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ORBITAL STABILITY OF A SUM OF SOLITONS AND BREATHERS OF THE MODIFIED KORTEWEG-DE VRIES EQUATION

ALEXANDER SEMENOV

ABSTRACT. In this article, we prove that a sum of solitons and breathers of the modified Korteweg-de Vries equation (mKdV) is orbitally stable. The orbital stability is shown in H^2 . More precisely, we will show that if a solution of (mKdV) is close enough to a sum of solitons and breathers with distinct velocities at $t = 0$ in the H^2 sense, then it stays close to this sum of solitons and breathers for any time $t \geq 0$ in the H^2 sense, up to space translations for solitons or space and phase translations for breathers, provided the condition that the considered solitons and breathers are sufficiently decoupled from each other and that the velocities of the considered breathers are all positive, except possibly one. The constants that appear in this stability result do not depend on translation parameters.

From this, we deduce the orbital stability of any multi-breather of (mKdV), provided the condition that the velocities of the considered breathers are all positive, except possibly one (the condition about the decoupling of the considered solitons and breathers between each other is not required in this setting). The constants that appear in this stability result depend on translation parameters of the considered solitons and breathers.

1. INTRODUCTION

1.1. Setting of the problem. We consider the modified Korteweg-de Vries equation:

$$(mKdV) \quad \begin{cases} u_t + (u_{xx} + u^3)_x = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

for $u_0 \in H^2(\mathbb{R})$.

(mKdV) appears as a good approximation of some physical problems as ferromagnetic vortices [44], fluid mechanics [22], electrodynamics [36], plasma physics [37, 9], etc.

The Cauchy problem for (mKdV) is locally well-posed in H^s for $s > -\frac{1}{2}$ [21]. For $s > \frac{1}{4}$, the Cauchy problem is globally well-posed [12]. In this paper, we will use only basic results about the Cauchy problem: the fact that it is globally well-posed in H^1 or H^2 .

Note that the set of solutions of (mKdV) is stable under space or time translations or under reflexions with respect to the x -axis.

We have following conservation laws for a solution $u(t)$ of (mKdV):

$$(1.1) \quad M[u](t) := \frac{1}{2} \int u^2(t, x) dx,$$

$$(1.2) \quad E[u](t) := \frac{1}{2} \int u_x^2(t, x) dx - \frac{1}{4} \int u^4(t, x) dx,$$

$$(1.3) \quad F[u](t) := \frac{1}{2} \int u_{xx}^2(t, x) dx - \frac{5}{2} \int u^2(t, x) u_x^2(t, x) dx + \frac{1}{4} \int u^6(t, x) dx.$$

Note that (mKdV) has actually infinitely many conservation laws, because it is integrable, like the original Korteweg-de Vries equation (KdV), which has quadratic nonlinearity [35, 1]. It is also a special case of the generalized Korteweg-de Vries equation (gKdV) [28].

(gKdV) belongs to the family of nonlinear (focusing) dispersive equations. Other examples of equations belonging to this family are the nonlinear Schrödinger equation (NLS) [32, 43, 42] and the nonlinear Klein-Gordon equation (KG) [13, 14]. They share a common property: they all admit special solutions called solitons, a bump that translates with a constant velocity without deformation. However, (mKdV) enjoys a specific feature: it admits another class of special solutions called

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breathers, which we will describe below. We will consider here both specific solutions of (mKdV) together: solitons and breathers.

For $c > 0$, $\kappa \in \{-1, 1\}$ and $x_0 \in \mathbb{R}$, a *soliton* parametrized by c , κ and x_0 is a solution of (mKdV) that corresponds to a bump of constant shape that translates with a constant velocity c without deformation, that is initially centered in x_0 and that has for sign κ . In other words, it is a solution of the form:

$$(1.4) \quad R_{c,\kappa}(t, x; x_0) := \kappa Q_c(x - ct - x_0),$$

where Q_c is the profile function that depends only on one variable and that is positive and pair. $Q_c \in H^1(\mathbb{R})$ should solve the following elliptic equation:

$$(1.5) \quad Q_c'' - cQ_c + Q_c^3 = 0.$$

It is possible to show that (1.5) has a unique solution in H^1 , up to translations and reflexion with respect to the x -axis. One can show that Q_c has the following expression:

$$(1.6) \quad Q_c(x) := \left(\frac{2c}{\cosh^2(\sqrt{c}x)} \right)^{\frac{1}{2}}.$$

We denote $Q := Q_1$ the basic ground state.

When $\kappa = -1$, the soliton is sometimes called an *antisoliton*.

The soliton $R_{c,\kappa}(x_0)$ travels with velocity c to the right, the position of its center at a time t is $x_0 + ct$. It is exponentially localized, depending on c and $x_0 + ct$ (the position of the bound depends on $x_0 + ct$, the amplitude and the exponential decay rate depend on c):

$$(1.7) \quad |R_{c,\kappa}(t, x; x_0)| \leq \sqrt{2c} \exp(-\sqrt{c}|x - x_0 - ct|).$$

Analogous bounds are valid for any derivative (with the same decay rate but different amplitude). This motivates the terminology “shape parameter” for c .

Solitons, in particular their stability, have been extensively studied: regarding orbital stability (in H^1), we refer to Cazenave, Lions [8] and Weinstein [43, 42] for (NLS) and Weinstein [43], Bona-Souganidis-Strauss [7] and Martel-Merle [30] for (mKdV), see also Grillakis-Shatah-Strauss [20] for a result in an abstract setting. Asymptotic stability (in H^1) of (mKdV) solitons was shown by Martel-Merle [29, 33, 31] and refined by Germain-Pusateri-Rousset [18].

For $\alpha, \beta > 0$ and $x_1, x_2 \in \mathbb{R}$, a *breather* parametrized by α, β, x_1, x_2 is a solution of (mKdV) that has the following expression:

$$(1.8) \quad B_{\alpha,\beta}(t, x; x_1, x_2) := 2\sqrt{2}\partial_x \left[\arctan \left(\frac{\beta \sin(\alpha y_1)}{\alpha \cosh(\beta y_2)} \right) \right],$$

where $y_1 := x + \delta t + x_1$, $y_2 := x + \gamma t + x_2$, $\delta := \alpha^2 - 3\beta^2$ and $\gamma := 3\alpha^2 - \beta^2$.

The breather $B_{\alpha,\beta}(x_1, x_2)$ travels with velocity $-\gamma$; the position of its center at a time t is $-x_2 - \gamma t$. It is exponentially localized, depending on α , β and $-x_2 - \gamma t$ (the position of the bound depends on $-x_2 - \gamma t$, the coefficient depends on α and β , and the exponential decay rate depends on β):

$$(1.9) \quad |B_{\alpha,\beta}(t, x; x_1, x_2)| \leq C(\alpha, \beta) \exp(-\beta|x + x_2 + \gamma t|),$$

Analogous bounds are valid for any derivative (with the same decay rate but different amplitude). This motivates the terminology “shape” and “frequency” parameters for β and α , respectively.

One doesn’t talk of “antibreathers”, because if we replace x_1 by $x_1 + \frac{\pi}{\alpha}$, then a breather is transformed in its opposite.

Similarly to (1.5), it is known that a breather $B = B_{\alpha,\beta}$ satisfies the following elliptic equation on \mathbb{R} :

$$(1.10) \quad B_{xxxx} + 5BB_x^2 + 5B^2B_{xx} + \frac{3}{2}B^5 - 2(\beta^2 - \alpha^2)(B_{xx} + B^3) + (\alpha^2 + \beta^2)^2 B = 0.$$

Note that in [39] a similar form of elliptic equation (with a fourth order derivative) is obtained for a soliton $R_{c,\kappa}$ from (1.5). This allows us to consider solitons and breathers at the same level of regularity, which is the key of the proof made in this article.

This object was first introduced by Wadati [40], and it was used by Kenig, Ponce and Vega in [25] for the ill-posedness for (mKdV) for rough data. Their properties, in particular their stability, are well studied by Alejo, Muñoz and co-authors [3, 2, 5, 6, 4]. We know that a breather is orbitally

stable in H^2 [3]. Afterwards, H^1 orbital stability was proved via Bäcklund transformation [4], and also H^1 asymptotic stability, for breathers moving to the right. Asymptotic stability of breathers in full generality is still an open problem.

When $\alpha \rightarrow 0$, $B_{\alpha,\beta}$ tends to a *double-pole solution* of (mKdV): it is a couple soliton-antisoliton that move with a constant velocity and that have a repulsive logarithmic interaction [41]. However, this limit, which is somehow at a boundary between solitons and breathers, is expected to be unstable [19]. We do not consider this object in this paper.

Solitons and breathers are important objects to study because of their stability properties and also because of the soliton-breather resolution. The latter is an important result about the long time dynamics of (mKdV), which asserts that any generic solution can be approached by a sum of solitons and breathers when $t \rightarrow +\infty$. It is established for initial conditions in a weighted Sobolev space in [10] (see also Schuur [38]) by inverse scattering method; see also [38] for the soliton resolution for (KdV).

Given a set of basic objects (solitons and breathers), we consider a solution that tends to this sum when $t \rightarrow +\infty$, called *multi-breather*. In [39], we have shown existence, regularity and uniqueness of multi-breathers of (mKdV). There is also a formula for multi-breathers of (mKdV) obtained by Wadati [41], which was derived as a consequence of the integrability of (mKdV); but is it not well suited for our purpose.

Martel, Merle and Tsai [34] proved that a sum of (decoupled and ordered) solitons is orbitally stable in H^1 for (mKdV), and actually asymptotically stable (in the region $x \geq \delta t$ for $\delta > 0$ small). Le Coz [26] has established stability of (mKdV) N -solitons in H^N by modifying the approach used by Maddocks and Sachs [27] for (KdV).

Inspired by [34], similar asymptotic stability results were obtained for sums of (decoupled) solitons for various nonlinear dispersive equations: we refer to El Dika [16] for the Benjamin-Bona-Mahony equation (BBM), Kenig and Martel [23] for the Benjamin-Ono equation (BO), El Dika-Molinet [17] for the Camassa-Holm multipeakon, and Côte-Muñoz-Pilod-Simpson [15] for the Zakharov-Kuznetsov (ZK) equation.

Because breathers have an H^2 structure, we prove orbital stability in H^2 in this paper for a sum of solitons and breathers. One of the difficulties is to obtain H^2 stability results for solitons too, i.e. to study solitons at a H^2 level.

1.2. Main results. We prove in this article that given any sum of solitons and breathers with distinct velocities and such that all these velocities except possibly one are positive, a solution u of (mKdV) that is initially close to this sum in H^2 stays close to this sum for any time of a considered time interval up to space translations for solitons or space and phase translations for breathers. This is orbital stability. Let us make the definition of orbital stability more precise.

Let $L \in \mathbb{N}$. We consider a set of L solitons: given $c_l^0 > 0$, $\kappa_l \in \{-1, 1\}$ and $x_{0,l}^0 \in \mathbb{R}$ for $1 \leq l \leq L$, we set, for $1 \leq l \leq L$,

$$(1.11) \quad R_l(t, x) := R_{c_l^0, \kappa_l}(t, x; x_{0,l}^0).$$

Let $K \in \mathbb{N}$. We consider a set of K breathers: given $\alpha_k > 0$, $\beta_k > 0$ and $x_{1,k}^0, x_{2,k}^0 \in \mathbb{R}$ for $1 \leq k \leq K$, we set, for $1 \leq k \leq K$,

$$(1.12) \quad B_k(t, x) := B_{\alpha_k, \beta_k}(t, x; x_{1,k}^0, x_{2,k}^0).$$

We now define important parameters for each of the objects of the problem. For $1 \leq l \leq L$, the velocity of the l th soliton is

$$(1.13) \quad v_l^s := c_l^0,$$

and for $1 \leq k \leq K$, the velocity of the k th breather is

$$(1.14) \quad v_k^b := \beta_k^2 - 3\alpha_k^2.$$

For $1 \leq l \leq L$, the center of the l th soliton is

$$(1.15) \quad x_l^s(t) := x_{0,l}^0 + v_l^s t,$$

and for $1 \leq k \leq K$, the center of the k th breather is

$$(1.16) \quad x_k^b(t) := -x_{2,k}^0 + v_k^b t.$$

We set $J := K + L$ the total number of objects in the problem. We assume that $J \geq 2$, because for $J = 1$ the proof is already done in [3] when the only object is a breather, and in [43] when the only object is a soliton.

We assume that the velocities of our objects are all distinct, this will imply that our objects are far from each other when time is large, which is essential for our analysis. More precisely,

$$(1.17) \quad \forall k \neq k', \quad v_k^b \neq v_{k'}^b, \quad \forall l \neq l', \quad v_l^s \neq v_{l'}^s, \quad \forall k, l, \quad v_k^b \neq v_l^s.$$

This allows us to define an increasing function

$$(1.18) \quad \underline{v} : \{1, \dots, J\} \rightarrow \{v_k^b, 1 \leq k \leq K\} \cup \{v_l^s, 1 \leq l \leq L\}.$$

The set $\{v_1, \dots, v_J\}$ is thus the set of all the possible velocities of our objects. We have

$$(1.19) \quad v_1 < v_2 < \dots < v_J.$$

We define, for $1 \leq j \leq J$, P_j as the object (R_l or B_k) that corresponds to the velocity v_j , i.e. if $v_j = v_l^s$, we set $P_j := R_l$, and if $v_j = v_k^b$, we set $P_j := B_k$. So, P_1, \dots, P_J are the considered objects that are ordered by increasing velocity.

We denote $x_j(t)$ the position (the center of mass) of $P_j(t)$. More precisely, if $P_j = R_l$, we set $x_j(t) := x_l^s(t)$; and if $P_j = B_k$, we set $x_j(t) := x_k^b(t)$.

We set

$$(1.20) \quad R := \sum_{l=1}^L R_l, \quad B := \sum_{k=1}^K B_k, \quad P := R + B = \sum_{j=1}^J P_j.$$

We need both notations: indexation by k and l , and indexation by j , and we keep these notations to avoid ambiguity.

The main result that we will prove in this article is the following: a sum of decoupled solitons and breathers, with $v_2 > 0$ (that is, all travel to the right, but at most one can be static or travel to the left), is orbitally stable. The precise statement is as follows.

Theorem 1.1. *Given a set of solitons and breathers (1.11), (1.12) whose velocities (1.13) and (1.14) satisfy (1.17), and whose positions are set by (1.15) and (1.16), we define the corresponding sum P in (1.20), and we define P_j, v_j, x_j a reindexation of the given set of solitons and breathers such that (1.19). We assume that*

$$(1.21) \quad v_2 > 0.$$

Then there exists $A_0, \theta_0, D_0, a_0 > 0$, constants (depending on c_l^0, α_k, β_k , but not on $x_{0,l}^0, x_{1,k}^0$ or $x_{2,k}^0$), such that the following is true. Let $D \geq D_0$ and $0 \leq a \leq a_0$ such that

$$(1.22) \quad \|u(0) - P(0)\|_{H^2} \leq a, \quad \text{and} \quad x_j(0) > x_{j-1}(0) + 2D, \quad \text{for all } j = 2, \dots, J,$$

for a solution $u \in \mathcal{C}(\mathbb{R}, H^2(\mathbb{R}))$ of (mKdV).

