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ON THE GLOBAL APPROXIMATE CONTROLLABILITY IN SMALL TIME OF SEMICLASSICAL 1-D SCHRÖDINGER EQUATIONS BETWEEN TWO STATES WITH POSITIVE QUANTUM DENSITIES

JEAN-MICHEL CORON, SHENGGUAN XIANG, AND PING ZHANG

ABSTRACT. In this paper, we study, in the semiclassical sense, the global approximate controllability in small time of the quantum density and quantum momentum of the 1-D semiclassical cubic Schrödinger equation with two controls between two states with positive quantum densities. We first control the asymptotic expansions of the zeroth and first order of the physical observables via Agrachev-Sarychev's method. Then we conclude the proof through techniques of semiclassical approximation of the nonlinear Schrödinger equation.

Keywords: the semiclassical Schrödinger equation, quantum density, quantum momentum, controllability, semiclassical limit.

AMS Subject Classification (2000): 35Q55, 35C20, 93C20.

1. INTRODUCTION

The purpose of this paper is to investigate the global approximate controllability of the physical observables of the following one-dimensional cubic Schrödinger equation in the semiclassical regime

$$(1.1) \quad \begin{cases} i\hbar\partial_t\psi^\hbar = -\frac{\hbar^2}{2}\partial_x^2\psi^\hbar + \left(F^\hbar + (|\psi^\hbar|^2 - 1)\right)\psi^\hbar, & (t, x) \in \mathbb{R}^+ \times \mathbb{T}, \\ \psi^\hbar|_{t=0} = \psi_0^\hbar = a^0(x; \hbar) \exp(iS(x)/\hbar), \end{cases}$$

where ψ^\hbar denotes the wave function in quantum mechanics so that $\int_{\mathbb{T}} |\psi^\hbar(t, x)|^2 dx = 1$, \hbar denotes the Planck constant, and $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. The function ψ^\hbar is complex-valued while the functions $S(x)$ and F^\hbar are real-valued. The potential $V \stackrel{\text{def}}{=} F^\hbar + (|\psi^\hbar|^2 - 1)$ is composed by two parts: $(|\psi^\hbar|^2 - 1)$ corresponds to the Gross-Pitaevskii equations, and F^\hbar denotes a background charge that can be controlled in semiconductor applications. In control theory, typically for the case $\hbar = 1$, the term $F^\hbar\psi^\hbar$ is called bilinear control.

We assume that, for some positive integer k ,

$$(1.2) \quad \begin{aligned} & S(x) \in H^{k+3}, \quad a^0(x; \hbar) = a_0^0(x) + \hbar a_1^0(x) + \hbar R(x; \hbar) \quad \text{with} \\ & a_0^0(x) \in H^{k+2}, \quad a_0^0(x) > 0, \quad a_1^0(x) \in H^k \quad \text{and} \quad \lim_{\hbar \rightarrow 0^+} \|R(x; \hbar)\|_{H^k} = 0. \end{aligned}$$

For (1.1), the well-known Madelung transform (see [22]) introduces two real variables $\rho^\hbar(t, x)$ and $S^\hbar(t, x)$ such that $\psi^\hbar = \sqrt{\rho^\hbar} \exp(iS^\hbar/\hbar)$. Then we can equivalently rewrite (1.1) as

$$(1.3) \quad \begin{cases} \partial_t \rho^\hbar + \partial_x (\rho^\hbar u^\hbar) = 0 & (t, x) \in \mathbb{R}^+ \times \mathbb{T}, \\ \partial_t (\rho^\hbar u^\hbar) + \partial_x (\rho^\hbar (u^\hbar)^2) + \frac{1}{2} \partial_x (\rho^\hbar)^2 + \partial_x (F^\hbar) \rho^\hbar = \frac{\hbar^2}{2} \rho^\hbar \left(\frac{\partial_x^2 \sqrt{\rho^\hbar}}{\sqrt{\rho^\hbar}} \right)_x, \\ \rho^\hbar|_{t=0} = |a^0(x, \hbar)|^2, \quad u^\hbar|_{t=0} = \partial_x S(x), \end{cases}$$

with

$$(1.4) \quad u^{\hbar}(t, x) \stackrel{\text{def}}{=} \partial_x S^{\hbar}(t, x).$$

The right-hand side of the second equation of (1.3) is called the quantum pressure.

Formally, by taking $\hbar \rightarrow 0$ in (1.3), we obtain the following compressible Euler equation in $\mathbb{R}^+ \times \mathbb{T}$ for quantities (ρ_0, u_0) with an additional force $\eta_0(t, x)$:

$$(1.5) \quad \begin{cases} \partial_t \rho_0 + \partial_x(u_0 \rho_0) = 0, \\ \partial_t u_0 + u_0 \partial_x u_0 + \partial_x \rho_0 = \eta_0, \\ (\rho_0, u_0)|_{t=0} = (|a_0^0|^2, \partial_x S), \end{cases}$$

where

$$(1.6) \quad \eta_0 \stackrel{\text{def}}{=} -\partial_x F_0,$$

and where the quantities ρ_0, u_0, F_0 correspond to the leading order of $\rho^{\hbar}, u^{\hbar}, F^{\hbar}$ under WKB sense. We refer to Section 2.1, especially to Equation (2.2)–(2.4) for details of this expansion and even for the related higher order expansions.

Since u_0 is the derivative of some function S_0 on the torus \mathbb{T} , it satisfies $\int_{\mathbb{T}} u_0 dx = 0$. For this purpose, we denote by H_0^s the codimension one subspace of H^s such that its elements satisfy $\int_{\mathbb{T}} f dx = 0$.

Corresponding to the Newtonian mechanics, we call ρ^{\hbar} and $\rho^{\hbar} u^{\hbar}$ the quantum density and quantum momentum respectively. In the whole space case and if there is no superfluid at infinity and $f'(0) > 0$, Grenier [16] solved the limit problem before the formation of singularity in the limit system with initial data in Sobolev spaces. The main idea in [16] is to use the symmetrizer of the limit system (1.5) to get H^s energy estimates which are uniform in ε for a singularly perturbed system. Nevertheless this method does not work for the semiclassical limit of Schrödinger-Poisson equations, as the resulting limit system is not a symmetric hyperbolic one. The third author of this manuscript [30] used the Wigner measure approach to study the semiclassical limit of Schrödinger-Poisson equation (see [29] for more general nonlinearity). The assumption that $f'(0) > 0$ in [29] was removed by Alazard and Carles in [3]. With superfluid at infinity to pass over an obstacle, the corresponding problem was solved by Lin and Zhang in [21].

The goal of this paper is to find control F^{\hbar}

$$(1.7) \quad F^{\hbar}(t, x) = F_0(t, x) + \hbar F_1(t, x) \text{ with } F_i(t, x) \in C([0, T]; E),$$

where

$$(1.8) \quad E \stackrel{\text{def}}{=} \text{span}\{\sin x, \cos x\} \text{ is a two-dimensional control space}$$

such that we are able to get the approximate controllability of the physical observables ρ^{\hbar} and u^{\hbar} .

More precisely, our main result states as follows.

Theorem 1.1. *Let $k \geq 3, T > 0$ and $\varepsilon > 0$. Let $(\rho_0^f, u_0^f) \in H^k \times H_0^k$ and $(\rho_1^f, u_1^f) \in H^{k-2} \times H_0^{k-2}$, which satisfy*

$$\int_{\mathbb{T}} \rho_0^f dx = \int_{\mathbb{T}} |a_0^0|^2 dx, \quad \int_{\mathbb{T}} \rho_1^f dx = \int_{\mathbb{T}} 2\text{Re}(\bar{a}_0^0 a_1^0) dx \quad \text{and} \quad \rho_0^f > 0.$$

There exist $(\hat{\rho}_0, \hat{u}_0) \in H^{k+2} \times H_0^{k+2}$, $(\hat{\rho}_1, \hat{u}_1) \in H^k \times H_0^k$ which satisfy

$$\int_{\mathbb{T}} \hat{\rho}_0 dx = \int_{\mathbb{T}} \rho_0^f dx, \quad \int_{\mathbb{T}} \hat{\rho}_1 dx = \int_{\mathbb{T}} \rho_1^f dx \quad \text{and} \quad \hat{\rho}_0 > 0,$$

$$\|(\hat{\rho}_0, \hat{u}_0) - (\rho_0^f, u_0^f)\|_{H^k \times H^k} \leq \varepsilon, \quad \|(\hat{\rho}_1, \hat{u}_1) - (\rho_1^f, u_1^f)\|_{H^{k-2} \times H^{k-2}} < \varepsilon,$$

and controls F^h in $C^\infty([0, T]; E)$ such that, the solution of equation (1.1) satisfies

$$\begin{aligned} \psi^h(T) &= (\hat{a}_0 + \hbar \hat{a}_1 + \hbar r_a^h) \exp\left(\frac{i}{\hbar} (\hat{S}_0 + \hbar \hat{S}_1 + \hbar r_S^h)\right) \\ &= (\hat{a}_0 e^{i\hat{S}_1} + s^h) \exp\left(\frac{i}{\hbar} \hat{S}_0\right), \end{aligned}$$

with

$$\begin{aligned} |\hat{a}_0(x)|^2 &= \hat{\rho}_0(x), \quad 2\text{Re}(\bar{\hat{a}}_0 \hat{a}_1) = \hat{\rho}_1(x), \\ \partial_x \hat{S}_0(x) &= \hat{u}_0(x), \quad \partial_x \hat{S}_1(x) = \hat{u}_1(x), \\ \lim_{\hbar \rightarrow 0^+} \|r_a^h\|_{H^{k-2}} &= 0, \quad \lim_{\hbar \rightarrow 0^+} \|r_S^h\|_{H^{k-1}} = 0, \quad \lim_{\hbar \rightarrow 0^+} \|s^h\|_{H^{k-2}} = 0. \end{aligned}$$

Remark 1.1. The proof of Theorem 1.1 can in fact ensure the same result for nonlinear Schrödinger equations with more general nonlinearity: $f(|\psi^h|^2)\psi^h$ which satisfies $f'(0) > 0$.

Remark 1.2. Let $N \geq 2$. Using the same approach, we are able to find control

$$F^h(t, x) = \sum_{i=0}^N \hbar^i F_i(t, x) \quad \text{with} \quad F_i \in C([0, T]; E), \quad i = 1, \dots, N,$$

so that we can get the approximate controllability of the physical observables up to order N .

Remark 1.3. Along the same line to the proof of Theorem 1.1, we can work out the same type of result in any space dimension, i.e. on \mathbb{T}^d for any $d \in \mathbb{N} \setminus \{0\}$, with controls be restricted to some finite dimensional space

$$(1.9) \quad \mathcal{E} := \text{span}\{\sin(\vec{l} \cdot \vec{x}), \cos(\vec{l} \cdot \vec{x}); \vec{l} \in \mathcal{K}_d\}.$$

An example is

$$(1.10) \quad \mathcal{K}_d := \{(l_1, \dots, l_d); l_j = 0 \text{ or } \pm 1\},$$

which also has been used by Duca and Nersesyan [12] for the study of approximate controllability between any pairs of eigenstates of NLS in \mathbb{T}^d .

Remark 1.4. The study of the controllability of the Schrödinger equation corresponds to the case that $\hbar = 1$ in (1.1) has been attracted by many authors. Let us mention, in particular, the survey paper [19] by Laurent and the references therein, [20] by Lebeau for internal controllability of the multidimensional Schrödinger equation using microlocal analysis. The local exact controllability of the one dimensional bilinear Schrödinger equation was first proved by Beauchard [4] using a Nash–Moser iteration process, and then simplified by Beauchard–Laurent in [5]. Finally, in [25] Sarychev also studied the approximate controllability of NLS in 1D and 2D torus based on Agrachev–Sarychev method, and again relying on this method Duca and Nersesyan [12] have studied the approximate controllability between any pairs of eigenstates of NLS in \mathbb{T}^d .

2. SKETCH OF THE PROOF

2.1. Semiclassical approximation in the general setting. Let us recall from [16] that instead of looking, as usual, for solutions ψ^{\hbar} to (1.1) of the form

$$\psi^{\hbar}(t, x) = a^{\hbar}(t, x) \exp(iS(t, x)/\hbar)$$

with $S(t, x)$ independent of \hbar and $a^{\hbar}(t, x)$ a real-valued function, we can look for solutions ψ^{\hbar} of (1.1) with the form:

$$(2.1) \quad \begin{aligned} \psi^{\hbar}(t, x) &= a^{\hbar}(t, x) \exp\left(iS^{\hbar}(t, x)/\hbar\right) \quad \text{with} \\ a^{\hbar}(t, x) &= a_0(t, x) + \hbar a_1(t, x) + \hbar r_a^{\hbar}(t, x), \\ S^{\hbar}(t, x) &= S_0(t, x) + \hbar S_1(t, x) + \hbar r_S^{\hbar}(t, x), \end{aligned}$$

where a^{\hbar} is a complex-valued function. By inserting (2.1) into (1.1), we obtain

$$(2.2) \quad \begin{cases} \partial_t S^{\hbar} + \frac{1}{2} (\partial_x S^{\hbar})^2 + F^{\hbar} + |a^{\hbar}|^2 - 1 = 0, \\ \partial_t a^{\hbar} + \partial_x S^{\hbar} \partial_x a^{\hbar} + \frac{1}{2} a^{\hbar} \partial_x^2 S^{\hbar} = i \frac{\hbar}{2} \partial_x^2 a^{\hbar}, \\ S^{\hbar}|_{t=0} = S(x), \quad a^{\hbar}|_{t=0} = a^0(x; \hbar). \end{cases}$$

Let us define

$$\eta_j(t, x) \stackrel{\text{def}}{=} -\partial_x F_j(t, x) \in E.$$

Since $F_j \in E$ we can determine F_j from η_j by

$$F_j(t, x) = -\int_0^x \eta_j(t, x) dx + (1/2\pi) \int_0^{2\pi} (2\pi - s) \eta_j(s) ds.$$

In view of (1.5), by comparing the coefficients of \hbar^j on both sides of (2.2), at least formally, we are able to obtain equations for the coefficients of the expansions in (2.1). The \hbar^0 order equation reads as

$$(2.3) \quad \begin{cases} \partial_t S_0 + \frac{1}{2} (\partial_x S_0)^2 + |a_0|^2 - 1 = -F_0(t, x), \\ \partial_t a_0 + \partial_x S_0 \partial_x a_0 + \frac{1}{2} a_0 \partial_x^2 S_0 = 0, \\ S_0|_{t=0} = S(x), \quad a_0|_{t=0} = a_0^0(x). \end{cases}$$

Let us denote $\rho_0 \stackrel{\text{def}}{=} |a_0|^2$ and $u_0 \stackrel{\text{def}}{=} \partial_x S_0$. We recover the compressible Euler system (1.5). However, (2.3) is not equivalent to (1.5) though it is derived from the previous one. In fact, let us denote a_0 as $a_0^r + i a_0^i$, then (2.3) can be reformulated as the following real-valued equation:

$$(2.4) \quad \begin{cases} \partial_t a_0^r + u_0 \partial_x a_0^r + \frac{1}{2} a_0^r \partial_x u_0 = 0, \\ \partial_t a_0^i + u_0 \partial_x a_0^i + \frac{1}{2} a_0^i \partial_x u_0 = 0, \\ \partial_t u_0 + u_0 \partial_x u_0 + \partial_x \rho_0 = \eta_0(t, x), \\ (a_0^r, a_0^i, u_0)|_{t=0} = (\text{Re}(a_0^0), \text{Im}(a_0^0), \partial_x S). \end{cases}$$

For \hbar^1 order of (2.2), we get the following equation for (a_1, S_1) :

$$(2.5) \quad \begin{cases} \partial_t S_1 + \partial_x S_0 \partial_x S_1 + 2\text{Re}(a_0 \bar{a}_1) = -F_1(t, x), \\ \partial_t a_1 + \partial_x S_0 \partial_x a_1 + \partial_x S_1 \partial_x a_0 + \frac{1}{2}(a_0 \partial_x^2 S_1 + a_1 \partial_x^2 S_0) = \frac{i}{2} \partial_x^2 a_0, \\ S_1|_{t=0} = 0, \quad a_1|_{t=0} = a_1^0(x). \end{cases}$$

Again by defining $a_1 \stackrel{\text{def}}{=} a_1^r + ia_1^i$ and $u_1 \stackrel{\text{def}}{=} \partial_x S_1$ we reformulate (2.5) as

$$(2.6) \quad \begin{cases} \partial_t a_1^r + u_0 \partial_x a_1^r + u_1 \partial_x a_0^r + \frac{1}{2}(a_0^r \partial_x u_1 + a_1^r \partial_x u_0) = -\frac{1}{2} \partial_x^2 a_0^i, \\ \partial_t a_1^i + u_0 \partial_x a_1^i + u_1 \partial_x a_0^i + \frac{1}{2}(a_0^i \partial_x u_1 + a_1^i \partial_x u_0) = \frac{1}{2} \partial_x^2 a_0^r, \\ \partial_t u_1 + u_0 \partial_x u_1 + u_1 \partial_x u_0 + 2\partial_x(a_0^r a_1^r + a_0^i a_1^i) = \eta_1(t, x), \\ (a_1^r, a_1^i, u_1)|_{t=0} = (\text{Re}(a_1^0), \text{Im}(a_1^0), 0). \end{cases}$$

2.2. On the control of physical observables. Following the idea of [16], for any time $T > 0$, it sounds interesting to let the final state at time T to approximate to $\bar{a} \exp(i\bar{S}/\hbar)$ with the help of some extra force as control, where \bar{a} and \bar{S} are target functions. Nevertheless, we are not able to control both a_0^r and a_0^i simultaneously. Indeed it is easy to observe from the zeroth-order approximate system (2.4) that if $\text{Im}(a_0^0) = 0$, we find that a_0^i is always zero in the future.

