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► **To cite this version:**

Alexander Semenov. ORBITAL STABILITY OF A SUM OF SOLITONS AND BREATHERS OF THE MODIFIED KORTEWEG-DE VRIES EQUATION. 2021. hal-03408004v2

**HAL Id: hal-03408004**

**<https://hal.science/hal-03408004v2>**

Preprint submitted on 3 Dec 2021 (v2), last revised 26 Jul 2022 (v3)

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# ORBITAL STABILITY OF A SUM OF SOLITONS AND BREATHERS OF THE MODIFIED KORTEWEG-DE VRIES EQUATION

ALEXANDER SEMENOV

**ABSTRACT.** In this article, we prove that a sum of solitons and breathers of the modified Korteweg-de Vries equation (mKdV) is orbitally stable. The orbital stability is shown in  $H^2$ . More precisely, we will show that if a solution of (mKdV) is close enough to a sum of solitons and breathers with distinct velocities at  $t = 0$  in the  $H^2$  sense, then it stays close to this sum of solitons and breathers, up to space translations for solitons and space or phase translations for breathers.

From this, we deduce the orbital stability of a multi-breather of (mKdV). As an application of the orbital stability and of the integrability of (mKdV), we deduce a result about uniqueness of multi-breathers.

## 1. INTRODUCTION

**1.1. Setting of the problem.** We consider the modified Korteweg-de Vries equation:

$$(mKdV) \quad \begin{cases} u_t + (u_{xx} + u^3)_x = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

for  $u_0 \in H^2(\mathbb{R})$ .

(mKdV) appears as a good approximation of some physical problems as ferromagnetic vortices [43], fluid mechanics [21], electrodynamics [35], plasma physics [36, 9], etc.

The Cauchy problem for (mKdV) is locally well-posed in  $H^s$  for  $s > -\frac{1}{2}$  [20]. For  $s > \frac{1}{4}$ , the Cauchy problem is globally well-posed [11]. In this paper, we will use only basic results about the Cauchy problem: the fact that it is globally well-posed in  $H^1$  or  $H^2$ .

Note that the set of solutions of (mKdV) is stable under space or time translations or under reflexions with respect to the  $x$ -axis.

We have following conservation laws for a solution  $u(t)$  of (mKdV):

$$(1) \quad M[u](t) := \frac{1}{2} \int u^2(t, x) dx,$$

$$(2) \quad E[u](t) := \frac{1}{2} \int u_x^2(t, x) dx - \frac{1}{4} \int u^4(t, x) dx,$$

$$(3) \quad F[u](t) := \frac{1}{2} \int u_{xx}^2(t, x) dx - \frac{5}{2} \int u^2(t, x) u_x^2(t, x) dx + \frac{1}{4} \int u^6(t, x) dx.$$

Note that (mKdV) has actually infinitely many conservation laws, because it is integrable, like the original Korteweg-de Vries equation (KdV), which has quadratic nonlinearity [34, 1]. It is also a special case of the generalized Korteweg-de Vries equation (gKdV) [27].

These equations are all nonlinear (focusing) dispersive equations, as are the nonlinear Schrödinger equation (NLS) [32, 41, 42] or the nonlinear Klein-Gordon equation (KG) [12, 13], and share a common property: they all admit special solutions called solitons, a bump that translates with a constant velocity without deformation. However, (mKdV) enjoys a specific feature: it admits another class of special solutions called breathers, which we will describe below. We will consider here both specific solutions of (mKdV) together : solitons and breathers.

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2020 *Mathematics Subject Classification.* Primary 35Q51, 35Q53, 35B40; Secondary 37K10, 37K40.

*Key words and phrases.* mKdV equation, breather, soliton, multi-breather, multi-soliton, N-breather, stability.

For  $c > 0$ ,  $\kappa \in \{-1, 1\}$  and  $x_0 \in \mathbb{R}$ , a *soliton* parametrized by  $c$ ,  $\kappa$  and  $x_0$  is a solution of (mKdV) that has the following expression:

$$(4) \quad R_{c,\kappa}(t, x; x_0) := \kappa \left( \frac{2c}{\cosh^2(c^{1/2}(x - x_0 - ct))} \right)^{\frac{1}{2}}.$$

When  $\kappa = -1$ , this solution is sometimes called an *antisoliton*.

The soliton  $R_{c,\kappa}(x_0)$  travels with velocity  $c$  to the right, the position of its center at a time  $t$  is  $x_0 + ct$ . It is exponentially localized, depending on  $c$  and  $x_0 + ct$  (the position of the bound depends on  $x_0 + ct$ , the amplitude and the exponential decay rate depend on  $c$ ):

$$(5) \quad |R_{c,\kappa}(t, x; x_0)| \leq \sqrt{2c} \exp\left(-\sqrt{c}|x - x_0 - ct|\right).$$

Analogous bounds are valid for any derivative (with the same decay rate but different amplitude). This motivates the terminology ‘‘shape parameter’’ for  $c$ .

We denote  $Q$  the basic ground state, more precisely:

$$(6) \quad Q(x) := R_{1,1}(0, x; 0).$$

It satisfies the following elliptic equation on  $\mathbb{R}$ :

$$(7) \quad Q'' - Q + Q^3 = 0.$$

Solitons, in particular their stability, have been extensively studied: regarding orbital stability (in  $H^1$ ), we refer to Cazenave, Lions [8] and Weinstein [41, 42] for (NLS) and Weinstein [41], Bona-Souganidis-Strauss [7] and Martel-Merle [31] for (mKdV), see also Grillakis-Shatah-Strauss [19] for a result in an abstract setting. Asymptotic stability (in  $H^1$ ) of (mKdV) solitons was shown by Martel-Merle [28, 29, 30] and refined by Germain-Pusateri-Rousset [17].

For  $\alpha, \beta > 0$  and  $x_1, x_2 \in \mathbb{R}$ , a *breather* parametrized by  $\alpha, \beta, x_1, x_2$  is a solution of (mKdV) that has the following expression:

$$(8) \quad B_{\alpha,\beta}(t, x; x_1, x_2) := 2\sqrt{2}\partial_x \left[ \arctan \left( \frac{\beta \sin(\alpha y_1)}{\alpha \cosh(\beta y_2)} \right) \right],$$

where  $y_1 := x + \delta t + x_1$ ,  $y_2 := x + \gamma t + x_2$ ,  $\delta := \alpha^2 - 3\beta^2$  and  $\gamma := 3\alpha^2 - \beta^2$ .

The breather  $B_{\alpha,\beta}(x_1, x_2)$  travels with velocity  $-\gamma$ ; the position of its center at a time  $t$  is  $-x_2 - \gamma t$ . It is exponentially localized, depending on  $\alpha, \beta$  and  $-x_2 - \gamma t$  (the position of the bound depends on  $-x_2 - \gamma t$ , the coefficient depends on  $\alpha$  and  $\beta$ , and the exponential decay rate depends on  $\beta$ ):

$$(9) \quad |B_{\alpha,\beta}(t, x; x_1, x_2)| \leq C(\alpha, \beta) \exp(-\beta|x + x_2 + \gamma t|),$$

Analogous bounds are valid for any derivative (with the same decay rate but different amplitude). This motivates the terminology ‘‘shape’’ and ‘‘frequency’’ parameters for  $\beta$  and  $\alpha$ , respectively.

One doesn't talk of ‘‘antibreathers’’, because if we replace  $x_1$  by  $x_1 + \frac{\pi}{\alpha}$ , then a breather is transformed in its opposite.

Similarly to (7), it is known that a breather  $B = B_{\alpha,\beta}$  satisfies the following elliptic equation on  $\mathbb{R}$ :

$$(10) \quad B_{xxxx} + 5BB_x^2 + 5B^2B_{xx} + \frac{3}{2}B^5 - 2(\beta^2 - \alpha^2)(B_{xx} + B^3) + (\alpha^2 + \beta^2)^2 B = 0.$$

This object was first introduced by Wadati [39], and it was used by Kenig, Ponce and Vega in [22] for the ill-posedness for (mKdV) for rough data. Their properties, in particular their stability, are well studied by Alejo, Muñoz and co-authors [3, 4, 5, 6, 2]. We know that a breather is orbitally stable in  $H^2$  [3]. Afterwards,  $H^1$  orbital stability was proved via Bäcklund transformation [2], and also  $H^1$  asymptotic stability, for breathers moving to the right. Asymptotic stability of breathers in full generality is still an open problem.

When  $\alpha \rightarrow 0$ ,  $B_{\alpha,\beta}$  tends to a *double-pole solution* of (mKdV): it is a couple soliton-antisoliton that move with a constant velocity and that have a repulsive logarithmic interaction [40]. However, this limit, which is somehow at a boundary between solitons and breathers, is expected to be unstable [18]. We do not consider this object in this paper.

Solitons and breathers are important objects to study because of their stability properties and also because of the soliton-breather resolution. The latter is an important result about the long time dynamics of (mKdV), which asserts that any generic solution can be approached by a sum of solitons and breathers when  $t \rightarrow +\infty$ . It is established for initial conditions in a weighted Sobolev space

in [10] (see also Schuur [37]) by inverse scattering method; see also [37] for the soliton resolution for (KdV).

Given a set of basic objects (solitons and breathers), we consider a solution that tends to this sum when  $t \rightarrow +\infty$ , called *multi-breather*. In [38], we have shown existence, regularity and uniqueness of multi-breathers of (mKdV). There is also a formula for multi-breathers of (mKdV) obtained by Wadati [40], which was derived as a consequence of the integrability of (mKdV); but is it not well suited for our purpose.

Martel, Merle and Tsai [33] proved that a sum of (decoupled and ordered) solitons is orbitally stable in  $H^1$  for (mKdV), and actually asymptotically stable (in the region  $x \geq \delta t$  for  $\delta > 0$  small). Le Coz [25] has established stability of (mKdV)  $N$ -solitons in  $H^N$  by modifying the approach used by Maddocks and Sachs [26] for (KdV).

Inspired by [33], similar asymptotic stability results were obtained for sums of (decoupled) solitons for various nonlinear dispersive equations: we refer to El Dika [15] for the Benjamin-Bona-Mahony equation (BBM), Kenig and Martel [24] for the Benjamin-Ono equation (BO), El Dika-Molinet [16] for the Camassa-Holm multipeakon, and Côte-Muñoz-Pilod-Simpson [14] for the Zakharov-Kuznetsov (ZK) equation.

Because breathers have an  $H^2$  structure, we prove orbital stability in  $H^2$  in this paper for a sum of solitons and breathers. One of the difficulties is to obtain  $H^2$  stability results for solitons too, i.e. to study solitons at a  $H^2$  level.

**1.2. Main results.** We prove in this article that given any sum of solitons and breathers with distinct velocities and such that all these velocities except possibly one are positive, a solution  $u$  of (mKdV) that is initially close to this sum in  $H^2$  stays close to this sum for any time up to space translations for solitons and space or phase translations for breathers. This is orbital stability. Let us make the definition of orbital stability more precise.

Let  $L \in \mathbb{N}$ . We consider a set of  $L$  solitons: given  $c_l^0 > 0$ ,  $\kappa_l \in \{-1, 1\}$  and  $x_{0,l}^0 \in \mathbb{R}$  for  $1 \leq l \leq L$ , we set, for  $1 \leq l \leq L$ ,

$$(11) \quad R_l(t, x) := R_{c_l^0, \kappa_l}(t, x; x_{0,l}^0).$$

Let  $K \in \mathbb{N}$ . We consider a set of  $K$  breathers: given  $\alpha_k > 0$ ,  $\beta_k > 0$  and  $x_{1,k}^0, x_{2,k}^0 \in \mathbb{R}$  for  $1 \leq k \leq K$ , we set, for  $1 \leq k \leq K$ ,

$$(12) \quad B_k(t, x) := B_{\alpha_k, \beta_k}(t, x; x_{1,k}^0, x_{2,k}^0).$$

We now define important parameters for each of the objects of the problem. For  $1 \leq l \leq L$ , the velocity of the  $l$ th soliton is

$$(13) \quad v_l^s := c_l^0,$$

and for  $1 \leq k \leq K$ , the velocity of the  $k$ th breather is

$$(14) \quad v_k^b := \beta_k^2 - 3\alpha_k^2.$$

For  $1 \leq l \leq L$ , the center of the  $l$ th soliton is

$$(15) \quad x_l^s(t) := x_{0,l}^0 + v_l^s t,$$

and for  $1 \leq k \leq K$ , the center of the  $k$ th breather is

$$(16) \quad x_k^b(t) := -x_{2,k}^0 + v_k^b t.$$

We set  $J := K + L$  the total number of objects in the problem. We assume that  $J \geq 2$ , because for  $J = 1$  the proof is already done in [3] when the only object is a breather, and in [41] when the only object is a soliton.

We assume that the velocities of our objects are all distinct, this will imply that our objects are far from each other when time is large, which is essential for our analysis. More precisely,

$$(17) \quad \forall k \neq k' \quad v_k^b \neq v_{k'}^b, \quad \forall l \neq l' \quad v_l^s \neq v_{l'}^s, \quad \forall k, l \quad v_k^b \neq v_l^s.$$

This allows us to define an increasing function

$$(18) \quad v : \{1, \dots, J\} \rightarrow \{v_k^b, 1 \leq k \leq K\} \cup \{v_l^s, 1 \leq l \leq L\}.$$

The set  $\{v_1, \dots, v_J\}$  is thus the set of all the possible velocities of our objects. We have

$$(19) \quad v_1 < v_2 < \dots < v_J.$$

We define, for  $1 \leq j \leq J$ ,  $P_j$  as the object ( $R_l$  or  $B_k$ ) that corresponds to the velocity  $v_j$ , i.e. if  $v_j = v_l^s$ , we set  $P_j := R_l$ , and if  $v_j = v_k^b$ , we set  $P_j := B_k$ . So,  $P_1, \dots, P_J$  are the considered objects that are ordered by increasing velocity.

We denote  $x_j(t)$  the position (the center of mass) of  $P_j(t)$ . More precisely, if  $P_j = R_l$ , we set  $x_j(t) := x_l^s(t)$ ; and if  $P_j = B_k$ , we set  $x_j(t) := x_k^b(t)$ .

We set

$$(20) \quad R := \sum_{l=1}^L R_l, \quad B := \sum_{k=1}^K B_k, \quad P := R + B = \sum_{j=1}^J P_j.$$

We need both notations: indexation by  $k$  and  $l$ , and indexation by  $j$ , and we keep these notations to avoid ambiguity.

The main result that we will prove in this article is the following: a sum of decoupled and ordered solitons and breathers, with  $v_2 > 0$  (that is, all but at most one travel to the right), is orbitally stable. The precise statement is as follows.