Then, there exist $x_{0,l}(t), x_{1,k}(t), x_{2,k}(t)$ defined for any $t \geq 0$ such that

$$(1.23) \quad \forall t \geq 0, \quad \left\| u(t) - \sum_{l=1}^L R_{c_l^0, \kappa_l}(t, \cdot; x_{0,l}(t)) - \sum_{k=1}^K B_{\alpha_k, \beta_k}(t, \cdot; x_{1,k}(t), x_{2,k}(t)) \right\|_{H^2} \leq A_0 \left(a + e^{-\theta_0 D} \right),$$

with

$$(1.24) \quad \forall t \geq 0, \quad \sum_{l=1}^L |x'_{0,l}(t)| + \sum_{k=1}^K (|x'_{1,k}(t)| + |x'_{2,k}(t)|) \leq CA_0 \left(a + e^{-\theta_0 D} \right),$$

for some constant $C > 0$.

Remark 1.2. Let us stress on the fact that the constant A_0 do not depend on the translation parameters of the considered objects.

We also deduce a consequence from Theorem 1.1 for multi-breathers.

Recall that multi-breathers were defined and constructed in [39], and their uniqueness was proved there when $v_2 > 0$. They can be also obtained from the formula from [41] obtained by inverse scattering method by Wadati. Let p be a multi-breather associated to P (in the setting $v_2 > 0$, we know that p is also unique). To emphasize the dependence of p with respect to the parameters, we will write

$$(1.25) \quad p(t, x) =: p(t, x; \alpha_k, \beta_k, x_{1,k}^0, x_{2,k}^0, \kappa_l, c_l^0, x_{0,l}^0).$$

The same way, to emphasize the dependence of P with respect to the parameters, we will write

$$(1.26) \quad P(t, x) =: P(t, x; \alpha_k, \beta_k, x_{1,k}^0, x_{2,k}^0, \kappa_l, c_l^0, x_{0,l}^0).$$

Theorem 1.1 can be recast as orbital stability of a *multi-breather*:

Theorem 1.3. *Let $\alpha_k, \beta_k, x_{1,k}^0, x_{2,k}^0, \kappa_l, c_l^0, x_{0,l}^0$ and p the multi-breather associated to these parameters with notations as in (1.25) given by [39, Theorem 1.2]. We assume that (1.17) holds and $v_2 > 0$. There exists $\eta_0 > 0$ small enough, $C_0 > 0$ large enough such that the following is true for $0 < \eta < \eta_0$.*

Let $u(t)$ be a solution of (mKdV), such that

$$(1.27) \quad \|u(0) - p(0; \alpha_k, \beta_k, x_{1,k}^0, x_{2,k}^0, c_l^0, x_{0,l}^0)\|_{H^2} \leq \eta.$$

Then there exist C^1 functions $x_{0,l}(t), x_{1,k}(t), x_{2,k}(t)$ defined for any $t \geq 0$ such that

$$(1.28) \quad \forall t \geq 0, \quad \|u(t) - p(t; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), c_l^0, x_{0,l}(t))\|_{H^2} \leq C_0 \eta,$$

with (1.24) that is satisfied.

Remark 1.4. Let us stress on the fact the C_0 do depend on the translation parameters of the considered solitons and breathers, i.e. on $x_{0,l}^0, x_{1,k}^0$ and $x_{2,k}^0$. More precisely, it depends on the time we need to wait until the collisions between the considered solitons and breathers are over. This is a fundamental difference between Theorem 1.3 and Theorem 1.1. In fact, in Theorem 1.1, we avoid collisions and this is the reason why we get a more uniform result. It is reasonable to expect that Theorem 1.3 can be improved so that C_0 do not depend on translation parameters.

In this paper, we adapt the arguments given by Martel, Merle and Tsai [34] to the context of breathers. To do so, it is needed to understand the variational structure of breathers, in the same manner as Weinstein did in [43] for (NLS) and (mKdV) solitons. Such results have been obtained by Alejo and Muñoz in [3]. When a soliton is a critical point of a Lyapunov functional at the H^1 level, whose Hessian is coercive up to two orthogonal conditions, a breather is a critical point of a Lyapunov functional at the H^2 level, whose Hessian is coercive up to three orthogonal conditions. One important issue that we address is to understand the variational structure of a soliton at the H^2 level. We do this by modifying the Lyapunov functional from [3], and we will also adapt it for a sum of solitons and breathers. We need to make assumptions on the velocities of our breathers (recall that the velocity of a soliton is always positive), because several arguments are based on monotonicity properties, which hold only on the right.

1.3. Organisation of the proof. The proof of Theorem 1.1 is based on two results: a modulation lemma and a bootstrap proposition. We give a detailed outline of both results in Section 2, and give in Section 3 the proof of the heart of the argument (Proposition 2.6), where we complete the bootstrap via a (finite) induction argument on the an improved bound localized on the last j objects.

In Section 4, we prove Theorem 1.3 as a quick consequence of Theorem 1.1.

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2. REDUCTION OF THE PROOF TO AN INDUCTION

For the proof, we assume the assumption of the Theorem 1.1 true (i.e. we assume (1.22) true) for a solution $u(t)$ and a choice of translation parameters, and the goal is to find the suitable constants A, θ, D_0, a_0 (that do not depend on u nor on translation parameters) so that the Theorem 1.1 holds.

2.1. Some useful notations. We set some useful constants for this paper. We define the worst exponential decay rate:

$$(2.1) \quad \beta := \min\{\beta_k, 1 \leq k \leq K\} \cup \{\sqrt{c_l}, 1 \leq l \leq L\},$$

and the worst distance between two consecutive velocities:

$$(2.2) \quad \tau := \min\{v_{j+1} - v_j, 1 \leq j \leq J - 1\}.$$

For any $j = 2, \dots, J$, we have that

$$(2.3) \quad \forall t \in I, \quad x_j(t) - x_{j-1}(t) \geq x_j(0) - x_{j-1}(0) + \tau t.$$

We introduce general parameters, for $j = 1, \dots, J$, (a_j, b_j) . If $P_j = B_k$ is a breather, we set $(a_j, b_j) := (\alpha_k, \beta_k)$. If $P_j = R_l$ is a soliton, we set $(a_j, b_j) := (0, \sqrt{c_l^0})$.

2.2. Modulation lemma. We will first state a standard modulation lemma, which can be proved similarly as the modulation lemma in [39]. We need it because we will construct translations of our objects so that they are near to u and some orthogonality conditions are satisfied. These orthogonality conditions will allow us to use coercivity of some quadratic forms in the following of the proof.

Lemma 2.1. *Let $A' > 0$, $\theta > 0$, $t' > 0$ and $y_{1,k}(t)$, $y_{2,k}(t)$, $y_{3,l}(t)$, $y_{0,l}(t)$ defined for $t \in [0, t']$ (with $y_{3,l}(t) > 0$) such that $\forall t \in [0, t']$, $|y_{3,l}(t) - c_l^0| \leq \min\{\frac{t}{8}\} \cup \{\frac{c_l^0}{4}, 1 \leq l \leq L\}$. If D_0 is large enough and a_0 is small enough (dependently on A'), there exists a constant $C_2 > 1$ such that the following holds. Let $u(t)$ be a solution of (mKdV) such that for any $t \in [0, t']$,*

$$(2.4) \quad \left\| u(t) - \sum_{l=1}^L \kappa_l Q_{y_{3,l}(t)}(\cdot + y_{0,l}(t) - c_l^0 t) - \sum_{k=1}^K B_{\alpha_k, \beta_k}(t, \cdot; y_{1,k}(t), y_{2,k}(t)) \right\|_{H^2} \leq A' (a + e^{-\theta D}),$$

$$(2.5) \quad \left\| u(0) - \sum_{l=1}^L \kappa_l Q_{y_{3,l}(0)}(\cdot + y_{0,l}(0)) - \sum_{k=1}^K B_{\alpha_k, \beta_k}(0, \cdot; y_{1,k}(0), y_{2,k}(0)) \right\|_{H^2} \leq a,$$

and if we set $y_j(t) := -y_{0,l}(t) + v_l^s t$ if $P_j = R_l$, and $y_j(t) := -y_{2,k}(t) + v_k^b t$ if $P_j = B_k$, we have, for any $2 \leq j \leq J$,

$$(2.6) \quad \forall t \in [0, t'], \quad y_j(t) - y_{j-1}(t) \geq D,$$

then, there exists C^1 functions $z_{1,k}(t)$, $z_{2,k}(t)$, $z_{3,l}(t)$, $z_{0,l}(t)$ defined for $t \in [0, t']$ (with $z_{3,l}(t) > 0$), such that if we set

$$(2.7) \quad \varepsilon(t) := u(t) - \tilde{P}(t),$$

where

$$(2.8) \quad \tilde{R}_l(t, x) := \kappa_l Q_{z_{3,l}(t)}(\cdot + z_{0,l}(t) - c_l^0 t) \quad \text{for } 1 \leq l \leq L,$$

$$(2.9) \quad \tilde{B}_k(t, x) := B_{\alpha_k, \beta_k}(t, x; z_{1,k}(t), z_{2,k}(t)) \quad \text{for } 1 \leq k \leq K,$$

$$(2.10) \quad \tilde{P}_j := \tilde{R}_l \quad \text{if } P_j = R_l, \quad \tilde{P}_j := \tilde{B}_k \quad \text{if } P_j = B_k,$$

$$(2.11) \quad \tilde{R} := \sum_{l=1}^L \tilde{R}_l, \quad \tilde{B} := \sum_{k=1}^K \tilde{B}_k, \quad \tilde{P} := \tilde{R} + \tilde{B} = \sum_{j=1}^J \tilde{P}_j,$$

then, for $t \in [0, t']$, for $l = 1, \dots, L$, for $k = 1, \dots, K$,

$$(2.12) \quad \int \tilde{R}_l(t) \varepsilon(t) = \int \partial_x \tilde{R}_l(t) \varepsilon(t) = \int \partial_{x_1} \tilde{B}_k(t) \varepsilon(t) = \int \partial_{x_2} \tilde{B}_k(t) \varepsilon(t) = 0.$$

Moreover, for $t \in [0, t']$, we have

$$(2.13) \quad \begin{aligned} & \|\varepsilon(t)\|_{H^2} + |z_{1,k}(t) - y_{1,k}(t)| + |z_{2,k}(t) - y_{2,k}(t)| \\ & + |z_{3,l}(t) - y_{3,l}(t)| + |z_{0,l}(t) - y_{0,l}(t)| \leq C_2 A' (a + e^{-\theta D}), \end{aligned}$$

and

$$(2.14) \quad \|\varepsilon(0)\|_{H^2} + |z_{1,k}(0) - y_{1,k}(0)| + |z_{2,k}(0) - y_{2,k}(0)| + |z_{3,l}(0) - y_{3,l}(0)| + |z_{0,l}(0) - y_{0,l}(0)| \leq C_2 a,$$

and for any $t \in [0, t']$, $(z_{1,k}(t), z_{2,k}(t), z_{3,l}(t), z_{0,l}(t)) \in \mathbb{R}^{2K+2L}$ is unique such that (2.12) is satisfied and $(z_{1,k}(t), z_{2,k}(t), z_{3,l}(t), z_{0,l}(t))$ is in a suitable neighbourhood of $(y_{1,k}(t), y_{2,k}(t), y_{3,l}(t), y_{0,l}(t))$ that depends only on $A' (a + e^{-\theta D})$.

We set, for $t \in [0, t']$, $z_j(t) := z_l^s(t) := -z_{0,l}(t) + v_l^s t$ if $P_j = R_l$, and $z_j(t) := z_k^b(t) := -z_{2,k}(t) + v_k^b t$ if $P_j = B_k$.

For D_0 large enough and a_0 small enough, if we assume that

$$(2.15) \quad \forall t \in [0, t'], \quad z_j(t) - z_{j-1}(t) \geq D,$$

(note that (2.15) is a consequence of $\forall t \in [0, t']$, $y_j(t) - y_{j-1}(t) \geq 2D$ if a_0 is small enough and D_0 is large enough), then for any $t \in [0, t']$, we have that

for $k = 1, \dots, K$,

$$(2.16) \quad |z'_{1,k}(t)| + |z'_{2,k}(t)| \leq C_2 \left(\int e^{-\frac{\beta}{2}|x-z_k^b(t)|} \varepsilon^2 \right)^{1/2} + C_2 e^{-\frac{\beta D}{8}},$$

for $l = 1, \dots, L$,

$$(2.17) \quad |z'_{3,l}(t)| + |z'_{0,l}(t)| \leq C_2 \left(\int e^{-\frac{\beta}{2}|x-z_l^s(t)|} \varepsilon^2 \right)^{1/2} + C_2 e^{-\frac{\beta D}{8}}.$$

Remark 2.2. We will also use generalized notations for $y_{1,k}(t)$, $y_{2,k}(t)$ and other object-specific notations, in the lemma above.

