V^k)''(z_0) \leq 0 \leq (V^k)''(z_1)$.

293 Then $-\frac{\sigma^2}{2} \left((V^k)''(z_1) - (V^k)''(z_0) \right) + \rho (V^k(z_1) - V^k(z_0)) - \frac{1}{B^{\alpha-1}} (z_1^\alpha - z_0^\alpha) < 0$,

294 which is a contradiction of (3.5b). Therefore, $(V^k)'(z) > 0$ for all $z \in \Omega$.

295 Property (3): Since $(V^k)'$ is continuous on $\bar{\Omega}$, $(V^k)' \geq 0$ and $(V^k)'(0) =$
 296 $(V^k)'(\bar{z}) = 0$, then $(V^k)'$ must have a maximum that it attains at some point $z^* \in \Omega$.
 297 Furthermore, since $(V^k)'$ is continuously differentiable in Ω , then $(V^k)''(z^*) = 0$. So,

298 using the bound on V^k found in Theorem 3.4

$$\begin{aligned}
0 \leq (V^k)'(z) &\leq (V^k)'(z^*) = \left[\frac{w^{\frac{\gamma}{1-\gamma}}}{(1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}} \left(\rho V^k(z^*) - k (V^k)'(z^*) - \frac{(z^*)^\alpha}{B^{\alpha-1}} \right) \right]^{1-\gamma} \\
&\leq \left[\frac{\bar{z}^\alpha}{(1-\gamma)B^{\alpha-1}} \right]^{1-\gamma} \left(\frac{w}{\gamma} \right)^\gamma.
\end{aligned}$$

300 Property (4): Take $k_1, k_2 \in [0, \infty)$ such that $k_1 < k_2$. First we show $V^{k_2} -$
301 $V^{k_1} \geq 0$, then we show $V^{k_2} - V^{k_1} \leq \frac{\|(V^{k_1})'\|_\infty}{\rho}(k_2 - k_1)$, and we can conclude using
302 Property (3). Let $u_1 = V^{k_2} - V^{k_1}$ and assume, for a contradiction, there exists $z \in \bar{\Omega}$
303 such that $u_1(z) < 0$. Then, u_1 attains a minimum at $z^* \in \bar{\Omega}$ and $u_1(z^*) < 0$. First
304 suppose $z^* \in \Omega$, then $u_1'(z^*) = 0$ and from (3.5b)

$$-\frac{\sigma^2}{2} u_1''(z^*) = -\rho u_1(z^*) + (k_2 - k_1) (V^{k_1})'(z^*) > 0,$$

306 since $(V^{k_1})' \geq 0$ and $u_1 < 0$. Then, by continuity of u_1'' , there exists an open bounded,
307 connected $\Omega' \subset \Omega$ such that $z^* \in \Omega'$ and $u_1'' < 0$ for all $z \in \Omega'$. So, by the strong
308 maximum principle, u_1 is constant in Ω' . In particular, $u_1' = 0$, which contradicts
309 $u_1'' < 0$ for all $z \in \Omega'$. So, $z^* \in \partial\Omega$ and $u_1(z) < u_1(z^*)$ for all $z \in \Omega$. However, from
310 Hopf's lemma $u_1'(z^*) \neq 0$, which contradicts (3.5c). So, we conclude that $u_1 \geq 0$.

311 Now let $u_2 = V^{k_2} - V^{k_1} - \epsilon$, with $\epsilon = \frac{\|(V^{k_1})'\|_\infty}{\rho}(k_2 - k_1) < \infty$. We assume, for a
312 contradiction, there exists $z \in \bar{\Omega}$ such that $u_2(z) > 0$. Then u_2 attains a maximum
313 at $z^* \in \bar{\Omega}$ and $u_2(z^*) > 0$. First suppose $z^* \in \Omega$, then $u_2'(z^*) = 0$ and from (3.5b)

$$-\frac{\sigma^2}{2} u_2''(z^*) = -\rho u_2(z^*) + (k_2 - k_1) (V^{k_1})'(z^*) - \rho\epsilon < 0,$$

315 since $u_2 > 0$ and $\rho\epsilon \geq (k_1 - k_2) (V^{k_1})'(z^*)$. Then, by continuity of u_2'' , there exists
316 an open bounded, connected $\Omega' \subset \Omega$ such that $z^* \in \Omega'$ and $u_2'' > 0$ for all $z \in \Omega'$.
317 So, by the strong maximum principle, u_2 is constant in Ω' . In particular, $u_2'' = 0$,
318 which contradicts $u_2'' > 0$ for all $z \in \Omega'$. So, $z^* \in \partial\Omega$ and $u_2(z) > u_2(z^*)$ for all
319 $z \in \Omega$. However, from Hopf's lemma $u_2'(z^*) \neq 0$, which contradicts (3.5c). So, we can
320 conclude that $u_2 \leq 0$.

321 Property (5): The proof of Property (4) shows V^k is increasing with respect
322 to k . Now suppose, for a contradiction, there exists $z^* \in \bar{\Omega}$ such that $k_1 < k_2$
323 but $V^{k_1}(z^*) = V^{k_2}(z^*)$. First, assume $z^* \in \Omega$ and define $u = V^{k_2} - V^{k_1}$. Then,
324 $u(z^*) = 0$, $u'(z^*) = 0$ and $u''(z^*) \geq 0$, since z^* is a minimum of u . Furthermore, from
325 Property (2), $(V^{k_1})'(z^*) > 0$. Therefore, using (3.5b), we get the contradiction

$$0 = -\frac{\sigma^2}{2} u'' + (k_1 - k_2) (V^{k_1})' < 0.$$

327 Hence, $z^* \in \partial\Omega$, so $u'(z^*) = 0$, using (3.5c). But, $u'(z^*) \neq 0$ by Hopf's lemma, which
328 is a contradiction. So, V^k is strictly increasing with respect to k .

329 Property (6): Fix $k_1, k_2 \in [0, \infty)$. Let $u = V^{k_1} - V^{k_2}$. Then, u satisfies

$$\frac{\sigma^2}{2} u'' = \rho u - k_1 u' + (k_2 - k_1) (V^{k_2})' - (1-\gamma) \left(\frac{\gamma}{w} \right)^{\frac{\gamma}{1-\gamma}} \left(\left((V^{k_1})' \right)^{\frac{1}{1-\gamma}} - \left((V^{k_2})' \right)^{\frac{1}{1-\gamma}} \right).$$

331 Suppose for $z \in \Omega$, $u'(z) \geq 0$. Then, since $u'(0) = 0$, there exists $z_0 \in [0, z]$ such that
 332 $u'(y) \geq 0$ for all $y \in [z_0, z]$ and $u'(z_0) = 0$. Therefore

$$333 \quad 0 \leq u'(z) = \int_{z_0}^z u''(y) dy \leq \frac{2\bar{z}}{\sigma^2} \left(\rho \|u\|_\infty + |k_2 - k_1| \left\| (V^{k_2})' \right\|_\infty \right) \\ \leq \frac{4\bar{z}}{\sigma^2} \left[\frac{\bar{z}^\alpha}{(1-\gamma)B^{\alpha-1}} \right]^{1-\gamma} \left(\frac{w}{\gamma} \right)^\gamma |k_2 - k_1|.$$

334 We can similarly show that $u'(z) \geq -\frac{4\bar{z}}{\sigma^2} \left[\frac{\bar{z}^\alpha}{(1-\gamma)B^{\alpha-1}} \right]^{1-\gamma} \left(\frac{w}{\gamma} \right)^\gamma |k_2 - k_1|$ if $u'(z) \leq 0$.

335 Hence, $(V^k)'$ is Lipschitz continuous with respect to k with the required constant.

336 Property (7): First, we will show $(V^k)''(0) > 0$ and then we will show $(V^k)''(\bar{z}) <$
 337 0 . Both steps use a similar method. Note that $(V^k)''(0) \geq 0$ and $(V^k)''(\bar{z}) \leq 0$,
 338 because $(V^k)'(0) = (V^k)'(\bar{z}) = 0$ and $(V^k)(z) > 0$ for all $z \in (0, \bar{z})$. So suppose,
 339 for a contradiction, that $(V^k)''(0) = 0$. Then, since $V^k \in C^{2,\tau}(\bar{\Omega})$, we can use
 340 continuity of V^k , $(V^k)'$, $(V^k)''$ and (3.5b), (3.5c) to show $V^k(0) = 0$. We can also
 341 use continuity of $(V^k)''$ to show that for every $C > 0$ there exists $\epsilon_1 > 0$ such that
 342 $z \in (0, \epsilon_1) \implies (V^k)''(z) < C$. Therefore, for any $z \in (0, \epsilon_1)$

$$343 \quad (3.7) \quad V^k(z) = \int_0^z \int_0^y (V^k)''(y') dy' dy - z (V^k)'(0) - V^k(0) < \frac{C}{2} z^2.$$

344 Since $(V^k)' > 0$ in Ω , there exists $\epsilon_2 > 0$ such that $(V^k)'$ increases on $(0, \epsilon_2)$. There-
 345 fore, $(V^k)'' \geq 0$ on $(0, \epsilon_2)$. Take $C = \frac{2}{\rho B^{\alpha-1}}$ and $\epsilon = \min(\epsilon_1, \epsilon_2, 1)$. Then, from (3.7)

$$346 \quad (3.8) \quad V^k(z) < \frac{z^2}{\rho B^{\alpha-1}} \leq \frac{z^\alpha}{\rho B^{\alpha-1}},$$

