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**Abstract**

In his paper from 1996 on quadratic forms Heath-Brown developed a version of the circle method to count points in the intersection of an unbounded quadric with a lattice of short period, if each point is given a weight, and approximated this quantity by the integral of the weight function against a measure on the quadric. The weight function is assumed to be $C_0^\infty$–smooth and vanish near the singularity of the quadric. In our work we allow the weight function to be finitely smooth and not vanish near the singularity and give an explicit dependence of the disparity on that function.

The paper uses only elementary results from the number theory and is available to readers without a number-theoretical background.

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# 1 Introduction

## 1.1 Setting and result

Let us consider a non-degenerate quadratic form with integer coefficients on $\mathbb{R}^d$, $d \geq 4$,

$$F(\mathbf{z}) = \tfrac{1}{2} A\mathbf{z} \cdot \mathbf{z} \ , \tag{1.1}$$

which implies that $A$ can be chosen as a non–degenerate symmetric matrix with integer elements whose diagonal elements are even. If $F$ is sign–definite, then for $t \in \mathbb{R}$ the quadric

$$\Sigma_t = \{\mathbf{z} : F^t(\mathbf{z}) = 0\}, \quad F^t = F - t, \tag{1.2}$$

is either an ellipsoid or an empty set, while in the non sign–definite case $\Sigma_t$ is an unbounded hyper-surface in $\mathbb{R}^d$, which is smooth if $t \neq 0$, while $\Sigma_0$ is a cone and has a locus at zero.

Let $\mathbb{Z}^d_L$ be the lattice of a small period $L^{-1}$,

$$\mathbb{Z}^d_L = L^{-1}\mathbb{Z}^d, \qquad L \geq 1,$$

and let $w$ be a *regular* real function on $\mathbb{R}^d$ which means that $w$ and its Fourier transform $\hat{w}(\xi)$ are continuous functions which decay at infinity sufficiently fast:

$$|w(\mathbf{z})| \leq C|\mathbf{z}|^{-d-\gamma}, \qquad |\hat{w}(\xi)| \leq C|\xi|^{-d-\gamma}, \tag{1.3}$$

for some $\gamma > 0$. Our goal is to study the behaviour of series

$$N_L(w; A, m) := \sum_{\mathbf{z} \in \Sigma_m \cap \mathbb{Z}^d_L} w(\mathbf{z}) \, ,$$

where $m \in \mathbb{R}$ is such that $L^2 m$ is an integer.[1] Let

$$w_L(\mathbf{z}) := w(\mathbf{z}/L).$$

Then, obviously,

$$N_L(w; A, m) = N_1(w_L; A, L^2 m) =: N(w_L; A, L^2 m) \, . \tag{1.4}$$

We will also write $N_L(w; A) := N_L(w; A, 0)$ and $N(w_L; A) := N(w_L; A, 0)$. To study $N_L(w; A, m)$ we closely follow the circle method in the form, given to it by Heath-Brown in [8]. Our notation differs a bit from that in [8].

[1] E.g., $m = 0$ – this case is the most important for us.

Namely, under the scaling $z = z'/L$, $z' \in \mathbb{Z}^d$, we count (with weights) solutions of equation $F(z') = mL^2$, $z' \in \mathbb{Z}^d$, while Heath-Brown writes the equation as $F(z') = m$, $z' \in \mathbb{Z}^d$, so that his $m$ corresponds to our $L^2 m$.

We start with a key theorem which expresses the analogue of Dirac's delta function on integers, i.e. the function $\delta : \mathbb{Z} \mapsto \mathbb{R}$ such that

$$\delta(n) := \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases} ,$$

through a sort of Fourier representation. This result goes back at least to Duke, Friedlander and Iwaniec [4] (cf. also [9]) , and we state it in the form, given in [8, Theorem 1]; basically, it replaces (a major arc decomposition of) the trivial identity $\delta(n) = \int_0^1 e^{2\pi i \alpha n} d\alpha$ employed in the usual circle method. In the theorem below for $q \in \mathbb{N}$ we denote by $e_q$ the exponential function $e_q(x) := e^{\frac{2\pi i x}{q}}$, and denote by $\sum_{a (\mathrm{mod}\, q)}^{*}$ the summation over residues $a$ with $(a, q) = 1$, i.e., over all integers $a \in [1, q-1]$, relatively prime with $q$.

**Theorem 1.1.** *For any $Q \geq 1$, there exists $c_Q > 0$ and a smooth function $h(x, y) : \mathbb{R}_{>0} \times \mathbb{R} \mapsto \mathbb{R}$, such that*

$$\delta(n) = c_Q Q^{-2} \sum_{q=1}^{\infty} \sum_{a (\mathrm{mod}\ q)}^{*} e_q(an) h\left(\frac{q}{Q}, \frac{n}{Q^2}\right) . \tag{1.5}$$

*The constant $c_Q$ satisfies $c_Q = 1 + O_N(Q^{-N})$ for any $N > 0$, while $h$ is such that $h(x, y) \leq c/x$ and $h(x, y) = 0$ for $x > \max(1, 2|y|)$ (so for each $n$ the sum in (1.5) contains finitely many non zero terms).*

Since $N(\tilde{w}; A, t)$ may be written as $\sum_{\mathbf{z} \in \mathbb{Z}^d} \tilde{w}(\mathbf{z}) \delta(F^t(\mathbf{z}))$, then Theorem 1.1 allows to represent series $N(\tilde{w}; A, t)$ as an iterated sum. Transforming that sum further using the Poisson summation formula as in [8, Theorem 2] we arrive at the following result: [2]

**Theorem 1.2** (Theorem 2 of [8])**.** *For any regular function $\tilde{w}$, any $t$ and any $Q \geq 1$ we have the expression*

$$N(\tilde{w}; A, t) = c_Q Q^{-2} \sum_{\mathbf{c} \in \mathbb{Z}^d} \sum_{q=1}^{\infty} q^{-d} S_q(\mathbf{c}) I_q^0(\mathbf{c}) , \tag{1.6}$$

[2] In [8] the result below is stated for $\tilde{w} \in C_0^\infty$. But the argument there, based on the Poisson summation, applies as well to regular functions $\tilde{w}$.

*with*

$$S_q(\mathbf{c}; A, t) := \sum_{a(\mathrm{mod}\ q)}{}^{*} \sum_{\mathbf{b}(\mathrm{mod}\ q)} e_q(aF^t(\mathbf{b}) + \mathbf{c}\cdot\mathbf{b}) \tag{1.7}$$

*and*

$$I_q^0(\mathbf{c}; A, t, Q) := \int_{\mathbb{R}^d} \tilde{w}(\mathbf{z}) h\left(\frac{q}{Q}, \frac{F^t(\mathbf{z})}{Q^2}\right) e_q(-\mathbf{z}\cdot\mathbf{c})\, d\mathbf{z}\,. \tag{1.8}$$

We will apply Theorem 1.2 to examine for large $L$ the sum $N(w_L; A, L^2 m)$ $= N_L(w; A, m)$, choosing $Q = L \geq 1$ and estimating explicitly the leading terms in $L$ of $S_q(\mathbf{c})$ and $I_q^0(\mathbf{c})$ as well as the remainders. The answer will be given in terms of the integral

$$\sigma_\infty(w) = \sigma_\infty(w; A, t) = \int_{\Sigma_t} w(\mathbf{z})\, \mu^{\Sigma_t}(d\mathbf{z}) \tag{1.9}$$

(which is singular if $t = 0$). Here $\mu^{\Sigma_t}(d\mathbf{z}) = |\nabla F(\mathbf{z})|^{-1} dz|_{\Sigma_t} = |A\mathbf{z}|^{-1} dz|_{\Sigma_t}$, with $dz|_{\Sigma_t}$ representing the volume element over $\Sigma_t$, induced from the standard euclidean structure on $\mathbb{R}^d$, and $A$ the symmetric matrix in (1.1). For regular functions $w$ this integral converges (see Section 7).

To write down the asymptotic for $N_L(w; A, m)$ we will need the following quantities, where $p$ ranges over all primes and $c \in \mathbb{Z}^d$:

$$\sigma_p^{\mathbf{c}} = \sigma_p^{\mathbf{c}}(A, L^2 m) := \sum_{l=0}^{\infty} p^{-dl} S_{p^l}(\mathbf{c}; A, L^2 m), \qquad \sigma_p := \sigma_p^{\mathbf{0}}, \tag{1.10}$$

where $S_1 \equiv 1$,

$$\sigma_{\mathbf{c}}^*(A) := \prod_p (1 - p^{-1}) \sigma_p^{\mathbf{c}}(A, 0), \qquad \sigma^*(A) := \sigma_{\mathbf{0}}^*(A) = \prod_p (1 - p^{-1}) \sigma_p(A, 0),$$

and

$$\sigma(A, L^2 m) = \prod_p \sigma_p^{\mathbf{0}}(A, L^2 m) = \prod_p \sigma_p(A, L^2 m). \tag{1.11}$$

The products in the formulas above are taken over all primes. In the asymptotics, where these quantities are used, they are bounded uniformly in $L$ (see below).

Our main results, stated below, specify Theorems 5, 6 and 7 from [8] in three respects: firstly, now the function $w$ has finite smoothness and sufficiently fast decays at infinity, while in [8] $w \in C_0^\infty$. Secondly, we specify how the remainder depends on $w$. Thirdly and the most importantly, we remove the imposed in [8] restriction that the support of $w$ does not contain

the origin (this improvement is crucial for us since in [6] the theorems are used in the situation when $w(0) \neq 0$).

We note that a similar specification of the Heath-Brown method was obtained in [1, Section 5] to study problems, related to those considered in [6].

Everywhere below for a function $f \in C^k(\mathbb{R}^N)$ we denote

$$\|f\|_{n_1,n_2} = \sup_{\mathbf{z}\in\mathbb{R}^N} \max_{|\alpha|\le n_1} |\partial^\alpha f(\mathbf{z})| \langle \mathbf{z}\rangle^{n_2},$$

where $n_1 \in \mathbb{N}\cup\{0\}$, $n_1 \le k$, and $n_2 \in \mathbb{R}$. Here

$$\langle \mathbf{x}\rangle := \max\{1, |\mathbf{x}|\} \quad \text{for } \mathbf{x} \in \mathbb{R}^l,$$

for any $l \ge 1$. By $\mathcal{C}^{n_1,n_2}(\mathbb{R}^N)$ we denote a linear space of $C^{n_1}$-smooth functions $f : \mathbb{R}^N \mapsto \mathbb{R}$, satisfying $\|f\|_{n_1,n_2} < \infty$.

Note that if $w \in \mathcal{C}^{d+1,d+1}$ then the function $w$ is regular, so Theorem 1.2 applies. Indeed, the first relation in (1.3) is obvious. To prove the second note that for any integer vector $\alpha \in (\mathbb{N}\cup\{0\})^d$, $\xi^\alpha \hat{w}(\xi) = \left(\frac{i}{2\pi}\right)^{|\alpha|} \widehat{\partial^\alpha_{\mathbf{x}} w}(\xi)$. But if $|\alpha| \equiv \sum \alpha_j \le d+1$, then $|\partial^\alpha_{\mathbf{x}} w| \le C\langle \mathbf{x}\rangle^{-d-1}$, so $\partial^\alpha_{\mathbf{x}} w$ is an $L_1$-function. Thus its Fourier transform $\widehat{\partial^\alpha_{\mathbf{x}} w}$ is a bounded continuous function for each $|\alpha| \le d+1$ and the second relation in (1.3) also holds.

Now we formulate our main results. First we treat the case $d \ge 5$.

**Theorem 1.3.** *Assume that $d \ge 5$. Then for any $0 < \varepsilon \le 1$ there exist positive constants $K_1(d,\varepsilon)$ and $K_2(d,\varepsilon) \le K_3(d,\varepsilon)$ such that if $w \in \mathcal{C}^{K_1,K_2}(\mathbb{R}^d) \cap \mathcal{C}^{0,K_3}(\mathbb{R}^d)$ and a real number $m$ satisfies $L^2 m \in \mathbb{Z}$, then*

$$\left|N_L(w; A, m) - \sigma_\infty(w)\sigma(A, L^2 m) L^{d-2}\right| \le C L^{d/2+\varepsilon} \left(\|w\|_{K_1,K_2} + \|w\|_{0,K_3}\right), \tag{1.12}$$

*where the constant $C$ depends on $d, \varepsilon$, $m$ and $A$. The constant $\sigma(A, L^2 m)$ is bounded uniformly in $L$ and $m$. In particular if $\varepsilon = 1/2$, then one can take $K_1 = 2d(d^2+d-1)$, $K_2 = 4(d+1)^2 + 3d + 1$ and $K_3 = K_1 + 3d + 4$.*

Next we study the case $d = 4$, restricting ourselves for the situation when $m = 0$.

**Theorem 1.4.** *Assume that $d = 4$ and $m = 0$. Then for any $0 < \varepsilon < 1/5$ there exist positive constants $K_1(d,\varepsilon)$ and $K_2(d,\varepsilon)$, such that for $w \in \mathcal{C}^{K_1,K_2}(\mathbb{R}^d)$*

$$\begin{aligned}\left|N_L(w; A, 0) - \eta(0)\sigma_\infty(w)\sigma^*(A) L^{d-2}\log L - \sigma_1(w; A, L) L^{d-2}\right| \\ \le C_0 L^{d-2-\varepsilon} \|w\|_{K_1,K_2},\end{aligned} \tag{1.13}$$

*where the constant $C_0$ depends on $d, \varepsilon$ and $A$. The constant $\eta(0)$ is 1 if the determinant* $\det A$ *is a square of an integer and is 0 otherwise. The $L$-independent constant $\sigma^*(A)$ is finite while the constant $\sigma_1$ satisfies*

$$|\sigma_1(w; A, L)| \le C_0 \|w\|_{K_1,K_2}$$

*uniformly in $L$. In the case of a square determinant* $\det A$*, when $\eta(0) = 1$, it is given by* (1.24)*. In the case of a non–square determinant* $\det A$*, when $\eta(0) = 0$ and the term $\sigma_1(w; A, L)L^{d-2}$ gives the asymptotic of the sum $N_L$, the constant $\sigma_1(w; A, L)$ does not depend on $L$ and has the form*

$$\sigma_1(w; A) = \sigma_\infty(w) L(1, \chi) \prod_p (1 - \chi(p)p^{-1})\sigma_p(A, 0)\,, \qquad (1.14)$$

*where $\chi$ is the Jacobi symbol $(\frac{\det(A)}{*})$ and $L(1, \chi)$ is the Dirichlet $L$–function.*

Concerning the classical notion of the Jacobi symbol and the Dirichlet $L$-function we refer a reader without number-theoretical background e.g. to [12] and [10].

If $\eta(0)\sigma^*(A) = 0$, then the asymptotic (1.13) degenerates. Similar (1.12) also degenerates to an upper bound on $N_L$, unless we know that $\sigma(A, L^2 m)$ admits a suitable positive lower bound, for all $L$. Luckily enough, the required lower bounds often exist, see Proposition 1.5 below.

*Remarks.* 1) Theorem 1.3 is a specification of Theorem 5 of [8], while Theorem 1.4 specifies Theorems 6 and 7 of [8]. In [8] also is available some asymptotic information about behaviour of the sums $N_L(w; A, m)$ when $d = 4$, $m \neq 0$ and $d = 3$, $m = 0$. Since our proof of Theorems 1.3 and 1.4 is based on ideas from [8], strengthened by Theorem 7.3, which is valid for $d \ge 3$, then most likely our approach allows to generalise the above-mentioned results of [8] for $d = 3, 4$ to the case when $w \in \mathcal{C}^{K_1,K_2}(\mathbb{R}^d)$ with suitable $K_1, K_2$.

2) Here and below the dependence of constants in estimates on $m$ is uniform on every compact interval, while the dependence on $A$ is via the norms of $A$ and $A^{-1}$.

3) The values of the constants $K_j(d, \varepsilon)$ in (1.12), given in Theorem 1.3, are far from optimal since it was not our goal to optimise them.

4) As the theorems' proof are based on the representation (1.6), then the function $w$ should be regular (see (1.3)). But this holds true if $w \in \mathcal{C}^{d+1,d+1}$ and so is valid if the constants $K_1, K_2$ are sufficiently big. E.g. if $K_1, K_2$ are as big as in the last line of the assertion of Theorem 1.3.

We present here in full only the proof of Theorem 1.3, which occupies the rest of the paper and follows that of [8, Theorem 5] with additional control how the constants depend on $w$. The only significant difference comes in Sections 3 and 4 below where we do not assume that the function $w$ vanishes near the origin, the last assumption being crucial in the analysis of integrals in Sections 6 and 7 of [8]. To cope with this difficulty, which becomes apparent e.g. in Proposition 3.8 below, we have to examine the smoothness at zero of the function

$$t \mapsto \sigma_\infty(w; A, t) \tag{1.15}$$

and its decay at infinity. The corresponding analysis is performed in Section 7. There, using the techniques, developed in [5] to study integrals (1.9), we prove that function (1.15) is $(d/2-2)$-smooth, but in general its derivative of order $(d/2-1)$ may have a logarithmic singularity at zero. There we also estimate the rate of decay of (1.15) at infinity.

The proof of Theorem 1.4 resembles the proof of Theorems 6 and 7 of [8] with a new addition given by Proposition 3.8, based on the result of Section 7. We thus limit ourselves to a sketch of its demonstration, given in Section 1.3 in parallel to that of Theorem 1.3, and point out the main differences between the two proofs. The demonstration of Theorem 1.3 is self-contained, while establishing Theorem 1.4 we use certain results from [8] (namely, Lemmas 30 and 31) without proof.

*Lower bounds for the constant from the asymptotics.* Let us now discuss lower bounds for the constants $\sigma(A, L^2 m)$ and $\sigma^*(A)$ from Theorems 1.3 and 1.4.

**Proposition 1.5.** *(i) If $d \geq 5$ then $0 < c(A) \leq \sigma(A, L^2 m) \leq C(A) < \infty$ for any non-degenerate matrix $A$, uniformly in $L$ and $m$.*

*(ii) If $d = 4$ and $m = 0$ we have $\sigma^*(A) > 0$ for any non-degenerate matrix $A$ such that the corresponding equation $2F(\mathbf{z}) = A\mathbf{z} \cdot \mathbf{z} = 0$ has non-trivial solutions in every p-adic field.*

See Theorems 4,6 and 7 of [8]. We do not prove this result, but just note that its demonstration uses a refinement of the calculation in the second part of the proof of Lemma 2.3. Namely, while the lemma gives an upper bound for the desired quantity, a more thorough analysis permits also to establish the claimed lower bounds.

Appendix B contains a brief discussion (for a non-specialist) of quadratic forms in 4 variables over $\mathbb{Q}$ and $\mathbb{Q}_p$ which gives an idea why the p-adic fields $\mathbb{Q}_p$ are involved in the study of integer points on the quadric $\Sigma_0$ and how

one could verify the conditions of Proposition 1.5 (ii). In Appendix C we give essentially a complete calculation, proving Proposition 1.5 in the case of the simplest quadratic form $F = \Sigma_{i=1}^{d/2} x_i y_i$, $d = 2s \geq 4$ and $t = 0$. For any $A$ the calculation may follow the same lines, replacing explicit formulas by some general results (e.g. Hensel's Lemma).

*Non-homogeneous quadratic forms.* Now consider a non-homogeneous quadratic form $\mathcal{F}$ with the second order part, equal to $F$ in (1.1):

$$\mathcal{F}(\mathbf{z}) = \tfrac{1}{2} A\mathbf{z} \cdot \mathbf{z} + \mathbf{z}_* \cdot \mathbf{z} + \tau, \qquad \mathbf{z} \in \mathbb{R}^d,\ \tau \in \mathbb{R},$$

and the corresponding set $\Sigma^{\mathcal{F}} = \{\mathbf{z} : \mathcal{F}(\mathbf{z}) = 0\}$, $N_L(w; \mathcal{F}) = \sum_{\mathbf{z} \in \Sigma^{\mathcal{F}} \cap \mathbb{Z}_L^d} w(\mathbf{z})$. Denote

$$\mathfrak{z} = A^{-1}\mathbf{z}_*, \quad \mathbf{z}' = \mathbf{z} + \mathfrak{z}, \quad m = \tfrac{1}{2}\mathfrak{z} \cdot A\mathfrak{z} - \tau,$$

and assume that $\mathfrak{z} \in \mathbb{Z}_L^d$ [3] and $L^2\tau \in \mathbb{Z}$. Then $L^2 m \in \mathbb{Z}$, $\mathbf{z}' \in \mathbb{Z}_L^d$ if and only if $\mathbf{z} \in \mathbb{Z}_L^d$, and $\mathcal{F}(\mathbf{z}) = F(\mathbf{z}') - m$. So setting $w^{\mathfrak{z}}(\mathbf{z}') = w(\mathbf{z}' - \mathfrak{z})$ we have $N_L(w; \mathcal{F}) = N_L(w^{\mathfrak{z}}; A, m)$. Since

$$\sigma_\infty(w^{\mathfrak{z}}; A, m) = \int_{\Sigma_m} w^{\mathfrak{z}}(\mathbf{z}') \frac{d\mathbf{z}'|_{\Sigma_m}}{|\nabla F(\mathbf{z}')|} = \int_{\Sigma^{\mathcal{F}}} w(\mathbf{z}) \frac{d\mathbf{z}|_{\Sigma^{\mathcal{F}}}}{|\nabla \mathcal{F}(\mathbf{z})|} =: \sigma_\infty(w; \mathcal{F}),$$

then we arrive at the following corollary from the theorem:

**Corollary 1.6.** *If $d \geq 5$, quadratic form $F$ is as in Theorem 1.3, $\mathcal{F}$ is a non-homogeneous quadratic form as above and $L$ is such that $\mathfrak{z} := A^{-1}\mathbf{z}_* \in \mathbb{Z}_L^d$, $\tau L^2 \in \mathbb{Z}$, then for any $0 < \varepsilon \leq 1$ we have*

$$\left| N_L(w; \mathcal{F}) - \sigma_\infty(w; \mathcal{F})\, \sigma(A, L^2 m) L^{d-2} \right| \leq C L^{d/2+\varepsilon} \left( \|w\|_{K_1, K_2} + \|w\|_{0, K_3} \right).$$

*Here the constants $K_1, K_2, K_3$ depend on $d$ and $\varepsilon$, while $C$ depends on $d, \varepsilon, A$ and $\tau, |\mathbf{z}_*|$.*

**Notation and agreements.** We write $A \lesssim_{a,b} B$ if $A \leq CB$, where the constant $C$ depends on $a$ and $b$. Similar, $O_{a,b}(\|w\|_{m_1,m_2})$ stands for a quantity, bounded in absolute value by $C(a,b)\|w\|_{m_1,m_2}$. We do not indicate the dependence on the norms $\|A\|$, $\|A^{-1}\|$ and on the dimension $d$ since most of our estimates depend on these quantities.