**Theorem 1.** *Given solitons and breathers (11), (12) whose velocities (13) and (14) satisfy (17), and whose positions are set by (15) and (16), we define the corresponding sum  $P$  in (20), and we define  $P_j, v_j, x_j$  a reindexation of the given set of solitons and breathers such that (19). We assume that*

$$(21) \quad v_2 > 0.$$

Then there exists  $A_0, \theta_0, D_0, a_0 > 0$ , constants (depending on  $c_l^0, \alpha_k, \beta_k$ , but not on  $x_j(0)$ ), such that the following is true. Let  $u_0 \in H^2(\mathbb{R})$ ,  $D \geq D_0$  and  $0 \leq a \leq a_0$ , such that

$$(22) \quad \|u_0 - P(0)\|_{H^2} \leq a, \quad \text{and} \quad x_j(0) > x_{j-1}(0) + 2D, \quad \text{for all } j = 2, \dots, J.$$

Let  $u(t)$  be a solution of (mKdV) such that  $u(0) = u_0$ . Then, there exists  $x_{0,l}(t), x_{1,k}(t), x_{2,k}(t)$  defined for any  $t \geq 0$  such that

$$(23) \quad \forall t \geq 0, \quad \left\| u(t) - \sum_{l=1}^L R_{c_l^0, x_l}(t, \cdot; x_{0,l}(t)) - \sum_{k=1}^K B_{\alpha_k, \beta_k}(t, \cdot; x_{1,k}(t), x_{2,k}(t)) \right\|_{H^2} \leq A_0 (a + e^{-\theta_0 D}),$$

with

$$(24) \quad \forall t \geq 0, \quad \sum_{l=1}^L |x'_{0,l}(t)| + \sum_{k=1}^K (|x'_{1,k}(t)| + |x'_{2,k}(t)|) \leq CA_0 (a + e^{-\theta_0 D}),$$

for some constant  $C > 0$ .

*Remark 2.* For  $1 \leq j \leq J$ , if  $P_j = R_l$  we denote  $\tilde{x}_j(t)$  the position of  $R_{c_l^0, x_l}(t, \cdot; x_{0,l}(t))$  and if  $P_j = B_k$  we denote  $\tilde{x}_j(t)$  the position of  $B_{\alpha_k, \beta_k}(t, \cdot; x_{1,k}(t), x_{2,k}(t))$ . (24) allows us to deduce that for any  $j = 1, \dots, J$ ,

$$(25) \quad \forall t \geq 0, \quad |\tilde{x}_j'(t) - v_j| \leq CA_0 (a + e^{-\theta_0 D}).$$

From the latter, we deduce that for any  $j = 2, \dots, J$ ,

$$(26) \quad \forall t \geq 0, \quad \tilde{x}_j'(t) - \widetilde{x_{j-1}}'(t) \geq \frac{\tau}{2},$$

for some suitable constant  $\tau > 0$  that will be precised later (for  $a_0$  small enough and  $D_0$  large enough). And, from the argument we make in order to prove Theorem 1, for any  $j = 2, \dots, J$ ,

$$(27) \quad \forall t \geq 0, \quad \tilde{x}_j(t) - \widetilde{x_{j-1}}(t) \geq D,$$

Thus, for any  $j = 2, \dots, J$ ,

$$(28) \quad \forall t \geq 0, \quad \tilde{x}_j(t) - \widetilde{x_{j-1}}(t) \geq D + \frac{\tau}{2}t.$$

In other words, our objects stay decoupled.

We also deduce some consequences from Theorem 1.

Recall that multi-breathers were defined and constructed in [38], and their uniqueness was proved there when all objects travel to the right (that is, if  $v_1 > 0$ ). Let  $p$  be a multi-breather associated to  $P$  (for the time being,  $p$  is possibly not unique, but we know for sure that it exists). To emphasize the dependence of  $p$  with respect to the parameters, we will write

$$(29) \quad p(t, x) =: p(t, x; \alpha_k, \beta_k, x_{1,k}^0, x_{2,k}^0, \kappa_l, c_l^0, x_{0,l}^0).$$

The same way, to emphasize the dependence of  $P$  with respect to the parameters, we will write

$$(30) \quad P(t, x) =: P(t, x; \alpha_k, \beta_k, x_{1,k}^0, x_{2,k}^0, \kappa_l, c_l^0, x_{0,l}^0).$$

Theorem 1 can be recast as orbital stability of a *multi-breather*:

**Theorem 3.** *Let  $\alpha_k, \beta_k, x_{1,k}^0, x_{2,k}^0, \kappa_l, c_l^0, x_{0,l}^0$  and  $p$  the multi-breather associated to these parameters with notations as in (29) given by [38, Theorem 2]. We assume that (17) holds and  $v_2 > 0$ . There exists  $\eta_0 > 0$  small enough,  $C_0 > 0$  large enough such that the following is true for  $0 < \eta < \eta_0$ .*

*Let  $u(t)$  be a solution of (mKdV), such that*

$$\left\| u(0) - p(0; \alpha_k, \beta_k, x_{1,k}^0, x_{2,k}^0, c_l^0, x_{0,l}^0) \right\|_{H^2} \leq \eta.$$

*Then there exists  $x_{0,l}(t), x_{1,k}(t), x_{2,k}(t)$  defined for any  $t \geq 0$  such that*

$$(31) \quad \forall t \geq 0, \quad \left\| u(t) - p(t; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), c_l^0, x_{0,l}(t)) \right\|_{H^2} \leq C_0 \eta.$$

As a corollary from Theorem 1 and Theorem 3, we can in turn improve the uniqueness result established in [38].

Indeed, in [38], the assumption is  $v_1 > 0$ : we can weaken it to  $v_2 > 0$ .

**Theorem 4.** *Given the same set of solitons and breathers as in Theorem 1 whose velocities satisfy (17) and  $v_2 > 0$  (all objects travel to the right except possibly one), the multi-breather  $p$  associated to  $P$  by [38, Theorem 2], in the sense of [38, Definition 1], is unique.*

In this paper, we adapt the arguments given by Martel, Merle and Tsai [33] to the context of breathers. To do so, it is needed to understand the variational structure of breathers, in the same manner as Weinstein did in [41] for (NLS) and (mKdV) solitons. Such results have been obtained by Alejo and Muñoz in [3]. When a soliton is a critical point of a Lyapunov functional at the  $H^1$  level, whose Hessian is coercive up to two orthogonal conditions; a breather is a critical point of a Lyapunov functional at the  $H^2$  level, whose Hessian is coercive up to three orthogonal conditions. One important issue that we address is to understand the variational structure of a soliton at the  $H^2$  level. We do this by modifying the Lyapunov functional from [3], and we will also adapt it for a sum of solitons and breathers. We need to make assumptions on the velocities of our breathers (recall that the velocity of a soliton is always positive), because several arguments are based on monotonicity properties, which hold only on the right.

**1.3. Organisation of the proof.** The proof of Theorem 1 is based on two results: a modulation lemma and a bootstrap proposition. We give a detailed outline of both results in Section 2, and give in Section 3 the proof of the heart of the argument (Proposition 7), where we complete the bootstrap via a (finite) induction argument on the an improved bound localized on the last  $j$  objects.

In Section 4, we prove Theorems 3 and 4 as quick consequences of Theorem 1.

**1.4. Acknowledgments.** The author would like to thank his supervisor Raphaël Côte for suggesting the idea of the work, for fruitful discussions and his useful advice.

## 2. REDUCTION OF THE PROOF TO AN INDUCTION

For the proof, we assume the assumption of the Theorem 1 true (i.e. we assume (22) true) and the goal is to find the suitable constants  $A, \theta, D_0, a_0$  so that the Theorem 1 holds.

**2.1. Some useful notations.** We set some useful constants for this paper. We define the worst exponential decay rate:

$$(32) \quad \beta := \min\{\beta_k, 1 \leq k \leq K\} \cup \{\sqrt{c_l}, 1 \leq l \leq L\},$$

and the worst distance between two consecutive velocities:

$$(33) \quad \tau := \min\{v_{j+1} - v_j, 1 \leq j \leq J-1\}.$$

We have

$$(34) \quad \forall t \geq 0 \quad x_j(t) - x_{j-1}(t) \geq x_j(0) - x_{j-1}(0) + \tau t.$$

We introduce general parameters, for  $j = 1, \dots, J$ ,  $(a_j, b_j)$ . If  $P_j = B_k$  is a breather, we set  $(a_j, b_j) := (\alpha_k, \beta_k)$ . If  $P_j = R_l$  is a soliton, we set  $(a_j, b_j) := (0, \sqrt{c_l^0})$ .

**2.2. Modulation lemma.** We will first state a standard modulation lemma, which can be proved similarly as the modulation lemma in [38]. We need it because we will construct translations of our objects so that they are near to  $u$  and some orthogonality conditions are satisfied. These orthogonality conditions will allow us to use coercivity of some quadratic forms in the following of the proof.

**Lemma 5.** *Let  $A' > 0$ ,  $\theta > 0$ ,  $t' > 0$  and  $C^1$  functions  $y_{1,k}(t), y_{2,k}(t), y_{3,l}(t), y_{0,l}(t)$  defined for  $t \in [0, t']$  (with  $y_{3,l}(t) > 0$ ) such that  $\forall t \in [0, t'] \quad |y_{3,l}(t) - c_l^0| \leq \min\{\frac{\tau}{8}\} \cup \{\frac{c_l^0}{4}, 1 \leq l \leq L\}$ . If  $D_0$  is large enough and  $a_0$  is small enough (dependently on  $A'$ ), there exists a constant  $C_2 > 1$  such that the following holds. Let  $u(t)$  be a solution of (mKdV) such that for any  $t \in [0, t']$ ,*

$$(35) \quad \left\| u(t) - \sum_{l=1}^L R_{y_{3,l}(t), \kappa_l}(t, \cdot; y_{0,l}(t)) - \sum_{k=1}^K B_{\alpha_k, \beta_k}(t, \cdot; y_{1,k}(t), y_{2,k}(t)) \right\|_{H^2} \leq A' (a + e^{-\theta D}),$$

$$(36) \quad \left\| u(0) - \sum_{l=1}^L R_{y_{3,l}(0), \kappa_l}(0, \cdot; y_{0,l}(0)) - \sum_{k=1}^K B_{\alpha_k, \beta_k}(0, \cdot; y_{1,k}(0), y_{2,k}(0)) \right\|_{H^2} \leq a,$$

and if we set  $y_j(t) := y_{0,l}(t) + y_{3,l}(t)$  if  $P_j = R_l$ , and  $y_j(t) := -y_{2,k}(t) + v_k^b t$  if  $P_j = B_k$ , we have

$$(37) \quad \forall t \in [0, t'] \quad y_j(t) - y_{j-1}(t) \geq D,$$

then, there exists  $C^1$  functions  $z_{1,k}(t), z_{2,k}(t), z_{3,l}(t), z_{0,l}(t)$  defined for  $t \in [0, t']$  (with  $z_{3,l}(t) > 0$ ), such that if we set

$$(38) \quad \varepsilon(t) := u(t) - \tilde{P}(t),$$

where

$$(39) \quad \tilde{R}_l(t, x) := R_{z_{3,l}(t), \kappa_l}(t, x; z_{0,l}(t)) \quad \text{for } 1 \leq l \leq L,$$

$$(40) \quad \tilde{B}_k(t, x) := B_{\alpha_k, \beta_k}(t, x; z_{1,k}(t), z_{2,k}(t)) \quad \text{for } 1 \leq k \leq K,$$

$$(41) \quad \tilde{P}_j := \tilde{R}_l \quad \text{if } P_j = R_l, \quad \tilde{P}_j := \tilde{B}_k \quad \text{if } P_j = B_k,$$

$$(42) \quad \tilde{R} := \sum_{l=1}^L \tilde{R}_l, \quad \tilde{B} := \sum_{k=1}^K \tilde{B}_k, \quad \tilde{P} := \tilde{R} + \tilde{B} = \sum_{j=1}^J \tilde{P}_j,$$

then, for  $t \in [0, t']$ , for  $l = 1, \dots, L$ , for  $k = 1, \dots, K$ ,

$$(43) \quad \int \tilde{R}_l(t) \varepsilon(t) = \int \partial_x \tilde{R}_l(t) \varepsilon(t) = \int \partial_{x_1} \tilde{B}_k(t) \varepsilon(t) = \int \partial_{x_2} \tilde{B}_k(t) \varepsilon(t) = 0.$$

Moreover, for  $t \in [0, t']$ , we have

$$(44) \quad \|\varepsilon(t)\|_{H^2} + |z_{1,k}(t) - y_{1,k}(t)| + |z_{2,k}(t) - y_{2,k}(t)| + |z_{3,l}(t) - y_{3,l}(t)| + |z_{0,l}(t) - y_{0,l}(t)| \leq C_2 A' (a + e^{-\theta D}),$$

and

$$(45) \quad \|\varepsilon(0)\|_{H^2} + |z_{1,k}(0) - y_{1,k}(0)| + |z_{2,k}(0) - y_{2,k}(0)| + |z_{3,l}(0) - y_{3,l}(0)| + |z_{0,l}(0) - y_{0,l}(0)| \leq C_2 a,$$

and for any  $t \in [0, t']$ ,  $(z_{1,k}(t), z_{2,k}(t), z_{3,l}(t), z_{0,l}(t)) \in \mathbb{R}^{2K+2L}$  is unique such that (43) is satisfied and  $(z_{1,k}(t), z_{2,k}(t), z_{3,l}(t), z_{0,l}(t))$  is in a suitable neighbourhood of  $(y_{1,k}(t), y_{2,k}(t), y_{3,l}(t), y_{0,l}(t))$  that depends only on  $A'$  ( $a + e^{-\theta D}$ ).

We set, for  $t \in [0, t']$ ,  $z_j^s(t) := z_{0,l}(t) + z_{3,l}(t)t$  and  $z_j(t) := z_j^s(t)$  if  $P_j = R_l$ , and  $z_k^b(t) := -z_{2,k}(t) + v_k^b t$  and  $z_j(t) := z_k^b(t)$  if  $P_j = B_k$ .