347 for all $z \in (0, \epsilon)$. But, by rearranging (3.5b) and using $(V^k)', (V^k)'' \geq 0$, we get
 348 $V^k(z) \geq \frac{z^\alpha}{\rho B^{\alpha-1}}$, which contradicts (3.8). Hence, $(V^k)''(0) > 0$. Now suppose, for a
 349 contradiction, that $(V^k)''(\bar{z}) = 0$. Then, since $V^k \in C^{2,\tau}(\bar{\Omega})$, we can use continuity
 350 of V^k , $(V^k)'$, $(V^k)''$ and (3.5b), (3.5c) to show $V^k(\bar{z}) = \frac{\bar{z}^\alpha}{\rho B^{\alpha-1}}$. We can also use
 351 continuity of $(V^k)''$ to show that for every $C > 0$ there exists $\epsilon_1 > 0$ such that
 352 $z \in (\bar{z} - \epsilon_1, \bar{z}) \implies (V^k)''(z) > -C$. Therefore, for any $z \in (\bar{z} - \epsilon_1, \bar{z})$

$$353 \quad (3.9) \quad V^k(z) = \int_z^{\bar{z}} \int_y^{\bar{z}} (V^k)''(y) dy + V^k(\bar{z}) - (\bar{z} - z) (V^k)'(\bar{z}) > \frac{\bar{z}^\alpha}{\rho B^{\alpha-1}} - \frac{C}{2} (\bar{z} - z)^2.$$

354 Since $(V^k)' > 0$ in Ω , there exists $\epsilon_2 > 0$ such that $(V^k)'$ decreases on $(\bar{z} - \epsilon_2, \bar{z})$.
 355 Therefore, $(V^k)'' \leq 0$ on $(\bar{z} - \epsilon_2, \bar{z})$. Take $C > 0$ such that

$$356 \quad \frac{kC}{\rho} + \frac{C}{2} + \frac{C^{\frac{1}{1-\gamma}}}{\rho} (1-\gamma) \left(\frac{\gamma}{w} \right)^{\frac{\gamma}{1-\gamma}} \leq \alpha \frac{\bar{z}^{\alpha-1}}{\rho B^{\alpha-1}},$$

357 and $\epsilon = \min(\epsilon_1, \epsilon_2, 1)$. Then, from (3.9)

$$358 \quad (3.10) \quad V^k(z) > \frac{\bar{z}^\alpha}{\rho B^{\alpha-1}} - \frac{C}{2} (\bar{z} - z)^2 \geq \frac{\bar{z}^\alpha}{\rho B^{\alpha-1}} - \frac{C}{2} (\bar{z} - z),$$

359 for all $z \in (\bar{z} - \epsilon, \bar{z})$. But, from (3.5b) and using $(V^k)' \leq C(\bar{z} - z)$, $(V^k)'' \leq 0$, we get

$$\begin{aligned}
V^k(z) &= \frac{1}{\rho} \left(\frac{\sigma^2}{2} (V^k)''(z) + k (V^k)'(z) + (1 - \gamma) \left(\frac{\gamma}{w} \right)^{\frac{1}{1-\gamma}} \left((V^k)' \right)^{\frac{1}{1-\gamma}} + \frac{z^\alpha}{B^{\alpha-1}} \right) \\
360 \quad &\leq \frac{kC}{\rho} (\bar{z} - z) + \frac{C^{\frac{1}{1-\gamma}}}{\rho} (1 - \gamma) \left(\frac{\gamma}{w} \right)^{\frac{1}{1-\gamma}} (\bar{z} - z)^{\frac{1}{1-\gamma}} + \frac{z^\alpha}{\rho B^{\alpha-1}} \\
&\leq \frac{z^\alpha}{\rho B^{\alpha-1}} + \left(\frac{kC}{\rho} + \frac{C^{\frac{1}{1-\gamma}}}{\rho} (1 - \gamma) \left(\frac{\gamma}{w} \right)^{\frac{1}{1-\gamma}} \right) (\bar{z} - z).
\end{aligned}$$

361 So, using (3.10), we get $\alpha \frac{\bar{z}^{\alpha-1}}{\rho B^{\alpha-1}} (\bar{z} - z) < \left(\frac{kC}{\rho} + \frac{C^{\frac{1}{1-\gamma}}}{\rho} (1 - \gamma) \left(\frac{\gamma}{w} \right)^{\frac{1}{1-\gamma}} \right) (\bar{z} - z)$, which
362 contradicts the definition of C . Hence, $(V^k)''(\bar{z}) < 0$. \square

363 The auxiliary Fokker–Planck equation.

364 DEFINITION 3.6. Fix $k \in [0, \infty)$ and let $V^k \in C^2(\Omega) \cap C^1(\bar{\Omega})$ denote the unique
365 solution to (3.5). Then, we define the function $m^k : \Omega \rightarrow (0, \infty)$ by

$$366 \quad (3.11a) \quad \bar{m}^k = e^{\frac{2}{\sigma^2} \left(kz + \int_0^z \left(\frac{\gamma}{w} (V^k)' \right)^{\frac{1}{1-\gamma}} dy \right)}$$

367

$$368 \quad (3.11b) \quad \|\bar{m}^k\|_1 = \int_{\Omega} \bar{m}^k dz$$

369

$$370 \quad (3.11c) \quad m^k = \frac{1}{\|\bar{m}^k\|_1} \bar{m}^k.$$

371 PROPOSITION 3.7. For every $k \in [0, \infty)$, $m^k \in C^2(\Omega) \cap C^1(\bar{\Omega})$ where m^k is
372 defined by (3.6).

373 *Proof.* First, note that m^k is well defined because $(V^k)' \geq 0$ and $(V^k)'$ is uni-
374 formly bounded. Hence, there exists $C \in (1, \infty)$ such that $\bar{m}^k(z) \in [1, C]$ and
375 $\|\bar{m}^k\|_1 \in [\bar{z}, C\bar{z}]$, so $m^k(z) \in [\frac{1}{C\bar{z}}, \frac{C}{\bar{z}}]$. Furthermore, $m^k \in C(\bar{\Omega})$ because $V^k \in$
376 $C^1(\bar{\Omega})$. Now, if $m^k \in C^2(\Omega) \cap C^1(\bar{\Omega})$, then its derivatives would be

$$377 \quad (3.12a) \quad (m^k)' = \frac{2}{\sigma^2} \left(k + \left(\frac{\gamma}{w} (V^k)' \right)^{\frac{1}{1-\gamma}} \right) m^k$$

378

$$\begin{aligned}
379 \quad (3.12b) \quad (m^k)'' &= \frac{2}{\sigma^2} \left(k + \left(\frac{\gamma}{w} (V^k)' \right)^{\frac{1}{1-\gamma}} \right) (m^k)' \\
&\quad + \frac{2\gamma^2}{\sigma^2 w (1-\gamma)} \left(\frac{\gamma}{w} (V^k)' \right)^{\frac{2\gamma-1}{1-\gamma}} (V^k)'' m^k.
\end{aligned}$$

380 But, since $V^k \in C^1(\bar{\Omega})$ and $m^k \in C(\bar{\Omega})$, then $\frac{2}{\sigma^2} \left(k + \left(\frac{\gamma}{w} (V^k)' \right)^{\frac{1}{1-\gamma}} \right) m^k$ is well-
381 defined and continuous for all $z \in \bar{\Omega}$. Hence, $m^k \in C^1(\bar{\Omega})$. Then, $(m^k)'$, (V^k) and
382 $(V^k)''$ are continuous in Ω and from Proposition 3.5 $(V^k)' > 0$ in Ω . Hence, $(m^k)''$
383 is well-defined in Ω , $(m^k)'' \in C(\Omega)$ and $m^k \in C^2(\Omega) \cap C^1(\bar{\Omega})$. \square

384 THEOREM 3.8. *There exists a unique solution $m^k \in C^2(\Omega) \cap C^1(\bar{\Omega})$ to the aux-*
 385 *iliary Fokker–Planck PDE (3.6) for any $k \in [0, \infty)$.*

386 *Proof.* Take m^k defined in Definition 3.6. Then, $m^k \in C^2(\Omega) \cap C^1(\bar{\Omega})$ by Propo-
 387 sition 3.7. Furthermore, from (3.12), m^k satisfies (3.6b), (3.6c). Finally, by con-
 388 struction, m^k satisfies (3.6d). Therefore, a solution to the auxiliary Fokker–Planck
 389 equation (3.6) exists, it is given by m^k , and $m^k \in C^2(\Omega) \cap C^1(\bar{\Omega})$. To prove unique-
 390 ness we follow the same proof in [6]. For brevity we only outline the argument here.
 391 First, with \bar{m}^k defined as in (3.11), we can use regularity of \bar{m}^k from Proposition 3.7
 392 to show (3.6) is equivalent to

$$393 \quad (3.13a) \quad m^k, \frac{m^k}{\bar{m}^k} \in H^1(\Omega)$$

$$394 \quad (3.13b) \quad \left(\bar{m}^k \left(\frac{m^k}{\bar{m}^k} \right)' \right)' = 0$$

$$395 \quad (3.13c) \quad \bar{m}^k \left(\frac{m^k}{\bar{m}^k} \right)' \Big|_{z=0, \bar{z}} = 0, \quad \int_{\Omega} m^k dz = 1.$$

396

397 Then, by multiplying (3.13b) by $\frac{m^k}{\bar{m}^k}$, integrating over Ω and using integration by
 398 parts, the system (3.13) is equivalent to

$$399 \quad (3.14) \quad m^k \in H^1(\Omega), \quad \text{there exists } Z > 0 \text{ such that } m^k = \frac{1}{Z} \bar{m}^k, \quad \int_{\Omega} m^k dz = 1.$$

