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# Computational determination of the largest lattice polytope diameter

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## Abstract

A lattice  $(d, k)$ -polytope is the convex hull of a set of points in dimension  $d$  whose coordinates are integers between 0 and  $k$ . Let  $\delta(d, k)$  be the largest diameter over all lattice  $(d, k)$ -polytopes. We develop a computational framework to determine  $\delta(d, k)$  for small instances. We show that  $\delta(3, 4) = 7$  and  $\delta(3, 5) = 9$ ; that is, we verify for  $(d, k) = (3, 4)$  and  $(3, 5)$  the conjecture whereby  $\delta(d, k)$  is at most  $\lfloor (k + 1)d/2 \rfloor$  and is achieved, up to translation, by a Minkowski sum of lattice vectors.

*Keywords:* Lattice polytope, edge-graph diameter, enumeration algorithm

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## 1. Introduction

Finding a good bound on the maximal edge-diameter of a polytope in terms of its dimension and the number of its facets is not only a natural question of discrete geometry, but also historically closely connected with the theory of the simplex method, as the diameter is a lower bound for the number of pivots required in the worst case. Considering bounded polytopes whose vertices are integer-valued, we investigate a similar question where the number of facets is replaced by the grid embedding size.

The convex hull of integer-valued points is called a lattice polytope and, if all the vertices are drawn from  $\{0, 1, \dots, k\}^d$ , it is referred to as a lattice  $(d, k)$ -polytope. Let  $\delta(d, k)$  be the largest edge-diameter over all lattice  $(d, k)$ -polytopes. Naddef [8] showed in 1989 that  $\delta(d, 1) = d$ , Kleinschmidt and

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Onn [7] generalized this result in 1992 showing that  $\delta(d, k) \leq kd$ . In 2016, Del Pia and Michini [3] strengthened the upper bound to  $\delta(d, k) \leq kd - \lceil d/2 \rceil$  for  $k \geq 2$ , and showed that  $\delta(d, 2) = \lfloor 3d/2 \rfloor$ . Pursuing Del Pia and Michini's approach, Deza and Pournin [6] showed that  $\delta(d, k) \leq kd - \lceil 2d/3 \rceil - (k - 2)$  for  $k \geq 4$ , and that  $\delta(4, 3) = 8$ . The determination of  $\delta(2, k)$  was investigated independently in the early nineties by Thiele [9], Balog and Bárány [2], and Acketa and Žunić [1] showing that  $\delta(2, k) = \frac{6}{(2\pi)^{2/3}} k^{2/3} + O(k^{1/3} \log k)$ .

Investigating the lower bound on  $\delta(d, k)$ , Deza, Manoussakis, and Onn [5] introduced the *primitive lattice polytope*  $H_1(d, 2)$  as the Minkowski sum of all the nonzero vectors  $v$  drawn from  $\{-1, 0, 1\}^d$  such that  $\|v\|_1 \leq 2$  and the first nonzero coordinate of  $v$  is positive. They showed that, for any  $k < 2d$ , there exists a subset of the generators of  $H_1(d, 2)$  whose Minkowski sum is, up to translation, a lattice  $(d, k)$ -polytope of diameter  $\lfloor (k + 1)d/2 \rfloor$ . Thus, they showed that  $\delta(d, k) \geq \lfloor (k + 1)d/2 \rfloor$  for all  $k < 2d$  and proposed Conjecture 1.

**Conjecture 1.** *For any  $d$  and  $k$ ,  $\delta(d, k)$  is achieved, up to translation, by a Minkowski sum of lattice vectors. In particular, when  $k < 2d$ ,  $\delta(d, k) = \lfloor (k + 1)d/2 \rfloor$ .*

In Section 2, we propose a computational framework to investigate Conjecture 1 by drastically reducing the search space for lattice  $(d, k)$ -polytopes achieving a large diameter. Applying this framework to  $(d, k) = (3, 4)$  and  $(3, 5)$ , we determine in Section 3 that  $\delta(3, 4) = 7$  and  $\delta(3, 5) = 9$ .