For  $D_0$  large enough and  $a_0$  small enough, if we assume that

$$(46) \quad \forall t \in [0, t'] \quad z_j(t) - z_{j-1}(t) \geq D,$$

(note that (46) is implied by  $\forall t \in [0, t'] \quad y_j(t) - y_{j-1}(t) \geq 2D$ ), then for any  $t \in [0, t']$ , we have that for  $k = 1, \dots, K$ ,

$$(47) \quad |z'_{1,k}(t)| + |z'_{2,k}(t)| \leq C_2 \left( \int e^{-\frac{\beta}{2}|x-z_k^b(t)|} \varepsilon^2 \right)^{1/2} + C_2 e^{-\frac{\beta D}{8}},$$

for  $l = 1, \dots, L$ ,

$$(48) \quad |z'_{3,l}(t)| + |z'_{0,l}(t) + z'_{3,l}(t)t| \leq C_2 \left( \int e^{-\frac{\beta}{2}|x-z_l^s(t)|} \varepsilon^2 \right)^{1/2} + C_2 e^{-\frac{\beta D}{8}}.$$

*Proof.* See [38, Lemma 18], for the proof of a similar result. We also refer to [33, 3].  $\square$

**2.3. Bootstrap.** Given Lemma 5, we reduce the proof of Theorem 1 to the following bootstrap proposition:

**Proposition 6.** *There exists  $A \geq C_2$  and  $\theta > 0$  such that, if  $D_0$  is large enough and  $a_0$  is small enough such that*

$$(49) \quad C_2 a \leq A (a + e^{-\theta D}) \leq \min\left\{\frac{\tau}{8}\right\} \cup \left\{\frac{c_p^0}{4}, 1 \leq p \leq L\right\},$$

for  $t^* > 0$ , we assume that there exist functions  $x_{1,k}(t), x_{2,k}(t), c_l(t), x_{0,l}(t) \in \mathbb{R}$  (with  $c_l(t) > 0$ ) defined for  $t \in [0, t^*]$  such that, if we denote

$$(50) \quad \forall t \in [0, t^*] \quad \varepsilon(t) := u(t) - \tilde{P}(t) := u(t) - \sum_{l=1}^L \tilde{R}_l(t) - \sum_{k=1}^K \tilde{B}_k(t),$$

where  $u$  is a solution of (mKdV),

$$(51) \quad \forall t \in [0, t^*] \quad \tilde{R}_l(t) := R_{c_l(t), \kappa_l}(t, \cdot; x_{0,l}(t)) \quad \text{for } 1 \leq l \leq L,$$

$$(52) \quad \forall t \in [0, t^*] \quad \tilde{B}_k(t) := B_{\alpha_k, \beta_k}(t, \cdot; x_{1,k}(t), x_{2,k}(t)),$$

$\tilde{P}_j := \tilde{R}_l$  if  $P_j = R_l$ ,  $\tilde{P}_j = \tilde{B}_k$  if  $P_j = B_k$ , and  $\tilde{x}_l^s(t) := x_{0,l}(t) + c_l(t)t$ ,  $\tilde{x}_k^b(t) := -x_{2,k}(t) + v_k^b t$ ,  $\tilde{x}_j(t) := \tilde{x}_l^s(t)$

if  $P_j = R_l$ , and  $\tilde{x}_j(t) := \tilde{x}_k^b(t)$  if  $P_j = B_k$ ,

and if we assume that

$$(53) \quad \forall t \in [0, t^*] \quad \|\varepsilon(t)\|_{H^2} \leq A (a + e^{-\theta D}), \quad \|\varepsilon(0)\|_{H^2} \leq C_2 a,$$

$$(54) \quad \forall t \in [0, t^*] \quad |c_l(t) - c_l^0| \leq A (a + e^{-\theta D}), \quad |c_l(0) - c_l^0| \leq C_2 a,$$

$$(55) \quad \forall j \geq 2, \forall t \in [0, t^*] \quad \tilde{x}_j(t) - \tilde{x}_{j-1}(t) \geq D, \quad \tilde{x}_j'(t) - \tilde{x}_{j-1}'(t) \geq \frac{\tau}{2}, \quad \frac{v_2}{2} \leq \tilde{x}_j'(t) \leq 2v_j,$$

$$(56) \quad \forall t \in [0, t^*] \quad \int \tilde{B}_{k1}(t) \varepsilon(t) = \int \tilde{B}_{k2}(t) \varepsilon(t) = \int \tilde{R}_l(t) \varepsilon(t) = \int \tilde{R}_{lx}(t) \varepsilon(t) = 0,$$

then

$$(57) \quad \forall t \in [0, t^*] \quad \|\varepsilon(t)\|_{H^2} \leq \frac{A}{2} (a + e^{-\theta D}),$$

$$(58) \quad \forall t \in [0, t^*] \quad |c_l(t) - c_l^0| \leq \frac{A}{2} (a + e^{-\theta D}).$$



The proof of this proposition will be the goal of the following. The proof of Theorem 1 then follows from a continuity argument.

*Proof of Theorem 1 assuming Proposition 6.* We take  $A, \theta, D_0, a_0$  that work for Proposition 6, we will show that they will also work for Theorem 1. We take  $D \geq D_0$  and  $0 \leq a \leq a_0$  and we assume that (22) is true for a solution  $u$  of (mKdV).

We assume that  $a_0$  is small enough and  $D_0$  is large enough such that  $A(a + e^{-\theta D}) \leq \min\{\frac{\tau}{8}\} \cup \{\frac{c_p^0}{4}, 1 \leq p \leq L\}$ .

Because  $\frac{A}{C_2} > 1$ , by continuity, there exists  $t_1 > 0$  such that

$$(59) \quad \forall t \in [0, t_1] \quad \|u(t) - P(t)\|_{H^2} \leq \frac{A}{C_2} (a + e^{-\theta D}).$$

Of course, we have that

$$(60) \quad \forall t \in [0, t_1] \quad x_j(t) - x_{j-1}(t) \geq 2D + \tau t.$$

We apply Lemma 5 on  $[0, t_1]$  with  $A' = \frac{A}{C_2}$ . We take  $D_0$  larger and  $a_0$  smaller if needed. So, there exist  $C^1$  functions  $x_{1,k}(t), x_{2,k}(t), c_l(t), x_{0,l}(t) \in \mathbb{R}$  with  $c_l(t) > 0$  defined for  $t \in [0, t_1]$ , such that if we set

$$(61) \quad \varepsilon(t) := u(t) - \widetilde{P}(t),$$

with the same notations as usual, we have

$$(62) \quad \forall t \in [0, t_1] \quad \|\varepsilon(t)\|_{H^2} \leq A(a + e^{-\theta D}), \quad \|\varepsilon(0)\|_{H^2} \leq C_2 a,$$

$$(63) \quad \forall t \in [0, t_1] \quad |c_l(t) - c_l^0| \leq A(a + e^{-\theta D}), \quad |c_l(0) - c_l^0| \leq C_2 a,$$

$$(64) \quad \forall t \in [0, t_1] \quad \widetilde{x}_j(t) - \widetilde{x}_{j-1}(t) \geq D,$$

$$(65) \quad \forall t \in [0, t_1] \quad \int \widetilde{B}_{k1}(t)\varepsilon(t) = \int \widetilde{B}_{k2}(t)\varepsilon(t) = \int \widetilde{R}_l(t)\varepsilon(t) = \int \widetilde{R}_{lx}(t)\varepsilon(t) = 0.$$

In order to minorate  $\widetilde{x}_j'(t) - \widetilde{x}_{j-1}'(t)$ , we first observe that  $\forall t \in [0, t_1] \quad x_j'(t) - x_{j-1}'(t) \geq \tau$ . Then, we see that  $x_k^b(t) - \widetilde{x}_k^b(t) = x_{2,k}(t) - x_{2,k}^0$ , and so  $(x_k^b)'(t) - \widetilde{x}_k^b'(t) = x_{2,k}'(t)$ , and Lemma 5 implies

$$(66) \quad |x_{2,k}'(t)| \leq C_2 \|\varepsilon(t)\|_{L^2} + C_2 e^{-\frac{\beta D}{8}} \leq C_2 A(a + e^{-\theta D}) + C_2 e^{-\frac{\beta D}{8}},$$

which implies that if we take  $a_0$  smaller and  $D_0$  larger if needed, we may have  $\left| (x_k^b)'(t) - \widetilde{x}_k^b'(t) \right| \leq \frac{\tau}{4}$  or may be bounded above by any other constant. After that, we see that  $x_l^s(t) - \widetilde{x}_l^s(t) = x_{0,l}^0 + v_l^s t - x_{0,l}(t) - c_l(t)t$ , and so  $(x_l^s)'(t) - \widetilde{x}_l^s'(t) = c_l^0 - c_l(t) - x_{0,l}'(t) - c_l'(t)t$ , and Lemma 5 implies

$$(67) \quad |c_l(t) - c_l^0| \leq A(a + e^{-\theta D}),$$

as seen above, and

$$(68) \quad \left| x_{0,l}'(t) + c_l'(t)t \right| \leq C_2 \|\varepsilon(t)\|_{L^2} + C_2 e^{-\frac{\beta D}{8}} \leq C_2 A(a + e^{-\theta D}) + C_2 e^{-\frac{\beta D}{8}},$$

which implies that if we take  $a_0$  smaller and  $D_0$  larger if needed, we may have  $\left| (x_l^s)'(t) - \widetilde{x}_l^s'(t) \right| \leq \frac{\tau}{4}$  or may be bounded above by any other constant. The latter implies that

$$(69) \quad \forall t \in [0, t_1] \quad \widetilde{x}_j'(t) - \widetilde{x}_{j-1}'(t) \geq \frac{\tau}{2},$$

and that for any  $j \geq 2$ ,

$$(70) \quad \frac{v_2}{2} \leq \widetilde{x}_j'(t) \leq 2v_j.$$

Let  $t^*$  be the supremum of  $t' \geq t_1$  such that the  $C^1$  functions  $x_{1,k}(t), x_{2,k}(t), c_l(t), x_{0,l}(t)$  may be extended for  $t \in [0, t']$  and (53), (54), (55) and (56) are still satisfied (with  $t'$  at the place of  $t^*$ ).

We argue by contradiction and suppose that  $t^* < +\infty$ .

By uniqueness in Lemma 5, we find that if we have two extensions on  $[0, t_1]$  and  $[0, t_2]$  with  $t_1 \leq t_2$ , we have that the extension on  $[0, t_2]$  is an extension of the extension on  $[0, t_1]$ .

When (53), (54), (55) and (56) are true for  $x_{1,k}(t), x_{2,k}(t), c_l(t), x_{0,l}(t)$  with  $\widehat{t}$  at the place of  $t^*$  (i.e. on  $[0, \widehat{t}]$ ), we may extend these implicit functions a bit further and have (53), (54), (55) and (56) that are still satisfied, but on the extended interval. We do it in the following way. First, we apply Proposition 6 on  $[0, \widehat{t}]$ , and that makes (53) and (54) a bit improved on  $[0, \widehat{t}]$  and become (57) and (58) on  $[0, \widehat{t}]$ . And so, we choose an extension of the implicit functions so that (53) and (54) are satisfied on the extension of  $[0, \widehat{t}]$ . This is not hard at all for (53) and (54), because the inequalities were improved. It is not hard to extend  $\widetilde{x}_j(t) - \widetilde{x}_{j-1}(t) \geq D$  of (55), because of  $\widetilde{x}_j'(t) - \widetilde{x}_{j-1}'(t) \geq \frac{\tau}{2}$  of (55), which makes the first inequality actually improved. And from Lemma 5, by taking  $a_0$  smaller and  $D_0$  larger if needed (depending only on  $\tau$ ), we may have  $\widetilde{x}_j'(t) - \widetilde{x}_{j-1}'(t) \geq \frac{3\tau}{4}$  for  $t \in [0, \widehat{t}]$ , this is why  $\widetilde{x}_j'(t) - \widetilde{x}_{j-1}'(t) \geq \frac{\tau}{2}$  may be extended. Similarly,  $\frac{v_2}{2} \leq \widetilde{x}_j'(t) \leq 2v_J$  may be extended. After application of Lemma 5 with  $A' = A$ , we see that (56) can be also extended. This is why, we may find an extension of  $[0, \widehat{t}]$  and extensions of  $x_{1,k}(t), x_{2,k}(t), c_l(t), x_{0,l}(t)$  on this extension such that (53), (54), (55) and (56) are satisfied.

But, there is a contradiction with the maximality of  $t^*$ , because after extending the implicit functions a bit further than  $t^*$ , using uniform continuity in parameters at a neighborhood of  $t^*$ , we may imply Lemma 5, which shows that these extensions are such that (53), (54), (55) and (56) are still satisfied.

So, we deduce that we have  $x_{1,k}(t), x_{2,k}(t), c_l(t), x_{0,l}(t)$  defined for any  $t \geq 0$ , such that for any  $t \geq 0$ ,

$$(71) \quad \|\varepsilon(t)\|_{H^2} \leq A(a + e^{-\theta D}), \quad \|\varepsilon(0)\|_{H^2} \leq C_2 a,$$

$$(72) \quad |c_l(t) - c_l^0| \leq A(a + e^{-\theta D}), \quad |c_l(0) - c_l^0| \leq C_2 a,$$

$$(73) \quad \forall j \geq 2 \quad \widetilde{x}_j(t) - \widetilde{x}_{j-1}(t) \geq D, \quad \widetilde{x}_j'(t) - \widetilde{x}_{j-1}'(t) \geq \frac{\tau}{2}, \quad \frac{v_2}{2} \leq \widetilde{x}_j'(t) \leq 2v_J,$$

$$(74) \quad \int \widetilde{B}_{k1}(t)\varepsilon(t) = \int \widetilde{B}_{k2}(t)\varepsilon(t) = \int \widetilde{R}_l(t)\varepsilon(t) = \int \widetilde{R}_{lx}(t)\varepsilon(t) = 0.$$

And we set

$$(75) \quad \varepsilon(t) := u(t) - \sum_{l=1}^{L'} R_{c_l^0, \kappa_l}(t, \cdot; x_{0,l}(t)) + (c_l(t) - c_l^0)t - \sum_{k=1}^K B_{\alpha_k, \beta_k}(t, \cdot; x_{1,k}(t), x_{2,k}(t)).$$

We need to bound  $\varepsilon$  to finish the proof. We have, for  $t \geq 0$ , and we bound the norm between two ground states centered at a same point,

$$(76) \quad \begin{aligned} \|\varepsilon(t)\|_{H^2} &\leq \|\varepsilon(t)\|_{H^2} + \sum_{l=1}^L \left\| R_{c_l(t), \kappa_l}(t, \cdot; x_{0,l}(t)) - R_{c_l^0, \kappa_l}(t, \cdot; x_{0,l}(t) + (c_l(t) - c_l^0)t) \right\|_{H^2} \\ &\leq \|\varepsilon(t)\|_{H^2} + C \sum_{l=1}^L |c_l(t) - c_l^0| \\ &\leq CA(a + e^{-\theta D}), \end{aligned}$$

and this is exactly what we wanted to prove. Moreover (24) is a straightforward consequence of Lemma 5.  $\square$

Hence, we are left to prove Proposition 6.

**2.4. Proof by induction.** We will prove Proposition 6 by induction. More precisely, we want to find  $A > C_2, \theta, D_0, a_0$  so that the Proposition 6 holds for any  $t^* > 0$ .