On the other hand, it is of physical importance and is natural to consider the controllability of the physical observables ρ^h and u^h . Hence, first we are going to control the compressible Euler system (1.5) through the control η_0 . As for the first order approximation, we observe from (2.6) that

$$\partial_t(\bar{a}_0 a_1) + u_0 \partial_x(\bar{a}_0 a_1) + u_1 \bar{a}_0 \partial_x a_0 + \frac{1}{2} \rho_0 \partial_x u_1 + \bar{a}_0 a_1 \partial_x u_0 = \frac{i}{2} \bar{a}_0 \partial_x^2 a_0.$$

Taking the real part of the above equation and defining $\rho_1 \stackrel{\text{def}}{=} 2\text{Re}(\bar{a}_0 a_1)$ give rise to

$$(2.7) \quad \partial_t \rho_1 + \partial_x(u_0 \rho_1 + u_1 \rho_0) = \frac{i}{2} \partial_x(\bar{a}_0 \partial_x a_0 - a_0 \partial_x \bar{a}_0).$$

This leads to the following control system:

$$(2.8) \quad \begin{cases} \partial_t \rho_1 + \partial_x(u_0 \rho_1 + u_1 \rho_0) = \frac{i}{2} \partial_x(\bar{a}_0 \partial_x a_0 - a_0 \partial_x \bar{a}_0) \\ \partial_t u_1 + \partial_x(u_0 u_1) + \partial_x \rho_1 = \eta_1. \end{cases}$$

As we mentioned before the compressible Euler system (1.5) is not equivalent to the zeroth-order approximate system (2.4). Hence in order to determine the right-hand side of the first equation of (2.8), let us define

$$(2.9) \quad A \stackrel{\text{def}}{=} \frac{i}{2} (\bar{a}_0 \partial_x a_0 - a_0 \partial_x \bar{a}_0).$$

Then it follows from (2.4) that

$$(2.10) \quad \begin{cases} \partial_t A + u_0 \partial_x A + 2\partial_x u_0 A = 0, \\ A(0) = A_0 \stackrel{\text{def}}{=} a_0^i(0) \partial_x a_0^r(0) - a_0^r(0) \partial_x a_0^i(0). \end{cases}$$

Once (ρ_0, u_0) is determined from (1.5), we can solve for (a_0^r, a_0^i) via (2.4). Similarly, with (ρ_1, u_1) being determined, we can solve for (a_1^r, a_1^i) via (2.6). Along the same line, we can control the physical observables of any k -th order, and then solve for (a_k^r, a_k^i) . For a concise presentation, we shall not pursue in this direction here.

Then the proof of Theorem 1.1 will be split in two parts: the controllability of the asymptotic expansions of the physical quantities, and then the justification of the semiclassical approximation to the system (1.1).

By combining the systems (1.5), (2.8) and (2.10), we obtain the following real valued controlled system in $[0, T] \times \mathbb{T}$ for the first two terms in the asymptotic expansions of physical quantities:

$$(2.11) \quad \begin{cases} \partial_t \rho_0 + \partial_x(u_0 \rho_0) = 0, \\ \partial_t u_0 + u_0 \partial_x u_0 + \partial_x \rho_0 = \eta_0, \\ \partial_t A + u_0 \partial_x A + 2 \partial_x u_0 A = 0, \\ \partial_t \rho_1 + \partial_x(u_0 \rho_1 + u_1 \rho_0) = \partial_x A, \\ \partial_t u_1 + \partial_x(u_0 u_1) + \partial_x \rho_1 = \eta_1. \end{cases}$$

Let $\vec{\rho} \stackrel{\text{def}}{=} \begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix}$ and $\vec{u} \stackrel{\text{def}}{=} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$. Given initial state $\vec{\rho}(0) = \vec{g} \stackrel{\text{def}}{=} \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}$ and $\vec{u}(0) = \vec{v} \stackrel{\text{def}}{=} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$, A_0 and the state at time T that $\vec{\rho}(T) = \widehat{\vec{g}} \stackrel{\text{def}}{=} \begin{pmatrix} \widehat{g}_0 \\ \widehat{g}_1 \end{pmatrix}$ and $\vec{u}(T) = \widehat{\vec{v}} \stackrel{\text{def}}{=} \begin{pmatrix} \widehat{v}_0 \\ \widehat{v}_1 \end{pmatrix}$, we are going to search for a control $\vec{\eta} \stackrel{\text{def}}{=} \begin{pmatrix} \eta_0 \\ \eta_1 \end{pmatrix}$ such that the solution of system (2.11) satisfying the initial condition

$$(2.12) \quad \vec{\rho}(0) = \vec{g}, \vec{u}(0) = \vec{v}$$

and $(u_0, u_1, \rho_0, \rho_1)(T)$ is closed to $(\widehat{v}_0, \widehat{v}_1, \widehat{g}_0, \widehat{g}_1)$.

Let us remark here that we will not control the quantity A , though it will be used to control the next order terms.

2.3. Approximate controllability of the physical observables. Therefore, we are going to control the limit system (2.11) for $(\vec{u}, \vec{\rho}, A)$. It is a complex system with cascade structure:

$$(\rho_0, u_0) \rightarrow A \rightarrow (\rho_1, u_1).$$

- Firstly, (ρ_0, u_0) is governed by the Saint-Venant equation with finite dimensional distributed control η that acts on the “velocity” part u_0 . Hence, it is rather natural to expect controllability properties of this part. Actually, in [24] Nersisyan has considered the approximate controllability of the 3D compressible Euler equations which is similar to this Saint-Venant equation, and we mimic his approach on the controllability of this part.

We comment here, as we shall see later on in Section 4, thanks to the “transport” nature of the equation on ρ_0 , during the whole controlling process the value of ρ_0 is always sufficiently close to some given trajectory that is strictly positive.

- Next, A verifies a transport equation that is influenced by the value of u_0 . However, there is no direct control mechanism on this quantity: therefore it is indirectly controlled by η_0 .

Moreover, we observe from the preceding section that the quantities ρ_0, ρ_1 and A all come from a^{\hbar} . Hence it is reasonable to expect that they will share certain similarities. As it is actually directly shown in Equation (2.11) that they all satisfy “transport” type equations. Consequently, similar to ρ_0 , the value of A is always sufficiently close to some given trajectory.

Finally, we remark here that we are not going to control A to some given final state \tilde{A} . Indeed, by looking at (2.4) for (a_0^r, a_0^i) , we observe that both terms satisfy the same equation, as a consequence, we are not able to control both items. If we further

look at the definition of ρ_0 and A , it becomes reasonable that we control ρ_0 as the “density” part of a_0 and “lose the control” of the “rotation” part A . But instead we will control the trajectory of $A(t)$ by making it to stay close to a given one. The value of $A(t)$ will be used to control (ρ_1, u_1) .

- Finally, (ρ_1, u_1) is a coupled systems with finite dimensional controlling terms that is also influenced by the value of (ρ_0, u_0, A) . The main feature on the controllability of this part, different from well-posedness issue which is interested in the uniqueness and existence issues, is that the value of (ρ_0, u_0, A) also plays a significant role in the controlling process. Therefore, it should be regarded as a bilinear system and the influence of the control term η_0 should not be ignored. This is actually one of the main novelties on the controllability part of this paper.

Similar to ρ_0 and A , the value of ρ_1 is also kept close to a constructed trajectory.

The main result lists as follows, while the detailed proof of which will be presented in Section 4.

Theorem 2.1. *Let $k \geq 3, T > 0, \vec{g} \in H^{k+2} \times H^k, \vec{v} \in H_0^{k+2} \times H_0^k, \hat{g} \in H^k \times H^{k-2}, \hat{v} \in H_0^k \times H_0^{k-2}$ and $A_0 \in H^{k+1}$, which satisfy*

$$\int_{\mathbb{T}} g_0 dx = \int_{\mathbb{T}} \hat{g}_0 dx = \alpha_0, \quad \int_{\mathbb{T}} g_1 dx = \int_{\mathbb{T}} \hat{g}_1 dx = \alpha_1 \quad \text{and} \quad g_0, \hat{g}_0 \geq c_0 > 0.$$

Then for any $\varepsilon > 0$, there exist controls $\eta_0, \eta_1 \in C^\infty([0, T]; E)$ such that the solution $(u_0, u_1, \rho_0, \rho_1, A)(t)$ to the system (2.11) satisfies the initial condition (2.12) and

$$\|(u_0, u_1, \rho_0, \rho_1)(T) - (\hat{v}_0, \hat{v}_1, \hat{g}_0, \hat{g}_1)\|_{H^k \times H^{k-2} \times H^k \times H^{k-2}} \leq \varepsilon.$$

Remark 2.1. *One can observe that we have assumed the value of density g_0, \hat{g}_0 being strictly positive in order to use the general well-posedness theory on symmetrizable hyperbolic systems. As we shall see later on, the constructed approximate solutions $\rho_0^l(t), l \in \{0, 1, \dots, N\}$ is always close to the first constructed solution $\rho_0^N(t)$ which is strictly positive. It will be interesting and also challenging to consider about the case that g_0, \hat{g}_0 are not strictly positive, for example admit finite many points such that $g_0(x)$ equal to zero.*

We remark that the main idea of the proof of Theorem 2.1 is directly motivated by [23, 24] on the controllability of the incompressible and compressible Euler system using the Agrachev–Sarychev method, though we are dealing with a coupled system that is more complicated. This approach originates from [1, 2] by Agrachev–Sarychev for the controllability of 2D Navier–Stokes and 2D Euler equations with finite-dimensional external control, and is further investigated on many other models, such as [25, 26] and among others. Shirikyan [27] wrote an instructive introduction on the application of this method upon the viscous Burgers equation. We refer to Section 4 on the detailed proof of Theorem 2.1, where we will also comment on each step how we get inspired from this method.

Let us also point out that this method emphasizing geometric control only leads to approximate controllability properties, while exact controllability can be only made on finite dimensional sets. It is natural to ask whether some local controllability result may further lead to global exact controllability. When dealing with boundary control or localized internal control problems, the exact controllability of the related models can be obtained using other methods notably the “return method” and “Carleman estimates”, see for example, [9, 10, 13, 28] on Navier–Stokes equations, and [7, 8, 14, 15] on Euler equations and similar Saint-Venant equations. However, there are two major difficulties that prevent us from getting local exact controllability of (2.11) using finite dimensional distributed controls. The

first difficulty is related to the fact that we are only controlling four of the five components of the state. The other difficulty is more systemic: it is reasonable to consider the moment theory for the linearized system of this nonlinear problem; however, so far, this method has not been adapted for first order quasi-linear hyperbolic systems. Indeed, a loss of derivative issue appears when we attempt to apply fixed point theorems, see [9, Chapter 4.2] for a simplified model that presents the same difficulty. Therefore, it is required to prove exact (partial) controllability of the linearized systems around a class of time-varying trajectories, see for example in [8, Lemma 17] the author has used this strategy to overcome the loss of derivative issue for Saint–Venant equation with boundary control, for which the proof relies on the fact that there are enough controllable linear systems close to the non linear control system (thanks to characteristic lines) to get the local controllability of the nonlinear control system by a suitable fixed point method. However, whether this idea can be adapted to distributed controlled systems by using the moment theory, it still remains open.

Finally, in view of the recent progresses in stabilization problems, see for instance [11, 17, 18], it also sounds interesting to have a look at the related stabilization problems.

2.4. Semiclassical limits. Armed with Theorem 2.1, we shall conclude the proof of Theorem 1.1 by

Theorem 2.2. *Let $k > 5/2, T > 0$. Suppose that $(\rho_0, u_0) \in C([0, T]; H^k)$ is the solution of (1.5) determined by Theorem 2.1, then under the assumptions of Theorem 1.1, for \hbar small enough, (1.1) has a solution of the form (2.1) on the interval $[0, T]$, which satisfies*

$$\psi^\hbar = a^\hbar \exp\left(\frac{i}{\hbar} S^\hbar\right) = (a_0 + \hbar a_1 + \hbar r_a^\hbar) \exp\left(\frac{i}{\hbar} (S_0 + \hbar S_1 + \hbar r_S^\hbar)\right),$$

with

$$\begin{aligned} & a^\hbar \text{ and } u^\hbar \text{ are uniformly bounded in } C([0, T]; H^k) \text{ as } \hbar \rightarrow 0^+ \\ & (a_0, S_0) \text{ the solution of (2.3), and } a_0, u_0 \in C([0, T]; H^{k+2}), \\ & (a_1, S_1) \text{ the solution of (2.5), and } a_1, u_1 \in C([0, T]; H^k), \\ & \lim_{\hbar \rightarrow 0^+} \|r_a^\hbar\|_{C([0, T]; H^{k-2})} = 0, \quad \lim_{\hbar \rightarrow 0^+} \|r_S^\hbar\|_{C([0, T]; H^{k-1})} = 0. \end{aligned}$$

3. THE WELL-POSEDNESS AND CONTINUOUS DEPENDENCE OF THE LIMIT SYSTEMS

This section is devoted to the study of the controlled system (2.11). In what follows, we shall always use the convention that $\vec{f} = (f_1, \dots, f_n) \in X$ means that each component, f_i , belongs to the space X , and we designate $\|\vec{f}\|_X \stackrel{\text{def}}{=} \sum_{i=1}^n \|f_i\|_X$.

