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# Yet another Proof of an old Hat

Roland Bacher

October 28, 2021

## Abstract

Every odd prime number  $p$  can be written in exactly  $(p+1)/2$  ways as a sum  $ab + cd$  with  $\min(a, b) > \max(c, d)$  of two ordered products. This gives a new proof Fermat's Theorem expressing primes of the form  $1 + 4\mathbb{N}$  as sums of two squares <sup>1</sup>.

**Theorem 0.1.** *For every odd prime number  $p$  there exist  $(p+1)/2$  ordered quadruplets  $(a, b, c, d)$  in  $\mathbb{N}$  such that  $p = a \cdot b + c \cdot d$  and  $\min(a, b) > \max(c, d)$ .*

As a consequence we obtain a new proof of the following result discovered by an old rascal who did not want to spoil his margins and left the proof to another chap who had no such qualms.

**Corollary 0.2.** *Every prime number of the form  $1 + 4\mathbb{N}$  is a sum of two squares.*

*Proof of Corollary 0.2.* If  $p$  is a prime-number congruent to 1 (mod 4), the number  $(p+1)/2$  of solutions  $(a, b, c, d)$  defined by Theorem 0.1 is odd. The involution  $(a, b, c, d) \mapsto (b, a, d, c)$  has thus a fixed point  $(a, a, c, c)$  expressing  $p$  as a sum of two squares.  $\square$

Corollary 0.2 has of course already quite a few proofs. Some are described in the entry "Fermat's theorem on sums of two squares" of [2]. The author enjoyed also the account given in [1].

The set  $\mathcal{S}_p$  of solutions defined by Theorem 0.1 is invariant under the action of Klein's Vierergruppe with non-trivial elements acting by

$$(a, b, c, d) \mapsto (b, a, c, d), (a, b, d, c), (b, a, d, c) .$$

The following tables list all elements  $(a, b, c, d)$  with  $a, b, c, d$  decreasing together with the size  $\#(\mathcal{O})$  of the corresponding orbit under Klein's Vier-

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ergruppe for the sets  $\mathcal{S}_{29}$  and  $\mathcal{S}_{31}$ :

$a$	$b$	$c$	$d$	$\#(\mathcal{O})$		$a$	$b$	$c$	$d$	$\#(\mathcal{O})$
29	1	0	0	2		31	1	0	0	2
14	2	1	1	2		15	2	1	1	2
7	4	1	1	2		10	3	1	1	2
9	3	2	1	4		6	5	1	1	2
5	5	4	1	2		7	4	3	1	4
5	5	2	2	1		9	3	2	2	2
5	4	3	3	2		5	5	3	2	2
				15						16

Establishing complete lists  $\mathcal{S}_p$  of solutions for small primes is rather pleasant and rates among the author's more confessable procrastinations.

We proceed now by giving an elementary proof of Theorem 0.1.

A last Section contains a few remarks and ends with a somewhat speculative part.

## 1 Proof of Theorem 0.1

We state the following reformulation of Pick's Theorem<sup>2</sup>:

**Lemma 1.1.** *Two linearly independent elements  $u, v$  of a 2-dimensional lattice  $\Lambda$  form a basis of the lattice  $\Lambda$  if and only if the triangle with vertices  $(0, 0), u, v$  contains no other elements of  $\Lambda$ .*

*Proof.* This is an easy corollary of Pick's Theorem.

It follows also from the observation that the parallelogram with vertices  $(0, 0), u, v, u + v$  is a fundamental domain of the sub-lattice  $\mathbb{Z}u + \mathbb{Z}v$  of  $\Lambda$  spanned by  $u$  and  $v$ .  $\square$

**Lemma 1.2.** *If  $f_1, f_2$  and  $g_1, g_2$  are two bases of a 2-dimensional lattice  $\Lambda = \mathbb{Z}f_1 + \mathbb{Z}f_2 = \mathbb{Z}g_1 + \mathbb{Z}g_2$  such that  $\{\pm f_1, \pm f_2\}$  and  $\{\pm g_1, \pm g_2\}$  do not intersect, then  $\{\pm g_1, \pm g_2\}$  is contained in a two opposite connected components of  $\mathbb{R}^2 \setminus (\mathbb{R}f_1 \cup \mathbb{R}f_2)$ .*

Lemma 1.2 can be remembered easily: The lines  $\mathbb{R}f_1, \mathbb{R}f_2$  and  $\mathbb{R}g_1, \mathbb{R}g_2$  defined by two generating sets  $f_1, f_2$  and  $g_1, g_2$  of a two-dimensional lattice are never intertwined.