We always assume that the function $w$ belongs to the space $\mathcal{C}^{m,n}(\mathbb{R}^d)$ with sufficiently large $m, n$. If in the statement of an assertion we employ the norm $\|w\|_{a,b}$ then we assume that $w \in \mathcal{C}^{a,b}(\mathbb{R}^d)$.

[3] This holds e.g. if $\det A = \pm 1$ and $\mathbf{z} \in \mathbb{Z}_L^d$.

We denote $e_q(x) = e^{2\pi i x/q}$ and abbreviate $e_1(x) =: e(x)$. By $\lceil\cdot\rceil$ we denote the ceiling function, $\lceil x\rceil = \min_{n\in\mathbb{Z}}\{n \ge x\}$. By $\mathbb{N}$ we denote the set of positive integers.

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### 1.2 Scheme of the proof of Theorem 1.3

Let $d \ge 5$. As it has been already discussed, if $w$ satisfies assumptions of the theorem with sufficiently large constants $K_i$ then $w$ is regular in the sense of Section 1.1, so Theorem 1.2 applies. Then, according to (1.6) and (1.4),

$$N_L(w; A, m) = c_L\, L^{-2} \sum_{\mathbf{c}\in\mathbb{Z}^d} \sum_{q=1}^{\infty} q^{-d} S_q(\mathbf{c}) I_q(\mathbf{c})\,, \tag{1.16}$$

where the sum $S_q(\mathbf{c}) = S_q(\mathbf{c}; A, L^2 m)$ is given by (1.7) with $t = L^2 m$ and the integral $I_q(\mathbf{c})$ – by (1.8) with $\tilde{w} = w_L$, $Q = L$ and $t = L^2 m$,

$$I_q(\mathbf{c}; A, m, L) := \int_{\mathbb{R}^d} w\left(\frac{\mathbf{z}}{L}\right) h\left(\frac{q}{L}, \frac{F^{L^2 m}(\mathbf{z})}{L^2}\right) e_q(-\mathbf{z}\cdot\mathbf{c})\, d\mathbf{z}\,. \tag{1.17}$$

Denoting

$$n(\mathbf{c}; A, m, L) = \sum_{q=1}^{\infty} q^{-d} S_q(\mathbf{c}) I_q(\mathbf{c})\,,$$

we have $N_L(w; A, m) = c_L L^{-2} \sum_{\mathbf{c}\in\mathbb{Z}^d} n(\mathbf{c})$. Then for an $\gamma_1 \in (0,1)$ we write $N_L$ as

$$N_L(w; A, m) = c_L L^{-2}\big(J_0 + J_<^{\gamma_1} + J_>^{\gamma_1}\big), \tag{1.18}$$

where

$$J_0 := n(0)\,, \quad J_<^{\gamma_1} := \sum_{\mathbf{c}\neq 0,\, |\mathbf{c}|\le L^{\gamma_1}} n(\mathbf{c})\,, \quad J_>^{\gamma_1} := \sum_{|\mathbf{c}|> L^{\gamma_1}} n(\mathbf{c})\,. \tag{1.19}$$

Proposition 5.1 (which is a modification of Lemmas 19 and 25 from [8]) implies that

$$|J_>^{\gamma_1}| \lesssim_{\gamma_1, m} \|w\|_{N_0, 2N_0+d+1}$$

with $N_0 := \lceil d + (d+1)/\gamma_1\rceil$ (see Corollary 5.2). In Proposition 6.1, following Lemmas 22 and 28 from [8], we show that

$$|J_<^{\gamma_1}| \lesssim_{\gamma_1, m} L^{d/2+2+\gamma_1(d+1)} \left(\|w\|_{\bar{N}, d+5} + \|w\|_{0, \bar{N}+3d+4}\right)\,, \tag{1.20}$$

$\bar{N} = \lceil d^2/\gamma_1 \rceil - 2d$.

To analyse $J_0$ we write it as $J_0 = J_0^+ + J_0^-$, where

$$J_0^+ := \sum_{q>\rho L} q^{-d} S_q(0) I_q(0)\,, \quad J_0^- := \sum_{q\le\rho L} q^{-d} S_q(0) I_q(0)\,, \tag{1.21}$$

with $\rho = L^{-\gamma_2}$ for some $0 < \gamma_2 < 1$ to be determined. Lemma 4.2, which is a combination of Lemmas 16 and 25 from [8], modified using the results from Section 7, implies that

$$\left|J_0^+\right| \lesssim L^{d/2+2+\gamma_2(d/2-1)} |w|_{L_1} \lesssim L^{d/2+2+\gamma_2(d/2-1)} \|w\|_{0,d+1}.$$

Finally Lemma 4.3, which is a combination of Lemma 13 and simplified Lemma 31 from [8] with the results from Section 7, establishes that $J_0^-$ equals

$$L^d \sigma_\infty(w)\sigma(A, L^2 m) + O_{\gamma_2,m}\Big(\big(\|w\|_{d/2-2,d-1} + \|w\|_{0,d+1}\big) L^{d/2+2+\gamma_2(d/2-2)}\Big)$$

(see (1.9) and (1.11)). Identity (1.18) together with the estimates above implies the desired result if we choose $\gamma_2 = \varepsilon/(d/2-1)$ and $\gamma_1 = \varepsilon/(d+1)$. Uniform in $L$ and $m$ boundedness of the product $\sigma(A, L^2 m)$ follows from Lemma 2.3.

### 1.3 Scheme of the proof of Theorem 1.4

In this section we assume that $d = 4$ and $m = 0$. The proof proceeds exactly as in the previous section up to formula (1.20), which is not sharp enough for the case $d = 4$ and should be replaced by

$$\left| J_<^{\gamma_1} - L^d \sum_{\mathbf{c}\neq 0} \eta(\mathbf{c}) \sigma_{\mathbf{c}}^*(A) \sigma_\infty^{\mathbf{c}}(w; A, L) \right| \lesssim_{\gamma_1} L^{7/2+(d+4)\gamma_1} \|w\|_{\tilde{K}_1, \tilde{K}_2} \tag{1.22}$$

for appropriate constants $\tilde{K}_1, \tilde{K}_2$, where the terms $\sigma_{\mathbf{c}}^*(A)$ are introduced in (1.10), terms $\sigma_\infty^{\mathbf{c}}(w; A)$ are given by

$$\sigma_\infty^{\mathbf{c}}(w; A, L) := L^{-d} \sum_{q=1}^{\infty} q^{-1} I_q(\mathbf{c}; A, 0, L)\,, \tag{1.23}$$

and the constants $\eta(\mathbf{c}) = \pm 1$ are defined in Lemma A.1. In particular, $\eta(0) = 1$ if the determinant $\det A$ is a square of an integer and $\eta(0) = 0$ otherwise.

The proof of the bound (1.22) makes use of Lemma A.1 (Lemma 30 of [8]), involving only minor modifications of the argument in [8] and is left to the reader.

The bound on $J_0$ must be refined too and this is done in Appendix A. We consider only the case when the determinant $\det A$ is a square of an integer, so in particular $\eta(0) = 1$. The opposite case can be obtained by minor modification of the latter, following [8] (see Appendix A for a discussion). In Proposition A.3, which is a combination of Lemmas 13, 16 and 31 of [8], modified using Proposition 3.8, we prove that in the case of square determinant $\det A$

$$\begin{aligned} J_0 =& \sigma_\infty(w)\sigma^*(A)L^d \log L + K(0)L^d \\ &+ O_\varepsilon\big(L^{d-\varepsilon}\big(\|w\|_{d/2-2,d-1} + \|w\|_{0,d+1}\big)\big)\,, \end{aligned}$$

where a constant $K(0) = K(0; w, A)$ is defined in Section A.1. Again, identity (1.18) together with the estimates above implies the desired result if we choose $\gamma_1 = (\frac{1}{2} - \varepsilon)/(d+4)$ and put

$$\sigma_1(w; A, L) := K(0) + \sum_{\mathbf{c}\neq 0} \eta(\mathbf{c})\sigma^*_{\mathbf{c}}(A)\sigma^{\mathbf{c}}_\infty(w; A, L)\,. \tag{1.24}$$

Finiteness of the products $\sigma^*_{\mathbf{c}}(A)$ follow from Lemma A.2 while the claimed in the theorem estimate for the constant $\sigma_1(w; A, L)$ is established in Section A.3.

## 2 Series $S_q$

Now we start to prove Theorem 1.3, following the scheme presented in Section 1.2. Part of the assertions, forming the proof, do not use that $d \geq 5$. So below in all assertion involving the dimension $d$, we indicate the real requirements on $d$. We recall that the constants in estimates may depend on $d$ and $A$, but this dependence is not indicated (see *Notation and agreements*).

In the present section we analyse the sums $S_q(\mathbf{c}) = S_q(\mathbf{c}; A, L^2 m)$ entering, in particular, the definitions of the singular series $\sigma(A, L^2 m)$ and $\sigma_p(A, L^2 m)$.

**Lemma 2.1** (Lemma 25 in [8]). *For any $d \geq 1$ we have $|S_q(\mathbf{c}; A, L^2 m)| \lesssim q^{d/2+1}$, uniformly in $\mathbf{c} \in \mathbb{Z}^d$.*

*Proof.* According to (1.7),

$$
\begin{aligned}
|S_q(\mathbf{c})|^2 &\le \phi(q) \sideset{}{^*}\sum_{a(\mathrm{mod}\ q)} \Big| \sum_{\mathbf{b}(\mathrm{mod}\ q)} e_q(aF^{L^2m}(\mathbf{b}) + \mathbf{c}\cdot\mathbf{b})\Big|^2 \\
&= \phi(q) \sideset{}{^*}\sum_{a(\mathrm{mod}\ q)} \sum_{\mathbf{u},\mathbf{v}(\mathrm{mod}\ q)} e_q\big(a(F^{L^2m}(\mathbf{u}) - F^{L^2m}(\mathbf{v})) + \mathbf{c}\cdot(\mathbf{u}-\mathbf{v})\big),
\end{aligned}
\tag{2.1}
$$

where $\phi(q)$ is the Euler totient function. Since $F^t(\mathbf{z}) = \frac{1}{2}A\mathbf{z}\cdot\mathbf{z} - t$, then

$$
F^{L^2m}(\mathbf{u}) - F^{L^2m}(\mathbf{v}) = (A\mathbf{v})\cdot\mathbf{w} + F(\mathbf{w}) = \mathbf{v}\cdot A\mathbf{w} + F(\mathbf{w}).
$$

So

$$
e_q\big(a(F^{L^2m}(\mathbf{u}) - F^{L^2m}(\mathbf{v})) + \mathbf{c}\cdot(\mathbf{u}-\mathbf{v})\big) = e_q\big(aF(\mathbf{w}) + \mathbf{c}\cdot\mathbf{w}\big)\, e_q(a\mathbf{v}\cdot A\mathbf{w}).
$$

Now we see that the summation over $\mathbf{v}$ in (2.1) produces a zero contribution, unless each component of the vector $A\mathbf{w}$ is divisible by $q$. This property holds for at most $N(\Delta)$ possible values of $\mathbf{w}$, where $\Delta = \det A$. Thus,

$$
|S_q(\mathbf{c})|^2 \lesssim \phi(q) \sideset{}{^*}\sum_{a(\mathrm{mod}\ q)} \sum_{\mathbf{v}(\mathrm{mod}\ q)} 1 \le \phi^2(q)\, q^d.
$$

□

The lemma's assertion shows that the sums $\sigma_p^{\mathbf{c}}$, defined in (1.10), are finite:

**Corollary 2.2.** *If $d \ge 3$, for any prime $p$ we have $\big|\sigma_p^{\mathbf{c}}(A, L^2m)\big| \lesssim 1$.*

Recall that $\sigma(A, L^2m) = \prod_p \sigma_p(A, L^2m)$ (see (1.11)).

**Lemma 2.3.** *For any $d \ge 5$ and $1 \le X \le \infty$ we have*

$$
\sum_{q\le X} q^{-d} S_q(0) = \sigma(A, L^2m) + O(X^{-d/2+2}).
$$

*In particular, $\sigma(A, L^2m) = \sum_{q=1}^{\infty} q^{-d} S_q(0)$. So $|\sigma(A, L^2m)| \lesssim 1$ in view Lemma 2.1.*

*Proof.* We start by showing that

$$
S_{qq'}(0) = S_q(0) S_{q'}(0)\,, \tag{2.2}
$$

whenever $(q,q')=1$ (cf. Lemma 23 from [8]). By definition

$$S_{qq'}(0)=\sideset{}{^*}\sum_{a(\mathrm{mod}\ qq')}\sum_{\mathbf{v}(\mathrm{mod}\ qq')}e_{qq'}(aF^{L^2m}(\mathbf{v}))\,.$$

When $(q,q')=1$ we can replace the summation on $a\ (\mathrm{mod}\ qq')$ by a double summation on $a_q$ modulo $q$ and $a_{q'}$ modulo $q'$ by writing $a=qa_{q'}+q'a_q$. Thus

$$S_{qq'}(0)=\sideset{}{^*}\sum_{a_q(\mathrm{mod}\ q)}\sideset{}{^*}\sum_{a_{q'}(\mathrm{mod}\ q')}\sum_{\mathbf{v}(\mathrm{mod}\ qq')}e_q(a_qF^{L^2m}(\mathbf{v}))e_{q'}(a_{q'}F^{L^2m}(\mathbf{v}))\,.$$

Then we replace the summation on $\mathbf{v}\ (\mathrm{mod}\ qq')$ with the double summation on $\mathbf{v}_q$ modulo $q$ and $\mathbf{v}_{q'}$ modulo $q'$ by writing $\mathbf{v}=q\bar{q}\mathbf{v}_{q'}+q'\bar{q}'\mathbf{v}_q$, where $\bar{q}$ and $\bar{q}'$ are defined through relations $q\bar{q}=1\,(\mathrm{mod}\ q')$ and $q'\bar{q}'=1\,(\mathrm{mod}\ q)$. We observe that

$$F^{L^2m}(\mathbf{v})=q^2\bar{q}^2F(\mathbf{v}_{q'})+q'^2\bar{q}'^2F(\mathbf{v}_q)+q\bar{q}q'\bar{q}'A\mathbf{v}_{q'}\cdot\mathbf{v}_q-L^2m\,,$$

so that

$$e_q(a_qF^{L^2m}(\mathbf{v}))=e_q(a_qq'^2\bar{q}'^2F(\mathbf{v}_q)-a_qL^2m)=e_q(a_qF^{L^2m}(\mathbf{v}_q)),$$

by the definition of $\bar{q}'$ and since $e_q(qN)=1$ for any integer $N$. Similar,

$$e_{q'}(a_{q'}F^{L^2m}(\mathbf{v}))=e_{q'}(a_{q'}F^{L^2m}(\mathbf{v}_{q'}))\,.$$

This gives (2.2).

Next we note that, due to Lemma 2.1,

$$\sum_{q\ge X}q^{-d}|S_q(0)|\lesssim\sum_{q\ge X}q^{-d/2+1}\lesssim X^{-d/2+2}. \tag{2.3}$$

By (2.2) and the definition of $\sigma$,

$$\sigma=\lim_{n\to\infty}\sigma^n,\qquad \sigma^n=\prod_{p\le n}\sum_{l=0}^{n}p^{-dl}S_{p^l}(0)=\sum_{q\in P_n}q^{-d}S_q(0),$$

where $p$ are primes and $P_n$ denotes the set of natural numbers $q$ with prime factorization of the form $q=p_1^{k_1}\cdots p_m^{k_m}$, where $2\le p_1<p_2\cdots<p_m\le n$, $k_j\le n$ and $m\ge 0$ ($m=0$ corresponds to $q=1$). Since any $q\le n$ belongs to $P_n$, then according to (2.3),

$$\Big|\sum_{q\in P_N}q^{-d}S_q(0)-\sum_{q\le X}q^{-d}S_q(0)\Big|\lesssim X^{-d/2+2}\quad\forall\,N\ge X,$$

for any finite $X>0$. Passing in this estimate to a limit as $N\to\infty$ we recover the assertion if $X<\infty$. Then the result with $X=\infty$ follows in an obvious way. □

# 3 Singular integrals $I^0_q$

## 3.1 Properties of $h(x,y)$

We construct the function $h(x,y) \in C^\infty(\mathbb{R}_>, \mathbb{R})$, entering Theorem 1.1, starting from the weight function $w_0 \in C^\infty_0(\mathbb{R})$, defined as

$$w_0(x) = \begin{cases} \exp\left(\frac{1}{x^2-1}\right) & \text{for } |x|<1 \\ 0 & \text{for } |x| \geq 1 \end{cases} . \tag{3.1}$$

We denote $c_0 := \int_{-\infty}^{\infty} w_0(x)\,dx$ and introduce the shifted weight function

$$\omega(x) = \tfrac{4}{c_0} w_0(4x-3)\,,$$

which of course belongs to $C^\infty_0(\mathbb{R})$. Obviously, $0 \leq \omega \leq 4e^{-1}/c_0$, $\omega$ is supported on $(1/2,1)$, and $\int_{-\infty}^{\infty} \omega(x)\,dx = 1$.

The required function $h : \mathbb{R}_{>0} \times \mathbb{R} \mapsto \mathbb{R}$ is defined in terms of $\omega$ as $h(x,y) := h_1(x) - h_2(x,y)$ with

$$h_1(x) := \sum_{j=1}^{\infty} \frac{1}{xj}\omega(xj)\,, \quad h_2(x,y) := \sum_{j=1}^{\infty} \frac{1}{xj}\omega\left(\frac{|y|}{xj}\right). \tag{3.2}$$

For any fixed pair $(x,y)$ each of the two sum in $j$ contains a finite number of nonzero terms. So $h$ is a smooth function.

In [8], Section 3, it is shown how to derive Theorem 1.1 from the definition (3.2).[4] Here we limit ourselves to providing some relevant properties of $h$, proved in Section 4 of [8]. In particular these properties imply that for small $x$, $h(x,y)$ behaves as the Dirac delta function in $y$

**Lemma 3.1** (Lemma 4 in [8])**.** *We have:*

1. $h(x,y) = 0$ *if* $x \geq 1$ *and* $|y| \leq x/2$.
2. *If* $x \leq 1$ *and* $|y| \leq x/2$, *then* $h(x,y) = h_1(x)$, *and for any* $m \geq 0$
$$\Big|\frac{\partial^m h(x,y)}{\partial x^m}\Big| \lesssim_m \frac{1}{x^{m+1}}\,.$$
3. *If* $|y| \geq x/2$, *then for any* $m,n \geq 0$
$$\Big|\frac{\partial^{m+n} h(x,y)}{\partial x^m \partial y^n}\Big| \lesssim_{m,n} \frac{1}{x^{m+1}|y|^n}\,.$$

[4] Actually it is proved there that any function $h$ defined through (3.2) with arbitrary weight function $\omega \in C^\infty_0(\mathbb{R})$, supported on $[1/2,1]$, may provide a representation of $\delta(n)$.

