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A GENERAL COMPARISON PRINCIPLE FOR HAMILTON JACOBI BELLMAN EQUATIONS ON STRATIFIED DOMAINS *

OTHMANE JERHAOUI † AND HASNAA ZIDANI†

Abstract. This manuscript aims to study finite horizon, first order Hamilton Jacobi Bellman (HJB) equations on stratified domains. This problem is related to optimal control problems with discontinuous dynamics. We use nonsmooth analysis techniques to derive a strong comparison principle as in the classical theory and deduce that the value function is the unique viscosity solution. Furthermore, we prove some stability results of the Hamilton Jacobi Bellman equation. Finally, we establish a general convergence result for monotone numerical schemes in the stratified case.

Key words. Nonsmooth analysis, Viscosity theory, Hamilton Jacobi equations.

AMS subject classifications. 49J53, 35F21, 49L25.

1 Introduction. In this paper, we study the well-posedness of a system of Hamilton Jacobi Bellman (HJB) equations defined on a stratification of \mathbb{R}^N . More precisely, let $n \geq 1$ and Ω_i , $i = 1, \dots, n$ be pairwise disjoint, connected open sets of \mathbb{R}^N such that $\mathbb{R}^N = \cup_i \bar{\Omega}_i$. A stratification of \mathbb{R}^N is the union of these open sets and their topological boundary, denoted $\Lambda = \mathbb{R}^N \setminus \cup_{i=1}^n \Omega_i$, supposed to be a finite union of embedded submanifolds \mathcal{M}_i of lower dimension. Each Ω_i or \mathcal{M}_i is called a domain of \mathbb{R}^N . Consider a system of finite horizon HJB equations

$$(1.1) \quad \begin{cases} -\partial_t u(t, x) + H_{F_i}(x, \partial_x u(t, x)) = 0 & \text{for } (t, x) \in (0, T) \times \Omega_i \\ u(T, x) = \psi(x), \end{cases}$$

where $T > 0$ is the final time and $\psi : \mathbb{R}^N \rightarrow \mathbb{R}$ is the final cost and is assumed to be Lipschitz continuous. $H_{F_i} : \Omega_i \times \mathbb{R}^N \rightarrow \mathbb{R}$ are Bellman Hamiltonians defined the following way:

$$H_{F_i}(x, p) = \sup_{q \in F_i(x)} \{-\langle p, q \rangle\}.$$

$F_i : \Omega_i \rightsquigarrow \mathbb{R}^N$ are set-valued maps, called the dynamics, that satisfy standard hypotheses. In the absence of a stratification, the HJB equation has a unique solution (in the viscosity sense) called the value function, which is a function linked to optimal control problems [16, 2]. The aim of this paper is to extend this result to the stratified setting. To do so, one needs to define the HJB equation (1.1) at the interface Λ , to ensure the well-posedness of the HJB equation and to ensure that the value function is its unique solution.

Bressan and Hong [8] were the first to provide a rather complete study of this problem. In particular, they used Filippov's ideas to define the dynamics at the interface and showed that the value function is the unique solution that satisfies two HJB inequalities on each domain. They even showed that a comparison result holds. However, in order to achieve that, an additional assumption had to be made to characterize the sub-solution and sub-optimality property. Loosely speaking, this assumption states that every sub-solution ought to be at least continuous at the interfaces in order for comparison-type results to hold.

These additional assumptions were used by some of the works that came after Bressan and Hong's paper. For instance, Barles and Chasseignes [5], required either the continuity of the sub-solution at the interfaces or an additional junction condition in order for the comparison principle to hold. For other publications following similar approaches see the papers by Barles, Briani and Chasseignes [4, 3] and references therein. Rao and Zidani [28] and Ghilli, Rao and Zidani [18] have a similar layout to ours, but they used different test functions of viscosity solution for the super-solution and the sub-solution together with the assumption that the sub-solution is continuous at the interfaces. Similarly, Rao, Zidani and Siconolfi [27] considered a stratification in the form of two disjoint open sets separated by an interface in the form, which is a special case of [18]. We also mention similar publications of Hamilton Jacobi equations on stratified networks that share the same kind

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of difficulties as our layout: Imbert and Monneau [23], Imbert, Monneau and Zidani [24], Achdou, Camilli, Cutri and Tchou [1], Camilli and Marchi [10]. Lions and Souganidis [25] also worked on Hamilton Jacobi equations on networks and considered a different class of Hamiltonians that are not necessarily convex, but only continuous and coercive.

HJB equations are also related to a geometric notion known as flow invariance in the theory of differential inclusions [11, chapter 12]. The classical case, meaning absence of stratification, has been treated thoroughly in the literature [11, 2]. In the case of stratified domains and discontinuous dynamics at the interfaces, Barnard and Wolenski [7] analyzed the characterization of weak and strong invariance principles with a new Hamiltonian that they called *the essential Hamiltonian*. However their statement of strong invariance was inaccurate. Despite their valid intuition regarding the choice of the Hamiltonian, the choice of the “test functions” (in analogy with the viscosity theory) did not take into account the singular geometry of the problem which turns out to be crucial for comparison type results.

A different approach is considered by Hermosilla and Zidani in [22] and by Hermosilla, Wolenski and Zidani in [21]. Indeed, in this point of view, one starts from an optimal control problem with state constraints in the form of a closed stratified subset of the euclidean space. The subtle difference here is that in their setting, the dynamics are defined first in the whole Euclidean space, then state constraints are imposed in the form of a closed stratified subset, whereas in our case, we consider the stratified domain first, then we define the dynamics on each region of the domain. Furthermore, the state constraints are only a subset of \mathbb{R}^N and they are in the form of a multidimensional network. This leads to the fact that the value function associated to the setting in [22, 21] might not even be continuous, thus requiring the use of a more generalized notion of solution to the HJB equation, called *bilateral viscosity solution*.

As for numerical schemes approximating this problem, the only known results of convergence of finite differences numerical schemes are in the setting of a one dimensional simple network and are due to Guerand and Koumaiha [19] and Morfe [26]. However, to the best of our knowledge, there aren't any known convergence results in our setting.

In this present work, we aim to prove the well-posedness of the HJB equation on stratified domains. We will first define a Hamiltonian H_Λ at the interface Λ and consider the HJB equation:

$$(1.2) \quad \begin{cases} -\partial_t u(t, x) + H_{F_i}(x, \partial_x u(t, x)) = 0 & \text{for } (t, x) \in (0, T) \times \Omega_i, \\ -\partial_t u(t, x) + H_\Lambda(x, \partial_x u(t, x)) = 0 & \text{for } (t, x) \in (0, T) \times \Lambda, \\ u(T, x) = \psi(x) & \text{for } x \in \mathbb{R}^N. \end{cases}$$

The Hamiltonian H_Λ is defined on the interface Λ in the form of a maximum of lower semicontinuous Hamiltonians obtained along the same lines of [7]. Then, we will revisit the definition of viscosity solutions and give a new one that encodes the nature of the singular geometry of the problem. This new definition of viscosity will allow us to extend the strong comparison type results known when the Hamiltonian is Lipschitz continuous to the present setting. More precisely, we prove the following result:

Let u and v be respectively upper semi-continuous and lower semi-continuous functions on $(0, T] \times \mathbb{R}^N$. If u is a sub-solution of (1.2), and if v is a super-solution of (1.2), then $u \leq v$ on $(0, T] \times \mathbb{R}^N$.

The proof of this result relies on nonsmooth analysis techniques. In particular, we introduce an optimal control problem whose value function is the solution of the HJB equation. In the classical case, the nonsmooth analysis approach consists in interpreting the sub-solution property of the value function as the strong invariance of the hypograph of the value function and the super-solution property as the weak invariance of its epigraph. We will extend the weak and strong invariance results to the stratified setting. We would like to emphasize that the extension of invariance principles is also a contribution of this paper.

The strong comparison principle will have two major consequences. First, it will allow to obtain some stability results in this setting in the presence of perturbations on the dynamics. We prove that if there exist sequences $(F_i^j)_j$ of set-valued maps such that $F_i^j \rightarrow F_i$ with respect to the Hausdorff distance, and a sequence $(v^j : \mathbb{R}^N \rightarrow \mathbb{R})_j$ of lower semicontinuous (respectively upper semicontinuous) functions such that $v^j \rightarrow v$ locally uniformly in \mathbb{R}^N and suppose for all j , v^j is a super-solution (respectively sub-solution) of

$$\begin{cases} -\partial_t u(t, x) + H_{F_i^j}(x, \partial_x u(t, x)) = 0 & \text{for } (t, x) \in (0, T) \times \Omega_i, \\ -\partial_t u(t, x) + H_\Lambda^j(x, \partial_x u(t, x)) = 0 & \text{for } (t, x) \in (0, T) \times \Lambda, \\ u(T, x) = \psi(x) & \text{for } x \in \mathbb{R}^N. \end{cases}$$

then v is a super-solution (respectively sub-solution) of (1.2).

Finally, we will extend the known result due to Barles and Souganidis [6] for the convergence of monotone numerical schemes to the stratified setting. The numerical scheme has the following form in each domain:

$$\begin{cases} S_i^h(t_h, x_h, u^h(t_h, x_h), [u^h]_{(t_h, x_h)}) = 0 & \text{for } (t_h, x_h) \in (\Pi^{\Delta t} \times \mathcal{G}^{\Delta x}) \cap ((0, T) \times \Omega_i), \\ S_\Lambda^h(t_h, x_h, u^h(t_h, x_h), [u^h]_{(t_h, x_h)}) = 0 & \text{for } (t_h, x_h) \in (\Pi^{\Delta t} \times \mathcal{G}^{\Delta x}) \cap ((0, T) \times \Lambda), \\ u^h(T, x_h) = \psi(x_h), & \text{for } (t_h, x_h) \in (\Pi^{\Delta t} \times \mathcal{G}^{\Delta x}) \cap \{t_h = T\}, \end{cases}$$

where $\Pi^{\Delta t}$ is a time grid, $\mathcal{G}^{\Delta x}$ is a spatial grid, $h = (\Delta t, \Delta x)$ is the step of the grid and $[u^h]_{(t_h, x_h)}$ are all the values of u^h on $\mathcal{G}^{\Delta x}$ at other points than (t_h, x_h) on the grid. We show that under the usual hypotheses of monotonicity, stability and consistency, the numerical scheme converges locally uniformly to the viscosity solution of (1.2).

The paper is organized as follows: in Section 2, we define the notations and conventions used throughout the paper. We also define the geometry of the problem, the dynamics of the HJB equation and we state the main results. Section 3 is devoted to the invariance principles, a nonsmooth analysis point of view of the HJB equation. In Section 4, we first define the optimal control problem associated to the HJB equation, we introduce the value function and we prove that the super-optimality and sub-optimality properties of the value function are equivalent to it being a viscosity super-solution and sub-solution respectively. Then we prove the strong comparison result. Section 5 is devoted to the proofs of the stability results. Finally, we prove in section 6 a general convergence result for monotone numerical schemes.

2 Main results.

2.1 Notations. Throughout the paper, we denote by \mathbb{R}^N the Euclidean space where the stratification is defined, \mathbb{B} the unit ball of center 0 of \mathbb{R}^N and $\mathbb{B}(x, r) = x + r\mathbb{B}$. For any set $S \subset \mathbb{R}^N$, we denote \bar{S} , ∂S its closure and topological boundary. We denote by $\text{co}(S)$ the convex hull of S and by \mathcal{L} , the Lebesgue measure on \mathbb{R} .

The distance function associated to S is $d_S(x) = \inf\{|x - y| : y \in S\}$ and the set of solutions where the infimum is attained is called the projection of x on S and denoted by $\text{proj}_S(x)$ (note that it might be empty). The Bouligand tangent cone of S at x , denoted $\mathcal{T}_S(x)$ is defined the following way:

$$\mathcal{T}_S(x) = \left\{ v \in \mathbb{R}^N : \liminf_{t \rightarrow 0^+} \frac{d_S(x + tv)}{t} = 0 \right\}.$$

If A and B are two sets of \mathbb{R}^N , we define a distance between them by $d(A, B) = \inf\{|a - b| : (a, b) \in A \times B\}$, with the convention $d(\emptyset, \emptyset) = 0$ and $d(\emptyset, B) = +\infty$ if $B \neq \emptyset$. For K_1 and K_2 two compact sets of \mathbb{R}^N , the Hausdorff distance is given by

$$d_{\mathcal{H}}(K_1, K_2) = \max \left\{ \sup_{x \in K_2} d_{K_1}(x), \sup_{x \in K_1} d_{K_2}(x) \right\},$$

with the convention $d_{\mathcal{H}}(\emptyset, \emptyset) = 0$ and $d_{\mathcal{H}}(\emptyset, S) = +\infty$ if $S \neq \emptyset$.

For a given function $f : \mathbb{R}^N \rightarrow \mathbb{R}$, $\text{epi}(f)$ and $\text{hyp}(f)$ denote respectively its epigraph and hypograph:

$$\text{epi}(f) = \{(x, r) \in \mathbb{R}^N \times \mathbb{R} : f(x) \leq r\}, \quad \text{hyp}(f) = \{(x, r) \in \mathbb{R}^N \times \mathbb{R} : f(x) \geq r\}.$$

If Γ is a set-valued map, then $\text{dom}(\Gamma)$ is the set of points x such that $\Gamma(x) \neq \emptyset$.

Let \mathcal{M} be a C^2 embedded manifold in \mathbb{R}^N and let $\Gamma : \mathcal{M} \rightsquigarrow \mathbb{R}^N$ be a set-valued map. For $T > 0$, we define the differential inclusion associated to Γ , with the initial condition $(t, x) \in (0, T) \times \mathbb{R}^N$, by

$$(DI)_\Gamma(t, x) = \begin{cases} \dot{y}(s) \in \Gamma(y(s)) & \text{a.e. } s \in [t, T] \\ y(t) = x. \end{cases}$$

The set of solutions of $(DI)_\Gamma(t, x)$ might be empty. Finally the abbreviations ‘u.s.c.’, ‘l.s.c.’, ‘HJB’ and ‘w.r.t’ respectively stand for: ‘upper semicontinuous’, ‘lower semicontinuous’, ‘Hamilton Jacobi Bellman’ and ‘with respect to’.

