



HAL
open science

The need for reconstruction: Students' learning of bivariate functions calculus

María Trigueros, Rafael Martínez-Planell

► **To cite this version:**

María Trigueros, Rafael Martínez-Planell. The need for reconstruction: Students' learning of bivariate functions calculus. 2021. hal-03409504v1

HAL Id: hal-03409504

<https://hal.science/hal-03409504v1>

Preprint submitted on 29 Oct 2021 (v1), last revised 23 Feb 2022 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The need for reconstruction: Students' learning of bivariate functions calculus

María Trigueros & Rafael Martínez-Planell

Abstract. A review of our research about student understanding of the calculus of bivariate functions is presented. It summarizes findings from studies conducted throughout 15 years of research activities on the topic. Its aim is to communicate our research results as a whole in an accessible way. The results we discuss underscore the new challenges that students meet when dealing with this new type of function and suggest that the belief that students can easily generalize from their knowledge of one-variable functions is not sustained by research, so different foundational notions necessary for the context of bivariate functions need to be explicitly considered during instruction. We include the presentation of research-based suggestions that have practical value for teaching these functions and bring to light the status of research in this important area of the didactics of mathematics, making apparent the need for further research.

Keywords. APOS, Calculus, Multivariable calculus, Functions of two variables, Bivariate functions.

Résumé. Une revue de nos recherches sur la compréhension par les étudiants du calcul des fonctions bivariées est présentée. Il résume les résultats des études menées tout au long de 15 ans d'activités de recherche sur le sujet. Son objectif est de communiquer nos résultats de recherche dans leur ensemble de manière accessible. Les résultats dont nous discutons soulignent les nouveaux défis que les étudiants rencontrent lorsqu'ils traitent de ce nouveau type de fonction, et suggèrent que la croyance que les étudiants peuvent facilement généraliser à partir de leur connaissance des fonctions à une variable, différentes notions fondamentales nécessaires au contexte des fonctions bivariées, n'est pas soutenue par la recherche. Ces fonctions doivent donc être explicitement prises en compte lors de l'enseignement. Nous incluons la présentation de suggestions, fondées sur la recherche, qui ont une valeur pratique pour l'enseignement des fonctions bivariées et peuvent éclairer l'état de la recherche dans ce domaine important de la didactique des mathématiques, ce qui rend évident la nécessité de poursuivre les recherches.

Mots-clés. APOS, Calcul, Calcul multivariable, Fonctions de deux variables, Fonctions bivariées.

Contents

1.A. Introduction	2
1.B. A brief introduction to APOS Theory	3
1.B.a. How can we generalize what we know about one variable functions to bivariate or multivariate functions?	4
1.B.b. First research cycle on the geometric aspects and the definition of two-variable functions	5
1.B.c. Second research cycle	7
1.B.d. Third research cycle	8
1.B.e. Future research on bivariate functions	8
1.C. Learning bivariate functions' differential calculus	8
1.C.a. First cycle of research: differential calculus of two-variable functions	10

1.C.b. Partial derivatives	10
1.C.c. Directional derivative	10
1.C.d. Total differential	11
1.D. On students' ideas about integral calculus	12
1.E. Final remarks	14
References	15

1.A. Introduction

Multivariable calculus is an important tool to model a wide variety of phenomena appearing in science, mathematics, engineering, economics and other disciplines. Even though one-variable functions and analysis have received much attention from researchers in mathematics education, that is not true in the case of multivariable functions.

The earliest research we found on this topic was published during the nineties by only two researchers. Tall (1992) discussed the particularities of the derivatives of two-variable functions. Yerushalmy (1997) described results obtained while working with functions of two variables with secondary school children. For many years these papers and the topics studied in them did not receive much attention. However, in the last fifteen years a wider interest for the calculus of multivariable functions, particularly two-variable functions, arose (for a survey see Martínez-Planell & Trigueros, 2021).

We have been working on students' understanding of two-variable functions continuously for the last fifteen years. In this paper we reflect on what we have learnt and what needs to be done. We consider this reflection important to promote more studies from different points of view and to offer people who teach courses on multivariable calculus information about research results concerning students' learning in this mathematical topic. In what follows we start by briefly discussing the theoretical framework that has guided our research. After that, we consider what we know about students' understanding of geometrical aspects and other basic properties linked to two-variable functions, including their definition. Then, we go on to consider students' learning about the differential calculus and finish by briefly introducing what we have studied so far about the integral calculus of these functions. We will focus on questions such as, how do students understand two-variable functions? Are the notions about real-valued functions directly generalized into the context of multivariable functions? How do students use what they have learnt before? We will also address important challenges that students face when encountering these new functions, such as the expansion of the range of possible contexts where algebraic expressions are interpreted; visualization of 3D mathematical objects; the relation between students' 3D mathematical schemas and their intuition about the world we live in; and the reinterpretation of some basic notions about one-variable functions in the new realm of two-variable or multivariable functions.

Throughout the article, the reader will notice that there are still many open questions that need attention from researchers and where help from experienced teachers can contribute in deepening the knowledge we have today.

How do we learn mathematics? Learning is a complex phenomenon. There are multiple important factors that influence what we learn and how we do it. Mathematical learning is not an exception. In order to understand its different facets, several theoretical frameworks have been proposed. In our research we have approached mathematics learning through a cognitive theory based on Piaget's epistemology (Piaget, 1970) and adapted by Ed Dubinsky and his collaborators in the Research in Undergraduate Mathematics Education Community (RUMEC). This theory, called APOS Theory proposes constructs that allow a description of advanced mathematics learning (Arnon et al., 2014; Dubinsky & McDonald, 2001). The main theoretical structures in this theory are Action, Process, Object and Schema. The acronym in the name of the theory APOS comes from them.