**Theorem 2.** *Conjecture 1 holds for  $(d, k) = (3, 4)$  and  $(3, 5)$ ; that is,  $\delta(3, 4) = 7$  and  $\delta(3, 5) = 9$ , and both diameters are achieved, up to translation, by a Minkowski sum of lattice vectors.*

Note that Conjecture 1 holds for all known values of  $\delta(d, k)$  given in Table 1, and hypothesizes, in particular, that  $\delta(d, 3) = 2d$ . The new entries corresponding to  $(d, k) = (3, 4)$  and  $(3, 5)$  are entered in bold.

$d \backslash k$	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	2	3	4	4	5	6	6	7	8	8
3	3	4	6	<b>7</b>	<b>9</b>					
4	4	6	8							
$\vdots$	$\vdots$	$\vdots$								
$d$	$d$	$\lfloor \frac{3d}{2} \rfloor$								

Table 1: The largest possible diameter  $\delta(d, k)$  of a lattice  $(d, k)$ -polytope

## 2. Theoretical and Computational Framework

Since  $\delta(2, k)$  and  $\delta(d, 2)$  are known, we consider in the remainder of the paper that  $d \geq 3$  and, with the exception of Section 3.2, that  $k \geq 3$ . While the number of lattice  $(d, k)$ -lattice polytopes is finite, a brute force search is typically intractable, even for small instances. Let  $d(u, v)$  denote the distance between two vertices  $u$  and  $v$  of a polytope  $P$  in the edge-graph of  $P$ . Theorem 3 considers a pair  $(u, v)$  of vertices of a lattice  $(d, k)$ -polytope such that  $d(u, v) = \delta(d, k)$ , and recalls conditions established in [6] that allow to drastically reduce the search space by exploiting integrality and convexity properties.

**Theorem 3.** *For  $d \geq 3$ , let  $(u, v)$  be a pair of vertices of a lattice  $(d, k)$ -polytope  $P$  such that  $d(u, v) = \delta(d, k)$ . For  $i = 1, \dots, d$ , let  $F_i^0$ , respectively  $F_i^k$ , denote the intersection of  $P$  with the facet of the cube  $[0, k]^d$  corresponding to  $x_i = 0$ , respectively  $x_i = k$ . Then,  $d(u, v) \leq \delta(d - 1, k) + k$ , and the following conditions are necessary for the inequality to hold with equality:*

- (1)  $u + v = (k, k, \dots, k)$ ,
- (2) any edge with  $u$  or  $v$  as vertex is drawn from  $\{-1, 0, 1\}^d$ ,
- (3) for  $i = 1, \dots, d$ , both  $F_i^0$  and  $F_i^k$ , are, up to an affine transformation, lattice  $(d - 1, k)$ -polytopes of diameter  $\delta(d - 1, k)$ .

Thus, to show that  $\delta(d, k) < \delta(d - 1, k) + k$ , it is enough to show that there is no lattice  $(d, k)$ -polytope  $P$  admitting a pair of vertices  $(u, v)$  such that  $d(u, v) = \delta(d, k)$  and the conditions (1), (2), and (3) are satisfied. Those conditions appear as items (i) and (ii) at the very end of [6] and are a

direct consequence of bounding  $d(u, v)$  by the length of path from  $u$  to  $v$  going through one the  $2d$  faces of  $P$  formed by the points of  $P$  maximizing, or minimizing, one of the  $d$  coordinates. The computational framework to determine, given  $(d, k)$ , whether  $\delta(d, k) = \delta(d - 1, k) + k$  is outlined below and illustrated for  $(d, k) = (3, 4)$  or  $(3, 5)$ .

