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# ON THE SPECTRAL THEORY OF GROUPS OF AUTOMORPHISMS OF $S$ -ADIC NILMANIFOLDS

BACHIR BEKKA AND YVES GUIVARC'H

ABSTRACT. Let  $S = \{p_1, \dots, p_r, \infty\}$  for prime integers  $p_1, \dots, p_r$ . Let  $X$  be an  $S$ -adic compact nilmanifold, equipped with the unique translation invariant probability measure  $\mu$ . We characterize the countable groups  $\Gamma$  of automorphisms of  $X$  for which the Koopman representation  $\kappa$  on  $L^2(X, \mu)$  has a spectral gap. More specifically, we show that  $\kappa$  does not have a spectral gap if and only if there exists a non-trivial  $\Gamma$ -invariant quotient solenoid (that is, a finite-dimensional, connected, compact abelian group) on which  $\Gamma$  acts as a virtually abelian group.

## 1. INTRODUCTION

Let  $\Gamma$  be a countable group acting measurably on a probability space  $(X, \mu)$  by measure preserving transformations. Let  $\kappa = \kappa_X$  denote the corresponding Koopman representation of  $\Gamma$ , that is, the unitary representation of  $\Gamma$  on  $L^2(X, \mu)$  given by

$$\kappa(\gamma)\xi(x) = \xi(\gamma^{-1}x) \quad \text{for all } \xi \in L^2(X, \mu), x \in X, \gamma \in \Gamma.$$

We say that the action  $\Gamma \curvearrowright (X, \mu)$  of  $\Gamma$  on  $(X, \mu)$  has a **spectral gap** if the restriction  $\kappa_0$  of  $\kappa$  to the  $\Gamma$ -invariant subspace

$$L_0^2(X, \mu) = \left\{ \xi \in L^2(X, \mu) : \int_X \xi(x) d\mu(x) = 0 \right\}$$

does not weakly contain the trivial representation  $1_\Gamma$ ; equivalently, if  $\kappa_0$  does not have almost invariant vectors, that is, there is **no** sequence  $(\xi_n)_n$  of unit vectors in  $L_0^2(X, \mu)$  such that

$$\lim_n \|\kappa_0(\gamma)\xi_n - \xi_n\| = 0 \quad \text{for all } \gamma \in \Gamma.$$

The existence of a spectral gap admits the following useful quantitative version. Let  $\nu$  be a probability measure on  $\Gamma$  and  $\kappa_0(\nu)$  the convolution operator defined on  $L_0^2(X, \mu)$  by

$$\kappa_0(\nu)\xi = \sum_{\gamma \in \Gamma} \nu(\gamma)\kappa_0(\gamma)\xi \quad \text{for all } \xi \in L_0^2(X, \mu).$$

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Observe that we have  $\|\kappa_0(\nu)\| \leq 1$  and hence  $r(\kappa_0(\nu)) \leq 1$  for the spectral radius  $r(\kappa_0(\nu))$  of  $\kappa_0(\nu)$ . Assume that  $\nu$  is aperiodic, that is, the support of  $\nu$  is not contained in the coset of a proper subgroup of  $\Gamma$ . Then the action of  $\Gamma$  on  $X$  has a spectral gap if and only if  $r(\kappa_0(\nu)) < 1$  and this is equivalent to  $\|\kappa_0(\nu)\| < 1$ ; for more details, see the survey [Bek16].

In this paper, we will be concerned with the case where  $X$  is an  $S$ -adic nilmanifold, to be introduced below, and  $\Gamma$  is a subgroup of automorphisms of  $X$ .

Fix a finite set  $\{p_1, \dots, p_r\}$  of integer primes and set  $S = \{p_1, \dots, p_r, \infty\}$ . For an integer  $d \geq 1$ , the product

$$\mathbf{Q}_S := \prod_{p \in S} \mathbf{Q}_p = \mathbf{Q}_\infty \times \mathbf{Q}_{p_1} \times \cdots \times \mathbf{Q}_{p_r}$$

is a locally compact ring, where  $\mathbf{Q}_\infty = \mathbf{R}$  and  $\mathbf{Q}_p$  is the field of  $p$ -adic numbers for a prime  $p$ . Let  $\mathbf{Z}[1/S] = \mathbf{Z}[1/p_1, \dots, 1/p_r]$  denote the subring of  $\mathbf{Q}$  generated by 1 and  $\{1/p_1, \dots, 1/p_r\}$ . Through the diagonal embedding

$$\mathbf{Z}[1/S] \rightarrow \mathbf{Q}_S, \quad b \mapsto (b, \dots, b),$$

we may identify  $\mathbf{Z}[1/S]$  with a discrete and cocompact subring of  $\mathbf{Q}_S$ .

If  $\mathbf{G}$  is a linear algebraic group defined over  $\mathbf{Q}$ , we denote by  $\mathbf{G}(R)$  the group of elements of  $\mathbf{G}$  with coefficients in  $R$  and determinant invertible in  $R$ , for every subring  $R$  of an overfield of  $\mathbf{Q}$ .

Let  $\mathbf{U}$  be a linear algebraic unipotent group defined over  $\mathbf{Q}$ , that is,  $\mathbf{U}$  is an algebraic subgroup of the group of  $n \times n$  upper triangular unipotent matrices for some  $n \geq 1$ . The group  $\mathbf{U}(\mathbf{Q}_S)$  is a locally compact group and  $\Lambda := \mathbf{U}(\mathbf{Z}[1/S])$  is a cocompact lattice in  $\mathbf{U}(\mathbf{Q}_S)$ . The corresponding  $S$ -adic compact nilmanifold

$$\mathbf{Nil}_S = \mathbf{U}(\mathbf{Q}_S)/\mathbf{U}(\mathbf{Z}[1/S])$$

will be equipped with the unique translation-invariant probability measure  $\mu$  on the Borel subsets of  $\mathbf{Nil}_S$ .

For  $p \in S$ , let  $\text{Aut}(\mathbf{U}(\mathbf{Q}_p))$  be the group of continuous automorphisms of  $\mathbf{U}(\mathbf{Q}_p)$ . Set

$$\text{Aut}(\mathbf{U}(\mathbf{Q}_S)) := \prod_{p \in S} \text{Aut}(\mathbf{U}(\mathbf{Q}_p))$$

and denote by  $\text{Aut}(\mathbf{Nil}_S)$  the subgroup

$$\{g \in \text{Aut}(\mathbf{U}(\mathbf{Q}_S)) \mid g(\Lambda) = \Lambda\}.$$

Every  $g \in \text{Aut}(\mathbf{Nil}_S)$  acts on  $\mathbf{Nil}_S$  preserving the probability measure  $\mu$ .

The abelian quotient group

$$\overline{\mathbf{U}(\mathbf{Q}_S)} := \mathbf{U}(\mathbf{Q}_S) / [\mathbf{U}(\mathbf{Q}_S), \mathbf{U}(\mathbf{Q}_S)]$$

can be identified with  $\mathbf{Q}_S^d$  for some  $d \geq 1$  and the image  $\Delta$  of  $\mathbf{U}(\mathbf{Z}[1/S])$  in  $\overline{\mathbf{U}(\mathbf{Q}_S)}$  is a cocompact and discrete subgroup of  $\overline{\mathbf{U}(\mathbf{Q}_S)}$ ; so,

$$\mathbf{Sol}_S := \overline{\mathbf{U}(\mathbf{Q}_S)} / \Delta$$

is a solenoid (that is, is a finite-dimensional, connected, compact abelian group; see [HeRo63, §25]). We refer to  $\mathbf{Sol}_S$  as the  $S$ -adic solenoid attached to the  $S$ -adic nilmanifold  $\mathbf{Nil}_S$ . We equip  $\mathbf{Sol}_S$  with the probability measure  $\nu$  which is the image of  $\mu$  under the canonical projection  $\mathbf{Nil}_S \rightarrow \mathbf{Sol}_S$ .

Observe that  $\text{Aut}(\mathbf{Q}_S^d)$  is canonically isomorphic to  $\prod_{s \in S} GL_d(\mathbf{Q}_s)$  and that  $\text{Aut}(\mathbf{Sol}_S)$  can be identified with the subgroup  $GL_d(\mathbf{Z}[1/S])$ . The group  $\text{Aut}(\mathbf{Nil}_S)$  acts naturally by automorphisms of  $\mathbf{Sol}_S$ ; we denote by

$$p_S : \text{Aut}(\mathbf{Nil}_S) \rightarrow GL_d(\mathbf{Z}[1/S]) \subset GL_d(\mathbf{Q})$$

the corresponding homomorphism.

**Theorem 1.** *Let  $\mathbf{U}$  be an algebraic unipotent group defined over  $\mathbf{Q}$  and  $S = \{p_1, \dots, p_r, \infty\}$ , where  $p_1, \dots, p_r$  are integer primes. Let  $\mathbf{Nil}_S = \mathbf{U}(\mathbf{Q}_S) / \mathbf{U}(\mathbf{Z}[1/S])$  be the associated  $S$ -adic nilmanifold and let  $\mathbf{Sol}_S$  be the corresponding  $S$ -adic solenoid, equipped with respectively the probability measures  $\mu$  and  $\nu$  as above. Let  $\Gamma$  be a countable subgroup  $\text{Aut}(\mathbf{Nil}_S)$ . The following properties are equivalent:*

- (i) *The action  $\Gamma \curvearrowright (\mathbf{Nil}_S, \mu)$  has a spectral gap.*
- (ii) *The action  $p_S(\Gamma) \curvearrowright (\mathbf{Sol}_S, \nu)$  has a spectral gap, where  $p_S : \text{Aut}(\mathbf{Nil}_S) \rightarrow GL_d(\mathbf{Z}[1/S])$  is the canonical homomorphism.*

Actions by groups of automorphisms (or more generally groups of by affine transformations) on the  $S$ -adic solenoid  $\mathbf{Sol}_S$  have been completely characterized in [BeFr20, Theorem 5]. The following result is an immediate consequence of this characterization and of Theorem 1. For a subset  $T$  of  $GL_d(\mathbf{K})$  for a field  $\mathbf{K}$ , we denote by  $T^t = \{g^t \mid g \in T\}$  the set of transposed matrices from  $T$ .

**Corollary 2.** *With the notation as in Theorem 1, the following properties are equivalent:*

- (i) *The action of  $\Gamma$  on the  $S$ -adic nilmanifold  $\mathbf{Nil}_S$  does not have a spectral gap.*
- (ii) *There exists a non-zero linear subspace  $W$  of  $\mathbf{Q}^d$  which is invariant under  $p_S(\Gamma)^t$  and such that the image of  $p_S(\Gamma)^t$  in  $GL(W)$  is a virtually abelian group.*

Here is an immediate consequence of Corollary 2.