For $j = 1, \dots, J$, either if $P_j = R_l$ is a soliton, we denote

$$(2.18) \quad y_{1,j}^*(t) := y_{3,l}(t), \quad z_{1,j}^*(t) := z_{3,l}(t), \quad y_{2,j}^*(t) := y_{0,l}(t), \quad z_{2,j}^*(t) := z_{0,l}(t),$$

or if $P_j = B_k$ is a breather, we denote

$$(2.19) \quad y_{1,j}^*(t) := y_{1,k}(t), \quad z_{1,j}^*(t) := z_{1,k}(t), \quad y_{2,j}^*(t) := y_{2,k}(t), \quad z_{2,j}^*(t) := z_{2,k}(t).$$

Proof. See [39, Lemma 2.8], for the proof of a similar result. We also refer to [34, 3, 11]. \square

2.3. Bootstrap. Given Lemma 2.1, we reduce the proof of Theorem 1.1 to the following bootstrap proposition (we will use the notations given in Section 1.2; in particular the position $x_j(t)$ of $P_j(t)$ is defined there):

Proposition 2.3. *There exists $A \geq C_2$ and $\theta > 0$ such that, if D_0 is large enough and a_0 is small enough such that*

$$(2.20) \quad C_2 a \leq A \left(a + e^{-\theta D} \right) \leq \min \left\{ \frac{\tau}{8} \right\} \cup \left\{ \frac{c_p^0}{4}, 1 \leq p \leq L \right\},$$

for $t^* > 0$, we assume that there exist C^1 functions $x_{1,k}(t), x_{2,k}(t), c_l(t), x_{0,l}(t) \in \mathbb{R}$ (with $c_l(t) > 0$) defined for $t \in [0, t^*]$ such that, if we denote

$$(2.21) \quad \forall t \in [0, t^*], \quad \varepsilon(t) := u(t) - \sum_{l=1}^L \tilde{R}_l(t) - \sum_{k=1}^K \tilde{B}_k(t),$$

where u is a solution of (mKdV),

$$(2.22) \quad \forall t \in [0, t^*], \quad \tilde{R}_l(t) := \kappa_l Q_{c_l(t)}(\cdot + x_{0,l}(t) - c_l^0 t) \quad \text{for } 1 \leq l \leq L,$$

and

$$(2.23) \quad \forall t \in [0, t^*], \quad \tilde{B}_k(t) := B_{\alpha_k, \beta_k}(t, \cdot; x_{1,k}(t), x_{2,k}(t)), \quad \text{for } 1 \leq k \leq K,$$

$\tilde{P}_j := \tilde{R}_l$ if $P_j = R_l$ and $\tilde{P}_j := \tilde{B}_k$ if $P_j = B_k$, and $\tilde{x}_l^s(t) := -x_{0,l}(t) + v_l^s t$, $\tilde{x}_k^b(t) := -x_{2,k}(t) + v_k^b t$, $\tilde{x}_j(t) := \tilde{x}_l^s(t)$ if $P_j = R_l$, and $\tilde{x}_j(t) := \tilde{x}_k^b(t)$ if $P_j = B_k$,

and if we assume that

$$(2.24) \quad \forall 1 \leq j \leq J-1, \quad x_{j+1}(0) - x_j(0) \geq 2D,$$

$$(2.25) \quad \forall t \in [0, t^*], \quad \|\varepsilon(t)\|_{H^2} \leq A \left(a + e^{-\theta D} \right), \quad \|\varepsilon(0)\|_{H^2} \leq C_2 a,$$

$$(2.26) \quad \forall t \in [0, t^*], \quad |c_l(t) - c_l^0| \leq A \left(a + e^{-\theta D} \right), \quad |c_l(0) - c_l^0| \leq C_2 a,$$

$$(2.27) \quad \sum_{l=1}^L |x_{0,l}(0) + x_{0,l}^0| + \sum_{k=1}^K (|x_{1,k}(0) - x_{1,k}^0| + |x_{2,k}(0) - x_{2,k}^0|) \leq Ca,$$

$$(2.28) \quad \forall t \in [0, t^*], \quad \sum_{l=1}^L (|c'_l(t)| + |x'_{0,l}(t)|) + \sum_{k=1}^K (|x'_{1,k}(t)| + |x'_{2,k}(t)|) \leq CA(a + e^{-\theta D}),$$

where $C > 0$ is a large enough constant, and

$$(2.29) \quad \forall t \in [0, t^*], \quad \int \tilde{B}_{k1}(t)\varepsilon(t) = \int \tilde{B}_{k2}(t)\varepsilon(t) = \int \tilde{R}_l(t)\varepsilon(t) = \int \tilde{R}_{lx}(t)\varepsilon(t) = 0,$$

then

$$(2.30) \quad \forall t \in [0, t^*], \quad \|\varepsilon(t)\|_{H^2} \leq \frac{A}{2} \left(a + e^{-\theta D} \right),$$

$$(2.31) \quad \forall t \in [0, t^*], \quad |c_l(t) - c_l^0| \leq \frac{A}{2} \left(a + e^{-\theta D} \right).$$

Remark 2.4. We will use generalized notations for $x_{1,k}(t)$, $x_{2,k}(t)$, $c_l(t)$ and $x_{0,l}(t)$.

For $j = 1, \dots, J$, either if $P_j = R_l$ is a soliton, we denote

$$(2.32) \quad x_{1,j}^*(t) := c_l(t), \quad x_{2,j}^*(t) := x_{0,l}(t),$$

or if $P_j = B_k$ is a breather, we denote

$$(2.33) \quad x_{1,j}^*(t) := x_{1,k}(t), \quad x_{2,j}^* := x_{2,k}(t).$$

We denote

$$(2.34) \quad \tilde{P}(t) := \sum_{j=1}^J \tilde{P}_j(t) = \sum_{k=1}^K \tilde{B}_k(t) + \sum_{l=1}^L \tilde{R}_l(t).$$

Remark 2.5. We note that, for any $j = 1, \dots, J-1$, (2.24) and (2.27), assuming that a_0 is chosen small enough and D_0 is chosen large enough, imply that

$$(2.35) \quad \widetilde{x_{j+1}}(0) - \tilde{x}_j(0) \geq D.$$

From definitions of $\tilde{x}_j(t)$, of τ and (2.28), assuming that a_0 is chosen small enough and D_0 is chosen large enough, we deduce that for any $t \in [0, t^*]$,

$$(2.36) \quad \forall 1 \leq j \leq J-1, \quad \widetilde{x_{j+1}}'(t) - \tilde{x}_j'(t) \geq \frac{\tau}{2},$$

and

$$(2.37) \quad \forall 2 \leq j \leq J, \quad \frac{v_2}{2} \leq \tilde{x}_j'(t) \leq 2v_J.$$

From (2.35) and (2.36), we may deduce that for any $t \in [0, t^*]$,

$$(2.38) \quad \forall 1 \leq j \leq J, \quad \widetilde{x_{j+1}}(t) - \tilde{x}_j(t) \geq D + \frac{\tau}{2}t.$$

The proof of this proposition will be the goal of the following. The proof of Theorem 1.1 then follows from a continuity argument.

Proof of Theorem 1.1 assuming Proposition 2.3. We take A, θ, D_0, a_0 that work for Proposition 2.3, we will show that they will also work for Theorem 1.1. We take $D \geq D_0$ and $0 \leq a \leq a_0$ and we assume that (1.22) is true for a solution u of (mKdV). This implies (2.24).

We assume that a_0 is small enough and D_0 is large enough such that $A(a + e^{-\theta D}) \leq \min\{\frac{\tau}{8}\} \cup \{\frac{c_p^0}{4}, 1 \leq p \leq L\}$.

Because $\frac{A}{C_2} > 1$, by continuity, there exists $t_1 > 0$ such that

$$(2.39) \quad \forall t \in [0, t_1], \quad \|u(t) - P(t)\|_{H^2} \leq \frac{A}{C_2} \left(a + e^{-\theta D} \right).$$

Of course, we have that

$$(2.40) \quad \forall t \in [0, t_1], \quad x_j(t) - x_{j-1}(t) \geq 2D + \tau t.$$

We apply Lemma 2.1 on $[0, t_1]$ with $A' = \frac{A}{C_2}$. We take D_0 larger and a_0 smaller if needed. So, there exist C^1 functions $x_{1,k}(t), x_{2,k}(t), c_l(t), x_{0,l}(t) \in \mathbb{R}$ with $c_l(t) > 0$ defined for $t \in [0, t_1]$, such that if we set

$$(2.41) \quad \varepsilon(t) := u(t) - \tilde{P}(t),$$

with the same notations as usual, we have that

$$(2.42) \quad \forall t \in [0, t_1], \quad \|\varepsilon(t)\|_{H^2} \leq A \left(a + e^{-\theta D} \right), \quad \|\varepsilon(0)\|_{H^2} \leq C_2 a,$$

$$(2.43) \quad \forall t \in [0, t_1], \quad |c_l(t) - c_l^0| \leq A \left(a + e^{-\theta D} \right), \quad |c_l(0) - c_l^0| \leq C_2 a,$$

$$(2.44) \quad \sum_{l=1}^L |x_{0,l}(0) + x_{0,l}^0| + \sum_{k=1}^K (|x_{1,k}(0) - x_{1,k}^0| + |x_{2,k}(0) - x_{2,k}^0|) \leq Ca,$$

$$(2.45) \quad \forall t \in [0, t_1], \quad \sum_{l=1}^L (|c'_l(t)| + |x'_{0,l}(t)|) + \sum_{k=1}^K (|x'_{1,k}(t)| + |x'_{2,k}(t)|) \leq CA(a + e^{-\theta D}),$$

if $\theta > 0$ is chosen small enough, and

$$(2.46) \quad \forall t \in [0, t_1], \quad \int \tilde{B}_{k_1}(t)\varepsilon(t) = \int \tilde{B}_{k_2}(t)\varepsilon(t) = \int \tilde{R}_l(t)\varepsilon(t) = \int \tilde{R}_{l_x}(t)\varepsilon(t) = 0.$$

Remember that (2.38) for any $t \in [0, t_1]$ is a consequence of what is written above.

Let $I^* \subset \mathbb{R}_+$ be the supremum of intervals $I' \supset [0, t_1]$ for inclusion in the set of intervals of \mathbb{R}_+ such that the C^1 functions $x_{1,k}(t)$, $x_{2,k}(t)$, $c_l(t)$, $x_{0,l}(t)$ may be extended on I' and such that (2.24), (2.25), (2.26), (2.27), (2.28) and (2.29) are still satisfied for any $t \in I'$.

In order to make the definition above licit, we need to point out the following fact: by uniqueness in Lemma 2.1, we find that if we have two extensions on I_2 and I_3 , then we find that those extensions coincide on $I_2 \cap I_3$ (we remind that $[0, t_1] \subset I_2 \cap I_3$). This is why, if we have suitable extensions on I_2 and I_3 , then we have a suitable extension on $I_2 \cup I_3$. And, the supremum I^* is simply the union of all the possible extensions I' and the implicit functions may be extended on I^* .

When (2.24), (2.25), (2.26), (2.27), (2.28) and (2.29) are true for $x_{1,k}(t)$, $x_{2,k}(t)$, $c_l(t)$, $x_{0,l}(t)$ with $I = [0, t_2] \subset I^*$ (where $t_2 \geq t_1$), we may extend these implicit functions a bit further (in a random way, that we denote $y_{1,k}(t)$, $y_{2,k}(t)$, $y_{3,l}(t)$ and $y_{0,l}(t)$, and we denote y_j their positions) and have (2.24), (2.25), (2.26), (2.27), (2.28) and (2.29) that are still satisfied, but on the extended interval. The extended interval will be an interval of the form $[0, t'_2]$, where $t'_2 > t_2$. We do it in the following way. First, we apply Proposition 2.3 on I , and that makes (2.25) and (2.26) a bit improved on I and become (2.30) and (2.31) on I . And so, (2.25) and (2.26) are satisfied on an extension of I . After application of Lemma 2.1 with $A' = A$ (where the notation A' is from Lemma 2.1), that we may apply thanks to (2.38) (more precisely, from Remark 2.5, (2.38) is satisfied on I , from what for any $1 \leq j \leq J-1$, $\widetilde{y_{j+1}}(t) - \widetilde{y_j}(t) \geq D$ is true on an extension of I) we see that (2.29) can be also extended (after modification of the implicit functions on the extension of I : we now denote them $x_{1,k}(t)$, $x_{2,k}(t)$, $x_{3,l}(t)$ and $x_{0,l}(t)$, as they coincide with implicit functions given at the beginning of this paragraph on I from uniqueness in Lemma 2.1). Note that after this modification, it is needed to reconsider all the previous sentences of this paragraph, in order to take a smaller extension if needed and have (2.25) and (2.26) satisfied. Because of Remark 2.5, (2.38) is satisfied on I , this is why for any $1 \leq j \leq J-1$, $\widetilde{x_{j+1}}(t) - \widetilde{x_j}(t) \geq D$ is satisfied on an extension of I (eventually smaller, but of the form $[0, t'_2]$ as specified at the beginning of the paragraph). This is why, we may apply consequences (2.16) and (2.17) of Lemma 2.1 and obtain that (2.28) is also satisfied on this extension of I . Thus, we may find an extension $[0, t'_2]$ (with $t'_2 > t_2$) of I and extensions of $x_{1,k}(t)$, $x_{2,k}(t)$, $c_l(t)$, $x_{0,l}(t)$ on this extension such that (2.24) (2.25), (2.26), (2.27) (2.28) and (2.29) are satisfied.

We deduce that the interval I^* is necessarily an open subset of \mathbb{R}_+ . Let us prove that I^* is also a closed subset of \mathbb{R}_+ . This will allow us to conclude that $I^* = \mathbb{R}_+$.

If I^* is not closed, then it is $[0, t_2)$, where $t_2 > 0$. Let (T_n) be a sequence of points of I^* that converges to t_2 . Then, $(x_{1,k}(T_n))$, $(x_{2,k}(T_n))$, $(c_l(T_n))$ and $(x_{0,l}(T_n))$ are Cauchy sequences because of (2.28). Thus, they converge and $x_{1,k}(t)$, $x_{2,k}(t)$, $c_l(t)$ and $x_{0,l}(t)$ may be extended continuously in $t = t_2$. By continuity, it is clear that (2.25), (2.26), (2.29) and (2.38) are still satisfied in $t = t_2$. This is why, we may apply Lemma 2.1 on $[0, t_2]$. From uniqueness in Lemma 2.1, we find that $x_{1,k}(t)$, $x_{2,k}(t)$, $c_l(t)$ and $x_{0,l}(t)$ are C^1 on $[0, t_2]$. Thus, (2.28) is also satisfied in $t = t_2$. That contradicts the maximality of I^* . Thus, I^* is closed in \mathbb{R}_+ .

So, we deduce that we have C^1 functions $x_{1,k}(t), x_{2,k}(t), c_l(t), x_{0,l}(t)$ defined for any $t \geq 0$ such that,

$$(2.47) \quad \forall t \geq 0, \quad \|\varepsilon(t)\|_{H^2} \leq A \left(a + e^{-\theta D} \right),$$

$$(2.48) \quad \forall t \geq 0, \quad |c_l(t) - c_l^0| \leq A \left(a + e^{-\theta D} \right),$$

$$(2.49) \quad \forall t \geq 0, \quad \sum_{l=1}^L (|c_l'(t)| + |x'_{0,l}(t)|) + \sum_{k=1}^K (|x'_{1,k}(t)| + |x'_{2,k}(t)|) \leq CA(a + e^{-\theta D}),$$

where $C > 0$ is a large enough constant, and

$$(2.50) \quad \forall t \geq 0, \quad \int \tilde{B}_{k1}(t)\varepsilon(t) = \int \tilde{B}_{k2}(t)\varepsilon(t) = \int \tilde{R}_l(t)\varepsilon(t) = \int \tilde{R}_{lx}(t)\varepsilon(t) = 0.$$

And we set

$$(2.51) \quad w(t) := u(t) - \sum_{l=1}^L \kappa_l Q_{c_l^0}(\cdot + x_{0,l}(t) - c_l^0 t) - \sum_{k=1}^K B_{\alpha_k, \beta_k}(t, \cdot; x_{1,k}(t), x_{2,k}(t)).$$

We need to bound w to finish the proof. For $t \geq 0$, we use triangular inequality and we bound the norm between two ground states centered at a same point,

$$(2.52) \quad \begin{aligned} \|w(t)\|_{H^2} &\leq \|\varepsilon(t)\|_{H^2} + \sum_{l=1}^L \left\| Q_{c_l(t)} - Q_{c_l^0} \right\|_{H^2} \\ &\leq \|\varepsilon(t)\|_{H^2} + C \sum_{l=1}^L |c_l(t) - c_l^0| \\ &\leq CA \left(a + e^{-\theta D} \right), \end{aligned}$$

and this is exactly what we wanted to prove. Moreover (1.24) is a straightforward consequence of (2.49). \square

Hence, we are left to prove Proposition 2.3.