1.B. A brief introduction to APOS Theory

In APOS theory, mathematical knowledge construction for any concept or topic starts from a generic student doing Actions on a mathematical Object he or she has already constructed. Examples of Actions are for example: comparing, calculating, following the steps of a given algorithm, representing, etc. For example, calculating the slope of a line using the formula $\frac{y_2 - y_1}{x_2 - x_1}$, using formulas to calculate partial derivatives, comparing similarities and differences when graphing two variable functions and one variable functions. As the student reflects on his Actions, they may be interiorized, which means that the student is able to do Actions without external help, and he or she may skip, justify or generalize them and it is considered that that the student shows the construction of a Process. For example, a student who can flexibly think of slope as independent of representation, including of slope as a rate of change, shows the construction of slope as a Process; a student who can generate the dynamical imagery necessary to think of partial derivative at a point as the slope of a tangent line obtained as a limit of slopes of secant lines, and equivalently, a limit of difference quotients, shows behavior consistent with the construction partial derivative at a point as a Process. The construction of Processes is also demonstrated by the possibility of the student to reverse the Process. Different Processes can be coordinated into a new Process. When the student needs to do a new Action on a Process, and is able to do it, he or she shows to have encapsulated the Process into an Object and this is demonstrated by the student's possibility to do Actions on it. For example, a student who can do Actions on the notion of slope to construct a Process of partial derivative, or a Process of directional derivative as suggested by Figure 5, shows evidence of having encapsulated slope as an Object; a student who demonstrates the possibility of comparing partial derivatives at a point in terms of their geometric and algebraic representation and to relate them to the total differential at a point, as suggested by Figure 6, gives evidence of construction of partial derivative at a point as an Object. Objects can be de-encapsulated into a Process when needed, and new Actions can be done on the Objects constructed. A Schema is an encompassing structure which is formed by Actions, Processes, Objects and other previously constructed Schema and the relations among them (Figure 1). When a student faces a mathematical problem, he or she uses a Schema to deal with it. As the student constructs new knowledge, the Schema develops by assimilation or accommodation of new structures and or by the construction of different types of relations among the components of the Schema.

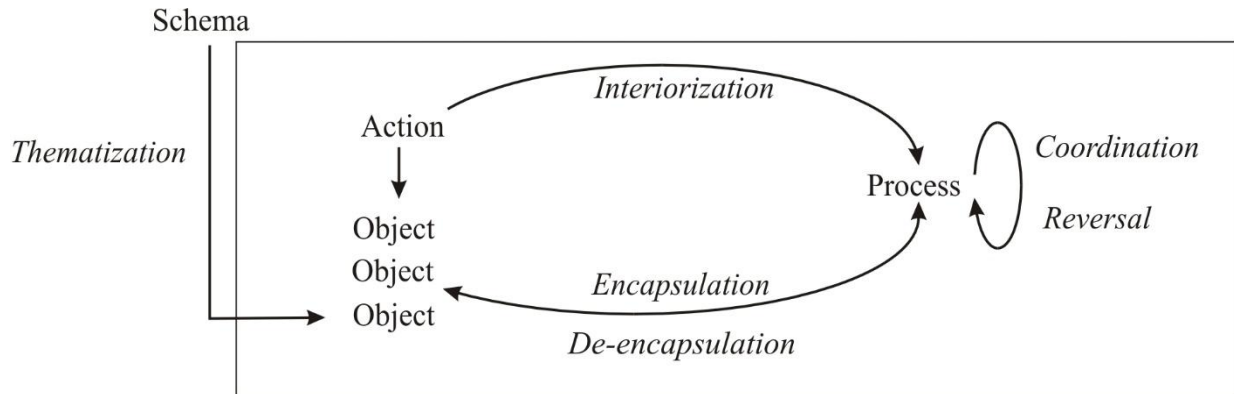


Figure 1 – APOS structures and mechanisms

Another important idea in APOS Theory is that of a genetic decomposition (GD). A GD is a model, expressed in terms of the structures and mechanisms of APOS, of how a student may construct a specific mathematical notion. A GD is not meant to be unique. Different researchers, or even the same researcher, may propose different GD's. It is important that a GD be tested with student data. To test a GD, one may start by researching what students who have studied a topic have learnt and comparing the construction shown by students to those proposed by the model or by designing activities to help students do the constructions that have been proposed in the GD. After implementing the activities, experimental data is obtained from the students. The analysis of that data is expected to result in a revision of the GD to take into account those unexpected constructions shown by students that are not included in the model and to delete those proposed constructions that seem not to be needed by students. Then, activities are designed or revised to reflect the refined GD. This opens the door to further research cycles testing the revised GD. Cycles may continue until a GD is found that is judged not to merit further revision. At that moment the GD will be considered to describe how most students may construct the mathematical notion or notions of interest, so the GD will also serve as a guide for instruction. In what follows we discuss what we have learnt about the learning and teaching of bivariate functions from our research using APOS theory.

APOS theory also includes a research methodology and a didactic methodology known as the ACE Cycle (Arnon et al. 2014) designed to foster students' reflection and their construction of knowledge.

1.B.a. How can we generalize what we know about one variable functions to bivariate or multivariate functions?

Our research has shown us that it is not easy for students to directly generalize what they know about one-variable functions to bivariate functions. Work with students is needed so that they can develop the tools that we have found are necessary for students to deeply understand these functions.

One important and unexpected finding of our research is that the construction of a coherent 3D Cartesian space Schema requires explicit attention during instruction. Results from posterior studies clearly showed that time devoted to help students understand the description of points,

curves and surfaces in 3D space contributes a lot to their understanding of bivariate functions, their graphs and their properties. As part of understanding 3D space, an important concept for students to construct is that of a fundamental plane. We consider what we and others call a “fundamental plane”, to be a plane in the form $x = c$, $y = c$ or $z = c$, where c is a constant, in three-dimensional space. It is important for students to be able to interpret equations in 3D space, and fundamental planes play a key role in this regard. Through cycles of research, we will later describe with care, we have found that an effective method for doing this is, given a bivariate function, to fix one of the variables, so that the problem is reduced to a problem in two dimensions where students can apply what they have previously learnt, and then to allow the fixed variable to change. Setting a variable constant in a given analytical expression for a surface is equivalent, from the geometrical point of view, to intersecting a fundamental plane with the graph of the surface. The plane can then be moved along an axis by changing the value of the constant c . Once students have done and reflected on these Actions by using different fundamental planes and different surfaces, they may construct a Process allowing them to imagine graphs of two-variable functions, cylinders, and other surfaces. These actions are essential. Fundamental planes play an important role throughout a course on Analysis of bivariate and multivariate functions and, in terms of APOS theory, this is reflected in their role in the genetic decomposition model for bivariate functions.