400 From the previous results in this section, we have shown there exists a unique solution
 401 to (3.14) given by m^k from Definition 3.6. Hence, existence and uniqueness of the
 402 auxiliary Fokker–Planck PDE follows from the equivalence between (3.6) and (3.14).□

403 The fixed point problem.

404 DEFINITION 3.9. Fix $k = (k_{\ell})_{\ell=1}^L \in [0, \infty)^L$. For $\ell = 1, \dots, L$, let $V^{k_{\ell}}$ be the
 405 unique solution to the auxiliary HJB PDE (3.5) with constant k_{ℓ} , and let $m^{k_{\ell}}$ be the
 406 unique solution to the auxiliary Fokker–Planck PDE (3.6) with constant k_{ℓ} . Then we
 407 define the function $\Phi : [0, \infty)^L \rightarrow [0, \infty)^L$ by

$$408 \quad \Phi_{\ell}(k) = \sum_{\ell'=1}^L A_{\ell'} p(\ell, \ell') \int_0^{\bar{z}} z m^{k_{\ell'}}(z) dz, \quad \ell = 1, \dots, L.$$

409 PROPOSITION 3.10. *The function Φ defined in Definition 3.9 is bounded. Fur-*
 410 *thermore, defining P as the $L \times L$ matrix with entries $P_{\ell, \ell'} = p(\ell, \ell')$ and A as the*
 411 *column vector $(A_1, \dots, A_L)^T$, then*

$$412 \quad 0 \leq \|\Phi(k)\|_1 \leq \bar{z} \|PA\|_1,$$

413 where the 1–norm $\|\cdot\|_1$ is defined as $\|x\|_1 = \sum_{\ell=1}^L |x_{\ell}|$ for any $x \in \mathbb{R}^L$.

414 Remark 3.11. Due to this proposition, we can define $\zeta = \bar{z} \|PA\|_1$ and consider
 415 only the restriction of Φ to $[0, \zeta]^L$, which we will still denote by Φ for convenience.

416 Proof. Take $\ell = 1, \dots, L$. Then $\Phi_{\ell}(k) = \sum_{\ell'=1}^L A_{\ell'} p(\ell, \ell') \int_0^{\bar{z}} z m^{k_{\ell'}}(z) dz \geq 0$,
 417 since $p(\ell, \ell') \geq 0$ and $m^{k_{\ell'}} \geq 0$. Similarly, since $m^{k_{\ell'}}$ is a probability distribution,

418 $\Phi_\ell(k) \leq \bar{z} \sum_{\ell'=1}^L A_{\ell'} p(\ell, \ell') \int_0^{\bar{z}} m^{k_{\ell'}}(z) dz = \bar{z} \sum_{\ell'=1}^L A_{\ell'} p(\ell, \ell')$. Therefore,

$$419 \quad 0 \leq \sum_{\ell=1}^L \Phi_\ell(k) \leq \bar{z} \sum_{\ell=1}^L \sum_{\ell'=1}^L A_{\ell'} p(\ell, \ell') = \bar{z} \|PA\|_1. \quad \square$$

420 **THEOREM 3.12.** *The function $\Phi : [0, \zeta]^L \rightarrow [0, \zeta]^L$ defined in Definition 3.9 is Lip-*
 421 *schitz in the 1-norm on \mathbb{R}^L . The Lipschitz constant is given by $C \max_{\ell=1, \dots, L} A_\ell P_\ell$,*
 422 *where $P_\ell = \sum_{\ell'=1}^L p(\ell', \ell)$ and C depends on $\|PA\|_1$, but not explicitly on P or A .*

423 *Proof.* First, fix $k \in [0, \zeta]$. From Property (5) of Proposition 3.5, the continuity
 424 of $V^k, (V^k)', (V^k)''$ with respect to z in $\bar{\Omega}$, and equations (3.5b), (3.5c), we find

$$425 \quad (V^k)''(0) = \frac{2\rho}{\sigma^2} V^k(0) \geq \frac{2\rho}{\sigma^2} V^0(0) = (V^0)''(0) > 0,$$

426 with the middle inequality an equality if and only if $k = 0$. Similarly, $(V^k)''(\bar{z}) \leq$
 427 $(V^\zeta)''(\bar{z}) < 0$ with the first inequality an equality if and only if $k = 0$. Moreover,
 428 $(V^k)''$ is continuous with respect to k due to (3.5b) and continuity of $V^k, (V^k)'$ with
 429 respect to k , which was proven in Proposition 3.5. Therefore, there exists $\epsilon_1, \epsilon_2 \in (0, 1)$
 430 and $C_1, C_2 > 0$, independent of k , such that

$$431 \quad \begin{aligned} (V^k)'(z) &= \int_0^z (V^k)''(y) dy \geq \int_0^z (V^0)''(y) dy \geq C_1 z, & \text{if } z \in [0, \epsilon_1] \\ (V^k)'(z) &= - \int_z^{\bar{z}} (V^k)''(y) dy \geq - \int_z^{\bar{z}} (V^\zeta)''(y) dy \geq C_2 z, & \text{if } z \in [\bar{z} - \epsilon_2, \bar{z}]. \end{aligned}$$

432 Furthermore, by continuity of $(V^k)'$ with respect to k and compactness of $[0, \zeta]$, there
 433 exists $C_3 > 0$ such that $\inf_{k \in [0, \zeta]} (V^k)'(z) \geq C_3$ if $z \in [\epsilon_1, \bar{z} - \epsilon_2]$. Note that C_j for
 434 $j = 1, 2, 3$ are all independent of $k \in [0, \zeta]$. Therefore, if $\gamma \leq \frac{1}{2}$, for any $k_1, k_2 \in [0, \zeta]$:

$$435 \quad \begin{aligned} (3.15) \quad & \int_0^{\bar{z}} \left[\min \left((V^{k_1})'(z), (V^{k_2})'(z) \right) \right]^{\frac{2\gamma-1}{1-\gamma}} dz \\ & \leq \int_0^{\epsilon_1} (C_1 z)^{\frac{2\gamma-1}{1-\gamma}} dz + \int_{\epsilon_1}^{\bar{z}-\epsilon_2} C_3^{\frac{2\gamma-1}{1-\gamma}} dz + \int_{\bar{z}-\epsilon_2}^{\bar{z}} (C_2(\bar{z}-z))^{\frac{2\gamma-1}{1-\gamma}} dz \\ & \leq \frac{1-\gamma}{\gamma} (C_1^{\frac{2\gamma-1}{1-\gamma}} + C_2^{\frac{2\gamma-1}{1-\gamma}}) + C_3^{\frac{2\gamma-1}{1-\gamma}} \bar{z}, \end{aligned}$$

436 while, using Proposition 3.5, if $\gamma \geq \frac{1}{2}$

$$437 \quad \begin{aligned} (3.16) \quad & \int_0^{\bar{z}} \left[\max \left((V^{k_1})'(z), (V^{k_2})'(z) \right) \right]^{\frac{2\gamma-1}{1-\gamma}} dz \\ & \leq \bar{z} \left(\frac{\bar{z}^\alpha}{(1-\gamma)B^{\alpha-1}} \right)^{2\gamma-1} \left(\frac{w}{\gamma} \right)^{\frac{\gamma(2\gamma-1)}{1-\gamma}}. \end{aligned}$$

438 Now, with the definition of \bar{m}^k in (3.11), for any $k_1, k_2 \in [0, \zeta]$ we have

$$439 \quad \begin{aligned} |\bar{m}^{k_1} - \bar{m}^{k_2}| &= \left| e^{\frac{2}{\sigma^2} \left(k_1 z + \int_0^z \left(\frac{\gamma}{w} (V^{k_1})'(y) \right)^{\frac{\gamma}{1-\gamma}} dy \right)} - e^{\frac{2}{\sigma^2} \left(k_2 z + \int_0^z \left(\frac{\gamma}{w} (V^{k_2})'(y) \right)^{\frac{\gamma}{1-\gamma}} dy \right)} \right| \\ &= \left| \frac{2}{\sigma^2} \int_{k_2 z + \int_0^z \left(\frac{\gamma}{w} (V^{k_2})'(y) \right)^{\frac{\gamma}{1-\gamma}} dy}^{k_1 z + \int_0^z \left(\frac{\gamma}{w} (V^{k_1})'(y) \right)^{\frac{\gamma}{1-\gamma}} dy} e^{\frac{2}{\sigma^2} u} du \right|. \end{aligned}$$

440 Then, using the uniform bound on $(V^k)'(y)$ with respect to k given by Proposition 3.5,
441 we get

$$\begin{aligned} |\bar{m}^{k_1} - \bar{m}^{k_2}| &\leq \frac{2\bar{C}_1}{\sigma^2} \left| (k_1 - k_2)z + \int_0^z \left[\left(\frac{\gamma}{w} (V^{k_1})'(y) \right)^{\frac{\gamma}{1-\gamma}} - \left(\frac{\gamma}{w} (V^{k_2})'(y) \right)^{\frac{\gamma}{1-\gamma}} \right] dy \right| \\ &\leq \frac{2\bar{C}_1}{\sigma^2} \left(|k_1 - k_2|z + \left(\frac{\gamma}{w} \right)^{\frac{\gamma}{1-\gamma}} \int_0^{\bar{z}} \int_{(V^{k_2})'(y)}^{(V^{k_1})'(y)} \frac{\gamma}{1-\gamma} u^{\frac{2\gamma-1}{1-\gamma}} du dy \right), \end{aligned}$$