*Proof.* Up to sign-changes and up to exchanging the roles of  $f_1$  and  $f_2$  we have otherwise  $f_1 = \alpha g_1 + \beta g_2$  and  $f_2 = \gamma g_1 - \delta g_2$  where  $\alpha, \beta, \gamma, \delta$  are strictly positive integers. This implies that  $g_1$  belongs to the segment joining  $\frac{1}{\alpha}f_1$  to  $\frac{1}{\gamma}f_2$  contained in the convex hull of  $(0, 0), f_1, f_2$ . The assumption  $g_1 \notin \{f_1, f_2\}$  shows that this contradicts Lemma 1.1.  $\square$

<sup>2</sup>Pick's theorem gives the area  $\frac{1}{2}b + i - 1$  of a closed lattice polygon  $P$  (with vertices in  $\mathbb{Z}^2$ ) containing  $b$  lattice points  $\partial P \cap \mathbb{Z}^2$  in its boundary and  $i$  lattice points in its interior.

We consider the eight open cones of  $\mathbb{R}^2$  forming the complement of the four lines defined by  $xy(x^2 - y^2) = 0$ . We call these eight open cones *windmill-cones* (mainly as an attempt to turn the content of this paper into a piece of loftier mathematics) and we colour them alternately black and white, starting with a black E-NE windmill-cone  $\{(x, y) \mid 0 < y < x\}$  (using the conventions of wind-roses).

A sub-lattice  $M$  of finite index in  $\mathbb{Z}^2$  has a *black (respectively white) monochromatic basis* if it is generated by two elements  $b_1, b_2$  such that the set  $\{\pm b_1, \pm b_2\}$  intersects all four open black (respectively white) windmill-cones.

**Lemma 1.3.** *All monochromatic bases of a lattice have the same colour.*

*Proof.* If  $b_1, b_2$ , respectively  $w_1, w_2$ , is a black, respectively white, monochromatic basis of  $\mathbb{Z}b_1 + \mathbb{Z}b_2 = \mathbb{Z}w_1 + \mathbb{Z}w_2$  then the four lines defined by  $\mathbb{R}b_1, \mathbb{R}b_2$  and  $\mathbb{R}w_1, \mathbb{R}w_2$  are intertwined in contradiction with Lemma 1.2.  $\square$

An odd prime-number  $p$  and an integer  $\mu$  define a sub-lattice

$$\Lambda_\mu(p) = \{(x, y) \in \mathbb{Z}^2 \mid x + \mu y \equiv 0 \pmod{p}\} \quad (1)$$

of index  $p$  in  $\mathbb{Z}^2$ .