2.2 Stratification. Let $N, n \geq 1$ be two integers. Let $\Omega_i, i = 1, \dots, n$ be pairwise disjoint, connected open sets of \mathbb{R}^N . We suppose that $\mathbb{R}^N = \cup_{i=1}^n \bar{\Omega}_i$ and we denote by $\Lambda := \mathbb{R}^N \setminus \cup_{i=1}^n \Omega_i$ the interfaces. It is more convenient in this setting to see the Ω_i as C^2 submanifolds of \mathbb{R}^N and we adopt the following notation throughout the paper:

$$\mathcal{M}_i := \Omega_i, \quad i = 1, \dots, n.$$

Furthermore, we suppose that Λ is equal to a union of l , pairwise disjoint, C^2 embedded submanifolds $\mathcal{M}_{n+1}, \dots, \mathcal{M}_{n+l}$ of lower dimension than N and with empty boundary, so that we have

$$\mathbb{R}^N = \bigcup_{i=1}^n \bar{\mathcal{M}}_i = \left(\bigcup_{i=1}^n \mathcal{M}_i \right) \cup \Lambda = \bigcup_{i=1}^{n+l} \mathcal{M}_i.$$

Finally, we suppose that each $\bar{\mathcal{M}}_i, i = 1, \dots, n+l$, is proximally smooth and relatively wedged. All these assumptions on the stratification are summarized as following:

$$(H_1) \begin{cases} (i) & \text{Each } \mathcal{M}_i \text{ is a } C^2 \text{ embedded submanifold, with empty boundary,} \\ (ii) & \dim(\mathcal{M}_1) = \dots = \dim(\mathcal{M}_n) = N, \quad \text{and} \quad \dim(\mathcal{M}_{n+1}), \dots, \dim(\mathcal{M}_{n+l}) < N, \\ (iii) & \mathbb{R}^N = \cup_{i=1}^n \bar{\mathcal{M}}_i = \bigcup_{i=1}^{n+l} \mathcal{M}_i, \\ (iv) & \forall i, j = 1, \dots, n+l, \quad \mathcal{M}_i \cap \mathcal{M}_j = \emptyset, \text{ if } i \neq j, \\ (v) & \text{if } \mathcal{M}_i \cap \bar{\mathcal{M}}_j \neq \emptyset, \text{ then } \mathcal{M}_i \subset \bar{\mathcal{M}}_j \text{ or } \mathcal{M}_j \subset \bar{\mathcal{M}}_i, \\ (vi) & \text{each } \bar{\mathcal{M}}_i \text{ is proximally smooth and relatively wedged.} \end{cases}$$

We call $\bigcup_{i=1}^n \mathcal{M}_i$ the regular part of the stratification and $\Lambda := \bigcup_{i=1}^l \mathcal{M}_{n+i}$ the singular part or the interfaces.

Comments on the hypothesis (H_1) . Hypotheses $(H_1)-(i)$ to $(H_1)-(v)$ are standard for a stratification of \mathbb{R}^N . As for $(H_1)-(vi)$, a closed set $X \subset \mathbb{R}^N$ is said to be proximally smooth if its distance function $d_X(\cdot)$ is differentiable on an open tube around X [14]. Relative wedgeness hypothesis was introduced in [7] for C^2 submanifolds of \mathbb{R}^N such that their closure is proximally smooth. Roughly speaking, relative wedgeness of $\bar{\mathcal{M}}_i$, with $i \in \{1, \dots, n+l\}$, means that the dimension of the Bouligand tangent cone at every point of $\bar{\mathcal{M}}_i$ is equal to the dimension of the manifold \mathcal{M}_i [7]. The precise definition of this property is presented in Appendix A.

EXAMPLE 1. Figure 2.1 shows an example of the stratified setting, where $N = 1, n = 2, l = 1$ and the domains

$$\mathcal{M}_1 = (0, +\infty)e_1, \quad \mathcal{M}_2 = (0, +\infty)e_2, \quad \mathcal{M}_3 = \{0\}.$$



FIG. 2.1. Example of a stratification of \mathbb{R} .

We set for any $x \in \mathbb{R}^N$, the index multifunction

$$I(x) := \{i \in \{1, \dots, n+l\} : x \in \bar{\mathcal{M}}_i\}.$$

Remark 2.1. It is clear from the definition of the stratification that for $x \in \mathbb{R}^N$ fixed, and $y \in \mathbb{R}^N$ close enough to x , we have $I(y) \subseteq I(x)$.

2.3 Setting of the problem. We begin by defining the dynamics for the Hamiltonians presented in the introduction. On each \mathcal{M}_i with $i = 1, \dots, n$, we are given a set-valued map $F_i : \mathcal{M}_i \rightsquigarrow \mathbb{R}^N$ that satisfies the standard hypotheses

$$(SH) \left\{ \begin{array}{l} (i) \quad x \rightsquigarrow F_i(x) \text{ has non empty convex and compact images,} \\ (ii) \quad \exists \lambda > 0 \text{ such that } \max\{|p|, p \in F_i(x)\} \leq \lambda(1+|x|), \\ (iii) \quad F_i \text{ is Lipschitz continuous on bounded sets of } \mathcal{M}_i \text{ w.r.t the Hausdorff metric,} \\ \text{i.e. for each } R > 0, \text{ there are constants } K_{1,R}, \dots, K_{n,R} > 0 \text{ such that} \\ d_{\mathcal{H}}(F_i(x), F_i(y)) \leq K_{i,R}|x - y| \quad \text{if } x, y \in \mathbb{B}(0, R) \cap \mathcal{M}_i, i \in \{1, \dots, n\}. \end{array} \right.$$

We are interested in studying the well-posedness of the following HJB equation.

$$(2.1) \quad \left\{ \begin{array}{l} -\partial_t u(t, x) + \sup_{\nu \in F_i(x)} \{-\langle \nu, \partial_x u(t, x) \rangle\} = 0 \quad \text{for } (t, x) \in (0, T) \times \mathcal{M}_i, i = 1, \dots, n, \\ u(T, x) = \psi(x), \end{array} \right.$$

where $T > 0$ is the final time and $\psi : \mathbb{R}^N \rightarrow \mathbb{R}$ is the final cost required to satisfy the following assumption

$$(H\psi) : \quad \psi \text{ is locally Lipschitz continuous.}$$

The study of HJB equations is done using a weak notion of solutions, called viscosity solutions. This setting requires the HJB equation to be defined at every point. Hence, we need to find suitable interfaces conditions in order to guarantee the well-posedness of the system. To do so, we aim to define the appropriate dynamics to consider at the interfaces.

Notice first that since the dynamics $F_i, i = 1, \dots, n$ verify hypothesis (SH), then they can be extended to $\overline{\mathcal{M}_i}$ while verifying the same hypothesis (SH). We denote this extension by F_i as well. In order to define the dynamics on the whole space, a classical idea is to consider the Filippov regularization $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ defined by

$$F(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} \bigcup_y \{ \cup_{i \in \{1 \dots n\}} F_i(y) : |x - y| \leq \varepsilon \}.$$

F is upper semicontinuous and satisfies (SH)–(i) and (SH)–(ii). However, the Lipschitz property (SH)–(iii) does not holds for F . By the nature of our problem, the Filippov regularization is equal to

$$F(x) = \text{co} \{ F_i(x) : i \in \{1 \dots n\} \}.$$

For $(x, p) \in \mathbb{R}^N \times \mathbb{R}^N$, we define the Hamiltonian associated to F by

$$H_F(x, p) = \sup_{q \in F(x)} \{-\langle p, q \rangle\}.$$

Since F is only upper semicontinuous, the Hamiltonian $H_F(\cdot, p)$ is also only upper semicontinuous. If $H_F(\cdot, p)$ were to be Lipschitz continuous, we would have defined our HJB equation using the Hamiltonian associated to F and the well-posedness of the HJB system would follow from the classical theory, see [11, 17]. This is generally not the case in a stratified domain. Furthermore, in the absence of stratification, the unique viscosity solution to HJB equation is the so called *value function* associated to the optimal control problem defined with dynamics F . The value function is characterized by this fact. It turns out that in the stratified case, this is no longer the case. Nevertheless, we use F to define a smaller set of dynamics on the singular part (the interfaces).

We define the dynamics $F_{n+i} : \mathcal{M}_{n+i} \rightsquigarrow \mathbb{R}^N$, for $i = 1, \dots, l$ on each interface \mathcal{M}_{n+i} by

$$F_{n+i}(x) = F(x) \cap \mathcal{T}_{\mathcal{M}_{n+i}}(x),$$

where $\mathcal{T}_{\mathcal{M}_{n+i}}(x)$ is the Bouligand tangent cone which coincides with the classical tangent space to \mathcal{M}_{n+i} at x since \mathcal{M}_{n+i} has empty boundary. All what follows is based upon the essential hypothesis that all the interface dynamics are Lipschitz continuous on bounded sets as well.

$$(H_D) \quad \text{for } i = n + 1, \dots, n + l, F_i(\cdot) \text{ is Lipschitz continuous on bounded sets in } \mathcal{M}_i.$$

We point out that since we have the convention $d(\emptyset, S) = +\infty$ if $S \neq \emptyset$, it follows that (H_D) implies that F_{n+i} is either identically the empty set or nonempty on the whole domain \mathcal{M}_{n+i} . Under assumption (H_D) , each $F_i : \mathcal{M}_i \rightsquigarrow \mathbb{R}^N$, ($i = n+1, \dots, n+l$) satisfies (SH) . Thus each F_i can be extended to $\overline{\mathcal{M}}_i$ while verifying the same hypothesis (SH) . We denote this extension by F_i as well.

A sufficient condition for (H_D) to be satisfied is full controllability near Λ . We mean by full controllability the following assumption:

$$(CH) \exists r > 0 : \text{ for all } i = 1, \dots, n, \text{ and } x \in \Lambda \cap \overline{\mathcal{M}}_i : \mathbb{B}(0, r) \subseteq F_i(x).$$

PROPOSITION 2.2. [28, Lemma 2.2.] Assume (CH) . Then, (H_D) holds.

For $x \in \overline{\mathcal{M}}_i$, $i = 1, \dots, n+l$, and $p \in \mathbb{R}^N$, we define the Hamiltonian

$$H_{F_i}(x, p) := \sup_{q \in F_i(x)} \{-\langle p, q \rangle\}.$$

At this point, we are tempted to define the HJB equation on the stratified domain using the dynamics $F_i(\cdot)$ (defined above), the following way:

$$(2.2) \quad \begin{cases} -\partial_t u(t, x) + \max_{i \in I(x)} \{H_{F_i}(x, \partial_x u(t, x))\} = 0 & \text{for } (t, x) \in (0, T) \times \Omega_i, \\ -\partial_t u(t, x) + \max_{i \in I(x)} \{H_{F_i}(x, \partial_x u(t, x))\} = 0 & \text{for } (t, x) \in (0, T) \times \Lambda, \\ u(T, x) = \psi(x). \end{cases}$$

However, it turns out that even these dynamics in equation (2.2) are too big to obtain comparison type results that are known in viscosity theory. Thus, we consider a smaller subset of these dynamics first proposed in [7].

The essential dynamics. We define the notion of *essential dynamics* F^\sharp , introduced in [7] for stratified Euclidean spaces.

DEFINITION 2.3. For any $x \in \mathbb{R}^N$, the set-valued map $F^\sharp : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ is defined by

$$F^\sharp(x) = \bigcup_{i=1}^{n+l} \{F_i(x) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(x) : x \in \overline{\mathcal{M}}_i\}.$$

The associated Hamiltonian is defined as usual. For $(x, p) \in \mathbb{R}^N \times \mathbb{R}^N$, we have

$$H_{F^\sharp}(x, p) = \sup_{q \in F^\sharp(x)} \{-\langle p, q \rangle\}.$$

We also denote $F_i^\sharp(x) = F_i(x) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(x)$ for all $x \in \overline{\mathcal{M}}_i$, with $i = 1, \dots, n+l$. Notice that if $x \in \mathcal{M}_i$, we have $F_i^\sharp(x) = F_i(x)$, for $i = 1, \dots, n+l$.

So far, two dynamics have been defined on each domain: $F_i(\cdot)$ and $F_i^\sharp(\cdot)$. The former is known to be used when partial differential equations techniques are used to prove existence and uniqueness, see [25, 24, 1, 5]. The latter is used more in the context of optimal control interpretation and was initially introduced in [7]. Examples of such a setting can be found in [28, 18, 7]. We will show that the essential dynamics are the ones that characterize sub-solutions without assuming continuity at the interfaces.

EXAMPLE 2. Let's go back to Example 2.1. Let $c_i \geq 0$ with $i = 1, 2$ be real positive constants. We define the following dynamics on each branch

$$F_i(x) = [-c_i, c_i], \quad i = 1, 2.$$

The resulting HJB system is the **Eikonal equation** on the stratification 2.1. The dynamics at the interface \mathcal{M}_3 and the essential dynamics are respectively equal to

$$F_3(\cdot) \equiv \{0\} \quad , \quad F^\sharp(x) = \begin{cases} [-c_i, c_i] & x \in \mathcal{M}_i \quad i = 1, 2, \\ [-c_2, c_1] & x = 0. \end{cases}$$

Let $T > 0$ be a given time horizon, and consider the following HJB associated to the dynamics F_i^\sharp

$$(2.3) \quad \begin{cases} -\partial_t u(t, x) + \max_{i \in I(x)} \{H_{F_i^\sharp}(x, \partial_x u(t, x))\} = 0 & \text{for } (t, x) \in (0, T) \times \mathbb{R}^N, \\ u(T, x) = \psi(x), \end{cases}$$

where $\psi : \mathbb{R}^N \rightarrow \mathbb{R}$ is the final cost and satisfies $(H\psi)$.