443 where $\bar{C}_1 = e^{\frac{2\bar{z}}{\sigma^2}} \left(\zeta + \left[\frac{\gamma \bar{z}^\alpha}{(1-\gamma)wB^{\alpha-1}} \right]^\gamma \right)$. Then, using Proposition 3.5 and either (3.15)
444 or (3.16), we get

$$\begin{aligned} |\bar{m}^{k_1} - \bar{m}^{k_2}| &\leq \frac{2\bar{C}_1}{\sigma^2} \left(|k_1 - k_2|z + \left(\frac{\gamma}{w} \right)^{\frac{\gamma}{1-\gamma}} \frac{\gamma}{1-\gamma} \| (V^{k_1})' - (V^{k_2})' \|_\infty \right. \\ &\quad \left. \int_0^z \max \left[\left((V^{k_1})'(y) \right)^{\frac{2\gamma-1}{1-\gamma}}, \left((V^{k_2})'(y) \right)^{\frac{2\gamma-1}{1-\gamma}} \right] dy \right) \\ &\leq \frac{2\bar{C}_1}{\sigma^2} (z + \bar{C}_2) |k_1 - k_2|, \end{aligned}$$

446 where $\bar{C}_2 = \frac{4\bar{z}}{\sigma^2} \left(\frac{\bar{z}}{(1-\gamma)B^{\alpha-1}} \right)^{1-\gamma} \left(\frac{\gamma}{w} \right)^{\frac{\gamma}{1-\gamma}} \left(C_1^{\frac{2\gamma-1}{1-\gamma}} + C_2^{\frac{2\gamma-1}{1-\gamma}} + \frac{\gamma}{1-\gamma} C_3^{\frac{2\gamma-1}{1-\gamma}} \bar{z} \right)$, if $\gamma < \frac{1}{2}$.

447 While $\bar{C}_2 = \frac{\gamma}{1-\gamma} \frac{4\bar{z}^2}{\sigma^2} \left(\frac{w\bar{z}^\alpha}{\gamma(1-\gamma)B^{\alpha-1}} \right)^\gamma$, if $\gamma \geq \frac{1}{2}$. Note that for any $k \in [0, \zeta]$, $\|\bar{m}^k\|_1$
448 satisfies

$$(3.18) \quad \|\bar{m}^k\|_1 = \int_0^{\bar{z}} e^{\frac{2}{\sigma^2} \left[kz + \int_0^z \left(\frac{\gamma}{w} (V^k)'(y) \right)^{\frac{\gamma}{1-\gamma}} dy \right]} dz \geq 1,$$

450 as $(V^k)' \geq 0$. So, for any $k_1, k_2 \in [0, \zeta]$, using (3.17) and (3.18), we have
451 (3.19)

$$\begin{aligned} \left| \int_0^{\bar{z}} z(m^{k_1} - m^{k_2}) dz \right| &\leq \frac{1}{\|\bar{m}^{k_1}\|_1} \left| \int_0^{\bar{z}} z(\bar{m}^{k_1} - \bar{m}^{k_2}) dz \right| \\ &\quad + \int_0^{\bar{z}} z \frac{\bar{m}^{k_2}}{\|\bar{m}^{k_1}\|_1 \|\bar{m}^{k_2}\|_1} dz \left| \|\bar{m}^{k_1}\|_1 - \|\bar{m}^{k_2}\|_1 \right| \\ &\leq 2\bar{z} \int_0^{\bar{z}} |\bar{m}^{k_1} - \bar{m}^{k_2}| dz \leq \frac{4\bar{C}_1 \bar{z}}{\sigma^2} \int_0^{\bar{z}} (z + \bar{C}_2) dz |k_1 - k_2| \\ &= \frac{2\bar{C}_1 \bar{z}^2 (\bar{z} + 2\bar{C}_2)}{\sigma^2} |k_1 - k_2| := C |k_1 - k_2|. \end{aligned}$$

452 Now take $k^{(1)}, k^{(2)} \in [0, \zeta]^L$. Define $P_\ell = \sum_{\ell'=1}^L p(\ell', \ell)$, then recalling the definition
453 of Φ given in Definition 3.9 and using (3.19)

$$\begin{aligned} \left\| \Phi(k^{(1)}) - \Phi(k^{(2)}) \right\|_1 &= \sum_{\ell=1}^L \sum_{\ell'=1}^L A_{\ell'} p(\ell, \ell') \left| \int_0^{\bar{z}} z(m^{k_{\ell'}^{(1)}} - m^{k_{\ell'}^{(2)}}) dz \right| \\ &\leq C \sum_{\ell'=1}^L A_{\ell'} P_{\ell'} \left| k_{\ell'}^{(1)} - k_{\ell'}^{(2)} \right| \leq C \max_{\ell=1, \dots, L} A_\ell P_\ell \left\| k^{(1)} - k^{(2)} \right\|_1, \end{aligned}$$

455 which concludes the proof. \square

456 **THEOREM 3.13.** *For any given data, there exists a solution to the innovation*
 457 *MFG (3.4). Furthermore, if $\|PA\|_1$ is fixed, this solution is unique provided $A_\ell P_\ell < \frac{1}{C}$*
 458 *for every $\ell = 1, \dots, L$.*

459 *Proof.* From Proposition 3.10 and Theorem 3.12, the function $\Phi : [0, \zeta]^L \rightarrow [0, \zeta]^L$
 460 is a continuous function from a convex compact subset of \mathbb{R}^L to itself. Therefore, by
 461 Brouwer's fixed point theorem, Φ has a fixed point. Furthermore, Theorem 3.12
 462 shows that Φ is a Lipschitz function in $\|\cdot\|_1$. The Lipschitz constant is given by
 463 $C \max_{\ell=1, \dots, L} A_\ell P_\ell$, where C depends on $\|PA\|_1$ but not directly on P_ℓ or A_ℓ .
 464 Therefore, for fixed $\|PA\|_1$, Φ is a contraction map provided $A_\ell P_\ell < \frac{1}{C}$ for every
 465 $\ell = 1, \dots, L$, and in this case the fixed point is unique.

466 Theorems 3.4 and 3.8 proved existence and uniqueness of solutions to equa-
 467 tions (3.5) and (3.6) respectively for any $k \in [0, \zeta]$. Now, if k^* is a fixed point
 468 of Φ then $(m^*, V^*) := (m^{k^*}, V^{k^*})_{\ell=1}^L$ is a solution to (3.4), which can be seen
 469 by replacing k_ℓ^* with $\Phi_\ell(k^*)$ in (3.5), (3.6) for every $\ell = 1, \dots, L$. Conversely, if
 470 (m^*, V^*) is a solution to (3.5), (3.6), then clearly, by defining k^* co-ordinate wise as
 471 $k_\ell^* = \sum_{\ell'=1}^L A_{\ell'} p(\ell, \ell') \int_0^z z m_{\ell'}(z) dz$, $k^* \in [0, \zeta]^L$ is a fixed point of Φ . Furthermore,
 472 by uniqueness of (3.5), (3.6), $(m^{k^*}, V^{k^*}) = (m^*, V^*)$. So, existence and uniqueness
 473 of solutions to the innovation MFG (3.4) is equivalent to existence and uniqueness of
 474 fixed points of Φ . Hence, there exists a solution to the innovation MFG. Furthermore,
 475 this solution is unique, provided $A_\ell P_\ell < \frac{1}{C}$ for every $\ell = 1, \dots, L$. \square

476 *Remark 3.14.* In practical terms we can guarantee the condition $A_\ell P_\ell < \frac{1}{C}$ holds
 477 for every $\ell = 1, \dots, L$ provided L is large enough. This is because $\sum_{\ell=1}^L A_\ell = 1$. So,
 478 for fixed $\|PA\|_1$, when L is sufficiently large we can take A_ℓ to be sufficiently small
 479 so that $A_\ell P_\ell < \frac{1}{C}$.