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**Algorithm to determine whether  $\delta(d, k) < \delta(d - 1, k) + k$**

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*Step 1: INITIALIZATION*

*Determine the set  $\mathcal{F}_{d-1,k}$  of all lattice  $(d - 1, k)$ -polytopes of diameter  $\delta(d - 1, k)$ . For example, for  $(d, k) = (3, 4)$ , the determination of all the 335 lattice  $(2, 4)$ -polygons of diameter 4 is straightforward.*

*Determine the set  $\mathcal{V}_{d-1,k}$  of all the vertices of all lattice  $(d - 1, k)$ -polytopes of diameter  $\delta(d - 1, k)$ . For example, for  $(d, k) = (3, 4)$ ,  $\mathcal{V}_{2,4}$  consists of all  $\{0, 1, \dots, 4\}$ -valued points except  $(2, 2)$ .*

*Determine the set  $\mathcal{P}_{d-1,k}$  of all the points with integer coordinates belonging to the intersection of all lattice  $(d - 1, k)$ -polytopes of diameter  $\delta(d - 1, k)$ . For example, for  $(d, k) = (3, 4)$ ,  $\mathcal{P}_{2,4} = \{(1, 2), (2, 1), (2, 2), (2, 3), (3, 2)\}$ .*

*Determine the convex hull  $\mathcal{C}_{d,k}$  of all the points  $x$  such that  $x_i = 0$  and  $\bar{x}_i \in \mathcal{P}_{d-1,k}$  for some  $1 \leq i \leq d$ . Here  $\bar{x}_i \in \mathbb{R}^{d-1}$  denotes the point consisting of all coordinates of  $x$  except  $x_i$ .*

*Step 2: SYMMETRIES AND OTHER REDUCTIONS*

*Up to the symmetries of the cube  $[0, k]^d$ , we can assume that the coordinates of  $u$  satisfy  $u_i \leq u_{i+1} \leq \lfloor k/2 \rfloor$  for  $i = 1, \dots, d - 1$ . For example, for  $(d, k) = (3, 4)$ , the following vertices cover all possibilities for  $u$ :  $(0, 0, 0)$ ,  $(0, 0, 1)$ ,  $(0, 0, 2)$ ,  $(0, 1, 1)$ ,  $(0, 1, 2)$ ,  $(0, 2, 2)$ ,  $(1, 1, 1)$ ,  $(1, 1, 2)$ ,  $(1, 2, 2)$ , and  $(2, 2, 2)$ .*

*If  $u_1 = 0$ , we can further assume that  $\bar{u}_i \in \mathcal{V}_{d-1,k}$  for all  $i$  such that  $u_i = 0$ , as otherwise  $u$  cannot be a vertex of a lattice  $(d, k)$ -polytope  $P$  of diameter  $\delta(d - 1, k) + k$  by item (3) of Theorem 3. For example, for  $(d, k) = (3, 4)$ , the point  $(0, 2, 2)$  can be removed as  $(2, 2) \notin \mathcal{V}_{2,4}$ .*

Similarly, if  $u_1 \neq 0$ , we can further assume that  $\bar{u}_i \notin \mathcal{P}_{d-1,k}$  for all  $i = 1, \dots, d$  and that  $u$  is not in the interior of  $\mathcal{C}_{d,k}$ , as otherwise  $u$  is in the interior of a convex combination of points belonging to any lattice  $(d, k)$ -polytope  $P$  of diameter  $\delta(d-1, k) + k$ . For example, for  $(d, k) = (3, 4)$ , the points  $(1, 1, 2)$ ,  $(1, 2, 2)$ , and  $(2, 2, 2)$  can be removed as  $(1, 2)$  and  $(2, 2)$  belong to  $\mathcal{P}_{2,4}$ , and the point  $(1, 1, 1)$  can be removed as  $(1, 1, 1)$  is in the interior of  $\mathcal{C}_{3,4}$  since  $(1, 1, 1)$  is a convex combination of  $(0, 1, 2)$ ,  $(2, 0, 1)$  and  $(1, 2, 0)$ .

For each remaining  $u$ , we proceed to Step 3 where, by item (1) of Theorem 3, we can assume that  $(u, v)$  forms a pair of vertices satisfying  $d(u, v) = \delta(d-1, k) + k$  and  $u + v = \{k, k, \dots, k\}$ . For example, for  $(d, k) = (3, 4)$ , we proceed to Step (3) for each of the 5 pairs  $(u, v)$  corresponding to  $u = (0, 0, 0)$ ,  $(0, 0, 1)$ ,  $(0, 0, 2)$ ,  $(0, 1, 1)$ , and  $(0, 1, 2)$ .