Notice that in the HJB equation (2.3), if x belongs to the regular part of the stratification (i.e. $x \in \bigcup_{i=1}^n \mathcal{M}_i$), then the HJB equation (2.3) is the same as the HJB equation (2.1). More precisely, when $x \in \mathcal{M}_i$, $i = 1, \dots, n$, we have

$$F_i^\sharp(x) = F_i(x) \quad \text{and} \quad I(x) = \{i\}.$$

So equation (2.3) has the following form

$$(2.4) \quad \begin{cases} -\partial_t u(t, x) + H_{F_i}(x, \partial_x u(t, x)) = 0 & \text{for } (t, x) \in (0, T) \times \mathcal{M}_i, \quad i = 1, \dots, n, \\ -\partial_t u(t, x) + \max_{i \in I(x)} \{H_{F_i^\sharp}(x, \partial_x u(t, x))\} = 0 & \text{for } (t, x) \in (0, T) \times \mathcal{M}_i, \quad i = n+1, \dots, n+l, \\ u(T, x) = \psi(x). \end{cases}$$

Given the singular nature of the stratification, one cannot use the classical notion of viscosity solutions. We are going to define a new one that will turn out to be appropriate for obtaining a strong comparison result.

DEFINITION 2.4. (*Viscosity super-solution*) Let $u : (0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a l.s.c function. We say that u is a **super-solution** of (2.3) at $(t, x) \in (0, T) \times \mathbb{R}^N$ if and only if there exists $i \in I(x)$ such that for all $\phi \in C^1((0, T) \times \mathbb{R}^N)$, $u - \phi$ attains a local minimum in $(0, T) \times \overline{\mathcal{M}}_i$ at (t, x) , we have

$$-\partial_t \phi + H_{F_i^\sharp}(x, \partial_x \phi) \geq 0.$$

DEFINITION 2.5. (*Viscosity sub-solution*) Let $u : (0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a u.s.c function. We say that u is a **sub-solution** of (2.3) at $(t, x) \in (0, T) \times \mathbb{R}^N$ if and only if for all $i \in I(x)$, for all $\phi \in C^1((0, T) \times \mathbb{R}^N)$, $u - \phi$ attains a local maximum in $(0, T) \times \overline{\mathcal{M}}_i$ at (t, x) , we have

$$-\partial_t \phi + H_{F_i^\sharp}(x, \partial_x \phi) \leq 0.$$

DEFINITION 2.6. (*Viscosity solution*) u is a viscosity solution of (2.3) if and only if it is both a super-solution and a sub-solution and satisfies the final condition $u(T, \cdot) = \psi(\cdot)$.

The above definitions of viscosity super and sub solutions can be rewritten using the viscosity sub-gradient and super-gradient (also known as the semijets [15, Page 10] or Dini sub/super gradient [11, Definition 11.18]).

DEFINITION 2.7. (*Viscosity sub/super gradient*)

- Let $u : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ be a l.s.c function (with some $d \geq 1$). The viscosity sub-gradient (or subjet) at a point $x \in \text{dom}(u)$ is defined the following way,

$$D^- u(x) := \left\{ p \in \mathbb{R}^d : \liminf_{y \rightarrow x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \geq 0 \right\}.$$

- Similarly, for an u.s.c function $u : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{-\infty\}$, the viscosity super-gradient (or superjet) at a point $x \in \text{dom}(u)$ is defined the following way,

$$D^+ u(x) := -D^-(-u)(x) = \left\{ p \in \mathbb{R}^d : \limsup_{y \rightarrow x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\}.$$

Remark 2.8. • Let $u : (0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a l.s.c function. We say that u is a **super-solution** of (2.3) at $(t, x) \in (0, T) \times \mathbb{R}^N$ if and only if there exists $i \in I(x)$, such that

$$-\theta + H_{F_i^\sharp}(x, \xi_i) \geq 0 \quad \forall (\theta, \xi_i) \in D^- u_i(t, x),$$

with $u_i \equiv u$ on $\overline{\mathcal{M}}_i$ and $u_i \equiv +\infty$ elsewhere.

- Let $u : (0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ be an u.s.c function. u is a **sub-solution** of (2.3) at $(t, x) \in (0, T) \times \mathbb{R}^N$ if and only if for all $i \in I(x)$, we have

$$-\theta + H_{F_i^\#}(x, \xi) \leq 0 \quad \forall (\theta, \xi) \in D^+ u_i(t, x),$$

with $u_i \equiv u$ on $\overline{\mathcal{M}}_i$ and $u_i \equiv -\infty$ elsewhere.

Indeed, if for example $u : (0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a l.s.c function, we set: $u_i \equiv u$ on $\overline{\mathcal{M}}_i$ and $u_i \equiv +\infty$ elsewhere, for some $i = 1, \dots, n + l$. Then, for $(t, x) \in (0, T) \times \overline{\mathcal{M}}_i$, we have

$$(\theta, \xi_i) \in D^- u_i(t, x) \iff \exists \phi^{(i)} \in C^1(\mathbb{R}^N), \text{ such that } u_i - \phi^{(i)} \text{ attains a local minimum at } (t, x).$$

Since $u_i - \phi^{(i)} \equiv -\infty$ whenever $x \notin \overline{\mathcal{M}}_i$, we get that $\phi^{(i)}$ satisfies the requirements of Definition 2.4. Conversely, if there exists such function ϕ in the sense of Definition 2.4, then $u_i - \phi$ attains a local maximum in \mathbb{R}^N at (t, x) . The exact same reasoning holds for sub-solutions.

Next we state the main results. We will show that equation (2.3) has a unique viscosity solution (following Definition 2.6).

2.4 Statement of the main results.

THEOREM 2.9. *Assume (H_1) , (SH) , $(H\psi)$ and (CH) . Then the HJB equation (2.3) has a unique locally Lipschitz viscosity solution in the sense of Definition 2.6.*

THEOREM 2.10. *(Strong comparison principle). Assume (H_1) , (SH) and (H_D) . Let $u_1, u_2 : (0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ be respectively a l.s.c super-solution and an u.s.c sub-solution in the sense of Definition 2.6 with $u_2(T, \cdot) \leq u_1(T, \cdot)$. Then*

$$u_2(t, x) \leq u_1(t, x) \quad \forall (t, x) \in (0, T] \times \mathbb{R}^N.$$

It is worth-noticing that, unlike the previous literature on the subject [27, 28, 18] or [5, 4, 3], the strong comparison principle stated in the above theorem does not require the sub-solution to be continuous nor to have any particular behavior on the interface. The proof of this result will clearly show the importance of the use of essential dynamics with the notion of viscosity as it is defined in Definitions 2.4-2.5 (and more precisely the choice of the test functions in those definitions).

The proofs of Theorems 2.9, 2.10 are given in Section 4. The proofs will heavily rely on invariance theorems stated in Section 3 and proven in Appendix B. Furthermore, we will establish stability results of the super-solution and sub-solution in presence of perturbations of the Hamiltonian in Section 5. Section 6 is devoted to stating and proving a general convergence result of monotone numerical schemes. The numerical scheme has the following form in each $\overline{\mathcal{M}}_i$, $i = 1, \dots, n + l$,

$$S_i^h(t_h, x_h, u^h(t_h, x_h), [u^h]_{(t_h, x_h)}) = 0 \quad \text{for } (t_h, x_h) \in (\Pi^{\Delta t} \times \mathcal{G}_i^{\Delta x}),$$

where $\Pi^{\Delta t}$ is time grid, $\mathcal{G}_i^{\Delta x}$ is a spatial grid of $\overline{\mathcal{M}}_i$ and $h = (\Delta t, \Delta x)$ is the step of the grid. We show that under the usual hypotheses of monotonicity, stability and consistency, the numerical scheme converges. This result generalizes the famous convergence theorem by Barles and Souganidis [6].

3 Invariance Principles. The following results are known as weak and strong invariance properties. They are known in the classical case, when the dynamics F is Lipschitz continuous, see [11, Chapter 11]. For the stratified case, the first attempt to prove these results was given in [7], along with the essential Hamiltonian. Although their intuition was correct, the proximal normal cone used (or equivalently the viscosity definition used) is the same as the one in the classical case (see [11, Chapter 11]), which does not take into account the geometry of the problem. We start by recalling the definitions of some nonsmooth analysis tools and the definitions of weak and strong invariance.

DEFINITION 3.1. *(Proximal sub-gradient and super-gradient).*

- Let $u : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ be a l.s.c function. We say that ζ is proximal sub-gradient at a point $x \in \text{dom}(u)$ for some $\sigma = \sigma(x, \zeta)$ and some neighborhood $V = V(x, \zeta)$ of x if we have

$$u(y) - u(x) + \sigma|y - x|^2 \geq \langle \zeta, y - x \rangle, \quad \forall y \in V.$$

The collection of such ζ form the proximal sub-gradient. It is denoted $\partial_p u(x)$.

- Similarly, Let $u : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{-\infty\}$ be an u.s.c function. We say that ζ is proximal super-gradient at a point $x \in \text{dom}(u)$ for some $\sigma = \sigma(x, \zeta)$ and some neighborhood $V = V(x, \zeta)$ of x if we have

$$u(y) - u(x) + \sigma|y - x|^2 \leq \langle \zeta, y - x \rangle, \quad \forall y \in V.$$

The collection of such ζ forms the proximal super-gradient. It is denoted $\partial^p u(x)$. We also have the property $\partial^p u(x) = -\partial_p(-u)(x)$.

DEFINITION 3.2. (Proximal normal cone).

Let $S \subseteq \mathbb{R}^N$ be a closed set and $x \in S$. A vector ζ is a proximal normal to the closed set S at the point x if there exists $\sigma > 0$ such that $\langle \zeta, y - x \rangle \leq \frac{|\zeta|}{2\sigma}|y - x|^2 \quad \forall y \in S$. The set of all proximal normal vectors at x is denoted by $N_S^p(x)$.

DEFINITION 3.3. (Weak invariance).

Let $\Gamma : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ be a set-valued map. Let $S \subseteq \mathbb{R}^N$ be a closed set. We say that (S, Γ) is weakly invariant provided that for $x \in S$, $t \in [0, T]$ there exists $y(\cdot)$ a solution of $(DI)_\Gamma(t, x)$ such that $y(\tau) \in S$ for all τ in $[t, T]$.

DEFINITION 3.4. (Strong invariance).

Let $\Gamma : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ be a set-valued map. Let $S \subseteq \mathbb{R}^N$ be a closed set. We say that (S, Γ) is strongly invariant provided that for $x \in S$, $t \in [0, T]$ and every $y(\cdot)$ a solution of $(DI)_\Gamma(t, x)$ we have $y(\tau) \in S$ for all τ in $[t, T]$.

THEOREM 3.5. Assume (H_1) , (SH) and (H_D) . Let S be a closed set of \mathbb{R}^N . We denote by $S_i := \overline{\mathcal{M}_i} \cap S$. The following assertions are equivalent:

- (i) (S, F) is weakly invariant,
- (ii) $\forall x \in S, \exists i \in I(x) : \forall \eta_i \in N_{S_i}^p(x), \quad H_{F_i}(x, \eta_i) \geq 0,$
- (iii) $\forall x \in S, \exists i \in I(x) : \forall \eta_i \in N_{S_i}^p(x), \quad H_{F_i^*}(x, \eta_i) \geq 0.$

THEOREM 3.6. Assume (H_1) , (SH) and (H_D) . Let S be a closed set of \mathbb{R}^N . We denote by $S_i := \overline{\mathcal{M}_i} \cap S$. The following assertions are equivalent:

- (i) (S, F) is strongly invariant,
- (ii) $\forall x \in S, \forall i \in I(x), \quad \forall \eta_i \in N_{S_i}^p(x), \quad H_{F_i^*}(x, -\eta_i) \leq 0,$

The complete proof of Theorems 3.5 and 3.6 are given in Appendix B.

In [7], Barnard and Wolenski tried to establish similar invariance principles in the stratified case using the essential Hamiltonian. In particular, for the strong invariance principle, if we assume that (H_1) , (SH) and (H_D) hold, [7, Theorem 5.1] states that (S, F) is strongly invariant for some closed set $S \subseteq \mathbb{R}^N$ if and only if

$$(3.1) \quad \forall x \in S, \forall \xi \in N_S^p, \quad H_{F^*}(x, -\xi) \leq 0.$$

However, the sufficient condition fails to be true in general. Here is a counterexample:

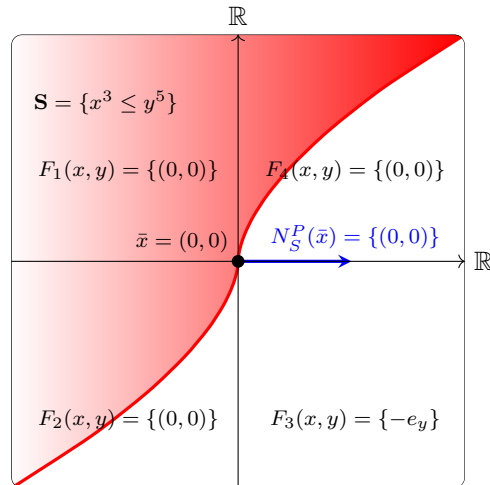


FIG. 3.1. Counterexample with a stratification in \mathbb{R}^N , with $N = 2$, $n = 4$ and $l = 5$.

EXAMPLE 3. We are given a stratification as follows

$$\mathcal{M}_1 = \{(x, y) \in \mathbb{R}^2 : x < 0 \ \& \ y > 0\} \quad \mathcal{M}_2 = \{(x, y) \in \mathbb{R}^2 : x < 0 \ \& \ y < 0\},$$

$$\mathcal{M}_3 = \{(x, y) \in \mathbb{R}^2 : x > 0 \ \& \ y < 0\} \quad \mathcal{M}_4 = \{(x, y) \in \mathbb{R}^2 : x > 0 \ \& \ y > 0\},$$

$$\mathcal{M}_5 = (0, +\infty)e_x \quad \mathcal{M}_6 = (-\infty, 0)e_x \quad \mathcal{M}_7 = (0, +\infty)e_y \quad \mathcal{M}_8 = (-\infty, 0)e_y \quad \mathcal{M}_9 = \{0\}.$$

Take S to be the closed set

$$S = \{(x, y) \in \mathbb{R}^2 : x^3 \leq y^5\},$$

and consider the following dynamics

$$F_1(x, y) = F_2(x, y) = F_4(x, y) = F_5(x, y) = F_6(x, y) = F_7(x, y) = F_9(x, y) = \{(0, 0)\},$$

$$F_3(x, y) = F_8(x, y) = \{-e_y\}.$$

Clearly, (S, F) is not strongly invariant since $F(0, y) = \text{co}\{0, -e_y\}$ if $y \leq 0$ and the trajectory

$$\tilde{Z}(s) = (0, t - s) \in S_{(t, T)}(0, 0)$$

is a trajectory of F that starts at $\bar{x} = (0, 0) \in S$, but $\tilde{Z}(\cdot) \not\subset S$. On the other hand, since the proximal normal cone to S at $\bar{x} = (0, 0)$ is equal to $N_S^p(\bar{x}) = \{(0, 0)\}$, the Hamiltonian inequality (3.1) is therefore verified at $\bar{x} = (0, 0)$. For the remaining points of S , the Hamiltonian inequality (3.1) is trivially verified since the dynamics F is reduced to $\{(0, 0)\}$. In conclusion, we have shown in this example that the Hamiltonian inequality

$$\forall x \in S, \forall \xi \in N_S^p, \quad H_{F^\sharp}(x, -\xi) \leq 0,$$

is verified for all the points in S . However, (S, F) is not strongly invariant.