Our studies on bivariate functions have gone through several research cycles in terms of APOS theory, where we have refined the GD model and designed and redesigned activities to teach topics usually covered in multivariate analysis. We will describe these cycles together with our findings and needs to reconsider some of our initial hypothesis until we could observe from the obtained data that most students showed evidence of having learnt the topic of interest.

1.B.b. First research cycle on the geometric aspects and the definition of two- variable functions

We started our research by analyzing if students who had followed a lecture-based multivariable calculus course had constructed the structures predicted by the GD, and if they did not show some of those constructions, to ask ourselves if they were necessary in their learning, or if other constructions that we had not considered before needed to be included. Our findings gave evidence that in order for these students to succeed in explaining what a bivariate function was, in describing lines and planes in Cartesian 3D space, in recognizing the difference between transversal sections and projections, and in recognizing equations for given geometrically represented surfaces or vice versa, students needed a solid construction of a 3D space Schema (Trigueros & Martínez-Planell, 2010). Also, students who recognized fundamental planes and were more confident on their understanding of space were able to do Actions on fundamental planes in order to solve problems related to two-variable functions. We concluded that the construction of fundamental plane as an Object was important.

Another interesting finding can be illustrated by the following example. When asking students to graph curves as the set of points satisfying two equations at the same time, for example, $z = x^2 + y^2$ and $y = 2$, we found that students gave different meanings to the resulting equation depending on the geometric context they assumed for the equation (Figure 2). Some students, after substituting 2 for y in the equation $z = x^2 + y^2$, obtained $z = x^2 + 4$, and seeing only the variables

x and z , assumed the graph was a parabola on the xz plane. Other students, as soon as they recognized that they must have $y = 2$, interpret the context as restricted to the given fundamental plane, and so obtain a parabola on the plane $y = 2$. Further, in situations where there is no restriction on the value of y , students would need to recognize that in 3D, the equation $z = x^2 + 4$ refers to a cylinder. The context of an algebraic equation, although dependent on the given problem situation, must be assumed by the student, and the tool the student has to determine appropriateness of an assumed context is geometric visualization. So, in our studies we learnt that visualization plays a very important role in allowing students to decide on the appropriate context for an algebraic equation, and thus on the possibility to graph and interpret two-variable functions and other surfaces.

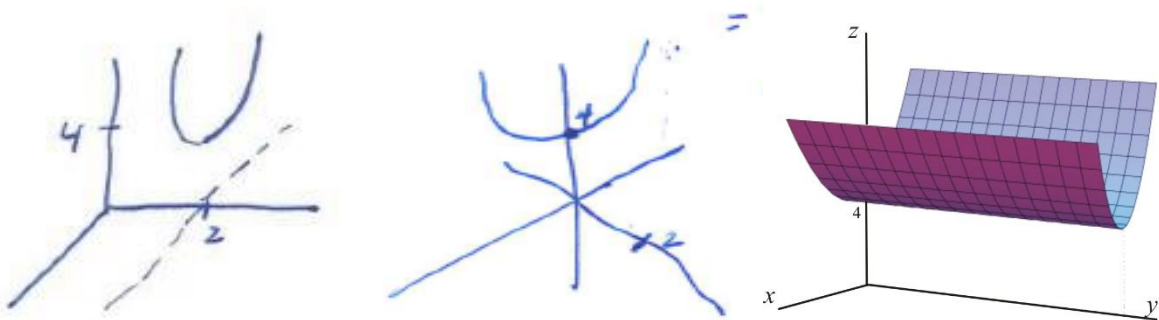


Figure 2 – Possible interpretations for $z = x^2 + 4$.

Moreover, we found that language and gestures used to describe mathematical situations can cause conflicts with the day to day use of language. For example, the phrase “lift a curve”, has been interpreted by some students as resulting in a surface, or the phrase “cut this surface with this plane”, has been interpreted by some students as resulting in two pieces of surface. Unless explicitly discussed with students, phrases used to intend a specific mathematical interpretation can conflict with students’ interpretations. This can be explained as a conflict between mathematical Schemes and our intuitions of space. For example, when some students were asked to “lift this circle up to $z = 3$ ”, referring to the circle in the xy plane in Figure 3, the researcher expected the students to produce only the circle at the height $z = 3$, however, the students explained that they would obtain the cylinder surface in the figure.

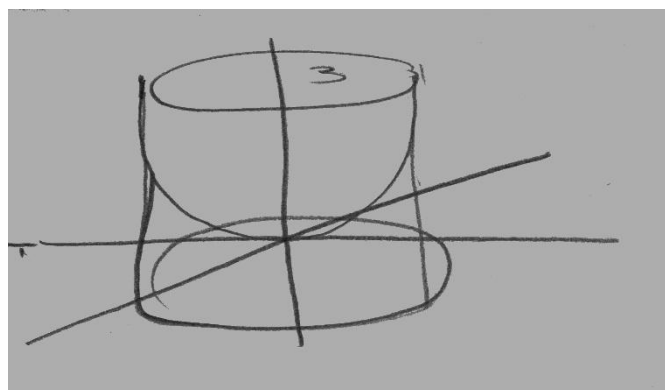


Figure 3 – A student’s view of points in R^2 that satisfy $x^2 + y^2 = 1$ and are assigned a height of 3

From our experience, now we can suggest that teaching students about objects in 3D space needs a careful use of language and explicit attention to the construction of an R^3 Schema. For

example, we have used with success the embodied language “move forward/backward” for “ x increases/decreases”, “move right/left” for “ y increases/decreases”, and “move up/down” for “ z increases/decreases”, consistently throughout a semester, while also consistently drawing the positive x axis coming out of the board (“forward”), the positive y axis pointing to the right, and the positive z axis pointing up. We also found out that the use of physical manipulatives can be of great help in this endeavor. Pedagogical strategies such as these can help students relate their mathematical 3D Schemas with their intuition of the space they live in.