**Proposition 1.4.** *Every lattice  $\Lambda_\mu(p)$  with  $2 \leq \mu \leq p - 2$  has a monochromatic basis.*

*Proof.*  $\Lambda_\mu(p)$  contains obviously no elements of the form  $(\pm m, 0)$  or  $(\pm m, \pm p)$  with  $m$  in  $\{1, 2, \dots, p - 1\}$ . Since  $p$  is prime,  $\Lambda_\mu(p)$  contains no elements of the form  $(0, \pm m), (\pm p, \pm m)$  with  $m$  in  $\{1, \dots, p - 1\}$ . Moreover, for  $\mu$  in  $\{2, \dots, p - 2\}$  considered as a subset of the finite field  $\mathbb{Z}/p\mathbb{Z}$ , the elements  $1 + \mu$  and  $1 - \mu$  are invertible in  $\mathbb{Z}/p\mathbb{Z}$ . This implies that  $\Lambda_\mu(p)$  contains also no elements of the form  $(\pm m, \pm m)$  with  $m$  in  $\{1, \dots, p - 1\}$ . The intersection of a (black or white) windmill-cone with  $[-p, p]^2$  defines thus a triangle of area  $p^2/2$  whose boundary contains no lattice-points of  $\Lambda_\mu(p)$  except for its three vertices. Lemma 1.1 implies now that every open (black or white) windmill-cone contains a non-zero element  $(x, y)$  of  $\Lambda_\mu(p)$  with coordinates  $x, y$  in  $\{\pm 1, \pm 2, \dots, \pm(p - 1)\}$ . Let  $b_1$  be such a point in the black E-NE windmill-cone and let  $b_2$  be such a point in the black N-NW windmill-cone. Let  $\mathcal{Q}$  be the parallelogram with vertices  $\pm b_1, \pm b_2$ . If the interior of  $\mathcal{Q}$  contains a non-zero element  $\tilde{b}$  of  $\Lambda_\mu(p)$  in a black windmill-cone, replacing  $b_1$  or  $b_2$  by  $\pm \tilde{b}$  yields a smaller parallelogram  $\mathcal{Q}'$  strictly contained in  $\mathcal{Q}$ . Iterating this construction leads finally to a parallelogram  $\tilde{\mathcal{Q}}$  such that  $\tilde{\mathcal{Q}} \cap \Lambda_\mu(p)$  intersects open black windmill-cones only in vertices. If  $\tilde{\mathcal{Q}}$  contains no elements of  $\Lambda_\mu(p)$  in open white windmill-cones we get a black monochromatic basis by Lemma 1.1. Otherwise  $\pm b_1, \pm b_2$  generate a strict sub-lattice of  $\Lambda_\mu(p)$  and Lemma 1.1 implies that  $\tilde{\mathcal{Q}} \cap \Lambda_\mu(p)$  intersects all four white windmill-cones

in non-zero elements  $\pm w_1, \pm w_2$  of  $\Lambda_\mu(p)$ . Switching colours and restarting the previous construction with the parallelogram spanned by  $\pm w_1, \pm w_2$  ends the proof.  $\square$

We call a black monochromatic basis  $u, v$  of a lattice  $\Lambda_\mu(p)$  (with  $\mu$  in  $\{2, \dots, p-2\}$ ) *reduced* if  $u = (a, c), v = (-d, b)$  with  $a, b, c, d \in \mathbb{N}$  such that  $\min(a, b) > \max(c, d)$ .

**Lemma 1.5.** *A reduced black monochromatic basis is uniquely defined by one of its elements.*

*Proof.* Let  $u = (a, c), v = (-d, b)$  be a reduced black monochromatic basis of  $\Lambda = \mathbb{Z}u + \mathbb{Z}v$ . The element  $v$  belongs necessarily to one of the two closest affine lines parallel to  $\mathbb{R}u$  which intersect  $\Lambda$ . Since  $v$  belongs to the open black N-NW windmill-cone,  $v$  belongs to the closest line  $L_+$  intersecting  $\Lambda$  which is parallel to  $\mathbb{R}u$  and strictly above  $\mathbb{R}u$ . Reducedness of the basis  $u, v$  shows that  $v$  is the rightmost element of the intersection of  $L_+ \cap \Lambda$  with the open black N-NW windmill-cone.

An analogous argument shows that  $v$  determines  $u$  uniquely.  $\square$

**Proposition 1.6.** *Given an odd prime number  $p$ , a lattice  $\Lambda_\mu(p)$  with  $\mu$  in  $\{2, \dots, p-2\}$  has either only white monochromatic bases or it has a unique reduced black monochromatic basis.*

*Proof.* Proposition 1.4 shows that such a lattice contains either black or white monochromatic bases. They are either all black or all white by Lemma 1.3. We can thus assume that  $\Lambda_\mu(p)$  has only black monochromatic bases. We chose such a basis with  $u$  and  $v$  respectively in the open black E-NE and N-NW windmill-cone. Replacing  $u$  if necessary with  $u - sv$  we can assume that  $u$  is the lowest element of the open E-NE windmill cone which belongs to  $u + \mathbb{R}v \cap \Lambda_\mu(p)$ . Replacing similarly  $v$  with  $v + tu$  we can similarly assume that  $v$  is the rightmost element of the open N-NW windmill-cone which belongs to  $v + \mathbb{R}u \cap \Lambda_\mu(p)$ .