**Corollary 3.** *With the notation as in Corollary 1, assume that the linear representation of  $p_S(\Gamma)^t$  in  $\mathbf{Q}^d$  is irreducible. Then the action  $\Gamma \curvearrowright (\mathbf{Nil}_S, \mu)$  has a spectral gap.*

Recall that the action of a countable group  $\Gamma$  by measure preserving transformations on a probability space  $(X, \mu)$  is **strongly ergodic** (see [Schm81]) if every sequence  $(B_n)_n$  of measurable subsets of  $X$  which is asymptotically invariant (that is, which is such that  $\lim_n \mu(\gamma B_n \Delta B_n) = 0$  for all  $\gamma \in \Gamma$ ) is trivial (that is,  $\lim_n \mu(B_n)(1 - \mu(B_n)) = 0$ ). It is straightforward to check that the spectral gap property implies strong ergodicity and it is known that the converse does not hold in general.

The following corollary is a direct consequence of Theorem 1 (compare with Corollary 2 in [BeGu15]).

**Corollary 4.** *With the notation as in 1, the following properties are equivalent:*

- (i) *The action  $\Gamma \curvearrowright (\mathbf{Nil}_S, \mu)$  has the spectral gap property.*
- (ii) *The action  $\Gamma \curvearrowright (\mathbf{Nil}_S, \mu)$  is strongly ergodic.*

Theorem 1 generalizes our previous work [BeGu15], where we treated the real case (that is, the case  $S = \infty$ ). We had to extend our methods to the  $S$ -adic setting. There are three main tools we use in the proof:

- a canonical decomposition of the Koopman representation of  $\Gamma$  in  $L^2(\mathbf{Nil}_S)$  as a direct sum of certain representations of  $\Gamma$  induced from stabilizers of representations of  $\mathbf{U}(\mathbf{Q}_S)$  (see Proposition 9);
- a result of Howe and Moore [HoMo79] about the decay of matrix coefficients of algebraic groups (see Proposition 11);
- the fact that the irreducible representations of  $\mathbf{U}(\mathbf{Q}_S)$  appearing in the decomposition of  $L^2(\mathbf{Nil}_S)$  are rational, in the sense that the Kirillov data associated to each one of them are defined over  $\mathbf{Q}$  (see Proposition 13).

Another tool we constantly use is a generalized version of Herz's majoration principle (see Lemma 7).

Given a probability measure  $\nu$  on  $\Gamma$ , our approach does not seem to provide quantitative estimates for the operator norm of the convolution operator  $\kappa_0(\nu)$  acting on  $L^2_0(\mathbf{Nil}_S, \mu)$  for a general unipotent group  $\mathbf{U}$ . However, using known bounds for the so-called metaplectic representation of  $Sp_{2n}(\mathbf{Q}_p)$ , we give such estimates in the case of  $S$ -adic Heisenberg nilmanifolds (see Section 11).

**Corollary 5.** *For an integer  $n \geq 1$ , let  $\mathbf{U} = \mathbf{H}_{2n+1}$  be the  $(2n+1)$ -dimensional Heisenberg group and  $\mathbf{Nil}_S = \mathbf{H}_{2n+1}(\mathbf{Q}_S)/\mathbf{H}_{2n+1}(\mathbf{Z}[1/S])$ . Let  $\nu$  be a probability measure on*

$$Sp_{2n}(\mathbf{Z}[1/S]) \subset \text{Aut}(\mathbf{Nil}_S).$$

Then

$$\|\kappa_0(\nu)\| \leq \max\{\|\lambda_\Gamma(\nu)\|^{1/2n+2}, \|\kappa_1(\nu)\|\},$$

where  $\kappa_1$  is the restriction of  $\kappa_0$  to  $L_0^2(\mathbf{Sol}_S)$  and  $\lambda_\Gamma$  is the regular representation of the group  $\Gamma$  generated by the support of  $\nu$ . In particular, in the case  $n = 1$ , the action of  $\Gamma$  on  $\mathbf{Nil}_S$  has a spectral gap if and only if  $\Gamma$  is non-amenable.

## 2. EXTENSION OF REPRESENTATIONS

Let  $G$  be a locally compact group which we assume to be second countable. We will need the notion of a projective representation. Recall that a mapping  $\pi : G \rightarrow U(\mathcal{H})$  from  $G$  to the unitary group of the Hilbert space  $\mathcal{H}$  is a **projective representation** of  $G$  if the following holds:

- $\pi(e) = I$ ,
- for all  $g_1, g_2 \in G$ , there exists  $c(g_1, g_2) \in \mathbf{C}$  such that

$$\pi(g_1 g_2) = c(g_1, g_2) \pi(g_1) \pi(g_2),$$

- the function  $g \mapsto \langle \pi(g)\xi, \eta \rangle$  is measurable for all  $\xi, \eta \in \mathcal{H}$ .

The mapping  $c : G \times G \rightarrow \mathbf{S}^1$  is a 2-cocycle with values in the unit circle  $\mathbf{S}^1$ . Every projective unitary representation of  $G$  can be lifted to an ordinary unitary representation of a central extension of  $G$  (for all this, see [Mack76] or [Mack58]).

Let  $N$  be a closed normal subgroup of  $G$ . Let  $\pi$  be an irreducible unitary representation of  $N$  on a Hilbert space  $\mathcal{H}$ . Consider the stabilizer

$$G_\pi = \{g \in G \mid \pi^g \text{ is equivalent to } \pi\}$$

of  $\pi$  in  $G$  for the natural action of  $G$  on the unitary dual  $\widehat{N}$  given by  $\pi^g(n) = \pi(g^{-1}ng)$ . Then  $G_\pi$  is a closed subgroup of  $G$  containing  $N$ . The following lemma is a well-known part of Mackey's theory of unitary representations of group extensions.

**Lemma 6.** *Let  $\pi$  be an irreducible unitary representation of  $N$  on the Hilbert space  $\mathcal{H}$ . There exists a projective unitary representation  $\tilde{\pi}$  of  $G_\pi$  on  $\mathcal{H}$  which extends  $\pi$ . Moreover,  $\tilde{\pi}$  is unique, up to scalars: any other projective unitary representation  $\tilde{\pi}'$  of  $G_\pi$  extending  $\pi$  is the form  $\tilde{\pi}' = \lambda \tilde{\pi}$  for a measurable function  $\lambda : G_\pi \rightarrow \mathbf{S}^1$ .*

**Proof** For every  $g \in G_\pi$ , there exists a unitary operator  $\tilde{\pi}(g)$  on  $\mathcal{H}$  such that

$$\pi(g(n)) = \tilde{\pi}(g)\pi(n)\tilde{\pi}(g)^{-1} \quad \text{for all } n \in N.$$

One can choose  $\tilde{\pi}(g)$  such that  $g \mapsto \tilde{\pi}(g)$  is a projective unitary representation of  $G_\pi$  which extends  $\pi$  (see Theorem 8.2 in [Mack58]). The uniqueness of  $\pi$  follows from the irreducibility of  $\pi$  and Schur's lemma. ■

### 3. A WEAK CONTAINMENT RESULT FOR INDUCED REPRESENTATIONS

Let  $G$  be a locally compact group with Haar measure  $\mu_G$ . Recall that a unitary representation  $(\rho, \mathcal{K})$  of  $G$  is weakly contained in another unitary representation  $(\pi, \mathcal{H})$  of  $G$ , if every matrix coefficient  $g \mapsto \langle \rho(g)\eta | \eta \rangle$  of  $\rho$  (for  $\eta \in \mathcal{K}$ ) is the limit, uniformly on compact subsets of  $G$ , of a finite sum of matrix coefficients of  $\pi$ . Equivalently, if  $\|\rho(f)\| \leq \|\pi(f)\|$  for every  $f \in C_c(G)$ , where  $C_c(G)$  is the space of continuous functions with compact support on  $G$  and where the operator  $\pi(f) \in \mathcal{B}(\mathcal{H})$  is defined by the integral

$$\pi(f)\xi = \int_G f(g)\pi(g)d\mu_G(g) \quad \text{for all } \xi \in \mathcal{H}.$$

The trivial representation  $1_G$  is weakly contained in  $\pi$  if and only if there exists, for every compact subset  $Q$  of  $G$  and every  $\varepsilon > 0$ , there exists a unit vector  $\xi \in \mathcal{H}$  which is  $(Q, \varepsilon)$ -invariant, that is such that

$$\sup_{g \in Q} \|\pi(g)\xi - \xi\| \leq \varepsilon.$$

Let  $H$  be a closed subgroup of  $G$ . We will always assume that the coset space  $H \backslash G$  admits a non-zero  $G$ -invariant (possibly infinite) measure on its Borel subsets. Let  $(\sigma, \mathcal{K})$  be a unitary representation of  $H$ . We will use the following model for the induced representation  $\pi := \text{Ind}_H^G \sigma$ . Choose a Borel fundamental domain  $X \subset G$  for the action of  $G$  on  $H \backslash G$ . For  $x \in X$  and  $g \in G$ , let  $x \cdot g \in X$  and  $c(x, g) \in H$  be defined by

$$xg = c(x, g)(x \cdot g).$$

There exists a non-zero  $G$ -invariant measure on  $X$  for the action  $(x, g) \mapsto x \cdot g$  of  $G$  on  $X$ . The Hilbert space of  $\pi$  is the space  $L^2(X, \mathcal{K}, \mu)$  of all square-integrable measurable mappings  $\xi : X \rightarrow \mathcal{K}$  and the action of  $G$  on  $L^2(X, \mathcal{K}, \mu)$  is given by

$$(\pi(g)\xi)(x) = \sigma(c(x, g))(\xi(x \cdot g)), \quad g \in G, \xi \in L^2(X, \mathcal{K}, \mu), x \in X.$$

Observe that, in the case where  $\sigma$  is the trivial representation  $1_H$ , the induced representation  $\text{Ind}_H^G 1_H$  is equivalent to **quasi-regular representation**  $\lambda_{H \backslash G}$ , that is the natural representation of  $G$  on  $L^2(H \backslash G, \mu)$  given by right translations.

We will use several times the following elementary but crucial lemma, which can be viewed as a generalization of Herz's majoration principle (see Proposition 17 in [BeGu15]).