When analyzing students’ descriptions about domain and range of bivariate functions we found out that some of them considered that “ x is the domain, y is the range”, others needed to graph the function in order to find its domain, and that when we asked them to describe functions of restricted domains such as $f(x, y) = x^2 + y^2$ in $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, they still needed to graph the function first in order to respond, or they insisted that the domain must be somehow or other obtained from the formula for the function, $f(x, y) = x^2 + y^2$, even though the domain is given explicitly as $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. We also observed that students struggled to describe what the unicity of z value meant, to give a correct definition of bivariate functions, and to relate a contour diagram with the functions’ graph (Martínez-Planell & Trigueros, 2012).

From our results, we concluded that bivariate functions are not a direct or expansive generalization (Harel & Tall, 1989) of one-variable functions. Students need to reconstruct their knowledge about one-variable function to be able to construct the notion of bivariate functions.

A replication study based on our first study was independently conducted in Turkey by different researchers (Şefik & Dost, 2020) using essentially the same interview instruments we used. Very similar results were found and the authors basically coincided with our conclusions. This gives further confidence that our results are not context dependent and can be generalized to other educational institutions, thus giving additional assurance of the implied generality (Sanchez, 2020) of the results that we have described thus far.

1.B.c. Second research cycle

The analysis of students’ responses put forward the need to refine our proposed GD. We did so, by considering what we learnt from the first experience. We then designed activities for students to work in teams with the goal of helping them reflect on their Actions so they would be able to do the constructions we considered were needed to understand two-variable functions.

APOS theory, as explained above, includes a theoretical framework, a research methodology and a teaching methodology. The last one, the ACE cycle (Arnon et al, 2014), involves a procedure to teach mathematical topics through the use of activities and collaborative work of students in teams, and to guide them in the construction of the structures described in the GD. The teacher acts as a guide to students by posing new questions related to those asked by students and promoting students’ attention on their reflective work. After some time, the teacher calls students to a whole-group discussion where teams present their work and new questions, which have arisen while working on the activities, offer new opportunities for students to reflect on what they have done. As part of the discussion, teacher and students, together, may define some concepts or work on proofs of problems according to what has been discussed. These two activities, student group work and

class discussion, are repeated throughout the lesson and the teacher leaves some activities or some new problems from texts or notes for students to work as homework.

The second research cycle (Martínez-Planell & Trigueros, 2013; Trigueros & Martínez-Planell, 2015) included research in two multivariable calculus course sections. In one of them the teacher used APOS-based activities and the ACE cycle to work with students (APOS section), while the other section used a lecture-based approach (lecture section). Research was conducted in the two groups, students were selected following the same criteria in both groups, interviews were recorded, data was analyzed by researchers, and they discussed and negotiated differences in analysis until they reached an agreement.

Results obtained showed that in the APOS section most students showed a better understanding of fundamental planes and their use in graphing or interpreting bivariate functions, while students from the lecture-section showed similar constructions as those found in the first research cycle. Also, new observations arose from the data: The importance of students understanding the role of free variables (e.g. $f(x, y) = x^2$; $f(x, y) = x \sin y$ when $x = 0$); and the fact that familiar algebraic expressions ($x^2 + y^2 = 1$, $z = x^2$) could inhibit the generalization of one-variable function properties to those of two-variable functions; the importance of students' possibilities to perform and recognize treatments applied to functions presented in specific representation registers (Duval, 2006) to interpret and better understand (e.g. some students could graph $z = x^2$ but not $f(x, y) = x^2$); the convenience of reviewing the topic of "transformation of one variable functions" in order for students to use transformations when needed; and the permanence of difficulties regarding the use of fundamental planes to graph cylinders, were found to play a very important role which was not considered in the refined genetic decomposition. So, although APOS-section students clearly showed a better understanding than those students in the lecture-section, research results indicated that the GD and the activities designed with it needed to be refined in order to include the newly found constructions. We decided to conduct a new research cycle when these changes were done.

1.B.d. Third research cycle

Again, two sections, an APOS section and a lecture section, as described above, were used in this research cycle. Data from this cycle were astounding; APOS-section students performed much better than those in the lecture-section while these students showed very similar results as those found in the previous research cycles. We concluded that it is possible to reconstruct the one-variable function notion to construct that of bivariate functions by using the designed activities which promoted successfully students' reflection on their Actions. Most students showed to have constructed a Process conception of bivariate functions and some students showed to have constructed an Object conception of these functions. A didactic approach, as the designed one, can successfully help students construct two-variable functions through their reflection on specific Actions and Processes to develop a 3D space Schema and fundamental plane as Object that can be used in interpreting, describing and graphing two-variable functions. The difficulties obtained in the second cycle were clearly overcome and APOS-sections students were able to work seamlessly with curves, cylinders and bivariate functions in general, they were also able to justify their work and showed the construction of a dynamic image of functions which they were able to use

independently of the representation register used. We thus validated the designed GD and activities (Martínez-Planell & Trigueros, 2019).

1.B.e. Future research on bivariate functions

Although we were satisfied with the results obtained in the third research cycle, the APOS-based section in that cycle was taught by a teacher that was familiar with APOS theory and with the ACE teaching cycle. We still wondered if it would be possible to obtain results as good as those obtained in the third research cycle in an experience conducted by other teachers in different conditions. We decided to conduct a reproducibility study (Sanchez, 2020) with an interested teacher-researcher at a university in Iran. He used all the material designed for the third research cycle and we are now involved in data analysis. Although we have not completed this study, we can anticipate that the results obtained were good, although there are substantial differences that we need to analyze in depth.

We also are in the process of using digital technologies, specifically the GeoGebra software, to help student's visualization of 3D space, fundamental planes, and two-variable functions. We will adapt the activities we have designed so far to this software, implement them in the classroom, and conduct research to find out if they influence students' understanding and how.