We assume that there exists functions  $x_{1,k}(t), x_{2,k}(t), c_l(t), x_{0,l}(t) \in \mathbb{R}$  defined for  $t \in [0, t^*]$  such that, with notations of Proposition 6,

$$(77) \quad \forall t \in [0, t^*] \quad \|\varepsilon(t)\|_{H^2} \leq A(a + e^{-\theta D}), \quad \|\varepsilon(0)\|_{H^2} \leq C_2 a,$$

$$(78) \quad \forall t \in [0, t^*] \quad |c_l(t) - c_l^0| \leq A(a + e^{-\theta D}) \leq \min \left\{ \frac{\tau}{8} \right\} \cup \left\{ \frac{c_p^0}{4}, 0 \leq p \leq L \right\}, \quad |c_l(0) - c_l^0| \leq C_2 a,$$

$$(79) \quad \forall t \in [0, t^*] \quad \tilde{x}_j(t) - \widetilde{x_{j-1}}(t) \geq D, \quad \tilde{x}_j'(t) - \widetilde{x_{j-1}}'(t) \geq \frac{\tau}{2},$$

$$(80) \quad \forall j \geq 2, \quad \forall t \in [0, t^*] \quad \frac{v_2}{2} \leq \tilde{x}_j'(t) \leq 2v_j,$$

$$(81) \quad \forall t \in [0, t^*] \quad \int \widetilde{B_{k1}}(t) \varepsilon(t) = \int \widetilde{B_{k2}}(t) \varepsilon(t) = \int \widetilde{R_l}(t) \varepsilon(t) = \int \widetilde{R_{lx}}(t) \varepsilon(t) = 0.$$

And the goal is to improve inequalities (77) and (78).

To improve them by induction, we introduce a cut-off function.

Let  $\sigma > 0$  be a constant small enough for which the conditions will be fixed in the following of the proof.

We denote:

$$(82) \quad \Psi(x) := \frac{2}{\pi} \arctan \left( \exp \left( \sqrt{\sigma} x / 2 \right) \right).$$

By direct calculations,

$$(83) \quad \Psi'(x) = \frac{\sqrt{\sigma}}{2\pi \cosh(\sqrt{\sigma} x / 2)},$$

and so,

$$(84) \quad |\Psi'(x)| \leq C \exp \left( -\sqrt{\sigma} |x| / 2 \right).$$

We have the following properties:  $\lim_{+\infty} \Psi = 1$ ,  $\lim_{-\infty} \Psi = 0$ , for all  $x \in \mathbb{R}$   $\Psi(-x) + \Psi(x) = 1$ ,  $\Psi'(x) > 0$ ,  $|\Psi'''(x)| \leq \frac{\sqrt{\sigma}}{2} |\Psi'(x)|$ ,  $|\Psi''''(x)| \leq \frac{\sqrt{\sigma}}{2} |\Psi'''(x)|$ ,  $|\Psi'(x)| \leq \frac{\sqrt{\sigma}}{2} \Psi$  and  $|\Psi'(x)| \leq \frac{\sqrt{\sigma}}{2} (1 - \Psi)$ .

We define the average between positions of two consecutive objects. For  $j = 3, \dots, J$ , we set

$$(85) \quad \forall t \in [0, t^*] \quad m_j(t) := \frac{\widetilde{x_{j-1}}(t) + \tilde{x}_j(t)}{2},$$

and we set

$$(86) \quad \forall t \in [0, t^*] \quad m_2(t) := \frac{\widetilde{x_1}(0) + \widetilde{x_2}(0)}{2} + \int_0^t \max \left( \frac{\widetilde{x_1}'(s) + \widetilde{x_2}'(s)}{2}, \frac{\widetilde{x_2}'(s)}{2} \right) ds.$$

By (79), for  $j \geq 3$ ,

$$(87) \quad \forall t \in [0, t^*] \quad \tilde{x}_j'(t) - m_j'(t) \geq \frac{\tau}{4}, \quad m_j'(t) - \widetilde{x_{j-1}}'(t) \geq \frac{\tau}{4},$$

and for  $j = 2$ , we have

$$(88) \quad \forall t \in [0, t^*] \quad m_2'(t) = \max \left( \frac{\widetilde{x_1}'(t) + \widetilde{x_2}'(t)}{2}, \frac{\widetilde{x_2}'(t)}{2} \right),$$

and so,

$$(89) \quad \forall t \in [0, t^*] \quad m_2'(t) - \widetilde{x_1}'(t) \geq \frac{\tau}{4}, \quad \widetilde{x_2}'(t) - m_2'(t) \geq \min \left( \frac{v_2}{4}, \frac{\tau}{4} \right).$$

This is why, we set

$$(90) \quad \zeta := \min \left( \frac{v_2}{4}, \frac{\tau}{4} \right),$$

a constant that depends only on problem data, and so for any  $j$ ,

$$(91) \quad \forall t \in [0, t^*] \quad \tilde{x}_j'(t) - m_j'(t) \geq \zeta, \quad m_j'(t) - \widetilde{x_{j-1}}'(t) \geq \zeta.$$

The latter implies that  $\forall j \geq 2 \quad \forall t \in [0, t^*] \quad \widetilde{x_{j-1}}(t) < m_j(t) < \tilde{x}_j(t)$ , and we may deduce by integration and by (79) and (91) that

$$\tilde{x}_j(t) - m_j(t) = \tilde{x}_j(0) - m_j(0) + \int_0^t \left( \tilde{x}_j'(s) - m_j'(s) \right) ds$$

$$\begin{aligned}
 &= \frac{\widetilde{x}_j(0) - \widetilde{x}_{j-1}(0)}{2} + \int_0^t \left( \widetilde{x}_j'(s) - m_j'(s) \right) ds \\
 (92) \quad &\geq \frac{D}{2} + \zeta t,
 \end{aligned}$$

and similarly,

$$(93) \quad m_j(t) - \widetilde{x}_{j-1}(t) \geq \frac{D}{2} + \zeta t.$$

We have that (the  $m_j$  are chosen for that) for any  $j \geq 2$ ,

$$(94) \quad \forall t \in [0, t^*] \quad 2v_j \geq \widetilde{x}_j'(t) \geq m_j'(t) \geq \frac{v_2}{4} \geq \zeta.$$

From this, we define for  $j = 2, \dots, J$ ,

$$(95) \quad \Phi_j(t, x) := \Psi(x - m_j(t)).$$

We have:

$$(96) \quad (\Phi_j)_t = -m_j'(\Phi_j)_x.$$

We may extend this definition to  $j = 1$  and  $j = J + 1$  in the following way:  $\Phi_1 := 1$  and  $\Phi_{J+1} := 0$ .

In order to prove Proposition 6 by induction, we will find an increasing sequence  $(Z_j)_{j=1, \dots, J+1}$  such that  $Z_1 := 2$  and  $Z_{J+1} := +\infty$  and such that we will be able to prove the following proposition for any  $j = 1, \dots, J$ . The goal is to obtain inequalities of Proposition 6 with better constants. So, in order to achieve this, we do the following induction: we suppose that localized inequalities around  $P_{j+1}, \dots, P_J$  are obtained with strongly improved constants (constants divided by  $Z_{j+1}$ ), and we deduce from them localized inequalities around  $P_j, \dots, P_1$  with improved constants, but a little bit less improved than earlier (constants divided by  $Z_j$ ). We will also assume the bootstrap assumption. This induction is sufficient, because it starts from an assumption on an empty set of objects and ends with a conclusion with inequalities localized around all the objects, i.e. global (not localized at all).

**Proposition 7.** *Assuming that*

$$(97) \quad \forall t \in [0, t^*] \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_{j+1} \leq \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

and for any  $j' \geq j + 1$  such that  $P_{j'} = R_l$  is a soliton,

$$(98) \quad \forall t \in [0, t^*] \quad |c_l(t) - c_l(0)| \leq \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

and for any  $j' \geq j + 1$  such that  $P_{j'}$  is a breather,

$$(99) \quad \forall t \in [0, t^*] \quad \left| \int \widetilde{P}_{j'} \varepsilon(t) - \int \widetilde{P}_{j'} \varepsilon(0) \right| \leq \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

and

$$(100) \quad \forall t \in [0, t^*] \quad \left| \int \left[ \widetilde{P}_{j'}'' + \widetilde{P}_{j'}^3 \right] \varepsilon(t) - \int \left[ \widetilde{P}_{j'}'' + \widetilde{P}_{j'}^3 \right] \varepsilon(0) \right| \leq \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

we have that

$$(101) \quad \forall t \in [0, t^*] \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq \left( \frac{A}{Z_j} \right)^2 (a^2 + e^{-2\theta D}),$$

and if  $P_j = R_l$  is a soliton,

$$(102) \quad \forall t \in [0, t^*] \quad |c_l(t) - c_l(0)| \leq \left( \frac{A}{Z_j} \right)^2 (a^2 + e^{-2\theta D}),$$

and if  $P_j$  is a breather,

$$(103) \quad \forall t \in [0, t^*] \quad \left| \int \widetilde{P}_j \varepsilon(t) - \int \widetilde{P}_j \varepsilon(0) \right| \leq \left( \frac{A}{Z_j} \right)^2 (a^2 + e^{-2\theta D}),$$

and

$$(104) \quad \forall t \in [0, t^*] \quad \left| \int [\tilde{P}_{jxx} + \tilde{P}_j^3] \varepsilon(t) - \int [\tilde{P}_{jxx} + \tilde{P}_j^3] \varepsilon(0) \right| \leq \left( \frac{A}{Z_j} \right)^2 (a^2 + e^{-2\theta D}).$$

*Remark 8.* For  $j = 1, \dots, J$ , we may denote by  $\mathcal{P}_j$  the following assertion: (101), (102), (103) and (104). The Proposition 7 may be reformulated in the following way:

There exists an increasing sequence  $(Z_j)_{j=1, \dots, J+1}$  with  $Z_1 = 2$  and  $Z_{J+1} = +\infty$ ,  $A$  large enough and  $\theta > 0$  such that for  $D_0$  large enough and for  $a_0$  small enough, we have the following: for any  $j = 1, \dots, J$ ,

$$(105) \quad \mathcal{P}_{j+1} \implies \mathcal{P}_j.$$

*Remark 9.* Note that the inequalities (103) and (104) imply the following inequality for any  $t \in [0, t^*]$  in the case when  $P_j$  is a breather:

$$(106) \quad \left| \int \left[ \tilde{P}_{jxxxx} + 5\tilde{P}_j \tilde{P}_{jx}^2 + 5\tilde{P}_j^2 \tilde{P}_{jxx} + \frac{3}{2} \tilde{P}_j^5 \right] \varepsilon(t) - \int \left[ \tilde{P}_{jxxxx} + 5\tilde{P}_j \tilde{P}_{jx}^2 + 5\tilde{P}_j^2 \tilde{P}_{jxx} + \frac{3}{2} \tilde{P}_j^5 \right] \varepsilon(0) \right| \leq C \left( \frac{A}{Z_j} \right)^2 (a^2 + e^{-2\theta D}),$$

because of the elliptic equation verified by  $\tilde{P}_j$ , which, in the case when  $\tilde{P}_j = \tilde{B}_k$  is a breather is the following:

$$(107) \quad \tilde{B}_{kxxxx} + 5\tilde{B}_k \tilde{B}_{kx}^2 + 5\tilde{B}_k^2 \tilde{B}_{kxx} + \frac{3}{2} \tilde{B}_k^5 - 2(\beta_k^2 - \alpha_k^2) (\tilde{B}_{kxx} + \tilde{B}_k^3) + (\alpha_k^2 + \beta_k^2) \tilde{B}_k = 0.$$

But the inequality (106) is also true in the case when  $\tilde{P}_j = \tilde{R}_l$  is a soliton, because we have even better in this case. There are two elliptic equations:

$$(108) \quad \tilde{R}_{lxxxx} + 5\tilde{R}_l \tilde{R}_{lx}^2 + 5\tilde{R}_l^2 \tilde{R}_{lxx} + \frac{3}{2} \tilde{R}_l^5 - 2c_l(t) (\tilde{R}_{lxx} + \tilde{R}_l^3) + c_l(t)^2 \tilde{R}_l = 0,$$

and

$$(109) \quad (\tilde{R}_{lxx} + \tilde{R}_l^3) - c_l(t) \tilde{R}_l = 0,$$

which implies that (81) implies

$$(110) \quad \int (\tilde{R}_{lxx} + \tilde{R}_l^3) \varepsilon = 0,$$

and

$$(111) \quad \int \left( \tilde{R}_{lxxxx} + 5\tilde{R}_l \tilde{R}_{lx}^2 + 5\tilde{R}_l^2 \tilde{R}_{lxx} + \frac{3}{2} \tilde{R}_l^5 \right) \varepsilon = 0,$$

which implies of course the inequality (106).

The proof of Proposition 7 will be the goal of the Section 3. The proof of Proposition 6 follows from it.

*Proof of Proposition 6 assuming Proposition 7.* We perform the induction in the decreasing order:  $j = J, J-1, \dots, 2, 1$ .  $\mathcal{P}_{J+1}$  is empty, and Proposition 7 gives the (decreasing) induction step. Hence,  $\mathcal{P}_1, \dots, \mathcal{P}_J$  are true. Due to  $\mathcal{P}_1$ :

$$(112) \quad \forall t \in [0, t^*] \quad \|\varepsilon(t)\|_{H^2} \leq \frac{A}{2} (a + e^{-\theta D}).$$

For  $l = 1, \dots, L$ , we have from  $\mathcal{P}_l$  that for any  $t \in [0, t^*]$ ,

$$(113) \quad \begin{aligned} |c_l(t) - c_l^0| &\leq |c_l(t) - c_l(0)| + |c_l(0) - c_l^0| \\ &\leq \left( \frac{A}{2} \right)^2 (a^2 + e^{-2\theta D}) + C_2 a \\ &\leq \left[ \left( \frac{A}{2} \right)^2 (a + e^{-\theta D}) + C_2 \right] (a + e^{-\theta D}). \end{aligned}$$

If we take  $A$  large enough with respect to  $C_2$ , and  $a_0$  smaller and  $D_0$  larger if needed with respect to  $A$  and  $\theta$ , then

$$(114) \quad \left(\frac{A}{2}\right)^2 (a + e^{-\theta D}) + C_2 \leq \frac{A}{2},$$

and that concludes the proof of Proposition 6.  $\square$

Hence, we are left to prove Proposition 7. We will write the proof for a fixed  $j \in \{1, \dots, J\}$ . We assume  $\mathcal{P}_{j+1}$  with a set of constants  $Z_1, \dots, Z_{J+1}, A, \theta, D_0, a_0$ . We will establish some conditions for these constants during the proof.

### 3. ORBITAL STABILITY OF A SUM OF SOLITONS AND BREATHERS IN $H^2(\mathbb{R})$

In this Section, we prove Proposition 7. We assume  $\mathcal{P}_{j+1}$  and we prove  $\mathcal{P}_j$ .