**Lemma 7.** *Let  $(H_i)_{i \in I}$  be a family of closed subgroups of  $G$  such that  $H_i \backslash G$  admits a non-zero  $G$ -invariant measure. Let  $(\sigma_i, \mathcal{K}_i)$  be a unitary representation of  $H_i$ . Assume that  $1_G$  is weakly contained in the direct sum  $\bigoplus_{i \in I} \text{Ind}_{H_i}^G \sigma_i$ . Then  $1_G$  is weakly contained in  $\bigoplus_{i \in I} \lambda_{H_i \backslash G}$ .*

**Proof** Let  $Q$  be a compact subset of  $G$  and  $\varepsilon$ . For every  $i \in I$ , let  $X_i \subset G$  be a Borel fundamental domain for the action of  $G$  on  $H_i \backslash G$  and  $\mu_i$  a non-zero  $G$ -invariant measure on  $X_i$ . There exists a family of vectors  $\xi_i \in L^2(X_i, \mathcal{K}_i, \mu_i)$  such that  $\sum_i \|\xi_i\|^2 = 1$  and

$$\sup_{g \in Q} \sum_i \|\text{Ind}_{H_i}^G \sigma_i(g) \xi_i - \xi_i\|^2 \leq \varepsilon,$$

Define  $\varphi_i$  in  $L^2(X_i, \mu_i)$  by  $\varphi_i(x) = \|\xi_i(x)\|$ . Then  $\sum_i \|\varphi_i\|^2 = 1$  and, for every  $g \in Q$ , we have

$$\begin{aligned} \|(\text{Ind}_{H_i}^G \sigma_i(g)) \xi_i - \xi_i\|^2 &= \int_{X_i} \|\sigma(c_i(x, g))(\xi_i(x \cdot_i g)) - \xi_i(x)\|^2 d\mu_i(x) \\ &\geq \int_{X_i} \left| \|\sigma(c_i(x, g))(\xi_i(x \cdot_i g))\| - \|\xi_i(x)\| \right|^2 d\mu_i(x) \\ &= \int_{X_i} \left| \|\xi_i(x \cdot_i g)\| - \|\xi_i(x)\| \right|^2 d\mu_i(x) \\ &= \int_{X_i} |\varphi_i(x \cdot_i g) - \varphi_i(x)|^2 d\mu_i(x) \\ &= \|\lambda_{H_i \backslash G}(g) \varphi_i - \varphi_i\|^2 \end{aligned}$$

and the claim follows. ■

#### 4. DECAY OF MATRIX COEFFICIENTS OF UNITARY REPRESENTATIONS

We recall a few general facts about decay of matrix coefficients of unitary representations, Recall that the projective kernel of a (genuine or projective) representation  $\pi$  of the locally compact group  $G$  is the closed normal subgroup  $P_\pi$  of  $G$  consisting of the elements  $g \in G$  such



that  $\pi(g)$  is a scalar multiple of the identity operator, that is such that  $\pi(g) = \lambda_\pi(g)I$  for some  $\lambda_\pi(g) \in \mathbf{S}^1$ .

Observe also that, for  $\xi, \eta \in \mathcal{H}$ , the absolute value of the matrix coefficient

$$C_{\xi, \eta}^\pi : g \mapsto \langle \pi(g)\xi, \eta \rangle$$

is constant on cosets modulo  $P_\pi$ . For a real number  $p$  with  $1 \leq p < +\infty$ , the representation  $\pi$  is said to be **strongly  $L^p$  modulo  $P_\pi$** , if there is dense subspace  $D \subset \mathcal{H}$  such that  $|C_{\xi, \eta}^\pi| \in L^p(G/P_\pi)$  for all  $\xi, \eta \in D$ .

**Proposition 8.** *Assume that the unitary representation  $\pi$  of the locally compact group  $G$  is strongly  $L^p$  modulo  $P_\pi$  for  $1 \leq p < +\infty$ . Let  $k$  be an integer  $k \geq p/2$ . Then the tensor power  $\pi^{\otimes k}$  is contained in an infinite multiple of  $\text{Ind}_{P_\pi}^G \lambda_\pi^k$ , where  $\lambda_\pi$  is the unitary character of  $P_\pi$  associated to  $\pi$ .*

**Proof** Observe  $\sigma := \pi^{\otimes k}$  is square-integrable modulo  $P_\pi$  for every integer  $k \geq p/2$ . It follows (see Proposition 4.2 in [HoMo79] or Proposition 1.2.3 in Chapter V of [HoTa92]) that  $\sigma$  is contained in an infinite multiple of  $\text{Ind}_{P_\sigma}^G \lambda_\sigma = \text{Ind}_{P_\pi}^G \lambda_\pi^k$ . ■

## 5. THE KOOPMAN REPRESENTATION OF THE AUTOMORPHISM GROUP OF A HOMOGENEOUS SPACE

We establish a decomposition result for the Koopman representation of a group of automorphisms of an  $S$ -adic compact nilmanifold. We will state the result in the general context of a compact homogeneous space.

Let  $G$  be a locally compact group and  $\Lambda$  a lattice in  $G$ . We assume that  $\Lambda$  is cocompact in  $G$ . The homogeneous space  $X := G/\Lambda$  carries a probability measure  $\mu$  on the Borel subsets  $X$  which is invariant by translations with elements from  $G$ . Every element from

$$\text{Aut}(X) := \{\gamma \in \text{Aut}(G) \mid \gamma(\Lambda) \subset \Lambda\}$$

induces a Borel isomorphism of  $X$ , which leaves  $\nu$  invariant, as follows from the uniqueness of  $\nu$ .

Given a subgroup  $\Gamma$  of  $\text{Aut}(X)$ , the following crucial proposition gives a decomposition of the associated Koopman  $\Gamma$  on  $L^2(X, \mu)$  as direct sum of certain induced representations of  $\Gamma$ .

**Proposition 9.** *Let  $G$  be a locally compact group and  $\Lambda$  a cocompact lattice in  $G$ , and let  $\Gamma$  be a countable subgroup of  $\text{Aut}(X)$  for  $X := G/\Lambda$ . Let  $\kappa$  be the Koopman representation of  $\Gamma$  associated to the action  $\Gamma \curvearrowright X$ . There exists a family  $(\pi_i)_{i \in I}$  of irreducible unitary representations*

of  $G$  such that  $\kappa$  is equivalent to a direct sum

$$\bigoplus_{i \in I} \text{Ind}_{\Gamma_i}^{\Gamma}(\tilde{\pi}_i|_{\Gamma_i} \otimes W_i),$$

where  $\tilde{\pi}_i$  is an irreducible projective representation of the stabilizer  $G_i$  of  $\pi_i$  in  $\text{Aut}(G) \rtimes G$  extending  $\pi_i$ , and where  $W_i$  is a finite dimensional projective unitary representation of  $\Gamma_i := \Gamma \cap G_i$ .

**Proof** We extend  $\kappa$  to a unitary representation, again denoted by  $\kappa$ , of  $\Gamma \rtimes G$  on  $L^2(X, \mu)$  given by

$$\kappa(\gamma, g)\xi(x) = \xi(\gamma^{-1}(gx)) \quad \text{for all } \gamma \in \Gamma, g \in G, \xi \in L^2(X, \mu), x \in X.$$

Identifying  $\Gamma$  and  $G$  with subgroups of  $\Gamma \rtimes G$ , we have

$$(*) \quad \kappa(\gamma^{-1})\kappa(g)\kappa(\gamma) = \kappa(\gamma^{-1}(g)) \quad \text{for all } \gamma \in \Gamma, n \in N.$$

Since  $\Lambda$  is cocompact in  $G$ , we can consider the decomposition of  $L^2(X, \mu)$  into  $G$ -isotypical components: we have (see Theorem in Chap. I, §3 of [GGPS69])

$$L^2(X, \mu) = \bigoplus_{\pi \in \Sigma} \mathcal{H}_{\pi},$$

where  $\Sigma$  is a certain set of pairwise non-equivalent irreducible unitary representations of  $G$ ; for every  $\pi \in \Sigma$ , the space  $\mathcal{H}_{\pi}$  is the union of the closed  $\kappa(G)$ -invariant subspaces  $\mathcal{K}$  of  $\mathcal{H}$  for which the corresponding representation of  $G$  in  $\mathcal{K}$  is equivalent to  $\pi$ ; moreover, the multiplicity of every  $\pi$  is finite, that is, every  $\mathcal{H}_{\pi}$  is a direct sum of finitely many irreducible unitary representations of  $G$ .

Let  $\gamma$  be a fixed automorphism in  $\Gamma$ . Let  $\kappa^{\gamma}$  be the conjugate representation of  $\kappa$  by  $\gamma$ , that is,  $\kappa^{\gamma}(g) = \kappa(\gamma g \gamma^{-1})$  for all  $g \in \Gamma \rtimes N$ . On the one hand, for every  $\pi \in \Sigma$ , the isotypical component of  $\kappa^{\gamma}|_G$  corresponding to  $\pi$  is  $\mathcal{H}_{\pi^{\gamma^{-1}}}$ . On the other hand, relation  $(*)$  shows that  $\kappa(\gamma)$  is a unitary equivalence between  $\kappa|_G$  and  $\kappa^{\gamma}|_G$ . It follows that

$$\kappa(\gamma)(\mathcal{H}_{\pi}) = \mathcal{H}_{\pi^{\gamma}} \quad \text{for all } \gamma \in \Gamma;$$

so,  $\Gamma$  permutes the  $\mathcal{H}_{\pi}$ 's among themselves according to its action on  $\widehat{G}$ .

Write  $\Sigma = \bigcup_{i \in I} \Sigma_i$ , where the  $\Sigma_i$ 's are the  $\Gamma$ -orbits in  $\Sigma$ , and set

$$\mathcal{H}_{\Sigma_i} = \bigoplus_{\pi \in \Sigma_i} \mathcal{H}_{\pi}.$$

Every  $\mathcal{H}_{\Sigma_i}$  is invariant under  $\Gamma \rtimes G$  and we have an orthogonal decomposition

$$\mathcal{H} = \bigoplus_i \mathcal{H}_{\Sigma_i}.$$

Fix  $i \in I$ . Choose a representation  $\pi_i$  in  $\Sigma_i$  and set  $\mathcal{H}_i = \mathcal{H}_{\pi_i}$ . Let  $\Gamma_i$  denote the stabilizer of  $\pi_i$  in  $\Gamma$ . The space  $\mathcal{H}_i$  is invariant under  $\Gamma_i$ . Let  $V_i$  be the corresponding representation of  $\Gamma_i$  on  $\mathcal{H}_i$ .

Choose a set  $S_i$  of representatives for the cosets in

$$\Gamma/\Gamma_i = (\Gamma \times G)/(\Gamma_i \times G)$$

with  $e \in S_i$ . Then  $\Sigma_i = \{\pi_i^s : s \in S_i\}$  and the Hilbert space  $\mathcal{H}_{\Sigma_i}$  is the sum of mutually orthogonal spaces:

$$\mathcal{H}_{\Sigma_i} = \bigoplus_{s \in S_i} \mathcal{H}_i^s.$$

Moreover,  $\mathcal{H}_i^s$  is the image under  $\kappa(s)$  of  $\mathcal{H}_i$  for every  $s \in S_i$ . This means that the restriction  $\kappa_i$  of  $\kappa$  to  $\mathcal{H}_{\Sigma_i}$  of the Koopman representation  $\kappa$  of  $\Gamma$  is equivalent to the induced representation  $\text{Ind}_{\Gamma_i}^{\Gamma} V_i$ .