1.C. Learning bivariate functions' differential calculus

Encouraged by the results obtained we decided to continue our research on the calculus of two-variable functions by following a similar research process as that described before. Again, we focused on the question of students' possibility to directly generalize what they know from real one-variable function differential calculus to two-variable functions.

We found previous studies on students' learning of differential calculus of bivariate functions where authors used theoretical frameworks other than APOS Theory (Yerushalmy, 1997; Weber, 2015). Both of these studies focus on the generalization of the rate of change notion for these functions. The main problem involved when facing differential calculus of bivariate function is that for one-variable functions, given a point in the domain of the function there is only one rate of change, while in the bivariate functions case, the rate of change in one point in the domain is not unique. Direction plays a very important role in this case. Both previous studies observed the fact that even good students who are able to construct rates of changes in the x and y directions for a bivariate function, struggle to construct what they call "the" rate of change which they hope must exist (this problem was called the "two change problem" by Weber, 2015; see also Moore-Russo, Conner, & Rugg, 2011). Students in those studies showed a tendency to look for an algebraic solution to the problem and they could not tell what exactly was that to be generalized.

In our studies, performed with the collaboration of Daniel McGee (2015, 2017, 2018), we have used a different approach: We consider the problem from the point of view of local linearity by adapting a suggestion made by David Tall (1992), which allows us to place the emphasis on the geometry of the problem. Thus, we focus on the notion of vertical change on a plane: $\Delta z = m_x \Delta x + m_y \Delta y$ (Figure 4), from an initial point (a, b, c) to a final generic point (x, y, z) .

We used vertical change on a plane to consider the point-slope equation for the plane $z - c = m_x(x - a) + m_y(y - b)$, which in turn is used to describe the tangent plane to a bivariate function at a point: $z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$. Vertical change on a plane was also used to discuss the total differential of a function f at a point (a, b) as $df(a, b) = f_x(a, b)dx + f_y(a, b)dy$, and the directional derivative as $D_{\langle \Delta x, \Delta y \rangle} f(a, b) = (f_x(a, b)\Delta x + f_y(a, b)\Delta y) / \|\langle \Delta x, \Delta y \rangle\|$. Note that vertical change on a plane is an idea that can bring coherence to the differential calculus Schema for functions of two variables, in the sense that this idea interrelates different important notions of the differential calculus thus making these notions easier to access and use in problem situations.

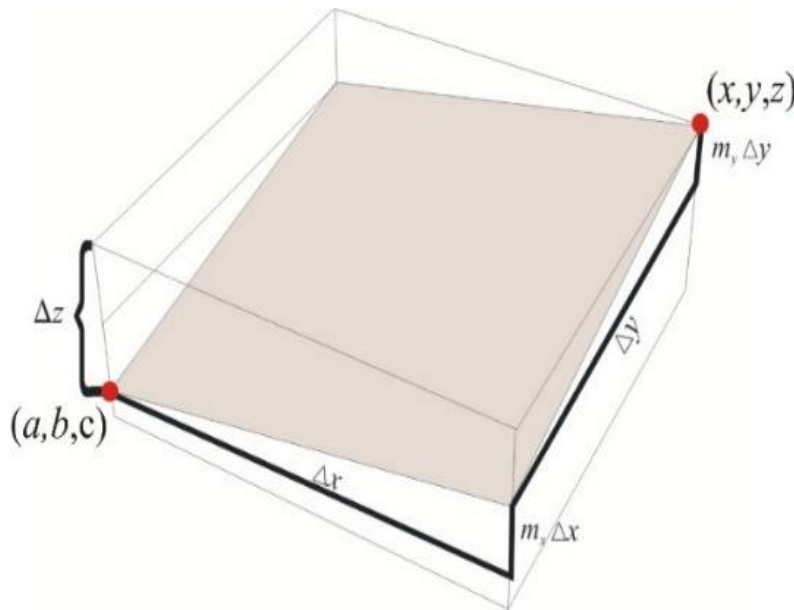


Figure 4 – Vertical change on a plane $\Delta z = m_x \Delta x + m_y \Delta y$

1.C.a. First cycle of research: differential calculus of two-variable functions

Taking into account the formerly discussed ideas about change in 3D space, we designed a genetic decomposition for the differential calculus of two-variable functions, using as main idea vertical change on a plane. Activities based on the genetic decomposition were designed independently by two professors, and were used in two sections of a course on multivariate functions calculus taught by the same professors, who also used the ACE cycle as didactic strategy. A research study was then designed to test the GD (Martínez-Planell, Trigueros, & McGee, 2015). The two APOS-based sections and one lecture-based section were included in the study in order to compare the obtained results.

Research results showed that the responses of students from the two sections following the APOS methodology were very similar, with students in both these sections showing better learning results than student in the lecture-based section. Most students from the lecture section were able to do some of the Actions predicted by the DG model while APOS sections students were able to do most Actions and some Processes predicted by the GD. Nevertheless, the analysis of students' responses related, for example, to the recognition that slope in the x or y direction on a non-vertical plane is independent of the initial point selected on the plane, or to the coordination of vertical change in the x direction to that in the y direction before determining the total vertical change on a

plane (Figure 4), included in the proposed GD, needed to be more carefully considered in the proposed activities to help students reflect on the basic construction related to the tangent plane to a surface.

1.C.b. Partial derivatives

The construction of the partial derivatives Process involves the coordination of a Process for derivative of one-variable function and a Process of fundamental plane. Results obtained revealed that several students in all the sections showed to have constructed an Action conception of derivative of functions of one variable, which suggested the need that it be reconsidered and reconstructed when teaching derivative of bivariate functions in 3D space. Also, the construction of relations between different representations of partial derivatives seemed, from our results, important for students to coordinate Processes involved in the construction of partial derivatives as a Process. Results obtained showed that students in both APOS sections evidenced more frequently the construction of the coordination needed to relate different representations for partial derivatives that those in the lecture-based section.