The set  $u = (a, c), v = (-d, b)$  is clearly still a black monochromatic basis of  $\Lambda_\mu(p)$ . We claim that  $u, v$  is reduced. Observe first that the inclusion of  $u$  in the open black E-NE windmill-cone implies  $0 < c < a$ . The inclusion of  $v$  in the open black N-NW windmill-cone implies similarly  $0 < d < b$ . Since  $\Lambda_\mu(p) \cap \mathbb{R}(1, 0) = \mathbb{Z}(p, 0)$ , we have either  $u - v = (p, 0)$  which leads to the contradiction  $\Lambda_\mu(p) = \mathbb{Z}(p, 0) + \mathbb{Z}(1, 0)$  or the vector  $u - v = (a + d, c - b)$  belongs to the lower half-plane and we have  $b > c$ . An analogous argument shows that  $v + u$  belongs to the half-plane  $\{(x, y), x > 0\}$ . This implies the inequality  $a > d$ . We have thus a black basis  $u = (a, c), v = (-d, b)$  with  $a, b, c, d$  in  $\mathbb{N}$  such that  $\min(a, b) > \max(b, c)$ . This shows that  $u, v$  is a reduced black monochromatic basis.

Assume now that  $\Lambda_\mu(p)$  has two reduced black monochromatic bases  $u = (a, c), v = (-d, b)$  and  $u' = (a', c'), v' = (-d', b')$ . Lemma 1.5 shows

that  $\mathbb{R}u, \mathbb{R}v$  and  $\mathbb{R}u', \mathbb{R}v'$  are four distinct lines which are not intertwined by Lemma 1.2. Up to exchanging  $u, v$  with  $u', v'$ , we can suppose that  $u', v'$  belong to the open cone  $(0, +\infty)u + (0, +\infty)v$  spanned by  $u$  and  $v$ . We have thus  $u' = \alpha u + \beta v$  and  $v' = \gamma u + \delta v$  with  $\alpha, \beta, \gamma, \delta$  strictly positive integers. Reducedness of the black monochromatic basis  $u, v$  implies that  $v + u$  does not belong to the black N-NW windmill-cone containing  $v$ . It is thus either an element of the closure of the white N-NE windmill-cone or it belongs to the black E-NE windmill-cone containing  $u$ .

Suppose first that  $u + v$  is an element of the closed white N-NE windmill-cone. Since  $\alpha + \beta$  and  $\gamma + \delta$  are both at least equal to 2 and since  $u', v'$  belong respectively to the black E-NE and the N-NW windmill cones, the element  $u + v$  belongs to the closed segment joining  $\frac{2}{\alpha + \beta}u'$  to  $\frac{2}{\gamma + \delta}v'$  contained in the convex hull of  $(0, 0), u', v'$ . This is a contradiction by Lemma 1.1.

All lattice points  $v + u, v + 2u, v + 3u, \dots$  are thus elements of the open black E-NE windmill-cone containing  $u$ . The affine line  $L = \gamma u + \mathbb{R}v$  intersects thus  $\Lambda_\mu(p)$  in at least two elements  $\gamma u, \gamma u + v$  of the E-NE windmill-cone. Since  $v$  has slope strictly smaller than  $-1$ , the intersection of  $L = \gamma u + \mathbb{R}v$  with the white N-NE windmill-cone is strictly longer than the intersection of  $L$  with the black E-NE windmill-cone. The intersection of  $L$  with the open white E-NE windmill-cone contains thus at least one element of  $\Lambda_\mu(p)$ . This implies  $\delta \geq 3$  where

$$v' = (-d', b') = \gamma u + \delta v = \gamma(a, c) + \delta(-d, b)$$

and we get  $b' = \gamma c + \delta b \geq c + 3b$ .