**Lemma 3.2** (Lemma 5 in [8]). *Let $m, n, N \geq 0$. Then for any $x, y$*

$$\Big|\frac{\partial^{m+n} h(x,y)}{\partial x^m \partial y^n}\Big| \lesssim_{N,m,n} \frac{1}{x^{1+m+n}} \left(\delta(n) x^N + \min\left\{1, (x/|y|)^N\right\}\right) .$$

Lemma 3.2 with $m = n = N = 0$ immediately implies

**Corollary 3.3.** *For any $x, y \in \mathbb{R}_> \times \mathbb{R}$ we have $|h(x,y)| \lesssim 1/x$.*

**Lemma 3.4** (Lemma 6 in [8]). *Fix $X \in \mathbb{R}_{>0}$ and $0 < x < C \min\{1, X\}$, for some $C > 0$. Then for any $N \geq 0$,*

$$\int_{-X}^{X} h(x,y)\, dy = 1 + O_{N,C}\left(X x^{N-1}\right) + O_{N,C}\left(\frac{x^N}{X^N}\right).$$

**Lemma 3.5** (Lemma 8 in [8]). *Fix $X \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}$. Let $x < C \min\{1, X\}$ for $C > 0$. Then*

$$\Big|\int_{-X}^{X} y^n h(x,y)\, dy\Big| \lesssim_{N,C} X^n \left(X x^{N-1} + \frac{x^N}{X^N}\right) .$$

The previous results are used to prove the key Lemma 9 of [8], which can be extended to the following

**Lemma 3.6.** *Let a function $f \in \mathcal{C}^{M-1,0}(\mathbb{R}) \cap L^1(\mathbb{R})$, $M \geq 1$, be such that its $(M-1)$-st derivative $f^{(M-1)}$ is absolutely continuous on $[-1, 1]$, and let $0 < x \leq C$ for some $C > 0$. Then for any $0 < \beta \leq 1$ and any $N \geq 0$,*

$$\begin{aligned}\int_{\mathbb{R}} f(y) h(x,y)\, dy =& f(0) + O_M\left(\frac{x^M}{\beta^{M+1}} \frac{1}{X} \int_{-X}^{X} |f^{(M)}(y)|\, dy\right) \\ &+ O_{N,C}\left((x^N + \beta^N)\left(\|f\|_{M-1,0} + x^{-1}|f|_{L_1}\right)\right) ,\end{aligned} \tag{3.3}$$

*where $X := \min\{1, x/\beta\}$.*

*Proof.* By Lemma 3.2 with $m = n = 0$, for any $N \geq 0$ we have $|h(x,y)| \lesssim_N (x^N + \beta^N) x^{-1}$ if $|y| \geq X$. So the tail-integral for $\int f h\, dy$ may be bounded as

$$\left|\int_{|y|\geq X} f(y) h(x,y)\, dy\right| \lesssim_N (x^N + \beta^N) x^{-1} \int_{\mathbb{R}} |f(y)| \lesssim_N (x^N + \beta^N) x^{-1} |f|_{L_1} . \tag{3.4}$$

For the integral in $|y| < X$, instead we take the Taylor expansion of $f(y)$ around zero and get that

$$\begin{aligned}\int_{-X}^{X} f(y)h(x,y)\,dy = \sum_{j=0}^{M-1} \frac{f^{(j)}(0)}{j!} \int_{-X}^{X} y^j h(x,y)\,dy \\ + O_M\left(\frac{X^M}{x}\int_{-X}^{X} |f^{(M)}(y)|\,dy\right),\end{aligned} \tag{3.5}$$

by Corollary 3.3. Next we use Lemma 3.4 with $N$ replaced by $N+1$ to get that

$$f(0)\int_{-X}^{X} h(x,y)\,dy = f(0) + O_{N,C}\left(\|f\|_{0,0}\left(Xx^N + \frac{x^{N+1}}{X^{N+1}}\right)\right), \tag{3.6}$$

while by Lemma 3.5, for any $j > 0$ we have

$$\left|\frac{f^{(j)}(0)}{j!}\int_{-X}^{X} y^j h(x,y)\,dy\right| \lesssim_{N,j,C} \|f\|_{j,0} X^j \left(Xx^N + \frac{x^{N+1}}{X^{N+1}}\right). \tag{3.7}$$

Putting together (3.4)–(3.7), we obtain the desired estimate. Indeed, since $X \le x/\beta$, then the term $O_M$ in (3.5) is bounded by that in (3.3). Moreover, as $(x/X)^{N+1} = \max(x^{N+1}, \beta^{N+1}) \lesssim_C Cx^N + \beta^N$, then the brackets in (3.6) and (3.7) are $\lesssim_C x^N + \beta^N$, where we also used that $X \le 1$. □

Lemma 3.6 is needed for the proof of Theorem 1.4, while for Theorem 1.3 we only need its simplified version:

**Corollary 3.7.** *Let an integrable function $f$ belongs to the class $\mathcal{C}^{M,0}(\mathbb{R})$, $M \in \mathbb{N}$, and $0 < x \le C$ for some $C > 0$. Then, for any $0 < \delta < 1$,*

$$\int_{\mathbb{R}} f(y)h(x,y)\,dy = f(0) + O_{M,C,\delta}\left(x^{M-\delta}\left(\|f\|_{M,0} + |f|_{L_1}\right)\right).$$

*Proof.* The assertion follows from Lemma 3.6 by choosing for any $0 < \delta < 1$, $\beta = x^{\delta/(M+1)}$ if $x \le 1$ and $\beta = 1$ if $x > 1$. Indeed, then for $0 < x \le 1$ we have that $x^M \beta^{-(M+1)} = x^{M-\delta}$, and that

$$(x^N + \beta^N)x^{-1} \le 2\beta^N x^{-1} \le 2x^{M-\delta} \qquad \text{if } N \ge N_\delta = (M-\delta+1)(M+1)/\delta.$$

While if $1 \le x \le C$, then $x^M \le C^\delta x^{M-\delta}$, and choosing $N = 0$ we get that $(x^N + 1) = 2 \le 2x^{M-\delta}$. The obtained relations imply the assertion. □

### 3.2 Approximation for $I_q(0)$

In what follows it is convenient to write the integrals $I_q(\mathbf{c}; A, L^2 m)$ as

$$I_q(\mathbf{c}) = L^d \tilde{I}_q(\mathbf{c}), \tag{3.8}$$

where

$$\tilde{I}_q(\mathbf{c}) = \tilde{I}_q(\mathbf{c}; A, m, L) = \int_{\mathbb{R}^d} w(\mathbf{z})\, h\left(\frac{q}{L}, F^m(\mathbf{z})\right) e_q(-\mathbf{z}\cdot\mathbf{c}L)\, d\mathbf{z}\,. \tag{3.9}$$

The proposition below replaces Lemmas 11, 13 and Theorem 3 of [8]. In difference with those results we do not assume that $0 \notin \operatorname{supp} w$. Since for $\mathbf{c} = 0$ the exponent $e_q$ in the definition of the integral $I_q(\mathbf{c})$ equals one, we can consider $I_q(0)$ as a function of a *real* argument $q \in \mathbb{R}$, and we do so in the proposition below; we will use this in Appendix A.

**Proposition 3.8.** *Let $q \in \mathbb{R}$, $q \le CL$ with some $C > 0$.*

*a) If $d \ge 5$ and $\mathbb{N} \ni M < d/2 - 1$, then for any $\delta > 0$,*

$$\begin{aligned} I_q(0; A, m, L) &= L^d \sigma_\infty(w; A, m) \\ &\quad + O_{m,M,C,\delta}\left(q^{M-\delta} L^{d-M+\delta} \|w\|_{M,d+1}\right). \end{aligned} \tag{3.10}$$

*b) If $d = 4$, $\mathbb{N} \ni M \le d/2 - 1$ and $m = 0$, then for any $0 < \beta \le 1$ and $N \ge 0$,*

$$\begin{aligned} I_q(0; A, 0, L) =& L^d \sigma_\infty(w; A, 0) + O\left(\beta^{-M-1} q^M L^{d-M} \left\langle \log\left(\frac{q}{L\beta}\right)\right\rangle \|w\|_{M,d+1}\right) \\ &+ O_{C,N}\left((q^N L^{d-N} + \beta^N)(\|w\|_{M-1,d+1} + Lq^{-1}\|w\|_{0,d+1})\right). \end{aligned} \tag{3.11}$$

*Proof.* For $d \ge 4$, applying the co-area formula (see e.g. [3], p.138) to the integral in (3.9) with $\mathbf{c} = 0$ we get that

$$\tilde{I}_q(0) = \int_{\mathbb{R}} \mathcal{I}(m+t) h(q/L, t)\, dt\,, \qquad \mathcal{I}(t) = \int_{\Sigma_t} w(\mathbf{z})\, \mu^{\Sigma_t}(d\mathbf{z})\,, \tag{3.12}$$

where the measure $\mu^{\Sigma_t}$ is the same as in (1.9). By Theorem 7.3,

$$\|\mathcal{I}\|_{k,\tilde{K}} \lesssim_{k,K,\tilde{K}} \|w\|_{k,K} \quad \text{if} \quad \tilde{K} < \frac{K+2-d}{2}, \qquad K > d, \tag{3.13}$$

and $k < d/2 - 1$. Denote $f^m(y) = \mathcal{I}(m + y)$. Then $\|f^m\|_{k,\tilde{K}} \lesssim_{m,\tilde{K}} \|\mathcal{I}\|_{k,\tilde{K}}$, and by (3.13)

$$|f^m|_{L_1} = |\mathcal{I}|_{L_1} \lesssim \|\mathcal{I}\|_{0,4/3} \lesssim \|w\|_{0,d+1}. \tag{3.14}$$

To prove a) we apply Corollary 3.7 with $f = f^m$ and $x = q/L$ to the first integral in (3.12). Note that $f^m(0) = \mathcal{I}(m) = \sigma_\infty(w; A, m)$. Then, using (3.13) with $\tilde{K} = 0$, $K = d+1$ and $k = M$ jointly with (3.14) we get that

$$\tilde{I}_q(0) = \sigma_\infty(w) + O_{M,m,C,\delta}\big(q^{M-\delta}L^{-M+\delta}\|w\|_{M,d+1}\big).$$

So (3.10) follows.

To establish (3.11), we apply Lemma 3.6 to write the integral in (3.12) with $m = 0$ as

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{I}(t)h(x,t)\,dt = \mathcal{I}(0) &+ O_M\left(\beta^{-M-1}x^M\left(\frac{1}{X}\int_{-X}^{X}|\mathcal{I}^{(M)}(t)|\,dt\right)\right)\\ &+ O_{C,N}\left((x^N+\beta^N)(\|\mathcal{I}\|_{M-1,0} + x^{-1}|\mathcal{I}|_{L_1})\right),\end{aligned}$$

where $x = q/L$ and $X = \min\{1, x/\beta\}$. By Theorem 7.3,

$$\int_{-X}^{X}|\mathcal{I}^{(M)}(t)|\,dt \lesssim X\langle\log X\rangle\|w\|_{M,d+1}\,.$$

Using this estimate jointly with (3.13) and (3.14) we arrive at (3.11).

# 4 The $J_0$ term

**Proposition 4.1.** *Let $d \geq 5$. Then for any $0 < \gamma_2 < 1$,*

$$\big|J_0 - L^d\sigma_\infty(w)\sigma(A, L^2m)\big| \lesssim_{\gamma_2,m} L^{\frac{d}{2}+2+\gamma_2(\frac{d}{2}-1)}\|w\|_{\lceil d/2\rceil-2,d+1}.$$

*Proof.* To establish the result we write $J_0$ in the form (1.21). Then the assertion follows from Lemmas 4.2 and 4.3 below which estimate the terms $J_0^+$ and $J_0^-$, noting that $|w|_{L_1} \lesssim \|w\|_{0,d+1}$. □

**Lemma 4.2.** *Assume that $w \in L_1(\mathbb{R}^d)$ and $d \geq 3$. Then we have the bound $|J_0^+| \lesssim L^{d/2+2+\gamma_2(d/2-1)}|w|_{L_1}$, for any $\gamma_2 \in (0,1)$.*

*Proof.* Since according to Lemma 2.1 $|S_q(0)| \lesssim q^{d/2+1}$, then

$$|J_0^+| \lesssim \sum_{q>L^{1-\gamma_2}} q^{-d/2+1}I_q(0).$$

Writing integral $I_q$ as in (3.8), by Corollary 3.3 we get $|I_q(0)| \lesssim \dfrac{L^{d+1}}{q}|w|_{L_1}$. Therefore,

$$\begin{aligned}|J_0^+| &\lesssim L^{d+1}|w|_{L_1}\sum_{q>L^{1-\gamma_2}} q^{-d/2} \lesssim L^{d+1}|w|_{L_1}L^{(-d/2+1)(1-\gamma_2)}\\ &= L^{d/2+2+\gamma_2(d/2-1)}|w|_{L_1}.\end{aligned}$$

□

**Lemma 4.3.** *Let $d \geq 5$. Then for any $\gamma_2 \in (0,1)$,*

$$J_0^- = L^d \sigma_\infty(w)\sigma(A, L^2 m) + O_{\gamma_2,m}\big(L^{d/2+2+\gamma_2(d/2-2)}\|w\|_{\lceil d/2\rceil-2,d+1}\big).$$

*Proof.* Inserting (3.10) with $C = 1$ into the definition of the term $J_0^-$, we get $J_0^- = I_A + I_B$, where

$$I_A := L^d \sigma_\infty(w) \sum_{q \leq L^{1-\gamma_2}} q^{-d} S_q(0),$$
$$|I_B| \lesssim_{M,\delta,m} L^{d-M+\delta}\|w\|_{M,d+1} \sum_{q \leq L^{1-\gamma_2}} S_q(0) q^{-d+M},$$

for $M < d/2 - 1$ and any $\delta > 0$. Lemma 2.3 implies that

$$\sum_{q \leq L^{1-\gamma_2}} q^{-d} S_q(0) = \sigma(A, L^2 m) + O(L^{(-d/2+2)(1-\gamma_2)}),$$

so

$$I_A = L^d \sigma_\infty(w)\sigma(A, L^2 m) + O(\sigma_\infty(w) L^{d/2+2+\gamma_2(d/2-2)}),$$

whereas $|\sigma_\infty(w)| = |\mathcal{I}(m)| \leq \|\mathcal{I}\|_{0,0} \leq \|w\|_{0,d+1}$ on account of (3.13). As for the term $I_B$, Lemma 2.1 implies that

$$|I_B| \lesssim_{M,\delta,m} L^{d-M+\delta}\|w\|_{M,d+1} \sum_{q \leq L^{1-\gamma_2}} q^{-d/2+1+M}.$$

Choosing $M = \lceil d/2 \rceil - 2$ and $\delta = \gamma_2/2$, we get

$$|I_B| \lesssim_{\delta,m} \|w\|_{\lceil d/2\rceil-2,d+1} L^{d/2+2+\delta} \ln L \lesssim_{\gamma_2,m} \|w\|_{\lceil d/2\rceil-2,d+1} L^{d/2+2+\gamma_2}.$$

□

# 5 The $J_>^{\gamma_1}$ term

We provide here an estimate of the term $J_>^{\gamma_1}$ defined in (1.19). The key point of the proof is an adaptation of Lemma 19 of [8] to our case. We recall the notation (3.8).

**Proposition 5.1.** *For any $d \geq 1$, $N > 0$ and $\mathbf{c} \neq 0$,*

$$|\tilde{I}_q(\mathbf{c})| \lesssim_{N,m} \frac{L}{q} |\mathbf{c}|^{-N} \|w\|_{N,2N+d+1} \tag{5.1}$$

*Proof.* Let $f_q(\mathbf{z}) := w(\mathbf{z})\, h\left(\frac{q}{L}, F^m(\mathbf{z})\right)$. Since

$$\frac{i}{2\pi}\frac{q}{L}|\mathbf{c}|^{-2}\,(\mathbf{c}\cdot\nabla_{\mathbf{z}})\, e_q(-\mathbf{z}\cdot\mathbf{c}L) = e_q(-\mathbf{z}\cdot\mathbf{c}L)\,,$$

then integrating by parts $N$ times the integral (3.9) we get that

$$\begin{aligned}\left|\tilde{I}_q(\mathbf{c})\right| &\le \left(\frac{q}{2\pi L}|\mathbf{c}|^{-2}\right)^N \int_{\mathbb{R}^d} \left|(\mathbf{c}\cdot\nabla_{\mathbf{z}})^N f_q(\mathbf{z})\right| d\mathbf{z} \\ &\lesssim_N \left(\frac{q}{L}\right)^N |\mathbf{c}|^{-N} \sum_{0\le n\le N} \int_{\mathbb{R}^d} \max_{0\le l\le n/2} \left|\frac{\partial^{n-l}}{\partial y^{n-l}} h\left(\frac{q}{L}, F^m(\mathbf{z})\right)\right| \\ &\qquad \times |\mathbf{z}|^{n-2l} \left|\nabla_{\mathbf{z}}^{N-n} w(\mathbf{z})\right| d\mathbf{z}\,,\end{aligned}$$

where $\dfrac{\partial}{\partial y}h$ stands for the derivative of $h$ with respect to the second argument.

Assume first that $q \le L$. Then, by Lemma 3.2 with $N = 0$,

$$\max_{0\le l\le n/2} \left|\frac{\partial^{n-l}}{\partial y^{n-l}} h\left(\frac{q}{L}, F^m(\mathbf{z})\right)\right| |\mathbf{z}|^{n-2l} \left|\nabla_{\mathbf{z}}^{N-n} w(\mathbf{z})\right| \le (L/q)^{n+1} \langle \mathbf{z}\rangle^{-d-1} \|w\|_{N-n,n+d+1}\,.$$

This implies (5.1) since $n \le N$. Let now $q > L$. Then, due to item 1 of Lemma 3.1, $h$ is different from zero only if

$$2|F^m(\mathbf{z})| > \frac{q}{L}. \tag{5.2}$$

Then for such $\mathbf{z}$ and for $l \le n$, item 3 of Lemma 3.1 implies that

$$\left|\frac{\partial^{n-l}}{\partial y^{n-l}} h\left(\frac{q}{L}, F^m(\mathbf{z})\right)\right| \lesssim_{n-l} \frac{L}{q}\frac{1}{|F^m(\mathbf{z})|^{n-l}} \lesssim_{n-l} \left(\frac{L}{q}\right)^{n-l+1}.$$

So

$$\max_{0\le l\le n/2} \left|\frac{\partial^{n-l}}{\partial y^{n-l}} h\left(\frac{q}{L}, F^m(\mathbf{z})\right)\right| |\mathbf{z}|^{n-2l} \left|\nabla_{\mathbf{z}}^{N-n} w(\mathbf{z})\right| \lesssim \max_{0\le l\le n} \frac{(L/q)^{n-l+1}}{\langle \mathbf{z}\rangle^{2(N-n+l)}} \frac{\|w\|_{N-n,2N-n+d+1}}{\langle \mathbf{z}\rangle^{d+1}}\,.$$

Since from (5.2) we have that $q/L \lesssim_m \langle \mathbf{z}\rangle^2$, then the first fraction above is bounded by $(L/q)^{N+1}$, and again (5.1) follows. □

As a corollary we get an estimate for $J_>^{\gamma_1}$:

**Corollary 5.2.** *For $J_{>}^{\gamma_1}$ defined in (1.19) with $\gamma_1 \in (0,1)$ and $d \geq 3$ we have*

$$|J_{>}^{\gamma_1}| \lesssim_{\gamma_1,m} \|w\|_{N_0,2N_0+d+1}\,,$$

*where $N_0 := \lceil d + (d+1)/\gamma_1 \rceil$.*

*Proof.* Denoting by $|\cdot|_1$ the $l^1$-norm, by the definition of $J_{>}^{\gamma_1}$ we have

$$\begin{aligned}
|J_{>}^{\gamma_1}| &\lesssim \sum_{s\geq L^{\gamma_1}} s^{d-1} \sum_{q=1}^{\infty} q^{-d} \sup_{|\mathbf{c}|_1=s} |S_q(\mathbf{c})||I_q(\mathbf{c})| \\
&\lesssim \sum_{s\geq L^{\gamma_1}} s^{d-1} \sum_{q=1}^{\infty} q^{1-d/2} L^d \sup_{|\mathbf{c}|_1=s} |\tilde{I}_q(\mathbf{c})| \\
&\lesssim_{N,m} \sum_{s\geq L^{\gamma_1}} s^{d-1} \sum_{q=1}^{\infty} q^{-d/2} s^{-N} L^{d+1} \|w\|_{N,2N+d+1}\,,
\end{aligned}$$

where the second line follows from Lemma 2.1, while the third one – from Proposition 5.1. The sum in $q$ is bounded by a constant. Choosing $N = N_0$ we get that

$$L^{d+1} \sum_{s\geq L^{\gamma_1}} s^{d-1} s^{-N} \leq L^{d+1} \sum_{s\geq L^{\gamma_1}} s^{-1-(d+1)/\gamma_1} \lesssim 1\,.$$

This concludes the proof. □

# 6 The $J_{<}^{\gamma_1}$ term

## 6.1 The estimate

Our next (and final) goal is to estimate the term $J_{<}^{\gamma_1}$ from (1.18).

**Proposition 6.1.** *For any $d \geq 3$ and $\gamma_1 \in (0,1)$,*

$$|J_{<}^{\gamma_1}| \lesssim_{\gamma_1,m} L^{d/2+2+\gamma_1(d+1)} \left( \|w\|_{\bar{N},d+5} + \|w\|_{0,\bar{N}+3d+4} \right)\,,$$

*where $\bar{N} = \bar{N}(d,\gamma_1) := \lceil d^2/\gamma_1 \rceil - 2d$.*

Proposition 6.1 will follow from the next lemma which is a modification of Lemma 22 in [8] and is proved in the next subsection:

**Lemma 6.2.** *For any $d \geq 1$ and $|\mathbf{c}| \leq L^{\gamma_1}$, $\mathbf{c} \neq 0$,*

$$|I_q(\mathbf{c})| \lesssim_{\gamma_1,m} L^{d/2+1+\gamma_1} q^{d/2-1} \left( \|w\|_{\bar{N},d+5} + \|w\|_{0,\bar{N}+3d+4} \right)\,,$$

*where $\bar{N}$ and $\gamma_1$ are the same as above.*

*Proof of Proposition 6.1.* Accordingly to Lemma 2.1,

$$|J^{\gamma_1}_<| \lesssim \sum_{\mathbf{c}\neq 0,\, |\mathbf{c}|\le L^{\gamma_1}} \sum_{q=1}^{\infty} q^{-d} q^{d/2+1} |I_q(\mathbf{c})| \lesssim L^{d\gamma_1} \max_{\mathbf{c}\neq 0:\, |\mathbf{c}|\le L^{\gamma_1}} |I_q(\mathbf{c})| \sum_{q=1}^{\infty} q^{-d/2+1}$$
$$= L^{d\gamma_1} \big( \sum_{q<L} + \sum_{q\ge L} \big) q^{-d/2+1} \max_{\mathbf{c}\neq 0:\, |\mathbf{c}|\le L^{\gamma_1}} |I_q(\mathbf{c})| =: J_- + J_+.$$

Corollary 3.3 together with (3.8), (3.9) implies

$$|I_q(\mathbf{c})| \lesssim \frac{L^{d+1}}{q} |w|_{L_1}\,, \tag{6.1}$$

so that

$$J_+ \lesssim L^{d\gamma_1} L^{d+1} |w|_{L_1} \sum_{q\ge L} q^{-d/2} \lesssim L^{d\gamma_1 + d/2+2} |w|_{L_1} \lesssim L^{d\gamma_1+d/2+2} \|w\|_{0,d+1}.$$

From other hand, from Lemma 6.2 we get

$$J_- \lesssim_{\gamma_1,m} L^{d\gamma_1} L^{d/2+1+\gamma_1} \left( \|w\|_{\bar N, d+5} + \|w\|_{0,\bar N+3d+4} \right) \sum_{q<L} 1$$
$$\le \left( \|w\|_{\bar N, d+5} + \|w\|_{0,\bar N+3d+4} \right) L^{\gamma_1(d+1)+d/2+2}\,.$$

□

## 6.2 Proof of Lemma 6.2

We begin with

### 6.2.1 Application of the inverse Fourier transform

Note that the proof is nontrivial only for $q \lesssim L$ since for any $\alpha > 0$, the bound (6.1) implies that

$$|I_q(\mathbf{c})| \lesssim_\alpha L^d |w|_{L_1} \lesssim_\alpha L^{d/2+1} q^{d/2-1} |w|_{L_1} \quad \text{if} \quad q \ge \alpha L$$

and, again, $|w|_{L_1} \lesssim \|w\|_{0,d+1}$.