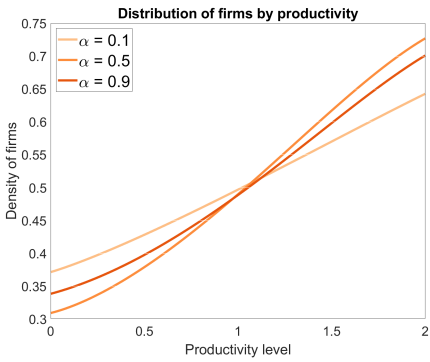
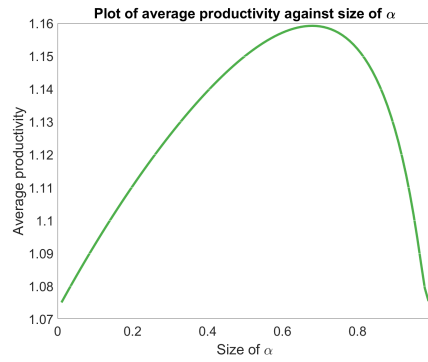
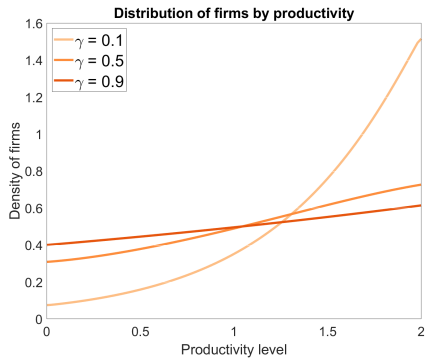
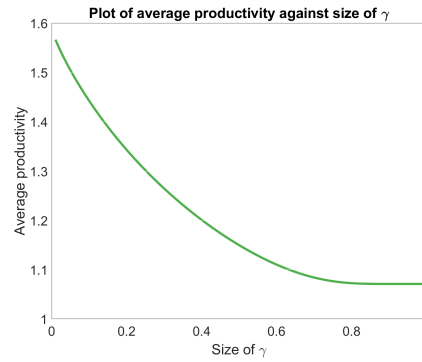
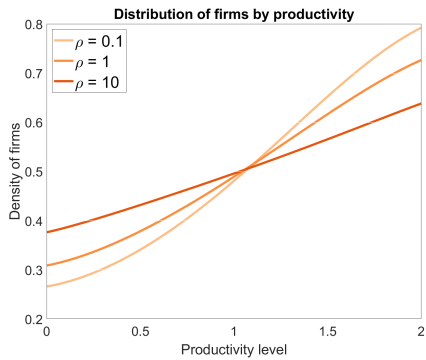
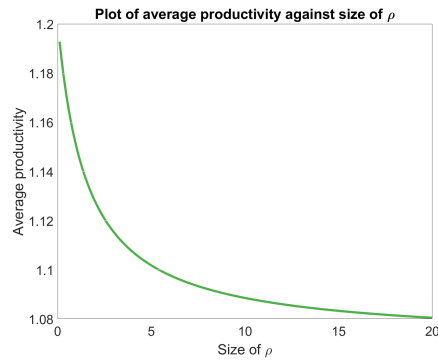
480 4. Numerical simulations.

481 **4.1. Consumers.** In the previous analysis, we assumed that consumers play a
 482 passive role in the model. In particular, the constant B has been fixed. However,
 483 in doing so we have not modelled the active nature of consumers in determining the
 484 price index R . To include this when implementing our numerical methods we return
 485 to (2.4) and we normalise economic output to $Y = 1$. Then, by rearranging (2.4) and
 486 using the production function $q_{\ell,i} = Z_{\ell,i}$, we get $B = \left[\frac{1}{N} \sum_{\ell=1}^L \sum_{i=1}^{N_\ell} Z_{\ell,i}^\alpha \right]^{\frac{1}{\alpha-1}}$. So, as
 487 the number of firms in each sector goes to infinity, $B = \left[\sum_{\ell=1}^L A_\ell \int_\Omega z^\alpha m_\ell(z) dz \right]^{\frac{1}{\alpha-1}}$.
 488 Note that this now needs to be solved as a fixed point, as m_ℓ itself depends on B .

489 **4.2. Simulations.** We computed simulations with synthetic data, using the nu-
 490 merical method outlined in Appendix A. From an economics perspective it is impor-
 491 tant to understand how the model affects the sector-level productivity. The purpose
 492 of the simulations is to provide initial insights into the role of the modelling parameters
 493 and of the network configuration.

494 **Parameter effects.** The MFG depends on the parameters σ , w , α , γ and ρ .
 495 Recall that $\sigma > 0$ is the strength of noise in an individual's dynamics, $w > 0$
 496 is the wage paid to employees, $\alpha \in (0, 1)$ is a parameter in the consumer optimisation
 497 problem which ensures convexity, and $\gamma \in (0, 1)$ is the returns to labour i.e. the
 498 inefficiency in converting one unit of labour to one unit of knowledge, it also ensures
 499 convexity of the firm-level optimisation problem.

500 In order to separate the parameter effects from any effects caused by the sector
 501 network, we ran simulations with just a single sector. We fixed $\bar{z} = 2$, $A = A_1 = 1$ and
 502 $P = 0.1$, where \bar{z} is the maximum productivity level, A_1 is the proportion of firms in
 503 sector 1 and P is the strength of connection from sector 1 to itself. For baseline values,

(a) Plot of firm distribution with varying α (b) Plot of average productivity against α (c) Plot of firm distribution with varying γ (d) Plot of average productivity against γ (e) Plot of firm distribution with varying ρ (f) Plot of average productivity against ρ Fig. 1: Simulations of MFG with varying α , γ , and ρ

504 we took $\sigma = 1$, $w = 1$, $\rho = 1$, $\gamma = 0.5$ and $\alpha = 0.5$. For each simulation, we varied one
 505 parameter while keeping all others at the baseline level. Figures 1a and 1b show that
 506 the relationship between α and the distribution of firms is a complex one. There is
 507 some $\alpha^* \in (0, 1)$ where the average productivity reaches a maximum, while on $(0, \alpha^*]$
 508 average productivity is monotonically increasing, and on $[\alpha^*, 1)$ average productivity is
 509 monotonically decreasing. Note that, for fixed productivity level and firm distribution,
 510 a firm's revenue is $r_{\ell} q_{\ell} = Z_{\ell}^{\alpha} \left[\frac{1}{N} \sum_{\ell'=1}^L A_{\ell'} \int_{\Omega} z^{\alpha} m_{\ell'}(z) dz \right]^{-1}$, which consists of a
 511 term that increases with respect to α multiplied by a term that decreases with respect
 512 to α . This results in a competing effect between α and a firm's revenue, which in
 513 turn affects a firm's return on investment, and therefore its level of investment in
 514 labour. Since labour investment has an increasing effect on average productivity,
 515 the competing terms in the revenue equation directly correspond to the behaviour
 516 exhibited in figure 1b.

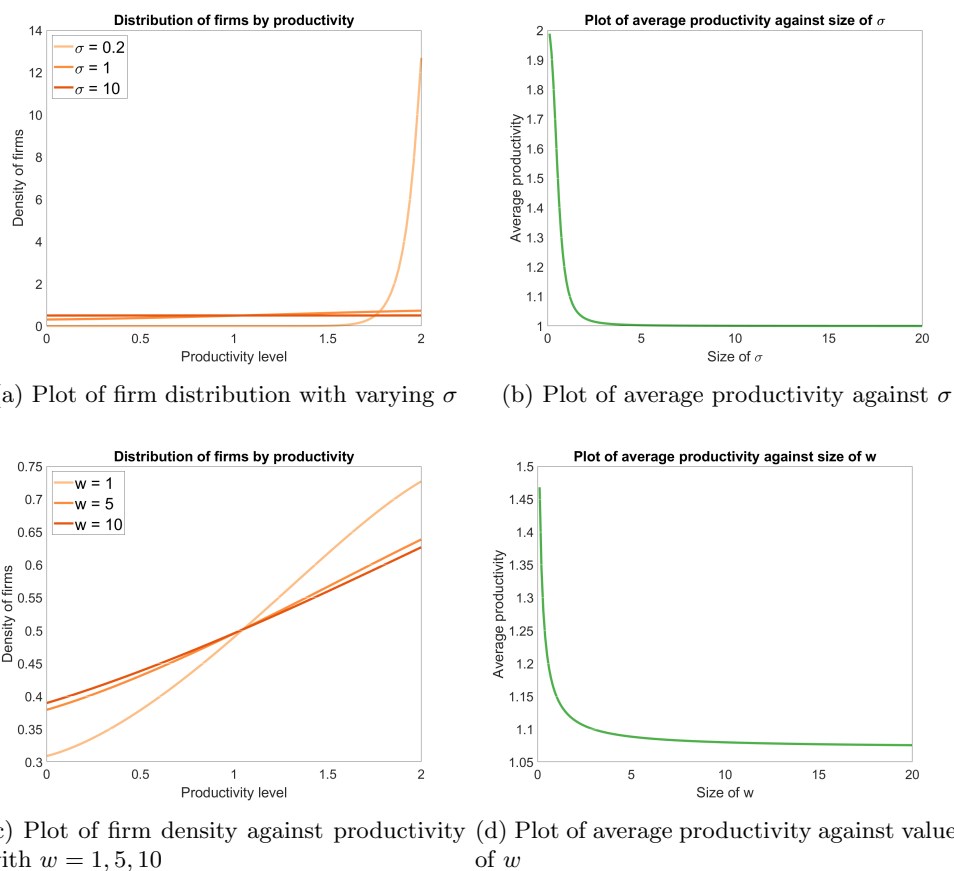
517 Figures 1c and 1d shows the effect of γ on the sector-level productivity. Figure 1d
 518 shows that as γ increases, the average productivity decreases. Since γ relates to the
 519 inefficiency of converting one unit of labour to one unit of productive work, it seems
 520 counter-intuitive at first that average productivity would be a decreasing function of
 521 γ . Recall that the optimal level of employment is given by $h^* = \left(\frac{\gamma}{w} \max(0, V') \right)^{\frac{1}{1-\gamma}}$,
 522 which increases productivity at a rate $(h^*)^{\gamma}$. Then, h^* is increasing with respect
 523 to γ for fixed V' if and only if $V' \geq \frac{w}{\gamma} e^{\frac{\gamma-1}{\gamma}}$ and $(h^*)^{\gamma}$ is increasing if and only if
 524 $V' \geq \frac{w}{\gamma} e^{\gamma-1}$. Hence, the effect of γ on the average productivity depends on V' and
 525 how it changes with respect to γ .

526 The effects of ρ and σ on the average productivity, shown in Figures 1e, 1f
 527 and 2a, 2b respectively, show the same trend: average productivity decreases as each
 528 parameter increases. The size of ρ is the extent to which a firm discounts future
 529 profits. As ρ increases, firms care less about the future state of the system and so
 530 they are less willing to invest in labour; it is an investment whose effect is only on the
 531 future value of productivity. This results in reduced average productivity in the long
 532 run, which can be seen in Figure 1f. As σ increases, the randomness in productivity
 533 evolution of each firm increases. So, the impact of labour on productivity decreases
 534 with increasing σ , and this is reflected in Figure 2b.