Since the open strip bounded by the two parallel lines  $\mathbb{R}v$  and  $(a, c) + \mathbb{R}v$  contains no elements of  $\Lambda_\mu(p)$  and since  $v$  has slope strictly smaller than  $-1$ , an element  $(x, y)$  of  $\Lambda_\mu(p)$  contained in the open black E-NE windmill-cone satisfies  $x \geq a/2$ . Applying this to  $u' = (a', c')$  we get  $a' \geq a/2$ . We have now

$$p = a'b' + c'd' > a'b' \geq \frac{a}{2}(c + 3b) = ab + a\frac{b+c}{2} > ab + ac > ab + cd = p$$

ending the proof.  $\square$

*Proof of Theorem 0.1.* Given an odd prime number  $p$ , we denote by  $\mathcal{S}_p$  be the set of all associated solutions  $(a, b, c, d)$  defined by Theorem 0.1.

We associate to a solution  $(a, b, c, d)$  in  $\mathcal{S}_p$  the two vectors  $u = (a, c)$ ,  $v = (-d, b)$  and we consider the sub-lattice  $\Lambda = \mathbb{Z}u + \mathbb{Z}v$  of index  $p = ab - c(-d)$  in  $\mathbb{Z}^2$  generated by  $u$  and  $v$ . Since  $p$  is prime, there are exactly two solutions with  $cd = 0$ , given by  $(p, 1, 0, 0)$  and  $(1, p, 0, 0)$  corresponding to the lattices  $\mathbb{Z}(p, 0) + \mathbb{Z}(0, 1)$  and  $\mathbb{Z}(1, 0) + \mathbb{Z}(0, p)$ .

We suppose henceforth  $cd > 0$ . The vectors  $u$  and  $v$  are contained respectively in the black E-NE and the black N-NW windmill-cone and form a reduced black monochromatic basis of the lattice  $\Lambda$ .

Sub-lattices of prime-index  $p$  in  $\mathbb{Z}^2$  are in bijection with the set of all  $p+1$  points on the projective line  $\mathbb{P}^2\mathbb{F}_p$  over the finite field  $\mathbb{F}_p$ . More precisely, a point  $[a : b]$  of the projective line defines the lattice

$$\Lambda_{[a:b]} = \{(x, y) \in \mathbb{Z}^2 \mid ax + by \equiv 0 \pmod{p}\}$$

corresponding to the lattice  $\Lambda_\mu$  (defined by (1)) with  $\mu \equiv b/a \pmod{p}$  for  $a$  invertible.

We have already considered lattices associated to the two solutions with  $cd = 0$ . The lattices corresponding to  $\mu \equiv \pm 1 \pmod{p}$  have no monochromatic basis and yield thus no solutions. All  $(p-3)$  lattices  $\Lambda_\mu$  with  $\mu \in \{2, \dots, p-2\}$  have monochromatic bases by Proposition 1.4.

Since  $\Lambda_\mu$  and  $\Lambda_{p-\mu}$  (respectively  $\Lambda_{\mu^{-1} \pmod{p}}$ ) differ by a horizontal (respectively diagonal) reflection, they have monochromatic bases of different colours. Proposition 1.6 shows thus that there are  $(p-3)/2$  different values of  $\mu$  in  $\{2, \dots, p-2\}$  which give rise to a lattice  $\Lambda_\mu$  corresponding to a solution in  $(a, b, c, d)$  in  $\mathcal{S}_p$  with  $cd \neq 0$ . The set  $\mathcal{S}_p$  contains thus exactly  $(p-3)/2 + 2 = (p+1)/2$  elements.  $\square$

**Remark 1.7.** *The lattice  $\Lambda = \mathbb{Z}(a, c) + \mathbb{Z}(-d, b)$  associated to a solution  $(a, b, c, d)$  in  $\mathcal{S}_p$  has a fundamental domain given by the union of the rectangle of size  $a \times b$  with vertices  $(0, 0), (a, 0), (a, c), (0, c)$  and of the rectangle of size  $d \times c$  with vertices  $(a, 0), (a+d, 0), (a+d, c), (a, c)$ .*