Let us take a small enough $\alpha = \alpha(d, \gamma_1, A) \in (0,1)$ and assume that $q < \alpha L$. Consider the function $w_2(x) = 1/(1+x^2)$ and set

$$\tilde w(\mathbf{z}) := \frac{w(\mathbf{z})}{w_2(F^m(\mathbf{z}))} = w(\mathbf{z})(1 + F^m(\mathbf{z})^2).$$

Let

$$p(t) := \int_{-\infty}^{+\infty} w_2(v) h(q/L, v) e(-tv)\, dv, \quad e(x) := e_1(x) = e^{2\pi i x}. \tag{6.2}$$

This is the Fourier transform of the function $w_2(\cdot)h(q/L, \cdot)$. Then, expressing $w_2 h$ via $p$ by the inverse Fourier transform and writing $w(\mathbf{z}) = \tilde{w}(\mathbf{z}) w_2(F^m(\mathbf{z}))$, we find that

$$w(\mathbf{z}) h(q/L, F^m(\mathbf{z})) = \tilde{w}(\mathbf{z}) \int_{-\infty}^{+\infty} p(t) e(tF^m(\mathbf{z}))\, dt.$$

Inserting this representation into (3.9) we get

$$\tilde{I}_q(\mathbf{c}) = \int_{-\infty}^{+\infty} dt\, p(t) e(-tm) \int_{\mathbb{R}^d} d\mathbf{z}\, \tilde{w}(\mathbf{z}) e\big(tF(\mathbf{z}) - \mathbf{u} \cdot \mathbf{z}\big), \quad \mathbf{u} := \mathbf{c}\, L/q.$$

Note that

$$|\mathbf{u}| \geq L/q \geq \alpha^{-1} > 1$$

since $\mathbf{c} \neq 0$ and $q < \alpha L$. Now let us denote $W_0(x) = c_0^{-d} \prod_{i=1}^d w_0(x_i)$ (see (3.1)). Then $W_0 \in C_0^\infty(\mathbb{R}^d)$, $W_0 \geq 0$ and

$$\operatorname{supp} W_0 = [-1, 1]^d \subset \{|x| \leq \sqrt{d}\}, \quad \int W_0 = 1. \tag{6.3}$$

Let us set $\delta = |\mathbf{u}|^{-1/2} < \sqrt{\alpha}$ and write $\tilde{w}$ as

$$\tilde{w}(\mathbf{z}) = \delta^{-d} \int W_0\left(\frac{\mathbf{z} - \mathbf{a}}{\delta}\right) \tilde{w}(\mathbf{z})\, d\mathbf{a}.$$

Then setting $\mathbf{b} := \dfrac{\mathbf{z} - \mathbf{a}}{\delta}$ we get that

$$|\tilde{I}_q(\mathbf{c})| \leq \int_{\mathbb{R}^d} d\mathbf{a} \int_{-\infty}^{+\infty} dt\, |p(t)| |I_{\mathbf{a},t}|,$$

where in view of (6.3),

$$I_{\mathbf{a},t} := \int_{\{|\mathbf{b}| \leq \sqrt{d}\}} W_0(\mathbf{b}) \tilde{w}(\mathbf{z})\, e(tF(\mathbf{z}) - \mathbf{u} \cdot \mathbf{z})\, d\mathbf{b}, \qquad \mathbf{z} := \mathbf{a} + \delta \mathbf{b}.$$

Consider the exponent in the integral $I_{\mathbf{a},t}$:

$$f(\mathbf{b}) = f_{\mathbf{a},t}(\mathbf{b}) := tF(\mathbf{a} + \delta\mathbf{b}) - \mathbf{u} \cdot (\mathbf{a} + \delta\mathbf{b}).$$

At the next step we will estimate integral $I_{\mathbf{a},t}$, regarding $(\mathbf{a},t)$ as a parameter. Consider another parameter $R$, satisfying

$$1 \le R \le |\mathbf{u}|^{1/3};$$

its value will be chosen later. Below we distinguish two cases:

1. $(\mathbf{a},t)$ belongs to the "good" domain $S_R$, where

$$S_R = \big\{(\mathbf{a},t) : |\nabla f(0)| = \delta|tA\mathbf{a} - \mathbf{u}| \ge R\big\langle t/|\mathbf{u}\big\rangle = R\langle \delta^2 t\rangle\big\};$$

2. $(\mathbf{a},t)$ belongs to the "bad" set ${S_R}^c = (\mathbb{R}^d \times \mathbb{R}) \setminus S_R$.

### 6.2.2 Integral over $S_R$.

We consider first the integral over the good set $S_R$:

**Lemma 6.3.** *For any $d \ge 1$, $N \ge 0$ and $R \ge 2\|A\|\sqrt{d}$ we have*

$$\int_{S_R} d\mathbf{a}\, dt |p(t)|\, |I_{\mathbf{a},t}| \lesssim_{N,m} \frac{L}{q} R^{-N} \|w\|_{N,d+5}\,. \tag{6.4}$$

*Proof.* Let $\mathbf{l} := \nabla f(0)/|\nabla f(0)|$ and $\mathcal{L} = \mathbf{l}\cdot\nabla_{\mathbf{b}}$. Then for $(\mathbf{a},t) \in S_R$ and $|\mathbf{b}| \le \sqrt{d}$ (see (6.3)),

$$|\mathcal{L}f(\mathbf{b})| \ge |\nabla f(0)| - \delta^2|t||A\mathbf{b}| \ge R\langle\delta^2 t\rangle - \delta^2|t|\|A\|\frac{R}{2\|A\|} \ge \tfrac{1}{2}R\langle\delta^2 t\rangle \ge R/2. \tag{6.5}$$

Since $(2\pi i\mathcal{L}f(\mathbf{b}))^{-1}\mathcal{L}e(f(\mathbf{b})) = e(f(\mathbf{b}))$, then integrating by parts $N$ times integral $I_{\mathbf{a},t}$ we get

$$|I_{\mathbf{a},t}| \lesssim_N \max_{|b_i|\le 1\,\forall i}\ \max_{0\le k\le N} \left| \mathcal{L}^{N-k}\tilde{w}(\delta\mathbf{b}+\mathbf{a}) \frac{\big(\mathcal{L}^2 f(\mathbf{b})\big)^k}{\big(\mathcal{L}f(\mathbf{b})\big)^{N+k}} \right|,$$

where we have used that $\mathcal{L}^m f(\mathbf{b}) = 0$ for $m \ge 3$. Since $|\mathcal{L}^2 f(\mathbf{b})| \le \delta^2|t||\mathbf{l}\cdot A\mathbf{l}| \le \delta^2|t|\|A\|$, then in view of (6.5)

$$\left|\frac{\mathcal{L}^2 f(\mathbf{b})}{\mathcal{L}f(\mathbf{b})}\right| \le \frac{\delta^2|t|\|A\|}{\frac{1}{2}R\langle\delta^2 t\rangle} = \frac{2\|A\|}{R} \le \frac{1}{\sqrt{d}}.$$

So using that $\left|\dfrac{1}{\mathcal{L}f(\mathbf{b})}\right| \le \dfrac{2}{R}$ by (6.5), we find

$$|I_{\mathbf{a},t}| \lesssim_N R^{-N} \max_{|b_i|\le 1\,\forall i}\ \max_{0\le k\le N} \left|\mathcal{L}^k\tilde{w}(\delta\mathbf{b}+\mathbf{a})\right|.$$

Thus, denoting by $\mathbf{1}_{S_R}$ the indicator function of the set $S_R$, we have

$$\int_{\mathbb{R}^d} |I_{\mathbf{a},t}| \mathbf{1}_{S_R}\, d\mathbf{a} \lesssim_N R^{-N} \int_{\mathbb{R}^d} \frac{d\mathbf{a}}{\langle \mathbf{a}\rangle^{d+1}} \Big( \langle \mathbf{a}\rangle^{d+1} \max_{|b_i|\le 1\,\forall i} \max_{0\le k\le N} \big|\mathcal{L}^k \tilde{w}(\delta \mathbf{b} + \mathbf{a})\big| \Big)$$
$$\lesssim_N R^{-N} \|\tilde{w}\|_{N,d+1} \lesssim_{N,m} R^{-N} \|w\|_{N,d+5}\,,$$

for every $t$. Then

$$\text{l.h.s. of (6.4)} \lesssim_{N,m} R^{-N} \|w\|_{N,d+5} \int_{-\infty}^{+\infty} |p(t)|\, dt. \tag{6.6}$$

To prove (6.4) it remains to show that

$$\int_{-\infty}^{\infty} |p(t)| dt \lesssim L/q\,. \tag{6.7}$$

In virtue of Lemma 3.2 with $N = 2$,

$$\Big| \frac{\partial^k}{\partial v^k} h(x,v) \Big| \lesssim_k x^{-k-1} \min\{1, x^2/v^2\}\,, \quad k \ge 1,$$

and by Corollary 3.3, $|h(x,v)| \lesssim x^{-1}$. Then an integration by parts in (6.2) shows that, for any $M \ge 0$,

$$|p(t)| \lesssim_M |t^{-M}| \Big( \int_{-\infty}^{\infty} |w_2^{(M)}(v)| x^{-1}\, dv$$
$$+ \max_{1\le k\le M} \int_{-\infty}^{\infty} |w_2^{(M-k)}(v)|\, x^{-k-1} \min\big\{1, \frac{x^2}{v^2}\big\}\, dv \Big),$$

where $x := q/L < \alpha$. Writing the latter integral as a sum $\int_{|v|\le x} + \int_{|v|>x}$ we see that

$$\int_{|v|\le x} = x^{-k-1} \int_{|v|\le x} |w_2^{(M-k)}(v)|\, dv \lesssim_M x^{-k}$$

and

$$\int_{|v|>x} = x^{-k+1} \int_{|v|>x} \frac{|w_2^{(M-k)}(v)|}{v^2}\, dv \lesssim_M x^{-k}.$$

Then, since $x = q/L < 1$,

$$|p(t)| \lesssim_M \Big( \frac{q}{L} |t| \Big)^{-M}\,, \quad M \ge 0. \tag{6.8}$$

Choosing $M = 2$ when $|t| > L/q$ and $M = 0$ when $|t| \le L/q$ we get (6.7). $\square$

### 6.2.3 Integral over $S_R{}^c$.

Now we study the integral over the bad set $S_R{}^c$.

**Lemma 6.4.** *For any* $d \geq 1$, $1 \leq R \leq |\mathbf{u}|^{1/3}$ *and* $0 < \beta < 1$ *we have*

$$\int_{S_R^c} d\mathbf{a}\, dt |p(t)||I_{\mathbf{a},t}| \lesssim_m R^d |\mathbf{u}|^{-d/2+1+\beta} \|w\|_{0,K(d,\beta)}\,,$$

*where* $K(d,\beta) = d + \lceil d^2/2\beta \rceil + 4$.

*Proof.* On $S_R{}^c$ we use for $I_{\mathbf{a},t}$ the easy upper bound

$$|I_{\mathbf{a},t}| \lesssim \max_{|b_i|\leq 1\, \forall i} |\tilde{w}(\delta\mathbf{b} + \mathbf{a})| \ \leq \|\tilde{w}\|_{0,0}. \tag{6.9}$$

The fact that $(\mathbf{a}, t) \in S_R{}^c$ implies that the integration in $d\mathbf{a}$ for a fixed $t$ is restricted to the region, where $\left|A\mathbf{a} - t^{-1}\mathbf{u}\right| \leq (R/\delta|t|)\langle t/|\mathbf{u}|\rangle$ , or

$$\left|\mathbf{a} - \frac{A^{-1}\mathbf{u}}{t}\right| \leq \|A^{-1}\| \, \frac{R}{\delta|t|} \, \langle t/|\mathbf{u}|\rangle\,. \tag{6.10}$$

We first consider the case $|t| \geq |\mathbf{u}|^{1-\beta/d}$. Since $|\mathbf{u}| > 1$, then considering separately the cases $|t| \leq |\mathbf{u}|$ and $|t| \geq |\mathbf{u}|$ we see that

$$\frac{R}{\delta|t|}\langle t/|\mathbf{u}|\rangle \leq R|\mathbf{u}|^{-1/2+\beta/d}\,. \tag{6.11}$$

In view of (6.9) -(6.11),

$$\left|\int_{\mathbb{R}^d} |I_{\mathbf{a},t}| \mathbf{1}_{S_R{}^c}(\mathbf{a},t) d\mathbf{a}\right| \lesssim R^d|\mathbf{u}|^{-d/2+\beta}\|\tilde{w}\|_{0,0} \lesssim_m R^d|\mathbf{u}|^{-d/2+\beta}\|w\|_{0,4}\,.$$

Since by (6.7) $\displaystyle\int_{|t|\geq|\mathbf{u}|^{1-\beta/d}} |p(t)|\, dt \lesssim \frac{L}{q} \leq |\mathbf{u}|$ , then

$$\int_{|t|\geq|\mathbf{u}|^{1-\beta/d}} dt \int_{\mathbb{R}^d} d\mathbf{a}\, |p(t)||I_{\mathbf{a},t}| \mathbf{1}_{S_R{}^c}(\mathbf{a},t) \lesssim_m R^d|\mathbf{u}|^{-d/2+1+\beta}\|w\|_{0,4}\,. \tag{6.12}$$

Now let $|t| \leq |\mathbf{u}|^{1-\beta/d}$. Then the r.h.s. of (6.10) is bounded by the quantity $\|A^{-1}\|R/(\delta|t|)$, so that $|\mathbf{a}| \gtrsim |A^{-1}\mathbf{u}|/t - \|A^{-1}\|R/(\delta|t|)$. Since $|A^{-1}\mathbf{u}| \geq C_A|\mathbf{u}|$ and $R \leq |\mathbf{u}|^{1/3}$, then

$$|\mathbf{a}| \gtrsim_A \frac{|\mathbf{u}| - RC'_A\sqrt{|\mathbf{u}|}}{|t|} \geq (1 - C'_A|\mathbf{u}|^{-1/6})\frac{|\mathbf{u}|}{|t|} \geq \frac{1}{2}\frac{|\mathbf{u}|}{|t|} \geq \frac{1}{2}|\mathbf{u}|^{\beta/d}$$

with $C'_A = C_A^{-1}\|A^{-1}\|$, since $|\mathbf{u}|^{-1} \le \alpha$, if $\alpha$ is so small that $1 - C'_A\alpha^{1/6} \ge 1/2$. Then $\mathbf{1}_{S_R{}^c}(\mathbf{a},t) \lesssim |\mathbf{u}|^{-d/2+\beta/d}|\mathbf{a}|^{d^2/2\beta-1}$, and we deduce from (6.9) that for such values of $t$

$$\left|\int_{\mathbb{R}^d} |I_{\mathbf{a},t}|\mathbf{1}_{S_R{}^c}(\mathbf{a},t)d\mathbf{a}\right| \lesssim |\mathbf{u}|^{-d/2+\beta/d}\int_{\mathbb{R}^d}|\mathbf{a}|^{d^2/2\beta-1}\max_{|b_i|\le 1\,\forall i}|\tilde{w}(\delta\mathbf{b}+\mathbf{a})|\,d\mathbf{a}$$
$$\lesssim_m |\mathbf{u}|^{-d/2+\beta/d}\|w\|_{0,K(d,\beta)}\,,$$

where $K(d,\beta) = d + \lceil d^2/2\beta\rceil + 4$. On the other hand, by (6.8) with $M = 0$, $\int_{|t|\le|\mathbf{u}|^{1-\beta/d}}|p(t)|dt \lesssim |\mathbf{u}|^{1-\beta/d}$, from which we obtain

$$\int_{|t|\le|\mathbf{u}|^{1-\beta/d}} dt\int_{\mathbb{R}^d} d\mathbf{a}\,|p(t)||I_{\mathbf{a},t}|\mathbf{1}_{S_R{}^c}(\mathbf{a},t) \lesssim_m |\mathbf{u}|^{-d/2+1}\|w\|_{0,K(d,\beta)}\,. \quad (6.13)$$

Putting together (6.12) and (6.13) we get the assertion. □

### 6.2.4 End of the proof

In order to complete the proof of Lemma 6.2 we combine Lemmas 6.3 and 6.4 to get that

$$|\tilde{I}_q(\mathbf{c})| \lesssim_{N,m} \left(\frac{L}{q}R^{-N} + R^d|\mathbf{u}|^{-d/2+1+\beta}\right)\left(\|w\|_{N,d+5} + \|w\|_{0,K(d,\beta)}\right)\,.$$

We fix here $\gamma_1 \in (0,1)$, $\beta = \gamma_1/2$, $R = |\mathbf{u}|^{\frac{\gamma_1}{2d}} \le |\mathbf{u}|^{\frac{1}{3}}$ and pick $N = \lceil\frac{d^2}{\gamma_1}\rceil - 2d > 0$ (notice that $R \ge \alpha^{-\gamma_1/2d} \ge 2\|A\|\sqrt{d}$ if $\alpha$ is small enough, so that assumption of Lemma 6.3 is satisfied). Then $K(d,\beta) = N + 3d + 4$, $R^{-N} \le |\mathbf{u}|^{-d/2+\gamma_1} \le (L/q)^{-d/2+\gamma_1}$ since $-d/2 + \gamma_1 < 0$ and $|\mathbf{u}| \ge L/q$. Moreover, $R^d|\mathbf{u}|^{-d/2+1+\beta} = |\mathbf{u}|^{-d/2+1+\gamma_1} \le (L/q)^{-d/2+1+\gamma_1}$. This concludes the proof. □

# 7 Integrals over quadrics

Our goal in this section is to study integrals $\mathcal{I}(t;w)$ over the quadrics $\Sigma_t$. We start with a case of quadratic forms $F$, written in a convenient normal form (Theorem 7.1), and show later in Section 7.4 (Theorem 7.3) how to reduce general integrals $\mathcal{I}(t;w)$ to those, corresponding to the quadratic forms like that. In this section we assume that

$$d \ge 3$$

and not use the bold font to denote vectors since most of variables we use are vectors.

## 7.1 Quadratic forms in normal form

On $\mathbb{R}^d = \mathbb{R}^n_u \times \mathbb{R}^{d_1}_x \times \mathbb{R}^{d_1}_y = \{z = (u, x, y)\}$, where $d \geq 3$, $n \geq 0$ and $d_1 \geq 1$, consider the quadratic form

$$F(z) = \tfrac{1}{2}|u|^2 + x \cdot y = \tfrac{1}{2} Az \cdot z\,, \quad A(u, x, y) = (u, y, x)\,. \tag{7.1}$$

Note that $A$ is an orthogonal operator, $|Az| = |z|$. As in Section 1.1 we define the quadrics $\Sigma_t = \{z : F(z) = t\}$, $t \in \mathbb{R}$. Note that for $t \neq 0$ $\Sigma_t$ is a smooth hypersurface, while $\Sigma_0$ is a cone with a singularity at the origin. We denote the volume element on $\Sigma_t$ (on $\Sigma_0 \backslash \{0\}$ if $t = 0$), induced from $\mathbb{R}^d$, as $dz\,|_{\Sigma_t}$ and set

$$\mu^{\Sigma_t}(dz) = |Az|^{-1} dz\,|_{\Sigma_t} \tag{7.2}$$

(see below concerning this measure when $t = 0$).