EXAMPLE 4. Let's go back to Figure 3.1. We have seen that (S, F) is not strongly invariant. We can also verify it using the new characterization of strong invariance. Indeed we have

$$S_8 := S \cap \overline{\mathcal{M}_8} = S \cap (-\infty, 0]e_y = \{(0, 0)\}.$$

Therefore, we have that the proximal normal to S_i at $\bar{x} := (0, 0)$ is equal to \mathbb{R}^2 . Moreover we have $F_8^\sharp(\bar{x}) = \text{co}\{0, -e_y\} = [0, -e_y]$. Thus

$$H_{F_8^\sharp}(\bar{x}, -N_{S_8}^p(\bar{x})) \geq \langle -e_y, -e_y \rangle = 1 > 0.$$

This shows that the sufficient condition in Theorem 3.6 is not satisfied in this example.

4 Proof of Theorems 2.9 and 2.10.

4.1 Optimal control problem and the value function. We denote by $S_{(t, T)}(x)$ the set of solutions of the differential inclusion associated to F :

$$S_{(t, T)}(x) := \left\{ y(\cdot) \in W^{1,1}([t, T]; \mathbb{R}^N) : \left\{ \begin{array}{l} \dot{y}(s) \in F(y(s)), \ s \in [t, T], \ \text{a.e.} \\ y(t) = x. \end{array} \right. \right\}.$$

We consider the optimal control problem defined for $(t, x) \in [0, T] \times \mathbb{R}^N$ by

$$\left\{ \begin{array}{l} \inf \quad \psi(y(T)) \\ \text{such that} \quad \dot{y}(s) \in F(y(s)), \ s \in [t, T] \\ y(t) = x, \end{array} \right.$$

where the infimum is taken over all trajectories $y(\cdot) \in S_{(t, T)}(x)$. The next proposition shows that F -trajectories are the same as F^\sharp -trajectories.

PROPOSITION 4.1. ([28, Proposition 3.4] and [7, Proposition 2.1].)

Let $(t, x) \in [0, T] \times \mathbb{R}^N$ and $y(\cdot) \in S_{(t, T)}(x)$. The following statements are equivalent:

- (i) $y(\cdot)$ is a solution of $(DI)_F(t, x)$,
- (ii) $y(\cdot)$ is a solution of $(DI)_{F^\sharp}(t, x)$,
- (iii) $y(\cdot)$ is a solution of $(DI)_{F_i}(t, x)$ whenever $y(\cdot) \in \mathcal{M}_i$.

Proof. (iii) \implies (ii) \implies (i) is obvious since $F_i(\cdot) \subseteq F^\sharp(\cdot) \subseteq F(\cdot)$. Now, suppose (i) and let $y(\cdot) \in S_{(t, T)}(x)$. For $k \in \{1, \dots, n+l\}$, let $J_k := \{s \in [t, T] : y(s) \in \mathcal{M}_k\}$. Without loss of generality, we suppose $\mathcal{L}(J_k) > 0$ (otherwise there is nothing to prove). We set

$$\tilde{J}_k := \{s \in J_k : \dot{y}(s) \text{ exists in } F(y(s)) \text{ and } s \text{ is a Lebesgue point of } J_k\}.$$

Clearly $\mathcal{L}(J_k) = \mathcal{L}(\tilde{J}_k)$ (\mathcal{L} stands for the Lebesgue measure). Let $s \in \tilde{J}_k$. So, there exists a sequence $(s_n)_n \subset J_k$ such that $s_n \rightarrow s$ and $s_n \neq s$ for all n . Since $y(s_n) \in \mathcal{M}_k$, we have

$$\dot{y}(s) = \lim_{s_n \rightarrow s} \frac{y(s_n) - y(s)}{s_n - s} \in \mathcal{T}_{\mathcal{M}_k}(y(s)),$$

which is the required result. \square

The above proposition shows in particular that the optimal control problem could be defined using F or F^\sharp or F_i^\sharp , $i = 1, \dots, n+l$. The latter dynamics are more suitable to our setting since it gives us the link with the HJB equation (2.3) we want to solve. Furthermore, thanks to hypothesis $(H_1) - (vi)$, the next proposition shows that they are l.s.c set-valued maps.

PROPOSITION 4.2. Suppose (H_1) , (H_D) hold. Then for all $i = 1, \dots, n+l$, $x \rightsquigarrow F_i^\sharp(x)$ is l.s.c on its domain $\overline{\mathcal{M}_i}$.

Proof. Let $i \in \{1, \dots, n+l\}$. We already have by (H_D) that F_i^\sharp is Lipschitz continuous on \mathcal{M}_i and therefore l.s.c on \mathcal{M}_i . Next, since $(H_1) - (vi)$ holds, we have that $\overline{\mathcal{M}_i}$ is a regular set (this means that the Bouligand tangent cone is equal to Clarke's tangent cone at every point of the set). Hence by [13, corollary 3.6.13], we have that $x \rightsquigarrow \mathcal{T}_{\overline{\mathcal{M}_i}}(x)$ is l.s.c. Let $x \in \partial \overline{\mathcal{M}_i}$. If $F_i^\sharp(x) = \emptyset$, then there is nothing to prove. Otherwise, let $p \in F_i^\sharp(x) = F_i(x) \cap \mathcal{T}_{\overline{\mathcal{M}_i}}(x)$. Let $(x_k)_k \subset \text{dom}(F_i^\sharp)$ such that $x_k \rightarrow x$. Then, since $F_i(\cdot)$ and $\mathcal{T}_{\overline{\mathcal{M}_i}}(\cdot)$ are l.s.c, there exist sequences $(p_k^1)_k$ and $(p_k^2)_k$ such that

$$p_k^1 \in F_i(x_k), \quad p_k^1 \rightarrow p, \quad p_k^2 \in \mathcal{T}_{\overline{\mathcal{M}_i}}(x_k), \quad p_k^2 \rightarrow p.$$

Let $(p_k)_k$ be a sequence such that for every k , $p_k \in F_i^\sharp(x_k)$ and satisfies

$$d([p_k^1, p_k^2], p_k) = d([p_k^1, p_k^2], F_i^\sharp(x_k)),$$

where the notation $[p_k^1, p_k^2]$ stands for the convex set $\text{co}(\{p_k^1, p_k^2\})$. Since $(p_k)_k$ is bounded, let $\nu \in \mathbb{R}^N$ be an accumulation point for $(p_k)_k$. So a subsequence of $(p_k)_k$, not relabeled here, converges to ν . Therefore, we get

$$|p - \nu| = \lim_{k \rightarrow \infty} d([p_k^1, p_k^2], F_i^\sharp(x_k)) \leq \lim_{k \rightarrow \infty} d(p_k^1, F_i^\sharp(x_k)) = 0.$$

Thus we have $p_k \in F_i^\sharp(x_k)$ and $p_k \rightarrow p$. This ends the proof. \square

Remark 4.3. Proposition 4.2 shows that assumption $(H_1) - (vi)$ is necessary. The extra regularity obtained for the essential dynamics, by assuming $(H_1) - (vi)$, is needed for the proof of Theorem 3.6.

Next, we consider the value function associated to the optimal control problem defined on $(t, x) \in [0, T] \times \mathbb{R}^N$

$$\vartheta(t, x) = \inf \{ \psi(y(T)) , y(\cdot) \in S_{(t, T)}(x) \}$$

Note that ϑ is well defined since, by Filippov's theorem, the set of solutions $S_{(t, T)}(x)$ is nonempty and compact in $C([t, T], \mathbb{R}^N)$.

We now proceed to define some properties of the value function:

DEFINITION 4.4. Let $u : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a function. u is said to enjoy:

- **The super-optimality property** if for all $(t, x) \in [0, T] \times \mathbb{R}^N$, there exists $y(\cdot) \in S_{(t,T)}(x)$ such that $u(t, y(t)) \geq u(s, y(s))$, $\forall s \in [t, T]$.
- **The sub-optimality property** if for all $(t, x) \in [0, T] \times \mathbb{R}^N$, for all $y(\cdot) \in S_{(t,T)}(x)$ we have $u(t, y(t)) \leq u(s, y(s))$, $\forall s \in [t, T]$.

LEMMA 4.5 ([20]). Let $u : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a function.

- If $u(T, x) \geq \psi(x)$ and u has the super-optimality property, then: $\vartheta(t, x) \leq u(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}^N$.
- If $u(T, x) \leq \psi(x)$ and u has the sub-optimality property, then: $\vartheta(t, x) \geq u(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}^N$.

Lemma 4.5 shows that the value function is the unique function that verifies both the super-optimality and the sub-optimality property and it solves the following functional equality known as *the dynamic programming principle*:

$$\vartheta(t, x) = \inf \{ \vartheta(s, y(s)) : y(\cdot) \in S_{(t,T)}(x) \}, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^N.$$

The next proposition states that the controllability hypothesis (CH) is a sufficient condition to ensure that the value function is locally Lipschitz continuous.

PROPOSITION 4.6. Suppose (H_1) , (CH) and $(H\psi)$ hold. Then, $\vartheta : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ is locally Lipschitz continuous.

Proof. From the controllability assumption (CH), there exists a neighborhood of Λ (the interfaces), denoted $V := \Lambda + \varepsilon\mathbb{B}$, and there exists $r > 0$, such that for all $x \in V$, we have $r\mathbb{B} \subset F(x)$.

First, we prove that $\vartheta(t, \cdot)$ is locally Lipschitz. Let $x, z \in \mathbb{R}^N$. First suppose that $x, z \in V$ close enough that $\frac{|x-z|}{r} \leq T-t$. Let M be the local supremum bound of F . Without loss of generality, we suppose $\vartheta(t, x) \geq \vartheta(t, z)$.

Let $y_{t,z}(\cdot) \in S_{(t,T)}(z)$ such that $\vartheta(t, z) = \psi(y_{t,z}(T))$. Set $h = \frac{|x-z|}{r}$ and $\xi(s) = x + r \frac{z-x}{|x-z|}(s-t)$ for $s \in [t, t+h]$. We define :

$$\tilde{y}(s) = \begin{cases} \xi(s) & \text{for } s \in [t, t+h] \\ y_{t,z}(s-h) & \text{for } s \in [t+h, T] \end{cases}$$

It is easy to see that $\tilde{y}(\cdot)$ is an F -trajectory. So we get

$$\begin{aligned} \vartheta(t, x) - \vartheta(t, z) &\leq \psi(\tilde{y}(T)) - \psi(y_{t,z}(T)) \leq L|\tilde{y}(T) - y_{t,z}(T)| = L|y_{t,z}(T-h) - y_{t,z}(T)| \\ &\leq LMh = L \frac{M}{r} |x-z|. \end{aligned}$$

Suppose now for example $z \notin V$ (we can do the same reasoning on x instead). Then by taking them close enough to each other, there exists $i \in \{1, \dots, n\}$ such that $x, z \in \mathcal{M}_i$. Let $y_{t,z}(\cdot) \in S_{(t,T)}(z)$ such that $\vartheta(t, z) = \psi(y_{t,z}(T))$. Suppose $y_{t,z}(\cdot)$ crosses the boundary of \mathcal{M}_i . Let $t_0 \in [t, T]$ be such that

$$y_{t,z}([t, t_0]) \subset \mathcal{M}_i, \text{ and } y_{t,z}(t_0) \in V.$$

Let $y_{t,x}(\cdot) \in S_{(t,T)}(x)$ such that $y_{t,x}([t, t_0]) \subset \mathcal{M}_i$ and $|y_{t,x}(t_0) - y_{t,z}(t_0)| \leq C|x-z|$. We can also suppose that

$$y_{t,x}(t_0) \in V \cap \mathcal{M}_i, \quad \text{and} \quad \frac{|y_{t,z}(t_0) - y_{t,x}(t_0)|}{r} \leq T - t_0.$$

We can always find such trajectories if we take x and z close enough, since F_i is Lipschitz. See [13, Theorem 4.3.11].

Set $h = \frac{|y_{t,z}(t_0) - y_{t,x}(t_0)|}{r}$ and $\xi(s) = y_{t,x}(t_0) + r \frac{y_{t,z}(t_0) - y_{t,x}(t_0)}{|y_{t,z}(t_0) - y_{t,x}(t_0)|}(s-t_0)$ for $s \in [t_0, t_0+h]$. We define

$$\tilde{y}(s) = \begin{cases} y_{t,x}(s) & \text{for } s \in [t, t_0] \\ \xi(s) & \text{for } s \in [t_0, t_0+h] \\ y_{t,z}(s-h) & \text{for } s \in [t_0+h, T] \end{cases}$$

It is easy to see that $\tilde{y}(\cdot)$ is an F -trajectory. So we get

$$\begin{aligned} \vartheta(t, x) - \vartheta(t, z) &\leq \psi(\tilde{y}(T)) - \psi(y_{t,z}(T)) \leq L|\tilde{y}(T) - y_{t,z}(T)| = L|y_{t,z}(T-h) - y_{t,z}(T)| \\ &\leq LMh \\ &= L\frac{M}{r}|y_{t,z}(t_0) - y_{t,x}(t_0)|, \\ &\leq LC\frac{M}{r}|x - z|. \end{aligned}$$

If the trajectories $y_{t,z}(\cdot)$ and $y_{t,x}(\cdot)$ don't cross the boundary of \mathcal{M}_i , then, they are both F_i -trajectories and the result follows again from the classical case, see [13, theorem 4.3.11]. This finishes the proof that $\vartheta(t, \cdot)$ is locally Lipschitz.