2.4. Proof by induction. We will prove Proposition 2.3 by induction. More precisely, we want to find $A > C_2, \theta, D_0, a_0$ so that the Proposition 2.3 holds for any $t^* > 0$.

We assume that there exists functions $x_{1,k}(t), x_{2,k}(t), c_l(t), x_{0,l}(t) \in \mathbb{R}$ defined for $t \in [0, t^*]$ such that, with notations of Proposition 2.3,

$$(2.53) \quad \forall t \in [0, t^*], \quad \|\varepsilon(t)\|_{H^2} \leq A(a + e^{-\theta D}), \quad \|\varepsilon(0)\|_{H^2} \leq C_2 a,$$

$$(2.54)$$

$$\forall t \in [0, t^*], \quad |c_l(t) - c_l^0| \leq A(a + e^{-\theta D}) \leq \min \left\{ \frac{\tau}{8} \right\} \cup \left\{ \frac{c_p^0}{4}, 0 \leq p \leq L \right\}, \quad |c_l(0) - c_l^0| \leq C_2 a,$$

$$(2.55) \quad \forall t \in [0, t^*], \quad \sum_{l=1}^L (|c_l'(t)| + |x'_{0,l}(t)|) + \sum_{k=1}^K (|x'_{1,k}(t)| + |x'_{2,k}(t)|) \leq CA(a + e^{-\theta D}),$$

$$(2.56) \quad \forall t \in [0, t^*], \quad \int \tilde{B}_{k1}(t)\varepsilon(t) = \int \tilde{B}_{k2}(t)\varepsilon(t) = \int \tilde{R}_l(t)\varepsilon(t) = \int \tilde{R}_{lx}(t)\varepsilon(t) = 0,$$

as well as (2.27) and (2.24).

As in Remark 2.5, we deduce that (2.36), (2.37) and (2.38) are satisfied.

And the goal is to improve inequalities (2.53) and (2.54).

We define the average between positions of two consecutive objects. For $j = 3, \dots, J$, we set

$$(2.57) \quad \forall t \in [0, t^*], \quad m_j(t) := \frac{\tilde{x}_{j-1}(t) + \tilde{x}_j(t)}{2},$$

and we set

$$(2.58) \quad \forall t \in [0, t^*], \quad m_2(t) := \frac{\tilde{x}_1(0) + \tilde{x}_2(0)}{2} + \int_0^t \max \left(\frac{\tilde{x}_1'(s) + \tilde{x}_2'(s)}{2}, \frac{\tilde{x}_2'(s)}{2} \right) ds.$$

With these definitions, we make sure that for any $j = 2, \dots, J$, $m_j'(t) > 0$, even if $v_1 < 0$.

By (2.36), for $j \geq 3$,

$$(2.59) \quad \forall t \in [0, t^*], \quad \tilde{x}_j'(t) - m_j'(t) \geq \frac{\tau}{4}, \quad m_j'(t) - \widetilde{x_{j-1}}'(t) \geq \frac{\tau}{4},$$

and for $j = 2$, we have

$$(2.60) \quad \forall t \in [0, t^*], \quad m_2'(t) = \max \left(\frac{\tilde{x}_1'(t) + \tilde{x}_2'(t)}{2}, \frac{\tilde{x}_2'(t)}{2} \right),$$

and so,

$$(2.61) \quad \forall t \in [0, t^*], \quad m_2'(t) - \tilde{x}_1'(t) \geq \frac{\tau}{4}, \quad \tilde{x}_2'(t) - m_2'(t) \geq \min \left(\frac{v_2}{4}, \frac{\tau}{4} \right).$$

This is why, we set

$$(2.62) \quad \zeta := \min \left(\frac{v_2}{4}, \frac{\tau}{4} \right),$$

a constant that depends only on problem data, and so for any $j \geq 2$,

$$(2.63) \quad \forall t \in [0, t^*], \quad \tilde{x}_j'(t) - m_j'(t) \geq \zeta, \quad m_j'(t) - \widetilde{x_{j-1}}'(t) \geq \zeta.$$

The latter implies that $\forall j \geq 2$, $\forall t \in [0, t^*]$, $\widetilde{x_{j-1}}(t) < m_j(t) < \tilde{x}_j(t)$, and we may deduce by integration and by (2.38) and (2.63) that

$$(2.64) \quad \begin{aligned} \tilde{x}_j(t) - m_j(t) &= \tilde{x}_j(0) - m_j(0) + \int_0^t (\tilde{x}_j'(s) - m_j'(s)) ds \\ &= \frac{\tilde{x}_j(0) - \widetilde{x_{j-1}}(0)}{2} + \int_0^t (\tilde{x}_j'(s) - m_j'(s)) ds \\ &\geq \frac{D}{2} + \zeta t, \end{aligned}$$

and similarly,

$$(2.65) \quad m_j(t) - \widetilde{x_{j-1}}(t) \geq \frac{D}{2} + \zeta t.$$

We have that (the m_j are chosen for that) for any $j \geq 2$,

$$(2.66) \quad \forall t \in [0, t^*], \quad 2v_j \geq \tilde{x}_j'(t) \geq m_j'(t) \geq \frac{v_2}{4} \geq \zeta.$$

We will reason by induction in order to improve (2.53) and (2.54). For this, we introduce a cut-off function.

Let $\sigma > 0$ be a constant small enough for which the conditions will be fixed in the following of the proof.

We denote:

$$(2.67) \quad \Psi(x) := \frac{2}{\pi} \arctan(\exp(\sqrt{\sigma}x/2)).$$

By direct calculations,

$$(2.68) \quad \Psi'(x) = \frac{\sqrt{\sigma}}{2\pi \cosh(\sqrt{\sigma}x/2)},$$

and so,

$$(2.69) \quad |\Psi'(x)| \leq C \exp(-\sqrt{\sigma}|x|/2).$$

We have the following properties: $\lim_{+\infty} \Psi = 1$, $\lim_{-\infty} \Psi = 0$, for all $x \in \mathbb{R}$ $\Psi(-x) + \Psi(x) = 1$, $\Psi'(x) > 0$, $|\Psi''(x)| \leq \frac{\sqrt{\sigma}}{2} |\Psi'(x)|$, $|\Psi'''(x)| \leq \frac{\sqrt{\sigma}}{2} |\Psi''(x)|$, $|\Psi'(x)| \leq \frac{\sqrt{\sigma}}{2} \Psi$ and $|\Psi'(x)| \leq \frac{\sqrt{\sigma}}{2} (1 - \Psi)$.

We define cut-off functions filtering P_j and all the objects faster than P_j : for $j = 2, \dots, J$,

$$(2.70) \quad \Phi_j(t, x) := \Psi(x - m_j(t)).$$

We have:

$$(2.71) \quad (\Phi_j)_t = -m_j'(\Phi_j)_x.$$

We may extend this definition to $j = 1$ and $j = J + 1$ in the following way: $\Phi_1 := 1$ and $\Phi_{J+1} := 0$.

In order to prove Proposition 2.3 by induction, we will find an increasing sequence $(Z_j)_{j=1,\dots,J+1}$ such that $Z_1 := 2$ and $Z_{J+1} := +\infty$ and such that we will be able to prove the following proposition for any $j = 1, \dots, J$. The goal is to obtain inequalities of Proposition 2.3 with better constants. So, in order to achieve this, we do the following induction: we suppose that localized inequalities around P_{j+1}, \dots, P_J are obtained with strongly improved constants (constants divided by Z_{j+1}), and we deduce from them localized inequalities around P_j, \dots, P_J with improved constants, but a little bit less improved than earlier (constants divided by Z_j). We will also assume the bootstrap assumption. This induction is sufficient, because it starts from an assumption on an empty set of objects and ends with a conclusion with inequalities localized around all the objects, i.e. global (not localized at all).

Proposition 2.6. *Assuming that*

$$(2.72) \quad \forall t \in [0, t^*], \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_{j+1} \leq \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

and for any $j' \geq j+1$ such that $P_{j'} = R_l$ is a soliton,

$$(2.73) \quad \forall t \in [0, t^*], \quad |c_l(t) - c_l(0)| \leq \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

and for any $j' \geq j+1$ such that $P_{j'}$ is a breather,

$$(2.74) \quad \forall t \in [0, t^*], \quad \left| \int \widetilde{P}_{j'} \varepsilon(t) - \int \widetilde{P}_{j'} \varepsilon(0) \right| \leq \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

and

$$(2.75) \quad \forall t \in [0, t^*], \quad \left| \int [\widetilde{P}_{j'}'_{xx} + \widetilde{P}_{j'}'^3] \varepsilon(t) - \int [\widetilde{P}_{j'}'_{xx} + \widetilde{P}_{j'}'^3] \varepsilon(0) \right| \leq \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

we have that

$$(2.76) \quad \forall t \in [0, t^*], \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq \left(\frac{A}{Z_j} \right)^2 (a^2 + e^{-2\theta D}),$$

and if $P_j = R_l$ is a soliton,

$$(2.77) \quad \forall t \in [0, t^*], \quad |c_l(t) - c_l(0)| \leq \left(\frac{A}{Z_j} \right)^2 (a^2 + e^{-2\theta D}),$$

and if P_j is a breather,

$$(2.78) \quad \forall t \in [0, t^*], \quad \left| \int \widetilde{P}_j \varepsilon(t) - \int \widetilde{P}_j \varepsilon(0) \right| \leq \left(\frac{A}{Z_j} \right)^2 (a^2 + e^{-2\theta D}),$$

and

$$(2.79) \quad \forall t \in [0, t^*], \quad \left| \int [\widetilde{P}_j'_{xx} + \widetilde{P}_j'^3] \varepsilon(t) - \int [\widetilde{P}_j'_{xx} + \widetilde{P}_j'^3] \varepsilon(0) \right| \leq \left(\frac{A}{Z_j} \right)^2 (a^2 + e^{-2\theta D}).$$

Remark 2.7. For $j = 1, \dots, J$, we may denote by \mathcal{P}_j the following assertion: (2.76), (2.77), (2.78) and (2.79). The Proposition 2.6 may be reformulated in the following way:

There exists an increasing sequence $(Z_j)_{j=1,\dots,J+1}$ with $Z_1 = 2$ and $Z_{J+1} = +\infty$, A large enough and $\theta > 0$ such that for D_0 large enough and for a_0 small enough, we have the following: for any $j = 1, \dots, J$,

$$(2.80) \quad \mathcal{P}_{j+1} \implies \mathcal{P}_j.$$

Remark 2.8. Note that the inequalities (2.78) and (2.79) imply the following inequality for any $t \in [0, t^*]$ in the case when P_j is a breather:

$$(2.81) \quad \left| \int \left[\widetilde{P}_j'_{xxxx} + 5\widetilde{P}_j' \widetilde{P}_j'^2 + 5\widetilde{P}_j'^2 \widetilde{P}_j'_{xx} + \frac{3}{2} \widetilde{P}_j'^5 \right] \varepsilon(t) - \int \left[\widetilde{P}_j'_{xxxx} + 5\widetilde{P}_j' \widetilde{P}_j'^2 + 5\widetilde{P}_j'^2 \widetilde{P}_j'_{xx} + \frac{3}{2} \widetilde{P}_j'^5 \right] \varepsilon(0) \right| \leq C \left(\frac{A}{Z_j} \right)^2 (a^2 + e^{-2\theta D}),$$

because of the elliptic equation verified by \tilde{P}_j , which, in the case when $\tilde{P}_j = \tilde{B}_k$ is a breather is the following:

$$(2.82) \quad \tilde{B}_{kxxxx} + 5\tilde{B}_k\tilde{B}_{kx}^2 + 5\tilde{B}_k^2\tilde{B}_{kxx} + \frac{3}{2}\tilde{B}_k^5 - 2(\beta_k^2 - \alpha_k^2)(\tilde{B}_{kxx} + \tilde{B}_k^3) + (\alpha_k^2 + \beta_k^2)^2\tilde{B}_k = 0.$$

But the inequality (2.81) is also true in the case when $\tilde{P}_j = \tilde{R}_l$ is a soliton, because we have even better in this case. There are two elliptic equations [39]:

$$(2.83) \quad \tilde{R}_{lxxxx} + 5\tilde{R}_l\tilde{R}_{lx}^2 + 5\tilde{R}_l^2\tilde{R}_{lxx} + \frac{3}{2}\tilde{R}_l^5 - 2c_l(t)(\tilde{R}_{lxx} + \tilde{R}_l^3) + c_l(t)^2\tilde{R}_l = 0,$$

and

$$(2.84) \quad (\tilde{R}_{lxx} + \tilde{R}_l^3) - c_l(t)\tilde{R}_l = 0,$$

which implies that (2.56) implies

$$(2.85) \quad \int (\tilde{R}_{lxx} + \tilde{R}_l^3) \varepsilon = 0,$$

and

$$(2.86) \quad \int \left(\tilde{R}_{lxxxx} + 5\tilde{R}_l\tilde{R}_{lx}^2 + 5\tilde{R}_l^2\tilde{R}_{lxx} + \frac{3}{2}\tilde{R}_l^5 \right) \varepsilon = 0,$$

which implies of course the inequality (2.81).

The proof of Proposition 2.6 will be the goal of the Section 3. The proof of Proposition 2.3 follows from it.

Proof of Proposition 2.3 assuming Proposition 2.6. We perform the induction in the decreasing order: $j = J, J-1, \dots, 2, 1$. \mathcal{P}_{J+1} is empty, and Proposition 2.6 gives the (decreasing) induction step. Hence, $\mathcal{P}_1, \dots, \mathcal{P}_J$ are true. Due to \mathcal{P}_1 :

$$(2.87) \quad \forall t \in [0, t^*], \quad \|\varepsilon(t)\|_{H^2} \leq \frac{A}{2} (a + e^{-\theta D}).$$

For $l = 1, \dots, L$, we have from \mathcal{P}_1 that for any $t \in [0, t^*]$,

$$(2.88) \quad \begin{aligned} |c_l(t) - c_l^0| &\leq |c_l(t) - c_l(0)| + |c_l(0) - c_l^0| \\ &\leq \left(\frac{A}{2}\right)^2 (a^2 + e^{-2\theta D}) + C_2 a \\ &\leq \left[\left(\frac{A}{2}\right)^2 (a + e^{-\theta D}) + C_2 \right] (a + e^{-\theta D}). \end{aligned}$$

If we take A large enough with respect to C_2 , and a_0 smaller and D_0 larger if needed with respect to A and θ , then

$$(2.89) \quad \left(\frac{A}{2}\right)^2 (a + e^{-\theta D}) + C_2 \leq \frac{A}{2},$$

and that concludes the proof of Proposition 2.3. □

Hence, we are left to prove Proposition 2.6. We will write the proof for a fixed $j \in \{1, \dots, J\}$. We assume \mathcal{P}_{j+1} with a set of constants $Z_1, \dots, Z_{j+1}, A, \theta, D_0, a_0$. We will establish some conditions for these constants during the proof.