535 Finally, Figures 2c and 2d shows that average productivity also decreases with
 536 increasing wage, w . The wage rate increases the cost of labour. So, we can directly see
 537 that as the wage increases, the optimal level of employment, and hence the average
 538 productivity, decreases.

539 **Spillover size effects.** The sector-level network, encoded by the vertex weights
 540 A_{ℓ} for sector ℓ , and the edge weights $p(\ell, \ell')$ for a transfer of knowledge from sector ℓ'
 541 to sector ℓ , is called the spillover network as it describes how knowledge and productivity
 542 spills over from one sector to another. A path in the spillover network is called a
 543 spillover path, or just spillover if there is no ambiguity. A path of length 1 from
 544 sector ℓ' to sector ℓ is called a direct spillover, a path of length 2 or greater from
 545 sector ℓ' to sector ℓ is called an indirect spillover, and in both cases sector ℓ is called
 546 the receiving sector and sector ℓ' is the originating sector.

547 In almost all economic literature, only direct spillovers have been modelled and
 548 we are aware of no models that pay attention to the effect indirect spillovers have on
 549 economic productivity. In this subsection, we begin investigating how the productiv-
 550 ity of a sector is affected by the structure of the spillover network, and in particular
 551 the effect of indirect spillovers on productivity. To undertake this investigation, we

Fig. 2: Simulations of MFG with varying σ and w

552 conducted three types of simulations. The first simulations were to model the six
 553 networks in Figure 3, to provide initial insight into how indirect spillover paths affect
 554 the distribution of firms. In the second simulations, we randomly generated spillover
 555 networks in models with three sectors and used the collected data to hypothesise a
 556 relationship between the average productivity of a sector and the size of spillovers
 557 (direct and indirect) it received. In the final simulations, we tested our hypothesis
 558 on more randomly generated spillover networks, this time for models with 10 sec-
 559 tors, which more closely resembles the number of sectors in the real economy. We
 560 showed that the hypothesis developed accurately describes the relationship between
 561 the spillover network and the average productivity of firms, moreover there was a
 562 20% reduction in error when direct and indirect spillovers were taken into account,
 563 compared with when only direct spillovers were considered. Therefore, our conclusion
 564 from this preliminary investigation is that indirect spillovers have a significant effect
 565 on economic productivity in our model and they should not be ignored.

566 The networks in Figure 3 provide insight into how indirect spillovers affect the
 567 distribution of firms, in comparison to direct spillovers. In network 1, sector C has one
 568 direct spillover, in network 2 it has one direct spillover and one indirect spillover of

length 2, and in network 3 it has two direct spillovers. So, the difference in productivity in sector C between network 2 and network 1 will show the effect of an indirect spillover compared with having no spillover, and the difference between networks 3 and 2 will show the effect of an indirect spillover compared with a direct spillover. The differences in density of sector C are plotted in Figure 4a. From the plots, it can be seen that the density of firms is larger at high productivity levels in network 2 compared with network 1 and the density is lower at low productivity levels. This means that the indirect spillover from sector A to sector C has a positive effect on sector C , skewing the distribution towards higher productivity levels. The same behaviour can be seen when we compare sector C in network 3 to network 2, however the effect is an order of magnitude larger. Therefore, although an indirect spillover path has some positive effect compared with no path at all, the effect is less strong than a direct spillover path.

In Figure 4b, sector D of networks four to six were modelled. For sector D : in network 4 there is one indirect spillover with path length 2; network 5 has one indirect spillover with path length 2 and one with path length 3; finally network 6 has an infinite number of indirect spillovers, one for every path length. We have plotted the difference in density of sector D between network 5 and network 4 and between networks 6 and 5. The difference between network 5 and network 4 shows the effect of an indirect spillover of length 3, while the difference between network 6 and network 5 shows the effect of indirect spillovers of all lengths greater than 3. For the difference between network 5 and network 4, the same qualitative result as the difference between network 2 and network 1, in Figure 4a, is observed. This suggests that having spillover paths of greater length do have positive impacts on productivity, but with reduced impact for increased path lengths. Interestingly, sector D in network 6 is less productive than sector D in sector 5. Further investigation showed that if B is fixed, rather than the solution of a fixed point problem, then the effect that more paths result in greater productivity returns, see figure 4c. The reason for this is not immediately obvious and warrants further study. Since the observed change is very small, it can't be ruled out that this result is an artefact from simplifications in the model.

In the second set of simulations, we took a closer look at how the spillover network structure affects the average productivity within each sector. Recall that if, given a network, we know the value of the fixed point, k^* , of the function Φ defined in Definition 3.9. Then the average productivity in sector ℓ is $\int_{\Omega} z m_{\ell}(z) dz = \int_{\Omega} z m^{k_{\ell}^*}(z) dz$. So, to understand the relationship between average productivity and the network, we first need to understand the relationship between $\int_{\Omega} z m^{k_{\ell}}(z) dz$ and k_{ℓ} , for any $k_{\ell} \geq 0$. Then, we also need to understand the relationship between k_{ℓ}^* and the $L \times L$ matrix S with entries defined by $S_{\ell, \ell'} = A_{\ell'} p(\ell, \ell')$, because

$$k^* = S \left(\int_{\Omega} z m^{k_{\ell}^*}(z) dz \right)_{\ell=1}^L.$$

In Figure 5, we have plotted $\int_{\Omega} z m^k(z) dz$ against k . The relationship appears to approximately follow

$$(4.1) \quad \int_{\Omega} z m^k(z) dz = \bar{z} - \frac{b_0}{k^{b_1} + b_2},$$

for some $b_0, b_1, b_2 > 0$, as can be seen by the second line in Figure 5. To understand the relationship between the fixed point of Φ and the matrix S , we considered networks

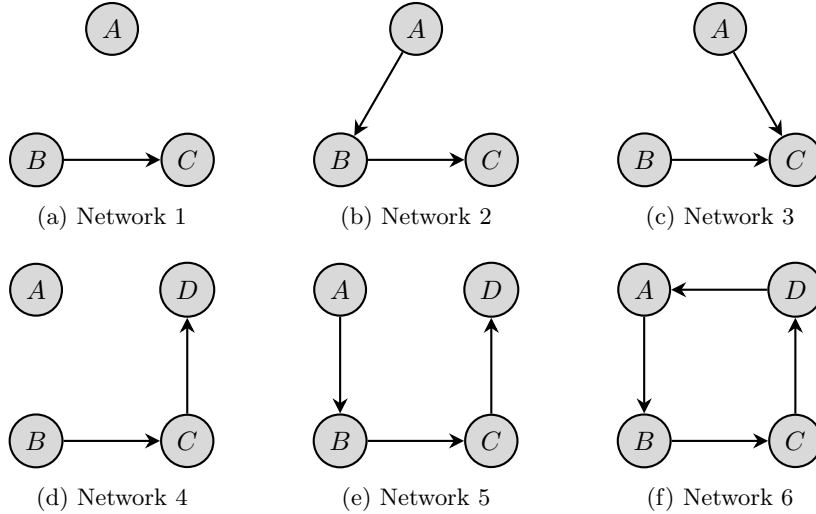


Fig. 3: Sector-level networks for simulations in Figures 4a and 4b

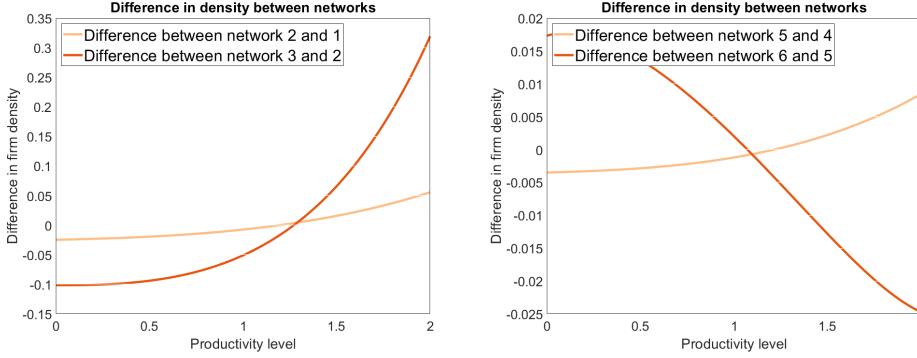
614 of three vertices, with $A_\ell = 1/3$ for all ℓ . We created a random network between the
 615 vertices by choosing a connection probability p , and making a directed edge between
 616 vertices with probability p . We then weighted each directed edge with a random
 617 weight, chosen from a uniform distribution on $[0, 1]$. We repeated this 100 times for
 618 each connection probability, and recorded both the size of direct spillovers to each
 619 sector and the value of the fixed point of Φ . Figures 6a and 6b shows a scatter plot
 620 of k_ℓ^* , the ℓ^{th} co-ordinate of the fixed point of Φ , against the sum of direct spillover
 621 strengths $\sum_{\ell'=1}^L S_{\ell,\ell'}$. In the simulations with a high connection probability, Figure 6a,
 622 there is a strong linear relationship between k_ℓ^* and $\sum_{\ell'=1}^L S_{\ell,\ell'}$. However, with low
 623 connection probabilities, Figure 6b, the simulations tend to follow one of two weaker
 624 linear relationships with the row sum.