Since every  $\mathcal{H}_i$  is a direct sum of finitely many irreducible unitary representations of  $G$ , we can assume that  $\mathcal{H}_i$  is the tensor product

$$\mathcal{H}_i = \mathcal{K}_i \otimes \mathcal{L}_i$$

of the Hilbert space  $\mathcal{K}_i$  of  $\pi_i$  with a finite dimensional Hilbert space  $\mathcal{L}_i$ , in such a way that

$$(**) \quad V_i(g) = \pi_i(g) \otimes I_{\mathcal{L}_i} \quad \text{for all } g \in G.$$

Let  $\gamma \in \Gamma_i$ . By (\*) and (\*\*) above, we have

$$(***) \quad V_i(\gamma) (\pi_i(g) \otimes I_{\mathcal{L}_i}) V_i(\gamma)^{-1} = \pi_i(\gamma g \gamma^{-1}) \otimes I_{\mathcal{L}_i} \quad \text{for all } g \in G.$$

On the other hand, let  $G_i$  be the stabilizer of  $\pi_i$  in  $\text{Aut}(G) \times G$ ; then  $\pi_i$  extends to an irreducible projective representation  $\tilde{\pi}_i$  of  $G_i$  (see Section 2). Since

$$\tilde{\pi}_i(\gamma) \pi_i(g) \tilde{\pi}_i(\gamma^{-1}) = \pi_i(\gamma g \gamma^{-1}) \quad \text{for all } g \in G,$$

it follows from (\*\*\*) that  $(\tilde{\pi}_i(\gamma^{-1}) \otimes I_{\mathcal{L}_i}) V_i(\gamma)$  commutes with  $\pi_i(g) \otimes I_{\mathcal{L}_i}$  for all  $g \in \tilde{G}_i$ . As  $\pi_i$  is irreducible, there exists a unitary operator  $W_i(\gamma)$  on  $\mathcal{L}_i$  such that

$$V_i(\gamma) = \tilde{\pi}_i(\gamma) \otimes W_i(\gamma).$$

It is clear that  $W_i$  is a projective unitary representation of  $\Gamma_i \times G$ , since  $V_i$  is a unitary representation of  $\Gamma_i \times G$ . ■

## 6. UNITARY DUAL OF SOLENOIDS

Let  $p$  be either a prime integer or  $p = \infty$ . Define an element  $e_p$  in the unitary dual group  $\widehat{\mathbf{Q}}_p$  of the additive group of  $\mathbf{Q}_p$  by  $e_p(x) = e^{2\pi i x}$  if  $p = \infty$  and  $e_p(x) = \exp(2\pi i \{x\})$ , where  $\{x\} = \sum_{j=m}^{-1} a_j p^j$  denotes the “fractional part” of a  $p$ -adic number  $x = \sum_{j=m}^{\infty} a_j p^j$  for integers  $m \in \mathbf{Z}$  and  $a_j \in \{0, \dots, p-1\}$ . Observe that  $\text{Ker}(e_p) = \mathbf{Z}$  in case  $p = \infty$  and that  $\text{Ker}(e_p) = \mathbf{Z}_p$  in case  $p$  is a prime integer, where  $\mathbf{Z}_p$  is the ring of  $p$ -adic integers. The map

$$\mathbf{Q}_p \rightarrow \widehat{\mathbf{Q}}_p, \quad y \mapsto (x \mapsto e_p(xy))$$

is an isomorphism of topological groups (see [BeHV08, Section D.4]).

Fix an integer  $d \geq 1$ . Then  $\widehat{\mathbf{Q}}_p^d$  will be identified with  $\mathbf{Q}_p^d$  by means of the map

$$\mathbf{Q}^d \rightarrow \widehat{\mathbf{Q}}_p^d, \quad y \mapsto x \mapsto e_p(x \cdot y),$$

where  $x \cdot y = \sum_{i=1}^d x_i y_i$  for  $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in \mathbf{Q}_p^d$ .

Let  $S = \{p_1, \dots, p_r, \infty\}$ , where  $p_1, \dots, p_r$  are integer primes. For an integer  $d \geq 1$ , consider the  $S$ -adic solenoid

$$\mathbf{Sol}_S = \mathbf{Q}_S^d / \mathbf{Z}[1/p_1, \dots, 1/p_r]^d,$$

where  $\mathbf{Z}[1/p_1, \dots, 1/p_r]^d$  is embedded diagonally in  $\mathbf{Q}_S = \prod_{p \in S} \mathbf{Q}_p$ .

Then  $\widehat{\mathbf{Sol}}_S$  is identified with the annihilator of  $\mathbf{Z}[1/p_1, \dots, 1/p_r]^d$  in  $\mathbf{Q}_S^d$ , that is, with  $\mathbf{Z}[1/p_1, \dots, 1/p_r]^d$  embedded in  $\mathbf{Q}_S^d$  via the map

$$\mathbf{Z}[1/p_1, \dots, 1/p_r]^d \rightarrow \mathbf{Q}_S^d, \quad b \mapsto (b, -b \cdots, -b).$$

Under this identification, the dual action of the automorphism group

$$\text{Aut}(\mathbf{Q}_S^d) \cong GL_d(\mathbf{R}) \times GL(\mathbf{Q}_{p_1}) \times \cdots \times GL(\mathbf{Q}_{p_r})$$

on  $\widehat{\mathbf{Q}}_S^d$  corresponds to the right action on  $\mathbf{R}^d \times \mathbf{Q}_{p_1}^d \times \cdots \times \mathbf{Q}_{p_r}^d$  given by

$$((g_\infty, g_1, \dots, g_r), (a_\infty, a_1, \dots, a_r)) \mapsto (g_\infty^t a_\infty, g_1^t a_1, \dots, g_r^t a_r),$$

where  $(g, a) \mapsto ga$  is the usual (left) linear action of  $GL_d(\mathbf{k})$  on  $\mathbf{k}^d$  for a field  $\mathbf{k}$ .

## 7. UNITARY REPRESENTATIONS OF UNIPOTENT GROUPS

Let  $\mathbf{U}$  be a linear algebraic unipotent group defined over  $\mathbf{Q}$ . The Lie algebra  $\mathfrak{u}$  is defined over  $\mathbf{Q}$  and the exponential map  $\exp : \mathfrak{u} \rightarrow U$  is a bijective morphism of algebraic varieties.

Let  $p$  be either a prime integer or  $p = \infty$ . The irreducible unitary representations of  $U_p := \mathbf{U}(\mathbf{Q}_p)$  are parametrized by Kirillov's theory as follows.

The Lie algebra of  $U_p$  is  $\mathfrak{u}_p = \mathfrak{u}(\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}_p$ , where  $\mathfrak{u}(\mathbf{Q})$  is the Lie algebra over  $\mathbf{Q}$  consisting of the  $\mathbf{Q}$ -points in  $\mathfrak{u}$ .

Fix an element  $f$  in the dual space  $\mathfrak{u}_p^* = \mathcal{H}om_{\mathbf{Q}_p}(\mathfrak{u}_p, \mathbf{Q}_p)$  of  $\mathfrak{u}_p$ . There exists a polarization  $\mathfrak{m}$  for  $f$ , that is, a Lie subalgebra  $\mathfrak{m}$  of  $\mathfrak{u}_p$  such that  $f([\mathfrak{m}, \mathfrak{m}]) = 0$  and which is of maximal dimension. The induced representation  $\text{Ind}_M^{U_p} \chi_f$  is irreducible, where  $M = \exp(\mathfrak{m})$  and  $\chi_f$  is the unitary character of  $M$  defined by

$$\chi_f(\exp X) = e_p(f(X)) \quad \text{for all } X \in \mathfrak{m},$$

where  $e_p \in \widehat{\mathbf{Q}_p}$  is as in Section 6. The unitary equivalence class of  $\text{Ind}_M^{U_p} \chi_f$  only depends on the co-adjoint orbit  $\text{Ad}^*(U_p)f$  of  $f$ . The map

$$\mathfrak{n}_p^*/\text{Ad}^*(U_p) \rightarrow \widehat{U}_p, \quad \mathcal{O} \mapsto \pi_{\mathcal{O}}$$

called the Kirillov map, from the orbit space  $\mathfrak{n}_p^*/\text{Ad}^*(U_p)$  of the co-adjoint representation to the unitary dual  $\widehat{U}_p$  of  $U_p$ , is a bijection. In particular,  $U_p$  is a so-called type I locally compact group. For all of this, see [Kiri62] or [CoGr89] in the case  $p = \infty$  and [Moor65] in the case of a prime integer  $p$ .

The group  $\text{Aut}(U_p)$  of continuous automorphisms of  $U$  can be identified with the group of  $\mathbf{Q}_p$ -points of the algebraic group  $\text{Aut}(\mathfrak{u})$  of automorphisms of the Lie algebra  $\mathfrak{u}$  of  $\mathbf{U}$ . Notice also that the natural action of  $\text{Aut}(U_p)$  on  $\mathfrak{u}_p$  as well as its dual action on  $\mathfrak{u}_p^*$  are algebraic.

Let  $\pi \in \widehat{U}_p$  with corresponding Kirillov orbit  $\mathcal{O}_\pi$  and  $g \in \text{Aut}(U_p)$ . Then  $g(\mathcal{O}_\pi)$  is the Kirillov orbit associated to the conjugate representation  $\pi^g$ .

**Lemma 10.** *Let  $\pi$  be an irreducible unitary representation of  $U_p$ . The stabilizer  $G_\pi$  of  $\pi$  in  $\text{Aut}(U_p)$  is an algebraic subgroup of  $\text{Aut}(U_p)$ .*

**Proof** Let  $\mathcal{O}_\pi \subset \mathfrak{u}_p^*$  be the Kirillov orbit corresponding to  $\pi$ . Then  $G_\pi$  is the set of  $g \in \text{Aut}(U_p)$  such that  $g(\mathcal{O}_\pi) = \mathcal{O}_\pi$ . As  $\mathcal{O}_\pi$  is an algebraic subvariety of  $\mathfrak{u}_p^*$ , the claim follows. ■

## 8. DECAY OF MATRIX COEFFICIENTS OF UNITARY REPRESENTATIONS OF $S$ -ADIC GROUPS

Let  $p$  be an integer prime or  $p = \infty$  and let  $\mathbf{U}$  be a linear algebraic unipotent group defined over  $\mathbf{Q}_p$ . Set  $U_p := \mathbf{U}(\mathbf{Q}_p)$ .

Let  $\pi$  be an irreducible unitary representation of  $U_p$ . Recall (see Lemma 10) that the stabilizer  $G_\pi$  of  $\pi$  in  $\text{Aut}(U_p)$  is an algebraic

subgroup of  $\text{Aut}(U_p)$ . Recall also (see Lemma 6) that  $\pi$  extends to a projective representation of  $G_\pi$ . The following result was proved in Proposition 22 of [BeGu15] in the case where  $p = \infty$ , using arguments from [HoMo79]. The proof in the case where  $p$  is a prime integer is along similar lines and will be omitted.

**Proposition 11.** *Let  $\pi$  be an irreducible unitary representation of  $U_p$  and let  $\tilde{\pi}$  be a projective unitary representation of  $G_\pi$  which extends  $\pi$ . There exists a real number  $r \geq 1$ , only depending on the dimension of  $G_\pi$ , such that  $\tilde{\pi}$  is strongly  $L^r$  modulo its projective kernel.*

We will need later a precise description of the projective kernel of a representation  $\tilde{\pi}$  as above.