### Step 3: SHELLING

For each of the possible pairs  $(u, v)$  determined during Step 2, the shelling step tries to embed elements of the set  $\mathcal{F}_{d-1,k}$  determined during Step 1 onto the ordered  $2d$  intersections of  $P$  with the facets of the cube  $[0, k]^d$ . We recall that  $P$  is assumed to be a lattice  $(d, k)$ -polytope of diameter  $\delta(d-1, k) + k$  with  $d(u, v) = \delta(d-1, k) + k$ .

If  $u_1 = 0$ , only elements of the set  $\mathcal{F}_{d-1,k}$  with  $\bar{u}_1$ , respectively  $\bar{v}_1$ , as vertex are considered to be embedded into the  $d$  facets of  $[0, k]^d$  with  $(0, 0, \dots, 0)$ , respectively  $(k, k, \dots, k)$ , as vertex. For example, for  $(d, k) = (3, 4)$ ,  $u = (0, 0, 0)$ , and  $v = (4, 4, 4)$ , the algorithm tries to embed 6 elements of  $\mathcal{F}_{2,4}$  into the 6 facets of  $[0, 4]^3$ ; 3 with  $(0, 0)$  as vertex, and 3 with  $(4, 4)$  as vertex.

These embeddings must be consistent; that is, given two embeddings  $E_1$  and  $E_2$ , the intersection of  $E_1$  with the facet of  $[0, k]^d$  containing  $E_2$  should be equal to the intersection of  $E_2$  with the facet of  $[0, k]^d$  containing  $E_1$ . In addition, by item (2) of Theorem 3, if an edge of an embedding of an element of  $\mathcal{F}_{d-1,k}$  with  $u$  or  $v$  as vertex is not drawn from  $\{-1, 0, 1\}^d$ , this embedding can be disregarded.

Let  $\Gamma$  denote the graph defined by the vertices and edges belonging to the embeddings of elements of  $\mathcal{F}_{d-1,k}$  considered to form a shelling, and let  $d_\Gamma(u, v)$

denote the distance in  $\Gamma$  between  $u$  and  $v$ . Since  $\Gamma$  is a subgraph of the edge-graph of a lattice  $(d, k)$ -polytope  $P$  of diameter assumed to be  $d(u, v)$ ,  $d_\Gamma(u, v)$  is an upper bound for  $d(u, v)$ ; that is, for  $\delta(d - 1, k) + k$ . Thus, if  $d_\Gamma(u, v) < \delta(d - 1, k) + k$ , a shortcut between  $u$  and  $v$  exists and the last embedding can be disregarded.

For example, for  $(d, k) = (3, 5)$ , all embeddings of 6 elements of  $\mathcal{F}_{2,5}$  forming a shelling yield a  $\Gamma$  such that  $d_\Gamma(u, v) < 10$ , and thus the algorithm terminates without Step 4 being executed.

#### Step 4. INNER POINTS

For each choice of  $2d$  embeddings of elements of  $\mathcal{F}_{d-1,k}$  forming a shelling obtained during Step 3, consider the  $\{1, 2, \dots, k-1\}$ -valued points not in the convex hull of the vertices of  $\Gamma$ ; that is, not in the convex hull of the vertices of the  $2d$  embeddings of elements of  $\mathcal{F}_{d-1,k}$  forming a shelling.

All subsets of such points are considered as potential vertices to be added to the vertices of  $\Gamma$  via a binary tree. A convex hull and diameter computation are performed for each node of the obtained binary tree. If there is a node yielding a diameter of  $\delta(d - 1, k) + k$  we can conclude that  $\delta(d, k) = \delta(d - 1, k) + k$ . Otherwise, we can conclude that  $\delta(d, k) < \delta(d - 1, k) + k$ .

