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# Comparing Cross Correlation-Based Similarities

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## Abstract

The common product between two multisets, or functions represented as multisets (multifunctions) can be understood as being analogue to the inner product in real vector or function spaces in spite of its non-linear nature. In addition to other interesting features, the common product also allows respective correlations to be derived which, in addition to their conceptual and computational simplicity, have been verified to be able to provide enhanced results in tasks such as template matching, tending to yield peaks that are sharper and narrower than those typically obtained by standard cross-correlation, while also attenuating substantially secondary matchings. The multiset-based correlations based on the real-valued multiset Jaccard and coincidence indices are compared in this work, with encouraging results which have immediate implications not only in pattern recognition and deep learning, but also in scientific modeling in general. As expected, the multiset correlation methods, and especially the coincidence index, presented remarkable performance characterized by sharper and narrower peaks while secondary peaks were attenuated, which was maintained even in presence of intense levels of noise. In particular, the two methods derived from the coincidence index led to the sharpest and narrowest peaks, as well as intense attenuation of the secondary peaks. The cross correlation, however, presented the best robustness to symmetric additive noise, which suggested a new combination of the considered approaches. After a preliminary investigation of the relative performance of the multiset approaches, as well as the classic cross-correlation, a systematic comparison framework is proposed and applied for the study of the aforementioned methods. Several interesting results are reported, including the confirmation, at least for the considered type of data, of the coincidence correlation as providing enhanced performance regarding detection of narrow, sharp peaks while secondary matches are duly attenuated. The combined method also confirmed its good performance for signals in presence of intense additive noise.

*‘Deep inside the mirror, a whole universe of similarities.’*

*LdaFC*

## 1 Introduction

Many concepts and methods in science, and in particular in Physics, rely on operations capable of expressing the similarity between two mathematical structures or data. Of particular interest and importance has been the inner product between two vectors or functions, which underlies a vast quantity of concepts and results in Physics, especially when incorporated into the convolution and correlation operations, which underly all linear systems.

The aforementioned concepts and operations play a particularly important role in optics, mathematical physics, semiconductors, complex systems, quantum mechanics and computing, magnetic resonance, astrophysics, and neuronal networks, to name but a few possibilities.

The Jaccard index (e.g. [1, 2, 3]) has been extensively used in many areas as an interesting and effective means for quantifying the similarity between any two sets. A multiset Jaccard index, in which multisets (e.g. [4, 5, 6, 7, 8, 9]) are taken instead of sets, has also been used in similar applications, with the ability to take into account the multiplicities of multiset elements. More recently [3, 10, 11], several further generalizations of the Jaccard index have been proposed based on respective motivations. By taking into account real-valued data, these developments also allowed the multiset concepts and operations to be extended to real functions and signals, which have been called multifunctions.

Reflecting an analogy with the traditional inner product of vectors or functions, which can also be understood as quantifications of similarity in the sense of multiset differences, a respective operator has been proposed, called the *common product* [3, 10]. Unlike the inner product in vector and function spaces, the common product is non-

linear and does not require algebraic products, and could be understood as a first degree binary operator, while the inner product would have a second degree characteristic.

The common product can be readily applied not only to quantify the similarity between two clusters in pattern recognition, but also be employed as a joint variation measurement similar to the Pearson correlation coefficient. Preliminary results [3] indicate that the common product provides a purportedly quantification of joint variation between two random variables that is more compatible with human intuition than product-based measurements such as the Pearson correlation coefficient, especially given the tendency of the latter to saturate as the correlation increases [3].

From the common product between two functions, several similarity indices can be derived, including the real-valued Jaccard, addition-based Jaccard, and coincidence indices [3]. Each of these indices lead to the definition of a respective convolution/correlation between two multifunctions [3, 10].

As with its real function counterpart, the multifunction convolution can be employed for typical signal processing tasks, including but not being limited to filtering, template matching (e.g. [12, 13]), and control theory. Indeed, preliminary results [3, 10] have been obtained that are particularly promising and encouraging. More specifically, when used for template matching, the obtained peaks corresponding to the maximum similarity between the object and template functions are not only substantially sharper and narrower, but also the secondary matching peaks result much more attenuated [10].

The combination of these two features suggests that the multifunction convolution has substantial potential for applications in pattern recognition, neuronal networks and deep learning, as well as in many other related areas involving estimations of similarity and/or convolutions. Actually, the linear part of neurons in artificial neuronal networks and deep learning can be made to correspond to common products, instead of the traditionally adopted inner product of inputs.

For all the above reasons, it becomes of particular important to compare, in a quantitative and objective manner, the performance of the several mentioned correlation methods based on the multiset-based similarity indices, including the real-valued Jaccard and coincidence indices [3]. The classic cross-correlation operation should also be taken into account as a reference method, given its extensive application in the most diverse areas. This constitutes the main purpose of the present work.

We start by reviewing the adopted indices, and then proceed to a preliminary comparison related to template matching between an object and a template function in presence of noise, in which it is identified that

the addition-based Jaccard and respectively associated addition-based coincidence present performance similar to the real-valued Jaccard and coincidence indices. Though the classic cross-correlation led to matching results that are worse than the original object function, it was verified to be more resilient to elevated levels of noise than the other considered cross-correlation methods.