**Lemma 12.** *Let  $\pi$  be an irreducible unitary representation of  $U_p$  and  $\tilde{\pi}$  a projective unitary representation of  $G_\pi$  which extends  $\pi$ . Let  $\mathcal{O}_\pi \subset \mathfrak{u}_p^*$  be the corresponding Kirillov orbit of  $\pi$ . For  $g \in \text{Aut}(U_p)$ , the following properties are equivalent:*

- (i)  $g$  belongs to the projective kernel  $P_{\tilde{\pi}}$  of  $\tilde{\pi}$ ;
- (ii) for every  $u \in U_p$ , we have

$$g(u)u^{-1} \in \bigcap_{f \in \mathcal{O}_\pi} \exp(\text{Ker}(f)).$$

**Proof** We can assume that  $\pi = \text{Ind}_M^{U_p} \chi_{f_0}$ , for  $f_0 \in \mathcal{O}_\pi$ , and  $M = \exp \mathfrak{m}$  for a polarization  $\mathfrak{m}$  of  $f_0$ . For  $g \in \text{Aut}(U_p)$ , we have  $g \in P_{\tilde{\pi}}$  if and only if

$$\pi(g(u)) = \pi(u) \quad \text{for all } u \in U_p$$

that is,

$$g(u)u^{-1} \in \text{Ker}(\pi) \quad \text{for all } u \in U_p.$$

Now, we have (see [BeGu15, Lemma 18])

$$\text{Ker}(\pi) = \bigcap_{f \in \mathcal{O}_\pi} \text{Ker}(\chi_f)$$

and so,  $g \in P_{\tilde{\pi}}$  if and only if

$$g(u)u^{-1} \in \bigcap_{f \in \mathcal{O}_\pi} \text{Ker}(\chi_f) \quad \text{for all } u \in U_p.$$

Let  $g \in P_{\tilde{\pi}}$ . Denote by  $X \mapsto g(X)$  the automorphism of  $\mathfrak{u}_p$  corresponding to  $g$ . Let  $u = \exp(X)$  for  $X \in \mathfrak{u}_p$  and  $f \in \mathcal{O}_\pi$ . Set  $u_t = \exp(tX)$ . By the Campbell Hausdorff formula, there exists  $Y_1, \dots, Y_r \in \mathfrak{u}_p$  such that

$$g(u_t)(u_t)^{-1} = \exp(tY_1 + t^2Y_2 + \dots + t^rY_r),$$

for every  $t \in \mathbf{Q}_p$ ; Since

$$(*) \quad 1 = \chi_f(g(u_t)(u_t)^{-1}) = e_p(f(tY_1 + t^2Y_2 + \cdots + t^rY_r)),$$

it follows that the polynomial

$$t \mapsto Q(t) = tf(Y_1) + t^2f(Y_2) + \cdots + t^rf(Y_r)$$

takes its values in  $\mathbf{Z}$  in case  $p = \infty$  and in  $\mathbf{Z}_p$  (and so  $Q$  has bounded image) otherwise. This clearly implies that  $Q(t) = 0$  for all  $t \in \mathbf{Q}_p$ ; in particular, we have

$$\log(g(u)u^{-1}) = Y_1 + Y_2 + \cdots + Y_r \in \text{Ker}(f).$$

This shows that (i) implies (ii).

Conversely, assume that (ii) holds. Then clearly

$$g(u)u^{-1} \in \bigcap_{f \in \mathcal{O}_\pi} \text{Ker}(\chi_f) \quad \text{for all } u \in U_p$$

and so  $g \in P_{\bar{\pi}}$ . ■

## 9. DECOMPOSITION OF THE KOOPMAN REPRESENTATION FOR A NILMANIFOLD

Let  $\mathbf{U}$  be a linear algebraic unipotent group defined over  $\mathbf{Q}$ . Let  $S = \{p_1, \dots, p_r, \infty\}$ , where  $p_1, \dots, p_r$  are integer primes. Set

$$U := \mathbf{U}(\mathbf{Q}_S) = \prod_{s \in S} U_p.$$

Since  $U$  is a type I group, the unitary dual  $\widehat{U}$  of  $U$  can be identified with the cartesian product  $\prod_{s \in S} \widehat{U}_p$  via the map

$$\prod_{s \in S} \widehat{U}_p \rightarrow \widehat{U}, \quad (\pi_p)_{p \in S} \mapsto \otimes_{s \in S} \pi_p,$$

where  $\otimes_{s \in S} \pi_p = \pi_\infty \otimes \pi_{p_1} \otimes \cdots \otimes \pi_{p_r}$  is the tensor product of the  $\pi_p$ 's.

Let  $\Lambda := \mathbf{U}(\mathbf{Z}[1/S])$  and consider the corresponding ***S*-adic compact nilmanifold**

$$\mathbf{Nil}_S := U/\Lambda,$$

equipped with the unique  $U$ -invariant probability measure  $\mu$  on its Borel subsets.

The associated ***S*-adic solenoid** is

$$\mathbf{Sol}_S = \overline{U}/\overline{\Lambda},$$

where  $\overline{U} := U/[U, U]$  is the quotient of  $U$  by its closed commutator subgroup  $[U, U]$  and where  $\overline{\Lambda}$  is the image of  $\mathbf{U}(\mathbf{Z}[1/S])$  in  $\overline{U}$ .

Set

$$\mathrm{Aut}(U) := \prod_{s \in S} \mathrm{Aut}(\mathbf{U}(\mathbf{Q}_s))$$

and denote by  $\mathrm{Aut}(\mathbf{Nil}_S)$  the subgroup of all  $g \in \mathrm{Aut}(U)$  with  $g(\Lambda) = \Lambda$ .

Let  $\Gamma$  be a subgroup of  $\mathrm{Aut}(\mathbf{Nil}_S)$ . Let  $\kappa$  be the Koopman representation of  $\Gamma \times U$  on  $L^2(\mathbf{Nil}_S)$  associated to the action  $\Gamma \times U \curvearrowright \mathbf{Nil}_S$ . By Proposition 9, there exists a family  $(\pi_i)_{i \in I}$  of irreducible representations of  $U$ , such that  $\kappa$  is equivalent to

$$\bigoplus_{i \in I} \mathrm{Ind}_{\Gamma_i \times U}^{\Gamma \times U} (\tilde{\pi}_i \otimes W_i),$$

where  $\tilde{\pi}_i$  is an irreducible projective representation  $\tilde{\pi}_i$  of the stabilizer  $G_i$  of  $\pi_i$  in  $\mathrm{Aut}(U) \times U$  extending  $\pi_i$ , and where  $W_i$  is a projective unitary representation of  $G_i \cap (\Gamma \times U)$ .

Fix  $i \in I$ . We have  $\pi_i = \otimes_{p \in S} \pi_{i,p}$  for irreducible representations  $\pi_{i,p}$  of  $U_p$ .

We will need the following more precise description of  $\pi_i$ . Recall that  $\mathfrak{u}$  is the Lie algebra of  $\mathbf{U}$  and that  $\mathfrak{u}(\mathbf{Q})$  denotes the Lie algebra over  $\mathbf{Q}$  consisting of the  $\mathbf{Q}$ -points in  $\mathfrak{u}$ . Let  $\mathfrak{u}^*(\mathbf{Q})$  be the set of  $\mathbf{Q}$ -rational points in the dual space  $\mathfrak{u}^*$ ; so,  $\mathfrak{u}^*(\mathbf{Q})$  is the subspace of  $f \in \mathfrak{u}^*$  with  $f(X) \in \mathbf{Q}$  for all  $X \in \mathfrak{u}(\mathbf{Q})$ . Observe that, for  $f \in \mathfrak{u}^*(\mathbf{Q})$ , we have  $f(X) \in \mathbf{Q}_p$  for all  $X \in \mathfrak{u}_p = \mathfrak{u}(\mathbf{Q}_p)$ .

A polarization for  $f \in \mathfrak{u}^*(\mathbf{Q})$  is a Lie subalgebra  $\mathfrak{m}$  of  $\mathfrak{u}(\mathbf{Q})$  such that  $f([\mathfrak{m}, \mathfrak{m}]) = 0$  and which is of maximal dimension with this property.

**Proposition 13.** *Let  $\pi_i = \otimes_{p \in S} \pi_{i,p}$  be one of the irreducible representations of  $U = \mathbf{U}(\mathbf{Q}_S)$  appearing in the decomposition  $L^2(\mathbf{Nil}_S)$  as above. There exist  $f_i \in \mathfrak{u}^*(\mathbf{Q})$  and a polarization  $\mathfrak{m}_i \subset \mathfrak{u}(\mathbf{Q})$  for  $f_i$  with the following property: for every  $p \in S$ , the representation  $\pi_{i,p}$  is equivalent to  $\mathrm{Ind}_{M_{i,p}}^U \chi_{f_i}$ , where  $M_{i,p} = \exp(\mathfrak{m}_{i,p})$  and  $\chi_{f_i}$  is the unitary character of  $M_{i,p}$  given by*

$$\chi_{f_i}(\exp X) = e_p(f_i(X)), \quad \text{for all } X \in \mathfrak{m}_{i,p} = \mathfrak{m}_i \otimes_{\mathbf{Q}} \mathbf{Q}_p,$$

where  $e_p \in \widehat{\mathbf{Q}_p}$  is as in Section 6.

**Proof** The same result is proved in Theorem 11 in [Moor65] (see also Theorem 1.2 in [Fox89]) for the Koopman representation of  $\mathbf{U}(\mathbf{A})$  in  $L^2(\mathbf{U}(\mathbf{A})/\mathbf{U}(\mathbf{Q}))$ , where  $\mathbf{A}$  is the ring of adèles of  $\mathbf{Q}$ . We could check that the proof, which proceeds by induction of the dimension of  $\mathbf{U}$ , carries over to the Koopman representation on  $L^2(\mathbf{U}(\mathbf{Q}_S)/\mathbf{U}(\mathbf{Z}[1/S]))$ , with the appropriate changes. We prefer to deduce our claim from the result for  $\mathbf{U}(\mathbf{A})$ , as follows.