### 3. Computational Results and Enhancements

#### 3.1. Determination of $\delta(3, 4)$ and $\delta(3, 5)$

For  $(d, k) = (3, 4)$ , a shelling exists for which path lengths are not decidable by the algorithm without convex hull computations. However, this shelling only achieves a diameter of 7. For  $(d, k) = (3, 5)$  the algorithm stops at Step 3, as there is no combination of 6 elements of  $\mathcal{F}_{2,5}$  which form a shelling such that  $d(u, v) = \delta(2, 5) + 5 = 10$ . Thus, no convex hull computations are required for  $(d, k) = (3, 5)$ . A shortcut from  $u$  to  $v$  is typically found early during the shelling step, which leads to the algorithm terminating quickly. Run on a 2009 Intel<sup>®</sup> Core<sup>™</sup>2 Duo 2.20GHz CPU, the algorithm is able to terminate for  $(d, k) = (3, 4)$  and  $(3, 5)$  in under a minute. Consequently,  $\delta(3, 4) < 8$  and  $\delta(3, 5) < 10$ . Since the Minkowski sum of  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$ ,  $(0, 1, 1)$ ,  $(1, 0, 1)$ ,  $(1, 1, 0)$ , and  $(1, 1, 1)$  forms

a lattice  $(3, 4)$ -polytope of diameter 7, we conclude that  $\delta(3, 4) = 7$ . Similarly, since  $H_1(3, 2)$ ; that is the Minkowski sum of  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$ ,  $(0, 1, 1)$ ,  $(1, 0, 1)$ ,  $(1, 1, 0)$ ,  $(0, 1, -1)$ ,  $(1, 0, -1)$ , and  $(1, -1, 0)$  forms, up to translation, a lattice  $(3, 5)$ -polytope of diameter 9, we conclude that  $\delta(3, 5) = 9$ .

Note that  $H_1(3, 2)$  is congruent to the truncated cuboctahedron; which is also called great rhombicuboctahedron, and is the Minkowski sum of an octahedron and a cuboctahedron, see Figure 1 for an illustration.

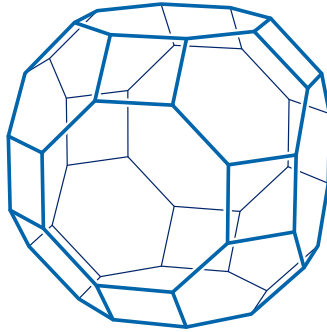


Figure 1:  $H_1(3, 2)$  is congruent to the truncated cuboctahedron and maximizes the diameter among all lattice  $(3, 5)$ -polytopes

Assuming that  $\delta(d - 1, k)$  is known and that the set  $\mathcal{F}_{d-1, k}$  of all lattice  $(d - 1, k)$ -polytopes of diameter  $\delta(d - 1, k)$  can be determined, the algorithm checks whether  $\delta(d, k) = \delta(d - 1, k) + k$  by performing a search over a highly constrained space. In case there exists a lattice  $(d, k)$ -polytope of diameter  $\delta(d, k) = \delta(d - 1, k) + k - 1$ , the algorithm allows the determination of  $\delta(d, k)$  as being equal to  $\delta(d, k) = \delta(d - 1, k) + k - 1$  or to  $\delta(d, k) = \delta(d - 1, k) + k$ . The question is significantly more challenging if  $\delta(d, k) - \delta(d - 1, k) < k - 1$  as the search space is much larger and additional structural properties are needed. The algorithm was enhanced in [4] to show that  $\delta(3, 6)$  and  $\delta(5, 3)$  are equal to 10.

### 3.2. Enumerating all lattice $(3, 2)$ -polytopes of diameter $\delta(3, 2)$

The algorithm can be adapted to enumerate all lattice  $(d, k)$ -polytopes of diameter  $\delta(d, k)$ . In this section we consider the case  $(d, k) = (3, 2)$ ; that is,



the determination of all lattice  $(3, 2)$ -polytopes of diameter 4.