Now we prove that $\vartheta(\cdot, x)$ is locally Lipschitz. Let $x \in \mathbb{R}^N$. Let $t, s \in [0, T]$ such that $t < s$. By the super-optimality property, there exists $y(\cdot) \in S_{(t,T)}(x)$ such that $\vartheta(t, x) = \vartheta(s, y(s))$. Then

$$|\vartheta(t, x) - \vartheta(s, x)| = |\vartheta(s, y(s)) - \vartheta(s, x)| \leq |\vartheta(s, y(s)) - \vartheta(s, y(t))|.$$

Since both $\vartheta(s, \cdot)$ and $y(\cdot)$ are locally Lipschitz, then from the expression above, $\vartheta(\cdot, x)$ is locally Lipschitz. This ends the proof. \square

4.2 The super-optimality and super-solution property. In this section, we characterize functions that are super-solutions of equation (2.3) with super-optimality property. The characterization using F is standard in the literature since it satisfies the usual hypotheses (upper semi-continuity with nonempty, convex and compact images). In our case, we use the fact F and F_i^\sharp , $i = 1, \dots, n+l$, give the same set of trajectories, as shown in Proposition 4.1, to prove this Hamiltonian inequality.

THEOREM 4.7. *Suppose (H_1) and (SH) . Let, $u : (0, T) \times \mathbb{R}^N \rightarrow \mathbb{R}$ be l.s.c. The following assertions are equivalent:*

- (i) u is a super-solution of (2.3),
- (ii) u satisfies the super-optimality principle.

Proof. The fact that (ii) is equivalent to

$$(4.1) \quad -\theta + H_F(x, \xi) \geq 0 \quad \forall (\theta, \xi) \in D^-u(t, x),$$

is well known since $F(\cdot)$ is u.s.c. For more on this, see [28, Proposition 3.5] or [11, Chapter 19]. Moreover, it is obvious from this that (i) \implies (ii) since $H_{F_i^\sharp}(\cdot, \cdot) \leq H_{F_i}(\cdot, \cdot) \leq H_F(\cdot, \cdot)$ and $D^-u(\cdot, \cdot) \subset D^-u_i(\cdot, \cdot)$, for all $i \in [1, n+l]$.

It remains to prove (ii) \implies (i). Let $y(\cdot) : [t, T] \rightarrow \mathbb{R}^N$ be a trajectory solution of $(DI)_F(t, x)$ such that the super-optimality property holds in $y(\cdot)$. We claim the following:

Claim: $\exists j \in I(x)$ such that there exists a sequence $(t_n)_n$, $t_n \downarrow t$ and $x_n := y(t_n) \in \mathcal{M}_j$, so that $\frac{x_n - x}{t_n - t} \rightarrow \nu$ and $\nu \in F_j^\sharp(x)$. Deferring the proof of the claim, let $\phi \in C^1((0, T) \times \mathbb{R}^N)$ such that $u_j - \phi$ attains a local minimum at (t, x) in $\overline{\mathcal{M}}_j$. For n big enough the super-optimality property gives

$$u_j(t, x) - u_j(t_n, x_n) \geq 0.$$

This inequality combined with the fact that $u_j(t_n, x_n) - \phi(t_n, x_n) \geq u_j(t, x) - \phi(t, x)$ lead to

$$\frac{1}{t_n - t}(\phi(t, x) - \phi(t_n, x_n)) \geq 0.$$

By letting n tend to $+\infty$, we obtain $-\partial_t \phi(t, x) - \langle \nu, \partial_x \phi(t, x) \rangle \geq 0$ and then

$$-\partial_t \phi(t, x) + H_{F_j^\sharp}(x, \partial_x \phi(t, x)) \geq 0.$$

This concludes the proof.

Now we turn our attention to the proof of the claim. We distinguish two cases: either there exists $r > 0$ such that $y([t, t+r])$ stays in one domain \mathcal{M}_j for some $j \in \{1, \dots, n\} \cap I(x)$, almost everywhere, or it touches or crosses the interface infinitely many times no matter how we are close to x . We begin with the first case: suppose there exists $r > 0$ such that $y([t, t+r]) \subset \mathcal{M}_j$ for some $j \in \{1, \dots, n\} \cap I(x)$ almost everywhere. So, there exists a sequence $(t_n)_n$, $t_n \downarrow t$ and $x_n := y(t_n) \in \mathcal{M}_j$, so that $\frac{x_n - x}{t_n - t} \rightarrow \nu$.

Notice that $\nu = \lim_{n \rightarrow +\infty} \frac{x_n - x}{t_n - t} \in \overline{\mathcal{T}_{\mathcal{M}_j}}(x)$, since $x_n \in \mathcal{M}_j$. It remains to prove that ν belongs to $F_j(x)$. Denote by κ and M respectively the Lipschitz constant of $F_j(\cdot)$ and the Lipschitz constant of $y(\cdot)$. We have

$$\begin{aligned} \nu &= \lim_{n \rightarrow +\infty} \frac{1}{t_n - t} \int_{[t, t_n]} \dot{y}(s) ds \in \lim_{n \rightarrow +\infty} \left(\frac{1}{t_n - t} \int_{[t, t_n]} \text{proj}_{F_j(x)}(\dot{y}(s)) ds + \frac{\kappa}{t_n - t} \int_{[t, t_n]} |y(s) - x| \mathbb{B} ds \right) \\ &\subseteq \lim_{n \rightarrow +\infty} \left(F_j(x) + \frac{\kappa M}{t_n - t} \left[\int_{[t, t_n]} (s - t) ds \right] \mathbb{B} \right) \\ &\subseteq \lim_{n \rightarrow +\infty} \left(F_j(x) + \kappa M \frac{|t_n - t|}{2} \mathbb{B} \right) = F_j(x). \end{aligned}$$

In conclusion, we get $\nu \in F_j(x) \cap \overline{\mathcal{T}_{\mathcal{M}_j}}(x) = F_j^\sharp(x)$.

Now we get to the second case. Since $y(\cdot)$ touches or crosses the interface infinitely many times no matter how we are close to x , then there exists $j \in \{n+1, \dots, n+l\} \cap I(x)$, a sequence $(t_n)_n$, $t_n \downarrow t$ and $x_n := y(t_n) \in \mathcal{M}_j$, so that $\frac{x_n - x}{t_n - t} \rightarrow \nu$.

Notice that $\nu = \lim_{n \rightarrow +\infty} \frac{x_n - x}{t_n - t} \in \overline{\mathcal{T}_{\mathcal{M}_j}}(x)$, since $x_n \in \mathcal{M}_j$. It remains to prove that ν belongs to $F_j(x)$. For $k = 1, \dots, n+l$, we set

$$J_n^k := \{s \in [t, t+t_n] : y(s) \in \mathcal{M}_k\}, \quad \mu_n^k := \mathcal{L}(J_n^k), \quad \mathbb{K}(x) := \{k : \mu_n^k > 0, \forall n \in \mathbb{N}\}.$$

We obviously have $\mathbb{K}(x) \subset I(x)$. Furthermore, up to a subsequence, there exist $0 \leq \lambda_k \leq 1$ and $p_k \in \mathbb{R}^N$ such that

$$\frac{\mu_n^k}{t_n - t} \rightarrow \lambda_k, \quad \sum_{k \in \mathbb{K}(x)} \lambda_k = 1, \quad \frac{1}{\mu_n^k} \int_{J_n^k} \dot{y}(s) ds \rightarrow p_k, \quad \text{as } n \rightarrow \infty.$$

Denote by κ and M respectively the Lipschitz constant of $F_k(\cdot)$ and the Lipschitz constant of $y(\cdot)$, we get

$$\begin{aligned} p_k &= \lim_{n \rightarrow +\infty} \frac{1}{\mu_n^k} \int_{J_n^k} \dot{y}(s) ds \\ &\in \lim_{n \rightarrow +\infty} \left(\frac{1}{\mu_n^k} \int_{J_n^k} \text{proj}_{F_k(x)}(\dot{y}(s)) ds + \frac{\kappa}{\mu_n^k} \int_{J_n^k} |y(s) - x| \mathbb{B} ds \right) \\ &\subseteq \lim_{n \rightarrow +\infty} \left(F_k(x) + \frac{\kappa M}{\mu_n^k} \left[\int_{J_n^k} (s - t) ds \right] \mathbb{B} \right) \\ &\subseteq \lim_{n \rightarrow +\infty} \left(F_k(x) + \kappa M |t_n - t| \mathbb{B} \right) = F_k(x). \end{aligned}$$

Therefore, we have

$$\nu = \lim_{n \rightarrow +\infty} \frac{1}{t_n - t} \int_{[t, t_n]} \dot{y}(s) ds = \sum_{k \in \mathbb{K}(x)} \lim_{n \rightarrow +\infty} \frac{\mu_n^k}{t_n - t} \left[\frac{1}{\mu_n^k} \int_{J_n^k} \dot{y}(s) ds \right] \subset \text{co} \{F_k(x) : k \in \mathbb{K}(x)\}.$$

Finally, we get $\nu \in \text{co} \{F_k(x) : k \in \mathbb{K}(x)\} \cap \overline{\mathcal{T}_{\mathcal{M}_j}}(x) \subset \text{co} \{F_k(x) : k \in I(x)\} \cap \overline{\mathcal{T}_{\mathcal{M}_j}}(x) = F_j^\sharp(x)$. This ends the proof of the claim. \square

Remark 4.8. By the arguments presented at the beginning of the above proof, it is easy to see that under the same assumptions of Theorem 4.7, the following statements are equivalent:

- (i) u satisfies the super-optimality principle,
- (ii) u is a super-solution of (2.3),
- (iii) u is a super-solution of (4.1),
- (iv) u is a super-solution of (2.2).

Remark 4.9. It was already known from [28], that the super-solution property is equivalent to the super-optimality property if we only use the classical definition of viscosity. The importance of this result lies in the fact that the equivalence is valid even if we take the notion of viscosity stated in Definition 2.4.

4.3 The sub-optimality and sub-solution property. This section aims at establishing the link between the sub-optimality principle and the the sub-solution property of (2.3).

THEOREM 4.10. *Suppose (H_1) , (SH) and (H_D) hold. Let $u : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ be an u.s.c function. The following assertions are equivalent:*

- (i) u satisfies the sub-optimality principle,
- (ii) u is a sub-solution of (2.3).

Proof. We prove (i) \implies (ii) first. Let $i \in \{1, \dots, n+l\}$. Let $(t, x) \in [0, T] \times \overline{\mathcal{M}}_i$. By [28, Lemma 3.9], for every $p \in F_i^\sharp(x)$, there exists a C^1 trajectory $y(\cdot)$ defined on some interval $[t, t + \varepsilon]$, with $\varepsilon > 0$, such that $y(t) = x$, $\dot{y}(t) = p$ and $y(\cdot) \subseteq \overline{\mathcal{M}}_i$.

Let $u_i \equiv u$ on $\overline{\mathcal{M}}_i$ and $u_i \equiv -\infty$ otherwise. Let (θ, ξ) be in $D^+u_i(t, x)$. For any sequence $((t_n, x_n))_n$ such that $(t_n, x_n) \in \text{dom}(u_i)$ and $(t_n, x_n) \rightarrow (t, x)$, we have

$$\limsup_{n \rightarrow \infty} \frac{u(t_n, x_n) - u(t, x) - \theta(t_n - t) - \langle \xi, x_n - x \rangle}{|x_n - x| + |t_n - t|} \leq 0.$$

Setting $x_n = y(t + \frac{\varepsilon}{n})$ and $t_n := t + \frac{\varepsilon}{n}$, we get by sub-optimality of u

$$\frac{-\theta(t_n - t) - \langle \xi, x_n - x \rangle}{|x_n - x| + |t_n - t|} \leq \frac{u(t_n, x_n) - u(t, x) - \theta(t_n - t) - \langle \xi, x_n - x \rangle}{|x_n - x| + |t_n - t|},$$

By letting $n \rightarrow \infty$ we get

$$\frac{-\theta - \langle \xi, p \rangle}{|p| + 1} \leq 0 \implies -\theta - \langle \xi, p \rangle \leq 0.$$

Since p is arbitrary, we get the result by taking the supremum over $F_i^\sharp(x)$.

It remains to prove (ii) \implies (i). We divide this proof into 2 steps.

Step 1. (Augmented dynamics). First, we establish the following claim:

Let $v := -u$ (so v is l.s.c), $M_i := \mathbb{R} \times \mathcal{M}_i \times \mathbb{R}$ for all $i = 1, \dots, n+l$. We denote by $v_i \equiv v$ on M_i and $v_i \equiv +\infty$ otherwise. Let $G_i^\sharp(t, x, z) := \{1\} \times F_i^\sharp(x) \times \{0\}$ for any $(t, x, z) \in M_i$. If we have

- (i) $-\theta + H_{F_i^\sharp}(x, \nu) \leq 0$ for all $(t, x) \in (0, T) \times \overline{\mathcal{M}}_i$, $(\theta, \nu) \in -D^-v_i(t, x)$.

Then it holds:

- (ii) $\sup_{\nu \in G_i^\sharp(t, x, z)} \{ \langle \eta, \nu \rangle \} \leq 0 \quad \forall (t, x, z) \in \text{epi}(v_i), \eta \in N_{\text{epi}(v_i)}^p(t, x, z)$.

Let us first point out the fact that (i) is equivalent to

$$-\theta + H_{F_i^\sharp}(x, \nu) \leq 0 \quad \text{for } (t, x) \in (0, T) \times \overline{\mathcal{M}}_i, (\theta, \nu) \in D^+u_i(t, x),$$

since $D^+u_i(t, x) = -D^-(-u_i)(t, x) = -D^-v_i(t, x)$.