It is well-known (see [Weil74]) that

$$\mathbf{A} = \left( \mathbf{Q}_S \times \prod_{p \notin S} \mathbf{Z}_p \right) + \mathbf{Q}$$

and that

$$\left( \mathbf{Q}_S \times \prod_{p \notin S} \mathbf{Z}_p \right) \cap \mathbf{Q} = \mathbf{Z}[1/S].$$

This gives rise to a well defined projection  $\varphi : \mathbf{A}/\mathbf{Q} \rightarrow \mathbf{Q}_S/\mathbf{Z}[1/S]$  given by

$$\varphi((a_S, (a_p)_{p \notin S}) + \mathbf{Q}) = a_S + \mathbf{Z}[1/S] \quad \text{for all } a_S \in \mathbf{Q}_S, (a_p)_{p \notin S} \in \prod_{p \notin S} \mathbf{Z}_p;$$

so the fiber over a point  $a_S + \mathbf{Z}[1/S] \in \mathbf{Q}_S/\mathbf{Z}[1/S]$  is

$$\varphi^{-1}(a_S + \mathbf{Z}[1/S]) = \{(a_S, (a_p)_{p \notin S}) + \mathbf{Q} \mid a_p \in \mathbf{Z}_p \text{ for all } p\}.$$

This induces an identification of  $\mathbf{U}(\mathbf{Q}_S)/\mathbf{U}(\mathbf{Z}[1/S]) = \mathbf{Nil}_S$  with the double coset space  $K_S \backslash \mathbf{U}(\mathbf{A})/\mathbf{U}(\mathbf{Q})$ , where  $K_S$  is the compact subgroup

$$K_S = \prod_{p \notin S} \mathbf{U}(\mathbf{Z}_p)$$

of  $\mathbf{U}(\mathbf{A})$ . Observe that this identification is equivariant under translation by elements from  $\mathbf{U}(\mathbf{Q}_S)$ . In this way, we can view  $L^2(\mathbf{Nil}_S)$  as the  $\mathbf{U}(\mathbf{Q}_S)$ -invariant subspace  $L^2(K_S \backslash \mathbf{U}(\mathbf{A})/\mathbf{U}(\mathbf{Q}))$  of  $L^2(\mathbf{U}(\mathbf{A})/\mathbf{U}(\mathbf{Q}))$ .

Choose a system  $T$  of representatives for the  $\text{Ad}^*(\mathbf{U}(\mathbf{Q}))$ -orbits in  $\mathfrak{u}^*(\mathbf{Q})$ . By [Moor65, Theorem 11], for every  $f \in T$ , we can find a polarization  $\mathfrak{m}_f \subset \mathfrak{u}(\mathbf{Q})$  for  $f$  with the following property: setting

$$\mathfrak{m}_f(\mathbf{A}) = \mathfrak{m}_f \otimes_{\mathbf{Q}} \mathbf{A},$$

we have a decomposition

$$L^2(\mathbf{U}(\mathbf{A})/\mathbf{U}(\mathbf{Q})) = \bigoplus_{f \in T} \mathcal{H}_f$$

into irreducible  $\mathbf{U}(\mathbf{A})$ -invariant subspaces  $\mathcal{H}_f$  such that the representation  $\pi_f$  of  $\mathbf{U}(\mathbf{A})$  in  $\mathcal{H}_f$  is equivalent to  $\text{Ind}_{M_f(\mathbf{A})}^{\mathbf{U}(\mathbf{A})} \chi_f$ , where

$$M_f(\mathbf{A}) = \exp(\mathfrak{m}_f(\mathbf{A}))$$

and  $\chi_{f,\mathbf{A}}$  is the unitary character of  $M_f(\mathbf{A})$  given by

$$\chi_{f,\mathbf{A}}(\exp X) = e(f(X)), \quad \text{for all } X \in \mathfrak{m}_f(\mathbf{A});$$

here,  $e$  is the unitary character of  $\mathbf{A}$  defined by

$$e((a_p)_p) = \prod_{p \in \mathcal{P} \cup \{\infty\}} e_p(a_p) \quad \text{for all } (a_p)_p \in \mathbf{A},$$

where  $\mathcal{P}$  is the set of integer primes.

We have

$$L^2(K_S \backslash \mathbf{U}(\mathbf{A}) / \mathbf{U}(\mathbf{Q})) = \bigoplus_{f \in T} \mathcal{H}_f^{K_S},$$

where  $\mathcal{H}_f^{K_S}$  is the space of  $K_S$ -fixed vectors in  $\mathcal{H}_f$ . It is clear that the representation of  $\mathbf{U}(\mathbf{Q}_S)$  in  $\mathcal{H}_f^{K_S}$  is equivalent to

$$\text{Ind}_{M_f(\mathbf{Q}_S)}^{\mathbf{U}(\mathbf{Q}_S)} (\otimes_{p \in S} \chi_{f,p}) = \otimes_{p \in S} \left( \text{Ind}_{M_f(\mathbf{Q}_p)}^{\mathbf{U}(\mathbf{Q}_p)} \chi_{f,p} \right),$$

where  $\chi_{f,p}$  is the unitary character of  $M_f(\mathbf{Q}_p)$  given by

$$\chi_{f,p}(\exp X) = e_p(f(X)), \quad \text{for all } X \in \mathfrak{m}_f(\mathbf{Q}_p).$$

Since  $M_f(\mathbf{Q}_p)$  is a polarization for  $f$ , each of the  $\mathbf{U}(\mathbf{Q}_p)$ -representations  $\text{Ind}_{M_f(\mathbf{Q}_p)}^{\mathbf{U}(\mathbf{Q}_p)} \chi_{f,p}$  and, hence, each of the  $\mathbf{U}(\mathbf{Q}_S)$ -representations

$$\text{Ind}_{M_f(\mathbf{Q}_S)}^{\mathbf{U}(\mathbf{Q}_S)} (\otimes_{p \in S} \chi_{f,p})$$

is irreducible. This proves the claim.  $\blacksquare$

We establish another crucial fact about the representations  $\pi_i$ 's in the following proposition.

**Proposition 14.** *With the notation of Proposition 13, let  $\mathcal{O}_{\mathbf{Q}}(f_i)$  be the co-adjoint orbit of  $f_i$  under  $\mathbf{U}(\mathbf{Q})$  and set*

$$\mathfrak{k}_{i,p} = \bigcap_{f \in \mathcal{O}_{\mathbf{Q}}(f_i)} \mathfrak{k}_p(f),$$

where  $\mathfrak{k}_p(f)$  is the kernel of  $f$  in  $\mathfrak{u}_p$ . Let  $K_{i,p} = \exp(\mathfrak{k}_{i,p})$  and  $K_i = \prod_{p \in S} K_{i,p}$ .

- (i)  $K_i$  is a closed normal subgroup of  $U$  and  $K_i \cap \Lambda = K_i \cap \mathbf{U}(\mathbf{Z}[1/S])$  is a lattice in  $K_i$ .
- (ii) Let  $P_{\tilde{\pi}_i}$  be the projective kernel of the extension  $\tilde{\pi}_i$  of  $\pi_i$  to the stabilizer  $G_i$  of  $\pi$  in  $\text{Aut}(U) \ltimes U$ . For  $g \in G_i$ , we have  $g \in P_{\tilde{\pi}_i}$  if and only if  $g(u) \in uK_i$  for every  $u \in U$ .

**Proof** (i) Let

$$\mathfrak{k}_{i,\mathbf{Q}} = \bigcap_{f \in \mathcal{O}_{\mathbf{Q}}(f_i)} \mathfrak{k}_{\mathbf{Q}}(f),$$

where  $\mathfrak{k}_{\mathbf{Q}}(f)$  is the kernel of  $f$  in  $\mathfrak{u}(\mathbf{Q})$ . Observe that  $\mathfrak{k}_{i,\mathbf{Q}}$  is an ideal in  $\mathfrak{u}(\mathbf{Q})$ , since it is  $\text{Ad}(\mathbf{U}(\mathbf{Q}))$ -invariant. So, we have

$$\mathfrak{k}_{i,\mathbf{Q}} = \mathfrak{k}_i(\mathbf{Q})$$

for an ideal  $\mathfrak{k}_i$  in  $\mathfrak{u}$ . Since  $f \in \mathfrak{u}^*(\mathbf{Q})$  for  $f \in \mathcal{O}_{\mathbf{Q}}(f_i)$ , we have

$$\mathfrak{k}_{i,p}(f) = \mathfrak{k}_{i,\mathbf{Q}}(f) \otimes_{\mathbf{Q}} \mathbf{Q}_p$$

and hence

$$\mathfrak{k}_{i,p} = \mathfrak{k}_i(\mathbf{Q}_p).$$

Let  $\mathbf{K}_i = \log(\mathfrak{k}_i)$ . Then  $\mathbf{K}_i$  is a normal algebraic  $\mathbf{Q}$ -subgroup of  $\mathbf{U}$  and we have  $K_{i,p} = \mathbf{K}_i(\mathbf{Q}_p)$  for every  $p$ ; so,

$$K_i = \prod_{s \in S} \mathbf{K}_i(\mathbf{Q}_p) = \mathbf{K}_i(\mathbf{Q}_S)$$

and  $K_i \cap \Lambda = \mathbf{K}_i(\mathbf{Z}[1/S])$  is a lattice in  $K_i$ . This proves Item (i).

To prove Item (ii), observe that

$$P_{\widetilde{\pi}_i} = \bigcap_{p \in S} P_{i,p},$$

where  $P_{i,p}$  is the projective kernel of  $\widetilde{\pi}_{i,p}$ .

Fix  $p \in S$  and let  $g \in G_i$ . By Lemma 12,  $g \in P_{i,p}$  if and only if  $g(u) \in uK_{i,p}$  for every  $u \in U_p = \mathbf{U}(\mathbf{Q}_p)$ . This finishes the proof. ■

## 10. PROOF OF THEOREM 1

Let  $\mathbf{U}$  be a linear algebraic unipotent group defined over  $\mathbf{Q}$  and  $S = \{p_1, \dots, p_r, \infty\}$ , where  $p_1, \dots, p_r$  are integer primes. Set  $U := \mathbf{U}(\mathbf{Q}_S)$  and  $\Lambda := \mathbf{U}(\mathbf{Z}[1/S])$ .

Let  $\mathbf{Nil}_S = U/\Lambda$  and  $\mathbf{Sol}_S$  be the  $S$ -adic nilmanifold and the associated  $S$ -adic solenoid as in Section 9. Denote by  $\mu$  the translation invariant probability measure on  $\mathbf{Nil}_S$  and let  $\nu$  be the image of  $\mu$  under the canonical projection  $\varphi : \mathbf{Nil}_S \rightarrow \mathbf{Sol}_S$ . We identify  $L^2(\mathbf{Sol}_S) = L^2(\mathbf{Sol}_S, \nu)$  with the closed  $\text{Aut}(\mathbf{Nil}_S)$ -invariant subspace

$$\{f \circ \varphi \mid f \in L^2(\mathbf{Sol}_S)\}$$

of  $L^2(\mathbf{Nil}_S) = L^2(\mathbf{Nil}_S, \mu)$ . We have an orthogonal decomposition into  $\text{Aut}(\mathbf{Nil}_S)$ -invariant subspaces

$$L^2(\mathbf{Nil}_S) = \mathbf{C}1_{\mathbf{Nil}_S} \oplus L_0^2(\mathbf{Sol}_S) \oplus \mathcal{H},$$

where

$$L_0^2(\mathbf{Sol}_S) = \{f \in L^2(\mathbf{Sol}_S) \mid \int_{\mathbf{Nil}_S} f d\mu = 0\}$$

and where  $\mathcal{H}$  is the orthogonal complement of  $L^2(\mathbf{Sol}_S)$  in  $L^2(\mathbf{Nil}_S)$ .

Let  $\Gamma$  be a subgroup of  $G := \text{Aut}(U)$ . Let  $\kappa$  be the Koopman representation of  $\Gamma$  on  $L^2(\mathbf{Nil}_S)$  and denote by  $\kappa_1$  and  $\kappa_2$  the restrictions of  $\kappa$  to respectively  $L_0^2(\mathbf{Sol}_S)$  and  $\mathcal{H}$ .