1.C.c. Directional derivative

Results obtained in the interviews of students of all three sections made us wonder what is directional derivative for students. Most students in the lecture section referred to it as a formula ($D_{\langle u_1, u_2 \rangle} f(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2$) that they could not even remember, and even those that were able to recall it could not explain its meaning geometrically. Students in the APOS sections, also related it to a formula, now given as $D_{\langle \Delta x, \Delta y \rangle} f(a, b) = (f_x(a, b)\Delta x + f_y(a, b)\Delta y) / \|\langle \Delta x, \Delta y \rangle\|$. While about half of them could remember and use this formula, when they were asked to give a geometrical justification for the formula, or to discuss the sign of a directional derivative on a graphical representation of a function, very few students succeeded in giving an appropriate description which can be geometrically interpreted by Figure 5. During the interviews we further inquired with them about the direction vector in 3D space, and found evidence that they showed difficulty representing it geometrically in space (Martínez-Planell, Trigueros, and McGee, 2017).

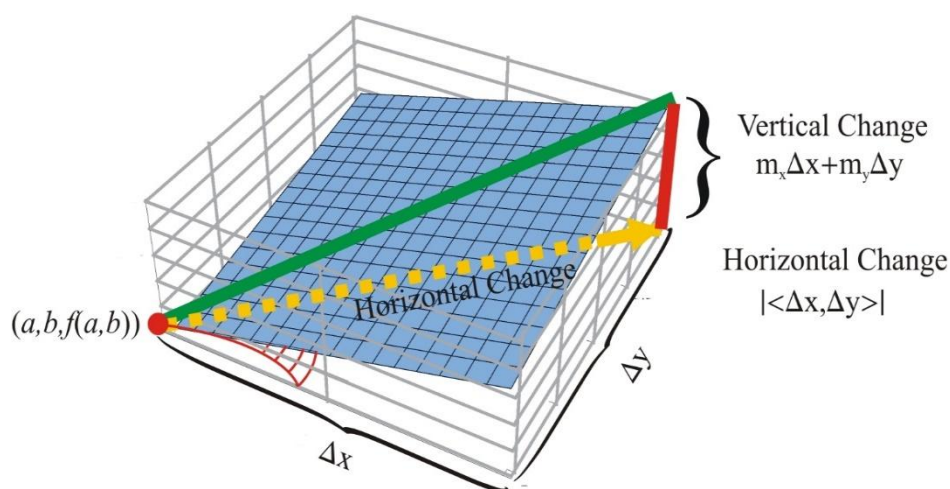


Figure 5 – Directional derivative at a point: $D_{\langle \Delta x, \Delta y \rangle} f(a, b) = \frac{f_x(a, b)\Delta x + f_y(a, b)\Delta y}{\|\langle \Delta x, \Delta y \rangle\|}$

1.C.d. Total differential

In our approach, using tangent plane at a point, and a function in a local neighborhood of the point, the total differential can be related to the plane as shown in Figure 6. Here, the total differential at a point gives the vertical change along the tangent plane as a function of the horizontal change (dx, dy) . In our study most students showed not even remembering what the total differential was. The few students showing an Action conception of total differential were from the APOS sections. With only one exception, students could at most repeat memorized Actions but were unable to justify them. For the most part, students did not relate the total differential at a point with the tangent plane at the point and the function at a neighborhood of the point. When helping them to think about the expression $df(a, b) = f_x(a, b)dx + f_y(a, b)dy$, they did not show evidence of considering it as a function of the independent variables dx, dy . For them dx and dy could only be considered as “very small numbers”. Clearly, this suggested the need to revise the GD and APOS-based activities dealing with the total differential (Martínez-Planell, Trigueros, and McGee, 2018).

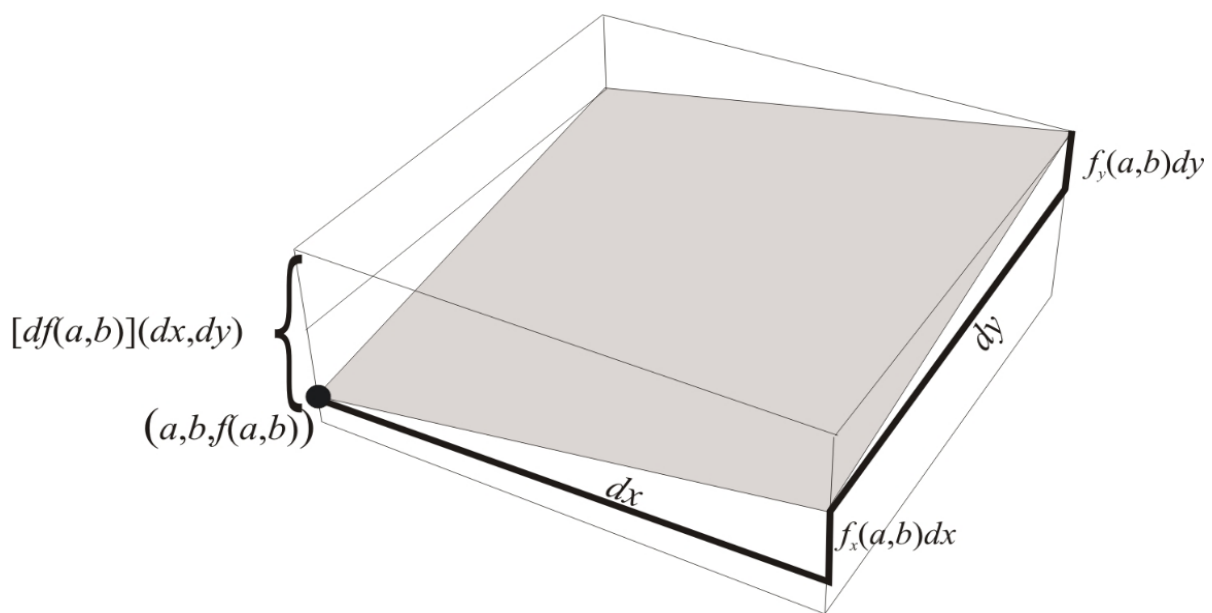


Figure 6 – The total differential at a point (a, b) : $[df(a, b)](dx, dy) = f_x(a, b)dx + f_y(a, b)dy$

When considering the overall results on the differential calculus it can be observed that, although students in the APOS section did show a better understanding than those in the lecture section, results obtained were not as good as expected. Students in the APOS section were more likely to be able to perform Actions to compute vertical change on a plane and the point-slopes equation of a plane, and could better interpret partial derivatives geometrically and numerically, but in the case of directional derivatives or the total differential of functions their constructions were not too different from those of students in the lecture section. These results clearly asked for a refinement of the proposed GD based on responses of participants in this study together with the design of new activities to be used and analyzed in a new research cycle on the topic of differential calculus.