**3.1. Almost conservation for conservation laws at the right.** We localize around the most right objects, starting from and including the  $j$ -th. We set:

$$(115) \quad M_j(t) := \frac{1}{2} \int u^2(t) \Phi_j(t) =: M_j[u](t),$$

$$(116) \quad E_j(t) := \int \left[ \frac{1}{2} u_x^2 - \frac{1}{4} u^4 \right] \Phi_j(t) =: E_j[u](t),$$

$$(117) \quad F_j(t) := \int \left[ \frac{1}{2} u_{xx}^2 - \frac{5}{2} u^2 u_x^2 + \frac{1}{4} u^6 \right] \Phi_j(t) =: F_j[u](t).$$

**Lemma 10.** *Let  $0 < \omega_1, \omega_2 < 1$ . If  $0 < \sigma < \zeta$ ,  $0 < \theta < \frac{\sqrt{\sigma}}{16}$ ,  $D_0$  is large enough and  $a_0$  is small enough (depending on  $A, \theta, \omega_1$  and  $\omega_2$ ), then for any  $t \in [0, t^*]$ ,*

$$(118) \quad M_j(t) - M_j(0) \leq C e^{-2\theta D},$$

$$(119) \quad (E_j(t) + \omega_1 M_j(t)) - (E_j(0) + \omega_1 M_j(0)) \leq C e^{-2\theta D},$$

$$(120) \quad (F_j(t) + \omega_2 M_j(t)) - (F_j(0) + \omega_2 M_j(0)) \leq C e^{-2\theta D}.$$

*Remark 11.* If  $j = 1$ , we have  $= 0$  at the place of  $\leq C e^{-2\theta D}$ , we will need it in the following of the proof.

*Proof.* If  $j = 1$ , we have exact conservation laws, so this Lemma is obvious. We assume that  $j \geq 2$  for the following of this proof. From Appendix and minoration of  $m'_j$ ,

$$(121) \quad \begin{aligned} \frac{d}{dt} M_j(t) &= \int \left[ -\frac{3}{2} u_x^2 + \frac{3}{4} u^4 \right] \Phi_{jx} + \frac{1}{2} \int u^2 \Phi_{jxxx} + \frac{1}{2} \int u^2 \Phi_{jt} \\ &= \int \left[ -\frac{3}{2} u_x^2 + \frac{3}{4} u^4 \right] \Phi_{jx} + \frac{1}{2} \int u^2 \Phi_{jxxx} - \frac{1}{2} m'_j \int u^2 \Phi_{jx} \\ &\leq \int \left[ -\frac{3}{2} u_x^2 + \frac{3}{4} u^4 - \frac{1}{2} \sigma u^2 \right] \Phi_{jx} + \frac{1}{2} \int u^2 \Phi_{jxxx}. \end{aligned}$$

Now, we use that  $|\Phi_{jxxx}| \leq \frac{\sigma}{4} \Phi_{jx}$ , and we obtain that

$$(122) \quad 2 \frac{d}{dt} M_j(t) \leq - \int \left[ -3u_x^2 + \frac{3\sigma}{4} u^2 - \frac{3}{2} u^4 \right] \Phi_{jx}.$$

Now, from Appendix, we know that for  $r > 0$ , if  $t, x$  satisfy  $\widetilde{x}_{j-1}(t) + r < x < \widetilde{x}_j(t) - r$ , then  $|\widetilde{P}(t, x)| \leq C e^{-\beta r}$ . And so, for  $t, x$  such that  $\widetilde{x}_{j-1}(t) + r < x < \widetilde{x}_j(t) - r$ , by Sobolev embedding,

$$(123) \quad \begin{aligned} |u(t, x)| &\leq \left| \widetilde{P}(t, x) \right| + C \|\varepsilon(t)\|_{H^2} \\ &\leq C e^{-\beta r} + CA (a + e^{-\theta D}). \end{aligned}$$

From that, we can deduce that for  $r$  large enough,  $a_0$  small enough and  $D_0$  large enough, for  $x \in [\widetilde{x}_{j-1}(t) + r, \widetilde{x}_j(t) - r]$ , we can obtain that  $|u(t, x)|$  is bounded by any chosen constant. Here, we will use that to bound  $\frac{3}{2}u^2$  by  $\frac{\sigma}{4}$ .

For  $t, x$  such that  $x < \widetilde{x}_{j-1}(t) + r$  or  $x > \widetilde{x}_j(t) - r$ :

$$\begin{aligned}
|\Phi_{jx}(t, x)| &\leq C \exp\left(-\frac{\sqrt{\sigma}}{2}|x - m_j(t)|\right) \\
&\leq C \exp\left(-\frac{\sqrt{\sigma}}{2}\left(\frac{D}{2} + \zeta t - r\right)\right) \\
(124) \quad &\leq C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{4}D\right) \exp\left(\frac{\sqrt{\sigma}}{2}r\right),
\end{aligned}$$

and so, if we choose  $D_0$  large enough (more precisely,  $D_0 \geq 4r$ ), we obtain for  $x \notin [\widetilde{x}_{j-1}(t) + r, \widetilde{x}_j(t) - r]$ :

$$(125) \quad |\Phi_{jx}(t, x)| \leq C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right).$$

Because  $\int u^4$  is bounded by a constant for any time (that depends only on problem data), we deduce that:

$$\begin{aligned}
\frac{d}{dt}M_j(t) &\leq -\int\left(\frac{3}{2}u_x^2 + \frac{\sigma}{4}u^2\right)\Phi_{jx}(t) + C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right) \\
(126) \quad &\leq C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right).
\end{aligned}$$

We deduce what we want to prove by integration.

For the second inequality, the argument is similar. We start by using Appendix:

$$\begin{aligned}
\frac{d}{dt}E_j(t) &= \int\left[-\frac{1}{2}(u_{xx} + u^3)^2 - u_{xx}^2 + 3u^2u_x^2\right]\Phi_{jx} + \frac{1}{2}\int u_x^2\Phi_{jxxx} - m'_j \int\left[\frac{1}{2}u_x^2 - \frac{1}{4}u^4\right]\Phi_{jx} \\
(127) \quad &\leq -\int\left[\frac{1}{2}(u_{xx} + u^3)^2 + u_{xx}^2 - 3u^2u_x^2 + \frac{\sigma}{2}u_x^2 - \frac{v_j}{2}u^4\right]\Phi_{jx} + \frac{1}{2}\int u_x^2\Phi_{jxxx},
\end{aligned}$$

and again, because  $|\Phi_{jxxx}| \leq \frac{\sigma}{4}\Phi_{jx}$ , we obtain that

$$(128) \quad 2\frac{d}{dt}E_j(t) \leq -\int\left[(u_{xx} + u^3)^2 + 2u_{xx}^2 - 6u^2u_x^2 + \frac{3\sigma}{4}u_x^2 - v_ju^4\right]\Phi_{jx},$$

and by doing a similar argument as for the mass, but to bound  $6u^2$  by  $\frac{\sigma}{4}$  and to bound  $v_ju^2$  by  $\omega_3 > 0$ , a constant as small as we need, we obtain

$$\begin{aligned}
\frac{d}{dt}E_j(t) &\leq -\int\left[\frac{1}{2}(u_{xx} + u^3)^2 + u_{xx}^2 + \frac{\sigma}{4}u_x^2 - \frac{\omega_3}{2}u^2\right]\Phi_{jx} + C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right) \\
(129) \quad &\leq \frac{\omega_3}{2}\int u^2\Phi_{jx} + C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right),
\end{aligned}$$

and so, if we choose  $\omega_3$  such that  $\frac{\omega_3}{2} \leq \omega_1\frac{\sigma}{4}$ ,

$$\begin{aligned}
\frac{d}{dt}(E_j(t) + \omega_1M_j(t)) &\leq \frac{\omega_3}{2}\int u^2\Phi_{jx} + C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right) - \frac{\omega_1\sigma}{4}\int u^2\Phi_{jx} \\
(130) \quad &\leq C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right),
\end{aligned}$$

and we deduce what we want to prove by integration.

For the third inequality, the argument is similar. We start by using Appendix:

$$\frac{d}{dt}F_j(t) = \int\left(-\frac{3}{2}u_{xxx}^2 + 9u_{xx}^2u^2 + 15u_x^2uu_{xx} + \frac{9}{16}u^8 + \frac{1}{4}u_x^4 + \frac{3}{2}u_{xx}u^5 - \frac{45}{4}u^4u_x^2\right)\Phi_{jx}$$

$$\begin{aligned}
 & + 5 \int u^2 u_x u_{xx} \Phi_{jxx} + \frac{1}{2} \int u_{xx}^2 \Phi_{jxxx} - m'_j \int \left[ \frac{1}{2} u_{xx}^2 - \frac{5}{2} u^2 u_x^2 + \frac{1}{4} u^6 \right] \Phi_{jx} \\
 & \leq - \int \left( \frac{3}{2} u_{xxx}^2 - 9 u_{xx}^2 u^2 - 15 u_x^2 u u_{xx} - \frac{9}{16} u^8 - \frac{1}{4} u_x^4 - \frac{3}{2} u_{xx} u^5 + \frac{45}{4} u^4 u_x^2 + \frac{\sigma}{2} u_{xx}^2 - 5 v_J u^2 u_x^2 + \frac{\sigma}{4} u^6 \right) \Phi_{jx} \\
 (131) \quad & + 5 \int u^2 u_x u_{xx} \Phi_{jxx} + \frac{1}{2} \int u_{xx}^2 \Phi_{jxxx},
 \end{aligned}$$

and again, because  $|\Phi_{jxx}| \leq \frac{\sqrt{\sigma}}{2} \Phi_{jx}$  and  $|\Phi_{jxxx}| \leq \frac{\sigma}{4} \Phi_{jx}$ , we obtain that

$$\begin{aligned}
 2 \frac{d}{dt} F_j(t) & \leq - \int \left( 3 u_{xxx}^2 - \left( 18 + \frac{5}{2} \sqrt{\sigma} \right) u_{xx}^2 u^2 - 30 u_x^2 u u_{xx} - \frac{9}{8} u^8 - \frac{1}{2} u_x^4 - 3 u_{xx} u^5 \right. \\
 (132) \quad & \left. + \frac{45}{2} u^4 u_x^2 + \frac{3\sigma}{4} u_{xx}^2 - \left( 10 v_J + \frac{5}{2} \sqrt{\sigma} \right) u^2 u_x^2 + \frac{\sigma}{2} u^6 \right) \Phi_{jx},
 \end{aligned}$$

and by doing a similar argument as for the mass, but to bound  $(18 + \frac{5}{2} \sqrt{\sigma}) u^2$  by  $\frac{\sigma}{4}$ , to bound  $30 u u_{xx}$  by  $\omega_4$ , to bound  $\frac{9}{8} u^6$  by  $\omega_5$ , to bound  $\frac{1}{2} u_x^2$  by  $\omega_4$ , to bound  $3 u_{xx} u^3$  by  $\omega_5$  and to bound  $(10 v_J + \frac{5}{2} \sqrt{\sigma}) u^2$  by  $\omega_4$ , where  $\omega_3, \omega_4 > 0$  are constants that we can choose as small as we want. And we obtain

$$\begin{aligned}
 \frac{d}{dt} F_j(t) & \leq - \int \left[ \frac{3}{2} u_{xxx}^2 + \frac{45}{4} u^4 u_x^2 + \frac{\sigma}{4} u^6 + \frac{\sigma}{4} u_{xx}^2 - C \omega_4 u_x^2 - C \omega_5 u^2 \right] \Phi_{jx} + C \exp \left( -\frac{\sqrt{\sigma} \zeta}{2} t \right) \exp \left( -\frac{\sqrt{\sigma}}{8} D \right) \\
 (133) \quad & \leq C \omega_4 \int u_x^2 \Phi_{jx} + C \omega_5 \int u^2 \Phi_{jx} + C \exp \left( -\frac{\sqrt{\sigma} \zeta}{2} t \right) \exp \left( -\frac{\sqrt{\sigma}}{8} D \right),
 \end{aligned}$$

and so, if we choose  $\omega_4, \omega_5$  such that  $C \omega_4 \leq \omega_1 \frac{3}{2}$  and  $C \omega_5 \leq \omega_1 \frac{\sigma}{4}$ ,

$$(134) \quad \frac{d}{dt} (F_j(t) + \omega_2 M_j(t)) \leq C \exp \left( -\frac{\sqrt{\sigma} \zeta}{2} t \right) \exp \left( -\frac{\sqrt{\sigma}}{8} D \right),$$

and we deduce what we want to prove by integration.  $\square$

**3.2. Quadratic approximation for conservation laws at the right.** We can write the following Taylor expansions with  $u = \tilde{P} + \varepsilon$  for any  $t \in [0, t^*]$ :

$$(135) \quad M_j(t) - M_j[\tilde{P}](t) - \int \tilde{P} \varepsilon \Phi_j - \frac{1}{2} \int \varepsilon^2 \Phi_j = 0,$$

$$(136) \quad \left| E_j(t) - E_j[\tilde{P}](t) - \int [\tilde{P}_x \varepsilon_x - \tilde{P}^3 \varepsilon] \Phi_j - \int \left[ \frac{1}{2} \varepsilon_x^2 - \frac{3}{2} \tilde{P}^2 \varepsilon^2 \right] \Phi_j \right| \leq CA (a + e^{-\theta D}) \int \varepsilon^2 \Phi_j,$$

$$\begin{aligned}
 (137) \quad & \left| F_j(t) - F_j[\tilde{P}](t) - \int \left[ \tilde{P}_{xx} \varepsilon_{xx} - 5 \tilde{P} \tilde{P}_x^2 \varepsilon - 5 \tilde{P}^2 \tilde{P}_x \varepsilon_x + \frac{3}{2} \tilde{P}^5 \varepsilon \right] \Phi_j \right. \\
 & \left. - \int \left[ \frac{1}{2} \varepsilon_{xx}^2 - \frac{5}{2} \tilde{P}_x^2 \varepsilon^2 - 10 \tilde{P} \tilde{P}_x \varepsilon \varepsilon_x - \frac{5}{2} \tilde{P}^2 \varepsilon_x^2 + \frac{15}{4} \tilde{P}^4 \varepsilon^2 \right] \Phi_j \right| \leq CA (a + e^{-\theta D}) \int (\varepsilon^2 + \varepsilon_x^2) \Phi_j.
 \end{aligned}$$

Now, we want to simplify each term of the Taylor expansion.

**3.2.1. Constant terms of the Taylor expansion.** We obtain the following lemma dealing with variations of the constant parts of each Taylor expansion of conservation laws at the right. We reduce each variation to the variation of each conservation law of  $\tilde{P}_j$ . Note that if  $P_j$  is a breather, the variation of any conservation law of  $\tilde{P}_j$  is 0. But, if  $P_j = R_l$ , we may express in the following way  $M[\tilde{R}_l]$ ,  $E[\tilde{R}_l]$  and  $F[\tilde{R}_l]$  with respect to  $Q$ , the ground state of parameter  $c = 1$  (the basic ground state):

$$(138) \quad M[\tilde{R}_l](t) = c_l(t)^{1/2} M[Q],$$

$$(139) \quad E[\tilde{R}_l](t) = c_l(t)^{3/2} E[Q],$$



$$(140) \quad F[\tilde{R}_l](t) = c_l(t)^{5/2} F[Q].$$

**Lemma 12.** *If  $\sigma < 4\beta^2$ ,  $\theta < \min\left(\frac{\beta}{4}, \frac{\sqrt{\sigma}}{8}\right)$ , then for any  $t \in [0, t^*]$ ,*

$$(141) \quad \left| M_j[\tilde{P}](t) - M_j[\tilde{P}](0) - \left( M[\tilde{P}_j](t) - M[\tilde{P}_j](0) \right) \right| \leq C e^{-2\theta D} + C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

$$(142) \quad \left| E_j[\tilde{P}](t) - E_j[\tilde{P}](0) - \left( E[\tilde{P}_j](t) - E[\tilde{P}_j](0) \right) \right| \leq C e^{-2\theta D} + C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

$$(143) \quad \left| F_j[\tilde{P}](t) - F_j[\tilde{P}](0) - \left( F[\tilde{P}_j](t) - F[\tilde{P}_j](0) \right) \right| \leq C e^{-2\theta D} + C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}).$$

*Remark 13.* In the case when  $P_j$  is a breather, we have that

$$(144) \quad M[\tilde{P}_j](t) - M[\tilde{P}_j](0) = E[\tilde{P}_j](t) - E[\tilde{P}_j](0) = F[\tilde{P}_j](t) - F[\tilde{P}_j](0) = 0.$$

It is not true in the case when  $P_j$  is a soliton.