For a $k_* \in \mathbb{N} \cup \{0\}$ and a function $f$ on $\mathbb{R}^d$ satisfying

$$f \in \mathcal{C}^{k_*, M}(\mathbb{R}^d)\,, \quad M > d\,, \tag{7.3}$$

we will study the integrals

$$\mathcal{I}(t) = \mathcal{I}(t; f) = \int_{\Sigma_t} f(z) \mu^{\Sigma_t}(dz)\,. \tag{7.4}$$

Our first goal is to demonstrate the following result:

**Theorem 7.1.** *For the quadratic form $F(z)$ as in* (7.1) *and a function $f \in \mathcal{C}^{k_*, M}(\mathbb{R}^d)$, $M > d$, consider integral $\mathcal{I}(t; f)$, defined in* (7.4)*. Then the function $\mathcal{I}(t)$, defined by* (7.4)*, is $C^k$–smooth if $k < d/2 - 1$, $k \leq k_*$, and is $C^k$–smooth outside zero if $k \leq \min(d/2 - 1, k_*)$. For $0 < |t| \leq 1$ we have*

$$\begin{aligned} &\left|\partial^k \mathcal{I}(t)\right| \lesssim_{k,M} \|f\|_{k,M} \quad \text{if } \ k < d/2 - 1, \\ &\left|\partial^k \mathcal{I}(t)\right| \lesssim_{k,M} \|f\|_{k,M} (1 - \ln|t|) \quad \text{if } \ k \leq d/2 - 1. \end{aligned} \tag{7.5}$$

*While for $|t| \geq 1$, denoting $\kappa = \frac{M+2-d}{2}$, we have*

$$\begin{aligned} &\left|\partial^k \mathcal{I}(t)\right| \lesssim_{k,M} \|f\|_{k,M} \langle t\rangle^{-\kappa} \quad \text{if } 1 \leq k \leq d/2 - 1,\ k \leq k_*, \\ &|\mathcal{I}(t)| \lesssim_{M,\kappa'} \|f\|_{0,M} \langle t\rangle^{-\kappa'} \quad \forall\, \kappa' < \kappa. \end{aligned} \tag{7.6}$$

An example, see [7, Example A.3], shows that in general the log-factor cannot be removed from the r.h.s. in (7.5).

The theorem is proved below in number of steps. In the proof for a given vector $x \in \mathbb{R}^{d_1}$ we consider its orthogonal complement in $\mathbb{R}^{d_1}$ – the hyperspace $x^\perp = \{\bar{x}\}$, provided with the Lebesgue measure $d\bar{x}$. If $d_1 = 1$, then $x^\perp$ degenerates to the space $\mathbb{R}^0 = \{0\}$, and $d\bar{x}$ – to the $\delta$-measure at 0. Practically it means that when $d_1 = 1$, the spaces $x^\perp$ and $y^\perp$ (and integrals over them) disappear from our construction. It makes the case $d_1 = 1$ easier, but notationally different from $d_1 \geq 2$. For example, in formula (7.8) with $d_1 = 1$ the affine space $\sigma_t^x(u', x')$ becomes the point $(u', x', (t - \frac{1}{2}|u'|^2)|x'|^{-2}x')$, the measure $d\mu^{\Sigma_t}\,|_{\Sigma_t^x}$ in (7.14) becomes $du\,|x|^{-1}dx$, etc. Accordingly, below we write the proof only for $d_1 \geq 2$, leaving the case $d_1 = 1$ as an easy exercise for the reader.

## 7.2 Disintegration of the two measures

Our goal in this subsection is to find a convenient disintegration of the measures $dz|_{\Sigma_t}$ and $\mu^{\Sigma_t}$, following the proof of Theorem 3.6 in [5].

Let us denote $\Sigma_t^x = \{(u, x, y) \in \Sigma_t : x \neq 0\}$ (if $t < 0$, then $\Sigma_t^x = \Sigma_t$). Then for any $t$ $\Sigma_t^x$ is a smooth hypersurface in $\mathbb{R}^d$, and the mapping

$$\Pi_t^x : \Sigma_t^x \to \mathbb{R}^n \times \mathbb{R}^{d_1} \backslash \{0\}\,, \quad (u, x, y) \mapsto (u, x)\,, \tag{7.7}$$

is a smooth affine euclidean vector bundle. Its fibers are

$$\sigma_t^x(u', x') := \left(\Pi_t^x\right)^{-1}(u', x') = \left(u', x', x'^\perp + \frac{t - \frac{1}{2}|u'|^2}{|x'|^2}x'\right), \tag{7.8}$$

where $x'^\perp$ is the orthogonal complement to $x'$ in $\mathbb{R}^{d_1}$. For any $x' \neq 0$ denote

$$U_{x'} = \left\{x : |x - x'| \leq \frac{1}{2}|x'|\right\}, \quad U = \mathbb{R}^n \times U_{x'} \times \mathbb{R}^{d_1}\,.$$

Now we construct a trivialisation of the bundle $\Pi_t^x$ over $U$. To do this we fix in $\mathbb{R}^{d_1}$ any orthonormal frame $(e_1, \ldots, e_{d_1})$ such that the ray $\mathbb{R}_+ e_1$ intersects $U_{x'}$. Then

$$x_1 \geq \kappa > 0 \quad \forall x = (x_1, \ldots, x_{d_1}) =: (x_1, \bar{x}) \in U_{x'}\,.$$

We wish to construct an affine in the third argument diffeomorphism

$$\Phi_t : \mathbb{R}^n \times U_{x'} \times \mathbb{R}^{d_1 - 1} \to U \cap \Sigma_t$$

of the form

$$\Phi_t(u, x, \bar{\eta}) = (u, x, \Phi_t^{u,x}(\bar{\eta}))\,, \quad \Phi_t^{u,x}(\bar{\eta}) = (\varphi_t(u, x, \bar{\eta}), \bar{\eta}) \in \mathbb{R}^{d_1}\,, \quad \bar{\eta} \in \mathbb{R}^{d_1 - 1}. \tag{7.9}$$

We easily see that $\Phi_t(u,x,\bar\eta)\in\Sigma_t$ if and only if

$$\varphi_t(u,x,\bar\eta)=\frac{t-\frac12|u|^2-\bar x\cdot\bar\eta}{x_1}\,. \tag{7.10}$$

The mapping $\bar\eta\to\Phi_t^{u,x}(\bar\eta)$ with this function $\varphi_t$ is affine, and the range of $\Phi_t$ equals $U\cap\Sigma_t$.

In the coordinates $(u,x,\eta_1,\bar\eta)\in\mathbb{R}^n\times U_{x'}\times\mathbb{R}\times\mathbb{R}^{d_1-1}$ on the domain $U\subset\mathbb{R}^d$ the hypersurface $\Sigma_t^x$ is embedded in $\mathbb{R}^d$ as a graph of the function $(u,x,\bar\eta)\mapsto\eta_1=\varphi_t$. Accordingly, in the coordinates $(u,x,\bar\eta)$ on $U\cap\Sigma_t$ the volume element on $\Sigma_t$ reads as $\bar\rho_t(u,x,\bar\eta)du\,dx\,d\bar\eta$, where

$$\bar\rho_t=\big(1+|\nabla\varphi_t|^2\big)^{1/2}=\Big(1+\frac{|u|^2+|\bar\eta|^2+|\bar x|^2+x_1^{-2}(t-\frac12|u|^2-\bar x\cdot\bar\eta)^2}{x_1^2}\Big)^{1/2}.$$

Passing from the variable $\bar\eta\in\mathbb{R}^{d_1-1}$ to $y=\Phi_t^{u,x}(\bar\eta)\in\sigma_t^x(u,x)$ we replace $d\bar\eta$ by $|\det\Phi_t^{u,x}(\bar\eta)|d_{\sigma_t^x(u,x)}y$. Here $d_{\sigma_t^x(u,x)}y$ is the Lebesgue measure on the $(d_1-1)$-dimensional affine euclidean space $\sigma_t^x(u,x)$ while by $\det\Phi_t^{u,x}$ we denote the determinant of the linear mapping $\Phi_t^{u,x}$, viewed as a linear isomorphism of the euclidean space $\mathbb{R}^{d_1-1}=\{\bar\eta\}$ and the tangent space to $\sigma_t^x(u,x)$, identified with the euclidean space $x^\perp\subset\mathbb{R}^{d_1}$. Accordingly we write the volume element on $\Sigma_t\cap U$ as $\rho_t(u,x,y)du\,dx\,d_{\sigma_t^x(u,x)}y$ with

$$\rho_t(u,x,y)=\bar\rho_t(u,x,\bar\eta)|\det\Phi_t^{u,x}(\bar\eta)|\,,\quad (u,x,y)\in\Sigma_t,\ \text{where }\Phi_t^{u,x}(\bar\eta)=y.$$

Now we will calculate the density $\rho_t$. Let us take any point $z_*=(u_*,x_*,y_*)\in U\cap\Sigma_t$ and choose a frame $(e_1,\dots,e_{d_1})$ such that $e_1=x_*/|x_*|$. Then

$$x_*=(|x_*|,0)\,,\ \ y_*=\big(y_{*1},\bar y_*\big)\,,\qquad y_{*1}=\Big(\frac{t-\frac12|u_*|^2}{|x_*|}\Big),\ \ \bar y_*\in\mathbb{R}^{d_1-1}\,.$$

So (see (7.9)–(7.10)) the mapping $\Phi_t$ is such that $\Phi_t^{u_*,x_*}(\bar\eta)=\big(y_{*1},\bar\eta\big)=\tilde y\in\sigma_t^x(u_*,x_*)$ (i.e. $\varphi_t(z_*)=y_{*1}$). In these coordinates $\rho_t(u_*,x_*,y_{*1},\bar y_*)=\bar\rho_t(u_*,x_*,\bar y_*)$, which equals

$$\big(1+|x_*|^{-2}\big(|u_*|^2+|\bar y_*|^2+|y_{*1}|^2\big)\big)^{1/2}=\frac{\big(|x_*|^2+|u_*|^2+|\bar y_*|^2+|y_{*1}|^2\big)^{1/2}}{|x_*|}\,.$$

That is, $\rho_t(z_*)=\frac{|z_*|}{|x_*|}$. Since $z_*$ is any point in $U\cap\Sigma_t$, then we have proved

**Proposition 7.2.** *The volume element $dz\,|_{\Sigma_t^x}$ with respect to the projection $\Pi_t^x$ disintegrates as follows:*

$$dz\,|_{\Sigma_t^x} = du\,|x|^{-1}dx\,|z|d_{\sigma_t^x(u,x)}y\,. \tag{7.11}$$

*That is, for any function $f \in C_0^0(\Sigma_t^x)$,*

$$\int f(z)dz\,|_{\Sigma_t^x} = \int_{\mathbb{R}^n} du \int_{\mathbb{R}^{d_1}} |x|^{-1}dx \int_{\sigma_t^x(u,x)} |z| f(z) d_{\sigma_t^x(u,x)}y\,.$$

Similarly, if we set $\Sigma_t^y = \{(u,x,y) \in \Sigma_t : y \neq 0\}$ and consider the projection

$$\Pi_t^y : \Sigma_t^y \to \mathbb{R}^n \times \mathbb{R}^{d_1}\backslash\{0\}\,, \quad (u,x,y) \mapsto (u,y)\,,$$

then

$$dz\,|_{\Sigma_t^y} = du\,|y|^{-1}dy\,|z|d_{\sigma_t^y(u,y)}x\,. \tag{7.12}$$

Let us denote $\Sigma_t^0 = \{(u,x,y) \in \Sigma_t : x = y = 0\}$. Then $\Sigma_t\backslash\Sigma_t^0$ is a smooth manifold and $dz\,|_{\Sigma_t}$ defines on it a smooth measure.

By (7.11) and (7.12), for any $t$ the volume of the set $\{z \in \Sigma_t\backslash\Sigma_t^0 : 0 < |x|^2 + |y|^2 \leq \varepsilon\}$ goes to zero with $\varepsilon$. So assigning to $\Sigma_t^0$ zero measure we exten $dz\,|_{\Sigma_t}$ to a Borel measure on $\Sigma_t$ such that each set $\{z \in \Sigma_t : |z| \leq R\}$ has a finite measure and

$$(dz\,|_{\Sigma_t})\left((\Sigma_t^x \cup \Sigma_t^y)^c\right) = 0\,. \tag{7.13}$$

By (7.11) and (7.12) the function $|z|^{-1}$ is locally integrable $\Sigma_t$ with respect to the measure $dz\,|_{\Sigma_t}$. So $\mu^{\Sigma_t}$ (see (7.2)) is a well defined Borel measure on $\Sigma_t$. Since $|Az| = |z|$, then, in view of (7.11) and (7.12),

$$d\mu^{\Sigma_t}\,|_{\Sigma_t^x} = du\,|x|^{-1}dx\,d_{\sigma_t^x(u,x)}y\,, \quad d\mu^{\Sigma_t}\,|_{\Sigma_t^y} = du\,|y|^{-1}dy\,d_{\sigma_t^y(u,y)}x\,. \tag{7.14}$$

The measure $\mu^{\Sigma_t}$ defines on $\mathbb{R}^d$ a Borel measure, supported by $\Sigma_t$. It will also be denoted $\mu^{\Sigma_t}$.

## 7.3 Analysis of the integral $\mathcal{I}(t;f)$

Note that for any $t$ the mapping

$$L_t : \Sigma_0^x \to \Sigma_t^x\,, \quad (u,x,y) \mapsto (u,x,y+t|x|^{-2}x)$$

defines an affine isomorphism of the bundles $\Pi_0\,|_{\Sigma_0^x}$ and $\Pi_t\,|_{\Sigma_t^x}$. Since $L_t$ preserves the Lebesgue measure on the fibers, then in view of (7.11) it sends

the measure $\mu^{\Sigma_0}$ to $\mu^{\Sigma_t}$. Using (7.14) we get that for any $t$ the integral $\mathcal{I}(t)$, defined in (7.4), may be written as

$$\begin{aligned}\mathcal{I}(t;f) &= \int_{\Sigma_0} f(L_t(z))\mu^{\Sigma_0}(dz)\\ &= \int_{\mathbb{R}^n\times\mathbb{R}^{d_1}} |x|^{-1}du\,dx \int_{\sigma(u,x)} f(u,x,y+t|x|^{-2}x)d_{\sigma^x(u,x)}y\,.\end{aligned} \tag{7.15}$$

Here $\sigma(u,x) := \sigma_0^x(u,x) = x^\perp - \frac{1}{2}|u|^2|x|^{-2}x$.

We recall that $f(u,x,y)$ satisfies (7.3). Taking any smooth function $\varphi(t)\geq 0$ on $\mathbb{R}$ which vanishes for $|t|\geq 2$ and equals one for $|t|\leq 1$ we write $f = f_{00}+f_1$, where $f_{00} = \varphi(|(x,y)|^2)f$ and $f_1 = (1-\varphi(|(x,y)|^2))f$. Denoting $B_r(\mathbb{R}^m) = \{\xi\in\mathbb{R}^m : |\xi|\leq r\}$ and $B^r(\mathbb{R}^m) = \{\xi\in\mathbb{R}^m : |\xi|\geq r\}$ we see that

$$\operatorname{supp} f_{00}\subset\mathbb{R}^n\times B_{\sqrt{2}}(\mathbb{R}^{2d_1})\,,\quad \operatorname{supp} f_1\subset\mathbb{R}^n\times B^1(\mathbb{R}^{2d_1})\,. \tag{7.16}$$

Setting next $f_{11}(z) = f_1(z)(1-\varphi(4|x|^2))$, $f_{10}(z) = f_1(z)\varphi(4|x|^2)$ we write

$$f = f_{00}+f_{11}+f_{10}\,.$$

Since $(x,y)\in B^1(\mathbb{R}^{2d_1})$ implies that $|x|\geq 1/\sqrt{2}$ or $|y|\geq 1/\sqrt{2}$, then in view of (7.16),

$$\begin{aligned}&\operatorname{supp} f_{11}\subset\mathbb{R}^n\times B^{1/2}(\mathbb{R}^{d_1}_x)\times\mathbb{R}^{d_1}_y\,,\\ &\operatorname{supp} f_{10}\subset\mathbb{R}^n\times\mathbb{R}^{d_1}_x\times B^{1/\sqrt{2}}(\mathbb{R}^{d_1}_y)\,.\end{aligned} \tag{7.17}$$

Obviously, for $i,j=0,1$ we have $\|f_{ij}\|_{k,m}\leq C_{k,m}\|f\|_{k,m}$, for all $k,m$.

Setting $\mathcal{I}_{ij}(t) = \mathcal{I}(t;f_{ij})$ we have:

$$\mathcal{I}(t;f) = \mathcal{I}_{00}(t)+\mathcal{I}_{10}(t)+\mathcal{I}_{11}(t)\,.$$

### 7.3.1 Integral $\mathcal{I}_{00}(t)$.

By (7.15) $\mathcal{I}_{00}(t)$ is a continuous function, and for $1\leq k\leq k_*$,

$$\begin{aligned}\partial^k\mathcal{I}_{00}(t) &= \int_{\mathbb{R}^n} du\int_{B_{\sqrt{2}}(\mathbb{R}^{d_1})}|x|^{-1}dx\\ &\quad\int_{y\in\sigma(u,x)}\left(d^k/dt^k\right)f_{00}(u,x,y+t|x|^{-2}x)\,d_{\sigma(u,x)}y\\ &= \int_{\mathbb{R}^n} du\int_{B_{\sqrt{2}}(\mathbb{R}^{d_1})}|x|^{-1}dx\int_{y\in\sigma(u,x)} d_y^k f_{00}(u,x,y+t|x|^{-2}x)\left[|x|^{-2}x\right]\,d_{\sigma(u,x)}y,\end{aligned} \tag{7.18}$$

where by $d_y^k f_{00}\left[|x|^{-2}x\right]$ we denote the action of the differential $d_y^k f_{00}$ on the set of $k$ vectors, each of which equals to $|x|^{-2}x$. Setting $\tau = t - \frac{1}{2}|u|^2$, for $y \in \sigma(u,x)$ we have

$$y + t|x|^{-2}x = \bar{y} + \tau|x|^{-2}x, \qquad \bar{y} \in x^{\perp}. \tag{7.19}$$

Then we write the integral over $y$ in (7.18) as

$$\int_{x^{\perp}} d_y^k f_{00}(u, x, \bar{y} + \tau|x|^{-2}x)\left[|x|^{-2}x\right]\, d\bar{y}. \tag{7.20}$$

Since $|\bar{y} + \tau x|x|^{-2}|^2 = |\bar{y}|^2 + \tau^2|x|^{-2}$, then on the support of the integrand

$$|x| \le \sqrt{2}, \qquad |\bar{y}|^2 + \tau^2|x|^{-2} \le 2. \tag{7.21}$$

In particular,

$$|\tau| = \left|t - \frac{1}{2}|u|^2\right| \le \sqrt{2}|x| \le 2 \quad \text{in (7.20)}. \tag{7.22}$$

By (7.16) the diameter of the domain of integration in (7.20) is bounded by $\sqrt{2}$. So, for any $m \ge 0$ integral (7.20) is bounded by $C_{k,m}|x|^{-k}\langle u\rangle^{-m}\|f\|_{k,m}$. Denoting $R = |u|$, $r = |x|$ we get that

$$|\partial^k \mathcal{I}_{00}(t)| \lesssim_{k,M} \|f\|_{k,M} \int_0^{\sqrt{2}} dr\, r^{d_1-k-2} \int_0^{\infty} dR\, R^{n-1}\langle R\rangle^{-M}\chi_{|\tau|\le\sqrt{2}r}\,. \tag{7.23}$$

If $n = 0$, then the integral over $R$ should be removed from the r.h.s.

a) If $n = 0$, then $\tau = t$, we get from (7.22) that $|x| \ge t/\sqrt{2}$ and see from (7.16) that, for $t \ne 0$, $\mathcal{I}_{00}(t)$ is $C^{k_*}$-smooth (since $f \in C^{k_*}$). Then from (7.23) we obtain

$$|\partial^k \mathcal{I}_{00}(t)| \lesssim_{k,M} \|f\|_{k,M} \int_{|t|/\sqrt{2}}^{\sqrt{2}} r^{d_1-k-2}\chi_{|t|\le 2}\, dr\,. \tag{7.24}$$

Obviously, $\mathcal{I}_{00}(t) = 0$ for $|t| \ge 2$. Next from (7.24) we obtain that

$$\begin{aligned} |\partial^k \mathcal{I}_{00}(t)| &\lesssim_k \|f\|_{k,0} \quad \text{if} \quad k \le \min(d_1 - 2, k_*), \\ |\partial^k \mathcal{I}_{00}(t)| &\lesssim_k \|f\|_{k,0}(1 + |\ln|t||) \quad \text{if} \quad k \le \min(d_1 - 1, k_*) \end{aligned} \tag{7.25}$$

b) If $n \ge 1$, then to estimate $\partial^k \mathcal{I}_{00}(t)$ we split the integral for $\mathcal{I}_{00}(t)$ in a sum of two. Namely, for a fixed $t \ne 0$ we write $f_{00}$ as $f_{00} = f_{00<} + f_{00>}$, with

$f_{00<} = f_{00}\varphi(8|x|^2/t^2)$, where $\varphi$ is the function, used to define the functions $f_{ij}$, $0 \le i, j \le 1$. Then

$$\operatorname{supp} f_{00<} \subset \{2|x| \le |t|\}, \quad \operatorname{supp} f_{00>} \subset \{2\sqrt{2}|x| \ge |t|\}. \tag{7.26}$$

With an obvious notation we have $\mathcal{I}_{00}(t) = \mathcal{I}_{00<}(t) + \mathcal{I}_{00>}(t)$, where

$$\begin{aligned}
\mathcal{I}_{00<}(t) = &\int_{\mathbb{R}^n} du \int_{B_{\sqrt{2}}(\mathbb{R}^{d_1}) \cap B_{|t|/2}(\mathbb{R}^{d_1})} |x|^{-1} dx \\
&\int_{\substack{y \in \sigma(u,x) \\ |x|^2 + |y+t|x|^{-2}x|^2 \le 2}} f_{00<}(u, x, y + t|x|^{-2}x)\, d_{\sigma(u,x)} y\,, \\
\mathcal{I}_{00>}(t) = &\int_{\mathbb{R}^n} du \int_{B_{\sqrt{2}}(\mathbb{R}^{d_1}) \cap B^{|t|/2\sqrt{2}}(\mathbb{R}^{d_1})} |x|^{-1} dx \\
&\int_{\substack{y \in \sigma(u,x) \\ |x|^2 + |y+t|x|^{-2}x|^2 \le 2}} f_{00>}(u, x, y + t|x|^{-2}x)\, d_{\sigma(u,x)} y \ .
\end{aligned}$$