Given $\vec{\zeta} = \begin{pmatrix} \zeta_0 \\ \zeta_1 \end{pmatrix}$, $\vec{\xi} = \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix}$ and $\vec{\eta} = \begin{pmatrix} \eta_0 \\ \eta_1 \end{pmatrix}$, we consider

$$(3.1) \quad \begin{cases} \partial_t \rho_0 + \partial_x((u_0 + \zeta_0)\rho_0) = 0, \\ \partial_t u_0 + (u_0 + \xi_0)\partial_x(u_0 + \xi_0) + \partial_x \rho_0 = \eta_0, \\ \partial_t A + (u_0 + \zeta_0)\partial_x A + 2\partial_x(u_0 + \zeta_0)A = 0 \quad \text{in } [0, T] \times \mathbb{T}, \\ \partial_t \rho_1 + \partial_x((u_0 + \zeta_0)\rho_1 + (u_1 + \zeta_1)\rho_0) = \partial_x A, \\ \partial_t u_1 + (u_0 + \xi_0)\partial_x(u_1 + \xi_1) + (u_1 + \xi_1)\partial_x(u_0 + \xi_0) + \partial_x \rho_1 = \eta_1, \\ \vec{\rho}(0) = (g_0, g_1), \quad A(0) = A_0 \quad \text{and} \quad \vec{u}(0) = (v_0, v_1), \end{cases}$$

where $\vec{\rho} = (\rho_0, \rho_1)$ and $\vec{u} = (u_0, u_1)$.

Let $U \stackrel{\text{def}}{=} (v_0, v_1, g_0, g_1, A_0, \xi_0, \xi_1, \zeta_0, \zeta_1, \eta_0, \eta_1)$, and

$$\begin{aligned} X^k(T) &\stackrel{\text{def}}{=} \{U \in H_0^k \times H_0^{k-2} \times H^k \times H^{k-2} \times H^{k-1} \times L_T^2(H_0^{k+1}) \times L_T^2(H_0^{k-1}) \\ &\quad \times L_T^2(H_0^{k+1}) \times L_T^2(H_0^{k-1}) \times L_T^2(H_0^k) \times L_T^2(H_0^{k-2}); g_0 > 0 \text{ in } \mathbb{T}\}; \\ Y^k(T) &\stackrel{\text{def}}{=} \{(u_0, u_1, \rho_0, \rho_1, A) \in C([0, T]; H_0^k) \times C([0, T]; H_0^{k-2}) \times C([0, T]; H^k) \\ &\quad \times C([0, T]; H^{k-2}) \times C([0, T]; H^{k-1}); \rho_0 > 0 \text{ in } [0, \tilde{T}] \times \mathbb{T}\}. \end{aligned}$$

Let us remark here that, concerning U , the natural convention should have been $(g_0, v_0, A_0, g_1, v_1, \xi_0, \xi_1, \zeta_0, \zeta_1, \eta_0, \eta_1)$. In this paper we choose a different order to simplify some notations for the presentation, for example we will denote (v_0, v_1) by \vec{v} . The same remark holds for the use of $(u_0, u_1, \rho_0, \rho_1, A)$.

The main results state as follows:

Theorem 3.1. *Let $k \geq 3$. Given $U \in X^k(T)$ the system (3.1) has a unique solution $(\vec{u}, \vec{\rho}, A)$ in $Y^k(T_0)$ for some $T_0 \in (0, T]$. Furthermore, if for some $U^1 \in X^k(T)$, the system (3.1) has a solution $(\vec{u}^1, \vec{\rho}^1, A) \in Y^k(T)$, then there exists δ and $C > 0$ which depend only on $\|U^1\|_{X^k(T)}$ and the uniform lower bound of ρ_0 such that*

(i) *if $U^2 \in X^k(T)$ satisfies*

$$(3.2) \quad \|U^1 - U^2\|_{X^k(T)} \leq \delta,$$

then (3.1) has a unique solution $(\vec{u}^2, \vec{\rho}^2, A^2) \in Y^k(T)$.

(ii) *We define $\mathcal{R}(U)$ to be the solution of (3.1). If U^2 satisfies (3.2), then*

$$\|\mathcal{R}(U^1) - \mathcal{R}(U^2)\|_{Y^{k-1}(T)} \leq C \|U^1 - U^2\|_{X^{k-1}(T)}.$$

(iii) *The operator $\mathcal{R} : X^k(T) \rightarrow Y^k(T)$ is continuous at U^1 .*

Theorem 3.2. *Let $k \geq 3$. If (ξ_0^n, ζ_0^n) and (ξ_1^n, ζ_1^n) are bounded sequences in $L_T^2(H^{k+3})$ and $L_T^2(H^{k+1})$ respectively, which satisfy, as $n \rightarrow +\infty$,*

$$\begin{aligned} \int_0^t \zeta_0^n(s) \chi_0^n(s) ds &\rightarrow 0 \text{ in } H^k, \\ \int_0^t \zeta_0^n(s) \tilde{\chi}_0^n(s) ds &\rightarrow 0 \text{ in } H^{k-1}, \\ \int_0^t \zeta_1^n(s) \chi_1^n(s) ds &\rightarrow 0 \text{ in } H^{k-2}, \end{aligned}$$

for any t and for any uniformly equicontinuous sequences $\chi^n = (\chi_0^n, \tilde{\chi}_0^n, \chi_1^n) : [0, T] \rightarrow H^k \times H^{k-1} \times H^{k-2}$. Let

$$U^n = (\vec{v}, \vec{g}, A_0, \vec{\xi}^n, 0, \vec{\eta}), \quad V^n = (\vec{v}, \vec{g}, A_0, \vec{\xi}^n, \vec{\zeta}^n, \vec{\eta}) \in X^{k+2}(T).$$

Suppose that corresponding to U^n , the equation (3.1) have solutions in $Y^{k+2}(T)$ being uniformly bounded in $Y^k(T)$. Then, for n sufficiently large the equation (3.1) corresponding to V^n have solutions which is also uniformly bounded in $Y^{k+2}(T)$ and, as $n \rightarrow +\infty$,

$$\mathcal{R}(U^n) - \mathcal{R}(V^n) \rightarrow 0 \text{ in } Y^k(T).$$

The same property holds if instead of U^n we assume the existence of solutions for V^n .

The rest part of this section is devoted to the proofs of the two preceding theorems.

Proof of Theorems 3.1 and 3.2. We split the proof of Theorems 3.1 and 3.2 into the following steps:

• **Step 1.** Given $(\zeta_0, \xi_0) \in L_T^2(H_0^{k+1})$ and $f \in L_T^2(H_0^k)$, we consider

$$(3.3) \quad \begin{cases} \partial_t \rho_0 + \partial_x((u_0 + \zeta_0)\rho_0) = 0, \\ \partial_t u_0 + (u_0 + \xi_0)\partial_x(u_0 + \xi_0) + \partial_x \rho_0 = f, \\ \rho_0(0) = g_0, u_0(0) = v_0. \end{cases}$$

Let $k \in \mathbb{N} \setminus \{0, 1\}$ and let $0 < \tilde{T} \leq T$,

$$Y_0^k(\tilde{T}) \stackrel{\text{def}}{=} \{(\rho_0, u_0) \in C([0, \tilde{T}]; H^k) \times C([0, \tilde{T}]; H^k); \rho_0 > 0 \text{ in } [0, \tilde{T}] \times \mathbb{T}\};$$

$$X_0^k(T) \stackrel{\text{def}}{=} \{U_0 \in H^k \times H^k \times L_T^2(H^{k+1}) \times L_T^2(H^{k+1}) \times L_T^2(H^k); g_0 > 0 \text{ in } \mathbb{T}\}.$$

Let $U_0 \stackrel{\text{def}}{=} (g_0, v_0, \zeta_0, \xi_0, f)$. We define the solution operator

$$\mathcal{R}_0 : X_0^k(T) \mapsto Y_0^k(\tilde{T}) \quad \text{so that} \quad \mathcal{R}_0(U_0) \stackrel{\text{def}}{=} (\rho_0, u_0).$$

The system (3.3) is a symmetric hyperbolic systems with symmetrizer $\begin{pmatrix} 1 & 0 \\ 0 & \rho_0 \end{pmatrix}$. Standard hyperbolic theory ([6]) ensures that, for every bounded set B of $X_0^k(T)$ such that, for some $c > 0$, $\rho_0 \geq c$ in \mathbb{T} for every $U_0 \in B$, there exists $\tilde{T} \in (0, T]$ such that the operator \mathcal{R}_0 is well-defined from B into $Y_0^k(\tilde{T})$. Furthermore, we have

Theorem 3.3 (Theorem 2.2 of [24]). *Let $k \geq 3$. Let $U_0^1 \in X_0^k(T)$. We suppose that the system (3.3) has a solution $(\rho_0^1, u_0^1) \in Y_0^k(T)$. Then there exists δ_0 and $C > 0$ which depend only on $\|U_0^1\|_{X_0^k}$ and the uniform lower bound of ρ_0^1 such that,*

(i) *if $U_0^2 \in X_0^k(T)$ satisfies*

$$\|U_0^1 - U_0^2\|_{X_0^k(T)} \leq \delta_0,$$

then (3.3) has a unique solution $(\rho_0^2, u_0^2) \in Y_0^k(T)$.

(ii)

$$\|\mathcal{R}_0(U_0^1) - \mathcal{R}_0(U_0^2)\|_{Y_0^{k-1}(T)} \leq C \|U_0^1 - U_0^2\|_{X_0^{k-1}(T)}.$$

(iii) *The operator $\mathcal{R}_0 : X_0^k(T) \rightarrow Y_0^k(T)$ is well defined and continuous on a neighborhood of U_0^1 .*

Another important property of \mathcal{R}_0 is the following oscillation type lemma.

Theorem 3.4 (Theorem 2.3 of [24]). *Let $k \geq 3$. Let ξ_0^n, ζ_0^n be bounded sequences in $L_T^2(H^{k+3})$ and ζ_0^n satisfy*

$$\int_0^t \zeta_0^n(s) \chi_0^n(s) ds \rightarrow 0 \text{ in } H^k \text{ as } n \rightarrow +\infty,$$

for any $t \in [0, T]$ and for any uniformly equicontinuous sequence $\chi_0^n : [0, T] \rightarrow H^k$. Let

$$U_0^n = (\rho_0, v_0, \xi_0^n, \zeta_0^n, \eta_0) \in X_0^{k+2}(T),$$

$$V_0^n = (\rho_0, v_0, \xi_0^n, 0, \eta_0) \in X_0^{k+2}(T).$$

Suppose that corresponding to U_0^n , the equation (3.3) have solutions in $Y_0^{k+2}(T)$ being uniformly bounded in $Y_0^k(T)$. Then, for n sufficiently large the equation (3.3) corresponding V_0^n have solutions in $Y_0^{k+2}(T)$, moreover, there holds

$$\mathcal{R}_0(U_0^n) - \mathcal{R}_0(V_0^n) \rightarrow 0 \text{ in } Y_0^k(T) \text{ as } n \rightarrow +\infty.$$

• **Step 2.** We investigate the linear transport equation

$$(3.4) \quad \begin{cases} \partial_t A + (u_A + \zeta_A) \partial_x A + 2\partial_x(u_A + \zeta_A)A = 0, \\ A(0) = A_0. \end{cases}$$

Let $U_A \stackrel{\text{def}}{=} (A_0, u_A, \zeta_A)$, and

$$X_A^k(T) \stackrel{\text{def}}{=} H^{k-1} \times L_T^2(H^k) \times L_T^2(H^k) \quad \text{and} \quad Y_A^k(T) = C([0, T]; H^{k-1}).$$

We define the solution operator

$$\mathcal{R}_A : X_A^k(T) \rightarrow Y_A^k(T) \quad \text{with} \quad \mathcal{R}_A(U_A) = A.$$

Theorem 3.5. *Let $l \geq 3$. For any $U_A \in X_A^l(T)$, (3.4) has a unique solution A in $Y_A^l(T)$. Moreover, for every bounded subset B of $X_A^l(T)$ there exists $C > 0$ so that*

$$\|\mathcal{R}_A(U_A^1) - \mathcal{R}_A(U_A^2)\|_{Y_A^{l-1}(T)} \leq C \|U_A^1 - U_A^2\|_{X_A^{l-1}(T)}, \quad \forall U_A^1 \in B, \forall U_A^2 \in B.$$

Furthermore, let $k \geq 3$, let ζ_A^n be a bounded sequences in $L_T^2(H^{k+2})$ such that

$$(3.5) \quad \int_0^t \zeta_A^n(s) \chi_A^n(s) ds \rightarrow 0 \text{ in } H^{k-1} \text{ as } n \rightarrow +\infty,$$

for any $t \in [0, T]$ and for any uniformly equicontinuous sequence $\chi_A^n : [0, T] \rightarrow H^{k-1}$. Let $U_A^n = (A_0, u_A, \zeta_A^n) \in X_A^{k+2}(T)$, we denote $V_A = (A_0, u_A, 0)$. Then, we have

$$(3.6) \quad \mathcal{R}_A(U_A^n) - \mathcal{R}_A(V_A) \rightarrow 0 \text{ in } Y_A^k(T).$$

Proof of Theorem 3.5. The first part of Theorem 3.5 follows from the explicit value of \mathcal{R}_A which can be obtained by using the characteristics method.

Let us present the proof of (3.6). Let us denote $A^n \stackrel{\text{def}}{=} \mathcal{R}_A(U_A^n)$, $A \stackrel{\text{def}}{=} \mathcal{R}_A(V_A)$ and $\delta_A^n \stackrel{\text{def}}{=} A^n - A$. Since ζ_A^n is a bounded sequences in $L_T^2(H^{k+2})$, and U_A^n is uniformly bounded in $X_A^{k+2}(T)$, we deduce from standard theory of transport equation that (3.4) has a unique solution A^n which is uniformly bounded in $C([0, T]; H^{k+1})$. The equation (3.4) implies that $\partial_t A^n$ which is uniformly bounded in $C([0, T]; H^k)$. This together with Aubin-Lions Lemma ensures that

$$(3.7) \quad A^n \text{ is equicontinuous in } C([0, T]; H^s) \quad \forall s < k + 1.$$

We know that there exists $f_A \in C([0, T]; H^k)$ such that as $n \rightarrow +\infty$,

$$\delta_A^n \rightarrow f_A \text{ uniformly in } C([0, T]; H^k).$$

Then one has

$$\partial_t \delta_A^n + u_A \partial_x \delta_A^n + 2\partial_x u_A \delta_A^n + \zeta_A^n \partial_x A^n + 2\partial_x \zeta_A^n A^n = 0.$$

By performing H^{k-1} energy estimate to the above equation, we obtain

$$(3.8) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\delta_A^n(t)\|_{H^{k-1}}^2 &= - (u_A \partial_x \delta_A^n, \delta_A^n)_{H^{k-1}} - 2 (\partial_x u_A \delta_A^n, \delta_A^n)_{H^{k-1}} \\ &\quad - (\zeta_A^n \partial_x A^n, \delta_A^n)_{H^{k-1}} - 2 (\partial_x \zeta_A^n A^n, \delta_A^n)_{H^{k-1}}. \end{aligned}$$

By applying Moser type inequality, we find

$$\begin{aligned}
|(u_A \partial_x \delta_A^n, \delta_A^n)_{H^{k-1}}| &\leq \sum_{\ell \leq k-1} \left(|(u_A \partial_x^{\ell+1} \delta_A^n, \partial_x^\ell \delta_A^n)_{L^2}| \right. \\
&\quad \left. + |(\partial_x^\ell (u_A \partial_x \delta_A^n) - u_A \partial_x^{\ell+1} \delta_A^n, \partial_x^\ell \delta_A^n)_{L^2}| \right) \\
&\leq C (\|\partial_x u_A\|_{L^\infty} \|\delta_A^n\|_{H^{k-1}} + \|\partial_x \delta_A^n\|_{L^\infty} \|u_A\|_{H^{k-1}}) \|\delta_A^n\|_{H^{k-1}} \\
&\leq C \|u_A\|_{H^{k-1}} \|\delta_A^n\|_{H^{k-1}}^2,
\end{aligned}$$

where in the last inequality, we used an integration by parts the assumption that $k \geq 3$ so that $H^{k-2} \hookrightarrow L^\infty$.