So far, we have refined the GD and the activities but are just starting to conduct a second cycle of the research. Since in the first research cycle we omitted some themes usually included in differential calculus for two-variable functions courses, we are also extending our research to

include topics such as the conceptualization of partial derivatives and directional derivatives as Objects, second partial derivatives and mixed partial derivatives, the gradient vector and the point-normal equation of a plane.

1.D. On students' ideas about integral calculus

There is scarce research on multivariable integral calculus learning and teaching. We started out research by asking these questions: What are the needed constructions for a student to relate a double integral with its Riemann sums? Which of those constructions can be evidenced by students who have finished a lecture-based multivariable calculus course?

According to APOS theory methodology, we started by designing a GD in order to explore students' basic understanding of Riemann sums. Then, we designed a semi-structured interview instrument to look at students' conceptions of very simple Riemann sums related to bivariate functions. Thus, we considered a continuous two-variable function on a rectangular domain and the simplest possible partition: the whole rectangle, that is, no partition at all, and a point in the rectangle (Figure 7). We aimed at examining students' constructions, described in the GD, in recognizing the rectangle and the function; forming a term of a Riemann sum; forming a more complex partition and a Riemann sum; and selecting a point in the rectangular domain to obtain a sub-estimation (Figure 7), an overestimation and the exact value of the integral of this two-variable function. We also aimed at finding out what they considered was the relation of Riemann sums and the function's double integral.

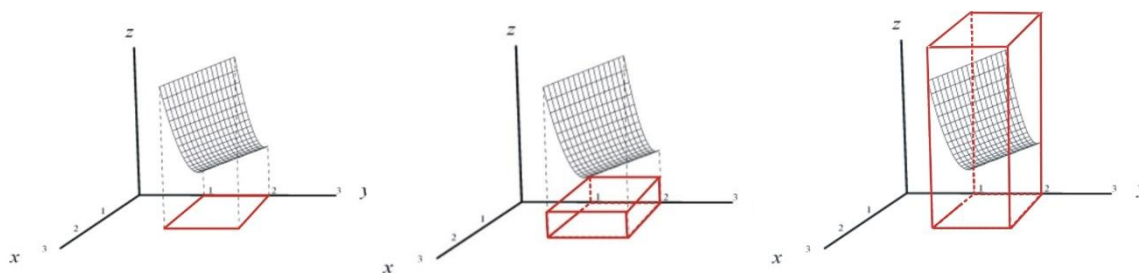


Figure 7 – Rectangle and function, sub-estimation, overestimation

For this first research cycle we used students from a lecture section, so they had not used any APOS-based activities in their classroom, and their construction of rectangular domain and function over the rectangle was similar to the constructions observed in the first round of geometric aspects and the definition of two-variable functions, discussed before. The analysis of students' responses on the meaning of a Riemann sum term, $f(a, b)\Delta x\Delta y$, evidenced that many of them had not yet constructed the meaning of this product, and this even included some students who were able to do Actions to interpret the meaning of each one of its components, i.e., $f(a, b)$, Δx , Δy , and $\Delta x\Delta y$. Some students interpreted $f(a, b)$ as a location or a point, so that for them $f(x, y)dxdy$ was “area of a small piece of surface at a point” and the double integral $\iint f(x, y)dA$ was surface area, rather than the volume between the rectangular domain and surface. Some students also showed they had yet to construct meaning for a double integral in a contextual situation even though some of them could discuss the contextual units corresponding to a term of a Riemann sum or to the expression $f(x, y)dxdy$ as an Action. Many students showed confusion between the meaning of area and

volume. The results suggest the need to explicitly discuss the different elements of this expression: the meaning of each factor $f(a, b)$, Δx , Δy , and $\Delta x \Delta y$ in a term of a Riemann sum; to multiply these factors to obtain and interpret a term of a Riemann sum $f(a, b) \Delta x \Delta y$, including in contextual situations; to form more complex Riemann sums; to understanding the existence of points where an underestimate, an overestimate, or the exact value of the double integral may be obtained in each sub-rectangle of a partition; and to relate Riemann sums to their double integrals. The results also suggest the importance of relating verbal, numeric, geometric representation of Riemann sums, their algebraic representations as extended sums, and as sums using sigma notation, before taking limits and representing the result as a double integral. Giving explicit attention to this chain of representations is also contemplated in the GD and has been shown to result in improved student understanding in other studies (McGee & Martínez-Planell, 2014).

The analysis of the data and the evidence gathered in students' responses were useful in reviewing the corresponding part of the GD and refining it (Martínez-Planell and Trigueros, 2020). We have designed teaching activities which will soon be used in the classroom so we can start a second cycle of research on integration of bivariate functions. We also expect to extend the study to other parts of the GD we did not test such as Riemann sums and their limits, how the students work with problems that have more complex domains, and double and triple integrals when other coordinates, such as polar, cylindrical, and spherical coordinates are introduced.

1.E. Final remarks

Results about the learning of bivariate function clearly show that their learning and the learning of the calculus associated with them require more than just generalizing what students know from one-variable functions. Learning these topics involves a reconstruction of the calculus of one-variable function. This reconstruction requires students to do specific Actions on two-variable functions, their domain and range, and their graph so that, when they reflect on them, they can construct Processes regarding their similarities and differences with one-variable functions. Doing so, together with collaborative work among students and discussion with the teacher can help students understand this new type of functions in depth.