Let  $\Sigma_1$  be a set of representatives for the  $\Gamma$ -orbits in  $\widehat{\mathbf{Sol}}_S \setminus \{\mathbf{1}_{\mathbf{Sol}_S}\}$ . We have

$$\kappa_1 \cong \bigoplus_{\chi \in \Sigma_1} \lambda_{\Gamma/\Gamma_\chi},$$

where  $\Gamma_\chi$  is the stabilizer of  $\chi$  in  $\Gamma$  and  $\lambda_{\Gamma/\Gamma_\chi}$  is the quasi-regular representation of  $\Gamma$  on  $\ell^2(\Gamma/\Gamma_\chi)$ .

By Proposition 9, there exists a family  $(\pi_i)_{i \in I}$  of irreducible representations of  $G$ , such that  $\kappa_2$  is equivalent to a direct sum

$$\bigoplus_{i \in I} \text{Ind}_{\Gamma_i}^\Gamma(\tilde{\pi}_i|_{\Gamma_i} \otimes W_i),$$

where  $\tilde{\pi}_i$  is an irreducible projective representation of the stabilizer  $G_i$  of  $\pi_i$  in  $\text{Aut}(G) \times G$  extending  $\pi_i$ , and where  $W_i$  is a projective unitary representation of  $\Gamma_i := \Gamma \cap G_i$ .

**Proposition 15.** *For  $i \in I$ , let  $\tilde{\pi}_i$  be the (projective) representation of  $G_i$  and let  $\Gamma_i$  be as above. There exists a real number  $r \geq 1$  such that  $\tilde{\pi}_i|_{\Gamma_i}$  is strongly  $L^r$  modulo  $P_{\tilde{\pi}_i} \cap \Gamma_i$ , where  $P_{\tilde{\pi}_i}$  is the projective kernel of  $\tilde{\pi}_i$ .*

**Proof** By Proposition 11, there exists a real number  $r \geq 1$  such that the representation  $\tilde{\pi}_i$  of the algebraic group  $G_i$  is strongly  $L^r$  modulo  $P_{\tilde{\pi}_i}$ . In order to show that  $\tilde{\pi}_i|_{\Gamma_i}$  is strongly  $L^r$  modulo  $P_{\tilde{\pi}_i} \cap \Gamma_i$ , it suffices to show that  $\Gamma_i P_{\tilde{\pi}_i}$  is closed in  $G_i$  (compare with the proof of Proposition 6.2 in [HoMo79]).

Let  $K_i$  be the the closed  $G_i$ -invariant normal subgroup  $K_i$  of  $U$  as described in Proposition 14. Then  $\bar{\Lambda} = K_i \Lambda / K_i$  is a lattice in the unipotent group  $\bar{U} = U / K_i$ . By Proposition 14.ii,  $P_{\tilde{\pi}_i}$  coincides with the kernel of the natural homomorphism  $\varphi : \text{Aut}(U) \rightarrow \text{Aut}(\bar{U})$ . Hence, we have

$$\Gamma_i P_{\tilde{\pi}_i} = \varphi^{-1}(\varphi(\Gamma_i)).$$

Now,  $\varphi(\Gamma_i)$  is a discrete (and hence closed) subgroup of  $\text{Aut}(\bar{U})$ , since  $\varphi(\Gamma_i)$  preserves  $\bar{\Lambda}$  (and so  $\varphi(\Gamma_i) \subset \text{Aut}(\bar{U}/\bar{\Lambda})$ ). It follows from the continuity of  $\varphi$  that  $\varphi^{-1}(\varphi(\Gamma_i))$  is closed in  $\text{Aut}(U)$ . ■

### Proof of Theorem 1

We have to show that, if  $1_\Gamma$  is weakly contained in  $\kappa_2$ , then  $1_\Gamma$  is weakly contained in  $\kappa_1$ . It suffices to show that, if  $1_\Gamma$  is weakly contained

in  $\kappa_2$ , then there exists a finite index subgroup  $H$  of  $\Gamma$  such that  $1_H$  is weakly contained in  $\kappa_1|_H$  (see Theorem 2 in [BeFr20]).

We proceed by induction on the integer

$$n(\Gamma) := \sum_{p \in S} \dim Z_{C_p}(\Gamma),$$

where  $Z_{C_p}(\Gamma)$  is the Zariski closure of the projection of  $\Gamma$  in  $GL_n(\mathbf{Q}_p)$ .

If  $n(\Gamma) = 0$ , then  $\Gamma$  is finite and there is nothing to prove.

Assume that  $n(\Gamma) \geq 1$  and that the claim above is proved for every countable subgroup  $H$  of  $\text{Aut}(\mathbf{Nil}_S)$  with  $n(H) < n(\Gamma)$ .

Let  $I_{\text{fin}} \subset I$  be the set of all  $i \in I$  such that  $\Gamma_i = G_i \cap \Gamma$  has finite index in  $\Gamma$  and set  $I_\infty = I \setminus I_{\text{fin}}$ . With  $V_i = (\tilde{\pi}_i|_{\Gamma_i} \otimes W_i)$ , set

$$\kappa_2^{\text{fin}} = \bigoplus_{i \in I_{\text{fin}}} \text{Ind}_{\Gamma_i}^\Gamma V_i \quad \text{and} \quad \kappa_2^\infty = \bigoplus_{i \in I_\infty} \text{Ind}_{\Gamma_i}^\Gamma V_i.$$

Two cases can occur.

- *First case:*  $1_\Gamma$  is weakly contained in  $\kappa_2^\infty$ .

Observe that  $n(\Gamma_i) < n(\Gamma)$  for  $i \in I_\infty$ . Indeed, otherwise  $Z_{C_p}(\Gamma_i)$  and  $Z_{C_p}(\Gamma)$  would have the same connected component  $C_p^0$  for every  $p \in S$ , since  $\Gamma_i \subset \Gamma$ . Then

$$C^0 := \bigcap_{p \in S} C_p^0$$

would stabilize  $\pi_i$  and  $\Gamma \cap C^0$  would therefore be contained in  $\Gamma_i$ . Since  $\Gamma \cap C^0$  has finite index in  $\Gamma$ , this would be a contradiction to the fact that  $\Gamma_i$  has infinite index in  $\Gamma$ .

By restriction,  $1_{\Gamma_i}$  is weakly contained in  $\kappa_2|_{\Gamma_i}$  for every  $i \in I$ . Hence, by the induction hypothesis,  $1_{\Gamma_i}$  is weakly contained in  $\kappa_1|_{\Gamma_i}$  for every  $i \in I_\infty$ . Now, on the one hand, we have

$$\kappa_1|_{\Gamma_i} \cong \bigoplus_{\chi \in T_i} \lambda_{\Gamma_i/\Gamma_\chi \cap \Gamma_i},$$

for a subset  $T_i$  of  $\widehat{\mathbf{Sol}}_S \setminus \{\mathbf{1}_{\mathbf{Sol}}\}$ . It follows that  $\text{Ind}_{\Gamma_i}^\Gamma 1_{\Gamma_i} = \lambda_{\Gamma/\Gamma_i}$  is weakly contained in

$$\bigoplus_{\chi \in T_i} \text{Ind}_{\Gamma_i}^\Gamma (\lambda_{\Gamma_i/\Gamma_\chi \cap \Gamma_i}) = \bigoplus_{\chi \in T_i} \lambda_{\Gamma/\Gamma_\chi \cap \Gamma_i},$$

for every  $i \in I_\infty$ . On the other hand, since  $1_\Gamma$  is weakly contained in

$$\kappa_2 \cong \bigoplus_{i \in I_\infty} \text{Ind}_{\Gamma_i}^\Gamma (\tilde{\pi}_i|_{\Gamma_i} \otimes W_i),$$

Lemma 7 shows that  $1_\Gamma$  is weakly contained in  $\bigoplus_{i \in I_\infty} \lambda_{\Gamma/\Gamma_i}$ . It follows that  $1_\Gamma$  is weakly contained in

$$\bigoplus_{i \in I_\infty} \bigoplus_{\chi \in T_i} \lambda_{\Gamma/\Gamma_\chi \cap \Gamma_i}.$$

Hence, by Lemma 7 again,  $1_\Gamma$  is weakly contained in

$$\bigoplus_{i \in I_\infty} \bigoplus_{\chi \in T_i} \lambda_{\Gamma/\Gamma_\chi}.$$

This shows that  $1_\Gamma$  is weakly contained in  $\kappa_1$ .

• *Second case:*  $1_\Gamma$  is weakly contained in  $\kappa_2^{\text{fin}}$ .

By the Noetherian property of the Zariski topology, we can find finitely many indices  $i_1, \dots, i_r$  in  $I_{\text{fin}}$  such that, for every  $p \in S$ , we have

$$Z_{c_p}(\Gamma_{i_1}) \cap \dots \cap Z_{c_p}(\Gamma_{i_r}) = \bigcap_{i \in I_{\text{fin}}} Z_{c_p}(\Gamma_i),$$

Set  $H := \Gamma_{i_1} \cap \dots \cap \Gamma_{i_r}$ . Observe that  $H$  has finite index in  $\Gamma$ . Moreover, it follows from Lemma 10 that  $Z_{c_p}(\Gamma_{i_1}) \cap \dots \cap Z_{c_p}(\Gamma_{i_r})$  stabilizes  $\pi_{i,p}$  for every  $i \in I_{\text{fin}}$  and  $p \in S$ . Hence,  $H$  is contained in  $\Gamma_i$  for every  $i \in I_{\text{fin}}$ .

By Proposition 9, we have a decomposition of  $\kappa_2^{\text{fin}}|_H$  into the direct sum

$$\bigoplus_{i \in I_{\text{fin}}} (\tilde{\pi}_i \otimes W_i)|_H.$$

By Proposition 8 and Proposition 15, there exists a real number  $r \geq 1$ , which is independent of  $i$ , such that  $(\tilde{\pi}_i \otimes W_i)|_H$  is a strongly  $L^r$  representation of  $H$  modulo its projective kernel  $P_i$ . Observe that  $P_i$  is contained in the projective kernel  $P_{\tilde{\pi}_i}$  of  $\tilde{\pi}_i$ . Hence (see Proposition 8), there exists an integer  $k \geq 1$  such that  $\kappa_2^{\text{fin}}|_H^{\otimes k}$  is contained in a multiple of the direct sum

$$\bigoplus_{i \in I_{\text{fin}}} \text{Ind}_{P_{\tilde{\pi}_i}}^H \rho_i,$$

for representations  $\rho_i$  of  $P_{\tilde{\pi}_i}$ . Since  $1_H$  is weakly contained in  $\kappa_2^{\text{fin}}|_H$  and hence in  $\kappa_2^{\text{fin}}|_H^{\otimes k}$ , using Lemma 7, it follows that  $1_H$  is weakly contained in

$$\bigoplus_{i \in I_{\text{fin}}} \lambda_{H/(H \cap P_{\tilde{\pi}_i})}.$$

Let  $i \in I$ . We claim that  $P_i$  is contained in  $\Gamma_\chi$  for some character  $\chi$  from  $\widehat{\mathbf{Sol}}_S \setminus \{\mathbf{1}_{\mathbf{Sol}_S}\}$ . Once proved, this will imply, again by Lemma 7, that  $\bigoplus_{i \in I_{\text{fin}}} \lambda_{H/(H \cap P_{\tilde{\pi}_i})}$  and, hence  $1_H$ , is weakly contained in  $\kappa_1|_H$ . Since  $H$  has finite index in  $\Gamma$ , this will show that  $1_\Gamma$  is weakly contained in  $\kappa_1$  and conclude the proof.