Since H^{k-1} is an algebra (note that $k \geq 2$), we have

$$|(\partial_x u_A \delta_A^n, \delta_A^n)_{H^{k-1}}| \leq C \|u_A\|_{H^k} \|\delta_A^n\|_{H^{k-1}}^2.$$

Finally, by using integration by parts, we have

$$\begin{aligned}
& - (\zeta_A^n \partial_x A^n, \delta_A^n)_{H^{k-1}} - 2 (\partial_x \zeta_A^n A^n, \delta_A^n)_{H^{k-1}} \\
&= - (\zeta_A^n \partial_x A^n, \delta_A^n)_{H^{k-1}} + 2 (\zeta_A^n A^n, \partial_x \delta_A^n)_{H^{k-1}}.
\end{aligned}$$

We focus on the estimate of $(\zeta_A^n \partial_x A^n, \delta_A^n)_{H^{k-1}}$ with the other one being similar. Since $\zeta_A^n \partial_x A^n$ is uniformly bounded in $L_T^2(H^{k-1})$ and that δ_A^n converges uniformly to f_A in $C([0, T]; H^k)$ sense, we have that for any $t \in [0, T]$, as $n \rightarrow +\infty$,

$$(3.9) \quad \int_0^t (\zeta_A^n \partial_x A^n, \delta_A^n)_{H^{k-1}} dt - \int_0^t (\zeta_A^n \partial_x A^n, f_A)_{H^{k-1}} dt \rightarrow 0.$$

For any $\delta > 0$ we can find a step function f_m in H^k such that $\|f_m(s) - f_A(s)\|_{H^k} \leq \delta$, $\forall s \in [0, T]$. Hence, as $n \rightarrow +\infty$,

$$(3.10) \quad \left| \int_0^t (\zeta_A^n \partial_x A^n, f_A)_{H^{k-1}} dt - \int_0^t (\zeta_A^n \partial_x A^n, f_m)_{H^{k-1}} dt \right| \leq C\delta$$

with C independent of δ and n . As $f_m(t)$ is a piecewise constant function in H^k , for instance assume that the discontinuous points are $\{t_1, t_2, \dots, t_M\}$, we can immediately conclude from the assumption, Equation (3.5), that

$$\int_0^{t_1} (\zeta_A^n \partial_x A^n, f_m)_{H^{k-1}} dt = \left(\int_0^{t_1} \zeta_A^n \partial_x A^n dt, f_m(0) \right)_{H^{k-1}} \rightarrow 0,$$

as $n \rightarrow +\infty$, which further implies that when n tends to $+\infty$,

$$\int_0^t (\zeta_A^n \partial_x A^n, f_m)_{H^{k-1}} dt \rightarrow 0,$$

thus

$$(3.11) \quad \int_0^t (\zeta_A^n \partial_x A^n, \delta_A^n)_{H^{k-1}} dt \rightarrow 0.$$

Therefore,

$$(3.12) \quad \frac{1}{2} \frac{d}{dt} \|\delta_A^n(t)\|_{H^{k-1}}^2 \leq C \|u_A\|_{H^k} \|\delta_A^n\|_{H^{k-1}}^2 + h^n(t),$$

with $h^n(t) \geq 0$ satisfying

$$(3.13) \quad \int_0^T h^n(s) ds \rightarrow 0.$$

Now let us present the following lemma,

Lemma 3.1. *Let $0 \leq f, g \in C[0, T]$ so that*

$$(3.14) \quad \frac{d}{dt}f(t) \leq g(t)f(t) + h(t) \quad \forall t \in [0, T].$$

Then one has

$$(3.15) \quad f(t) \leq f(0)e^{\int_0^t g(t') dt'} + \int_0^t h(t') dt' + \int_0^t g(t')e^{\int_{t'}^t g(s) ds} \int_0^{t'} h(s) ds dt'.$$

By inserting equations (3.12)–(3.13) into Lemma 3.1 we get (3.6). \square

Proof of Lemma 3.1. We first get, by integrating (3.14) over $[0, t]$, that

$$(3.16) \quad f(t) \leq f(0) + \int_0^t g(t')f(t') dt' + \int_0^t h(t') dt',$$

from which, we infer

$$\frac{d}{dt} \int_0^t g(t')f(t') dt' = g(t)f(t) \leq g(t) \left(f(0) + \int_0^t g(t')f(t') dt' + \int_0^t h(t') dt' \right),$$

so that there holds

$$\frac{d}{dt} \left(e^{-\int_0^t g(t') dt'} \int_0^t g(t')f(t') dt' \right) \leq g(t)e^{-\int_0^t g(t') dt'} \left(f(0) + \int_0^t h(t') dt' \right).$$

Integrating the above inequality over $[0, t]$ gives rise to

$$\int_0^t g(t')f(t') dt' \leq \int_0^t g(t')e^{\int_{t'}^t g(s) ds} \left(f(0) + \int_0^{t'} h(s) ds \right) dt'$$

Inserting the above inequality into (3.16) and using the fact that

$$\int_0^t g(t')e^{\int_{t'}^t g(s) ds} dt' = e^{\int_0^t g(t') dt'} - 1$$

leads to (3.15). \square

• **Step 3.** We investigate the linear hyperbolic system:

$$(3.17) \quad \begin{cases} \partial_t \rho_1 + \partial_x \left((u_0 + \zeta_0)\rho_1 + (u_1 + \zeta_1)\rho_0 \right) = f, \\ \partial_t u_1 + (u_0 + \xi_0)\partial_x (u_1 + \xi_1) + (u_1 + \xi_1)\partial_x (u_0 + \xi_0) + \partial_x \rho_1 = \eta_1, \\ \rho_1(0) = g_1, u_1(0) = v_1. \end{cases}$$

Let us denote

$$U_1 \stackrel{\text{def}}{=} (g_1, v_1, \xi_1, \zeta_1, f, \eta_1, \rho_0, u_0, \xi_0, \zeta_0) \quad \text{and} \quad \mathcal{R}_1(U_1) \stackrel{\text{def}}{=} (\rho_1, u_1),$$

and

$$X_1^k(T) \stackrel{\text{def}}{=} \{U_1 \in H^{k-2} \times H_0^{k-2} \times L_T^2(H_0^{k-1}) \times L_T^2(H_0^{k-1}) \times L_T^2(H_0^{k-2}) \times L_T^2(H_0^{k-2}) \\ \times L_T^2(H^{k-1}) \times L_T^2(H_0^{k-1}) \times L_T^2(H_0^{k-1}) \times L_T^2(H_0^{k-1}); \rho_0 > 0 \text{ in } \mathbb{T}\},$$

$$Y_1^k(T) \stackrel{\text{def}}{=} C([0, T]; H^{k-2}) \times C([0, T]; H_0^{k-2}).$$

Due to (3.17) is a linear symmetric hyperbolic system, similar well-posedness theorem as Theorem 3.3 holds for $\mathcal{R}_1(U_1)$. Moreover, there holds

Theorem 3.6. *Let ζ_0^n, ζ_1^n be bounded sequences in $L_T^2(H_0^{k+1})$ and ζ_0^n, ζ_1^n satisfy*

$$\int_0^t \zeta_0^n(s) \chi_0^n(s) ds \quad \text{and} \quad \int_0^t \zeta_1^n(s) \chi_0^n(s) ds \rightarrow 0 \text{ in } H^{k-2}$$

for any $t \in [0, T]$ and for any uniformly equicontinuous sequence $\chi_0^n : [0, T] \rightarrow H^{k-2}$. Let

$$U_1^n = (g_1, v_1, \xi_1, 0, f, \eta_1, \rho_0, u_0, \xi_0, 0) \in X_1^{k+2}(T),$$

$$V_1^n = (g_1, v_1, \xi_1, \zeta_1^n, f, \eta_1, \rho_0, u_0, \xi_0, \zeta_0^n) \in X_1^{k+2}(T).$$

Suppose that corresponding to U_1^n , the equation (3.17) have solutions in $Y_1^{k+2}(T)$ being uniformly bounded in $Y_1^k(T)$. Then, for n sufficiently large the equation (3.17) corresponding V_1^n have solutions in $Y_1^{k+2}(T)$, moreover, there holds Then there holds

$$\mathcal{R}_1(U_1^n) - \mathcal{R}_1(V_1^n) \rightarrow 0 \text{ in } Y_1^k(T) \text{ as } n \rightarrow +\infty.$$

By summarizing the above steps, we conclude the proof of Theorems 3.1 and 3.2. \square

4. THE CONTROL OF THE LIMIT SYSTEMS

The goal of this section is to present the proof of Theorem 2.1. Before going into the detailed proof, here we heuristically outline the main ideas.

- 1) In Section 4.1, we construct a sequence of finite dimensional spaces

$$E_0 \subset E_1 \subset E_2 \subset \dots \subset E_n \dots$$

whose union is dense in $H^s(\mathbb{T})$.

- 2) Thanks to the preceding step, for $\forall \varepsilon > 0$, for any given initial data $(\vec{v}, \vec{g}) \in H_0^{k+2} \times H^k \times H_0^{k+2} \times H^k$, $A_0 \in H^{k+1}$, and final state $(\widehat{\vec{v}}, \widehat{\vec{g}}) \in H_0^{k+2} \times H^k \times H_0^{k+2} \times H^k$ satisfying the conservation of mass, there exists E_N -valued control $\vec{\eta}^{(N)}$ such that the solution

$$(\vec{u}^{(N)}, \vec{\rho}^{(N)}, A^{(N)}) \stackrel{\text{def}}{=} \mathcal{R}(\vec{v}, \vec{g}, A_0, 0, 0, \vec{\eta}^{(N)}) \in Y^{k+2}(T),$$

verifies that in the $H_0^k \times H^{k-2} \times H_0^k \times H^{k-2}$ space

$$\|(\vec{u}^{(N)}(T), \vec{\rho}^{(N)}(T)) - (\widehat{\vec{v}}, \widehat{\vec{g}})\| < \varepsilon.$$

Section 4.2, more precisely, Theorem 4.1, is devoted to this step.

- 3) Next, as the most important step of the proof, in Section 4.3 we show that there exists E_{N-1} -valued control $\vec{\eta}^{(N-1)}$ such that the solution

$$(\vec{u}^{(N-1)}, \vec{\rho}^{(N-1)}, A^{(N-1)}) \stackrel{\text{def}}{=} \mathcal{R}(\vec{v}, \vec{g}, A_0, 0, 0, \vec{\eta}^{(N-1)}) \in Y^{k+2}(T),$$

$$\|(\vec{u}^{(N)}(T), \vec{\rho}^{(N)}(T)) - (\vec{u}^{(N-1)}(T), \vec{\rho}^{(N-1)}(T))\| < \varepsilon,$$

$$\|\vec{\rho}^{(N)} - \vec{\rho}^{(N-1)}\|_{C([0, T]; H^k \times H^{k-2})} < \varepsilon,$$

then we can further find E_{N-2} -valued control $\vec{\eta}^{(N-2)}$... E_0 -valued control $\vec{\eta}^{(0)}$ to approximate the first constructed trajectory $(\vec{u}^{(N)}, \vec{\rho}^{(N)}, A^{(N)})$. In the terminology of Agrachev–Sarychev method, this procedure is called “extension” or “saturation”, see for instance [1, Section 5.2].

More precisely, this is again divided into several steps. At first, in Section 4.3.1–4.3.3, we find $\vec{\eta}, \vec{\mu}_n \in E^{N-1}$ such that

$$(\vec{v}, \vec{g}, A_0, \vec{\mu}_n, 0, \vec{\eta}) \text{ is uniformly bounded in } X^{k+2}(T),$$

$\mathcal{R}(\vec{v}, \vec{g}, A_0, \vec{\mu}_n, 0, \vec{\eta})$ is uniformly bounded in $Y^{k+2}(T)$,

$$\|\mathcal{R}(\vec{v}, \vec{g}, A_0, \vec{\mu}_n, 0, \vec{\eta}) - \mathcal{R}(\vec{v}, \vec{g}, A_0, 0, 0, \vec{\eta}^{(N)})\|_{Y^k(T)} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Then in Section 4.3.4 we show that thanks to Theorem 3.2,

$\mathcal{R}(\vec{v}, \vec{g}, A_0, \vec{\mu}_n, \vec{\mu}_n, \vec{\eta})$ is uniformly bounded in $Y^{k+2}(T)$,

$$\|\mathcal{R}(\vec{v}, \vec{g}, A_0, \vec{\mu}_n, \vec{\mu}_n, \vec{\eta}) - \mathcal{R}(\vec{v}, \vec{g}, A_0, \vec{\mu}_n, \vec{\mu}_n, \vec{\eta})\|_{Y^k(T)} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Next, in Section 4.3.5 we approximate $\vec{\mu}_n$ by smooth (in time) functions $\vec{\mu}_n^m$. Finally, in Section 4.3.6 we construct the required $\vec{\eta}^{(N-1)}$.

- 4) Let us emphasize here that during the whole process $\vec{\rho}$ is close to $\vec{\rho}^{(N)}$ in H^k , thus in $C(\mathbb{T})$ and being uniformly away from 0, while \vec{u} is not necessarily always close to $\vec{u}^{(N)}$ though $\vec{u}(T)$ is close to $\vec{u}^{(N)}(T)$ in H^k sense.

4.1. Introduce the controlling space. We define

$$(4.1) \quad \begin{aligned} E_0 &\stackrel{\text{def}}{=} \text{span}\{\sin x, \cos x\}, \\ E_n &\stackrel{\text{def}}{=} \text{span}\{\sin x, \cos x, \sin 2x, \cos 2x, \dots, \sin(n+1)x, \cos(n+1)x\}, \\ \mathbf{E}_n &\stackrel{\text{def}}{=} E_n \times E_n. \end{aligned}$$

It is obvious that $E_n \subset C^\infty(\mathbb{T})$ and

Lemma 4.1. $\mathbf{E}_\infty \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} \mathbf{E}_k$ is dense in $H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$, $\forall s > 0$.

Moreover, one has the following ‘‘Lie bracket’’ type argument.