Let $(t, x, z) \in \text{epi}(v_i)$. If $F_i^\sharp(x) = \emptyset$ then the result holds by vacuity. Otherwise, let $(\xi, -\lambda) \in N_{\text{epi}(v_i)}^p(t, x, z)$. So we have $\lambda \geq 0$. Suppose we have $\lambda > 0$, then $z = v_i(t, x)$ and there exists $(\theta, \zeta) \in -\partial_p v_i(t, x)$ such that $\xi = (-\lambda\theta, -\lambda\zeta)$. Hence, by [11, Theorem 11.32], for any $\nu \in G_i^\sharp(t, x, z)$ we have, for some $p \in F_i^\sharp(x)$:

$$\langle (\xi, -\lambda), \nu \rangle = -\lambda(\theta + \langle \zeta, p \rangle) \leq \lambda(-\theta + H_{F_i^\sharp}(x, \zeta)) \leq 0$$

We take the supremum over ν and we get the result. Now, if $\lambda = 0$, then by Rockafellar theorem [11, theorem 11.31], there exist sequences $((t_n, x_n))_n \subseteq [0, T] \times \overline{\mathcal{M}}_i$, $(\xi_n)_n \subseteq \mathbb{R}^{N+1}$ and $(\lambda_n)_n \subseteq (0, \infty)$ such that

$$(t_n, x_n, \lambda_n) \rightarrow (t, x, 0), \quad v(t_n, x_n) \rightarrow z \quad \xi_n \rightarrow \xi, \quad \frac{1}{\lambda_n} \xi_n \in -\partial_p v_i(t_n, x_n).$$

Thus the argument above shows that:

$$\langle (\xi_n, -\lambda_n), \nu_n \rangle \leq 0 \quad \forall \nu_n \in G_i^\sharp(t_n, x_n, u_i(t_n, x_n)), \forall n \in \mathbb{N}.$$

Furthermore, by Proposition 4.2 we have that $G_i^\sharp(\cdot)$ is lower semicontinuous. So there exists a sequence $(\nu_n)_n \rightarrow \nu$ such that $\nu_n \in G_i^\sharp(t_n, x_n, v_i(t_n, x_n))$. By evaluating the last inequality at this sequence and letting $n \rightarrow +\infty$, then taking the supremum over ν , we get the result.

Step 2: We show that

$$\forall i \in \{1, \dots, n+l\}, \text{epi}(v_i) = \text{epi}(v) \cap \overline{M}_i.$$

Let $(t, x, r) \in \text{epi}(v_i)$. So $v_i(t, x) \leq r$. Hence $x \in \overline{M}_i$ and $v(t, x) \leq r$. Thus we get $(t, x, r) \in \text{epi}(v) \cap \overline{M}_i$.

Conversely, if $(t, x, r) \in \text{epi}(v) \cap \overline{M}_i$, then $v(t, x) \leq r$ and $x \in \overline{M}_i$. So $v_i(t, x) = v(t, x)$, whence $v_i(t, x) \leq r$, which finishes the proof of step 2.

We apply Theorem 3.6 to the augmented dynamics $G(\cdot) := \{1\} \times F(\cdot) \times \{0\}$, the stratification $\mathbb{R}^{N+2} = \cup_{i=1}^{n+l} M_i$ and the set $S = \text{epi}(v)$. Thus we get that $(\text{epi}(v), G)$ is strongly invariant. Let $(t, x) \in (0, T) \times \mathbb{R}^N$ and $y(\cdot)$ be a solution of $(DI)_F(t, x)$. So we have that

$$Y(s) = (s, y(s), v(t, y(t))) \quad s \in [t, T],$$

is a solution of the differential inclusion with the augmented dynamics $G(\cdot)$ and initial condition $(t, x, v(t, x)) = (t, y(t), v(t, y(t)))$.

we have $(t, y(t), v(t, y(t))) \in \text{epi}(v)$ and equation (ii) from step 1 is verified. Thus, by Theorem 3.6, we get $(t+h, y(t+h), v(t, y(t))) \in \text{epi}(v)$ for all $h \in [0, T-t]$. Hence

$$v(t+h, y(t+h)) \leq v(t, y(t)) \iff u(t, y(t)) \leq u(t+h, y(t+h)) \quad (\text{since } u = -v).$$

This ends the proof of (ii) \implies (i) and Theorem 4.10. \square

4.4 Proof of Theorems 2.9 and 2.10.

Proof. (Theorem 2.9). Proposition 4.6 shows that the value function ϑ is locally Lipschitz continuous. Furthermore, we have $\vartheta(T, x) = \psi(x)$. Theorem 4.7 shows that ϑ is a super-solution since it enjoys the super-optimality property. In addition, Theorem 4.10 shows that ϑ is a viscosity sub-solution since it enjoys the sub-optimality property. Finally, uniqueness comes from the global comparison result in Theorem 2.10.

Proof. (Theorem 2.10). Let u_1 and u_2 respectively be a l.s.c super-solution and an u.s.c sub-solution of equation (2.3). By Theorem 4.7, we conclude that u_1 satisfies the super-optimality principle which means that for all $(t, x) \in [0, T) \times \mathbb{R}^N$, there exists a trajectory $y(\cdot) \in S_{(t, T)}(x)$ such that

$$u_1(t, x) \geq u_1(T, y(T)).$$

Likewise, by Theorem 4.10, we conclude that u_2 satisfies the sub-optimality principle. Then for the same trajectory $y(\cdot)$ we have

$$\forall (t, x) \in [0, T) \times \mathbb{R}^N, \quad u_2(t, x) \leq u_2(T, y(T)).$$

Henceforth, using the fact that $u_2(T, \cdot) \leq u_1(T, \cdot)$, we get the desired result $u_2(t, x) \leq u_1(t, x)$ for any $(t, x) \in [0, T) \times \mathbb{R}^N$. \square

5 Stability.

THEOREM 5.1. For $i = 1, \dots, n+l$, let $(F_i^j : \overline{M}_i \rightsquigarrow \mathbb{R}^N)_j$ be a sequence of set-valued maps satisfying (SH) and such that $F_i^j \rightarrow F_i$ w.r.t the Hausdorff metric (i.e. uniform convergence). Let $(v^j : \mathbb{R}^N \rightarrow \mathbb{R})_j$ be a sequence of l.s.c functions such that $v^j \rightarrow v$ locally uniformly in \mathbb{R}^N . Suppose in addition that for all j , v^j is a super-solution of

$$-\partial_t v^j(t, x) + \max_{i \in I(x)} \{H_{F_i^j}(x, \partial_x v^j(t, x))\} = 0 \quad \text{for all } (t, x) \in (0, T) \times \mathbb{R}^N,$$

in the sense of Definition 2.4. Then v is a super-solution of (2.3).

Proof. (Theorem 5.1). Let $(t, x) \in (0, T) \times \mathbb{R}^N$. Using Remark 4.8, it suffices to prove that v is a super-solution of (4.1). Let $\phi \in C^1((0, T) \times \mathbb{R}^N)$ such that $u - \phi$ attains a local minimum at (t, x) . Then, there exists $(t^j, x^j) \in (0, T) \times \mathbb{R}^N$ such that $v^j - \phi$ attains local minimum and such that $(t^j, x^j) \rightarrow (t, x)$. Since the stratification is finite and v^j is a super-solution of (2.3), then up to a subsequence (not relabeled), there exists $i_0 \in [1, n + l]$ such that for all j , we have

$$-\partial_t \phi(t^j, x^j) + H_{F_{i_0}^{j\sharp}}(x^j, \partial_x \phi(t^j, x^j)) \geq 0.$$

Since $F_{i_0}^{j\sharp}(\cdot) \subseteq F_{i_0}^j(\cdot)$, we get

$$-\partial_t \phi(t^j, x^j) + H_{F_{i_0}^j}(x^j, \partial_x \phi(t^j, x^j)) \geq 0.$$

So by letting j tend to infinity, we get

$$-\partial_t \phi(t, x) + H_{F_{i_0}}(x, \partial_x \phi(t, x)) \geq 0.$$

Finally, since $F_{i_0}(\cdot) \subseteq F(\cdot)$, then we get

$$-\partial_t \phi(t, x) + H_F(x, \partial_x \phi(t, x)) \geq 0,$$

which is the required result by Remark 4.8. \square

THEOREM 5.2. *For $i = 1, \dots, n + l$, let $(F_i^j : \overline{\mathcal{M}}_i \rightsquigarrow \mathbb{R}^N)_j$ be a sequence of set-valued maps satisfying (SH) and such that $F_i^j \rightarrow F_i$ w.r.t the Hausdorff metric (i.e. uniform convergence). Let $(v^j : \mathbb{R}^N \rightarrow \mathbb{R})_j$ be a sequence of u.s.c functions such that $v^j \rightarrow v$ locally uniformly in \mathbb{R}^N . Suppose in addition that for all j , v^j is a sub-solution of*

$$-\partial_t v^j(t, x) + H_{F_i^{j\sharp}}(x, \partial_x v^j(t, x)) = 0 \quad \text{for all } (t, x) \in (0, T) \times \mathbb{R}^N, \quad i \in I(x),$$

in the sense of definition 2.5. Then v is a sub-solution of (2.3).

Proof. (Theorem 5.2). Let $(t, x) \in (0, T) \times \mathbb{R}^N$ and $i \in I(x)$. Let $\phi \in C^1((0, T) \times \mathbb{R}^N)$, such that $u - \phi$ attains a local maximum in $(0, T) \times \overline{\mathcal{M}}_i$, at (t, x) . Then there exists $(t^j, x^j) \in (0, T) \times \overline{\mathcal{M}}_i$, such that $v^j - \phi$ attains local maximum in $(0, T) \times \overline{\mathcal{M}}_i$ at (t^j, x^j) , and such that $(t^j, x^j) \rightarrow (t, x)$. Since v^j is a subsolution, we get

$$-\partial_t \phi(t^j, x^j) + H_{F_i^{j\sharp}}(x^j, \partial_x \phi(t^j, x^j)) \leq 0.$$

Since $F_i^j \rightarrow F_i$ w.r.t the Hausdorff metric, we have

$$F_i^j(\cdot) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(\cdot) = F_i^{j\sharp}(\cdot) \rightarrow F_i^\sharp(\cdot) = F_i(\cdot) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(\cdot), \quad \text{w.r.t the Hausdorff metric.}$$

To see this, we recall an equivalent definition of the Hausdorff metric [9, Definition 7.3.1]:

$$d_{\mathcal{H}}(A, B) = \inf\{r > 0 : A \subset U_r(B) \text{ and } B \subset U_r(A)\}, \quad A, B \text{ compact subsets of } \mathbb{R}^N,$$

with $U_r(A) := \cup_{a \in A} \mathbb{B}(a, r)$ and $U_r(B) := \cup_{b \in B} \mathbb{B}(b, r)$ are the r -neighborhoods of A and B respectively. From this one gets immediately

$$d_{\mathcal{H}}(F_i^j(\cdot) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(\cdot), F_i^\sharp(\cdot)) \leq d_{\mathcal{H}}(F_i^j(\cdot), F_i(\cdot)).$$

Now let $\nu \in F_i^\sharp(x)$. Then by the Hausdorff convergence there exists a sequence $\nu^j \in F_i^{j\sharp}(x^j)$ such that $\nu^j \rightarrow \nu$. Finally, we arrive at

$$-\partial_t \phi(t^j, x^j) + \langle -\nu^j, \partial_x \phi(t^j, x^j) \rangle \leq -\partial_t \phi(t^j, x^j) + H_{F_i^{j\sharp}}(x^j, \partial_x \phi(t^j, x^j)) \leq 0.$$

By letting j tend to infinity, we get

$$-\partial_t \phi(t, x) + \langle -\nu, \partial_x \phi(t, x) \rangle \leq 0.$$

Lastly, we take the supremum over ν and we get the required result. \square

6 General convergence result for monotone schemes. In this section, we aim at studying the convergence of monotone numerical schemes approximating the HJB equation (2.3).

Let $\mathcal{G}^{\Delta x} = (\mathcal{G}_i^{\Delta x})_i$ be a spatial grid of \mathbb{R}^N of step Δx , compatible with the stratification $(\mathcal{M}_i)_i$ in the following sense:

$$(CC): \begin{cases} (i) \text{ For all } i, j = 1, \dots, n+l, \text{ such that } \mathcal{M}_j \subset \overline{\mathcal{M}}_i, \mathcal{G}_i^{\Delta x} \text{ and } \mathcal{G}_j^{\Delta x} \text{ coincide on } \mathcal{G}_j^{\Delta x}, \\ (ii) \forall R > 0, \forall i = 1, \dots, n+l, \lim_{\Delta x \rightarrow 0} d_{\mathcal{H}}(\overline{\mathcal{M}}_i \cap \overline{\mathbb{B}}(0, R), \mathcal{G}_i^{\Delta x} \cap \overline{\mathbb{B}}(0, R)) = 0. \end{cases}$$

Comments on the hypothesis (CC) . Hypothesis (CC)-(i) implies that the grid $\mathcal{G}^{\Delta x}$ is divided into $n+l$ subgrids $(\mathcal{G}_i^{\Delta x})_i$ with a partial order relation that ensures compatibility with the stratification. Hypothesis (CC)-(ii) asserts that for each $i = 1, \dots, n+l$, the subgrid $\mathcal{G}_i^{\Delta x}$ approaches $\overline{\mathcal{M}}_i$ in the sense of Hausdorff convergence for locally compact sets. Notice that this implies in particular that the points of a subgrid $\mathcal{G}_i^{\Delta x}$ don't have to belong to $\overline{\mathcal{M}}_i$. What is important here is that the grid $\mathcal{G}^{\Delta x}$ is divided into $n+l$ subgrids compatible with the stratification, and each subgrid converges in the Hausdorff sense to its corresponding domain.