625 To understand the relationships further, we can look at the equation that $k_\ell^* \in$
 626 $[0, \infty)$ implicitly satisfies: $k_\ell^* = \sum_{\ell'=1}^L S_{\ell,\ell'} \int_{\Omega} z m^{k_{\ell'}}$ dz , where $m^{k_{\ell'}}$ is defined by (3.11).
 627 So, if sector ℓ receives no spillovers then $k_\ell^* = 0$. If it has only direct spillovers, then
 628 it is only connected to sectors with no spillovers. So, by defining $f(k) = \int_{\Omega} z m^k dz$

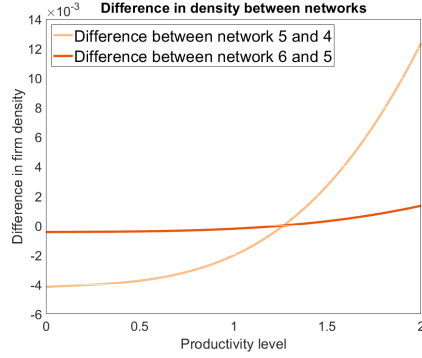
$$629 \quad (4.2) \quad k_\ell^* = f(0) \sum_{\ell'=1}^L S_{\ell,\ell'}.$$

630 We can see this linear relationship between k_ℓ^* and $\sum_{\ell'=1}^L S_{\ell,\ell'}$ in Figures 6c and 6d,
 631 where we have taken the simulated points in Figure 6b, and split the data into those
 632 points which have only direct spillovers and those that have indirect spillovers as well.
 633 In Figure 6c, where sectors with only direct spillover paths are considered, the linear
 634 relationship described by (4.2) can be clearly seen.

635 To understand how the value of k_ℓ^* depends on the matrix S in the case of indirect
 636 spillovers, we can return to the definition of the spillover size and $f(k)$. If we assume
 637 that f is approximately linear for sectors with indirect spillovers, i.e. $f(k) = f_0 + f_1 k$,



(a) Difference in density of sector C between networks 2 and 1, and networks 3 and 2 (b) Difference in density of sector D between networks 5 and 4, and networks 6 and 5



(c) Difference in density of sector D between networks 5 and 4, and networks 6 and 5, with fixed $B = 1$

Fig. 4: Simulations of MFG comparing distribution of firms in sectors C and D with respect to productivity in networks 1, 2 and 3, and networks 4, 5 and 6 from Figure 3

638 then

639 (4.3)
$$k_\ell^* = (S(f_0 \mathbf{1} + f_1 k^*))_\ell,$$

640 where $\mathbf{1}$ is the vector of length L with ones in every entry. Using the identity $(I +$
 641 $f_1 S)^{-1} = \sum_{n=0}^{\infty} f_1^n S^n$, we can rearrange (4.3)

642 (4.4)
$$k_\ell^* = f_0 \sum_{n=0}^{\infty} f_1^n (S^{n+1} \mathbf{1})_\ell,$$

643 which gives a way to estimate the value k_ℓ^* directly from the initial data. Therefore,
 644 combining estimates (4.1) and (4.4), we can estimate the value of average productivity
 645 from the matrix S by

646 (4.5)
$$\int_{\Omega} z m_\ell(z) dz = \bar{z} - \frac{b_0}{\left(f_0 \sum_{\ell'=1}^L \sum_{n=0}^{\infty} f_1^n (S^{n+1})_{\ell, \ell'} \right)^{b_1} + b_2}.$$

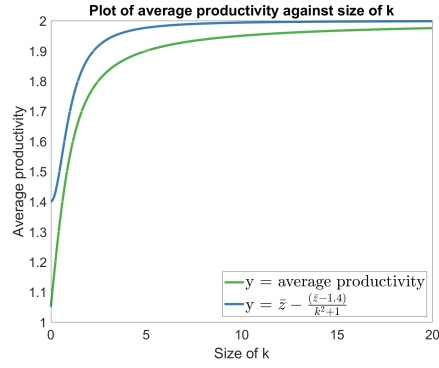
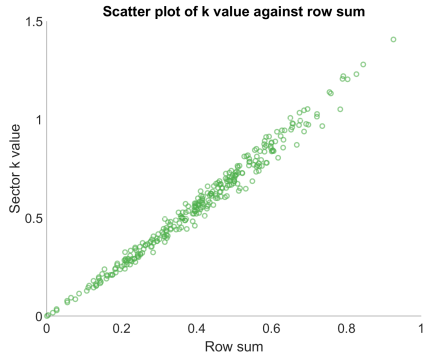
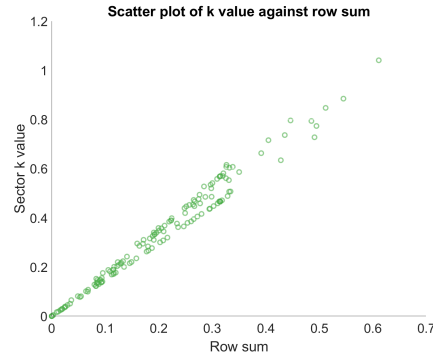


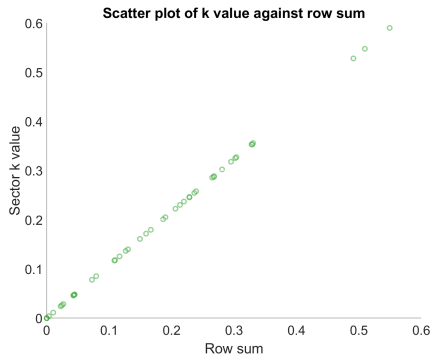
Fig. 5: Plot of average productivity against size of k in auxiliary Fokker–Planck equation (3.6) and plot of $y = \bar{z} - \frac{(\bar{z}-1.4)}{k^2+1}$ for comparison



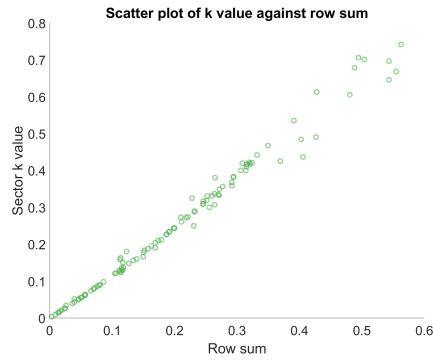
(a) Probability of directed edge = 0.8



(b) Probability of directed edge = 0.2



(c) Probability of directed edge = 0.2, and length of longest path to sector = 1



length of longest path to sector > 1

Fig. 6: Relationship between k_ℓ^* and sum of direct spillovers $\sum_{\ell'=1}^L S_{\ell,\ell'}$

647 The relationship suggests that the average productivity depends on S^n for every n
 648 i.e. on indirect spillovers of every path length. Moreover, if f_1 is small enough, the
 649 effect of a spillover path is decreasing by an order of magnitude for every increase in
 650 path length, which agrees with our initial simulations of networks 1–6.

651 In order to verify the hypothesis, in the final simulations we ran a regression to es-
 652 timate the parameters f_0, f_1, b_0, b_1, b_2 and provide evidence that approximation (4.5)
 653 is accurate. We performed 1000 simulations on networks of ten vertices, with connec-
 654 tion probability chosen randomly and uniformly distributed in $[0, 1]$, with connection
 655 strength chosen randomly and uniformly in $[0, 3]$, and with sector sizes A_ℓ also ran-
 656 domly chosen. We ran a nonlinear regression, of the form (4.4), on sectors with
 657 indirect spillovers, to obtain optimal values of f_0 and f_1 . Then, using the optimal
 658 values of f_0 and f_1 we ran a second nonlinear regression, of the form (4.5), to find the
 659 optimal values of b_0, b_1 and b_2 . Table 1 gives estimates for the parameters f_i and b_i .
 660 We found that average productivity does behave approximately according to (4.5),
 661 with table 1 suggesting a statistically significant result. Visually, this can be seen
 662 in Figure 7b, where we plotted (4.5) using the optimal values of f_i and b_i . We also
 663 computed estimates for the model

$$664 \quad (4.6) \quad \int_{\Omega} z m_\ell(z) dz = \bar{z} - \frac{\bar{b}_0}{\left(\bar{f}_0 \sum_{\ell'=1}^L S_{\ell,\ell'}\right)^{\bar{b}_1} + \bar{b}_2},$$

665 which assumes average productivity depend on direct spillovers only, and plotted the
 666 result in Figure 7b. Comparing plots 7a and 7b shows that the model (4.5), which
 667 includes the effects of indirect spillovers, provides a more accurate estimate for average
 668 productivity than model (4.6), which only accounts for the effect of direct spillovers.
 669 This is reconfirmed by the 20% reduction in R-squared error when indirect spillover
 670 paths are included in the model. Therefore, indirect spillover paths can not be ignored
 671 as a factor determining a sector's productivity.

672 **5. Conclusion and future research.** We have developed an MFG model of
 673 firm-level innovation from a microscopic formulation. The model can be calibrated to
 674 fit economic data of spillovers, so its economic validity can be verified. We have been
 675 able to prove existence of solutions and, under a smallness assumption on the data,
 676 uniqueness. We have investigated numerically how the modelling parameters and the
 677 spillover network affects the sector-level productivity, through the development of a
 678 simple algorithm that takes advantage of the structure of the proof of existence and
 679 uniqueness.