3. ORBITAL STABILITY OF A SUM OF SOLITONS AND BREATHERS IN $H^2(\mathbb{R})$

In this Section, we prove Proposition 2.6. We assume \mathcal{P}_{j+1} and we prove \mathcal{P}_j .

3.1. Almost decay of conservation laws at the right. We localize around the most right objects, starting from and including the j -th. We set:

$$(3.1) \quad M_j(t) := \frac{1}{2} \int u^2(t) \Phi_j(t) =: M_j[u](t),$$

$$(3.2) \quad E_j(t) := \int \left[\frac{1}{2} u_x^2 - \frac{1}{4} u^4 \right] \Phi_j(t) =: E_j[u](t),$$

$$(3.3) \quad F_j(t) := \int \left[\frac{1}{2} u_{xx}^2 - \frac{5}{2} u^2 u_x^2 + \frac{1}{4} u^6 \right] \Phi_j(t) =: F_j[u](t).$$

Lemma 3.1. *Let $0 < \omega_1, \omega_2 < 1$. If $0 < \sigma < \zeta$, $0 < \theta < \frac{\sqrt{\sigma}}{16}$, D_0 is large enough and a_0 is small enough (depending on A, θ, ω_1 and ω_2), then for any $t \in [0, t^*]$,*

$$(3.4) \quad M_j(t) - M_j(0) \leq C e^{-2\theta D},$$

$$(3.5) \quad (E_j(t) + \omega_1 M_j(t)) - (E_j(0) + \omega_1 M_j(0)) \leq C e^{-2\theta D},$$

$$(3.6) \quad (F_j(t) + \omega_2 M_j(t)) - (F_j(0) + \omega_2 M_j(0)) \leq C e^{-2\theta D}.$$

Remark 3.2. If $j = 1$, we have $= 0$ at the place of $\leq C e^{-2\theta D}$, we will need it in the following of the proof.

Proof. If $j = 1$, we have exact conservation laws, so this Lemma is obvious. We assume that $j \geq 2$ for the following of this proof. From Appendix and minoration of m'_j ,

$$(3.7) \quad \begin{aligned} \frac{d}{dt} M_j(t) &= \int \left[-\frac{3}{2} u_x^2 + \frac{3}{4} u^4 \right] \Phi_{jx} + \frac{1}{2} \int u^2 \Phi_{jxxx} + \frac{1}{2} \int u^2 \Phi_{jt} \\ &= \int \left[-\frac{3}{2} u_x^2 + \frac{3}{4} u^4 \right] \Phi_{jx} + \frac{1}{2} \int u^2 \Phi_{jxxx} - \frac{1}{2} m'_j \int u^2 \Phi_{jx} \\ &\leq \int \left[-\frac{3}{2} u_x^2 + \frac{3}{4} u^4 - \frac{1}{2} \sigma u^2 \right] \Phi_{jx} + \frac{1}{2} \int u^2 \Phi_{jxxx}. \end{aligned}$$

Now, we use that $|\Phi_{jxxx}| \leq \frac{\sigma}{4} \Phi_{jx}$, and we obtain that

$$(3.8) \quad 2 \frac{d}{dt} M_j(t) \leq - \int \left[-3u_x^2 + \frac{3\sigma}{4} u^2 - \frac{3}{2} u^4 \right] \Phi_{jx}.$$

Now, from Appendix, we know that for $r > 0$, if t, x satisfy $\widetilde{x}_{j-1}(t) + r < x < \widetilde{x}_j(t) - r$, then $|\widetilde{P}(t, x)| \leq C e^{-\beta r}$. And so, for t, x such that $\widetilde{x}_{j-1}(t) + r < x < \widetilde{x}_j(t) - r$, by Sobolev embedding,

$$(3.9) \quad \begin{aligned} |u(t, x)| &\leq |\widetilde{P}(t, x)| + C \|\varepsilon(t)\|_{H^2} \\ &\leq C e^{-\beta r} + CA \left(a + e^{-\theta D} \right). \end{aligned}$$

From that, we can deduce that for r large enough, a_0 small enough and D_0 large enough, for $x \in [\widetilde{x}_{j-1}(t) + r, \widetilde{x}_j(t) - r]$, we can obtain that $|u(t, x)|$ is bounded by any chosen constant. Here, we will use that to bound $\frac{3}{2} u^2$ by $\frac{\sigma}{4}$.

For t, x such that $x < \widetilde{x}_{j-1}(t) + r$ or $x > \widetilde{x}_j(t) - r$:

$$(3.10) \quad \begin{aligned} |\Phi_{jx}(t, x)| &\leq C \exp \left(-\frac{\sqrt{\sigma}}{2} |x - m_j(t)| \right) \\ &\leq C \exp \left(-\frac{\sqrt{\sigma}}{2} \left(\frac{D}{2} + \zeta t - r \right) \right) \\ &\leq C \exp \left(-\frac{\sqrt{\sigma} \zeta}{2} t \right) \exp \left(-\frac{\sqrt{\sigma}}{4} D \right) \exp \left(\frac{\sqrt{\sigma}}{2} r \right), \end{aligned}$$

and so, if we choose D_0 large enough (more precisely, $D_0 \geq 4r$), we obtain for $x \notin [\widetilde{x}_{j-1}(t) + r, \widetilde{x}_j(t) - r]$:

$$(3.11) \quad |\Phi_{jx}(t, x)| \leq C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right).$$

Because $\int u^4$ is bounded by a constant for any time (that depends only on problem data), we deduce that:

$$(3.12) \quad \begin{aligned} \frac{d}{dt}M_j(t) &\leq -\int \left(\frac{3}{2}u_x^2 + \frac{\sigma}{4}u^2\right) \Phi_{jx}(t) + C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right) \\ &\leq C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right). \end{aligned}$$

We deduce what we want to prove by integration.

For the second inequality, the argument is similar. We start by using Appendix:

$$(3.13) \quad \begin{aligned} \frac{d}{dt}E_j(t) &= \int \left[-\frac{1}{2}(u_{xx} + u^3)^2 - u_{xx}^2 + 3u^2u_x^2\right] \Phi_{jx} + \frac{1}{2} \int u_x^2 \Phi_{jxxx} - m'_j \int \left[\frac{1}{2}u_x^2 - \frac{1}{4}u^4\right] \Phi_{jx} \\ &\leq -\int \left[\frac{1}{2}(u_{xx} + u^3)^2 + u_{xx}^2 - 3u^2u_x^2 + \frac{\sigma}{2}u_x^2 - \frac{v_J}{2}u^4\right] \Phi_{jx} + \frac{1}{2} \int u_x^2 \Phi_{jxxx}, \end{aligned}$$

and again, because $|\Phi_{jxxx}| \leq \frac{\sigma}{4}\Phi_{jx}$, we obtain that

$$(3.14) \quad 2\frac{d}{dt}E_j(t) \leq -\int \left[(u_{xx} + u^3)^2 + 2u_{xx}^2 - 6u^2u_x^2 + \frac{3\sigma}{4}u_x^2 - v_Ju^4\right] \Phi_{jx},$$

and by doing a similar argument as for the mass, but to bound $6u^2$ by $\frac{\sigma}{4}$ and to bound v_Ju^2 by $\omega_3 > 0$, a constant as small as we need, we obtain

$$(3.15) \quad \begin{aligned} \frac{d}{dt}E_j(t) &\leq -\int \left[\frac{1}{2}(u_{xx} + u^3)^2 + u_{xx}^2 + \frac{\sigma}{4}u_x^2 - \frac{\omega_3}{2}u^2\right] \Phi_{jx} + C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right) \\ &\leq \frac{\omega_3}{2} \int u^2 \Phi_{jx} + C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right), \end{aligned}$$

and so, if we choose ω_3 such that $\frac{\omega_3}{2} \leq \omega_1\frac{\sigma}{4}$,

$$(3.16) \quad \begin{aligned} \frac{d}{dt}(E_j(t) + \omega_1M_j(t)) &\leq \frac{\omega_3}{2} \int u^2 \Phi_{jx} + C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right) - \frac{\omega_1\sigma}{4} \int u^2 \Phi_{jx} \\ &\leq C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right), \end{aligned}$$

and we deduce what we want to prove by integration.

For the third inequality, the argument is similar. We start by using Appendix:

$$(3.17) \quad \begin{aligned} \frac{d}{dt}F_j(t) &= \int \left(-\frac{3}{2}u_{xxx}^2 + 9u_{xx}^2u^2 + 15u_x^2uu_{xx} + \frac{9}{16}u^8 + \frac{1}{4}u_x^4 + \frac{3}{2}u_{xx}u^5 - \frac{45}{4}u^4u_x^2\right) \Phi_{jx} \\ &\quad + 5 \int u^2u_xu_{xx}\Phi_{jxx} + \frac{1}{2} \int u_{xx}^2\Phi_{jxxx} - m'_j \int \left[\frac{1}{2}u_{xx}^2 - \frac{5}{2}u^2u_x^2 + \frac{1}{4}u^6\right] \Phi_{jx} \\ &\leq -\int \left(\frac{3}{2}u_{xxx}^2 - 9u_{xx}^2u^2 - 15u_x^2uu_{xx} - \frac{9}{16}u^8 - \frac{1}{4}u_x^4 - \frac{3}{2}u_{xx}u^5 + \frac{45}{4}u^4u_x^2 + \frac{\sigma}{2}u_{xx}^2\right. \\ &\quad \left.- 5v_Ju^2u_x^2 + \frac{\sigma}{4}u^6\right) \Phi_{jx} + 5 \int u^2u_xu_{xx}\Phi_{jxx} + \frac{1}{2} \int u_{xx}^2\Phi_{jxxx}, \end{aligned}$$

and again, because $|\Phi_{jxx}| \leq \frac{\sqrt{\sigma}}{2}\Phi_{jx}$ and $|\Phi_{jxxx}| \leq \frac{\sigma}{4}\Phi_{jx}$, we obtain that

$$(3.18) \quad \begin{aligned} 2\frac{d}{dt}F_j(t) &\leq -\int \left(3u_{xxx}^2 - \left(18 + \frac{5}{2}\sqrt{\sigma}\right)u_{xx}^2u^2 - 30u_x^2uu_{xx} - \frac{9}{8}u^8 - \frac{1}{2}u_x^4 - 3u_{xx}u^5\right. \\ &\quad \left.+ \frac{45}{2}u^4u_x^2 + \frac{3\sigma}{4}u_{xx}^2 - \left(10v_J + \frac{5}{2}\sqrt{\sigma}\right)u^2u_x^2 + \frac{\sigma}{2}u^6\right) \Phi_{jx}, \end{aligned}$$

and by doing a similar argument as for the mass, but to bound $(18 + \frac{5}{2}\sqrt{\sigma})u^2$ by $\frac{\sigma}{4}$, to bound $30uu_{xx}$ by ω_4 , to bound $\frac{9}{8}u^6$ by ω_5 , to bound $\frac{1}{2}u_x^2$ by ω_4 , to bound $3u_{xx}u^3$ by ω_5 and to bound $(10v_J + \frac{5}{2}\sqrt{\sigma})u^2$ by ω_4 , where $\omega_3, \omega_4 > 0$ are constants that we can choose as small as we want. And we obtain

$$\begin{aligned} \frac{d}{dt}F_j(t) &\leq - \int \left[\frac{3}{2}u_{xxx}^2 + \frac{45}{4}u^4u_x^2 + \frac{\sigma}{4}u^6 + \frac{\sigma}{4}u_{xx}^2 - C\omega_4u_x^2 - C\omega_5u^2 \right] \Phi_{jx} \\ &\quad + C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right) \\ (3.19) \quad &\leq C\omega_4 \int u_x^2 \Phi_{jx} + C\omega_5 \int u^2 \Phi_{jx} + C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right), \end{aligned}$$

and so, if we choose ω_4, ω_5 such that $C\omega_4 \leq \omega_1 \frac{3}{2}$ and $C\omega_5 \leq \omega_1 \frac{\sigma}{4}$,

$$(3.20) \quad \frac{d}{dt}(F_j(t) + \omega_2 M_j(t)) \leq C \exp\left(-\frac{\sqrt{\sigma}\zeta}{2}t\right) \exp\left(-\frac{\sqrt{\sigma}}{8}D\right),$$

and we deduce what we want to prove by integration. \square

3.2. Quadratic approximation for conservation laws at the right. We can write the following Taylor expansions with $u = \tilde{P} + \varepsilon$ for any $t \in [0, t^*]$:

$$(3.21) \quad M_j(t) - M_j[\tilde{P}](t) - \int \tilde{P}_\varepsilon \Phi_j - \frac{1}{2} \int \varepsilon^2 \Phi_j = 0,$$

$$(3.22) \quad \left| E_j(t) - E_j[\tilde{P}](t) - \int [\tilde{P}_x \varepsilon_x - \tilde{P}^3 \varepsilon] \Phi_j - \int \left[\frac{1}{2} \varepsilon_x^2 - \frac{3}{2} \tilde{P}^2 \varepsilon^2 \right] \Phi_j \right| \leq CA \left(a + e^{-\theta D} \right) \int \varepsilon^2 \Phi_j,$$

$$(3.23) \quad \left| F_j(t) - F_j[\tilde{P}](t) - \int \left[\tilde{P}_{xx} \varepsilon_{xx} - 5\tilde{P} \tilde{P}_x^2 \varepsilon - 5\tilde{P}^2 \tilde{P}_x \varepsilon_x + \frac{3}{2} \tilde{P}^5 \varepsilon \right] \Phi_j \right. \\ \left. - \int \left[\frac{1}{2} \varepsilon_{xx}^2 - \frac{5}{2} \tilde{P}_x^2 \varepsilon^2 - 10\tilde{P} \tilde{P}_x \varepsilon \varepsilon_x - \frac{5}{2} \tilde{P}^2 \varepsilon_x^2 + \frac{15}{4} \tilde{P}^4 \varepsilon^2 \right] \Phi_j \right| \leq CA \left(a + e^{-\theta D} \right) \int (\varepsilon^2 + \varepsilon_x^2) \Phi_j.$$

Now, we want to simplify each term of the Taylor expansion.