Consider first the function $\mathcal{I}_{00<}(t)$. We observe that, by (7.19), for $y \in \sigma(u,x)$ and $|x| \le |t|/2$ (cf. (7.26))

$$|y + t|x|^{-2}x| \ge |\tau||x|^{-1} = \left| t - \frac{1}{2}|u|^2 \right| |x|^{-1} \ge -t|x|^{-1} > \sqrt{2}\,, \qquad t < 0\,,$$

so that $\mathcal{I}_{00<}(t) = 0$ for $t < 0$. For $t > 0$, performing the change of variables $\sqrt{t}u' = u$, $tx' = x$, we get

$$\begin{aligned}
\mathcal{I}_{00<}(t) = &t^{d/2-1} \int_{\mathbb{R}^n} du' \int_{B_{\sqrt{2}/t}(\mathbb{R}^{d_1}) \cap B_{1/2}(\mathbb{R}^{d_1})} |x'|^{-1} \varphi(8|x'|^2) dx' \\
&\int_{\substack{y \in \sigma(u',x') \\ |x'|^2 t^2 + |y+|x'|^{-2}x'|^2 \le 2}} f_{00}(\sqrt{t}u', tx', y + |x'|^{-2}x')\, d_{\sigma(u',x')} y\,,
\end{aligned}$$

where we notice that $\sigma(u', x') = \sigma(u, x)$. We differentiate with respect to $t$, observing that, by induction in $k$, for any $l$ and $k$ we have

$$\begin{aligned}
\frac{d^k}{dt^k} t^l g(\sqrt{t}u', tx') = &\sum_{l_1 + l_2 + l_3 = k} c_{l_1,l_2,l_3} t^{l - l_1 - l_2/2} \left( u'^{l_2} \cdot \nabla_u \right)^{l_2} \\
&\left( x'^{l_3} \cdot \nabla_x \right)^{l_3} g(\sqrt{t}u', tx')\,.
\end{aligned}$$

From this we get

$$\begin{aligned}
\left| \partial^k \mathcal{I}_{00<}(t) \right| \lesssim_{k,M} &\max_{l_1+l_2+l_3=k} t^{d/2-1-l_1-l_2/2} \|f\|_{k,M} \int_{\mathbb{R}^n} |u'|^{l_2} \langle u'\sqrt{t} \rangle^{-M} du' \\
&\int_{B_{\sqrt{2}/t}(\mathbb{R}^{d_1}) \cap B_{1/2}(\mathbb{R}^{d_1})} |x'|^{l_3 - 1} dx' \int_{\substack{y \in \sigma(u',x') \\ |x'|^2 t^2 + |y+|x'|^{-2}x'|^2 \le 2}} d_{\sigma(u',x')} y\,.
\end{aligned}$$

Denoting points of the space $x^\perp$ as $\bar y$, we see that the integral over $d_{\sigma(u',x')}y$ is bounded by

$$\int_{\substack{\bar y\in x^\perp\\ |x'|^2t^2+|\bar y+\tau'|x'|^{-2}x'|^2\le 2}} 1\, d\bar y, \qquad \tau' = 1-\frac{1}{2}|u'|^2\,. \tag{7.27}$$

By (7.22), on the support of the integrand $|\tau'|\le\sqrt{2}|x'|$. So there

$$1-\sqrt{2}|x'|\le\frac{|u'|^2}{2}\le 1+\sqrt{2}|x'|\,. \tag{7.28}$$

As the the domain of integration in $\bar y$ is bounded, then integral (7.27) is bounded by a constant. So putting $|x'|=r'$, $|u'|=R'$ and using (7.28) we have

$$\begin{aligned}\left|\partial^k\mathcal{I}_{00<}(t)\right| \lesssim_{k,M} \max_{l_1+l_2+l_3=k}\|f\|_{k,M}t^{d/2-l_1-l_2/2-1}\int_0^{1/2} dr'\, r'^{d_1-2+l_3}\\ \int_{\sqrt{2}\sqrt{1-\sqrt{2}r'}}^{\sqrt{2}\sqrt{1+\sqrt{2}r'}} dR'\, R'^{n-1+l_2}\langle R'^2t\rangle^{-M/2}\,.\end{aligned}$$

Since $r'\le 1/2$, then on the domain of integration $\sqrt{2-\sqrt{2}}\le R'\le\sqrt{2+\sqrt{2}}$, while $\sqrt{2}\sqrt{1+\sqrt{2}r'}-\sqrt{2}\sqrt{1-\sqrt{2}r'}\lesssim r'$. So the integral in $dR'$ is bounded by $C\langle t\rangle^{-M/2}r'$. Therefore

$$\left|\partial^k\mathcal{I}_{00<}(t)\right| \lesssim_{k,M} \max_{l_1+l_2+l_3=k}\|f\|_{k,M}t^{d/2-l_1-l_2/2-1}\langle t\rangle^{-M/2}\int_0^{1/2} dr'\, r'^{d_1-1+l_3}\,.$$

This implies that for $0<t\le 4$, for any $k\le k_*$ and any $d_1\ge 1$ we have

$$|\partial^k\mathcal{I}_{00<}(t)|\lesssim_k \|f\|_{k,0}t^{d/2-k-1}\,. \tag{7.29}$$

While for any $t\ge 4$ and any $k\le k_*$,

$$\begin{aligned}|\partial^k\mathcal{I}_{00<}(t)| \lesssim_{k,M} \max_{l_1+l_2+l_3=k}\|f\|_{k,M,d}t^{d/2-M/2-l_1-l_2/2-1}\\ \times\int_0^{\sqrt{2}/t} dr'\, r'^{d_1-1+l_3}\lesssim_{k,M}\|f\|_{k,M}t^{-(M+2+k+2d_1-d)/2}\,.\end{aligned} \tag{7.30}$$

We recall that $\mathcal{I}_{00<}(t)$ vanishes for $t<0$.

For $\mathcal{I}_{00>}(t)$ we first note that by (7.21) and (7.26) the function $\mathcal{I}_{00>}(t)$ vanishes if $|t|>4$. Next, by induction in $k$, we observe that

$$\begin{aligned}\frac{d^k}{dt^k}g(tx|x|^{-2})(1-\varphi(8|x|^2/t^2)) = \sum_{l_1+l_2+l_3=k} c_{l_1,l_2,l_3}|x|^{2(l_2-l_1)}t^{-3l_2-l_3}\\ \times\left((x\cdot\nabla)^{l_1}\, g\right)\,\frac{d^{l_2}}{dy^{l_2}}(1-\varphi)\,,\end{aligned} \tag{7.31}$$

where $c_{l_1,l_2,l_3} = 0$ if $l_3 > 0$ and $l_2 = 0$. Since $\varphi' \neq 0$ only for $|t|/2\sqrt{2} \le |x| \le |t|/2$, then

$$\frac{d^{l_2}}{dy^{l_2}}(1-\varphi)t^{-3l_2-l_3} \lesssim_{l_2,l_3} |x|^{-3l_2-l_3}\,, \quad l_2 > 0\,,$$

so that

$$\left|\frac{d^k}{dt^k} g(tx|x|^{-2})(1-\varphi(8|x|^2/t^2))\right| \lesssim_k |x|^{-k}\|g\|_{k,0}\,.$$

From here, in a way analogous to (7.23), putting again $|x| = r$ and $|u| = R$, we get that

$$|\partial^k \mathcal{I}_{00>}(t)| \lesssim_{k,M} \|f\|_{k,M} \int_{|t|/2\sqrt{2}}^{\sqrt{2}} dr\, r^{d_1-k-2} \int_0^\infty dR\, R^{n-1}\langle R\rangle^{-M} \chi_{|\tau|\le\sqrt{2}r}$$

(here and below $\int_a^b dr = 0$ if $b \le a$). Since on the integration domain, due to (7.26) and the indicator function $\chi_{|\tau|\le\sqrt{2}r}$, we have $R^2 \le 6\sqrt{2}r$, then

$$\begin{aligned} |\partial^k \mathcal{I}_{00>}(t)| &\lesssim_{k,M,n} \|f\|_{k,M} \int_{|t|/2\sqrt{2}}^{\sqrt{2}} dr\, r^{d/2-k-2} \\ &\lesssim_{k,M} \begin{cases} \|f\|_{k,M}\,, & k < d/2-1, \\ \|f\|_{k,M}(1+|\ln|t||)\,, & k \le d/2-1. \end{cases} \end{aligned} \tag{7.32}$$

If $k < d/2-1$, then by the above $\partial^k \mathcal{I}_{00}(t)$ is bounded for all $t$. In this case, modifying the integrand in (7.18) by the factor $\chi_{|x|\ge\varepsilon}$, we see that $\partial^k \mathcal{I}_{00}(t)$ is a uniform on compact sets limit of continuous functions, so that itself is continuous. Similar $\partial^k \mathcal{I}_{00}(t)$ with $k = d/2-1$ is continuous for $t \neq 0$.

### 7.3.2 Integral $\mathcal{I}_{11}(t)$.

Due to (7.17) and similar to (7.18), (7.20), for any $k \le k_*$ we have

$$\partial^k \mathcal{I}_{11}(t) = \int_{\mathbb{R}^n} du \int_{|x|\ge 1/2} |x|^{-1}\,dx \int_{x^\perp} d_y^k\, f_{11}(u, x, \bar{y} + \tau x|x|^{-2})[x|x|^{-2}]\,d\bar{y}.$$

We easily see that $\mathcal{I}_{11}(t)$ is a $C^k$-smooth function and, since $M > d$ and $\left|\bar{y} + \tau x|x|^{-2}\right| \ge |\bar{y}|$, then

$$\left|\partial^k \mathcal{I}_{11}(t)\right| \lesssim_{k,M} \|f\|_{k,M} \quad \forall t. \tag{7.33}$$

Now let $|t| \ge 1$. Let us write $\partial^k \mathcal{I}_{11}$ as

$$\partial^k \mathcal{I}_{11}(t) = \int_{\mathbb{R}^n} du \int_{|x|\ge 1/2} |x|^{-k-1}\, dx \int_{x^\perp} \Phi_k(\bar z)\, d\bar y, \tag{7.34}$$

where $\bar z = (u, x, \bar y)$, $\bar y \in x^\perp$, and

$$|\Phi_k(\bar z)| \lesssim_k \|f\|_{k,M} \langle \hat z\rangle^{-M}, \quad \hat z = (u, x, \bar y + \tau x|x|^{-2}). \tag{7.35}$$

Obviously,

$$|\hat z| \ge |\bar z|\,, \quad |\hat z| \ge 2^{-1/2}\big(|\bar z| + |\tau||x|^{-1}\big). \tag{7.36}$$

1) Let $n \ge 1$.

a) We first integrate in (7.34) over $u$ in the spherical layer

$$O := \{u : |\tau| = \big|t - \tfrac12 |u|^2\big| \le \tfrac12 t\}\,.$$

It is empty if $t < 0$, while for $t \ge 0$, $O = \{u : t \le |u|^2 \le 3t\}$. By (7.35) and the first relation in (7.36), for $t \ge 0$ the part of the integral in (7.34) with $u \in O$ is bounded by

$$K := C_k \|f\|_{k,M} \int_O du \int_{|x|\ge 1/2} |x|^{-k-1}\, dx \int_{x^\perp} \big(|t| + |x|^2 + |\bar y|^2\big)^{-M/2}\, d\bar y\,.$$

Since $\int_O 1\, du \le C t^{n/2}$, then by putting $r = |x|$, $|t| + r^2 = T^2$ and $R = |\bar y|/T$ we find that

$$K \lesssim_k \|f\|_{k,M} t^{n/2} \int_{1/2}^\infty r^{d_1-2-k}\, dr\, T^{d_1-1-M} \int_0^\infty R^{d_1-2} \big(1 + R^2\big)^{-M/2}\, dR\,.$$

The integral in $dR$ is bounded since $M > d_1$, so that

$$K \lesssim_{k,M} \|f\|_{k,M} t^{n/2} \int_{1/2}^\infty r^{d_1-2-k}\, \big(|t| + r^2\big)^{(d_1-1-M)/2}\, dr\,.$$

Recalling that we are considering the case $t \ge 1$, we put $r = \sqrt{t}\, l$. Then

$$K \lesssim_{kM} \|f\|_{k,M} t^{\frac{n+1+d_1-2-k+d_1-1-M}{2}} \int_{t^{-1/2}/2}^\infty l^{d_1-2-k} (1 + l^2)^{\frac{d_1-1-M}{2}}\, dl\,.$$

Since $M > 2d_1$, the integral over $l$ converges and we get

$$K \lesssim_{k,M} \|f\|_{k,M} |t|^{-(M+2-d+k)/2} |t|^{\max(0,\vee k+1-d_1)/2} Y(t)\,,$$

with $Y = \ln t$ if $k = d_1 - 1$ and $Y = 1$ otherwise. Then, in the case $Y = 1$ the component of (7.34), corresponding to $u \in O$, is bounded by

$$C(k, M, d)\|f\|_{k,M}|t|^{-\kappa}\,, \qquad \kappa = \frac{M + 2 - d}{2}, \tag{7.37}$$

for all $|t| \geq 1$, since $0 \vee k + 1 - d_1 \leq k$. If $Y = \ln t$ the same estimate holds in the case $d_1 \geq 2$ since $0 \vee k + 1 - d_1 < k$. In the case $d_1 = 1$ and $Y = \ln t$ (i.e. $k = 0$) we get (7.37) with $\kappa$ replaced by any $\kappa' < \kappa$ (and the constant $C$ depending on $\kappa'$).

b) Now consider the integral for $u \in O^c = \mathbb{R}^n \backslash O$. There $|\tau| = |t - \frac{1}{2}|u|^2| \geq \frac{1}{2}|t|$. So, by inequalities (7.35) and (7.36), $|\Phi_k(\bar{z})| \lesssim_k \langle (u, \bar{y})\rangle^{-M}$ and $|\Phi_k(\bar{z})| \lesssim_k (|t||x|^{-1} + |x|)^{-M}$. Let $M = M_1 + M_2$, $M_j \geq 0$. Then the part of the integral (7.34) for $u \in O^c$ is bounded by

$$C\|f\|_{k,M} \int_{|x|\geq 1/2} |x|^{-1-k} \left(t|x|^{-1} + |x|\right)^{-M_1} dx \int_{\mathbb{R}^n} du \int_{x^\perp} d\bar{y} \langle (u, \bar{y})\rangle^{-M_2}\,.$$

Choosing $M_2 = n + d_1 - 1 + \gamma$ with $0 < \gamma < 1$ (then $M_1, M_2 > 0$ since $M > d$) we achieve that the integral over $du\, d\bar{y}$ is bounded by $C(\gamma)$, for any $\gamma$. Since by Young's inequality

$$(A + B)^{-1} \leq C_a A^{-a} B^{a-1}\,, \quad 0 < a < 1\,,$$

for any $A, B > 0$, then $\left(t|x|^{-1} + |x|\right)^{-M_1} \leq C_a |x|^{(2a-1)M_1} |t|^{-aM_1}$ $(0 < a < 1)$. So the integral above is bounded by

$$C(\gamma)\|f\|_{k,M}|t|^{-aM_1} \int_{|x|\geq 1/2} |x|^{-1-k+bM_1}\, dx\,, \quad b = 2a - 1 \in (-1, 1)\,.$$

Denote $b_* = \frac{1+k-d_1}{M_1}$. Then for $b = b_*$ the exponent for $|x|$ in the formula above equals $-d_1$, and $b_* > -1$ if $\gamma$ is sufficiently small, since $M > d$. Noting that

$$a(b_*)M_1 = \frac{b_* + 1}{2} M_1 = \frac{M + 2 + k - d - \gamma}{2} = \kappa + \frac{k}{2} - \frac{\gamma}{2}$$

($\kappa$ was defined in (7.37)), we see that the part of integral (7.34), corresponding to $u \in O^c$,

$$\begin{aligned} &\text{is bounded by (7.37) if } k \geq 1, \text{ while for } k = 0 \text{ it is bounded by} \\ &\qquad\qquad \text{(7.37) with } \kappa \text{ replaced by any } \kappa' < \kappa. \end{aligned} \tag{7.38}$$

2) Now let $n = 0$. Then

$$\left|\partial^k \mathcal{I}_{11}(t)\right| \leq \int_{|x|\geq 1/2} |x|^{-1-k}\, dx \int_{x^\perp} \Phi_k(\bar{z}) d\bar{y}\,, \quad \bar{z} = (x, \bar{y})\,, \tag{7.39}$$

where $|\Phi_k(\bar z)|\lesssim_k\langle\hat z\rangle^{-M}$ with $\hat z=(x,\bar y+tx|x|^{-2})$. Repeating literally the step 1b) above with $n=0$ we get that for $|t|\ge 1$ the integral in (7.39) may also be bounded by (7.37). We recall that for $|t|\le 1$ the derivative $\partial^k\mathcal{I}_{11}(t)$ was estimated in (7.33).

### 7.3.3 Integral $\mathcal{I}_{10}(t)$.

Now we use the second disintegration in (7.14) instead of the first. Since by (7.17) on the support of the integrand $|y|\ge 1/\sqrt{2}$, then repeating the argument above with $x$ and $y$ swapped we get that $\mathcal{I}_{10}(t)$ meets the same estimates as $\mathcal{I}_{11}(t)$.

### 7.3.4 End of the proof of Theorem 7.1

Finally,
– combining together relations (7.25), (7.29), (7.32) and (7.33) we estimate $\partial^k\mathcal{I}(t)$ for $0<|t|\le 4$,

while
– combining together (7.30), (7.37), (7.38) and using the fact that $\partial^k\mathcal{I}_{00>}(t)$ and $\partial^k\mathcal{I}_{00}(t)$ vanish for $|t|\ge 4$ when $n=0$, we estimate $\partial^k\mathcal{I}(t)$ for $t\ge 4$.

For the reason, explained at the end of Section 7.3.1, the involved derivatives are continuous functions. This proves the theorem.

## 7.4 Linear transformations of quadrics

In this subsection we denote by $C_0$ spaces of continuous functions with compact support.

In $\mathbb{R}^d=\{z\}$ let us consider a quadratic form with real coefficients [5] $F(z)=\frac12 Az\cdot z$ of signature $(n_0,n_+,n_-)$ such that $n_0=0,\ n_+\ge n_-=:d_1\ge 1$. Denote $n=n_+-n_-$. Then there exists a normalising linear transformation

$$L:\mathbb{R}^d\to\mathbb{R}^d,\quad z\mapsto Z=(u,x,y),\quad u\in\mathbb{R}^n,\quad x,y\in\mathbb{R}^{d_1},$$

such that $Q(L(z))=F(z)$, where $Q(Z)=\frac12|u|^2+x\cdot y$. Consider the corresponding quadrics $\Sigma_t^Q=\{Z:Q(Z)=t\}$, $\Sigma_t^F=\{z:F(z)=t\}$, and the $\delta$-measures $\mu_t^Q,\mu_t^F$ on them (e.g. see [11, Section II.7]):

$$\langle\mu_t^Q,f^Q\rangle=\lim_{\varepsilon\to0}\frac{1}{2\varepsilon}\int_{t-\varepsilon\le Q(Z)\le t+\varepsilon}f^Q(Z)\,dZ,\tag{7.40}$$

[5] Section 7.4-7.5 is the only part of our work, where quadratic forms are allowed to have non-rational coefficients.

$$\langle \mu_t^F, f^F \rangle = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{t-\varepsilon \le F(z) \le t+\varepsilon} f^F(z)\, dz,$$

where $f^Q, f^F \in C_0(\mathbb{R}^d)$ and $\langle \mu, f \rangle$ signifies the integral of a function $f$ against a measure $\mu$. Then $\mu_t^Q$ and $\mu_t^F$ and Borel measures in $\mathbb{R}^d$, supported by $\Sigma_t^Q$ and $\Sigma_t^F$ respectively, and for $f^Q \in C_0\big(\Sigma_t^Q \setminus \{0\}\big)$ and $f^F \in C_0\big(\Sigma_t^F \setminus \{0\}\big)$ we have

$$\langle \mu_t^Q, f^Q \rangle = \int_{\Sigma_t^Q} \frac{f^Q(Z)}{|\nabla Q(Z)|}\, dZ\,|_{\Sigma_t^Q}, \quad \langle \mu_t^F, f^F \rangle = \int_{\Sigma_t^F} \frac{f^F(z)}{|\nabla F(z)|}\, dz\,|_{\Sigma_t^F},$$

where $dZ\,|_{\Sigma_t^{Q(\text{or } F)}}$ is the volume element on $\Sigma_t^{Q(\text{or } F)} \setminus \{0\}$, induced from $\mathbb{R}^d$, see [11]. Now let $f^F = f^Q \circ L$. Then the integral in (7.40) equals

$$\int_{t-\varepsilon \le Q(Z) \le t+\varepsilon} f^Q(Z)\, dZ = |\det(L)| \int_{t-\varepsilon \le F(z) \le t+\varepsilon} f^F(z)\, dz,$$

so passing to the limit we get that

$$L \circ \big(|\det(L)| \mu_t^F\big) = \mu_t^Q. \tag{7.41}$$

Thus,
*to examine the function*

$$t \mapsto \mathcal{I}^F(t; f) = \langle \mu_t^F, f \rangle, \qquad \mu_t^F = |\nabla F(z)|^{-1} dz\,|_{\Sigma_t^F}, \tag{7.42}$$

*we are free to use any linear coordinate system in $\mathbb{R}^d$ since changing the coordinates we only modify the function $\mathcal{I}^F$ by a constant factor.*

## 7.5 Sign definite forms

Finally let us consider the case when $n_0 = 0$ and $\min(n_+, n_-) = 0$, i.e. when the form $F(z) = \frac{1}{2} Az \cdot z$ is sign–definite and non degenerate. Suppose for definitenes that $n_- = 0$. Then there exists a linear transformation $L$ such that $F(z) = Q(L(z))$, where $Q(Z) = \frac{1}{2}|Z|^2$, $Z \in \mathbb{R}^d$. The quadric $\Sigma_t$ reduces to the empty set for $t < 0$, so the function $\mathcal{I}^F(t)$ (see (7.42)) vanishes for $t < 0$. The calculation of previous subsection remains true in this case, so (7.41) and the change of coordinates $Z = \sqrt{2t}\, Z'$ show that

$$\begin{aligned} \mathcal{I}^F(t; f) &= C(d, L) t^{-1} \int_{|Z| = \sqrt{2t}} f^Q(Z) \mu_{S^{d-1}_{\sqrt{2t}}}(dZ) \\ &= C(d, L) t^{d/2 - 1} \int_{|Z'|=1} f^Q(\sqrt{2t} Z') \mu_{S_1^{d-1}}(dZ'), \qquad t > 0, \quad f^Q = f \circ L^{-1}, \end{aligned}$$

where $\mu_{S_r^{d-1}}$ is the volume element on the $d-1$ sphere of radius $r$. From this relation we immediately get that for any $k \le \min(d/2-1, k_*)$,

$$\left|\partial^k \mathcal{I}^F(t)\right| \lesssim_k \|f\|_{k,0} \quad \text{if} \quad 0 \le t \le 1,$$

and

$$\left|\partial^k \mathcal{I}^F(t)\right| \lesssim_{k,M} \|f\|_{k,M} t^{-(M+2+k-d)/2} \quad \text{if} \quad t \ge 1.$$

### 7.6 General result

We sum up the obtained results in the following

**Theorem 7.3.** *Consider any nondegenerate quadratic form $F(z) = \frac{1}{2}Az \cdot z$ on $\mathbb{R}^d$, $d \ge 3$, and a function $f \in \mathcal{C}^{k_*,M}(\mathbb{R}^d)$, $M > d$. Then the corresponding integral $\mathcal{I}^F(t;f) = \langle \mu_t^F, f\rangle$ (see (7.42)) meets the assertions of Theorem 7.1.*

*Proof.* i) If $n_+ \ge n_-$, then by means of a linear change of variable $F$ may be put to the normal form (7.1), where $d_1 \ge 0$. Now the assertion follows from the argument in Subsections 7.4, 7.5 and Theorem 7.1.

ii) If $n_- > n_+$, then the quadratic form $-F$ is as in i), and the assertion follows again since obviously $\mathcal{I}^{-F}(t;f) = \mathcal{I}^F(-t;f)$. □

## A The $J_0$ term: case $d = 4$

In this section we find asymptotic for the term $J_0$ from (1.19) in the case

$$d = 4 \quad \text{and} \quad m = 0. \tag{A.1}$$

Below in this section we always assume (A.1).