Note first that a lattice  $(3, 2)$ -polytope with an empty intersection with at least one of the facet of  $[0, 2]^3$  is either a hexagonal prism or a lattice  $(3, 2)$ -polytope of diameter at most 3. Thus, the hexagonal prism depicted in Figure 2 in the middle of the top row is, up to the symmetries of  $[0, 2]^3$ , the unique lattice  $(3, 2)$ -polytope of diameter 4 with an empty intersection with at least one facet of  $[0, 2]^3$ .

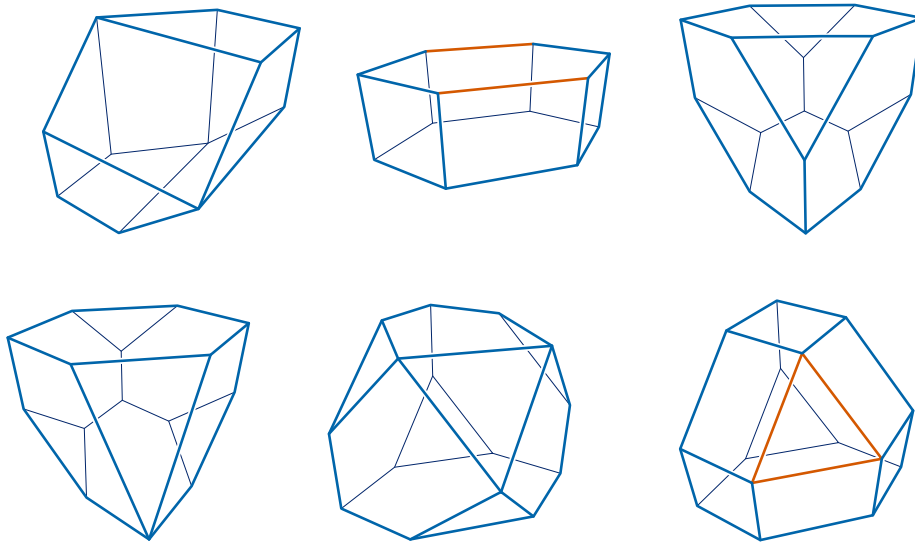


Figure 2: All, up to the symmetries of  $[0, 2]^3$ , lattice  $(3, 2)$ -polytopes of diameter 4 with 11, 15, or 16 vertices, or with an empty intersection with at least one facet of  $[0, 2]^3$

For a lattice  $(3, 2)$ -polytope of diameter 4 with a nonempty intersection with each facet of  $[0, 2]^3$ , Theorem 3 can be adapted as follows. Let  $u$  and  $v$  be two vertices of a lattice  $(3, 2)$ -polytope such that  $d(u, v) = 4$ , then  $(u, v)$  must satisfy  $1 \leq u_i + v_i \leq 3$  for  $i = 1, 2, 3$ , and the intersection of the lattice  $(3, 2)$ -polytope with any facet of  $[0, 2]^3$  must contain at least 2 vertices. The computational results show that there are, up to the symmetries of  $[0, 2]^3$ , hundreds of lattice  $(3, 2)$ -polytopes of diameter 4 with a nonempty intersection with each facet of  $[0, 2]^3$ . The set  $\mathcal{V}_{3,2}$  of all the vertices of all the lattice  $(3, 2)$ -polytopes of diameter 4 consists of all  $\{0, 1, 2\}$ -valued points except  $(1, 1, 1)$ . This point forms the intersection of all lattice  $(3, 2)$ -polytopes

of diameter 4; that is,  $\mathcal{P}_{3,2} = \{(1, 1, 1)\}$ .

There are 3, up to the symmetries of  $[0, 2]^3$ , lattice  $(3, 2)$ -polytopes of diameter 4 with 15 vertices which are depicted in the bottom row of Figure 2 where the edges of the intersections with the facets of  $[0, 2]^3$  are shown in blue. The unique, up to the symmetries of  $[0, 2]^3$ , lattice  $(3, 2)$ -polytope of diameter 4 with 11, respectively 16, vertices is represented leftmost, respectively rightmost, in the top row.

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