3.2.1. Constant terms of the Taylor expansion. We obtain the following lemma dealing with variations of the constant parts of each Taylor expansion of conservation laws at the right. We reduce each variation to the variation of each conservation law of \tilde{P}_j . Note that if P_j is a breather, the variation of any conservation law of \tilde{P}_j is 0. But, if $P_j = R_l$, we may express in the following way $M[\tilde{R}_l]$, $E[\tilde{R}_l]$ and $F[\tilde{R}_l]$ with respect to Q , the ground state of parameter $c = 1$ (the basic ground state):

$$(3.24) \quad M[\tilde{R}_l](t) = c_l(t)^{1/2} M[Q],$$

$$(3.25) \quad E[\tilde{R}_l](t) = c_l(t)^{3/2} E[Q],$$

$$(3.26) \quad F[\tilde{R}_l](t) = c_l(t)^{5/2} F[Q].$$

Lemma 3.3. *If $\sigma < 4\beta^2, \theta < \min\left(\frac{\beta}{4}, \frac{\sqrt{\sigma}}{8}\right)$, then for any $t \in [0, t^*]$,*

$$(3.27) \quad \left| M_j[\tilde{P}](t) - M_j[\tilde{P}](0) - \left(M[\tilde{P}_j](t) - M[\tilde{P}_j](0) \right) \right| \leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right),$$

$$(3.28) \quad \left| E_j[\tilde{P}](t) - E_j[\tilde{P}](0) - \left(E[\tilde{P}_j](t) - E[\tilde{P}_j](0) \right) \right| \leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right),$$

$$(3.29) \quad \left| F_j[\tilde{P}](t) - F_j[\tilde{P}](0) - \left(F[\tilde{P}_j](t) - F[\tilde{P}_j](0) \right) \right| \leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right).$$

Remark 3.4. In the case when P_j is a breather, we have that

$$(3.30) \quad M[\tilde{P}_j](t) - M[\tilde{P}_j](0) = E[\tilde{P}_j](t) - E[\tilde{P}_j](0) = F[\tilde{P}_j](t) - F[\tilde{P}_j](0) = 0.$$

It is not true in the case when P_j is a soliton.

Proof. When we develop using $\tilde{P} = \sum_{i=1}^J \tilde{P}_i$, we obtain terms with $\tilde{P}_i \tilde{P}_j$ with $i \neq j$ and the other terms that have all the same index. For the first type of terms, it is enough to bound $\int \tilde{P}_i \tilde{P}_j$ for $i \neq j$:

$$(3.31) \quad \left| \int \tilde{P}_i \tilde{P}_j \right| \leq C e^{-\frac{\beta D}{2}}.$$

Now, we look on the terms with the same index, for example $\int \tilde{P}_i^2 \Phi_j$. We will distinguish several cases. If $i < j$,

$$(3.32) \quad \begin{aligned} \int \tilde{P}_i^2 \Phi_j &\leq C \int e^{-2\beta|x-\tilde{x}_i(t)|} e^{\frac{\sqrt{\sigma}}{2}(x-m_j(t))} dx \\ &= C \int_{-\infty}^{\tilde{x}_i(t)} e^{(2\beta+\sqrt{\sigma}/2)x-2\beta\tilde{x}_i(t)-\frac{\sqrt{\sigma}}{2}m_j(t)} dx + C \int_{\tilde{x}_i(t)}^{+\infty} e^{(-2\beta+\sqrt{\sigma}/2)x+2\beta\tilde{x}_i(t)-\frac{\sqrt{\sigma}}{2}m_j(t)} dx \\ &\leq \frac{C}{\sqrt{\sigma}} e^{\frac{\sqrt{\sigma}}{2}(\tilde{x}_i(t)-m_j(t))} \\ &\leq C e^{-\frac{\sqrt{\sigma}D}{4}}. \end{aligned}$$

For the same reason and properties of Ψ , for $i \geq j$,

$$(3.33) \quad \int \tilde{P}_i^2 (1 - \Phi_j) \leq C e^{-\frac{\sqrt{\sigma}D}{4}}.$$

For $i > j$, we may use \mathcal{P}_i ; and for $i = j$, we cannot. So, if for $i \geq j + 1$, $\tilde{P}_i = \tilde{R}_l$ is a soliton, we have by the mean-value theorem, using (2.73),

$$(3.34) \quad \begin{aligned} \left| M[\tilde{P}_i](t) - M[\tilde{P}_i](0) \right| &= \left| c_l(t)^{1/2} - c_l(0)^{1/2} \right| |M[Q]| \\ &\leq C |c_l(t) - c_l(0)| \\ &\leq C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}), \end{aligned}$$

and, by the same way, we may obtain,

$$(3.35) \quad \left| E[\tilde{P}_i](t) - E[\tilde{P}_i](0) \right| \leq C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

and

$$(3.36) \quad \left| F[\tilde{P}_i](t) - F[\tilde{P}_i](0) \right| \leq C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}).$$

□

3.2.2. Linear terms of the Taylor expansion. We denote:

$$(3.37) \quad m[X] := \int X \varepsilon,$$

$$(3.38) \quad m_j[X] := \int X \varepsilon \Phi_j,$$

$$(3.39) \quad e[X] := \int [X_x \varepsilon_x - X^3 \varepsilon] = - \int [X_{xx} + X^3] \varepsilon,$$

$$(3.40) \quad e_j[X] := \int [X_x \varepsilon_x - X^3 \varepsilon] \Phi_j,$$

$$(3.41) \quad f[X] := \int \left[X_{xx} \varepsilon_{xx} - 5X X_x^2 \varepsilon - 5X^2 X_x \varepsilon_x + \frac{3}{2} X^5 \varepsilon \right] = \int \left[X_{xxxx} + 5X X_x^2 + 5X^2 X_{xx} + \frac{3}{2} X^5 \right] \varepsilon,$$

$$(3.42) \quad f_j[X] := \int \left[X_{xx} \varepsilon_{xx} - 5XX_x^2 \varepsilon - 5X^2 X_x \varepsilon_x + \frac{3}{2} X^5 \varepsilon \right] \Phi_j.$$

Lemma 3.5. *If $\sigma < \beta^2, \theta < \min\left(\frac{\beta}{4}, \frac{\sqrt{\sigma}}{8}\right)$, if D_0 is large enough and a_0 is small enough, then for any $t \in [0, t^*]$,*

$$(3.43) \quad \left| m_j[\tilde{P}](t) - m_j[\tilde{P}](0) - \left(m[\tilde{P}_j](t) - m[\tilde{P}_j](0) \right) \right| \leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right),$$

$$(3.44) \quad \left| e_j[\tilde{P}](t) - e_j[\tilde{P}](0) - \left(e[\tilde{P}_j](t) - e[\tilde{P}_j](0) \right) \right| \leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right),$$

$$(3.45) \quad \left| f_j[\tilde{P}](t) - f_j[\tilde{P}](0) - \left(f[\tilde{P}_j](t) - f[\tilde{P}_j](0) \right) \right| \leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right).$$

Proof. We develop using $\tilde{P} = \sum_{i=1}^J \tilde{P}_i$. We obtain terms with $\tilde{P}_i \tilde{P}_j$ with $i \neq j$ and the other terms that have all the same index. Knowing that ε bounded for D_0 large enough and a_0 small enough (with respect to A), we obtain the same bounds in the same way as for the constant part.

Now, if, for $i \geq j+1$, \tilde{P}_i is a soliton, then we have simply: $m[\tilde{P}_i] = e[\tilde{P}_i] = f[\tilde{P}_i] = 0$.

If, for $i \geq j+1$, \tilde{P}_i is a breather, we have a bound for the variation of these quantities by \mathcal{P}_i . \square

3.2.3. *Quadratic part of the Taylor expansion.* We set:

$$(3.46) \quad \mathcal{M}_j[X] := \frac{1}{2} \int \varepsilon^2 \Phi_j,$$

$$(3.47) \quad \mathcal{E}_j[X] := \int \left[\frac{1}{2} \varepsilon_x^2 - \frac{3}{2} X^2 \varepsilon^2 \right] \Phi_j,$$

$$(3.48) \quad \mathcal{F}_j[X] := \int \left[\frac{1}{2} \varepsilon_{xx}^2 - \frac{5}{2} X_x^2 \varepsilon^2 - 10 X X_x \varepsilon \varepsilon_x - \frac{5}{2} X^2 \varepsilon_x^2 + \frac{15}{4} X^4 \varepsilon^2 \right] \Phi_j,$$

and $\mathcal{M}_j(t) := \mathcal{M}_j[\tilde{P}](t), \dots$

Lemma 3.6. *If $\sigma < \beta^2, \theta < \frac{\sqrt{\sigma}}{8}$, if D_0 is large enough and a_0 is small enough, then for any $t \in [0, t^*]$,*

$$(3.49) \quad \left| \mathcal{M}_j[\tilde{P}](t) - \mathcal{M}_j[\tilde{P}_j](t) \right| \leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right),$$

$$(3.50) \quad \left| \mathcal{E}_j[\tilde{P}](t) - \mathcal{E}_j[\tilde{P}_j](t) \right| \leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right),$$

$$(3.51) \quad \left| \mathcal{F}_j[\tilde{P}](t) - \mathcal{F}_j[\tilde{P}_j](t) \right| \leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right).$$

Proof. We develop using $\tilde{P} = \sum_{i=1}^J \tilde{P}_i$. For terms with \tilde{P}_i , with $i > j$, we use the induction assumption for ε . For terms with \tilde{P}_i , with $i < j$, we do as in the previous sections. \square

Note that (3.49) is useless, because $\mathcal{M}_j[\tilde{P}](t) - \mathcal{M}_j[\tilde{P}_j](t) = 0$ since $\mathcal{M}_j[X]$ do not depend on X , but we write it in order to argue in the same way for the three conserved quantities.

3.3. Lyapunov functional and simplifications. We introduce the following Lyapunov functional:

$$(3.52) \quad \mathcal{H}_j(t) := F_j(t) + 2 \left(b_j^2 - a_j^2 \right) E_j(t) + \left(a_j^2 + b_j^2 \right)^2 M_j(t).$$

We set:

$$(3.53) \quad \mathcal{K}(t) := F[\tilde{P}_j](t) + 2 \left(b_j^2 - a_j^2 \right) E[\tilde{P}_j](t) + \left(a_j^2 + b_j^2 \right)^2 M[\tilde{P}_j](t),$$

$$(3.54) \quad \mathcal{L}(t) := f[\tilde{P}_j](t) + 2 \left(b_j^2 - a_j^2 \right) e[\tilde{P}_j](t) + \left(a_j^2 + b_j^2 \right)^2 m[\tilde{P}_j](t),$$

$$(3.55) \quad \mathcal{Q}(t) := \mathcal{F}_j[\tilde{P}_j](t) + 2 \left(b_j^2 - a_j^2 \right) \mathcal{E}_j[\tilde{P}_j](t) + \left(a_j^2 + b_j^2 \right)^2 \mathcal{M}_j[\tilde{P}_j](t).$$

We have the following:

Lemma 3.7. *If $\sigma < \beta^2, \theta < \min \left(\frac{\beta}{4}, \frac{\sqrt{\sigma}}{8} \right)$, if D_0 is large enough and a_0 is small enough, then for any $t \in [0, t^*]$,*

$$(3.56) \quad \left| \mathcal{H}_j(t) - \mathcal{H}_j(0) - (\mathcal{Q}(t) - \mathcal{Q}(0)) \right| \leq C e^{-2\theta D} + C \left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right).$$

Proof. From (3.21), (3.22), (3.23), Lemmas 3.3, 3.5 and 3.6 we deduce that:

if $\sigma < \beta^2, \theta > \min \left(\frac{\beta}{4}, \frac{\sqrt{\sigma}}{8} \right)$, if D_0 is large enough and a_0 is small enough, then for any $t \in [0, t^*]$,

$$(3.57) \quad \begin{aligned} \mathcal{H}_j(t) - \mathcal{H}_j(0) &= \mathcal{K}(t) - \mathcal{K}(0) + \mathcal{L}(t) - \mathcal{L}(0) + \mathcal{Q}(t) - \mathcal{Q}(0) + O \left(A \left(a + e^{-\theta D} \right) \int (\varepsilon^2 + \varepsilon_x^2) \Phi_j \right) \\ &+ O \left(e^{-2\theta D} \right) + O \left(\left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right) \right). \end{aligned}$$

Now, from the bootstrap assumption (2.25), we see that if we take D_0 large enough and a_0 small enough, then we can bound $A \left(a + e^{-\theta D} \right) \int (\varepsilon^2 + \varepsilon_x^2) \Phi_j$ by $\left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right)$. So, for any $t \in [0, t^*]$,

$$(3.58) \quad \begin{aligned} \mathcal{H}_j(t) - \mathcal{H}_j(0) &= \mathcal{K}(t) - \mathcal{K}(0) + \mathcal{L}(t) - \mathcal{L}(0) + \mathcal{Q}(t) - \mathcal{Q}(0) + O \left(e^{-2\theta D} \right) + O \left(\left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right) \right). \end{aligned}$$

Now, we will simplify $\mathcal{K}(t) - \mathcal{K}(0)$ and $\mathcal{L}(t) - \mathcal{L}(0)$.

Simplification of $\mathcal{K}(t) - \mathcal{K}(0)$:

If \tilde{P}_j is a breather, then $\mathcal{K}(t) - \mathcal{K}(0) = 0$. If $\tilde{P}_j = \tilde{R}_l$ is a soliton, then we have

$$(3.59) \quad \begin{aligned} \mathcal{K}(t) &= F[\tilde{R}_l](t) + 2c_l^0 E[\tilde{R}_l](t) + (c_l^0)^2 M[\tilde{R}_l](t) \\ &= c_l(t)^{5/2} F[Q] + 2c_l^0 c_l(t)^{3/2} E[Q] + (c_l^0)^2 c_l(t)^{1/2} M[Q]. \end{aligned}$$

Now, we observe that $c_l(t) = c_l^0 + (c_l(t) - c_l^0)$, this is why we want to do a Taylor expansion for each power function. We recall that by (2.54), for D_0 large enough and a_0 small enough (with respect to Z_{j+1}), $|c_l(t) - c_l^0|^3$ is bounded by $C \left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right)$. This is why we may approximate $\mathcal{K}(t)$ by a Taylor expansion of order 2:

$$\begin{aligned} &(c_l^0)^{5/2} \left(1 + \frac{5}{2} \frac{c_l(t) - c_l^0}{c_l^0} + \frac{15}{8} \left(\frac{c_l(t) - c_l^0}{c_l^0} \right)^2 \right) F[Q] \\ &+ 2 (c_l^0)^{5/2} \left(1 + \frac{3}{2} \frac{c_l(t) - c_l^0}{c_l^0} + \frac{3}{8} \left(\frac{c_l(t) - c_l^0}{c_l^0} \right)^2 \right) E[Q] \end{aligned}$$

$$\begin{aligned}
(3.60) \quad & + (c_l^0)^{5/2} \left(1 + \frac{1}{2} \frac{c_l(t) - c_l^0}{c_l^0} - \frac{1}{8} \left(\frac{c_l(t) - c_l^0}{c_l^0} \right)^2 \right) M[Q] \\
& = (c_l^0)^{5/2} (F[Q] + 2E[Q] + M[Q]) + (c_l(t) - c_l^0) (c_l^0)^{3/2} \left(\frac{5}{2} F[Q] + 3E[Q] + \frac{1}{2} M[Q] \right) \\
& \quad + (c_l(t) - c_l^0)^2 (c_l^0)^{1/2} \left(\frac{15}{8} F[Q] + \frac{3}{4} E[Q] - \frac{1}{8} M[Q] \right).
\end{aligned}$$

Now, we use the fact that $M[Q] = 2$, $E[Q] = -\frac{2}{3}$ and $F[Q] = \frac{2}{5}$, and we obtain that the Taylor expression of order 2 is in fact:

$$(3.61) \quad \frac{16}{15} (c_l^0)^{5/2} + 0 (c_l(t) - c_l^0) (c_l^0)^{3/2} + 0 (c_l(t) - c_l^0)^2 (c_l^0)^{1/2} = \frac{16}{15} (c_l^0)^{5/2}.$$

Thus,

$$(3.62) \quad \left| \mathcal{K}(t) - \frac{16}{15} (c_l^0)^{5/2} \right| \leq C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}),$$

and so,

$$(3.63) \quad |\mathcal{K}(t) - \mathcal{K}(0)| \leq C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}).$$

Simplification of $\mathcal{L}(t) - \mathcal{L}(0)$:

If \tilde{P}_j is a breather, we have, by the elliptic equation verified by a breather, that $\mathcal{L}(t) = 0$.