Lemma 4.2. Let $n \in \mathbb{N}$. For any $\psi \in E_{n+1}$, there exists $\phi, \varphi^1, \dots, \varphi^p \in E_n$ such that

$$(4.2) \quad \psi = \phi - \sum_{i=1}^p \varphi^i \partial_x \varphi^i.$$

Proof. Notice that $E_{n+1} = E_n \oplus \text{span}\{\pm \sin(n+2)x, \pm \cos(n+2)x\}$. It suffices to prove (4.2) with $\psi = \pm \sin(n+2)x$ or $\pm \cos(n+2)x$. In the case when $n+2 = 2m$, we have

$$\begin{aligned} +\sin 2mx &= -\frac{2}{m} \cos mx \partial_x \cos mx, \\ -\sin 2mx &= -\frac{2}{m} \sin mx \partial_x \sin mx, \\ +\cos 2mx &= -\frac{1}{m} (\sin mx - \cos mx) \partial_x (\sin mx - \cos mx), \\ -\cos 2mx &= -\frac{1}{m} (\sin mx + \cos mx) \partial_x (\sin mx + \cos mx). \end{aligned}$$

While when $n+2 = 2m+1$, we observe that

$$\begin{aligned} &(\sin(m+1)x \pm \sin mx) \partial_x (\sin(m+1)x \pm \sin mx) \\ &= \frac{m+1}{2} \sin(2m+2)x + \frac{m}{2} \sin(2m)x \pm (m+1) \sin mx \cos(m+1)x \\ &\quad \pm m \sin(m+1)x \cos mx \\ &= \frac{m+1}{2} \sin(2m+2)x + \frac{m}{2} \sin(2m)x \pm \frac{2m+1}{2} \sin(2m+1)x \mp \frac{1}{2} \sin x, \end{aligned}$$

which gives

$$\begin{aligned}
-\frac{2m+1}{2} \sin(2m+1)x &= -(\sin(m+1)x + \sin mx) \partial_x (\sin(m+1)x + \sin mx) \\
&\quad + \frac{m+1}{2} \sin(2m+2)x + \frac{m}{2} \sin 2mx - \frac{1}{2} \sin x \\
&= -(\sin(m+1)x + \sin mx) \partial_x (\sin(m+1)x + \sin mx) \\
&\quad - \cos(m+1)x \partial_x \cos(m+1)x + \left(\frac{m}{2} \sin 2mx - \frac{1}{2} \sin x \right),
\end{aligned}$$

and

$$\begin{aligned}
+\frac{2m+1}{2} \sin(2m+1)x &= -(\sin(m+1)x - \sin mx) \partial_x (\sin(m+1)x - \sin mx) \\
&\quad + \frac{m+1}{2} \sin(2m+2)x + \frac{m}{2} \sin 2mx + \frac{1}{2} \sin x \\
&= -(\sin(m+1)x - \sin mx) \partial_x (\sin(m+1)x - \sin mx) \\
&\quad - \cos(m+1)x \partial_x \cos(m+1)x + \left(\frac{m}{2} \sin 2mx + \frac{1}{2} \sin x \right).
\end{aligned}$$

Thanks to the fact that $\sin(m+1)x + \sin mx$, $\cos(m+1)x$, $\frac{m}{2} \sin 2mx - \frac{1}{2} \sin x \in E_n$, we obtain (4.2) for $\psi = \pm \sin(2m+1)x$.

Along the same line, we deduce from the fact

$$\begin{aligned}
&(\sin(m+1)x \pm \cos mx) \partial_x (\sin(m+1)x \pm \cos mx) \\
&= \frac{m+1}{2} \sin(2m+2)x - \frac{m}{2} \sin(2m)x \pm (m+1) \cos mx \cos(m+1)x \\
&\quad \mp m \sin(m+1)x \sin mx \\
&= \frac{m+1}{2} \sin(2m+2)x - \frac{m}{2} \sin(2m)x \pm \frac{2m+1}{2} \cos(2m+1)x \pm \frac{1}{2} \cos x,
\end{aligned}$$

that

$$\begin{aligned}
-\frac{2m+1}{2} \cos(2m+1)x &= -(\sin(m+1)x + \cos mx) \partial_x (\sin(m+1)x + \cos mx) \\
&\quad + \frac{m+1}{2} \sin(2m+2)x - \frac{m}{2} \sin 2mx + \frac{1}{2} \cos x \\
&= -(\sin(m+1)x + \cos mx) \partial_x (\sin(m+1)x + \cos mx) \\
&\quad - \cos(m+1)x \partial_x \cos(m+1)x + \left(-\frac{m}{2} \sin 2mx + \frac{1}{2} \cos x \right),
\end{aligned}$$

and

$$\begin{aligned}
+\frac{2m+1}{2} \cos(2m+1)x &= -(\sin(m+1)x - \cos mx) \partial_x (\sin(m+1)x - \cos mx) \\
&\quad + \frac{m+1}{2} \sin(2m+2)x - \frac{m}{2} \sin 2mx - \frac{1}{2} \cos x \\
&= -(\sin(m+1)x - \cos mx) \partial_x (\sin(m+1)x - \cos mx) \\
&\quad - \cos(m+1)x \partial_x \cos(m+1)x + \left(-\frac{m}{2} \sin 2mx - \frac{1}{2} \cos x \right).
\end{aligned}$$

This shows that (4.2) holds for $\psi = \pm \cos(2m+1)x$. This ends the proof of the lemma. \square

4.2. Controllability of (2.11) with (\mathbf{E}_N) -valued controls. The main result states as follows:

Theorem 4.1. *Let $T > 0$, $\vec{v}, \widehat{v} \in H_0^{k+1} \times H_0^{k-1}$, $\vec{g}, \widehat{g} \in H^{k+1} \times H^{k-1}$ and $A_0 \in H^k$, which satisfy*

$$(4.3) \quad \int_{\mathbb{T}} g_0 dx = \int_{\mathbb{T}} \widehat{g}_0 dx = \alpha_0, \quad \int_{\mathbb{T}} g_1 dx = \int_{\mathbb{T}} \widehat{g}_1 dx = \alpha_1 \quad \text{and} \quad g_0, \widehat{g}_0 \geq c_0 > 0.$$

Then for any $\varepsilon > 0$, there exists an some $N \in \mathbb{N}^$, some E_N -valued control $\vec{\eta}^{(N)}$, and some $\hat{A}(T) \in H^k$ such that*

$$(4.4) \quad \left\| (\widehat{v}, \widehat{g}, \hat{A}(T)) - \mathcal{R}_T(\vec{v}, \vec{g}, A_0, 0, 0, \vec{\eta}^{(N)}) \right\|_{Z^k} \leq \varepsilon,$$

where $Z^k \stackrel{\text{def}}{=} H_0^k \times H_0^{k-2} \times H^k \times H^{k-2} \times H^{k-1}$.

We remark here that in this theorem we are dealing with the case that initial and final values belong to $X^{k+1}(T)$ to get approximation in Z^k sense. Eventually, in order to prove Theorem 2.1 it suffices to let the initial and final states in the more regular space $X^{k+2}(T)$.

In what follows, we always denote this solution by

$$(\vec{u}^{(N)}, \vec{\rho}^{(N)}, A^{(N)}) \stackrel{\text{def}}{=} \mathcal{R}(\vec{v}, \vec{g}, A_0, 0, 0, \vec{\eta}^{(N)}).$$

Proof. Let us define

$$(4.5) \quad \vec{\rho} = \begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix} \stackrel{\text{def}}{=} T^{-1} \left(t\widehat{g} + (T-t)\vec{g} \right) \quad \text{and} \quad \vec{u} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \stackrel{\text{def}}{=} T^{-1} (t\widehat{v} + (T-t)\mathbf{v}).$$

Then due to (4.5), we have $\int_{\mathbb{T}} \partial_t \rho_0 dx = 0$ so that we can solve $\xi_0 \in C^\infty([0, T]; H_0^{k+1}(\mathbb{T}))$ from the equation

$$(4.6) \quad \partial_x(\rho_0 \xi_0) = -\partial_t \rho_0 - \partial_x(\rho_0 u_0).$$

Indeed, since $1/\rho_0$ is a solution of $\partial_x(\rho_0 f) = 0$, we are able to find a solution $\xi_0 \in H_0^{k+1}(\mathbb{T})$ of (4.6). With such ξ_0 , we can solve for $A \in C^\infty([0, T]; H^k(\mathbb{T}))$ via

$$\begin{cases} \partial_t A + (u_0 + \xi_0) \partial_x A + 2\partial_x(u_0 + \xi_0)A = 0 & \text{in } [0, T] \times \mathbb{T}, \\ A(0) = A_0. \end{cases}$$

Let us remark here that the function $\hat{A}(T)$ stated in the theorem is exactly the solution $A(T)$.

Again due to (4.5), we have $\int_{\mathbb{T}} \partial_t \rho_1 dx = 0$ so that we can solve $\xi_1 \in C^\infty([0, T]; H_0^{k-1}(\mathbb{T}))$ from the equation

$$\partial_x(\xi_1 \rho_0) = \partial_x A - (\partial_t \rho_1 + \partial_x((u_0 + \xi_0)\rho_1 + u_1 \rho_0)).$$

We now define

$$\eta_0 \stackrel{\text{def}}{=} \partial_t u_0 + (u_0 + \xi_0) \partial_x(u_0 + \xi_0) + \partial_x \rho_0 \in C^\infty([0, T]; H_0^k),$$

$$\eta_1 \stackrel{\text{def}}{=} \partial_t u_1 + (u_0 + \xi_0) \partial_x(u_1 + \xi_1) + (u_1 + \xi_1) \partial_x(u_0 + \xi_0) + \partial_x \rho_1 \in C^\infty([0, T]; H_0^{k-2}).$$

Let us take $\vec{\xi}^\delta = \begin{pmatrix} \xi_0^\delta \\ \xi_1^\delta \end{pmatrix} \in C^\infty([0, T]; H_0^{k+1} \times H_0^{k-1})$ so that $\vec{\xi}^\delta(0, x) = \vec{\xi}^\delta(T, x) = 0$ and

$$\lim_{\delta \rightarrow 0} \|\vec{\xi}^\delta - \vec{\xi}\|_{L_T^2(H^{k+1} \times H^{k-1})} = 0.$$

It is easy to observe that

$$\mathcal{R}_T(\vec{v}, \vec{g}, A_0, \vec{\xi}^\delta, \xi^\delta, \eta) = \mathcal{R}_T(\vec{v}, \vec{g}, A_0, 0, 0, \vec{\eta} - \partial_t \vec{\xi}^\delta).$$

Then in view of Theorem 3.1, we deduce that

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \left\| \left(\widehat{\vec{v}}, \widehat{\vec{g}}, A(T) \right) - \mathcal{R}_T(\vec{v}, \vec{g}, A_0, 0, 0, \vec{\eta} - \partial_t \vec{\xi}^\delta) \right\|_{Z^k} \\ &= \lim_{\delta \rightarrow 0} \left\| \mathcal{R}_T(\vec{v}, \vec{g}, A_0, \vec{\xi}, \vec{\xi}, \vec{\eta}) - \mathcal{R}_T(\vec{v}, \vec{g}, A_0, \vec{\xi}^\delta, \xi^\delta, \eta) \right\|_{Z^k} = 0. \end{aligned}$$

On the other hand, due to \mathbf{E}_∞ is dense in $H_0^k \times H_0^{k-2}$, we have

$$\lim_{N \rightarrow +\infty} \left\| \mathbb{P}_{\mathbf{E}_N}(\vec{\eta} - \partial_t \vec{\xi}^\delta) - (\vec{\eta} - \partial_t \vec{\xi}^\delta) \right\|_{L_T^2(H^k \times H^{k-2})} = 0,$$

which together with Theorem 3.1 ensures (4.4) with $\vec{\eta}^{(N)} \stackrel{\text{def}}{=} \mathbb{P}_{\mathbf{E}_N}(\vec{\eta} - \partial_t \vec{\xi}^\delta)$. This completes the proof of the theorem. \square

4.3. Reduction of dimension of control space. The aim of this section is to look for $\vec{\eta}^{(N-1)} \in \mathbf{E}_{N-1}$ so that

$$\left\| \left(\widehat{\vec{v}}, \widehat{\vec{g}}, \widehat{A}(T) \right) - \mathcal{R}_T(\vec{v}, \vec{g}, A_0, 0, 0, \vec{\eta}^{(N-1)}) \right\|_{Z^k} \leq \varepsilon.$$

Let us start by stating the following lemma which is also called ‘‘convexification’’ by Agrachev–Sarychev [1, Section 5.3]. The main novelty for this lemma is on the treatment of (ζ_0, ζ_1) involving both nonlinear terms $\xi_0^i \partial_x \xi_0^i$ and bilinear terms $\partial_x(\xi_0^i \xi_1^i)$.

Lemma 4.3. *Let $n \in \mathbb{N}$. For any $\vec{\zeta} = \begin{pmatrix} \zeta_0 \\ \zeta_1 \end{pmatrix} \in \mathbf{E}_{n+1}$, there exists $\vec{\eta}, \vec{\xi}^1, \dots, \vec{\xi}^m \in \mathbf{E}_n$ such that*

$$(4.7) \quad \zeta_0 = \eta_0 - \sum_{i=1}^m \xi_0^i \partial_x \xi_0^i \quad \text{and} \quad \zeta_1 = \eta_1 - \sum_{i=1}^m \partial_x(\xi_0^i \xi_1^i).$$

Proof. Since we can extend the length (m) of the sequence, we only need to prove the cases when $\vec{\eta} = (\vec{\eta}_0, 0)$ or $\vec{\eta} = (0, \vec{\eta}_1)$, Furthermore, we can assume that $\vec{\eta}_0, \vec{\eta}_1 = \pm \sin(n+2)x$ or $\pm \cos(n+2)x$.

The case when $\vec{\eta} = (\vec{\eta}_0, 0)$ is a direct consequence of Lemma 4.2, We can take $\eta_1, \xi_1^i = 0$.

Let us turn to the case when $\vec{\eta} = (0, \vec{\eta}_1)$ with $\vec{\eta}_1 = \pm \sin(n+2)x$ or $\pm \cos(n+2)x$. Indeed it follows from Lemma 4.2 that there exist of $\eta_{1+}, \eta_{1-}, \varphi_{1+}^i, \varphi_{1-}^i \in E_n$ such that

$$\begin{aligned} +\zeta_1 &= \eta_{1+} - \sum_{i=1}^p \varphi_{1+}^i \partial_x \varphi_{1+}^i, \\ -\zeta_1 &= \eta_{1-} - \sum_{i=1}^p \varphi_{1-}^i \partial_x \varphi_{1-}^i, \end{aligned}$$

which implies

$$\begin{aligned} 0 &= (\eta_{1+} + \eta_{1-}) - \sum_{i=1}^p \varphi_{1+}^i \partial_x \varphi_{1+}^i - \sum_{i=1}^p \varphi_{1-}^i \partial_x \varphi_{1-}^i, \\ \zeta_1 &= \eta_{1+} - \sum_{i=1}^p \partial_x(\varphi_{1+}^i \cdot \frac{1}{2} \varphi_{1+}^i) - \sum_{i=1}^p \partial_x(\varphi_{1-}^i \cdot 0). \end{aligned}$$

Then (4.7) is proved by taking

$$\begin{aligned} m &\stackrel{\text{def}}{=} 2p, \\ \eta_0 &\stackrel{\text{def}}{=} \eta_{1+} + \eta_{1-}, \end{aligned}$$

$$\begin{aligned}
\eta_1 &\stackrel{\text{def}}{=} \eta_{1+}, \\
\xi_0^i &\stackrel{\text{def}}{=} \varphi_{1+}^i, \text{ when } i \in \{1, \dots, p\}, \\
\xi_0^i &\stackrel{\text{def}}{=} \varphi_{1-}^{i-k}, \text{ when } i \in \{p+1, \dots, 2p\}, \\
\xi_1^i &\stackrel{\text{def}}{=} \frac{1}{2} \varphi_{1+}^i, \text{ when } i \in \{1, \dots, p\}, \\
\xi_1^i &\stackrel{\text{def}}{=} 0, \text{ when } i \in \{p+1, \dots, 2p\}.
\end{aligned}$$

This completes the proof of Lemma 4.3. \square

With the above key lemma, we are going to prove the following proposition. In fact this proposition together with Theorem 4.1 immediately lead to Theorem 2.1.