To prove the claim, recall from Proposition 14 that there exists a closed normal subgroup  $K_i$  of  $U$  with the following properties:  $K_i \Lambda / K_i$

is a lattice in the unipotent algebraic group  $U/K_i$ ,  $K_i$  is invariant under  $P_{\tilde{\pi}_i}$  and  $P_{\tilde{\pi}_i}$  acts as the identity on  $U/K_i$ . Observe that  $K_i \neq U$ , since  $\pi_i$  is not a unitary character of  $U$ . We can find a non-trivial unitary character  $\chi$  of  $U/K_i$  which is trivial on  $K_i\Lambda/K_i$ . Then  $\chi$  lifts to a non-trivial unitary character which is fixed by  $P_{\tilde{\pi}_i}$  and hence by  $P_i$ . ■

## 11. AN EXAMPLE: THE $S$ -ADIC HEISENBERG NILMANIFOLD

As an example, we study the spectral gap property for group of automorphisms of the  $S$ -adic Heisenberg nilmanifold. We will give a quantitative estimate for the norm of associated convolution operators, as we did in [BeHe11] in the case of real Heisenberg nilmanifolds (that is, in the case  $S = \{\infty\}$ ).

Let  $\mathbf{K}$  be an algebraically closed field containing  $\mathbf{Q}_p$  for  $p = \infty$  and for all prime integers  $p$ . For an integer  $n \geq 1$ , consider the symplectic form  $\beta$  on  $\mathbf{K}^{2n}$  given by

$$\beta((x, y), (x', y')) = (x, y)^t J (x', y') \quad \text{for all } (x, y), (x', y') \in \mathbf{K}^{2n},$$

where  $J$  is the  $(2n \times 2n)$ -matrix

$$J = \begin{pmatrix} 0 & I \\ -I_n & 0 \end{pmatrix}.$$

The symplectic group

$$Sp_{2n} = \{g \in GL_{2n}(\mathbf{K}) \mid {}^t g J g = J\}$$

is an algebraic group defined over  $\mathbf{Q}$ .

The  $(2n+1)$ -dimensional Heisenberg group is the unipotent algebraic group  $\mathbf{H}$  defined over  $\mathbf{Q}$ , with underlying set  $\mathbf{K}^{2n} \times \mathbf{K}$  and product

$$((x, y), s)((x', y'), t) = ((x + x', y + y'), s + t + \beta((x, y), (x', y'))),$$

for  $(x, y), (x', y') \in \mathbf{K}^{2n}$ ,  $s, t \in \mathbf{K}$ .

The group  $Sp_{2n}$  acts by rational automorphisms of  $H_{2n+1}$ , given by

$$g((x, y), t) = (g(x, y), t) \quad \text{for all } g \in Sp_{2n}, (x, y) \in \mathbf{K}^{2n}, t \in \mathbf{K}.$$

Let  $p$  be either an integer prime or  $p = \infty$ . Set  $H_p = \mathbf{H}(\mathbf{Q}_p)$ . The center  $Z$  of  $H_p$  is  $\{(0, 0, t) \mid t \in \mathbf{Q}_p\}$ . The unitary dual  $\widehat{H}$  of  $H$  consists of the equivalence classes of the following representations:

- the unitary characters of the abelianized group  $H/Z$ ;
- for every  $t \in \mathbf{Q}_p \setminus \{0\}$ , the infinite dimensional representation  $\pi_t$  defined on  $L^2(\mathbf{Q}_p^n)$  by the formula

$$\pi_t((a, b), s)\xi(x) = e_p(ts)e_p(\langle a, x - b \rangle)\xi(x - b)$$

for  $((a, b), s) \in H$ ,  $\xi \in L^2(\mathbf{Q}_p^n)$ , and  $x \in \mathbf{Q}_p^n$ , where  $e_p \in \widehat{\mathbf{Q}_p}$  is as in Section 6.

For  $t \neq 0$ , the representation  $\pi_t$  is, up to unitary equivalence, the unique irreducible unitary representation of  $H$  whose restriction to the centre  $Z$  is a multiple of the unitary character  $s \mapsto e_p(ts)$ .

For  $g \in Sp_{2n}(\mathbf{Q}_p)$  and  $t \in \mathbf{Q}_p \setminus \{0\}$ , the representation  $\pi_t^g$  is unitary equivalent to  $\pi_t$ , since both representations have the same restriction to  $Z$ . This shows that  $Sp_{2n}(\mathbf{Q}_p)$  stabilizes  $\pi_t$ . We denote the corresponding projective representation of  $Sp_{2n}(\mathbf{Q}_p)$  by  $\omega_t^{(p)}$ . The representation  $\omega_t^{(p)}$  has different names: it is called **metaplectic representation**, **Weil's representation**, or **oscillator representation**. The projective kernel of  $\omega_t^{(p)}$  coincides with the (finite) center of  $Sp_{2n}(\mathbf{Q}_p)$  and  $\omega_t^{(p)}$  is strongly  $L^{4n+2+\varepsilon}$  on  $Sp_{2n}(\mathbf{Q}_p)$  for every  $\varepsilon > 0$  (see Proposition 6.4 in [HoMo79] or Proposition 8.1 in [Howe82]).

Let  $S = \{p_1, \dots, p_r, \infty\}$ , where  $p_1, \dots, p_r$  are integer primes. Set  $U := \mathbf{H}(\mathbf{Q}_S)$  and

$$\Lambda := \mathbf{H}(\mathbf{Z}[1/S]) = \{((x, y), s) : x, y \in \mathbf{Z}^n[1/S], s \in \mathbf{Z}[1/S]\}.$$

Let  $\mathbf{Nil}_S = U/\Lambda$ ; the associated  $S$ -adic solenoid is  $\mathbf{Sol}_S = \mathbf{Q}_S^{2n}/\mathbf{Z}[1/S]^{2n}$ . The group  $Sp_{2n}(\mathbf{Z}[1/S])$  is a subgroup of  $\text{Aut}(\mathbf{Nil}_S)$ . The action of  $Sp_{2n}(\mathbf{Z}[1/S])$  on  $\mathbf{Sol}_S$  is induced by its linear representation by linear bijections on  $\mathbf{Q}_S$ .

Let  $\Gamma$  be a subgroup of  $Sp_{2n}(\mathbf{Z}[1/S])$ . The Koopman representation  $\kappa$  of  $\Gamma$  on  $L^2(\mathbf{Nil}_S)$  decomposes as

$$\kappa = \mathbf{1}_{\mathbf{Nil}_S} \oplus \kappa_1 \oplus \kappa_2,$$

where  $\kappa_1$  is the restriction of  $\kappa$  to  $L_0^2(\mathbf{Sol}_S)$  and  $\kappa_2$  the restriction of  $\kappa$  to the orthogonal complement of  $L_0^2(\mathbf{Sol}_S)$  in  $L^2(\mathbf{Nil}_S)$ . Since  $Sp_{2n}(\mathbf{Q}_p)$  stabilizes every infinite dimensional representation of  $H_p$ , it follows from Proposition 14 that there exists a subset  $I \subset \mathbf{Q}$  such that  $\kappa_2$  is equivalent to a direct sum

$$\bigoplus_{t \in I} \left( \otimes_{p \in S} (\omega_t^{(p)}|_{\Gamma} \otimes W_i) \right),$$

where  $W_i$  is an projective representation  $\Gamma$ .

Let  $\nu$  be a probability measure on  $\Gamma$ . We can give an estimate of the norm of  $\kappa_2(\nu)$  as in [BeHe11] in the case of  $S = \{\infty\}$ . Indeed, by a general inequality (see Proposition 30 in [BeGu15]), we have

$$\|\kappa_2(\nu)\| \leq \|(\kappa_2 \otimes \overline{\kappa_2})^{\otimes k}(\nu)\|^{1/2k},$$

for every integer  $k \geq 1$ , where  $\overline{\kappa_2}$  denotes the representation conjugate to  $\kappa_2$ . Since  $\omega_t^{(p)}$  is strongly  $L^{4n+2+\varepsilon}$  on  $Sp_{2n}(\mathbf{Q}_p)$  for any  $t \in I$  and



$p \in S$ , Proposition 8 implies that  $(\kappa_2 \otimes \overline{\kappa_2})^{\otimes(n+1)}$  is contained in an infinite multiple of the regular representation  $\lambda_\Gamma$  of  $\Gamma$ . Hence,

$$\|\kappa_2(\nu)\| \leq \|\lambda_\Gamma(\nu)\|^{1/2n+2}$$

and so,

$$\|\kappa_0(\nu)\| \leq \max\{\|\lambda_\Gamma(\nu)\|^{1/2n+2}, \|\kappa_1(\nu)\|\},$$

where  $\kappa_0$  is the restriction of  $\kappa$  to  $L_0^2(\mathbf{Nil}_S)$ .

Assume that the subgroup generated by the support of  $\nu$  coincides with  $\Gamma$ . If  $\Gamma$  is not amenable then  $\|\lambda_\Gamma(\nu)\| < 1$  by Kersten's theorem (see [BeHV08, Appendix G]); so, in this case, the action of  $\Gamma$  on  $\mathbf{Nil}_S$  has a spectral gap if and only if  $\|\kappa_1(\nu)\| < 1$ , as stated in Theorem 1.

Observe that, if  $\Gamma$  is amenable, then the action of  $\Gamma$  on  $\mathbf{Nil}_S$  or  $\mathbf{Sol}_S$  does not have a spectral gap; indeed, by a general result (see [JuRo79, Theorem 2.4]), no action of a countable amenable group by measure preserving transformations on a non-atomic probability space has a spectral gap.

Let us look more closely to the case  $n = 1$ . We have  $Sp_2(\mathbf{Z}[1/S]) = SL_2(\mathbf{Z}[1/S])$  and the stabilizer of every element in  $\widehat{\mathbf{Sol}}_S \setminus \{\mathbf{1}_{\mathbf{sol}_S}\}$  is conjugate to the group of unipotent matrices in  $SL_2(\mathbf{Z}[1/S])$  and hence amenable. This implies that  $\kappa_1$  is weakly contained in  $\lambda_\Gamma$ ; so, we have

$$\|\kappa_1(\nu)\| < 1 \iff \Gamma \text{ is not amenable.}$$

As a consequence, we see that the action of  $\Gamma$  on  $\mathbf{Nil}_S$  has a spectral gap if and only if  $\Gamma$  is not amenable.

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