Given that the classic cross-correlation allowed good robustness to high levels of noise relatively to the similarity index-based methods, we also propose a combine method in which the former approach is applied prior to the latter indices.

Guided by the aforementioned preliminary comparison, attention is subsequently focused on the real-valued Jaccard and coincidence indices, as well as on the classic cross-correlation product. A systematic framework for comparing similarity cross-correlations is then proposed that can be understood as an ancillary contribution of the present work. Several interesting results are obtained, including the identification of the coincidence method as the most effective, at least for the type of signals considered here. Then, we also found that the classic cross-correlation turned out to be nearly hopeless regarding as far as the width and relative height of the principal matched peak detection is concerned. In addition, the combined method confirmed its potential for enhanced performance when the signals contain high levels of noise, especially for smooth template functions.

We start by presenting the considered cross-correlation methods, and then report on the preliminary comparison. The formal, systematic framework for comparing cross-correlation methods is then presented and applied for respective characterization of the considered cross-correlation approaches.

## 2 The Considered Similarity Indices

We start with the *classic cross-correlation* between two functions  $f(x)$  and  $g(x)$ :

$$\text{Corr}(f, g)[y] = \int_{-\infty}^{\infty} f(x)g(x - y)dx \quad (1)$$

The *basic Jaccard index* between two sets with non-negative multiplicity can be defined as:

$$\mathcal{J}(A, B) = \frac{|A \cap B|}{|A \cup B|} \quad (2)$$

where  $A$  and  $B$  are any two sets to be compared. It can be verified that  $0 \leq \mathcal{J}(A, B) \leq 1$ .

The *homogeneity* or *interiority index* can be expressed

as:

$$\mathcal{I}(A, B) = \frac{|A \cap B|}{\min\{|A|, |B|\}} \quad (3)$$

with  $0 \leq \mathcal{I}(A, B) \leq 1$  and  $0 \leq \mathcal{J}(A, B) \leq \mathcal{H}(A, B) \leq 1$ .

The combination of the basic Jaccard and the interiority indices yields the *coincidence index*, proposed in [3] as:

$$\mathcal{C}(A, B) = \mathcal{J}(A, B) \mathcal{I}(A, B) \quad (4)$$

The Jaccard index extended to multisets with non-negative multiplicities can be expressed as:

$$\mathcal{J}_M(A, B) = \frac{\sum_{i=1}^N \min(a_i, b_i)}{\sum_{i=1}^N \max(a_i, b_i)} \quad (5)$$

with  $0 \leq \mathcal{J}_M(A, B) \leq 1$ . Therefore, this index is capable of taking into account the multiplicity of the elements in the involved multisets.

The extension of the Jaccard index to real, possibly negative-valued multiplicities, involves the following developments [3, 10].

In particular, the possibility to have positive and/or negative multiplicity values now requires the application of a binary operator analogous to the inner product in function spaces, in which the multiplicities are properly mirrored among the four quadrants depending on their signs [3, 10]. This operator, which will be referenced here as *common product*, can be expressed as:

$$\ll f(x), g(x) \gg = \int_{-\infty}^{\infty} s_f s_g \min(s_f f(x), s_g g(x)) dx \quad (6)$$

This binary operator (in the sense of taking two arguments) corresponds to the direct counterpart of inner product between multisets with non-negative multiplicities.

We shall also adopt the following operator:

$$f(x) \circ g(x) = \int_{-\infty}^{\infty} \max(s_f f(x), s_g g(x)) dx \quad (7)$$

From the above results, the *multifunction convolution* [3, 10] (*mconvolution*) of two functions can now be derived:

$$f(x) \blacksquare g(x)[y] = \int_{-\infty}^{\infty} \ll f(x)g(y-x) \gg dx \quad (8)$$

Similarly, we derive the *multifunction correlation*, or *mcorrelation* as:

$$f(x) \square g(x)[y] = \int_{-\infty}^{\infty} \ll f(x)g(x-y) \gg dx \quad (9)$$

It is also possible [10] to define:

$$f(x) \circledast g(x)[y] = \int_{-\infty}^{\infty} [f(x) \circ g(x-y)] dx \quad (10)$$

We also have:

$$A_f = \int_{-\infty}^{\infty} |f(x)| dx \quad (11)$$

$$A_g = \int_{-\infty}^{\infty} |g(x)| dx \quad (12)$$

Now, the multifunction correlation, when normalized by the above function, yields:

$$f(x) \square g(x)[y] = \frac{\int_{-\infty}^{\infty} \ll f(x), g(x-y) \gg dx}{\int_{-\infty}^{\infty} f(x) \circledast g(x-y) dx} \quad (13)$$

observe that the integrand corresponds to the generalized Jaccard index adapted to cope also with negative multiplicities, here called *negative multiset Jaccard index*.