Proposition 4.1. *Let $T > 0$, $m \in \mathbb{N}$, and (\vec{v}, \vec{g}, A_0) belong to $H_0^{k+2} \times H_0^k \times H^{k+2} \times H^k \times H^{k+1}$ and $\vec{\eta}_1 \in C^\infty([0, T]; \mathbf{E}_{m+1})$. For any $\varepsilon > 0$, there exists $\vec{\eta} \in C^\infty([0, T]; \mathbf{E}_m)$ such that,*

$$(4.8) \quad \|\mathcal{R}_T(\vec{v}, \vec{g}, A_0, 0, 0, \vec{\eta}) - \mathcal{R}_T(\vec{v}, \vec{g}, A_0, 0, 0, \vec{\eta}_1)\|_{Z^k} \leq \varepsilon.$$

The rest part of this section is devoted to the proof of Proposition 4.1. Since $\vec{\eta}_1 \in C^\infty([0, T]; \mathbf{E}_1)$ can be approximated by simple functions (piecewise constant with respect to time),

$$\vec{\eta}_1^m \rightarrow \vec{\eta}_1 \text{ in } L_T^\infty(\mathbf{E}_1) \text{ as } m \rightarrow +\infty,$$

it suffices to consider simple functions. Furthermore, thanks to a simple iteration argument and the continuity of \mathcal{R} (Theorem 3.1 (iii)), it suffices to consider the case when $\vec{\eta}_1$ is independent of the time.

From now on we assume that $\vec{\eta}_1(t) \equiv \vec{\eta}_1 \in \mathbf{E}_1$. Let us denote

$$(4.9) \quad (\vec{u}, \vec{\rho}, A) \stackrel{\text{def}}{=} \mathcal{R}(\vec{v}, \vec{g}, A_0, 0, 0, \vec{\eta}_1) \in Y^{k+2}(T).$$

4.3.1. **Well-chosen ξ_i^j .** We want to use Lemma 4.3 in order to reduce the control of $\vec{\eta}_1 \in \mathbf{E}_1$ by the control of $\vec{\eta}_0 \in \mathbf{E}$.

• **Use of Lemma 4.3**

Thanks to Lemma 4.3, there exists $\vec{\eta}, \vec{\xi}^j \in \mathbf{E}, j \in \{1, \dots, m\}$ such that

$$(4.10) \quad (\eta_1)_0 = \bar{\eta}_0 - \frac{1}{m} \sum_{j=1}^m \xi_0^j \partial_x \xi_0^j,$$

$$(4.11) \quad (\eta_1)_1 = \bar{\eta}_1 - \frac{1}{m} \sum_{j=1}^m \partial_x (\xi_0^j \xi_1^j).$$

However, equation (4.10) **does not** mean that

$$u_0 \partial_x u_0 - (\eta_1)_0 = \frac{1}{m} \sum_{j=1}^m (u_0 + \xi_0^j) \partial_x (u_0 + \xi_0^j) - \bar{\eta}_0,$$

because of linear terms $\xi_0^j \partial_x u_0$ and $u_0 \partial_x \xi_0^j$.

• **Adding the adjoint terms**

The way to fix such a linear error is to add the opposite quadratic terms, which is motivated by a simple formula

$$(a+b)^2 + (a-b)^2 = 2(a^2 + b^2).$$

More precisely, let

$$(4.12) \quad \lambda \stackrel{\text{def}}{=} \frac{1}{2m} \text{ and } \xi^{j+m} \stackrel{\text{def}}{=} -\xi^j \text{ for } j = 1, 2, \dots, m.$$

Then it is easy to observe that

$$(4.13) \quad \begin{aligned} u_0 \partial_x u_0 - (\eta_1)_0 &= \sum_{j=1}^{2m} \lambda (u_0 + \xi_0^j) \partial_x (u_0 + \xi_0^j) - \bar{\eta}_0, \\ \partial_x (u_0 u_1) - (\eta_1)_1 &= \sum_{j=1}^{2m} \lambda \partial_x ((u_0 + \xi_0^j)(u_1 + \xi_1^j)) - \bar{\eta}_1^\ell. \end{aligned}$$

Hence, the following two systems in $[0, T] \times \mathbb{T}$ are equivalent:

$$\begin{cases} \partial_t \rho_0 + \partial_x (u_0 \rho_0) = 0, \\ \partial_t u_0 + u_0 \partial_x u_0 + \partial_x \rho_0 = (\eta_1)_0, \\ \partial_t A + u_0 \partial_x A + 2 \partial_x u_0 A = 0, \\ \partial_t \rho_1 + \partial_x (u_0 \rho_1 + u_1 \rho_0) = \partial_x A, \\ \partial_t u_1 + \partial_x (u_0 u_1) + \partial_x \rho_1 = (\eta_1)_1, \\ \bar{\rho}(0) = \bar{g}, \quad A(0) = A_0 \quad \text{and} \quad \bar{u}(0) = \bar{v}, \end{cases}$$

and

$$(4.14) \quad \begin{cases} \partial_t \rho_0 + \partial_x (u_0 \rho_0) = 0, \\ \partial_t u_0 + \sum_{j=1}^{2n} \lambda (u_0 + \xi_0^j) \partial_x (u_0 + \xi_0^j) + \partial_x \rho_0 = \bar{\eta}_0, \\ \partial_t A + u_0 \partial_x A + 2 \partial_x u_0 A = 0, \\ \partial_t \rho_1 + \partial_x (u_0 \rho_1 + u_1 \rho_0) = \partial_x A, \\ \partial_t u_1 + \sum_{j=1}^{2n} \lambda \partial_x ((u_0 + \xi_0^j)(u_1 + \xi_1^j)) + \partial_x \rho_1 = \bar{\eta}_1, \\ \bar{\rho}(0) = \bar{g}, \quad A(0) = A_0 \quad \text{and} \quad \bar{u}(0) = \bar{v}. \end{cases}$$

The preceding equivalent

4.3.2. From stationary sequence $\{\xi^j\}_{j=1}^{2m}$ to curves $\{\mu_n(t)\}_n$. Let us define a periodic function $\mu_i(t)$ with period 1 as

$$(4.15) \quad \begin{aligned} \mu_i(t) &\stackrel{\text{def}}{=} \xi_i^j \quad \text{if } t \in \left[\frac{j-1}{2m}, \frac{j}{2m} \right) \\ \mu_i(t) &\stackrel{\text{def}}{=} -\mu_i \left(t - \frac{1}{2} \right) \quad \text{if } t \in (1/2, 1] \end{aligned}$$

for $i = 0, 1$, and for $j \in \{1, \dots, m\}$. Notice that

$$(4.16) \quad \sum_{j=1}^{2m} \xi_i^j = 0 \text{ for } i = 0, 1,$$

$$(4.17) \quad \int_0^1 \mu_i(t) dt = 0 \text{ for } i = 0, 1.$$

For any $n \in \mathbb{N}^*$, we also define a periodic function with period T/n as

$$(4.18) \quad \mu_{i,n}(t) \stackrel{\text{def}}{=} \mu_i \left(\frac{nt}{T} \right) \in L_T^\infty(E) \text{ for } i = 0, 1,$$

which is uniformly bounded in $L_T^\infty(E)$.

Then we can rewrite Equation (4.14) in $[0, T] \times \mathbb{T}$ as

$$(4.19) \quad \begin{cases} \partial_t \rho_0 + \partial_x(u_0 \rho_0) = 0, \\ \partial_t u_0 + (u_0 + \mu_{0,n}) \partial_x(u_0 + \mu_{0,n}) + \partial_x \rho_0 = \bar{\eta}_0 + f_{0,n}, \\ \partial_t A + u_0 \partial_x A + 2 \partial_x u_0 A = 0 \\ \partial_t \rho_1 + \partial_x(u_0 \rho_1 + u_1 \rho_0) = \partial_x A, \\ \partial_t u_1 + \partial_x((u_0 + \mu_{0,n})(u_1 + \mu_{1,n})) + \partial_x \rho_1 = \bar{\eta}_1 + f_{1,n}, \\ \vec{\rho}(0) = \vec{g}, \quad A(0) = A_0 \quad \text{and} \quad \vec{u}(0) = \vec{v}, \end{cases}$$

where

$$f_{0,n}(t) \stackrel{\text{def}}{=} \frac{1}{2} \partial_x (u_0 + \mu_{0,n}(t))^2 - \frac{1}{2} \sum_{j=1}^{2m} \lambda \partial_x (u_0 + \xi_0^j)^2,$$

$$f_{1,n}(t) \stackrel{\text{def}}{=} \partial_x \left((u_0 + \mu_{0,n}(t))(u_1 + \mu_{1,n}(t)) \right) - \sum_{j=1}^{2m} \lambda \partial_x \left((u_0 + \xi_0^j)(u_1 + \xi_1^j) \right).$$

Therefore,

$$(4.20) \quad (\vec{u}, \vec{\rho}, A) = \mathcal{R}(\vec{v}, \vec{g}, A_0, \vec{\mu}_n, 0, \vec{\eta}_0 + \vec{f}_n).$$

Because $\vec{u}_0, \vec{\mu}_n, \vec{\xi}^j$ are uniformly bounded in $L_T^\infty(H_0^{k+2} \times H_0^k)$, we know that \vec{f}_n are uniformly bounded in $L_T^\infty(H_0^{k+1} \times H_0^{k-1})$.

4.3.3. Removing the extra source term \vec{f}_n . We can see from (4.19) and (4.20) that the “extra” source term \vec{f}_n does not belong to \mathbf{E} . In order to remove \vec{f}_n , a natural idea is to regard it as a perturbation term, and to use the continuity of the mapping \mathcal{R} . However, we only know the uniform boundedness of \vec{f}_n in $L_T^\infty(H_0^{k+1} \times H_0^{k-1})$, which is not assumed to be small. In such a case we prove that with the help of a lemma f_n can be removed. More precisely, we have the following lemma concerning “relaxation metric” (name according to [1, Section 4.1]).

In the next lemma and in the following, for $\psi \in L^1((0, T); \mathbb{R}^l)$, $l \in \mathbb{N} \setminus \{0\}$, we define $\mathcal{K}\psi \in W^{1,1}((0, T); \mathbb{R}^l)$ by

$$(4.21) \quad \mathcal{K}\psi(t) \stackrel{\text{def}}{=} \int_0^t \psi(t') dt'.$$

Lemma 4.4. *One has*

$$(4.22) \quad \lim_{n \rightarrow \infty} \left\| \mathcal{K} \vec{f}_n \right\|_{C([0, T]; H_0^{k+1} \times H_0^{k-1})} = 0.$$

Proof. We know that $\vec{u} \in H_0^{k+2} \times H_0^k$, $\vec{\mu}_n \in L_T^\infty(\mathbf{E})$, $\vec{\xi}^k \in \mathbf{E}$.

• **Step 1.** We only prove the above limit for $f_{0,n}$, as the same proof holds for $f_{1,n}$. More generally, we prove the following lemma.

Lemma 4.5. *Let ξ_0^j be given. Let $\mu_{0,n}$ be generated by ξ_0^j following (4.15). For any $u_0(t) \in L_T^2(H_0^{k+2})$, we define*

$$\tilde{f}_{0,n}(u_0) \stackrel{\text{def}}{=} \frac{1}{2} \partial_x (u_0(t) + \mu_{0,n}(t))^2 - \frac{1}{2} \sum_{j=1}^{2m} \lambda \partial_x (u_0(t) + \xi_0^j)^2.$$

Then, for any given $u_0(t) \in L_T^2(H_0^{k+2})$, we have

$$(4.23) \quad \lim_{n \rightarrow +\infty} \left\| \mathcal{K} \tilde{f}_{0,n}(u_0) \right\|_{C([0,T]; H^{k+1} \times H^{k-1})} = 0.$$

We notice that, by recalling (4.12), (4.15) and (4.16),

$$\begin{aligned} & \|\mathcal{K} \tilde{f}_{0,n}(u) - \mathcal{K} \tilde{f}_{0,n}(v)\|_{C([0,T]; H^{k+1})} \\ & \leq \|\tilde{f}_{0,n}(u) - \tilde{f}_{0,n}(v)\|_{L_T^1(H^{k+1})} \\ & \leq \frac{1}{2} \|(u(t) + \mu_{0,n}(t))^2 - (v(t) + \mu_{0,n}(t))^2\|_{L_T^1(H^{k+2})} \\ & \quad + \frac{1}{4m} \left\| \sum_{j=1}^{2m} (u(t) + \xi_0^j)^2 - (v(t) + \xi_0^j)^2 \right\|_{L_T^1(H^{k+2})} \\ & = \frac{1}{2} \|(u(t) + \mu_{0,n}(t) + v(t) + \mu_{0,n}(t))(u(t) - v(t))\|_{L_T^1(H^{k+2})} \\ & \quad + \frac{1}{2} \|(u(t) + v(t))(u(t) - v(t))\|_{L_T^1(H^{k+2})} \\ & \leq (\|u\|_{L_T^2(H^{k+2})} + \|v\|_{L_T^2(H^{k+2})} + \|\mu_{0,n}\|_{L_T^2(H^{k+2})}) \|u - v\|_{L_T^2(H^{k+2})} \\ & \leq (\|u\|_{L_T^2(H^{k+2})} + \|v\|_{L_T^2(H^{k+2})} + C) \|u - v\|_{L_T^2(H^{k+2})}, \end{aligned}$$

where C is independent of n . Thus, it suffices to prove (4.23) for **simple functions** $u_0(t)$.

The reason why we introduce $\tilde{f}_{0,n}$ is that in $f_{0,n}$ the functions ξ_0^k depend on $u_0(t)$.

• **Step 2.** Since $u_0, \mu_0(t), \xi_0^j$ are chosen from a finite set, we know that $\mathcal{K} \tilde{f}_{0,n}(t)$ is included in a bounded finite dimensional set. Therefore, for all t , $\mathcal{K} f_{0,n}(t)$ is relatively compact. On the other hand, as $\tilde{f}_{0,n}$ is uniformly bounded in $C([0, T]; H^{k+1})$, we know that $\{\mathcal{K} \tilde{f}_{0,n}\}$ is equicontinuous on $[0, T]$. Hence, by Arzelà-Ascoli theorem we only need to prove that

$$(4.24) \quad \lim_{n \rightarrow +\infty} \mathcal{K} \tilde{f}_{0,n}(t) = 0, \text{ in } H_0^{k+1},$$

for any $t \in [0, T]$.