*Proof.* When we develop using  $\tilde{P} = \sum_{i=1}^J \tilde{P}_i$ , we obtain terms with  $\tilde{P}_i \tilde{P}_j$  with  $i \neq j$  and the other terms that have all the same index. For the first type of terms, it is enough to bound  $\int \tilde{P}_i \tilde{P}_j$  for  $i \neq j$ :

$$(145) \quad \left| \int \tilde{P}_i \tilde{P}_j \right| \leq C e^{-\frac{\beta D}{2}}.$$

Now, we look on the terms with the same index, for example  $\int \tilde{P}_i^2 \Phi_j$ . We will distinguish several cases. If  $i < j$ ,

$$(146) \quad \begin{aligned} \int \tilde{P}_i^2 \Phi_j &\leq C \int e^{-2\beta|x-\tilde{x}_i(t)|} e^{\frac{\sqrt{\sigma}}{2}(x-m_j(t))} dx \\ &= C \int_{-\infty}^{\tilde{x}_i(t)} e^{(2\beta+\sqrt{\sigma}/2)x-2\beta\tilde{x}_i(t)-\frac{\sqrt{\sigma}}{2}m_j(t)} dx + C \int_{\tilde{x}_i(t)}^{+\infty} e^{(-2\beta+\sqrt{\sigma}/2)x+2\beta\tilde{x}_i(t)-\frac{\sqrt{\sigma}}{2}m_j(t)} dx \\ &\leq \frac{C}{\sqrt{\sigma}} e^{\frac{\sqrt{\sigma}}{2}(\tilde{x}_i(t)-m_j(t))} \\ &\leq C e^{-\frac{\sqrt{\sigma}D}{4}}. \end{aligned}$$

For the same reason and properties of  $\Psi$ , for  $i \geq j$ ,

$$(147) \quad \int \tilde{P}_i^2 (1 - \Phi_j) \leq C e^{-\frac{\sqrt{\sigma}D}{4}}.$$

For  $i > j$ , we may use  $\mathcal{P}_i$ ; and for  $i = j$ , we cannot. So, if for  $i \geq j+1$ ,  $\tilde{P}_i = \tilde{R}_l$  is a soliton, we have by the mean-value theorem, using (98),

$$(148) \quad \begin{aligned} \left| M[\tilde{P}_i](t) - M[\tilde{P}_i](0) \right| &= |c_l(t)^{1/2} - c_l(0)^{1/2}| |M[Q]| \\ &\leq C |c_l(t) - c_l(0)| \\ &\leq C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}), \end{aligned}$$

and, by the same way, we may obtain,

$$(149) \quad \left| E[\tilde{P}_i](t) - E[\tilde{P}_i](0) \right| \leq C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

and

$$(150) \quad \left| F[\tilde{P}_i](t) - F[\tilde{P}_i](0) \right| \leq C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}).$$

□

3.2.2. *Linear terms of the Taylor expansion.* We denote:

$$(151) \quad m[X] := \int X \varepsilon,$$

$$(152) \quad m_j[X] := \int X \varepsilon \Phi_j,$$

$$(153) \quad e[X] := \int [X_x \varepsilon_x - X^3 \varepsilon] = - \int [X_{xx} + X^3] \varepsilon,$$

$$(154) \quad e_j[X] := \int [X_x \varepsilon_x - X^3 \varepsilon] \Phi_j,$$

$$(155) \quad f[X] := \int \left[ X_{xx} \varepsilon_{xx} - 5X X_x^2 \varepsilon - 5X^2 X_x \varepsilon_x + \frac{3}{2} X^5 \varepsilon \right] = \int \left[ X_{xxxx} + 5X X_x^2 + 5X^2 X_{xx} + \frac{3}{2} X^5 \right] \varepsilon,$$

$$(156) \quad f_j[X] := \int \left[ X_{xx} \varepsilon_{xx} - 5X X_x^2 \varepsilon - 5X^2 X_x \varepsilon_x + \frac{3}{2} X^5 \varepsilon \right] \Phi_j.$$

**Lemma 14.** *If  $\sigma < \beta^2$ ,  $\theta < \min\left(\frac{\beta}{4}, \frac{\sqrt{\sigma}}{8}\right)$ , if  $D_0$  is large enough and  $a_0$  is small enough, then for any  $t \in [0, t^*]$ ,*

$$(157) \quad \left| m_j[\tilde{P}](t) - m_j[\tilde{P}](0) - \left( m[\tilde{P}_j](t) - m[\tilde{P}_j](0) \right) \right| \leq C e^{-2\theta D} + C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

$$(158) \quad \left| e_j[\tilde{P}](t) - e_j[\tilde{P}](0) - \left( e[\tilde{P}_j](t) - e[\tilde{P}_j](0) \right) \right| \leq C e^{-2\theta D} + C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

$$(159) \quad \left| f_j[\tilde{P}](t) - f_j[\tilde{P}](0) - \left( f[\tilde{P}_j](t) - f[\tilde{P}_j](0) \right) \right| \leq C e^{-2\theta D} + C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}).$$

*Proof.* We develop using  $\tilde{P} = \sum_{i=1}^J \tilde{P}_i$ . We obtain terms with  $\tilde{P}_i \tilde{P}_j$  with  $i \neq j$  and the other terms that have all the same index. Knowing that  $\varepsilon$  bounded for  $D_0$  large enough and  $a_0$  small enough (with respect to  $A$ ), we obtain the same bounds in the same way as for the constant part.

Now, if, for  $i \geq j+1$ ,  $\tilde{P}_i$  is a soliton, then we have simply:  $m[\tilde{P}_i] = e[\tilde{P}_i] = f[\tilde{P}_i] = 0$ .

If, for  $i \geq j+1$ ,  $\tilde{P}_i$  is a breather, we have a bound for the variation of these quantities by  $\mathcal{P}_i$ .  $\square$

3.2.3. *Quadratic part of the Taylor expansion.* We set:

$$(160) \quad \mathcal{M}_j[X] := \frac{1}{2} \int \varepsilon^2 \Phi_j,$$

$$(161) \quad \mathcal{E}_j[X] := \int \left[ \frac{1}{2} \varepsilon_x^2 - \frac{3}{2} X^2 \varepsilon^2 \right] \Phi_j,$$

$$(162) \quad \mathcal{F}_j[X] := \int \left[ \frac{1}{2} \varepsilon_{xx}^2 - \frac{5}{2} X_x^2 \varepsilon^2 - 10X X_x \varepsilon \varepsilon_x - \frac{5}{2} X^2 \varepsilon_x^2 + \frac{15}{4} X^4 \varepsilon^2 \right] \Phi_j,$$

and  $\mathcal{M}_j(t) := \mathcal{M}_j[\tilde{P}](t), \dots$

**Lemma 15.** *If  $\sigma < \beta^2$ ,  $\theta < \frac{\sqrt{\sigma}}{8}$ , if  $D_0$  is large enough and  $a_0$  is small enough, then for any  $t \in [0, t^*]$ ,*

$$(163) \quad \left| \mathcal{M}_j[\tilde{P}](t) - \mathcal{M}_j[\tilde{P}_j](t) \right| \leq C e^{-2\theta D} + C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

$$(164) \quad \left| \mathcal{E}_j[\tilde{P}](t) - \mathcal{E}_j[\tilde{P}_j](t) \right| \leq C e^{-2\theta D} + C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

$$(165) \quad \left| \mathcal{F}_j[\tilde{P}](t) - \mathcal{F}_j[\tilde{P}_j](t) \right| \leq C e^{-2\theta D} + C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}).$$

*Proof.* We develop using  $\tilde{P} = \sum_{i=1}^J \tilde{P}_i$ . For terms with  $\tilde{P}_i$ , with  $i > j$ , we use the induction assumption for  $\varepsilon$ . For terms with  $\tilde{P}_i$ , with  $i < j$ , we do as in the previous sections.  $\square$

Note that (163) is useless, because  $\mathcal{M}_j[\tilde{P}](t) - \mathcal{M}_j[\tilde{P}_j](t) = 0$  since  $\mathcal{M}_j[X]$  do not depend on  $X$ , but we write it in order to argue in the same way for the three conserved quantities.

**3.3. Lyapunov functional and simplifications.** We introduce the following Lyapunov functional:

$$(166) \quad \mathcal{H}_j(t) := F_j(t) + 2(b_j^2 - a_j^2)E_j(t) + (a_j^2 + b_j^2)^2 M_j(t).$$

We set:

$$(167) \quad \mathcal{K}(t) := F[\tilde{P}_j](t) + 2(b_j^2 - a_j^2)E[\tilde{P}_j](t) + (a_j^2 + b_j^2)^2 M[\tilde{P}_j](t),$$

$$(168) \quad \mathcal{L}(t) := f[\tilde{P}_j](t) + 2(b_j^2 - a_j^2)e[\tilde{P}_j](t) + (a_j^2 + b_j^2)^2 m[\tilde{P}_j](t),$$

$$(169) \quad \mathcal{Q}(t) := \mathcal{F}_j[\tilde{P}_j](t) + 2(b_j^2 - a_j^2)\mathcal{E}_j[\tilde{P}_j](t) + (a_j^2 + b_j^2)^2 \mathcal{M}_j[\tilde{P}_j](t).$$

We have the following:

**Lemma 16.** *If  $\sigma < \beta^2$ ,  $\theta < \min\left(\frac{\beta}{4}, \frac{\sqrt{\sigma}}{8}\right)$ , if  $D_0$  is large enough and  $a_0$  is small enough, then for any  $t \in [0, t^*]$ ,*

$$(170) \quad |\mathcal{H}_j(t) - \mathcal{H}_j(0) - (\mathcal{Q}(t) - \mathcal{Q}(0))| \leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}).$$

*Proof.* From (135), (136), (137), Lemmas 12, 14 and 15 we deduce that:

if  $\sigma < \beta^2$ ,  $\theta > \min\left(\frac{\beta}{4}, \frac{\sqrt{\sigma}}{8}\right)$ , if  $D_0$  is large enough and  $a_0$  is small enough, then for any  $t \in [0, t^*]$ ,

$$(171) \quad \mathcal{H}_j(t) - \mathcal{H}_j(0) = \mathcal{K}(t) - \mathcal{K}(0) + \mathcal{L}(t) - \mathcal{L}(0) + \mathcal{Q}(t) - \mathcal{Q}(0) + O\left(A(a + e^{-\theta D}) \int (\varepsilon^2 + \varepsilon_x^2) \Phi_j\right)$$

$$(172) \quad + O(e^{-2\theta D}) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})\right).$$

Now, from the bootstrap assumption (53), we see that if we take  $D_0$  large enough and  $a_0$  small enough, then we can bound  $A(a + e^{-\theta D}) \int (\varepsilon^2 + \varepsilon_x^2) \Phi_j$  by  $\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})$ . So, for any  $t \in [0, t^*]$ ,

$$(173) \quad \mathcal{H}_j(t) - \mathcal{H}_j(0) = \mathcal{K}(t) - \mathcal{K}(0) + \mathcal{L}(t) - \mathcal{L}(0) + \mathcal{Q}(t) - \mathcal{Q}(0) + O(e^{-2\theta D}) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})\right).$$

Now, we will simplify  $\mathcal{K}(t) - \mathcal{K}(0)$  and  $\mathcal{L}(t) - \mathcal{L}(0)$ .

**Simplification of  $\mathcal{K}(t) - \mathcal{K}(0)$ :**

If  $\tilde{P}_j$  is a breather, then  $\mathcal{K}(t) - \mathcal{K}(0) = 0$ . If  $\tilde{P}_j = \tilde{R}_l$  is a soliton, then we have

$$(174) \quad \begin{aligned} \mathcal{K}(t) &= F[\tilde{R}_l](t) + 2c_l^0 E[\tilde{R}_l](t) + (c_l^0)^2 M[\tilde{R}_l](t) \\ &= c_l(t)^{5/2} F[Q] + 2c_l^0 c_l(t)^{3/2} E[Q] + (c_l^0)^2 c_l(t)^{1/2} M[Q]. \end{aligned}$$

Now, we observe that  $c_l(t) = c_l^0 + (c_l(t) - c_l^0)$ , this is why we want to do a Taylor expansion for each power function. We recall that by (78), for  $D_0$  large enough and  $a_0$  small enough (with respect to  $Z_{j+1}$ ),  $|c_l(t) - c_l^0|^3$  is bounded by  $C \left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})$ . This is why we may approximate  $\mathcal{K}(t)$  by a Taylor expansion of order 2:

$$(c_l^0)^{5/2} \left(1 + \frac{5}{2} \frac{c_l(t) - c_l^0}{c_l^0} + \frac{15}{8} \left(\frac{c_l(t) - c_l^0}{c_l^0}\right)^2\right) F[Q] + 2(c_l^0)^{5/2} \left(1 + \frac{3}{2} \frac{c_l(t) - c_l^0}{c_l^0} + \frac{3}{8} \left(\frac{c_l(t) - c_l^0}{c_l^0}\right)^2\right) E[Q]$$

$$\begin{aligned}
 (175) \quad & + (c_l^0)^{5/2} \left( 1 + \frac{1}{2} \frac{c_l(t) - c_l^0}{c_l^0} - \frac{1}{8} \left( \frac{c_l(t) - c_l^0}{c_l^0} \right)^2 \right) M[Q] \\
 & = (c_l^0)^{5/2} (F[Q] + 2E[Q] + M[Q]) + (c_l(t) - c_l^0) (c_l^0)^{3/2} \left( \frac{5}{2} F[Q] + 3E[Q] + \frac{1}{2} M[Q] \right) \\
 & \quad + (c_l(t) - c_l^0)^2 (c_l^0)^{1/2} \left( \frac{15}{8} F[Q] + \frac{3}{4} E[Q] - \frac{1}{8} M[Q] \right).
 \end{aligned}$$

Now, we use the fact that  $M[Q] = 2$ ,  $E[Q] = -\frac{2}{3}$  and  $F[Q] = \frac{2}{5}$ , and we obtain that the Taylor expression of order 2 is in fact:

$$(176) \quad \frac{16}{15} (c_l^0)^{5/2} + 0 (c_l(t) - c_l^0) (c_l^0)^{3/2} + 0 (c_l(t) - c_l^0)^2 (c_l^0)^{1/2} = \frac{16}{15} (c_l^0)^{5/2}.$$

Thus,

$$(177) \quad \left| \mathcal{K}(t) - \frac{16}{15} (c_l^0)^{5/2} \right| \leq C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

and so,

$$(178) \quad |\mathcal{K}(t) - \mathcal{K}(0)| \leq C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}).$$

**Simplification of  $\mathcal{L}(t) - \mathcal{L}(0)$ :**

If  $\tilde{P}_j$  is a breather, we have, by the elliptic equation verified by a breather, that  $\mathcal{L}(t) = 0$ .