### A.1 Preliminary results and definitions

We will need Lemmas 30 and 31 of [8], restricted for the case $m = 0$ and $d = 4$, which we state below without a proof. Recall that the constants $\sigma_{\mathbf{c}}^*(A)$ are defined in (1.10) and $\sigma^*(A) = \sigma_{\mathbf{0}}^*(A)$. Set $\alpha := 7/2$ and recall (A.1).

**Lemma A.1** (Lemma 30 of [8])**.** *For any $\varepsilon > 0$ and $X \in \mathbb{N}$,*

$$\sum_{q \le X} S_q(\mathbf{c}; A, 0) = \eta(\mathbf{c})\sigma_{\mathbf{c}}^*(A) \sum_{q \le X} q^{d-1} + O_\varepsilon(X^{\alpha+\varepsilon}(1+|\mathbf{c}|)), \tag{A.2}$$

*where $\eta(\mathbf{c}) = 1$ if $\mathbf{c} \cdot A^{-1}\mathbf{c} = 0$ and at the same time $\det A$ is a square of an integer, and $\eta(\mathbf{c}) = 0$ otherwise. Moreover, $|\sigma_{\mathbf{c}}^*(A)| \lesssim_\varepsilon 1 + |\mathbf{c}|^\varepsilon$ when $\eta(\mathbf{c}) \ne 0$.*

**Lemma A.2** (Lemma 31 of [8]). *Assume that the determinant* $\det A$ *is a square of an integer. Then for any* $\varepsilon > 0$ *and* $X \in \mathbb{N}$,

$$\sum_{q \le X} q^{-d} S_q(0; A, 0) = \sigma^*(A) \log X + \hat{C}_A + O_\varepsilon(X^{\alpha+\varepsilon-d}),$$

*where* $\hat{C}_A$ *is a constant depending only on* $A$. *Otherwise, if* $\det A$ *is not a square of an integer, then for any* $\varepsilon > 0$ *and* $X \in \mathbb{N}$

$$\sum_{q \le X} q^{-d} S_q(0; A, 0) = L(1, \chi) \prod_p (1 - \chi(p) p^{-1}) \sigma_p(A, 0) + O_\varepsilon(X^{-1/2+\varepsilon}),$$

*where* $\chi$ *is the Jacobi symbol* $(\frac{\det(A)}{*})$ *and* $L(1, \chi)$ *is the Dirichlet* $L$*–function.*

We will also need the following construction. Let us define for $r \in \mathbb{R}_{>0}$

$$I^*(r) := \tilde{I}_{rL}(0) = \int_{\mathbb{R}^d} w(\mathbf{z})\, h\left(r, F^0(\mathbf{z})\right)\, d\mathbf{z}\,. \tag{A.3}$$

Consider a function $K(\rho; w, A)$, $\rho \in \mathbb{R}_{>0}$, given by

$$K(\rho) := \eta(0)\sigma^*(A) \left( \sigma_\infty(w; A, 0) \log \rho + \int_\rho^\infty r^{-1} I^*(r)\, dr \right) + \sigma_\infty(w; A, 0) \hat{C}_A\,, \tag{A.4}$$

where the constant $\eta(0)$ is defined according to Lemma A.1 and $\hat{C}_A$ — according to Lemma A.2. Note that the functions $I^*(r)$ and $K(\rho)$ do not depend on $L$.

We claim that the function $K(\rho)$, $\rho > 0$, can be extended at $\rho = 0$ by continuity. Indeed, for $0 < \rho_1 < \rho_2 \le 1$

$$K(\rho_2) - K(\rho_1) = \eta(0)\sigma^*(A) \left( \sigma_\infty(w; A, 0) \log(\rho_2/\rho_1) - \int_{\rho_1}^{\rho_2} r^{-1} I^*(r)\, dr \right). \tag{A.5}$$

Using that $I^*(r) = L^{-d} I_{rL}(0)$ (see (3.8)), we write the term $I^*(r)$ from (A.5) in the form, given by Proposition 3.8 b). Then $I^*(r)$ takes the form of the r.h.s. of (3.11), divided by $L^d$, with $q = rL$. The leading term in the obtained formula for $I^*(r)$ is $\sigma_\infty(w; A, 0)$ and the corresponding integral $\int_{\rho_1}^{\rho_2} r^{-1} \sigma_\infty\, dr$ in (A.5) cancels the first term in the brackets of (A.5). Then, setting $M = d/2 - 1$, $\beta = r^{\bar\gamma}$, $\bar\gamma = \gamma/d$ and $0 < \gamma < 1$ in the just discussed formula for $I^*(r)$, obtained from (3.11), we get the estimate

$$\begin{aligned} |K(\rho_2) - K(\rho_1)| &\lesssim_N \|w\|_{d/2-1,d+1} \int_{\rho_1}^{\rho_2} \left( r^{d/2(1-\bar\gamma)-2} \langle \log r \rangle + r^{N-2} + r^{\bar\gamma N - 2} \right) dr \\ &\lesssim_\gamma \rho_2^{d/2-1-\gamma} \|w\|_{d/2-1,d+1}\,. \end{aligned}$$

The last inequality here is obtained by choosing $N = N(\gamma)$ to be sufficiently large. Therefore $K(\rho)$ extends at $\rho = 0$ by continuity and

$$|K(\rho) - K(0)| \lesssim_\gamma \rho^{d/2-1-\gamma} \|w\|_{d/2-1,d+1}\,, \tag{A.6}$$

so the function $K$ is $(d/2-1-\gamma)$-Hölder continuous at zero, for any $\gamma > 0$.

## A.2 Estimate for $J_0$

In this section we restrict ourselves for the case when the determinant $\det A$ is a square of an integer, so in particular $\eta(0) = 1$. We use this specification only in the proof of Lemma A.5, when applying Lemma A.2. The case of non-square determinant is easier and can be obtained similarly, using the second assertion of Lemma A.2.

**Proposition A.3.** *Assume that the determinant* $\det A$ *is a square of an integer. Then for any* $0 < \varepsilon < 1/5$*,*

$$\begin{aligned} J_0 =& \sigma_\infty(w; A, 0)\sigma^*(A) L^d \log L + K(0; w, A) L^d \\ &+ O_\varepsilon(L^{d-\varepsilon} \left( \|w\|_{d/2-1,d-1} + \|w\|_{0,d+1} \right)). \end{aligned}$$

*Proof.* To establish Proposition A.3 we write $J_0$ in the form (1.21), $J_0 = J_0^+ + J_0^-$, where

$$J_0^+ := \sum_{q > \rho L} q^{-d} S_q(0) I_q(0) \quad \text{and} \quad J_0^- := \sum_{q \le \rho L} q^{-d} S_q(0) I_q(0)\,,$$

with $\rho \le 1$. Then the assertion follows from Lemmas A.4 and A.5 below. Recall that $\alpha = 7/2$.

**Lemma A.4.** *Let* $w \in L_1(\mathbb{R}^d)$*. Then for any* $\gamma > 0$*, any* $\rho \le 1$ *and* $L$ *satisfying* $\rho L > 1$*,*

$$\left| J_0^+ - L^d \eta(0) \sigma^*(A) \int_\rho^\infty r^{-1} I^*(r)\, dr \right| \lesssim_\gamma (\rho^{\alpha+\gamma-d-1} L^{\alpha+\gamma} + \rho^{-2} L^{d-1}) |w|_{L_1}\,.$$

*Proof.* To simplify the notation, in this proof we denote $I_q := I_q(0)$ and $S_q := S_q(0)$. Let us recall the summation by parts formula for sequences $(f_q)$ and $(g_q)$:

$$\sum_{m < q \le n} f_q (g_q - g_{q-1}) = f_n g_n - f_{m+1} g_m - \sum_{m < q < n} (f_{q+1} - f_q) g_q.$$

We take arbitrary $R \in \mathbb{N}$ and apply the latter with $m = R$, $n = 2R$, $f_q = q^{-d} I_q$ and $g_q = \sum_{R<q'\le q} S_q$, so that $g_R = 0$ and $S_q = g_q - g_{q-1}$ for $q > R$. We find

$$\begin{aligned}\sum_{R<q\le 2R} q^{-d} S_q I_q =& (2R)^{-d} I_{2R} \sum_{R<q\le 2R} S_q \\ &- \sum_{R<q<2R} \tilde\partial_q (q^{-d} I_q) \sum_{R<q'\le q} S_{q'}\,,\end{aligned} \tag{A.7}$$

where for a sequence $(a_q)$ we denote $\tilde\partial_q a_q := a_{q+1} - a_q$. By (3.8)–(3.9),

$$I_q = L^d \int_{\mathbb{R}^d} w(\mathbf{z}) h(q/L, F^0(\mathbf{z}))\, d\mathbf{z}\,.$$

So,

$$|I_q| \lesssim \frac{L^{d+1}}{q} |w|_{L_1} \qquad \text{and} \qquad |\partial_q I_q| \lesssim \frac{L^{d+1}}{q^2} |w|_{L_1}\,, \tag{A.8}$$

where the first estimate above follows from Corollary 3.3 while the second one — from Lemma 3.2 with $m = 1, n = N = 0$. Then, $|\tilde\partial_q (q^{-d} I_q)| \lesssim L^{d+1} q^{-d-2} |w|_{L_1}$. According to (A.2) with $\varepsilon$ replaced by $\gamma$, for $R' \le 2R$

$$\sum_{R<q\le R'} S_q = \eta(0)\sigma^*(A) \sum_{R<q\le R'} q^{d-1} + O_\gamma(R^{\alpha+\gamma})\,, \tag{A.9}$$

where we recall that $\sigma^*_{\mathbf{0}}(A) = \sigma^*(A)$. Let us view the r.h.s. of (A.7) as a linear functional $G\big((S_q)\big)$ on the space of sequences $(S_q)$. Then, inserting formula (A.9) in the r.h.s. of (A.7), we get

$$\begin{aligned}\sum_{R<q\le 2R} q^{-d} S_q I_q &= \eta(0)\sigma^*(A) G\big((q^{d-1})\big) \\ &\quad + O_\gamma\Big(L^{d+1}|w|_{L_1}\big(R^{-d-1+\alpha+\gamma} + \sum_{R<q\le 2R} q^{-d-2+\alpha+\gamma}\big)\Big),\end{aligned} \tag{A.10}$$

where the $O_\gamma$ term is obtained by applying (A.8) together with the estimate for $\tilde\partial_q(q^{-d} I_q)$ above and replacing the sums $\sum S_q$, $\sum S_{q'}$ in the r.h.s. of (A.7) by $O_\gamma(R^{\alpha+\gamma})$. According to the summation by parts formula (A.7) with $S_q$ replaced by $q^{d-1}$, we have $\sum_{R<q\le 2R} q^{-d} q^{d-1} I_q = G\big((q^{d-1})\big)$. Thus, by (A.10),

$$\sum_{R<q\le 2R} q^{-d} S_q I_q = \eta(0)\sigma^*(A) \sum_{R<q\le 2R} q^{-1} I_q + O_\gamma\Big(L^{d+1} R^{-d-1+\alpha+\gamma} |w|_{L_1}\Big)\,.$$

Then, setting $R_l = \lfloor 2^l \rho L \rfloor$ we get

$$\begin{aligned} J_0^+ =& \sum_{l=0}^{\infty} \sum_{R_l < q \le R_{l+1}} q^{-d} I_q S_q \\ =& \eta(0)\sigma^*(A) \sum_{q>\rho L} q^{-1} I_q + O_\gamma \left( \rho^{\alpha+\gamma-d-1} L^{\alpha+\gamma} |w|_{L_1} \sum_{l=0}^{\infty} 2^{-l(d+1-\alpha-\gamma)} \right) \\ =& \eta(0)\sigma^*(A) \sum_{q>\rho L} q^{-1} I_q + O_\gamma \left( \rho^{\alpha+\gamma-d-1} L^{\alpha+\gamma} |w|_{L_1} \right). \end{aligned}$$

It remains to compare the sum $A := \sum_{q>\rho L} q^{-1} I_q$ with the integral $B := L^d \int_\rho^\infty r^{-1} I^*(r)\, dr$. Since $L^d I^*(r) = I_{rL}$, then changing the variable of integration $r$ to $q = rL$, $B$ takes the form $\int_{\rho L}^\infty q^{-1} I_q \, dq$. Then,

$$|A - B| \le \Big| \sum_{q>\rho L} q^{-1} I_q - \int_{\lfloor \rho L \rfloor + 1}^{\infty} q^{-1} I_q \, dq \Big| + \Big| \int_{\rho L}^{\lfloor \rho L \rfloor + 1} q^{-1} I_q \, dq \Big|. \quad \text{(A.11)}$$

Due to (A.8), $|q^{-1} I_q| \lesssim q^{-2} L^{d+1} |w|_{L^1}$ and $|\partial_q (q^{-1} I_q)| \lesssim q^{-3} L^{d+1} |w|_{L^1}$. Thus,the both terms in the r.h.s. of (A.11) are bounded by $(\rho L)^{-2} L^{d+1} |w|_{L^1} = \rho^{-2} L^{d-1} |w|_{L^1}$. □

Recall that $\hat{C}_A$ is a constant arising in Lemma A.2.

**Lemma A.5.** *Assume that the determinant* $\det A$ *is a square of an integer. Then for any* $\gamma > 0$, $N > 1$, *any* $\rho \le 1$ *and* $L$ *satisfying* $\rho L > 1$,

$$\begin{aligned} J_0^- =& L^d \sigma_\infty(w; A, 0) \left( \sigma^*(A) \log(\rho L) + \hat{C}_A \right) + O_{\gamma,N} \Big( \Big( \rho^{\alpha+\gamma-d} L^{\alpha+\gamma} \\ &+ L^d \big( \rho \log L + \rho^{N-1} + L^{1-d} \big) \Big) \|w\|_{d/2-1,d+1} \Big). \end{aligned}$$

*Proof.* Inserting Proposition 3.8 b) with $M = d/2 - 1 = 1$ and $\beta = 1$ into the definition of the term $J_0^-$, we get $J_0^- = I_A + I_B$, where

$$I_A := L^d \sigma_\infty(w) \sum_{q \le \rho L} q^{-d} S_q(0), \qquad I_B := \sum_{q \le \rho L} S_q(0) q^{-d} (f_q + g_q),$$

with

$$\begin{aligned} |f_q| &\lesssim q L^{d-1} \left\langle \log(\frac{q}{L}) \right\rangle \|w\|_{d/2-1,d+1}, \\ |g_q| &\lesssim_N \left( q^N L^{d-N} + 1 \right) L q^{-1} \|w\|_{0,d+1}. \end{aligned}$$

By Lemma A.2,

$$\sum_{q\le\rho L} q^{-d}S_q(0) = \sigma^*(A)\log(\rho L) + \hat{C}_A + O_\gamma((\rho L)^{\alpha+\gamma-d}).$$

So,

$$I_A = L^d\sigma_\infty(w)\left(\sigma^*(A)\log(\rho L) + \hat{C}_A\right) + O_\gamma(\sigma_\infty(w)L^{\alpha+\gamma}\rho^{\alpha+\gamma-d}),$$

whereas

$$|\sigma_\infty(w)| = |\sigma_\infty(w;A,0)| = |\mathcal{I}(0)| \le \|\mathcal{I}\|_{0,0} \lesssim_A \|w\|_{0,d+1} \tag{A.12}$$

on account of (3.13). As for the term $I_B$, since $d=4$, Lemma 2.1 implies that

$$|I_B| \lesssim \sum_{q\le\rho L} q^{-1}(|f_q|+|g_q|) \lesssim_N L^d\left(\rho\log L + \rho^{N-1} + L^{1-d}\right)\|w\|_{d/2-1,d+1},$$

for $N\ge 2$. The obtained estimates on $I_A$ and $I_B$ imply the assertion. □

Now we conclude the proof of Proposition A.3. The leading term of $J_0$ is given by the sum of leading terms from formulas for $J_0^+$ and $J_0^-$ in Lemmas A.4 and A.5. Since $\eta(0)=1$, it takes the form

$$\begin{aligned}
&L^d\sigma^*(A)\Big(\int_\rho^\infty r^{-1}I^*(r)\,dr + \sigma_\infty(w)\log(\rho L)\Big) + L^d\sigma_\infty(w)\hat{C}_A\\
&\quad=\sigma_\infty(w)\sigma^*(A)L^d\log L + K(0)L^d\\
&\quad+O_\gamma\big(L^d\rho^{d/2-1-\gamma}\|w\|_{d/2-1,d+1}\big),
\end{aligned}$$

where in the last equality we used (A.4) and (A.6). Then we find

$$\begin{aligned}
J_0 =&\sigma_\infty(w)\sigma^*(A)L^d\log L + K(0)L^d + O_{\gamma,N}\Big(\big(\rho^{\alpha+\gamma-d-1}L^{\alpha+\gamma} + \rho^{-2}L^{d-1}\\
&+ L^d(\rho^{d/2-1-\gamma} + \rho\log L + \rho^{N-1} + L^{1-d})\big)\|w\|_{d/2-1,d+1}\Big),
\end{aligned}$$

since $|w|_{L_1} \lesssim \|w\|_{0,d+1}$. We now pick $\rho = L^{-1/5}$ and $N=2$, and, using that $d=4$, get the assertion of proposition. □

### A.3 Estimate for $\sigma_1(w;A,L)$

In this section we get an upper bound for the subleading order term $\sigma_1$ of the asymptotics from Theorem 1.4.

In the case when the determinant $\det A$ is not a square of an integer, $\sigma_1$ is given by (1.14) and the task is not complicated. Indeed, according to Lemma A.2, the product $\prod_p(1-\chi(p)p^{-1})\sigma_p(A,0)$ is finite (and independent from $L$). On the other hand, by (A.12), $|\sigma_\infty(w;A,0)| \lesssim \|w\|_{0,d+1}$. Thus,

$$|\sigma_1(w;A,L)| \lesssim \|w\|_{0,d+1}.$$

In the case when $\det A$ is a square, $\sigma_1$ is given by (1.24) and the required estimate is less trivial.

**Proposition A.6.** *Assume that* $\det A$ *is a square of an integer. Then*

$$|\sigma_1(w;A,L)| \lesssim \|w\|_{\tilde N,\tilde N+3d+4}, \qquad \text{where} \qquad \tilde N := d^2(d+3)-2d.$$

Proof of the proposition is based on the given below refinement of Lemma 6.2, which is obtained with help of Lemma A.1. Proof of this result follows the lines of the proofs of Lemma 6.2 and Lemma 22 from [8] and we omit it. Recall (A.1).