Using APOS theory as a framework helps researchers and teachers find out what constructions can play a fundamental role in students learning, when doing the reconstruction needed to fully understand these functions. Through the observation of teachers and the analysis of textbooks we have found evidence that it is frequently assumed that students can do generalizations without much help. However, our research and that of other researchers using different theoretical frameworks and doing studies in different countries such as the United States, Turkey, Iran, Malaysia, and Ethiopia, clearly shows that this is not the case (Martínez-Planell & Trigueros, 2021).

A solid construction of bivariate functions is indispensable for students to be able to understand and learn the notions involved in the differential and integral calculus of two-variable functions. The construction of these notions involves coordination of different Processes and the construction of relations between different representation registers. It is important for teachers to be aware of the fact that making an effort in helping students do the necessary constructions of the geometric aspects and definition of two-variable function plays a key role in helping them deeply learn the notions involved in the differential and integral calculus.

Our research results show only a small portion of all the notions involved in the study of the learning and teaching of two-variable functions. Much more research is needed; there are many areas of multivariable calculus that need attention from researchers. Research on the teaching of bivariate and multivariate functions from different theoretical perspectives is needed, as well as the teaching and learning possibilities that the use of technology in the classroom may open for students' learning. However, we strongly believe that results obtained by research on bivariate functions conducted so far offers university teachers important ideas to approach this subject from a richer perspective that can propose better possibilities for more students with different interests to significantly learn these fundamental topics.

References

- Arnon, I., Cotrill, J., Dubinsky, E., Octaç, A., Roa Fuentes, S., Trigueros, M., & Weller, K. (2014). *APOS Theory: A framework for research and curriculum development in mathematics education*. Springer Verlag: New York.
- Dubinsky, E., & McDonald, M. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 273–280). Dordrecht, The Netherlands: Kluwer.
- Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61(1), 103-131.
- Harel, G., & Tall, D. (1989). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics*, 11(1), 38-42.
- McGee, D., & Martínez-Planell, R. (2014). A study of semiotic registers in the development of the definite integral of functions of two and three variables. *International Journal of Science and Mathematics Education*, 12(4), 883-916.
- McGee, D., Moore-Russo, D., Ebersole, D., Lomen, D., & Marin, M. (2012). Using physical manipulatives in the multivariable calculus classroom, *PRIMUS*, 22(4), 265-283.
- Martínez-Planell, R. & Trigueros Gaisman, M. (2012). Students' understanding of the general notion of a function of two variables. *Educational Studies in Mathematics*, 81(3), 365-384.
- Martínez-Planell, R. & Trigueros, M. (2013). Graphs of functions of two variables: Results from the design of instruction. *International Journal of Mathematical Education in Science and Technology*, 44(5), pp. 663-672.
- Martínez-Planell, R. & Trigueros, M. (2019). Using cycles of research in APOS: The case of functions of two variables. *The Journal of Mathematical Behavior*, V. 55, September 2019, 100687. <https://doi.org/10.1016/j.jmathb.2019.01.003>
- Martínez-Planell, R. & Trigueros, M. (2020). Students' understanding of Riemann sums for integrals of functions of two variables. *The Journal of Mathematical Behavior*, V. 59, September 2020, 100791. <https://doi.org/10.1016/j.jmathb.2020.100791>

- Martínez-Planell, R. & Trigueros, M. (2021). Multivariable calculus results in different countries. *ZDM-Mathematics Education* 3. DOI 10.1007/s11858-021-01233-6
- Martínez-Planell, R., Trigueros, M. & McGee, D., (2015). On students' understanding of the differential calculus of functions of two variables. *Journal of Mathematical Behavior*, V. 38, pp. 57-86.
- Martínez-Planell, R., Trigueros, M., & McGee, D. (2017). Students' understanding of the relation between tangent plane and directional derivatives of functions of two variables. *The Journal of Mathematical Behavior*, V. 46, pp. 13-41.
- Moore-Russo, D., Conner, A., & Rugg, K. (2011). Can slope be negative in 3-space? Studying concept image of slope through collective definition construction. *Educational Studies in Mathematics*, 76(1), 3-21.
- Piaget, J. (1970). *L'épistémologie génétique*. Paris: Presses universitaires de France.
- Şefik, Ö & Dost, Ş. (2020). The Analysis of the understanding of three dimensional Euclidian Space and the two-variable function concept by university students. *The Journal of Mathematical Behavior*, 57. 100697. DOI [10.1016/j.jmathb.2019.03.004](https://doi.org/10.1016/j.jmathb.2019.03.004)
- Tall, D. (1992). Visualizing differentials in two and three dimensions. *Teaching Mathematics and its Applications*, 11(1), 1-7.
- Trigueros, M. and Martínez-Planell, R. (2010). Geometrical representations in the learning of two variable functions. *Educational Studies in Mathematics* 73(1), 3-19.
- Trigueros, M. & Martínez-Planell, R. (2015) Las funciones de dos variables: Análisis mediante los resultados del diálogo entre la teoría APOS y la TAD. *Enseñanza de las Ciencias*, 33(2), pp. 157-171.
- Trigueros, M., Martínez-Planell, R., & McGee, D. (2018). Student understanding of the relation between tangent plane and the total differential of two-variable functions. *International Journal of Research in Undergraduate Mathematics Education*, 4(1), 181-197.
- Wangberg, A., & Johnson, B. (2013). Discovering calculus on the surface. *PRIMUS*, 23(7), 627-639.
- Weber, E. D. (2015). The two-change problem and calculus students' thinking about direction and path. *The Journal of Mathematical Behavior*, 37, 83-93.
- Yerushalmy, M. (1997). Designing representations: Reasoning about functions of two variables. *Journal of Research in Mathematics Education*, 28, 431-466.

María Trigueros
Instituto Tecnológico Autónomo de México
e-mail: mtriguerosg@gmail.com

Rafael Martínez-Planell
University of Puerto Rico at Mayagüez, Department of Mathematical Sciences
e-mail: rmplanell@gmail.com