If \tilde{P}_j is a soliton, we have, by (2.56), (2.85) and (2.86), that $\mathcal{L}(t) = 0$ (we have simply $m[\tilde{P}_j] = e[\tilde{P}_j] = f[\tilde{P}_j] = 0$). \square

3.4. Coercivity.

Lemma 3.8. *If σ is small enough (with respect to constants that depend only on problem data),*

$$(3.64) \quad \theta < \min \left(\frac{\beta}{4}, \frac{\sqrt{\sigma}}{8} \right),$$

if D_0 is large enough and a_0 is small enough, then for any $t \in [0, t^]$,*

$$(3.65) \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq CQ(t) + C \left(\int \varepsilon \tilde{P}_j \right)^2 + Ce^{-2\theta D}.$$

Proof. We notice that $\int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j$ is $\|\varepsilon \sqrt{\Phi_j}\|_{H^2}^2$ modulo some terms that can be bounded by

$$(3.66) \quad C\sqrt{\sigma} \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j.$$

This is why we will bound $\|\varepsilon \sqrt{\Phi_j}\|_{H^2}^2$. We can bound it by the canonical quadratic form associated to \tilde{P}_j and evaluated in $\varepsilon \sqrt{\Phi_j}$, if $\varepsilon \sqrt{\Phi_j}$ satisfies quite well the orthogonality conditions, which is the case (see Sections 5.4 and (372) in [39]).

So, we obtain that, if \tilde{P}_j is a breather, the canonical quadratic form is $Q(t)$ modulo (3.66), and we have

$$(3.67) \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq CQ(t) + C \left(\int \varepsilon \sqrt{\Phi_j} \tilde{P}_j \right)^2 + C\sqrt{\sigma} \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j,$$

which means that if σ is small enough,

$$(3.68) \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq CQ(t) + C \left(\int \varepsilon \sqrt{\Phi_j} \tilde{P}_j \right)^2,$$

and we check that $\int \varepsilon \sqrt{\Phi_j} \tilde{P}_j$ is $\int \varepsilon \tilde{P}_j$ modulo $Ce^{-\theta D}$. So,

$$(3.69) \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq CQ(t) + C \left(\int \varepsilon \tilde{P}_j \right)^2 + Ce^{-2\theta D}.$$

If $\tilde{P}_j = \tilde{R}_l$ is a soliton, the canonical quadratic form is, modulo $C\sqrt{\sigma} \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j$,

$$(3.70) \quad \mathcal{Q}_0(t) := \mathcal{F}_j[\tilde{R}_l](t) + 2c_l(t)\mathcal{E}_j[\tilde{R}_l](t) + c_l(t)^2\mathcal{M}_j[\tilde{R}_l](t).$$

This is why, for the same reasons as above, we have

$$(3.71) \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq C\mathcal{Q}_0(t).$$

This is why, we need to be able to bound $|\mathcal{Q}_0(t) - \mathcal{Q}(t)|$. We have

$$(3.72) \quad \mathcal{Q}_0(t) - \mathcal{Q}(t) = 2(c_l(t) - c_l^0)\mathcal{E}_j[\tilde{R}_l](t) + 2c_l^0(c_l(t) - c_l^0)\mathcal{M}_j[\tilde{R}_l](t) + (c_l(t) - c_l^0)^2\mathcal{M}_j[\tilde{R}_l](t),$$

and so, because \mathcal{M}_j and \mathcal{E}_j are quadratic in ε , we have that

$$(3.73) \quad |\mathcal{Q}_0(t) - \mathcal{Q}(t)| \leq CA \left(a + e^{-\theta D}\right) \int (\varepsilon^2 + \varepsilon_x^2) \Phi_j,$$

and so, if we take a_0 small enough and D_0 large enough, we obtain

$$(3.74) \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq C\mathcal{Q}(t).$$

□

3.5. Proof of \mathcal{P}_j . Now, we are left to prove \mathcal{P}_j . More precisely, if P_j is a soliton, we will prove (2.76) and (2.77); if P_j is a breather, we will prove (2.76), (2.78) and (2.79). We will distinguish several cases. We assume that $\sigma \leq \min(\zeta, \beta^2)$, $\theta \leq \min\left(\frac{\beta}{4}, \frac{\sqrt{\sigma}}{16}\right)$, and that D_0 is large enough and a_0 is small enough (depending on A, θ), so that all the previous lemmas are verified.

3.5.1. Case when P_j is a soliton.

Proof of (2.76). By Lemma 3.1 and the fact that $b_j^2 - a_j^2 > 0$ for $j \geq 2$ (which is a direct consequence of $v_2 > 0$), we have that, if $j \geq 2$, for any $t \in [0, t^*]$,

$$(3.75) \quad \mathcal{H}_j(t) - \mathcal{H}_j(0) \leq Ce^{-2\theta D}.$$

(3.75) is also true for $j = 1$, because of Remark 3.2.

We have, by (3.56), (3.75), the definition of $\mathcal{Q}(0)$ and (2.53) for $\varepsilon(0)$, for any $t \in [0, t^*]$,

$$(3.76) \quad \mathcal{Q}(t) = [\mathcal{Q}(t) - \mathcal{Q}(0) - (\mathcal{H}_j(t) - \mathcal{H}_j(0))] + [\mathcal{H}_j(t) - \mathcal{H}_j(0)] + \mathcal{Q}(0)$$

$$(3.77) \quad \leq Ce^{-2\theta D} + C \left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}) + C\|\varepsilon(0)\|_{H^2}^2$$

$$(3.78) \quad \leq C(a^2 + e^{-2\theta D}) + C \left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}).$$

And so, by (3.65) and (3.78), we have for any $t \in [0, t^*]$,

$$(3.79) \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq C\mathcal{Q}(t) + C \left(\int \varepsilon \tilde{P}_j\right)^2 + Ce^{-2\theta D}$$

$$(3.80) \quad \leq C(a^2 + e^{-2\theta D}) + C \left(\frac{A}{Z_{j+1}}\right)^2 (a^2 + e^{-2\theta D}) + C \left(\int \varepsilon \tilde{P}_j\right)^2.$$

Because of (2.56) and \tilde{P}_j is a soliton, we have that

$$(3.81) \quad \int \varepsilon \tilde{P}_j = 0.$$

So, the proof of (2.76) is completed for a suitable constant Z_j that will be precised later.

Proof of (2.77). From (3.21), (3.24), (3.27), (3.43), (2.56) and (3.80), we have for any $t \in [0, t^*]$:

(3.82)

$$\begin{aligned} M_j(0) - M_j(t) &= M_j[\tilde{P}](0) - M_j[\tilde{P}](t) + \int \tilde{P}\varepsilon\Phi_j(0) - \int \tilde{P}\varepsilon\Phi_j(t) + \frac{1}{2} \int \varepsilon^2\Phi_j(0) - \frac{1}{2} \int \varepsilon^2\Phi_j(t) \\ &= \left(c_l(0)^{1/2} - c_l(t)^{1/2}\right) M[Q] + O\left(\left(a^2 + e^{-2\theta D}\right)\right) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 \left(a^2 + e^{-2\theta D}\right)\right). \end{aligned}$$

Similarly, from (3.22), (3.25), (3.28), (3.44), (2.85) and (3.80), by taking a_0 smaller and D_0 larger with respect to A if needed, we have for any $t \in [0, t^*]$:

(3.83)

$$E_j(0) - E_j(t) = \left(c_l(0)^{3/2} - c_l(t)^{3/2}\right) E[Q] + O\left(\left(a^2 + e^{-2\theta D}\right)\right) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 \left(a^2 + e^{-2\theta D}\right)\right).$$

For $\eta = \frac{1}{2}, \frac{3}{2}$, because we know that $c_l(t)$ is not too far from $c_l(0)$ (and the both are not too far from c_l^0 , by (2.54)), we can write for any $t \in [0, t^*]$:

$$(3.84) \quad c_l(t)^\eta = (c_l(0)^\eta) \left(1 + \eta \frac{c_l(t) - c_l(0)}{c_l(0)} + O\left((c_l(t) - c_l(0))^2\right)\right),$$

and so for any $t \in [0, t^*]$,

$$(3.85) \quad c_l(t)^\eta - c_l(0)^\eta = \eta c_l(0)^{\eta-1} (c_l(t) - c_l(0)) + O\left((c_l(t) - c_l(0))^2\right),$$

and if a_0 is small enough and D_0 is large enough, we have by (2.26), for any $t \in [0, t^*]$:

$$(3.86) \quad \left|O\left((c_l(t) - c_l(0))^2\right)\right| \leq \frac{1}{2} \eta c_l(0)^{\eta-1} |c_l(t) - c_l(0)|.$$

Thus, for any $t \in [0, t^*]$,

$$(3.87) \quad 2\eta c_l(0)^{\eta-1} |c_l(t) - c_l(0)| \geq |c_l(t)^\eta - c_l(0)^\eta| \geq \frac{\eta c_l(0)^{\eta-1}}{2} |c_l(t) - c_l(0)|,$$

where $c_l(0)^{\eta-1}$ is between $\min\{\frac{c_p^0}{2}, 1 \leq p \leq L\}^{\eta-1}$ and $\max\{2c_p^0, 1 \leq p \leq L\}^{\eta-1}$ by (2.54), and so is bounded above and below by a constant that depends only on the shape parameters of the solitons. In order to bound $|c_l(t) - c_l(0)|$ for a given $t \in [0, t^*]$, we will distinguish two cases.

Case when $c_l(t) - c_l(0) \geq 0$. From (3.82) and (3.87) for $\eta = \frac{1}{2}$, we can say that

$$(3.88) \quad \begin{aligned} |c_l(t) - c_l(0)| &\leq C \left(c_l(t)^{1/2} - c_l(0)^{1/2}\right) \\ &\leq C \frac{M_j(t) - M_j(0)}{M[Q]} + C \left(a^2 + e^{-2\theta D}\right) + C \left(\frac{A}{Z_{j+1}}\right)^2 \left(a^2 + e^{-2\theta D}\right). \end{aligned}$$

Now, $M[Q] > 0$ and from Lemma 3.1,

$$(3.89) \quad |c_l(t) - c_l(0)| \leq C \left(a^2 + e^{-2\theta D}\right) + C \left(\frac{A}{Z_{j+1}}\right)^2 \left(a^2 + e^{-2\theta D}\right).$$

Thus, (2.77) is established for a suitable constant Z_j that will be precised later.

Case when $c_l(t) - c_l(0) \leq 0$. From (3.83) and (3.87) for $\eta = \frac{3}{2}$, we can say that

$$(3.90) \quad \begin{aligned} |c_l(t) - c_l(0)| &\leq C \left(c_l(0)^{3/2} - c_l(t)^{3/2}\right) \\ &\leq C \frac{E_j(0) - E_j(t)}{E[Q]} + C \left(a^2 + e^{-2\theta D}\right) + C \left(\frac{A}{Z_{j+1}}\right)^2 \left(a^2 + e^{-2\theta D}\right). \end{aligned}$$

Now, $E[Q] < 0$ and from Lemma 3.1, (3.82) and (3.87) for $\eta = \frac{1}{2}$,

$$(3.91) \quad \begin{aligned} |c_l(t) - c_l(0)| &\leq C\omega_1 (M_j(0) - M_j(t)) + C \left(a^2 + e^{-2\theta D}\right) + C \left(\frac{A}{Z_{j+1}}\right)^2 \left(a^2 + e^{-2\theta D}\right) \\ &\leq C\omega_1 \left(c_l(0)^{1/2} - c_l(t)^{1/2}\right) + C \left(a^2 + e^{-2\theta D}\right) + C \left(\frac{A}{Z_{j+1}}\right)^2 \left(a^2 + e^{-2\theta D}\right) \end{aligned}$$

$$\leq C\omega_1 |c_l(t) - c_l(0)| + C \left(a^2 + e^{-2\theta D} \right) + C \left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right),$$

and so, by taking ω_1 small enough, we may deduce the desired inequality:

$$(3.92) \quad |c_l(t) - c_l(0)| \leq C \left(a^2 + e^{-2\theta D} \right) + C \left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right).$$

Thus, (2.77) is established for a suitable constant Z_j that will be precised later.

3.5.2. *Case when P_j is a breather.*

Preliminaries. By the same argument as in the case when P_j is a soliton, we establish (3.80). However, we are not able to prove (2.76) immediately, because $\int \tilde{P}_j \varepsilon$ is not necessarily equal to 0 in the case when \tilde{P}_j is a breather.