**Lemma A.7.** *For any* $\mathbf{c}\neq 0$ *and* $\gamma\in(0,1)$,

$$|I_q(\mathbf{c})| \lesssim_\gamma L^{d/2+1+\gamma}q^{d/2-1-\gamma}|\mathbf{c}|^{-d/2+1+\gamma}\big(\|w\|_{\bar N,d+5}+\|w\|_{0,\bar N+3d+4}\big),$$

*where* $\bar N := \lceil d^2/\gamma\rceil - 2d$.

*Proof of Proposition A.6.* Since $\eta(\mathbf{c})$ takes values 0 or 1, then according to the definition (1.24) of $\sigma_1$, we have

$$|\sigma_1(w)| \le |K(0)| + \sum_{\mathbf{c}\neq 0:\,\eta(\mathbf{c})=1} |\sigma^*_{\mathbf{c}}(A)\sigma^{\mathbf{c}}_\infty(w)|. \tag{A.13}$$

Let us first estimate the term $K(0)$. According to (A.6),

$$|K(1)-K(0)| \lesssim \|w\|_{d/2-1,d+1}. \tag{A.14}$$

On the other hand, $\sigma^*(A)$ is independent from $L$ and, in view of Lemma A.2 is finite. Then, by the definition (A.4) of $K(\rho)$,

$$|K(1)| \lesssim \int_1^\infty r^{-1}|I^*(r)|\,dr + |\sigma_\infty(w;A,0)\hat C_A|.$$

Due to the definition (A.3) of the integral $I^*(r)$ and Corollary 3.3, $|I^*(r)| \lesssim r^{-1}|w|_{L_1} \lesssim r^{-1}\|w\|_{0,d+1}$. Then, in v iew of (A.12), $|K(1)| \lesssim \|w\|_{0,d+1}$, so that, by (A.14),

$$|K(0)| \lesssim \|w\|_{d/2-1,d+1}. \tag{A.15}$$

Let us now estimate the terms $\sigma_\infty^{\mathbf{c}}(w)$, which are given by (1.23):

$$\sigma_\infty^{\mathbf{c}}(w) = L^{-d}\sum_{q=1}^{\infty} q^{-1} I_q(\mathbf{c}; A, 0, L) = Y_1(\mathbf{c}) + Y_2(\mathbf{c}),$$

where $Y_1 = L^{-d}\sum_{q=1}^{L|\mathbf{c}|^{-M}}$, $Y_2 = L^{-d}\sum_{q>L|\mathbf{c}|^{-M}}$ and $M \in \mathbb{N}$ will be chosen later. Using that $d = 4$, according to Lemma A.7,

$$|Y_1(\mathbf{c})| \lesssim_\gamma L^{-1+\gamma}|\mathbf{c}|^{-1+\gamma} C(w) \sum_{q=1}^{L|\mathbf{c}|^{-M}} q^{-\gamma} \lesssim |\mathbf{c}|^{-(1-\gamma)(M+1)} C(w),$$

where we denoted $C(w) := \|w\|_{\bar{N},d+5} + \|w\|_{0,\bar{N}+3d+4}$. On the other hand, by Proposition 5.1, $|I_q(\mathbf{c})| \lesssim_N L^{d+1} q^{-1} |\mathbf{c}|^{-N} \|w\|_{N,2N+d+1}$ for every $N \in \mathbb{N}$. So,

$$|Y_2(\mathbf{c})| \lesssim_N L|\mathbf{c}|^{-N}\|w\|_{N,2N+d+1} \sum_{q>L|\mathbf{c}|^{-M}} q^{-2} \lesssim |\mathbf{c}|^{-N+M}\|w\|_{N,2N+d+1}.$$

Thus,

$$|\sigma_\infty^{\mathbf{c}}(w)| \lesssim_{\gamma,N} \big(|\mathbf{c}|^{-(1-\gamma)(M+1)} + |\mathbf{c}|^{-N+M}\big)\big(\|w\|_{\bar{N},\bar{N}+3d+4} + \|w\|_{N,2N+d+1}\big).$$

By Lemma A.1, $|\sigma_{\mathbf{c}}^*(A)| \lesssim_\gamma 1 + |\mathbf{c}|^\gamma$ if $\eta(\mathbf{c}) = 1$, so we get

$$\sum_{\mathbf{c}\neq 0:\, \eta(\mathbf{c})=1} |\sigma_{\mathbf{c}}^*(A)\sigma_\infty^{\mathbf{c}}(w)| \lesssim_{\gamma,N} \|w\|_{\bar{N},\bar{N}+3d+4} + \|w\|_{N,2N+d+1},$$

once $M$ and $N-M$ are sufficiently large and $\gamma$ is sufficiently small. Choosing $M = d$, $N = 2d+1$ and $\gamma = 1/(d+3)$, we get $\bar{N} = d^2(d+3) - 2d$. Together with (A.13) and (A.15), this implies the assertion of the proposition.

# B Quadratic forms in four variables over $\mathbb{Q}$ and $\mathbb{Q}_p$

Item (ii) of Proposition 1.5 treats the case of quadric $F$ in dimension $d = 4$. We give here some basic facts about such forms. All results of this appendix can be found in Sections IV.2 and IV.3 of [12].

Let

$$F(\mathbf{x}) = \Sigma_{i,j=1}^4 a_{ij} x_i x_j$$

be a non-degenerate quadratic form with integer coefficients. We are interested whether the equation

$$F(\mathbf{x}) = \Sigma_{i=1}^4 a_{ij} x_i x_j = 0 \tag{B.1}$$

has nontrivial ($\mathbf{x} \neq 0$) solutions in $\mathbb{Z}^4$; due to homogeneity the existence of solutions in $\mathbb{Z}^4$ is equivalent to that in $\mathbb{Q}^4$. If nontrivial solutions exist, the equation is called solvable in $\mathbb{Z}$ (and in $\mathbb{Q}$). The solvability depends only on the class of $\mathbb{Q}$-equivalence of $F$, where $F(\mathbf{x})$ is equivalent to $F'(\mathbf{x})$ if $F'(\mathbf{x}) = F(M\mathbf{x})$ for some $M \in \mathrm{GL}_4(\mathbb{Q})$.

We have evident necessary conditions: to be solvable in $\mathbb{Z}$, equation (B.1) should be solvable in any p-adic field $\mathbb{Q}_p \supset \mathbb{Q}$, where $p$ are primes, and also in $\mathbb{R} \supset \mathbb{Q}$. In fact these two conditions are also sufficient.

**Theorem B.1.** *("the Hasse principle") If* (B.1) *has a nontrivial solution in each of* $\mathbb{Q}_p^4$ *and in* $\mathbb{R}^4$ *then it has a nontrivial solution in* $\mathbb{Q}^4$.

Passing to an equivalent form, we can suppose that $F$ is diagonal:

$$F(\mathbf{x}) = \Sigma_{i=1}^4 a_i x_i^2, \quad a_j \in \mathbb{Q}.$$

We set $D(F) = \Pi_{i=1}^4 a_i \neq 0$ which is correctly defined as an element of the multiplicative group $\mathbb{Q}^*/(\mathbb{Q}^*)^2$, i.e. modulo rational squares, where $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$.

The equation (B.1) obviously is not solvable in $\mathbb{R}$ if $F$ is definite; therefore we suppose here that $F$ is indefinite. Let now $p$ be a prime. Then the image $D_p(F)$ of $D(F)$ in $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$ is well defined.

Let then $(a, b) \in \mathbb{Q}_p^* \times \mathbb{Q}_p^*$ be a pair of non-zero elements. Their *Hilbert symbol* $(a, b)_p \in \{\pm 1\}$ is defined as follows. We set $(a, b)_p = 1$ if the equation $x^2 - ay^2 - bz^2 = 0$ admits a nontrivial solution in $\mathbb{Q}_p^3$, and $(a, b)_p = -1$ otherwise. This symbol can be expressed as a product of Legendre symbols (see [12], Chapter III, Theorem 1), but we do not need it here. We can now define the following invariant of $F$ as a form over $\mathbb{Q}_p$:

$$\varepsilon_p(F) = \Pi_{1 \leq i < j \leq 4}(a_i, a_j)_p \in \{\pm 1\}.$$

**Theorem B.2.** *Eq.* (B.1) *is solvable in* $\mathbb{Q}_p$ *if and only if one of the following conditions holds:*

1. *either* $D_p(F) \neq 1$ *or*
2. $D_p(F) = 1$ *and* $\varepsilon_p(F) = (-1, -1)_p \in \{\pm 1\}$.

It is possible to deduce the following corollary from Theorem B.2.

**Corollary B.3.** *Over* $\mathbb{Q}_p$ *there exists only one, up to* $\mathbb{Q}_p$*-equvalence (and proportionality), non-degenerate quadratic form* $F_1$ *in 4 variables such that eq.* (B.1) *is not solvable in* $\mathbb{Q}_p$. *Namely,*

$$F_1(x, y, z, t) = x^2 - ay^2 - bz^2 + abt^2 \tag{B.2}$$

*with* $(a, b)_p = -1$.

This result is effective: let us take a non-degenerate quadratic form $F(x,y,z,t)$ and, diagonalizing and dividing by the coefficient of $x^2$, write it as $x^2 - ay^2 - bz^2 + ct^2$. Then $F \neq F_1$ for any $F_1$ as in (B.2) if $abc$ is not a square in $\mathbb{Q}_p$. Otherwise, if $abc \in (\mathbb{Q}_p^*)^2$, then it is sufficient to calculate $(a,b)_p$ (note that all these calculations reduce to calculating certain Legendre symbols).

# C Constants $\sigma(A,0)$ and $\sigma^*(A)$

In this section we consider the case when the quadratic form reads as

$$F(x,y) = \Sigma_{i=1}^{d/2} x_i y_i =: F_d(x,y) \qquad \text{where} \qquad d = 2s \geq 4 \tag{C.1}$$

and $x = (x_1, \dots, x_s)$, $y = (y_1, \dots, y_s)$. Our goal is to evaluate the constants $\sigma(A,0)$ for $d \geq 5$ and $\sigma^*(A)$ for $d = 4$. Below we use the usual notation for the relation that an integer $m$ divides or non-divides an integer vector $s$ (e.g. $2|(8,6)$ and $2 \nmid (8,7)$).

In view of the definitions (1.10)–(1.11), our first aim is to compute the constants $\sigma_p(A,0)$. For a prime $p$ and $k \in \mathbb{N}$ let consider the set

$$S_p(d;k) = \{(x,y) \mod p^k : \ F_d(x,y) = 0 \mod p^k\}$$

and denote $N_p(d;k) := \sharp S_p(d;k)$. Then the constants $\sigma_p$ can be rewritten as

$$\sigma_p(d) := \sigma_p(A,0) = \lim_{k\to\infty} \frac{N_p(d;k)}{p^{(d-1)k}}. \tag{C.2}$$

This relation is mentioned in [8], p. 50, without a proof; we provide a sketch of its rigorous derivation at the end of this appendix.

Let $\mathcal{N}_p(d) := N_p(d;1)$ be the number of $\mathbb{F}_p$–points on $\{F_d = 0 \mod p\}$.

**Lemma C.1.** *For any prime $p$,*

$$\sigma_p(d) = \frac{\mathcal{N}_p(d) - 1}{p^{d-1} - p^{1-d}}. \tag{C.3}$$

*Proof.* Since in the proof the dimension $d$ is fixed, we skip the dependence on it and write simply $S_p(k)$, $N_p(k)$ and $\mathcal{N}_p$ (so $\mathcal{N}_p = N_p(1)$). For $j = 0, 1, \dots, k$ we define $S_p(k,j)$ as a set of $(x,y) \in S_p(k)$ such that

$$(x,y) = p^j(x',y') \mod p^k, \ \text{ where } \ p \nmid (x',y').$$

So $S_p(k,0) = \{(x,y) \in S_p(k) : p \nmid (x,y)\}$ and $S_p(k,k) = \{(0,0)\}$. Sets $S_p(k,j)$ and $S_p(k,j')$ with $j \neq j'$ do not intersect, and denoting $N_p(k,j) = \sharp S_p(k,j)$ we have

$$S_p(k) = \bigcup\nolimits_{j=0}^{k} S_p(k,j), \quad N_p(k) = \sum\nolimits_{j=0}^{k} N_p(k,j)\,.$$

In particular, $N_p(1,0) = \mathcal{N}_p - 1$ since $N_p(1,1) = 1$. We claim that

$$N_p(k,0) = N_p(k-1,0)p^{(d-1)},$$

and thus

$$N_p(k,0) = N_p(1,0)p^{(d-1)(k-1)} = (\mathcal{N}_p - 1)\, p^{(d-1)(k-1)}. \tag{C.4}$$

Indeed, we argue by induction in $k$. Let $k = 2$ and $(x,y) \in S_p(2,0)$. Let us write $(x,y)$ as $(x_0 + pa, y_0 + pb)$ with $(x_0,y_0)$, $(a,b) \in \mathbb{F}_p^d$. Then $p \nmid (x_0,y_0)$, so $(x_0,y_0) \in S_p(1,0)$. Let us now fix any $(x_0,y_0) \in S_p(1,0)$ and look for $(a,b) \in \mathbb{F}_p^d$ such that $(x_0 + pa, y_0 + pb) \in S_p(2,0)$. Since $p^2 F(a,b) = 0 \mod p^2$ and $p \nmid (x_0,y_0)$, then relation $F(x,y) = 0 \mod p^2$ implies a non-trivial linear equation on $(a,b) \in \mathbb{F}_p^d$. So each $(x_0,y_0) \in S_p(1,0)$ generates exactly $p^{d-1}$ vectors $(x,y) \in S_p(2,0)$, which proves the formula for $k = 2$. This argument remains valid for any $k \geq 2$, by representing $(x,y) \bmod p^k$ in the form $(x_0 + p^{k-1}a, y_0 + p^{k-1}b)$ with $(x_0,y_0) \in \mathbb{F}_{p^{k-1}}^d$ and $(a,b) \in \mathbb{F}_p^d$.

Let now $(x,y) \in S_p(k,j)$ with $j \geq 1$. Then $(x,y) = p^j(x',y') \mod p^k$, where $p \nmid (x',y')$ and $(x',y')$ satisfies $p^{2j}F(x',y') = 0 \mod p^k$. Thus $(x',y') \in S_p(k-2j,0)$, if $j \leq \frac{k-1}{2}$, i.e. $j \leq \lfloor \frac{k-1}{2} \rfloor =: j_k$. The correspondence $(x,y) \mapsto (x',y')$ is a well defined mapping from $S_p(k,j)$ to $S_p(k-2j,0)$. Indeed, if $(x_1,y_1) \sim (x,y)$ in $S_p(k,j)$, then $p^{k-j} | \big((x_1',y_1') - (x',y')\big)$, so $(x_1',y_1') \sim (x',y')$ in $S_p(k-2j,0)$. Since this map is obviously surjective, then it is a bijection of $S_p(k,j)$ onto $S_p(k-2j,0)$, which in view of (C.4) implies

$$N_p(k,j) = N_p(k-2j,0) = (\mathcal{N}_p - 1)\, p^{(d-1)(k-2j-1)}.$$

By (C.4) this formula as well holds for $j = 0$.

Any $(x,y)$ such that $p^j | (x,y)$ with $j \geq j_k + 1$ satisfies $F(x,y) = 0 \mod p^k$. Thus

$$\sum_{j=j_k+1}^{k} N_p(k,j) = \sharp\{(x,y) \bmod p^k : (x,y) = 0 \bmod p^{j_k+1}\} = p^{d(k-j_k-1)} \leq p^{dk/2}.$$

Therefore

$$N_p(k) = (\mathcal{N}_p - 1)\, p^{(d-1)(k-1)} \sum_{j=0}^{j_k} p^{-2j(d-1)} + O(p^{dk/2}).$$

So

$$\sigma_p = \lim_{k\to\infty} \frac{N_p(k)}{p^{(d-1)k}} = (\mathcal{N}_p - 1)\, p^{1-d} \sum_{j=0}^{\infty} p^{-2j(d-1)} = \frac{p^{1-d}(\mathcal{N}_p - 1)}{1 - p^{2-2d}}\,,$$

which proves (C.3). □

Let then deduce a formula for $\mathcal{N}_p(d)$ using induction in $d/2 = s$. For $d = 2$ we have $\mathcal{N}_p(2) = \sharp\{(x, y) \in \mathbb{F}_p^2 : xy = 0 \mod p\} = 2p - 1$. Next,

$$\begin{aligned}\mathcal{N}_p(d+2) &= \sharp\{\text{solutions with } x_{s+1} = 0\} + \sharp\{\text{solutions with } x_{s+1} \neq 0\} \\ &= p\mathcal{N}_p(d) + (p-1)p^d.\end{aligned}$$

Therefore for any even $d = 2s \geq 2$,

$$\mathcal{N}_p(d) = p^{d-1} + p^s - p^{s-1},$$

and thus

$$\sigma_p(d) = \frac{1 + p^{1-s} - p^{-s} - p^{1-d}}{1 - p^{2-2d}} = \frac{(1 + p^{1-s})(1 - p^{-s})}{1 - p^{2-2d}}.$$

Since by Euler's formula $\prod_p (1 - p^{-l}) = 1/\zeta(l)$ for any $l > 1$, then in the case $d = 4$ we get from (1.11) and the obtained formula for $\sigma_p(d)$ that

$$\sigma(A, 0; d = 4) = \prod_p \sigma_p(4) = \frac{\zeta(6)}{\zeta(2)} \prod_p \left(1 + p^{-1}\right).$$

This does not converge, but

$$\sigma^*(A; d = 4) = \prod_p (1 - p^{-1})\sigma_p(4) = \frac{\zeta(6)}{\zeta(2)^2} = \frac{4\pi^2}{105} \simeq 0.376,$$

converges. Further,

$$\sigma(A, 0; d = 6) = \frac{\zeta(2)\zeta(10)}{\zeta(3)\zeta(4)} \simeq 1.265, \quad \sigma(A, 0; d = 8) = \frac{\zeta(3)\zeta(14)}{\zeta(4)\zeta(6)} \simeq 1.092,$$

whereas

$$1 < \sigma(A,0;d) = \frac{\zeta(s-1)\zeta(2d-2)}{\zeta(s)\zeta(d-2)} < 1 + 2^{2-s}$$

tends to 1 when $d = 2s \geq 10$ grows.

It remains to prove (C.2). By definition (1.10), $\sigma_p = \sum_{t=0}^{\infty} p^{-dt} S_{p^t}(\mathbf{0})$, where

$$S_{p^t}(\mathbf{0}) = \sum_{a \bmod p^t}^{*} \sum_{\mathbf{b} \bmod p^t} e_{p^t}(aF(\mathbf{b})).$$

Note that $p^{-dt} S_{p^t}(\mathbf{0}) = 1$ for $t = 0$, while for $t = 1$:

$$\begin{aligned} p^{-d} S_p(\mathbf{0}) &= p^{-d} \sum_{a=1}^{p-1} \sum_{\mathbf{b} \bmod p} e_p(aF(\mathbf{b})) \\ &= p^{-d} \sum_{a=1}^{p-1} \sum_{\mathbf{b} \bmod p,\, p \mid F(\mathbf{b})} 1 + \sum_{a=1}^{p-1} \sum_{\mathbf{b} \bmod p,\, p \nmid F(\mathbf{b})} e_p(aF(\mathbf{b})) \\ &= p^{-d}(p-1)\mathcal{N}_p(d) + p^{-d}(-1)(p^d - \mathcal{N}_p(d)) = p^{1-d}\mathcal{N}_p(d) - 1 \ , \end{aligned}$$

since

$$\sum_{a=1}^{m-1} e_m(an) = -1 \ , \tag{C.5}$$

for any $n, m \neq 0$ such that $m \nmid n$. Therefore $\sum_{t=0}^{1} p^{-dt} S_{p^t}(\mathbf{0}) = p^{1-d} N_p(d;1)$.

We proceed now by induction, supposing that, for $k \geq 1$,

$$\sum_{t=0}^{k} p^{-dt} S_{p^t}(\mathbf{0}) = p^{(1-d)k} N_p(d;k) \ .$$

Then we write

$$S_{p^{k+1}}(\mathbf{0}) = \sum_{a \bmod p^{k+1}}^{*} \sum_{\mathbf{b} \bmod p^{k+1}} e_{p^{k+1}}(aF(\mathbf{b})) = \Sigma_1 + \Sigma_2 + \Sigma_3 \ ,$$

where we have defined

$$\begin{aligned} \Sigma_1 &:= \sum_{a \bmod p^{k+1}}^{*} \sum_{p^{k+1} \mid F(\mathbf{b})} 1 = p^k(p-1)N_p(d;k+1) \ , \\ \Sigma_2 &:= \sum_{a \bmod p^{k+1}}^{*} \sum_{F(\mathbf{b}) = lp^k} e_p(al) = -p^k(p^d N_p(d;k) - N_p(d;k+1)) \ , \\ \Sigma_3 &:= \sum_{a \bmod p^{k+1}}^{*} \sum_{s=0}^{k-1} \sum_{F(\mathbf{b}) = lp^s} e_{p^{k+1-s}}(al) = 0 \ , \end{aligned}$$

with a non-zero $l = l(b)$ such that $p \nmid l$. The equalities above essentially follow by a repeated application of (C.5).

This way we have got

$$\frac{S_{p^{k+1}}(\mathbf{0})}{p^{d(k+1)}} = \frac{p^{k+1}N_p(d;k+1) - p^{d+k}N_p(d;k)}{p^{d(k+1)}} = \frac{N_p(d;k+1)}{p^{(d-1)(k+1)}} - \frac{N_p(d;k)}{p^{(d-1)k}} \ ,$$

which completes the induction step, thus proving (C.2).

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