• **Step 3.** It further suffices to prove (4.24) for time-independent u_0 . The general case (when $u_0(t)$ is timely piecewise constant) can be proved by applying the same approach. Let us therefore suppose that u_0 is a constant function in E : $u_0(t, x) = \varphi(x)$ for some $\varphi \in E$. We define the quadratic structure by

$$B(y) \stackrel{\text{def}}{=} \frac{1}{2} \partial_x y^2.$$

Thus

$$\tilde{f}_{0,n}(t) = B(u_0 + \mu_{0,n}) - \sum_{j=1}^{2m} \lambda B(u_0 + \xi_0^j).$$

The T/n -periodicity of $\mu_{0,n}$ tells us that $(u_0 + \mu_{0,n})$ is periodic with period T/n , which to be combined with the fact that $B(u_0 + \xi_0^j)$ is stationary with respect to time, implies that

$$\tilde{f}_{0,n}(t + \frac{T}{n}) = \tilde{f}_{0,n}(t).$$

Moreover, thanks to the construction of $\mu_{0,n}$, we also have that

$$\begin{aligned}
\int_0^{T/n} \tilde{f}_{0,n}(t) dt &= \int_0^{T/n} \left(B(u_0 + \mu_{0,n}(t)) - \sum_{k=1}^{2m} \lambda B(u_0 + \xi_0^k) \right) dt \\
&= \frac{1}{2} \partial_x \left(\int_0^{T/n} (u_0 + \mu_{0,n}(t))^2 - \lambda \sum_{k=1}^{2m} (u_0 + \xi_0^k)^2 dt \right) \\
&= \frac{1}{2} \partial_x \left(\int_0^{T/n} (\mu_{0,n}(t))^2 - \lambda \sum_{k=1}^{2m} (\xi_0^k)^2 dt \right) \\
&= \frac{1}{2} \partial_x \left(\frac{T}{2nm} \sum_{k=1}^{2m} (\xi_0^k)^2 - \lambda \frac{T}{n} \sum_{k=1}^{2m} (\xi_0^k)^2 \right) \\
&= 0.
\end{aligned}$$

It is known that $\tilde{f}_{0,n}$ is uniformly bounded in $L^\infty(H^{k+1})$. Thus, for every $t \in [0, T]$,

$$\|\mathcal{K}\tilde{f}_{0,n}(t)\|_{H_0^{k+1}} \leq C \frac{T}{n},$$

which completes the proof of Lemma 4.5. \square

Let us define

$$(4.25) \quad w_{i,n} \stackrel{\text{def}}{=} u_i - \mathcal{K}f_{i,n},$$

for $i = 0, 1$. Then (4.19) can be equivalently written as

$$(4.26) \quad \begin{cases} \partial_t \rho_0 + \partial_x((w_{0,n} + \mathcal{K}f_{0,n})\rho_0) = 0, \\ \partial_t w_{0,n} + \frac{1}{2} \partial_x(w_{0,n} + \mu_{0,n} + \mathcal{K}f_{0,n})^2 + \partial_x \rho_0 = \bar{\eta}_0, \\ \partial_t A + (w_{0,n} + \mathcal{K}f_{0,n})\partial_x A + 2\partial_x(w_{0,n} + \mathcal{K}f_{0,n})A = 0, \\ \partial_t \rho_1 + \partial_x((w_{0,n} + \mathcal{K}f_{0,n})\rho_1 + (w_{1,n} + \mathcal{K}f_{1,n})\rho_0) = \partial_x A, \\ \partial_t w_{1,n} + \partial_x((w_{0,n} + \mu_{0,n} + \mathcal{K}f_{0,n})(w_{1,n} + \mu_{1,n} + \mathcal{K}f_{1,n})) + \partial_x \rho_1 = \bar{\eta}_1, \\ \bar{\rho}(0) = \bar{g}, \quad A(0) = A_0 \quad \text{and} \quad \bar{u}(0) = \bar{v}, \end{cases}$$

which means that

$$(\bar{w}_n, \bar{\rho}, A) = \mathcal{R}(\bar{v}, \bar{g}, A_0, \bar{\mu}_n + \mathcal{K}\bar{f}_n, \mathcal{K}\bar{f}_n, \bar{\eta}).$$

Equations (4.9), (4.22) and (4.25) tell us that $(\bar{w}_n, \bar{\rho}, A) \in Y^{k+1}(T)$. It further reduces to the following problem:

$$(4.27) \quad \begin{cases} \partial_t \bar{\rho}_{0,n} + \partial_x(\bar{w}_{0,n}\bar{\rho}_{0,n}) = 0, \\ \partial_t \bar{w}_{0,n} + \frac{1}{2} \partial_x(\bar{w}_{0,n} + \mu_{0,n})^2 + \partial_x \bar{\rho}_{0,n} = \bar{\eta}_0, \\ \partial_t \bar{A}_n + \bar{w}_{0,n}\partial_x \bar{A}_n + 2\partial_x \bar{w}_{0,n}\bar{A}_n = 0, \\ \partial_t \bar{\rho}_{1,n} + \partial_x(\bar{w}_{0,n}\bar{\rho}_{1,n} + \bar{w}_{1,n}\bar{\rho}_{0,n}) = \partial_x \bar{A}_n, \\ \partial_t \bar{w}_{1,n} + \partial_x((\bar{w}_{0,n} + \mu_{0,n})(\bar{w}_{1,n} + \mu_{1,n})) + \partial_x \bar{\rho}_{1,n} = \bar{\eta}_1, \\ \bar{\rho}(0) = \bar{g}, \quad \bar{A}_n(0) = A_0 \quad \text{and} \quad \bar{u}(0) = \bar{v}, \end{cases}$$

which means

$$(\bar{w}_n, \bar{\rho}_n, \bar{A}_n) = \mathcal{R}(\bar{v}, \bar{g}, A_0, \bar{\mu}_n, 0, \bar{\eta}).$$

We know from Theorem 3.1, Lemma 4.5 and the fact that both $\|(\bar{w}_n, \bar{\rho}, A)\|_{Y^{k+1}(T)}$ and $\|(\bar{v}, \bar{g}, A_0, \bar{\mu}_n + \mathcal{K}\bar{f}_n, \mathcal{K}\bar{f}_n, \bar{\eta})\|_{X^k(T)}$ are uniformly bounded, that $\|\mathcal{R}(\bar{v}, \bar{g}, A_0, \bar{\mu}_n, 0, \bar{\eta})\|_{Y^k(T)}$

is uniformly bounded for large n and that $\vec{\rho}_{0,n}(x)$ are uniformly away from 0. Since the norm of $\|(\vec{v}, \vec{g}, A_0, \vec{\mu}_n, 0, \vec{\eta})\|_{X^{k+2}(T)}$ is uniformly bounded, thanks to the blow up criteria we know that $(\vec{w}_n, \vec{\rho}_n, \vec{A}_n)$ are uniformly bounded in $Y^{k+2}(T)$ for large n .

Because for n large enough $\|(\vec{w}_n, \vec{\rho}, A)\|_{Y^{k+1}(T)}$, $\|(\vec{w}_n, \vec{\rho}_n, \vec{A}_n)\|_{Y^{k+1}(T)}$, $\|\mathcal{K}\vec{f}_n\|_{C([0,T];H_0^{k+1}\times H_0^{k-1})}$ and $\|(\vec{v}, \vec{g}, A_0, \vec{\mu}_n, 0, \vec{\eta})\|_{X^{k+1}(T)}$ are uniformly bounded, and $\vec{\rho}_{0,n}$ are uniformly away from 0, we have Lipschitz property in lower regularity spaces, namely in $X^k(T)$ and $Y^k(T)$ space,

$$\|(\vec{w}_n, \vec{\rho}_n, \vec{A}_n) - (\vec{w}_n, \vec{\rho}, A)\|_{Y^k(T)} \leq C\|(\vec{v}, \vec{g}, A_0, \vec{\mu}_n + \mathcal{K}\vec{f}_n, \mathcal{K}\vec{f}_n, \vec{\eta}) - (\vec{v}, \vec{g}, A_0, \vec{\mu}_n, 0, \vec{\eta})\|_{X^k(T)},$$

which, together with Lemma 4.4, implies that

$$(4.28) \quad \lim_{n \rightarrow +\infty} \|(\vec{w}_n, \vec{\rho}_n, \vec{A}_n) - (\vec{w}_n, \vec{\rho}, A)\|_{Y^k(T)} = 0.$$

Finally, (4.22), (4.25) and (4.28) imply that

$$\lim_{n \rightarrow +\infty} \|(\vec{w}_n, \vec{\rho}_n, \vec{A}_n) - (\vec{u}, \vec{\rho}, A)\|_{Y^k(T)} = 0.$$

4.3.4. Continuity property: on the use of Theorem 3.2. At first we prove the following lemma.

Lemma 4.6. *For any $t_0 > 0$, any uniformly equicontinuous function $\vec{\chi}_n(t) \in C([0, T]; H^k \times H^{k-2})$, and any uniformly equicontinuous function $\tilde{\chi}_n(t) \in C([0, T]; H^{k-1})$, we have*

$$\lim_{n \rightarrow +\infty} \left\| \int_0^{t_0} \mu_{0,n}(t) \chi_{0,n}(t) dt \right\|_{H^k} = 0,$$

$$\lim_{n \rightarrow +\infty} \left\| \int_0^{t_0} \mu_{0,n}(t) \tilde{\chi}_n(t) dt \right\|_{H^{k-1}} = 0,$$

$$\lim_{n \rightarrow +\infty} \left\| \int_0^{t_0} \mu_{1,n}(t) \chi_{1,n}(t) dt \right\|_{H^{k-2}} = 0.$$

Proof. The proof is straightforward, thanks to the fact that μ_n behave like an oscillation. Indeed,

$$\begin{aligned} \int_0^{t_0} \mu_{0,n} \chi_{0,n}(t) dt &= \int_0^{t_0} \mu_0\left(\frac{nt}{T}\right)(t) \chi_{0,n}(t) dt \\ &= \frac{T}{n} \int_0^{\frac{nt_0}{T}} \mu_0(t) \chi_{0,n}\left(\frac{tT}{n}\right) dt \\ &= \sum_{i=0}^{\lfloor \frac{nt_0}{T} \rfloor - 1} \frac{T}{n} \int_i^{i+1} \mu_0(t) \chi_{0,n}\left(\frac{tT}{n}\right) dt + \frac{T}{n} \int_{\lfloor \frac{nt_0}{T} \rfloor}^{\frac{nt_0}{T}} \mu_0(t) \chi_{0,n}\left(\frac{tT}{n}\right) dt. \end{aligned}$$

Since $\vec{\chi}_n(t) \in C([0, T]; H^k \times H^{k-2})$ is equi-continuous, and $\mu_0(t)$ is in $L_T^\infty(\mathbf{E})$ (hence uniformly bounded in $L_T^\infty(H^s)$, $\forall s > 0$), we know that

$$\int_{\lfloor \frac{nt_0}{T} \rfloor}^{\frac{nt_0}{T}} \mu_0(t) \chi_{0,n}\left(\frac{tT}{n}\right) dt$$

is uniformly bounded in H^k for $t_0 \in [0, T]$ and for n . On the other hand, we know from the construction of $\mu_0(t)$ that

$$\int_i^{i+1} \mu_0(t) \chi_{0,n}\left(\frac{tT}{n}\right) dt = \int_i^{i+1/2} \mu_0(t) \chi_{0,n}\left(\frac{tT}{n}\right) dt + \int_{i+1/2}^{i+1} -\mu_0\left(t - \frac{1}{2}\right) \chi_{0,n}\left(\frac{tT}{n}\right) dt$$

$$= \int_i^{i+1/2} \mu_0(t) \left(\chi_{0,n} \left(\frac{tT}{n} \right) - \chi_{0,n} \left(\frac{tT}{n} + \frac{T}{2n} \right) \right) dt.$$

Since $\tilde{\chi}_n : [0, T] \rightarrow H^k \times H^{k-2}$ is uniformly equicontinuous, for any $\delta > 0$ there exists M such that when $n > M$ we have

$$\left\| \chi_{0,n} \left(\frac{tT}{n} \right) - \chi_{0,n} \left(\frac{tT}{n} + \frac{T}{2n} \right) \right\|_{H^k} < \delta.$$

Hence

$$\left\| \mu_0(t) \left(\chi_{0,n} \left(\frac{tT}{n} \right) - \chi_{0,n} \left(\frac{tT}{n} + \frac{T}{2n} \right) \right) \right\|_{H^k} < \delta, \forall 0 < t < n.$$

Therefore,

$$\sum_{i=0}^{\lfloor \frac{nt_0}{T} \rfloor - 1} \frac{T}{n} \int_i^{i+1} \mu_0(t) \chi_{0,n} \left(\frac{tT}{n} \right) dt < \left[\frac{nt_0}{T} \right] \frac{T}{n} \delta < T\delta.$$

The same proof shows that

$$\lim_{n \rightarrow +\infty} \left\| \int_0^{t_0} \mu_{0,n} \tilde{\chi}_n(0, t) dt \right\|_{H^{k-1}} = 0.$$

The same proof also holds for μ_1 . □

Following Theorem 3.2, we compare (4.27) to the following equation in $[0, T] \times \mathbb{T}$,

$$(4.29) \quad \begin{cases} \partial_t \tilde{\rho}_{0,n} + \partial_x ((\tilde{w}_{0,n} + \mu_{0,n}) \tilde{\rho}_{0,n}) = 0, \\ \partial_t \tilde{w}_{0,n} + \frac{1}{2} \partial_x (\tilde{w}_{0,n} + \mu_{0,n})^2 + \partial_x \tilde{\rho}_{0,n} = \tilde{\eta}_0, \\ \partial_t \tilde{A}_n + (\tilde{w}_{0,n} + \mu_{0,n}) \partial_x \tilde{A}_n + 2 \partial_x (\tilde{w}_{0,n} + \mu_{0,n}) \tilde{A}_n = 0 \\ \partial_t \tilde{\rho}_{1,n} + \partial_x ((\tilde{w}_{0,n} + \mu_{0,n}) \tilde{\rho}_{1,n} + (\tilde{w}_{1,n} + \mu_{1,n}) \tilde{\rho}_{0,n}) = \partial_x \tilde{A}_n, \\ \partial_t \tilde{w}_{1,n} + \partial_x ((\tilde{w}_{0,n} + \mu_{0,n}) (\tilde{w}_{1,n} + \mu_{1,n})) + \partial_x \tilde{\rho}_{1,n} = \tilde{\eta}_1, \\ \tilde{\rho}_n(0) = \tilde{g}, \quad \tilde{A}_n(0) = A_0 \quad \text{and} \quad \tilde{w}_n(0) = \tilde{v}. \end{cases}$$

As $(\tilde{w}_n, \tilde{\rho}_n, \tilde{A}_n)$ are uniformly bounded in $Y^{k+2}(T)$, thanks to Theorem 3.2 and the blow up criteria, we know that

$$\lim_{n \rightarrow +\infty} \|(\tilde{w}_n, \tilde{\rho}_n, \tilde{A}_n) - (\tilde{w}_n, \tilde{\rho}_n, \tilde{A}_n)\|_{Y^k(T)} = 0,$$

$(\tilde{w}_n, \tilde{\rho}_n, \tilde{A}_n) = \mathcal{R}(\tilde{v}, \tilde{g}, A_0, \tilde{\mu}_n, \tilde{\mu}_n, \tilde{\eta})$ is uniformly bounded in $Y^{k+2}(T)$ for large n .

Remark 4.1. We observe that the $L_T^\infty(H^s \times H^{s-2})$ norm of $\tilde{\mu}_n$ (with $s > 0$) and the source term $\tilde{\eta}$ only depend on $\tilde{\eta}_1$. This point probably could be used to reduce the regularity of the initial state. However, this is not the main purpose of this paper.