From Lemma 3.5, (3.21), Lemma 3.3 and Remark 3.4, we have that for any $t \in [0, t^*]$,

$$(3.93) \quad \begin{aligned} \int \tilde{P}_j \varepsilon(t) - \int \tilde{P}_j \varepsilon(0) &= M_j(t) - M_j(0) - \frac{1}{2} \int \varepsilon^2 \Phi_j(t) + \frac{1}{2} \int \varepsilon^2 \Phi_j(0) \\ &+ O \left(e^{-2\theta D} \right) + O \left(\left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right) \right). \end{aligned}$$

then we use (3.4) and (2.53) for $\varepsilon(0)$, we have that for any $t \in [0, t^*]$,

$$(3.94) \quad \begin{aligned} \int \tilde{P}_j \varepsilon(t) - \int \tilde{P}_j \varepsilon(0) &\leq -\frac{1}{2} \int \varepsilon^2 \Phi_j(t) + \frac{1}{2} \int \varepsilon^2 \Phi_j(0) + O \left(e^{-2\theta D} \right) + O \left(\left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right) \right) \\ &\leq C \left(a^2 + e^{-2\theta D} \right) + C \left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right). \end{aligned}$$

Now, from (2.82), Lemma 3.5, (3.21), Lemma 3.3, Remark 3.4 and (2.25), we have for any $t \in [0, t^*]$,

$$(3.95) \quad \begin{aligned} \left((a_j^2 + b_j^2)^2 \right) \left(\int \tilde{P}_j \varepsilon(0) - \int \tilde{P}_j \varepsilon(t) \right) &= 2 \left(b_j^2 - a_j^2 \right) \left(e[\tilde{P}_j](t) - e[\tilde{P}_j](0) \right) + \left(f[\tilde{P}_j](t) - f[\tilde{P}_j](0) \right) \\ &= 2 \left(b_j^2 - a_j^2 \right) \left(E_j(t) - E_j(0) - \mathcal{E}_j(t) + \mathcal{E}_j(0) \right) \\ &+ \left(F_j(t) - F_j(0) - \mathcal{F}_j(t) + \mathcal{F}_j(0) \right) \\ &+ O \left(e^{-2\theta D} \right) + O \left(\left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right) \right). \end{aligned}$$

Recall that if $j \geq 2$, $b_j^2 - a_j^2 > 0$. This is why, from Lemma 3.1 and (2.53) for $\varepsilon(0)$, we have that, if $j \geq 2$, for any $t \in [0, t^*]$:

$$(3.96) \quad \begin{aligned} \int \tilde{P}_j \varepsilon(0) - \int \tilde{P}_j \varepsilon(t) &\leq \frac{-2 \left(b_j^2 - a_j^2 \right) \mathcal{E}_j(t) - \mathcal{F}_j(t)}{\left(a_j^2 + b_j^2 \right)^2} \\ &+ C \left(a^2 + e^{-2\theta D} \right) + C \left(\frac{A}{Z_{j+1}} \right)^2 \left(a^2 + e^{-2\theta D} \right) \\ &+ \frac{2 \left(b_j^2 - a_j^2 \right) \omega_1 + \omega_2}{\left(a_j^2 + b_j^2 \right)^2} \left(M_j(0) - M_j(t) \right). \end{aligned}$$

(3.96) is also true if $j = 1$ because of Remark 3.2.

And, from (3.21), Lemma 3.3, Remark 3.4, Lemma 3.5 and (2.53) for $\varepsilon(0)$, we have for any $t \in [0, t^*]$,

$$M_j(0) - M_j(t) = M_j[\tilde{P}](0) - M_j[\tilde{P}](t) + \int \tilde{P} \varepsilon \Phi_j(0) - \int \tilde{P} \varepsilon \Phi_j(t) + \frac{1}{2} \int \varepsilon^2 \Phi_j(0) - \frac{1}{2} \int \varepsilon^2 \Phi_j(t)$$

$$(3.97) \quad = O\left(a^2 + e^{-2\theta D}\right) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 \left(a^2 + e^{-2\theta D}\right)\right) + \int \tilde{P}_j \varepsilon(0) - \int \tilde{P}_j \varepsilon(t) - \mathcal{M}_j(t).$$

And so, if we choose ω_1 and ω_2 small enough with respect to the problem constants, we obtain for any $t \in [0, t^*]$:

$$(3.98) \quad \int \tilde{P}_j \varepsilon(0) - \int \tilde{P}_j \varepsilon(t) \leq \frac{-\left(2\left(b_j^2 - a_j^2\right)\omega_1 + \omega_2\right)\mathcal{M}_j(t) - 2\left(b_j^2 - a_j^2\right)\mathcal{E}_j(t) - \mathcal{F}_j(t)}{\left(a_j^2 + b_j^2\right)^2} \\ + C\left(a^2 + e^{-2\theta D}\right) + C\left(\frac{A}{Z_{j+1}}\right)^2 \left(a^2 + e^{-2\theta D}\right).$$

Because $|\mathcal{M}_j(t)| + |\mathcal{E}_j(t)| + |\mathcal{F}_j(t)| \leq \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j$, we deduce that for any $t \in [0, t^*]$,

$$(3.99) \quad \int \tilde{P}_j \varepsilon(0) - \int \tilde{P}_j \varepsilon(t) \leq C \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j + C\left(a^2 + e^{-2\theta D}\right) + C\left(\frac{A}{Z_{j+1}}\right)^2 \left(a^2 + e^{-2\theta D}\right).$$

And so, by putting (3.94) and (3.99) together, we have that for any $t \in [0, t^*]$:

$$(3.100) \quad \left| \int \tilde{P}_j \varepsilon(0) - \int \tilde{P}_j \varepsilon(t) \right| \leq C \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j + C\left(a^2 + e^{-2\theta D}\right) + C\left(\frac{A}{Z_{j+1}}\right)^2 \left(a^2 + e^{-2\theta D}\right).$$

Proof of (2.76). From (2.53), (3.100), we deduce for any $t \in [0, t^*]$:

$$(3.101) \quad \left| \int \tilde{P}_j \varepsilon(t) \right| \leq \left| \int \tilde{P}_j \varepsilon(0) - \int \tilde{P}_j \varepsilon(t) \right| + \left| \int \tilde{P}_j \varepsilon(0) \right| \\ \leq C \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j + C\left(a^2 + e^{-2\theta D}\right) + C\left(\frac{A}{Z_{j+1}}\right)^2 \left(a^2 + e^{-2\theta D}\right) \\ \leq CA^2 \left(a^2 + e^{-2\theta D}\right) + C\left(a^2 + e^{-2\theta D}\right).$$

And so, if a_0 is small enough and D_0 is large enough, for any $t \in [0, t^*]$,

$$(3.102) \quad \left(\int \tilde{P}_j \varepsilon(t) \right)^2 \leq C\left(\frac{A}{Z_{j+1}}\right)^2 \left(a^2 + e^{-2\theta D}\right) + C\left(a^2 + e^{-2\theta D}\right).$$

This is why, from (3.80), for any $t \in [0, t^*]$,

$$(3.103) \quad \int (\varepsilon^2 + \varepsilon_x^2 + \varepsilon_{xx}^2) \Phi_j \leq C\left(a^2 + e^{-2\theta D}\right) + C\left(\frac{A}{Z_{j+1}}\right)^2 \left(a^2 + e^{-2\theta D}\right),$$

and (2.76) is established for a suitable constant Z_j that will be precised later.

Proof of (2.78). From (3.103) and (3.100), for any $t \in [0, t^*]$:

$$(3.104) \quad \left| \int \tilde{P}_j \varepsilon(0) - \int \tilde{P}_j \varepsilon(t) \right| \leq C\left(\frac{A}{Z_{j+1}}\right)^2 \left(a^2 + e^{-2\theta D}\right) + C\left(a^2 + e^{-2\theta D}\right).$$

Thus, (2.78) is established for a suitable constant Z_j that will be precised later.

Proof of (2.79). From Lemma 3.5, (3.22), Lemma 3.3, Remark 3.4, (3.100) and (3.103), for any $t \in [0, t^*]$,

$$(3.105) \quad e[\tilde{P}_j](t) - e[\tilde{P}_j](0) = E_j(t) - E_j(0) - \mathcal{E}_j(t) + \mathcal{E}_j(0) \\ + O\left(e^{-2\theta D}\right) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 \left(a^2 + e^{-2\theta D}\right)\right) \\ = E_j(t) - E_j(0) + O\left(a^2 + e^{-2\theta D}\right) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 \left(a^2 + e^{-2\theta D}\right)\right).$$

Now, from Lemma 3.1, for any $t \in [0, t^*]$,

$$(3.106) \quad e[\tilde{P}_j](t) - e[\tilde{P}_j](0) \leq \omega_1 (M_j(0) - M_j(t)) + O\left(a^2 + e^{-2\theta D}\right) + O\left(\left(\frac{A}{Z_{j+1}}\right)^2 \left(a^2 + e^{-2\theta D}\right)\right).$$

And from (3.97) and (3.104), for any $t \in [0, t^*]$,

$$(3.107) \quad e[\tilde{P}_j](t) - e[\tilde{P}_j](0) \leq C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) + C (a^2 + e^{-2\theta D}).$$

To bound above $e[\tilde{P}_j](0) - e[\tilde{P}_j](t)$, we do as in (3.95), (3.96), (3.97), (3.98) and (3.99), but with the energy instead of the mass, and after using (3.103), we obtain for any $t \in [0, t^*]$,

$$(3.108) \quad \left| e[\tilde{P}_j](t) - e[\tilde{P}_j](0) \right| \leq C \left(\frac{A}{Z_{j+1}} \right)^2 (a^2 + e^{-2\theta D}) + C (a^2 + e^{-2\theta D}).$$

Thus, (2.79) is established for a suitable constant Z_j that will be precised later.

3.6. Choice of suitable A and Z_j . The induction holds if

$$(3.109) \quad C \left(1 + \left(\frac{A}{Z_{j+1}} \right)^2 \right) \leq \left(\frac{A}{Z_j} \right)^2.$$

We can set

$$(3.110) \quad A := \sqrt{2} (2C)^{\frac{1}{2}},$$

and for $1 \leq j \leq J$,

$$(3.111) \quad Z_j := 2 (2C)^{\frac{j-1}{2}}.$$

And, if $C > 1$, the induction holds.

The proof of Proposition 2.6 is now complete.

4. A CONSEQUENCE OF THEOREM 1.1: ORBITAL STABILITY OF A MULTI-BREATHER

We assume the Theorem 1.1 proved, let us prove Theorem 1.3.

Proof of Theorem 1.3. Let $A_0 > 0$, $\theta_0 > 0$, $D_0 > 0$ and $a_0 > 0$ from Theorem 1.1 (these constants do only depend on the shape/frequency parameters of our objects and not on their initial positions). Let $\eta_0 > \eta > 0$ with $\eta_0 < \frac{a_0}{2C_1}$ and C_1 defined in the following. Let $a < a_0$ and $D > D_0$ such that $A_0 (a + e^{-\theta_0 D}) < 4A_0 C_1 \eta$ and $a = 2C_1 \eta$. We may take D even larger so that $T^* = 0$ where T^* is defined in [39, Theorem 1.2]. Let θ_1 associated to shape/frequency parameters $\alpha_k, \beta_k, \kappa_l, c_l^0$ by [39, Theorem 1.2]. Let A_2 associated to D by [39, Theorem 1.2].

Let $\tau > 0$ be the minimal difference between two velocities.

Let p be the multi-breather associated to $\alpha_k, \beta_k, x_{1,k}^0, x_{2,k}^0, c_l^0, x_{0,l}^0$ by [39, Theorem 1.2] with notations as in (1.25) and P the corresponding sum with notations as in (1.26). We may choose $T \geq 0$ large enough such that

$$(4.1) \quad \forall t \geq T, \quad \|p(t) - P(t)\|_{H^2} \leq a/2,$$

which is possible from [39], and such that

$$(4.2) \quad \forall j \geq 2, \quad x_j(T) - x_{j-1}(T) > 2D,$$

which is possible because the distance between two objects is increasing with a speed that is at least τ .

By [24] we know that we have continuous dependence of the solution of (mKdV) with respect to the initial data. And so, there exists $C_1 > 0$ (that depends on T) such that if $\|u(0) - p(0)\|_{H^2} \leq \eta$ and η_0 is small enough, then

$$(4.3) \quad \forall t \in [0, T], \quad \|u(t) - p(t)\|_{H^2} \leq C_1 \eta.$$

Therefore, by triangular inequality,

$$(4.4) \quad \|u(T) - P(T)\|_{H^2} \leq a = 2C_1 \eta.$$

This means that the assumptions of Theorem 1.1 are all satisfied in T instead of 0. And so, this means that there exists $x_{0,l}(t), x_{1,k}(t), x_{2,k}(t)$ defined for any $t \geq T$ such that

$$(4.5) \quad \forall t \geq T, \quad \|u(t) - P(t; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t))\|_{H^2} \leq A_0 (a + e^{-\theta_0 D}) < 4A_0 C_1 \eta.$$

Now, we see that the assumptions of [39, Theorem 1.2, Remark 1.3] are all satisfied in any $t \geq T$ instead of 0 for the sum $P(\alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t))$. Indeed, if we denote $\tilde{x}_j(t)$ the position of

$$(4.6) \quad P_j(t; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t)),$$

we know from Remark 2.38 that for any $t \geq T$ and $j \geq 2$,

$$(4.7) \quad \tilde{x}_j(t) - \widetilde{x}_{j-1}(t) \geq D.$$

By taking T larger if needed, we may insure that $e^{-\theta_1 T} < C_1 \eta$.

Therefore, for any $t \geq T$,

$$\begin{aligned} & \|p(t; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t)) - P(t; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t))\|_{H^2} \\ & \leq A_2 e^{-\theta_1 t} \leq A_2 e^{-\theta_1 T} \leq A_2 C_1 \eta. \end{aligned}$$

And so, by triangular inequality,

$$(4.8) \quad \forall t \geq T, \quad \|u(t) - p(t; \alpha_k, \beta_k, x_{1,k}(t), x_{2,k}(t), \kappa_l, c_l^0, x_{0,l}(t))\|_{H^2} \leq (4A_0 + A_2)C_1 \eta.$$

The latter proves the Theorem for $t \geq T$ with $C_0 = (4A_0 + A_2)C_1$. For $0 \leq t \leq T$, it is enough to use (4.3). \square

APPENDIX: EQUATIONS FOR LOCALIZED CONSERVATION LAWS

Lemma 4.1. *Let f be a C^3 function that do not depend on time and u a solution of (mKdV). Then,*

$$(4.9) \quad \frac{d}{dt} \frac{1}{2} \int u^2 f = \int \left[-\frac{3}{2} u_x^2 + \frac{3}{4} u^4 \right] f_x + \frac{1}{2} \int u^2 f_{xxx},$$

$$(4.10) \quad \frac{d}{dt} \int \left[\frac{1}{2} u_x^2 - \frac{1}{4} u^4 \right] f = \int \left[-\frac{1}{2} (u_{xx} + u^3)^2 - u_{xx}^2 + 3u^2 u_x^2 \right] f_x + \frac{1}{2} \int u_x^2 f_{xxx},$$

$$\begin{aligned} & \frac{d}{dt} \int \left[\frac{1}{2} u_{xx}^2 - \frac{5}{2} u^2 u_x^2 + \frac{1}{4} u^6 \right] f \\ & = \int \left(-\frac{3}{2} u_{xxx}^2 + 9u_{xx}^2 u^2 + 15u_x^2 u u_{xx} + \frac{9}{16} u^8 + \frac{1}{4} u_x^4 + \frac{3}{2} u_{xx} u^5 - \frac{45}{4} u^4 u_x^2 \right) f' \\ (4.11) \quad & + 5 \int u^2 u_x u_{xx} f'' + \frac{1}{2} \int u_{xx}^2 f'''. \end{aligned}$$

Proof. see the bottom of the page 1115 and the bottom of the page 1116 of [28] and Section 5.5 in [39]. \square

Lemma 4.2. *Let $r > 0$. If t, x satisfy $\widetilde{x}_{j-1}(t) + r < x < \tilde{x}_j(t) - r$, then*

$$(4.12) \quad \left| \tilde{P}(t, x) \right| \leq C e^{-\beta r}.$$

Proof. immediate consequence of the exponential majoration of each object. \square

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