We define for any $x \in \mathcal{G}^{\Delta x}$, the index multifunction of the grid

$$I_{\mathcal{G}^{\Delta x}}(x) := \{i \in \{1, \dots, n+l\} : x \in \mathcal{G}_i^{\Delta x}\}.$$

Let Δt be a constant time step of a regular grid $\Pi^{\Delta t}$ of $[0, T]$. We denote $h = (\Delta t, \Delta x)$. We consider the following numerical scheme:

$$\begin{cases} \max_{i \in I_{\mathcal{G}^{\Delta x}}(x_h)} \{S_i^h(t_h, x_h, u^h(t_h, x_h), [u^h]_{(t_h, x_h)})\} = 0 & \text{for } (t_h, x_h) \in (\Pi^{\Delta t} \times \mathcal{G}^{\Delta x}) \setminus \{t_h = T\}, \\ u^h(T, x_h) = \psi(x_h), & \text{for } (t_h, x_h) \in (\Pi^{\Delta t} \times \mathcal{G}^{\Delta x}) \cap \{t_h = T\} \end{cases}$$

with $[u^h]_{(t_h, x_h)}$ are all the values of u^h on $\mathcal{G}^{\Delta x}$ at other points than (t_h, x_h) . Each S_i^h , $i = 1, \dots, n+l$ is supposed to verify the following hypotheses:

- **Monotonicity** : $S_i^h(s, z, u, w_1) \leq S_i^h(s, z, u, w_2)$, if $w_1 \geq w_2$.
- **Stability** :
 - each u^h is locally bounded independently from h , for h small enough, i.e. for all $R > 0$, there is a $C_R > 0$, independent of h , such that

$$|u^h(t_i^h, x_i^h)| \leq C_R \quad \text{if } (t_i^h, x_i^h) \in \left([0, T] \times \mathbb{B}(0, R)\right) \cap \left(\Pi^{\Delta t} \times \mathcal{G}^{\Delta x}\right).$$

- u^h verifies the following inequality on a neighborhood of the interfaces: there exists $r > 0$ and $C_r > 0$, independent of h , such that

$$|u^h(t_h, x_h) - u^h(s_h, y_h)| \leq C_r(|t_h - s_h| + |x_h - y_h|),$$

$$\text{if } (t_h, x_h), (s_h, y_h) \in \left([0, T] \times (\Lambda + \mathbb{B}(0, r))\right) \cap \left(\Pi^{\Delta t} \times \mathcal{G}^{\Delta x}\right).$$

- **Consistency** : for all $\phi \in C^1((0, T) \times \mathbb{R}^N)$ and $(t, x) \in (0, T) \times \mathbb{R}^N$, we have

$$\lim_{\substack{\Pi^{\Delta t} \times \mathcal{G}_i^{\Delta x} \ni (s, z) \rightarrow (t, x) \\ h \rightarrow 0 \\ \zeta \rightarrow 0}} S_i^h(s, z, \phi(s, z) + \zeta, [\phi + \zeta]_{(s, z)}^h) = -\partial_t \phi(t, x) + H_{F_i^\#}(x, \partial_x \phi(t, x)),$$

where $[\phi + \zeta]_{(s, z)}^h$, is a function representing the values of $\phi + \zeta$ on the grid at other points than (s, z) . The following theorem is an extension of the result by Barles and Souganidis [6], to the case of HJB equations on stratified domains.

THEOREM 6.1. *Suppose that*

- *the HJB equation (2.3) admits a continuous viscosity solution u ,*
- *for all $h > 0$ small enough, the numerical scheme admits a solution u^h ,*
- *the spatial grid verifies hypothesis (CC),*
- *each S_i^h verifies the monotonicity, stability and consistency hypotheses,*

then, u^h converges locally uniformly to u .

Proof. First, we begin by defining the following functions

$$\bar{u}(t, x) := \limsup_{\substack{\Pi^{\Delta t} \times \mathcal{G}^{\Delta x} \ni (s, z) \rightarrow (t, x) \\ h \rightarrow 0}} u^h(s, z), \quad \underline{u}(t, x) := \liminf_{\substack{\Pi^{\Delta t} \times \mathcal{G}^{\Delta x} \ni (s, z) \rightarrow (t, x) \\ h \rightarrow 0}} u^h(s, z).$$

We also have that $\underline{u} \leq \bar{u}$. It suffices to prove $\underline{u} \geq \bar{u}$. For that we will prove that \underline{u} is a l.s.c super-solution and \bar{u} is an u.s.c sub-solution. We first prove that \bar{u} is a sub-solution.

Let $(t, x) \in (0, T) \times \mathbb{R}^N$. Without loss of generality, we suppose that $x \in \mathbb{B}(0, \rho)$ for some $\rho > 0$ big enough and we restrict our analysis on this bounded open set. Let $i \in I(x)$ and let $\phi \in C^1((0, T) \times \mathbb{R}^N)$ such that $\bar{u}_i - \phi$ attains its local maximum at $(t, x) \in (0, T) \times \bar{\mathcal{M}}_i$. We recall that $\bar{u}_i = u$ in $(0, T) \times \bar{\mathcal{M}}_i$ and $u_i \equiv -\infty$ otherwise.

Without loss of generality, we can suppose that

$$\bar{u}_i(t, x) = \phi(t, x), \quad \bar{u}_i(s, z) < \phi(s, z) \text{ if } (s, z) \neq (t, x),$$

$$\phi \geq C_R + 1 \text{ outside of a neighborhood } \Omega \text{ of } (t, x), \quad \Omega \subsetneq (0, T) \times \mathbb{B}(0, \rho),$$

where C_R is defined from the first part of the stability assumption. We have, $\bar{u}_i - \phi$ attains a local maximum at (t, x) . Furthermore, from the second part of the stability assumption, \bar{u} is Lipschitz continuous in a neighborhood of the interfaces. So we get

$$0 = \bar{u}_i(t, x) - \phi(t, x) = \limsup_{\substack{\Pi^{\Delta t} \times \mathcal{G}^{\Delta x} \ni (s, z) \rightarrow (t, x) \\ h \rightarrow 0}} u^h(s, z) - \phi(s, z) = \limsup_{\substack{\Pi^{\Delta t} \times \mathcal{G}_i^{\Delta x} \ni (s, z) \rightarrow (t, x) \\ h \rightarrow 0}} u^h(s, z) - \phi(s, z).$$

The second part of the stability assumption is essential here to get the last equality since the limit sup might not be reached from every subgrid $\mathcal{G}_i^{\Delta x}$. Moreover, outside of Ω , we have $\bar{u}_i - \phi \leq -1$. So, there exists $r > 0$, such that

$$0 \geq u^h(s, z) - \phi(s, z) \geq -1 \text{ for all } (s, z) \in ([t-r, t+r] \cap \Pi^{\Delta t}) \times (\bar{\mathbb{B}}(0, r) \cap \mathcal{G}_i^{\Delta x}) \subset \Omega.$$

So, the maximum is attained in the compact set $([t-r, t+r] \cap \Pi^{\Delta t}) \times (\bar{\mathbb{B}}(0, r) \cap \mathcal{G}_i^{\Delta x}) \subset \Omega$. Let (t_i^h, x_i^h) be the maximum and (t_i, x_i) the limit when $h \rightarrow 0$ of a subsequence not relabeled. we have

$$\lim_{h \rightarrow 0} u^h(t_i^h, x_i^h) - \phi(t_i^h, x_i^h) \geq \limsup_{\substack{\Pi^{\Delta t} \times \mathcal{G}_i^{\Delta x} \ni (s, z) \rightarrow (t, x) \\ h \rightarrow 0}} u^h(s, z) - \phi(s, z) = \bar{u}_i(t, x) - \phi(t, x) = 0.$$

On the other hand, since $\bar{u} - \phi$ is u.s.c, we get

$$0 \geq \bar{u}_i(t_i, x_i) - \phi(t_i, x_i) = \bar{u}_i(t_i, x_i) - \phi(t_i, x_i) \geq \lim_{h \rightarrow 0} u^h(t_i^h, x_i^h) - \phi(t_i^h, x_i^h).$$

Thus, we conclude

$$(t_i, x_i) = (t, x), \quad u^h(t_i^h, x_i^h) \rightarrow \bar{u}_i(t, x).$$

Let $\zeta_h := u^h(t_i^h, x_i^h) - \phi(t_i^h, x_i^h)$. We get

$$u^h(t_i^h, x_i^h) = \phi(t_i^h, x_i^h) + \zeta_h, \quad u^h(s, z) \leq \phi(s, z) + \zeta_h, \quad (t_i^h, x_i^h) \neq (s, z) \in \Pi^{\Delta t} \times \mathcal{G}_i^{\Delta x}.$$

From the monotonicity of the scheme and u^h being a solution, we get

$$S_i^h(t_i^h, x_i^h, u^h(t_i^h, x_i^h), [\phi + \zeta_h]_{(t_i^h, x_i^h)}^h) \leq S_i^h(t_i^h, x_i^h, u^h(t_i^h, x_i^h), [u^h]_{(t_i^h, x_i^h)}^h) \leq 0,$$

and by the consistency hypothesis and passing to the limit, we get

$$\lim_{\substack{(t_i^h, x_i^h) \rightarrow (t, x) \\ h \rightarrow 0 \\ \zeta_h \rightarrow 0}} S_i^h(t_i^h, x_i^h, \phi(t_i^h, x_i^h) + \zeta_h, [\phi + \zeta_h]_{(t_i^h, x_i^h)}^h) = -\partial_t \phi(t, x) + H_{F_i^\#}(x, \partial_x \phi(t, x)) \leq 0.$$

This is true for any $i \in I(x)$, which ends the proof.

Now we prove that \underline{u} is a super-solution. Following Remark 4.8, it suffices to prove that \underline{u} is a super-solution of (4.1).

Let $(t, x) \in (0, T) \times \mathbb{R}^N$. Let $\phi \in C^1((0, T) \times \mathbb{R}^N)$ such that $\underline{u} - \phi$ attains its local minimum in $(0, T) \times \mathbb{R}^N$ at (t, x) and $\phi(t, x) = \underline{u}(t, x)$. Using the same arguments as in the first part of the proof, we get a sequence $(h_n)_n$ such that

$$\begin{aligned} h_n \downarrow 0, \quad u^{h_n} - \phi \text{ attains a local maximum at } (t_n, x_n) \in \Pi^{\Delta t} \times \mathcal{G}^{\Delta x}, \text{ and} \\ (t_n, x_n) \rightarrow (t, x), \quad u^{h_n}(t_n, x_n) \rightarrow \underline{u}(t, x). \end{aligned}$$

Furthermore, since the stratification is finite and u^{h_n} is a solution to the numerical scheme, there exists a subsequence (not relabeled) of $(h_n)_n$ such that there exists $i_0 \in I_{\mathcal{G}^{\Delta x}}(x_n)$, for all $n \in \mathbb{N}$ and $(x_n)_n \subseteq \mathcal{G}_{i_0}^{\Delta x}$ such that we have

$$\max_{i \in I_{\mathcal{G}^{\Delta x}}(x_n)} S_i^{h_n}(t_n, x_n, u^{h_n}(t_n, x_n), u^{h_n}) = S_{i_0}^{h_n}(t_n, x_n, u^{h_n}(t_n, x_n), u^{h_n}) \geq 0.$$

Let $\zeta_n := u^{h_n}(t_n, x_n) - \phi(t_n, x_n)$. So

$$u^{h_n}(t_n, x_n) = \phi(t_n, x_n) + \zeta_n, \quad u^{h_n}(s, z) \geq \phi(s, z) + \zeta_h, \quad (t_i^h, x_i^h) \neq (s, z) \in \Pi^{\Delta t} \times \mathcal{G}^{\Delta x}.$$

From the monotonicity assumption and u^{h_n} being a solution, we get

$$S_{i_0}^{h_n}(t_n, x_n, \phi(t_n, x_n) + \zeta_h, [\phi + \zeta_h]_{(t_n^h, x_n^h)}^{h_n}) \geq S_{i_0}^{h_n}(t_n, x_n, u^{h_n}(t_n, x_n), [u^{h_n}]_{(t_n^h, x_n^h)}^{h_n}) \geq 0.$$

and by the consistency hypothesis and passing to the limit, we get

$$\lim_{\substack{(t_n, x_n) \rightarrow (t, x) \\ h_n \rightarrow 0 \\ \zeta_n \rightarrow 0}} S_{i_0}^{h_n}(t_n, x_n, \phi(t_n, x_n) + \zeta_h, [\phi + \zeta_h]_{(t_n^h, x_n^h)}^{h_n}) = -\partial_t \phi(t, x) + H_{F_i^{\sharp}}(x, \partial_x \phi(t, x)) \geq 0.$$

By Remark 4.8 and the fact that $F_i^{\sharp}(\cdot) \subseteq F(\cdot)$, we get the required result.

In conclusion, $\bar{u}(\cdot, \cdot) = \underline{u}(\cdot, \cdot) = u(\cdot, \cdot)$ and u^h converges locally uniformly to u , which ends the proof. \square

Appendix A. Relative wedgeness. This appendix presents the concept of relative wedgeness first introduced in [7].

Let $S \subset \mathbb{R}^N$ be a closed set. S is said to be proximally smooth if there exists $R > 0$ such that the distance function $d_S(\cdot)$ is continuously differentiable on the set $\{x \in \mathbb{R}^N : 0 < d_S(x) < R\}$. If S is proximally smooth, then its Clarke tangent cone is equal to its Bouligand tangent cone $\mathcal{T}_S(\cdot)$. Clarke tangent cone is always closed and convex.

Now, let \mathcal{M} be a C^2 embedded manifold in \mathbb{R}^N such that $\overline{\mathcal{M}}$ is proximally smooth and let $d = \dim(\mathcal{M})$ be its dimension. Then, for every $x \in \overline{\mathcal{M}}$, the tangent cone $\mathcal{T}_{\overline{\mathcal{M}}}(x)$ is closed and convex, hence it has a relative interior (in the sense of convex analysis), denoted by $r\text{-int}(\mathcal{T}_{\overline{\mathcal{M}}}(x))$.

The set $\overline{\mathcal{M}}$ is said to be relatively wedged if for every $x \in \overline{\mathcal{M}}$, the dimension of $r\text{-int}(\mathcal{T}_{\overline{\mathcal{M}}}(x))$ (in the sense of convex analysis) is equal to the dimension of \mathcal{M} :

$$\dim(r\text{-int}(\mathcal{T}_{\overline{\mathcal{M}}}(x))) = \dim(\mathcal{M}) = d.$$

Appendix B. Proof of invariance theorems.