4.3.5. Approximation of μ_n by smooth (with respect to time) functions. In this step we approximate $\tilde{\mu}_n$ by smooth $\tilde{\mu}_n^m$ such that their values at the end points are 0. Let us take $\tilde{\mu}_n^m \in C^\infty([0, T]; \mathbf{E})$ with $\tilde{\mu}_n^m(0) = \tilde{\mu}_n^m(T) = \mathbf{0}$ and

$$\lim_{m \rightarrow \infty} \|\tilde{\mu}_n^m - \tilde{\mu}_n\|_{L_T^2(H_0^{k+3} \times H_0^{k+1})} = 0.$$

Let

$$(\tilde{w}_n^m, \tilde{\rho}_n^m, \tilde{A}_n^m) \stackrel{\text{def}}{=} \mathcal{R}(\tilde{v}, \tilde{g}, A_0, \tilde{\mu}_n^m, \tilde{\mu}_n^m, \tilde{\eta}) \in Y^{k+2}(T).$$

Then, for any n , thanks to the continuity property of \mathcal{R} stated in Theorem 3.1, we have

$$\lim_{m \rightarrow \infty} \|(\tilde{w}_n^m, \tilde{\rho}_n^m, \tilde{A}_n^m) - (\tilde{w}_n, \tilde{\rho}_n, \tilde{A}_n)\|_{Y^{k+2}(T)} = 0.$$

Consequently, we are able to select a sequence $\{m(n)\}_{\mathbb{N}}$ such that

$$\lim_{n \rightarrow +\infty} \|(\tilde{w}_n^{m(n)}, \tilde{\rho}_n^{m(n)}, \tilde{A}_n^{m(n)}) - (\tilde{w}_n, \tilde{\rho}_n, \tilde{A}_n)\|_{Y^{k+2}(T)} = 0 \text{ as } n \rightarrow +\infty.$$

4.3.6. Extension technique. By taking $\bar{w}_n^{m(n)} = \tilde{w}_n + \bar{\mu}_n^{m(n)}$, $\bar{\rho}_n^{m(n)} = \tilde{\rho}_n^{m(n)}$ and $A_n^m = \tilde{A}_n^{m(n)}$, we get that in $[0, T] \times \mathbb{T}$

$$(4.30) \quad \begin{cases} \partial_t \rho_{0,n}^{m(n)} + \partial_x (w_{0,n}^{m(n)} \rho_{0,n}^{m(n)}) = 0, \\ \partial_t w_{0,n}^{m(n)} + \frac{1}{2} \partial_x (w_{0,n}^{m(n)})^2 + \partial_x \rho_{0,n}^{m(n)} = \bar{\eta}_0 + \partial_t \mu_{0,n}^{m(n)}, \\ \partial_t A_n^{m(n)} + w_{0,n}^{m(n)} \partial_x A_n^{m(n)} + 2 \partial_x w_{0,n}^{m(n)} A_n^{m(n)} = 0, \\ \partial_t \rho_{1,n}^{m(n)} + \partial_x (w_{0,n}^{m(n)} \rho_{1,n}^{m(n)} + w_{1,n}^{m(n)} \rho_{0,n}^{m(n)}) = \partial_x A_n^{m(n)}, \\ \partial_t w_{1,n}^{m(n)} + \partial_x (w_{0,n}^{m(n)} w_{1,n}^{m(n)}) + \partial_x \rho_{1,n}^{m(n)} = \bar{\eta}_1 + \partial_t \mu_{0,n}^{m(n)}, \\ \bar{\rho}_n^{m(n)}(0) = \bar{g}, \quad A_n^{m(n)}(0) = A_0 \quad \text{and} \quad \bar{u}_n^{m(n)}(0) = \bar{v}. \end{cases}$$

Hence, as n tends to $+\infty$,

$$\begin{aligned} (\bar{w}_n^{m(n)}, \bar{\rho}_n^{m(n)}, A_n^{m(n)}) &\stackrel{\text{def}}{=} \mathcal{R}(\bar{v}, \bar{g}, A_0, 0, 0, \bar{\eta} + \partial_t \bar{\mu}_n^{m(n)}) \in Y^{k+2}(T), \\ (\bar{w}_n^{m(n)}, \bar{\rho}_n^{m(n)}, A_n^{m(n)})(T) &= (\tilde{w}_n^{m(n)}, \tilde{\rho}_n^{m(n)}, \tilde{A}_n^{m(n)})(T) \rightarrow (\bar{u}, \bar{\rho}, A)(T) \text{ in } Z^k. \end{aligned}$$

5. SEMICLASSICAL LIMIT OF THE CONTROLLED SYSTEM

In the end, this section is devoted to the proof of Theorem 2.2.

Proof of Theorem 2.2. The main idea of the proof basically follows from [16], which we outline as the following three steps:

• **Step 1.** *There exists $T_0 > 0$ such that (1.1)–(1.2) have solutions which are uniformly bounded for a^{\hbar} in $C([0, T_0]; H^k)$ and for S^{\hbar} in $C([0, T_0]; H^{k+1})$.*

At first we study the system of u^{\hbar} and $(a^{r,\hbar}, a^{i,\hbar})$:

$$(5.1) \quad \begin{cases} \partial_t a^{r,\hbar} + u^{\hbar} \partial_x a^{r,\hbar} + \frac{1}{2} a^{r,\hbar} \partial_x u^{\hbar} = -\frac{\hbar}{2} \partial_x^2 a^{i,\hbar}, \\ \partial_t a^{i,\hbar} + u^{\hbar} \partial_x a^{i,\hbar} + \frac{1}{2} a^{i,\hbar} \partial_x u^{\hbar} = \frac{\hbar}{2} \partial_x^2 a^{r,\hbar}, \\ \partial_t u^{\hbar} + u^{\hbar} \partial_x u^{\hbar} + 2a^{r,\hbar} \partial_x a^{r,\hbar} + 2a^{i,\hbar} \partial_x a^{i,\hbar} = -\eta^{\hbar}(t, x), \\ (a^{r,\hbar}, a^{i,\hbar}, u^{\hbar})|_{t=0} = (\text{Re}(a^0(\hbar)), \text{Im}(a^0(\hbar)), \partial_x S). \end{cases}$$

This system can be written in the form

$$(5.2) \quad \partial_t w^{\hbar} + A(w^{\hbar}) \partial_x w^{\hbar} = \hbar H(w^{\hbar}) + f^{\hbar},$$

$$(5.3) \quad w^{\hbar} = \begin{pmatrix} a^{r,\hbar} \\ a^{i,\hbar} \\ u^{\hbar} \end{pmatrix}, \quad A(w^{\hbar}) = \begin{pmatrix} u^{\hbar} & 0 & \frac{1}{2} a^{r,\hbar} \\ 0 & u^{\hbar} & \frac{1}{2} a^{i,\hbar} \\ 2a^{r,\hbar} & 2a^{i,\hbar} & u^{\hbar} \end{pmatrix},$$

$$(5.4) \quad H(w^{\hbar}) = \begin{pmatrix} -\frac{1}{2} \partial_x^2 a^{i,\hbar} \\ \frac{1}{2} \partial_x^2 a^{r,\hbar} \\ 0 \end{pmatrix}, \quad f^{\hbar}(t, x) = f_0(t, x) + \hbar f_1(t, x) = \begin{pmatrix} 0 \\ 0 \\ -\eta^{\hbar}(t, x) \end{pmatrix}.$$

We observe that this system shares the following properties.

- (i) The maps $w \mapsto A(w)$ and $w \mapsto H(w)$ are linear.
- (ii) The left hand side of (5.2) is symmetrizable with diagonal matrix $A_0 := \text{diag}(1, 1, 1/4)$.
- (iii) $H(w^{\hbar})$ has an Hamiltonian structure.

(iv) $f^{\hbar}(t, x)$ is uniformly bounded in $C([0, T]; H^{k+4})$.

Therefore, for $\alpha \leq k$ classical energy estimates lead to

$$(5.5) \quad \begin{aligned} \partial_t(A_0 \partial_x^\alpha w^{\hbar}, \partial_x^\alpha w^{\hbar}) &= 2\hbar(A_0 H(\partial_x^\alpha w^{\hbar}), \partial_x^\alpha w^{\hbar}) \\ &\quad - 2(A_0 \partial_x^\alpha (A(w^{\hbar}) \partial_x w^{\hbar}), \partial_x^\alpha w^{\hbar}) + 2(A_0 \partial_x^\alpha f^{\hbar}, \partial_x^\alpha w^{\hbar}). \end{aligned}$$

Thanks to the special structure of $H(w^{\hbar})$, an integration by parts gives

$$(A_0 H(\partial_x^\alpha w^{\hbar}), \partial_x^\alpha w^{\hbar}) = 0.$$

Moreover, since $\partial_x^k f^{\hbar}(t, x)$ are uniformly bounded on $[0, T] \times \mathbb{T}$ and using the fact that A_0 is a symmetrizer for (5.2), classical estimates on the right hand side of (5.5) lead to

$$\partial_t \sum_{\alpha \leq k} (A_0 \partial_x^\alpha w^{\hbar}, \partial_x^\alpha w^{\hbar}) \leq C \|w^{\hbar}\|_{C^1} \|w^{\hbar}\|_{H^k}^2 + C \|w^{\hbar}\|_{H^k},$$

which $C > 0$ is a constant independent of $\hbar \in (0, 1]$. As the initial data are uniformly bounded in H^k (see (1.2)), by Gronwall's lemma we get the existence of T_0 such that a^{\hbar}, u^{\hbar} are uniformly bounded in $C([0, T_0]; H^k)$.

Next, we solve equation (2.2) via (5.1) and $w^{\hbar} = \partial_x S^{\hbar}$. Then it suffices to prove that $\int_{\mathbb{T}} S^{\hbar}(t)$ are uniformly bounded on $[0, T_0]$. We know from (2.2) that

$$\partial_t S^{\hbar} + \frac{1}{2}(u^{\hbar})^2 + f^{\hbar}(t, x) + |a^{\hbar}|^2 - 1 = 0.$$

Since $\int_{\mathbb{T}} f^{\hbar}(t) = 0$ and $\int_{\mathbb{T}} \frac{1}{2}(u^{\hbar})^2 + |a^{\hbar}|^2 - 1$ are uniformly bounded, this concludes the proof.

• **Step 2.** For \hbar small enough, the solutions (a^{\hbar}, S^{\hbar}) are uniformly bounded in $C([0, T]; H^k \times H^{k+1})$. Moreover, (a^{\hbar}, S^{\hbar}) tends to (a_0, S_0) in $C([0, T]; H^k \times H^{k+1})$ as $\hbar \rightarrow 0^+$.

It is known from the assumption of Theorem 2.2 that (1.5) admits a solution (ρ_0, u_0) in $C([0, T]; H^k)$. At first, by denoting $w^T(t) := (a_0^r(t), a_0^i(t), u_0(t))$ we prove that the zeroth order limit system (2.4),

$$(5.6) \quad \partial_t w + A(w) \partial_x w = f_0,$$

$$(5.7) \quad w(0) = (a_0^r, a_0^i, u_0)|_{t=0} = (\operatorname{Re}(a_0^0), \operatorname{Im}(a_0^0), \partial_x S),$$

has a solution $w_0 \in C([0, T]; H^{k+2})$. Indeed, suppose that the maximal solution is on the interval $[0, s)$ with $s < T$, then ρ_0, u_0 are bounded in $C([0, s]; H^{k+2})$. We know from (2.4) that a_0 belongs to $C([0, s]; H^{k+1})$, which is in contradiction of the definition of s .

Let us define $v^{\hbar} = w^{\hbar} - w_0$. Then comparing (5.6) and (5.2) we get

$$(5.8) \quad \partial_t v^{\hbar} + (A(v^{\hbar}) + A(w_0)) \partial_x v^{\hbar} + A(v^{\hbar}) \partial_x w_0 = \hbar(H(w_0) + H(v^{\hbar}) + f_1),$$

$$(5.9) \quad v^{\hbar}(0) = (\operatorname{Re}(a^0(\hbar)) - \operatorname{Re}(a_0^0), \operatorname{Im}(a^0(\hbar)) - \operatorname{Im}(a_0^0), 0),$$

As w_0 belongs to $C([0, T]; H^{k+2})$, we get that

$$|(\hbar H(\partial_x^\alpha w), \partial_x^\alpha v^{\hbar})| \leq \hbar C \|w_0\|_{H^{k+2}} \|v^{\hbar}\|_{H^k}, \forall \alpha \leq k,$$

which together with similar energy estimates as in the previous step lead to

$$\partial_t \sum_{\alpha \leq k} (A_0 \partial_x^\alpha v^{\hbar}, \partial_x^\alpha v^{\hbar}) \leq C \|v^{\hbar}\|_{H^k}^2 (\|v^{\hbar}\|_{C^1} + 1) + \hbar C \|v^{\hbar}\|_{H^k} + \hbar C \|v^{\hbar}\|_{H^k}.$$

Since $\|v^{\hbar}(0)\|_{H^k}$ tends to 0 as \hbar tends to 0, we are able to find positive \hbar_0 such that

$$\|v^{\hbar}\|_{C([0, T]; H^k)} \leq C \hbar + C \|v^{\hbar}(0)\|_{H^k} \leq C \hbar, \forall 0 < \hbar < \hbar_0.$$

Therefore, the solutions (a^h, u^h) converge uniformly to (a_0, u_0) in $C([0, T]; H^k \times H^k)$ as $\hbar \rightarrow 0^+$. Meanwhile we know that S^h (resp. S_0) satisfies (2.2) (resp. (2.3)), thus direct calculations as above show that

$$\int_{\mathbb{T}} S^h(t) \rightarrow \int_{\mathbb{T}} S_0(t) \text{ in } C([0, T]) \text{ as } \hbar \rightarrow 0^+.$$

Hence

$$(a^h, S^h) \text{ tends to } (a_0, S_0) \text{ in } C([0, T]; H^k \times H^{k+1}) \text{ as } \hbar \rightarrow 0^+.$$

• **Step 3.** Let $\tilde{S}^h = (S^h - S_0)/\hbar$, $\tilde{v}^h = v^h/\hbar$. Then $(\tilde{a}^h, \tilde{S}^h)$ tends to (a_1, S_1) in $C([0, T]; H^{k-2} \times H^{k-1})$.

We deduce from (5.8) and the last equation of (5.1) that

$$(5.10) \quad \partial_t \tilde{v}^h + (\hbar A(\tilde{v}^h) + A(w_0)) \partial_x \tilde{v}^h + A(\tilde{v}^h) \partial_x w_0 = H(w_0) + \hbar H(\tilde{v}^h) + f_1,$$

$$(5.11) \quad \tilde{v}^h(0) = (\operatorname{Re}(a_1^0 + R(\hbar)), \operatorname{Im}(a_1^0 + R(\hbar)), 0).$$

The limit system of (5.10)–(5.11) reads

$$(5.12) \quad \partial_t \tilde{v} + A(w_0) \partial_x \tilde{v} + A(\tilde{v}^h) \partial_x w_0 = H(w_0) + f_1,$$

$$(5.13) \quad \tilde{v}(0) = (\operatorname{Re}(a_1^0), \operatorname{Im}(a_1^0), 0),$$

which is exactly the first order system (2.6). Let us denote the solution of this linear system by $v_1 = (a_1^r, a_1^i, u_1)$.

Thanks to (5.10), \tilde{v}^h are uniformly bounded in $C([0, T]; H^k) \subset L^\infty([0, T]; H^k)$. Then by plugging this bound into (5.10)–(5.11), we obtain the uniform boundedness of $\partial_t \tilde{v}^h$ in $C([0, T]; H^{k-2}) \subset L^\infty([0, T]; H^{k-2})$. Therefore, up to a subsequence, \tilde{v}^h converges to some function v' in $C([0, T]; H^{k-2})$. By taking the limit, we find that v' solves the equation (5.12)–(5.13). Hence v' coincidence with v_1 . To this end, we deduce from

$$\partial_t \tilde{S}^h + u_0 \tilde{u}^h + \frac{1}{2} \hbar (\tilde{u}^h)^2 + F_2 + (\tilde{a}^h \bar{a}_0 + \bar{a}^h a_0) + \hbar |\tilde{a}^h|^2 = 0,$$

$$\partial_t S_1 + u_0 u_1 + 2\operatorname{Re}(a_0 \bar{a}_1) + F_2 = 0,$$

that

$$\int_{\mathbb{T}} \tilde{S}^h(t) \rightarrow \int_{\mathbb{T}} S_1(t) \text{ in } C([0, T]) \text{ as } \hbar \rightarrow 0^+.$$

Therefore

$$(\tilde{a}^h, \tilde{S}^h) \text{ tends to } (a_1, S_1) \text{ in } C([0, T]; H^{k-2} \times H^{k-1}) \text{ as } \hbar \rightarrow 0^+.$$

More precisely, we get the first order expansion of ψ^h :

$$a^h = a_0 + \hbar a_1 + \hbar r_a^h, \quad S^h = S_0 + \hbar S_1 + \hbar r_S^h,$$

$$\lim_{\hbar \rightarrow 0^+} \|r_a^h\|_{C([0, T]; H^{k-2})} = 0, \quad \lim_{\hbar \rightarrow 0^+} \|r_S^h\|_{C([0, T]; H^{k-1})} = 0,$$

and complete the proof of Theorem 2.2. \square

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