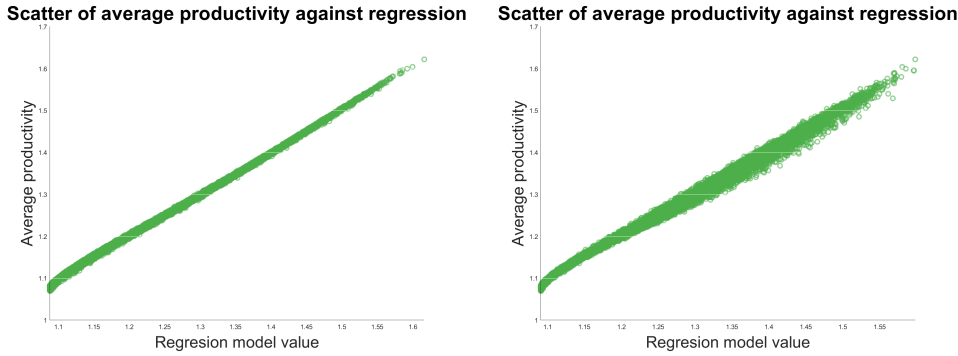
680 In future work, we hope to compare the MFG model with the socially optimal
 681 behaviour, as described by the mean field optimal control problem. We will also use
 682 patent-level data to calibrate and test the two models for their accuracy. We hope the
 683 comparison between the social optimum and the competitive equilibrium will suggest
 684 a method for implementing socially optimal subsidy policies for R&D.

685 **Appendix A. Numerical Methods.** The numerical method we designed to
 686 solve (3.4) is informed by the structure of the proof of existence and uniqueness. The
 687 method of proof relies on the contraction mapping theorem to find a fixed point of the
 688 map Φ , defined in Definition 3.9. We are also required to solve a fixed point problem
 689 to find the value of the parameter B . In light of this, our numerical method proceeds
 690 as follows, after choosing an initial guess $k^0 \in [0, \infty)^L$, $B^0 \in [0, \infty)$ and tolerances
 691 δ_1, δ_2 .

- 692 1. Given $k^i \in [0, \infty)^L$ and $B^i \in [0, \infty)$, solve (3.5b), (3.5c) using the following
693 method, based on a Newton–Raphson method in a Banach space.
- 694 (a) Define $F(v) = -\frac{\sigma^2}{2}v'' + \rho v - kv' - (1 - \gamma) \left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}} (v')^{\frac{1}{1-\gamma}} - \frac{z^\alpha}{B^{\alpha-1}}$. We
695 want to find zeros of $F(v)$.
- 696 (b) We define $dF(v)(u) = -\frac{\sigma^2}{2}u'' + \rho u - ku' - \left(\frac{\gamma}{w}v'\right)^{\frac{\gamma}{1-\gamma}} u'$, which is the
697 Fréchet derivative of F .
- 698 (c) Denote by $V_0^{k_\ell^i, B^i}$ the initial guess for the ℓ th component of the solution
699 to (3.5b), (3.5c) with $k = k_\ell^i$ and $B = B^i$.
- 700 (d) Given $V_n^{k_\ell^i, B^i}$, we compute the next iteration, $V_{n+1}^{k_\ell^i, B^i}$, using a Newton–
701 Raphson method: $V_{n+1}^{k_\ell^i, B^i} = V_n^{k_\ell^i, B^i} - dF \left(V_n^{k_\ell^i, B^i} \right)^{-1} \left(F \left(V_n^{k_\ell^i, B^i} \right) \right)$.
- 702 (e) Continue iteratively until $\left\| F \left(V_n^{k_\ell^i, B^i} \right) \right\|_1 \leq \delta_1$ and define $V^{i, \ell} = V_n^{k_\ell^i, B^i}$
- 703 2. Given V^i , compute the solution to (3.6b) using (3.11) and denote it by m^i
- 704 3. Define $k^{i+1} = \Phi(k^i)$ and $B^{i+1} = \left[\sum_{\ell=1}^L A_\ell \int_\Omega z^\alpha m_\ell(z) dz \right]^{\frac{1}{1-\alpha}}$
- 705 4. If $\|k^{i+1} - k^i\|_1 + |B^{i+1} - B^i| \leq \delta_2$ then stop the iteration process and define
706 the MFG solution $(m, V) = (m^i, V^i)$. Otherwise return to Step 1.

Variable	Coefficient estimate	Standard error	t stat	p value
f_0	2.29	2.13×10^{-3}	1080	0
f_1	0.483	3.64×10^{-4}	1330	0
b_0	0.892	4.37×10^{-4}	2040	0
b_1	1.27	9.49×10^{-4}	1330	0
b_2	0.978	4.71×10^{-4}	2080	0

Table 1: Table of regression results related to linear regression (4.5)



(a) Plot of average productivity against right hand side of (4.5), with optimal values for f_i, b_i
(b) Plot of average productivity against right hand side of (4.6), with optimal values for \bar{f}_0, \bar{b}_i

Fig. 7: Plots of average productivity against (4.5) with models for k_ℓ given by considering direct and indirect spillovers (4.4) or only direct spillovers (4.6)

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REFERENCES

- 708 [1] D. ACEMOGLU, *Introduction to Modern Economic Growth*, Princeton University Press, Prince-
709 ton, NJ, 2009.
- 710 [2] Z. J. ACS AND M. SANDERS, *Patents, knowledge spillovers, and entrepreneurship*, Small Busi-
711 ness Economics, 39 (2012), pp. 801–817.
- 712 [3] P. AGHION, P. HOWITT, AND D. MAYER-FOULKES, *The effect of financial development on*
713 *convergence: Theory and evidence*, Quarterly Journal of Economics, 120 (2005), pp. 173–
714 222.
- 715 [4] D. M. AMBROSE, *Strong solutions for time-dependent mean field games with non-separable*
716 *Hamiltonians*, Journal de Mathématiques Pures et Appliquées, 113 (2018), pp. 141–154.
- 717 [5] R. J. AUMANN, *Markets with a continuum of traders*, Econometrica, 32 (1964), pp. 39–50.
- 718 [6] M. BARKER, P. DEGOND, AND M.-T. WOLFRAM, *Comparing the best-reply strategy and mean-*
719 *field games: The stationary case*, Eur. J. Appl. Math., 6 (2020), pp. 1–32.
- 720 [7] M. BURGER, A. LORZ, AND M.-T. WOLFRAM, *On a Boltzmann mean field model for knowledge*
721 *growth*, SIAM J. Appl. Math, 76 (2016), pp. 1799–1818.
- 722 [8] R. CARMONA AND F. DELARUE, *Probabilistic Theory of Mean Field Games with Applications*
723 *I*, Springer International Publishing, Cham, 2018.
- 724 [9] M. CIRANT, *Multi-population mean field games systems with Neumann boundary conditions*,
725 Journal de Mathématiques Pures et Appliquées, 103 (2015), pp. 1294–1315.
- 726 [10] W. M. COHEN AND D. A. LEVINTHAL, *Innovation and learning: the two faces of R & D*, The
727 Economic Journal, 99 (1989), pp. 569–596.
- 728 [11] A. DECHEZLEPRÊTRE, R. MARTIN, AND M. MOHNEN, *Knowledge spillovers from clean and dirty*
729 *technologies: a patent citation analysis*, 2013.
- 730 [12] J. EECKHOUT AND B. JOVANOVIĆ, *Knowledge spillovers and inequality*, The American Economic
731 Review, 92 (2002), pp. 1290–1307.
- 732 [13] L. C. EVANS, *Partial Differential Equations*, American Mathematical Society, USA, 1998.
- 733 [14] R. FERREIRA AND D. GOMES, *Existence of weak solutions to stationary mean-field games*
734 *through variational inequalities*, SIAM J. Math. Anal, 50 (2018), pp. 5969–6006.
- 735 [15] M. K. FUNG, *Are knowledge spillovers driving the convergence of productivity among firms?*,
736 *Economica*, 72 (2005), pp. 287–305.
- 737 [16] P. HOWITT AND D. MAYER-FOULKES, *R&D, implementation and stagnation: a Schumpeterian*
738 *theory of convergence clubs*, 2002.
- 739 [17] M. HUANG, P. E. CAINES, AND R. P. MALHAMÉ, *Distributed multi-agent decision-making with*
740 *partial observations: asymptotic Nash equilibria*, in Proceedings of the 17th International
741 Symposium on Mathematical Theory on Networks and Systems, Kyoto, Japan, July 2006,
742 pp. 2725–2730.
- 743 [18] J.-M. LASRY AND P.-L. LIONS, *Jeux à champ moyen. I — Le cas stationnaire*, C. R. Math.
744 Acad. Sci. Paris, 343 (2006), pp. 619–625.
- 745 [19] J.-M. LASRY AND P.-L. LIONS, *Mean field games*, Jpn. J. Math., 2 (2007), pp. 229–260.
- 746 [20] N. H. LOC AND K. SCHMITT, *Bernstein–Nagumo conditions and solutions to nonlinear differ-*
747 *ential inequalities*, Nonlinear Anal., 75 (2012), pp. 4664–4671.
- 748 [21] R. E. LUCAS AND B. MOLL, *Knowledge growth and the allocation of time*, Journal of Political
749 Economy, 122 (2014), pp. 1–51.
- 750 [22] D. SCHMEIDLER, *Equilibrium points of nonatomic games*, J. Stat. Phys., 7 (1973), pp. 295–300.
- 751 [23] K. SCHMITT, *Boundary value problems for quasilinear second order elliptic equations*, Nonlinear
752 Anal., 2 (1978), pp. 263–309.
- 753 [24] M. E. TAYLOR, *Partial Differential Equations I*, Springer New York, New York, NY, 2011.
- 754 [25] H. TEMBINE, *Mean-field-type games*, AIMS Mathematics, 2 (2017), pp. 706–735.