Proof. (Theorem 3.5). Since F is an u.s.c set-valued map with convex, compact non-empty images, and since S is a closed set of \mathbb{R}^N , it is known that assertion (i) is equivalent to (see for instance [11, Theorem 12.11])

$$H_F(x, \eta) \geq 0 \quad \forall \eta \in N_S^p(x), \quad \forall x \in \mathbb{R}^N.$$

Let $x \in \mathbb{R}^N$. We have $F_i^\sharp(\cdot) \subseteq F_i(\cdot) \subseteq F(\cdot)$. We have

$$(B.1) \quad H_{F_i}(x, \eta_i) \geq H_{F_i^\sharp}(x, \eta_i), \quad \forall \eta_i \in N_{S_i}^p(x).$$

From this inequality, we deduce easily the implication (iii) \implies (ii). Moreover, since $N_S^p(x) \subseteq N_{S_i}^p(x)$ for all $i \in I(x)$, then

$$H_F(x, \eta) \geq \max_{i \in I(x)} H_{F_i}(x, \eta) \geq \max_{i \in I(x)} H_{F_i^\sharp}(x, \eta), \quad \forall \eta \in N_S^p(x).$$

Hence, the implication (ii) \implies (i) holds. It remains to prove the implication (i) \implies (iii). Suppose (S, F) is weakly invariant. Let $x \in \mathbb{R}^N$ and $t \in [0, T]$. So there exists a trajectory $y(\cdot)$ solution of $(DI)_F(t, x)$ such that $y(\cdot) \subset S$. We claim the following:

Claim: $\exists j \in I(x)$ such that there exists a sequence $(t_n)_n$, $t_n \downarrow t$ and $x_n := y(t_n) \in \mathcal{M}_j$, so that $\frac{x_n - x}{t_n - t} \rightarrow \nu$ and $\nu \in F_j^\sharp(x)$.

The proof of the claim is the same as the proof of the same claim in Proposition 4.7. With this claim, we are almost done. Indeed let $\eta_j \in N_{S_j}^p(x)$ be such that the proximal normal inequality is satisfied with $\sigma > 0$. we get

$$\langle \nu, \eta_j \rangle = \lim_{n \rightarrow +\infty} \left\langle \frac{x_n - x}{t_n - t}, \eta_j \right\rangle \leq \lim_{n \rightarrow +\infty} \frac{1}{2\sigma(t_n - t)} \|x_n - x\|^2 = 0.$$

Thus, we have

$$H_{F_j^\sharp}(x, \eta_j) \geq -\langle \nu, \eta_j \rangle \geq 0,$$

Which is the required result. □

Proof. (Theorem 3.6). The implication (ii) \implies (i) is proven first. We separate the proof into 3 parts. First we prove the result for every trajectory that lies entirely in one of the domains \mathcal{M}_i , $i = 1, \dots, n+l$. Then, we prove the result for every regular trajectory using an induction argument. Finally we prove the result for every trajectory using Filippov's theorem [12, Theorem 3.1.6].

Step 1. (Inspired from [11, Theorem 12.15]). Let $y(\cdot)$ be a trajectory of F such that $y((t, T)) \subset \mathcal{M}_i$ and such that $y(t) = \alpha \in S \cap \overline{\mathcal{M}}_i$. We show that for some $\varepsilon \in (0, T - t)$, we have $y([t, t + \varepsilon]) \subset S$ which is sufficient to conclude.

Let $r > 0$ small enough such that $\mathbb{B}(\alpha, r) \cap \mathcal{M}_i$ is a relative neighborhood in \mathcal{M}_i . From $(SH) - (ii)$ (linear growth hypothesis), we deduce that there exists $M > 0$ such that $\|F_i\| \leq M$ on the ball $\mathbb{B}(\alpha, r) \cap \mathcal{M}_i$. So $y(\cdot)$ is Lipschitz continuous on $[t, t + \varepsilon]$ with constant M . There exists $\varepsilon \in (0, T - t)$ such that

$$\forall \tau \in [t, t + \varepsilon], \quad s \in \text{proj}_{S_i}(y(\tau)) \implies y(\tau) \in \mathbb{B}(\alpha, r) \cap \mathcal{M}_i, \quad s \in \mathbb{B}(\alpha, r) \cap \mathcal{M}_i.$$

We define $f(\tau) := d_{S_i}(y(\tau))$. f is Lipschitz continuous on $[t, t + \varepsilon]$. We prove the following Lemma:

LEMMA B.1. $f'(\tau) \leq \kappa f(\tau)$ for almost all $\tau \in (t, t + \varepsilon)$.

Proof. Let $\tau_* \in (t, t + \varepsilon)$ such that $f'(\tau_*)$ exists, $y'(\tau_*)$ exists and $y'(\tau_*) \in F_i(y(\tau_*))$ (almost all points satisfy those conditions). If $f(\tau_*) = 0$ then f attains a minimum at τ_* and therefore $f'(\tau_*) = 0$ and the inequality holds. Suppose now $f(\tau_*) > 0$ and let $s \in \text{proj}_{S_i}(y(\tau_*))$. Then by [11], proposition 11.29, we have

$$\eta := \frac{y(\tau_*) - s}{|y(\tau_*) - s|} \in N_{S_i}^p(s).$$

Since $F_i(= F_i^\sharp)$ on \mathcal{M}_i is Lipschitz continuous on $\mathbb{B}(\alpha, r)$ with constant κ , there exists $\nu \in F_i(s)$ such that

$$|y'(\tau_*) - \nu| \leq \kappa |y(\tau_*) - s|.$$

Therefore we get

$$\langle \eta, y'(\tau_*) \rangle = \langle \eta, \nu \rangle + \langle \eta, y'(\tau_*) - \nu \rangle \leq H_{F_i^\#}(s, -\eta) + \kappa |y(\tau_*) - s| \leq \kappa |y(\tau_*) - s|.$$

Hence, we get

$$\begin{aligned} f'(\tau_*) &= \lim_{\delta \rightarrow 0} \frac{d_{S_i}(y(\tau_* + \delta)) - d_{S_i}(y(\tau_*))}{\delta} \leq \lim_{\delta \rightarrow 0} \frac{|y(\tau_* + \delta) - s| - |y(\tau_*) - s|}{\delta} \\ &= \langle \eta, y'(\tau_*) \rangle \leq \kappa |y(\tau_*) - s| = kf(\tau_*). \end{aligned} \quad \square$$

Since f is Lipschitz, positive and $f(t) = 0$ ($y(t) = \alpha \in S_i$), then by using the lemma above and Gronwall lemma, we get that $f \equiv 0$ on $[t, t + \varepsilon]$, which finishes the proof of step 1.

Notice that, every trajectory that lies entirely in $\cup_{i=1}^n \mathcal{M}_i$ also verifies step 1 since it is a disjoint union.

Step 2. Let \mathcal{M} be union of subdomains such that $\cup_{i=1}^n \mathcal{M}_i \subseteq \mathcal{M}$ and denote by $\delta_{\mathcal{M}}$ the minimum dimension of the subdomains of \mathcal{M} . Let \mathcal{M}_{k_0} be a subdomain such that $\mathcal{M}_{k_0} \subset \overline{\mathcal{M}} \setminus \mathcal{M}$ and its dimension is inferior or equal to $\delta_{\mathcal{M}}$. We show the following proposition:

PROPOSITION B.2. *If we have (ii) \implies (i) for every trajectory that lies entirely in \mathcal{M} or lies entirely in \mathcal{M}_{k_0} , then (ii) \implies (i) holds true for every trajectory that lies in $\mathcal{M} \cup \mathcal{M}_{k_0}$.*

Proof. Let $y(\cdot) \subseteq \mathcal{M} \cup \mathcal{M}_{k_0}$ be a trajectory of F on $[t, T]$ such that $y(t) \in S$. We define

$$J = \{\tau \in [t, T] : y(\tau) \notin \mathcal{M}_{k_0}\}.$$

The set J is open since \mathcal{M}_{k_0} is of inferior dimension than \mathcal{M} , then it is a closed set relative to $\mathcal{M} \cup \mathcal{M}_{k_0}$ (equipped with inherited topology from \mathbb{R}^N). Thus J can be written as a countable union of open intervals in the following way:

$$J = \bigcup_{i=1}^{\infty} (a_i, b_i),$$

such that the open sets (a_i, b_i) are pairwise disjoint and $a_i < b_i \leq a_{i+1}$, $\forall i \geq 1$. Notice that we necessarily have $y(a_i), y(b_i) \in \mathcal{M}_{k_0}$. Set $b_0 = t$. First, we prove that

$$\forall i \in \mathbb{N}, y(b_i), y(a_{i+1}) \in S \text{ and } y((a_{i+1}, b_{i+1})) \subset S.$$

We have $y(b_0) \in S$ by assumption. If $b_0 = a_1$ then we have $y(a_1) \in S$. If $b_0 < a_1$ then $y(b_0) \in S \cap \mathcal{M}_{k_0}$ and $y((b_0, a_1)) \subset \mathcal{M}_{k_0}$. Hence, by Step 1, we have $y((b_0, a_1)) \subset S$ and therefore $y(a_1) \in S$ since S is a closed set. Moreover, since $y(a_1) \in S \cap \overline{\mathcal{M}}$ and $y((a_1, b_1)) \subset \mathcal{M}$, then by Step 1, we have that $y((a_1, b_1)) \subset S$ and therefore $y(b_1) \in S$ since S is closed. By induction, following the same argument, we get that

$$\forall i \in \mathbb{N}, y(b_i), y(a_{i+1}) \in S \text{ and } y((a_{i+1}, b_{i+1})) \subset S.$$

It remains to prove $y([t, T] \setminus J) \subset S$. If the set J was equal to a finite union of open intervals then the above argument would have been sufficient to prove that $y([t, T] \setminus J) \subset S$. However, this is not the case for all trajectories $y(\cdot)$. The trajectories $y(\cdot)$ can move in and out of \mathcal{M}_{k_0} infinitely many times exhibiting the phenomenon known as the *Zeno effect*. To deal with this case, we will approximate such trajectories $y(\cdot)$ by ones such that the set J is equal to a finite union of open intervals.

We fix $m \geq 1$ and set

$$J_m = \bigcup_{k=1}^m (a_k, b_k),$$

Which we can assume to satisfy

$$t = b_0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_m < b_m \leq a_{m+1} := T.$$

We choose m large enough such that

$$\mathcal{L}(J \setminus J_m) < \frac{r}{2e^{\kappa T} \|F\|},$$

with κ being the Lipschitz constant of F_{k_0} and r is equal to

$$r := \inf_{\substack{w \in \mathcal{M}_{k_0} \setminus \mathcal{M}_{k_0} \\ s \in [t, T]}} |y(s) - w|,$$

(notice that r is strictly positive and can be infinite). The choice of m is made in such a way to be able to apply [12, Theorem 3.1.6] on manifolds (see [7, Remark 3.1]). We will approximate the arc $y([b_i, a_{i+1}])$, for some $i = 0, \dots, m$, by trajectories that remain entirely in \mathcal{M}_{k_0} . By Filippov theorem ([12, Theorem 3.1.6]) and [7, Proposition 3.2], there exists $z_i(\cdot)$ a trajectory of F_{k_0} on $[b_i, a_{i+1}]$ such that $z_i(b_i) = y(b_i) \in \mathcal{M}_{k_0} \cap S$, $z_i(\cdot) \subset \mathcal{M}_{k_0}$ and

$$\|y(\cdot) - z_i(\cdot)\|_{L^\infty[b_i, a_{i+1}]} \leq e^{\kappa(a_{i+1} - b_i)} \rho_i \leq 2 e^{\kappa T} \|F\| \varepsilon_i,$$

where we denote $\varepsilon_i = \mathcal{L}(J \cap (b_i, a_{i+1}))$ and

$$\rho_i := \int_{b_i}^{a_{i+1}} d_{\mathcal{H}}(\{\dot{y}(s)\}, F_{k_0}(z_i(s))) ds \leq 2 \|F\| \varepsilon_i.$$

Since $\varepsilon_i \leq \mathcal{L}(J \setminus J_m)$, we get

$$\|y(\cdot) - z_i(\cdot)\|_{L^\infty[b_i, a_{i+1}]} \leq 2 e^{\kappa T} \|F\| \mathcal{L}(J \setminus J_m).$$

Furthermore, from Step 1 we have $z_i(\cdot) \subset S$. Thus we get

$$d_S(y(\cdot)) \leq \|y(\cdot) - z_i(\cdot)\|_{L^\infty[b_i, a_{i+1}]} \leq 2 e^{\kappa T} \|F\| \mathcal{L}(J \setminus J_m), \quad \forall m \geq 1.$$

By letting $m \rightarrow \infty$, we have $\mathcal{L}(J \setminus J_m) \rightarrow 0$. Therefore $y(\cdot) \subset S$, which is the required result. □

From the above proposition, we deduce that $(ii) \implies (i)$ by a simple finite induction argument starting from $\mathcal{M} = \cup_{i=1}^n \mathcal{M}_i$ and adding an interface $\mathcal{M}_{k_0} \subset \Lambda$, with $k_0 \in \{n+1, \dots, n+l\}$, in such a way that decreases the dimension of \mathcal{M}_{k_0} at each iteration.

Now we prove the direct implication $(i) \implies (ii)$. For that, we use [28, Lemma 3.9]. See also [7, Proposition 5.1 and Lemma 5.2]. Suppose (S, F) is strongly invariant. Let $x \in S \cap \overline{\mathcal{M}}_i$, $\nu \in F_i^\sharp(x)$ and $\eta \in N_{S_i}^p(x)$ such that $\|\eta\| = 1$ a proximal normal realised at $\sigma > 0$ (from the definition of the proximal normal).

Since $\nu \in F_i^\sharp(x)$, then by [28, Lemma 3.9], there exists a C^1 trajectory $y(\cdot)$ of F_i^\sharp , defined on some interval $[t, t + \varepsilon]$ with $\varepsilon > 0$ such that $y(t) = x$ and $\dot{y}(t) = \nu$ and $y(\cdot) \subseteq \overline{\mathcal{M}}_i$. By the strong invariance hypothesis, we have $y(\cdot) \subseteq S_i$. So we get

$$\langle \nu, \eta \rangle = \lim_{\tau \downarrow t} \left\langle \frac{y(\tau) - x}{\tau - t}, \eta \right\rangle \leq \lim_{\tau \downarrow t} \frac{1}{2\sigma(\tau - t)} |y(\tau) - x|^2 = 0.$$

By taking the supremum over $F_i^\sharp(x)$, we obtain the desired result. □

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