If  $\tilde{P}_j$  is a soliton, we have, by (81), (110) and (111), that  $\mathcal{L}(t) = 0$  (we have simply  $m[\tilde{P}_j] = e[\tilde{P}_j] = f[\tilde{P}_j] = 0$ ).  $\square$

### 3.4. Coercivity.

**Lemma 17.** *If  $\sigma$  is small enough (with respect to constants that depend only on problem data),  $\theta < \min\left(\frac{\beta}{4}, \frac{\sqrt{\sigma}}{8}\right)$ , if  $D_0$  is large enough and  $a_0$  is small enough, then for any  $t \in [0, t^*]$ ,*

$$(179) \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq CQ(t) + C \left( \int \varepsilon \tilde{P}_j \right)^2 + Ce^{-2\theta D}.$$

*Proof.* We notice that  $\int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j$  is  $\|\varepsilon \sqrt{\Phi_j}\|_{H^2}^2$  modulo some terms that can be bounded by  $C\sqrt{\sigma} \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j$ . This is why we will bound  $\|\varepsilon \sqrt{\Phi_j}\|_{H^2}^2$ . We can bound it by the canonical quadratic form associated to  $\tilde{P}_j$  and evaluated in  $\varepsilon \sqrt{\Phi_j}$ , if  $\varepsilon \sqrt{\Phi_j}$  satisfies quite well the orthogonality conditions, which is the case (see Sections 5.4 and (372) in [38]).

So, we obtain that, if  $\tilde{P}_j$  is a breather, the canonical quadratic form is  $Q(t)$  modulo  $C\sqrt{\sigma} \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j$ , and we have

$$(180) \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq CQ(t) + C \left( \int \varepsilon \sqrt{\Phi_j} \tilde{P}_j \right)^2 + C\sqrt{\sigma} \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j,$$

which means that if  $\sigma$  is small enough,

$$(181) \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq CQ(t) + C \left( \int \varepsilon \sqrt{\Phi_j} \tilde{P}_j \right)^2,$$

and we check that  $\int \varepsilon \sqrt{\Phi_j} \tilde{P}_j$  is  $\int \varepsilon \tilde{P}_j$  modulo  $Ce^{-\theta D}$ . So,

$$(182) \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq CQ(t) + C \left( \int \varepsilon \tilde{P}_j \right)^2 + Ce^{-2\theta D}.$$

If  $\tilde{P}_j = \tilde{R}_l$  is a soliton, the canonical quadratic form is, modulo  $C\sqrt{\sigma} \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j$ ,

$$(183) \quad Q_0(t) := \mathcal{F}_j[\tilde{R}_l](t) + 2c_l(t)\mathcal{E}_j[\tilde{R}_l](t) + c_l(t)^2 \mathcal{M}_j[\tilde{R}_l](t).$$

This is why, for the same reasons as above, we have

$$(184) \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq CQ_0(t).$$

This is why, we need to be able to bound  $|Q_0(t) - Q(t)|$ . We have

$$(185) \quad Q_0(t) - Q(t) = 2(c_l(t) - c_l^0) \mathcal{E}_j[\tilde{R}_l](t) + 2c_l^0(c_l(t) - c_l^0) \mathcal{M}_j[\tilde{R}_l](t) + (c_l(t) - c_l^0)^2 \mathcal{M}_j[\tilde{R}_l](t),$$

and so, because  $\mathcal{M}_j$  and  $\mathcal{E}_j$  are quadratic in  $\varepsilon$ , we have that

$$(186) \quad |Q_0(t) - Q(t)| \leq CA(a + e^{-\theta D}) \int (\varepsilon^2 + \varepsilon_x^2) \Phi_j,$$

and so, if we take  $a_0$  small enough and  $D_0$  large enough, we obtain

$$(187) \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq CQ(t).$$

□

**3.5. Proof of  $\mathcal{P}_j$ .** Now, we are left to prove  $\mathcal{P}_j$ . More precisely, if  $P_j$  is a soliton, we will prove (101) and (102); if  $P_j$  is a breather, we will prove (101), (103) and (104). We will distinguish the cases. We assume that  $\sigma \leq \min(\zeta, \beta^2)$ ,  $\theta \leq \min\left(\frac{\beta}{4}, \frac{\sqrt{\sigma}}{16}\right)$ , and that  $D_0$  is large enough and  $a_0$  is small enough (depending on  $A, \theta$ ), so that all the previous lemmas are verified.

**3.5.1. Case when  $P_j$  is a soliton.**

*Proof of (101).* By Lemma 10, we have for any  $t \in [0, t^*]$ ,

$$(188) \quad \mathcal{H}_j(t) - \mathcal{H}_j(0) \leq Ce^{-2\theta D}.$$

We have, by (170), (188), the definition of  $Q(0)$  and (77) for  $\varepsilon(0)$ , for any  $t \in [0, t^*]$ ,

$$(189) \quad Q(t) = [Q(t) - Q(0) - (\mathcal{H}_j(t) - \mathcal{H}_j(0))] + [\mathcal{H}_j(t) - \mathcal{H}_j(0)] + Q(0)$$

$$(190) \quad \leq Ce^{-2\theta D} + C\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}) + C\|\varepsilon(0)\|_{H^2}^2$$

$$(191) \quad \leq C(a^2 + e^{-2\theta D}) + C\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}).$$

And so, by (179) and (191), we have for any  $t \in [0, t^*]$ ,

$$(192) \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq CQ(t) + C\left(\int \varepsilon \tilde{P}_j\right)^2 + Ce^{-2\theta D}$$

$$(193) \quad \leq C(a^2 + e^{-2\theta D}) + C\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}) + C\left(\int \varepsilon \tilde{P}_j\right)^2.$$

Because of (81) and  $\tilde{P}_j$  is a soliton, we have that

$$(194) \quad \int \varepsilon \tilde{P}_j = 0.$$

So, the proof of (101) is completed for a suitable constant  $Z_j$  that will be precised later.

*Proof of (102).* From (135), (138), (141), (157), (81) and (193), we have for any  $t \in [0, t^*]$ :

$$(195) \quad \begin{aligned} M_j(0) - M_j(t) &= M_j[\tilde{P}](0) - M_j[\tilde{P}](t) + \int \tilde{P} \varepsilon \Phi_j(0) - \int \tilde{P} \varepsilon \Phi_j(t) + \frac{1}{2} \int \varepsilon^2 \Phi_j(0) - \frac{1}{2} \int \varepsilon^2 \Phi_j(t) \\ &= (c_l(0)^{1/2} - c_l(t)^{1/2}) M[Q] + O((a^2 + e^{-2\theta D})) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})\right). \end{aligned}$$

Similarly, from (136), (139), (142), (158), (110) and (193), by taking  $a_0$  smaller and  $D_0$  larger with respect to  $A$  if needed, we have for any  $t \in [0, t^*]$ :

$$(196) \quad E_j(0) - E_j(t) = \left( c_l(0)^{3/2} - c_l(t)^{3/2} \right) E[Q] + O\left( (a^2 + e^{-2\theta D}) \right) + O\left( \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) \right).$$

For  $\eta = \frac{1}{2}, \frac{3}{2}$ , because we know that  $c_l(t)$  is not too far from  $c_l(0)$  (and the both are not too far from  $c_l^0$ , by (78)), we can write for any  $t \in [0, t^*]$ :

$$(197) \quad c_l(t)^\eta = (c_l(0))^\eta \left( 1 + \eta \frac{c_l(t) - c_l(0)}{c_l(0)} + O\left( (c_l(t) - c_l(0))^2 \right) \right),$$

and so for any  $t \in [0, t^*]$ ,

$$(198) \quad c_l(t)^\eta - c_l(0)^\eta = \eta c_l(0)^{\eta-1} (c_l(t) - c_l(0)) + O\left( (c_l(t) - c_l(0))^2 \right),$$

and if  $a_0$  is small enough and  $D_0$  is large enough, we have by (54), for any  $t \in [0, t^*]$ :

$$(199) \quad \left| O\left( (c_l(t) - c_l(0))^2 \right) \right| \leq \frac{1}{2} \eta c_l(0)^{\eta-1} |c_l(t) - c_l(0)|.$$

Thus, for any  $t \in [0, t^*]$ ,

$$(200) \quad 2\eta c_l(0)^{\eta-1} |c_l(t) - c_l(0)| \geq |c_l(t)^\eta - c_l(0)^\eta| \geq \frac{\eta c_l(0)^{\eta-1}}{2} |c_l(t) - c_l(0)|,$$

where  $c_l(0)^{\eta-1}$  is between  $\min\{\frac{c_p^0}{2}, 1 \leq p \leq L\}^{\eta-1}$  and  $\max\{2c_p^0, 1 \leq p \leq L\}^{\eta-1}$  by (78), and so is bounded above and below by a constant that depends only on the shape parameters of the solitons. In order to bound  $|c_l(t) - c_l(0)|$  for a given  $t \in [0, t^*]$ , we will distinguish two cases.

Case when  $c_l(t) - c_l(0) \geq 0$ . From (195) and (200) for  $\eta = \frac{1}{2}$ , we can say that

$$(201) \quad |c_l(t) - c_l(0)| \leq C \left( c_l(t)^{1/2} - c_l(0)^{1/2} \right) \\ \leq C \frac{M_j(t) - M_j(0)}{M[Q]} + C (a^2 + e^{-2\theta D}) + C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}).$$

Now,  $M[Q] > 0$  and from Lemma 10,

$$(202) \quad |c_l(t) - c_l(0)| \leq C (a^2 + e^{-2\theta D}) + C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}).$$

Thus, (102) is established for a suitable constant  $Z_j$  that will be precised later.

Case when  $c_l(t) - c_l(0) \leq 0$ . From (196) and (200) for  $\eta = \frac{3}{2}$ , we can say that

$$(203) \quad |c_l(t) - c_l(0)| \leq C \left( c_l(0)^{3/2} - c_l(t)^{3/2} \right) \\ \leq C \frac{E_j(0) - E_j(t)}{E[Q]} + C (a^2 + e^{-2\theta D}) + C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}).$$

Now,  $E[Q] < 0$  and from Lemma 10, (195) and (200) for  $\eta = \frac{1}{2}$ ,

$$(204) \quad |c_l(t) - c_l(0)| \leq C \omega_1 (M_j(0) - M_j(t)) + C (a^2 + e^{-2\theta D}) + C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) \\ \leq C \omega_1 \left( c_l(0)^{1/2} - c_l(t)^{1/2} \right) + C (a^2 + e^{-2\theta D}) + C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) \\ \leq C \omega_1 |c_l(t) - c_l(0)| + C (a^2 + e^{-2\theta D}) + C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

and so, by taking  $\omega_1$  small enough, we may deduce the desired inequality:

$$(205) \quad |c_l(t) - c_l(0)| \leq C (a^2 + e^{-2\theta D}) + C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}).$$

Thus, (102) is established for a suitable constant  $Z_j$  that will be precised later.

### 3.5.2. Case when $P_j$ is a breather.

*Preliminaries.* By the same argument as in the case when  $P_j$  is a soliton, we establish (193). However, we are not able to prove (101) immediately, because  $\int \tilde{P}_j \varepsilon$  is not necessarily equal to 0 in the case when  $\tilde{P}_j$  is a breather.

From Lemma 14, (135), Lemma 12 and Remark 13, we have that for any  $t \in [0, t^*]$ ,

$$(206) \quad \int \tilde{P}_j \varepsilon(t) - \int \tilde{P}_j \varepsilon(0) = M_j(t) - M_j(0) - \frac{1}{2} \int \varepsilon^2 \Phi_j(t) + \frac{1}{2} \int \varepsilon^2 \Phi_j(0) + O(e^{-2\theta D}) + O \left( \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) \right).$$

then we use (118) and (77) for  $\varepsilon(0)$ , we have that for any  $t \in [0, t^*]$ ,

$$(207) \quad \int \tilde{P}_j \varepsilon(t) - \int \tilde{P}_j \varepsilon(0) \leq -\frac{1}{2} \int \varepsilon^2 \Phi_j(t) + \frac{1}{2} \int \varepsilon^2 \Phi_j(0) + O(e^{-2\theta D}) + O \left( \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) \right) \\ \leq C (a^2 + e^{-2\theta D}) + C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}).$$

Now, from (107), Lemma 14, (135), Lemma 12, Remark 13 and (53), we have for any  $t \in [0, t^*]$ ,

$$(208) \quad \left( (a_j^2 + b_j^2)^2 \right) \left( \int \tilde{P}_j \varepsilon(0) - \int \tilde{P}_j \varepsilon(t) \right) = 2 (b_j^2 - a_j^2) \left( e[\tilde{P}_j](t) - e[\tilde{P}_j](0) \right) + \left( f[\tilde{P}_j](t) - f[\tilde{P}_j](0) \right) \\ = 2 (b_j^2 - a_j^2) (E_j(t) - E_j(0) - \mathcal{E}_j(t) + \mathcal{E}_j(0)) \\ + (F_j(t) - F_j(0) - \mathcal{F}_j(t) + \mathcal{F}_j(0)) \\ + O(e^{-2\theta D}) + O \left( \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) \right).$$

We note this is  $j \geq 2$ ,  $b_j^2 - a_j^2 \geq b_j^2 - 3a_j^2 \geq v_2 > 0$ ; and if  $j = 1$ , we have simply  $E_1(t) - E_1(0) = F_1(t) - F_1(0) = 0$  from Remark 11. This is why, from Lemma 10 and (77) for  $\varepsilon(0)$ , we have for any  $j$ , for any  $t \in [0, t^*]$ :

$$(209) \quad \int \tilde{P}_j \varepsilon(0) - \int \tilde{P}_j \varepsilon(t) \leq \frac{-2 (b_j^2 - a_j^2) \mathcal{E}_j(t) - \mathcal{F}_j(t)}{(a_j^2 + b_j^2)^2} \\ + C (a^2 + e^{-2\theta D}) + C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) \\ + \frac{2 (b_j^2 - a_j^2) \omega_1 + \omega_2}{(a_j^2 + b_j^2)^2} (M_j(0) - M_j(t)).$$