## 2 Complements

### 2.1 Constructing the solution associated to $\pm\mu$ in $\{2, \dots, p-2\}$

Every pair of opposite elements  $\pm\mu$  represented by an integer  $\mu \in \{2, \dots, p-2\}$  defines exactly one solution in  $\mathcal{S}_p$  and all solutions except  $(p, 1, 0, 0)$  and  $(1, p, 0, 0)$  are of this form. The associated solution can be constructed as follows: Gaußian lattice-reduction applied to

$$\Lambda_\mu(p) = \mathbb{Z}(p, 0) + \mathbb{Z}(-\mu, 1) = \{(x, y) \mid x + \mu y \equiv 0 \pmod{p}\}$$

yields a basis containing a shortest vector  $w$  in  $\Lambda_\mu(p)$ . Proposition 2.1 below shows how to deduce from this a black monochromatic basis either of  $\Lambda_\mu(p)$  or of  $\Lambda_{-\mu}(p)$ . The first part of the proof of Proposition 1.6 shows how to construct a reduced basis  $(a, c), (-d, b)$  (associated to the solution  $(a, b, c, d)$  defined by  $\Lambda_{\pm\mu}(p)$ ) from a black monochromatic basis.

**Proposition 2.1.** *Given an odd prime number  $p$  and  $\mu$  in  $\{2, \dots, p-2\}$ , let  $w$  be a shortest non-zero element of  $\Lambda_\mu(p)$ . There exists a monochromatic basis of  $\Lambda_\mu(p)$  which contains either  $w$  or a shortest element of  $\Lambda_\mu(p) \setminus \mathbb{Z}w$ .*

*Proof.* After a rotation by a suitable angle  $k\pi/2$  and perhaps a horizontal reflection, we end up with a lattice  $\Lambda$  having a shortest non-zero element  $w$  in the open black E-NE windmill-cone. Let  $L_+$  be the closest affine line above  $\mathbb{R}w$  which is parallel to  $\mathbb{R}w$  and intersects  $\Lambda \setminus \mathbb{Z}w$ . If the intersection of  $L_+$  with the open black N-NW windmill-cone contains an element  $r$  of  $\Lambda$ , we get a black monochromatic basis by considering  $w, r$ .

Otherwise an easy geometric argument shows that  $L_+$  intersects  $\Lambda$  in a rightmost point  $v$  of the open white W-NW windmill-cone and in a leftmost point  $u$  of the open white N-NE windmill-cone and we get a white monochromatic basis by considering  $u, v$ . Since  $u, v$  are separated by the black N-NW windmill-cone containing the orthogonal line to  $\mathbb{R}w$ , either  $u$  or  $v$  is a shortest element of  $\Lambda \setminus \mathbb{Z}w$ .  $\square$

## 2.2 Projective statistics

We say that a function  $f : \mathbb{N}^4 \setminus \{0, 0, 0, 0\} \rightarrow \mathbb{R}$  defines a *projective statistic* if  $f(\lambda a, \lambda b, \lambda c, \lambda d) = f(a, b, c, d)$  for all  $\lambda \geq 1$  (i.e. if  $f$  factorises through the projection of  $\mathbb{N}^4$  into  $\mathbb{P}^3\mathbb{R}$ ). Interesting examples when studying the sets  $\mathcal{S}_p$  of solutions to Theorem 0.1 are perhaps

$$\frac{c+d}{a+b}, \frac{cd}{ab}, \frac{c+d}{\sqrt{ab}}, \frac{\min(c,d)}{\max(a,b)}, \frac{\min(a,b)}{\max(a,b)}, \text{ etc..}$$

We assume henceforth  $f$  continuous on (an open set of)  $\mathbb{P}^3\mathbb{R}$  and we are interested in the asymptotic (with respect to  $p \rightarrow \infty$ ) proportion  $\mu_f(\Omega)$  of elements in  $\mathcal{S}_p$  given by the preimage  $f^{-1}(\Omega) \subset \mathcal{S}_p$  of an open set  $\Omega$  in  $\mathbb{R}$ . (Equivalently, one can also consider the asymptotic proportion of all elements in  $\mathcal{S}_p$  projecting on an open set  $\mathcal{O}$  of  $\mathbb{P}^3\mathbb{R}$ . Observe however that the closure of  $\cup_p \mathcal{S}_p$  avoids most of  $\mathbb{P}^3\mathbb{R}$ .)

If  $\mu_f(\Omega)$  exists (which should be the case for all reasonable continuous projective statistics  $f$ ) the probability measure  $\mu_f(\Omega)$  is perhaps equal to an integral explained below.

The value  $\mu_f(\Omega)$  can of course also be approximated almost surely, either by computing the set  $f^{-1}(\Omega) \subset \mathcal{S}_p$  for a large prime  $p$ , or by choosing a large number of pairs  $(p_i, \mu_i)$  with  $p_i$  large primes and  $\mu_i$  chosen uniformly among  $\{2, \dots, p-2\}$  leading to a lattice  $\Lambda_{\pm\mu_i}(p_i)$  having a reduced black basis and estimating  $\mu_f(\Omega)$  as the proportion of choices which lead to solutions  $(a, b, c, d)$  (computed using for example Section 2.1) in  $\mathcal{S}_{p_i}$  with  $f(a, b, c, d)$  in  $\Omega$ .

We are now going to explain a computation of  $\mu_f(\Omega)$  which is exact under an assumption of equidistribution. The famous modular curve  $\mathcal{M} = \mathbb{H}/\text{PSL}_2(\mathbb{Z})$  is the moduli space for rank 2 lattices in  $\mathbb{C}$ , up to orientation-preserving similitudes. An obvious quotient  $\tilde{U}$  of the unitary tangent bundle of  $\mathcal{M}$  corresponds to sub-lattices of  $\mathbb{C}$  up to positive real scalings (or equivalently to geodesics of the orbifold  $\mathcal{M}$  containing a marked point: Given a



sublattice  $\Lambda$  of  $\mathbb{C}$  with shortest non-zero vector  $w$  consider the corresponding point on the standard fundamental domain for  $\mathcal{M}$  together with the geodesic having the slope of  $w$  at this point). It can thus be identified with the set of all sub-lattices of  $\mathbb{C}$  with a given determinant.  $\mathcal{U}$  has a natural finite probability measure  $\nu_{\mathcal{U}}$ . We denote by  $\mathcal{U}_b$ , respectively by  $\mathcal{U}_w$  the subset of all elements of  $\mathcal{U}$  corresponding to sub-lattices of  $\mathbb{C}$  having a black, respectively white, monochromatic basis. The complement  $\mathcal{U} \setminus (\mathcal{U}_b \cup \mathcal{U}_w)$  is of measure zero and can be neglected. The function  $f$  defines now a continuous function  $\tilde{f}$  on an open subset of  $\mathcal{U}_b$ . Assuming asymptotical equidistribution in  $\mathcal{U}$  with respect to  $\nu_{\mathcal{U}}$  of sub-lattices of prime index  $p$  in  $\mathbb{Z}^2$ , we have  $\mu_f(\Omega) = \nu_{\mathcal{U}}(\tilde{f}^{-1}(\Omega))/\nu_{\mathcal{U}}(\mathcal{U}_B)$  (where  $\nu_{\mathcal{U}}(\mathcal{U}_b) = 1/2$  if  $\nu_{\mathcal{U}}$  is a probability measure with total measure 1 on  $\mathcal{U}$ ). This reduces the computation of  $\mu_f(\Omega)$  to an integration (of a complicated function on a complicated subset of  $\mathcal{U}$ ).

### 2.3 Variations

One can of course also consider the equation  $n = ab + cd$  for arbitrary  $n$ . There are two possibilities when  $n = ab$ : either require  $c = d = 0$  or accept solutions with  $cd = 0$  but  $c + d \in \{0, 1, \dots, \min(a, b) - 1\}$ . Both problems can be solved by the techniques of this paper, up to technicalities.

Also interesting is the equation  $n = ab - cd$  with  $\min(a, b) > \max(c, d)$ . The number of solutions in  $\mathbb{N}^4$  can be shown to be

$$\sum_{d|n, d^2 \geq n} \left( d + 1 + \frac{n}{d} \right).$$

Details will hopefully be provided ulteriorly in another paper.

## References

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