And, from (135), Lemma 12, Remark 13, Lemma 14 and (77) for  $\varepsilon(0)$ , we have for any  $t \in [0, t^*]$ ,

$$(210) \quad M_j(0) - M_j(t) = M_j[\tilde{P}](0) - M_j[\tilde{P}](t) + \int \tilde{P} \varepsilon \Phi_j(0) - \int \tilde{P} \varepsilon \Phi_j(t) + \frac{1}{2} \int \varepsilon^2 \Phi_j(0) - \frac{1}{2} \int \varepsilon^2 \Phi_j(t) \\ = O(a^2 + e^{-2\theta D}) + O \left( \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) \right) + \int \tilde{P}_j \varepsilon(0) - \int \tilde{P}_j \varepsilon(t) - M_j(t).$$

And so, if we choose  $\omega_1$  and  $\omega_2$  small enough with respect to the problem constants, we obtain for any  $t \in [0, t^*]$ :

$$(211) \quad \int \tilde{P}_j \varepsilon(0) - \int \tilde{P}_j \varepsilon(t) \leq \frac{-\left(2(b_j^2 - a_j^2)\omega_1 + \omega_2\right) \mathcal{M}_j(t) - 2(b_j^2 - a_j^2) \mathcal{E}_j(t) - \mathcal{F}_j(t)}{(a_j^2 + b_j^2)^2} + C(a^2 + e^{-2\theta D}) + C\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}).$$

Because  $|\mathcal{M}_j(t)| + |\mathcal{E}_j(t)| + |\mathcal{F}_j(t)| \leq \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j$ , we deduce that for any  $t \in [0, t^*]$ ,

$$(212) \quad \int \tilde{P}_j \varepsilon(0) - \int \tilde{P}_j \varepsilon(t) \leq C \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j + C(a^2 + e^{-2\theta D}) + C\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}).$$

And so, by putting (207) and (212) together, we have that for any  $t \in [0, t^*]$ :

$$(213) \quad \left| \int \tilde{P}_j \varepsilon(0) - \int \tilde{P}_j \varepsilon(t) \right| \leq C \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j + C(a^2 + e^{-2\theta D}) + C\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}).$$

*Proof of (101).* From (77), (213), we deduce for any  $t \in [0, t^*]$ :

$$(214) \quad \left| \int \tilde{P}_j \varepsilon(t) \right| \leq \left| \int \tilde{P}_j \varepsilon(0) - \int \tilde{P}_j \varepsilon(t) \right| + \left| \int \tilde{P}_j \varepsilon(0) \right| \leq C \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j + C(a^2 + e^{-2\theta D}) + C\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}) \leq CA^2 (a^2 + e^{-2\theta D}) + C(a^2 + e^{-2\theta D}).$$

And so, if  $a_0$  is small enough and  $D_0$  is large enough, for any  $t \in [0, t^*]$ ,

$$(215) \quad \left( \int \tilde{P}_j \varepsilon(t) \right)^2 \leq C\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}) + C(a^2 + e^{-2\theta D}).$$

This is why, from (193), for any  $t \in [0, t^*]$ ,

$$(216) \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq C(a^2 + e^{-2\theta D}) + C\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}),$$

and (101) is established for a suitable constant  $Z_j$  that will be precised later.

*Proof of (103).* From (216) and (213), for any  $t \in [0, t^*]$ :

$$(217) \quad \left| \int \tilde{P}_j \varepsilon(0) - \int \tilde{P}_j \varepsilon(t) \right| \leq C\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}) + C(a^2 + e^{-2\theta D}).$$

Thus, (103) is established for a suitable constant  $Z_j$  that will be precised later.

*Proof of (104).* From Lemma 14, (136), Lemma 12, Remark 13, (213) and (216), for any  $t \in [0, t^*]$ ,

$$(218) \quad \begin{aligned} e[\tilde{P}_j](t) - e[\tilde{P}_j](0) &= E_j(t) - E_j(0) - \mathcal{E}_j(t) + \mathcal{E}_j(0) \\ &\quad + O(e^{-2\theta D}) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})\right) \\ &= E_j(t) - E_j(0) + O(a^2 + e^{-2\theta D}) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})\right). \end{aligned}$$

Now, from Lemma 10, for any  $t \in [0, t^*]$ ,

$$(219) \quad e[\tilde{P}_j](t) - e[\tilde{P}_j](0) \leq \omega_1 (M_j(0) - M_j(t)) + O(a^2 + e^{-2\theta D}) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D})\right).$$



And from (210) and (217), for any  $t \in [0, t^*]$ ,

$$(220) \quad e[\tilde{P}_j](t) - e[\tilde{P}_j](0) \leq C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) + C (a^2 + e^{-2\theta D}).$$

To majorate  $e[\tilde{P}_j](0) - e[\tilde{P}_j](t)$ , we do as in (208), (209), (210), (211) and (212), but with the energy instead of the mass, and after using (216), we obtain for any  $t \in [0, t^*]$ ,

$$(221) \quad \left| e[\tilde{P}_j](t) - e[\tilde{P}_j](0) \right| \leq C \left( \frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) + C (a^2 + e^{-2\theta D}).$$

Thus, (104) is established for a suitable constant  $Z_j$  that will be precised later.

**3.6. Choice of suitable  $A$  and  $Z_j$ .** The induction holds if

$$(222) \quad C \left( 1 + \left( \frac{A}{Z_{j+1}} \right)^2 \right) \leq \left( \frac{A}{Z_j} \right)^2.$$

We can set

$$(223) \quad A := \sqrt{2} (2C)^{\frac{1}{2}},$$

and for  $1 \leq j \leq J$ ,

$$(224) \quad Z_j := 2 (2C)^{\frac{j-1}{2}}.$$

And, if  $C > 1$ , the induction holds.

The proof of Proposition 7 is now complete.

#### 4. CONSEQUENCES OF THEOREM 1

**4.1. Orbital stability of a multi-breather.** We assume the Theorem 1 proved, let us prove Theorem 3.

*Proof of Theorem 3.* Let  $A_0, \theta_0, D_0, a_0 > 0$  from Theorem 1 (these constants do only depend on the shape/frequency parameters of our objects and not on their initial positions). Let  $\eta_0 > \eta > 0$  with  $\eta_0 < \frac{a_0}{2C_1}$  and  $C_1$  defined in the following. Let  $a < a_0$  and  $D > D_0$  such that  $A_0 (a + e^{-\theta_0 D}) < 4A_0 C_1 \eta$  and  $a = 2C_1 \eta$ . We may take  $D$  even larger so that  $T^* = 0$  where  $T^*$  is defined in [38, Theorem 2]. Let  $\theta_1$  associated to shape/frequency parameters  $\alpha_k, \beta_k, \kappa_l, c_l^l$  by [38, Theorem 2]. Let  $A_2$  associated to  $D$  by [38, Theorem 2].

Let  $\tau > 0$  be the minimal difference between two velocities.

Let  $p$  be the multi-breather associated to  $\alpha_k, \beta_k, x_{1,k}^0, x_{2,k}^0, c_l^0, x_{0,l}^0$  by [38, Theorem 2] with notations as in (29) and  $P$  the corresponding sum with notations as in (30). We may choose  $T \geq 0$  large enough such that

$$(225) \quad \forall t \geq T, \quad \|p(t) - P(t)\|_{H^2} \leq a/2,$$

which is possible from [38], and such that

$$(226) \quad \forall j \geq 2, \quad x_j(T) - x_{j-1}(T) > 2D,$$

which is possible because the distance between two objects is increasing with a speed that is at least  $\tau$ .

By [23] we know that we have continuous dependence of the solution of (mKdV) with respect to the initial data. And so, there exists  $C_1 > 0$  such that if  $\|u(0) - p(0)\|_{H^2} \leq \eta$  and  $\eta_0$  is small enough, then

$$(227) \quad \forall t \in [0, T], \quad \|u(t) - p(t)\|_{H^2} \leq C_1 \eta.$$

Therefore, by triangular inequality,

$$(228) \quad \|u(T) - P(T)\|_{H^2} \leq a = 2C_1 \eta.$$

This means that the assumptions of Theorem 1 are all satisfied in  $T$  instead of 0. And so, this means that there exists  $x_{0,l}(t), x_{1,k}(t), x_{2,k}(t)$  defined for any  $t \geq T$  such that

$$(229) \quad \forall t \geq T, \quad \|u(t) - P(t; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t))\|_{H^2} \leq A_0 (a + e^{-\theta_0 D}) < 4A_0 C_1 \eta.$$

Now, we see that the assumptions of [38, Theorem 2, Remark 3] are all satisfied in any  $t \geq T$  instead of 0 for the sum  $P(\alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t))$ . Indeed, if we denote  $\tilde{x}_j(t)$  the position of  $P_j(t; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t))$ , we know from Remark 2 that for any  $t \geq T$  and  $j \geq 2$ ,

$$\tilde{x}_j(t) - \tilde{x}_{j-1}(t) \geq D.$$

By taking  $T$  larger if needed, we may insure that  $e^{-\theta_1 T} < C_1 \eta$ . Therefore, for any  $t \geq T$ ,

$$\begin{aligned} \left\| p(t; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t)) - P(t; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t)) \right\|_{H^2} \\ \leq A_2 e^{-\theta_1 t} \leq A_2 e^{-\theta_1 T} \leq A_2 C_1 \eta. \end{aligned}$$

And so, by triangular inequality,

$$(230) \quad \forall t \geq T, \quad \left\| u(t) - p(t; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t)) \right\|_{H^2} \leq (4A_0 + A_2)C_1 \eta.$$

The latter proves the Theorem for  $t \geq T$  with  $C_0 = (4A_0 + A_2)C_1$ . For  $0 \leq t \leq T$ , it is enough to use (227).  $\square$

**4.2. Uniqueness of a multi-breather.** We will prove that, under the condition  $v_2 > 0$ , the multi-breather constructed in [38] is unique. This uniqueness result is better than the result that was obtained in [38], but it requires to use the formula for multi-breathers obtained in [40] by inverse scattering method. More precisely, we will need to use that for a set of parameters (a set of solitons and breathers), there exists a solution of (mKdV) that is a multi-breather associated to this set when  $t \rightarrow +\infty$  and also when  $t \rightarrow -\infty$  for the same set of objects but with different translation parameters.

*Proof of Theorem 4.* The existence is already established without the assumption  $v_2 > 0$ . It gives us a special solution  $p$  of (mKdV) such that

$$(231) \quad \left\| p(t) - P(t) \right\|_{H^2} \rightarrow_{t \rightarrow +\infty} 0.$$

Here, we take  $p$  given by the formula in [40], so that it is also a multi-breather when  $t \rightarrow -\infty$ . Let  $u$  be a solution of (mKdV) such that

$$(232) \quad \left\| u(t) - p(t) \right\|_{H^2} \rightarrow_{t \rightarrow +\infty} 0.$$

We can write

$$(233) \quad p(t) = p(t; \alpha_k, \beta_k, x_{1,k}^0, x_{2,k}^0, \kappa_l, c_l^0, x_{0,l}^0),$$

and

$$(234) \quad P(t) = P(t; \alpha_k, \beta_k, x_{1,k}^0, x_{2,k}^0, \kappa_l, c_l^0, x_{0,l}^0).$$

We remark that if  $w(t, x)$  is a solution of (mKdV), then  $w(-t, -x)$  is also a solution of (mKdV). So,  $u(-t, -x)$  is a solution of (mKdV) and  $p(-t, -x)$  too. Note that  $p(-t, -x)$  is also a multi-breather given by the formula from [40] with the same breathers and with solitons replaced by their opposite.

Let  $0 < \eta < \eta_0$  where  $\eta_0$  is given by Theorem 3. By (232), there exists  $t_0 > 0$  such that

$$(235) \quad \forall t \geq t_0 \quad \left\| u(t) - p(t) \right\|_{H^2} \leq \eta.$$

So,

$$(236) \quad \left\| u(-(-t_0), -x) - p(-(-t_0), -x) \right\|_{H^2} \leq \eta.$$

And, by Theorem 3, there exists  $x_{0,l}(t), x_{1,k}(t), x_{2,k}(t)$  and  $C_0 > 0$  such that for any  $t \geq -t_0$ ,

$$(237) \quad \left\| u(-t, -x) - p(-t, -x; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t)) \right\|_{H^2} \leq C_0 \eta.$$

In particular, we have for  $t = 0$ ,

$$(238) \quad \left\| u(0) - p(0; \alpha_k, \beta_k, x_{1,k}[\eta], x_{2,k}[\eta], \kappa_l, c_l^0, x_{0,l}[\eta]) \right\|_{H^2} \leq C_0 \eta.$$

The last expression can be established for any  $\eta > 0$ . So, we can find a sequence  $\eta_n \rightarrow_{n \rightarrow +\infty} 0$  such that  $x_{1,k}[\eta_n], x_{2,k}[\eta_n]$  and  $x_{0,l}[\eta_n]$  converge (this comes from the fact that  $p(0; \alpha_k, \beta_k, x_{1,k}[\eta], x_{2,k}[\eta], \kappa_l, c_l^0, x_{0,l}[\eta])$  is necessarily in a compact set). Thus,  $u(0)$  is a multi-breather.  $\square$

## APPENDIX: EQUATIONS FOR LOCALIZED CONSERVATION LAWS

**Lemma 18.** Let  $f$  be a  $C^3$  function that do not depend on time and  $u$  a solution of (mKdV). Then,

$$(239) \quad \frac{d}{dt} \frac{1}{2} \int u^2 f = \int \left[ -\frac{3}{2} u_x^2 + \frac{3}{4} u^4 \right] f_x + \frac{1}{2} \int u^2 f_{xxx},$$

$$(240) \quad \frac{d}{dt} \int \left[ \frac{1}{2} u_x^2 - \frac{1}{4} u^4 \right] f = \int \left[ -\frac{1}{2} (u_{xx} + u^3)^2 - u_{xx}^2 + 3u^2 u_x^2 \right] f_x + \frac{1}{2} \int u_x^2 f_{xxx},$$

$$(241) \quad \begin{aligned} & \frac{d}{dt} \int \left[ \frac{1}{2} u_{xx}^2 - \frac{5}{2} u^2 u_x^2 + \frac{1}{4} u^6 \right] f \\ &= \int \left( -\frac{3}{2} u_{xxx}^2 + 9u_{xx}^2 u^2 + 15u_x^2 u u_{xx} + \frac{9}{16} u^8 + \frac{1}{4} u_x^4 + \frac{3}{2} u_{xx} u^5 - \frac{45}{4} u^4 u_x^2 \right) f' \\ &+ 5 \int u^2 u_x u_{xx} f'' + \frac{1}{2} \int u_{xx}^2 f'''. \end{aligned}$$

*Proof.* see the bottom of the page 1115 and the bottom of the page 1116 of [27] and Section 5.5 in [38].  $\square$

**Lemma 19.** Let  $r > 0$ . If  $t, x$  satisfy  $\widetilde{x}_{j-1}(t) + r < x < \widetilde{x}_j(t) - r$ , then

$$(242) \quad \left| \widetilde{P}(t, x) \right| \leq C e^{-\beta r}.$$

*Proof.* immediate consequence of the exponential majoration of each object.  $\square$

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