The interiority index also needs to be adapted to possibly negative multiplicities. First, we make:

$$\ll f(x), g(x) \gg_{+} = \int_{S_{+}} \min(s_f f(x), s_g g(x)) dx \quad (14)$$

where  $S_{+} = \{x | s_f(x)s_g(x) > 0\}$ , and then we can write:

$$\mathcal{I}(f(x), g(x)) = \frac{\ll f(x), g(x) \gg_{+}}{\min\{A_f, A_g\}} \quad (15)$$

The respective combination with the interiority index yields the *coincidence index* [3, 10], written as:

$$\mathcal{C}_N(f(x), g(x)) = \mathcal{J}_N(f(x), g(x)) \mathcal{I}(f(x), g(x)) \quad (16)$$

Now, when performing template matching between two functions, it may be interesting to consider the interiority also when  $s_f(x)s_g(x) < 0$ . In this case, it is possible to take the absolute value of the object function as argument of the interiority index above, which is the case considered henceforth in this work.

The real-valued Jaccard index can be adapted for taking into account the *sum* of the two sets  $A$  and  $B$ , instead of their respective union, which leads to the *addition-based real-valued Jaccard index*:

$$\mathcal{J}_A(f(x), g(x)) = \frac{2 \int_{-\infty}^{\infty} \min(f(x), g(x)) dx}{\int_{-\infty}^{\infty} (f(x) + g(x)) dx} \quad (17)$$

with  $0 \leq \mathcal{J}_S(f(x), g(x)) \leq 1$ .

The addition-based real-valued Jaccard index can also be combined with the interiority index, leading to the respective *addition-based coincidence index*:

$$\mathcal{C}_A(f(x), g(x)) = \mathcal{J}_A(f(x), g(x)) \mathcal{I}(f(x), g(x)) \quad (18)$$

Each of these indices lead to respective multifunction convolutions and correlations, which involve sliding one function with respect to the other while calculating the respective index, followed by the respective integration.

Several other indices can be derived from those described above by choosing other functionals for numerator and denominator of the Jaccard index, as well as by taking into account other product combinations of those indices.

Table 1 summarizes the several similarity index underlying the correlation/convolution methods that will be compared in the present work.

method	definition
Classic cross-correlation	Eq. 1
Real-valued Jaccard	Eq. 13
Interiority	Eq. 14
Coincidence	Eq. 16
Addition-based real-valued Jaccard	Eq. 17
Addition-based coincidence	Eq. 18

Table 1: A summary of the similarity correlation/convolution methods adopted in this work respectively to their underlying indices, also including their abbreviations and respective equations.

### 3 Preliminary Comparison

In order to have a first indication about the relative performance of all the methods in Table 1 in presence of increasing noise levels, we performed a respective preliminary comparison regarding the object and template functions shown in Figure 1(a) and (b), respectively. Also shown in that figure are the results of results of applying the several types of correlations/convolutions considered in the present work in the case of null noise.

As expected, the multiset-based correlations yielded substantially enhanced results regarding the identification of the peaks corresponding to the matches, leading to sharper and narrower peaks and attenuation of secondary matches, while the standard cross correlation resulted with peaks that are even wider than in the original object.

Of particular interest is to observe the results of the interiority-based convolution, yielding two sharp positive peaks that are, however, less narrow than those obtained for the other multifunction convolutions. This interesting property contributed to the verified enhanced performance of the two coincidence index-based multifunction convolution methods, which yielded the sharpest and narrowest matching peaks while almost eliminating the secondary matches.

Subsequent results considered the incorporation of progressive noise levels into the object function. More specifically, we added noised points at each of the  $x$  values drawn

from the symmetric, uniform density:

$$n(x) = L(u(x) - 0.5) \tag{19}$$

where  $u()$  is the uniform random distribution in the interval  $[0, 1]$  and  $L$  is the *noise level*.

The best results, in the sense of enhancing the main peak while attenuating the secondary matches, resulted from the application of the coincidence and addition-based coincidence correlations.

It follows from the analysis of these results that all considered methods are robust to substantially high levels of noise. Actually, the multiset based approaches results combine a high-pass action in sharpening and enhancing the peaks with a low-pass effect in substantially reducing the high-frequency noise.

However, two effects are of particular importance. First, we have that the added noised implied in slight loss of sharpness in the case of all the multiset methods. At the same time, the classic cross correlation accounted for the best robustness to noise. These complementary features motivated the combination of these two types of methods as described in the following section.

### 4 Combining Cross Correlation and Multifunction Correlations

We have seen that, while the multiset-based cross-correlation methods allow enhanced performance compared the classic cross-correlation, the latter is more robust to the highest considered noise levels. These complementary features of these two families of methods motivate us to propose a combined method that would take advantage of both interesting features, to be applied in elevated levels of noise.

This can be immediately achieved by applying the multiset methods *after* the two signals have been cross-correlated, with the result being taken as the next object function. However, the noise tolerance will depend on the smoothness of the template function. It is also possible to apply low-pass filtering on the noise functions, but this may not be necessary in case the template function is itself smooth as in the case of the current examples.

Figure 2 illustrates the results obtained by application of the above proposed methodology on the functions in Figure 1, a situation which corresponds to the highest levels of noise considered in this work. The obtained results corroborate the potential of the approach.

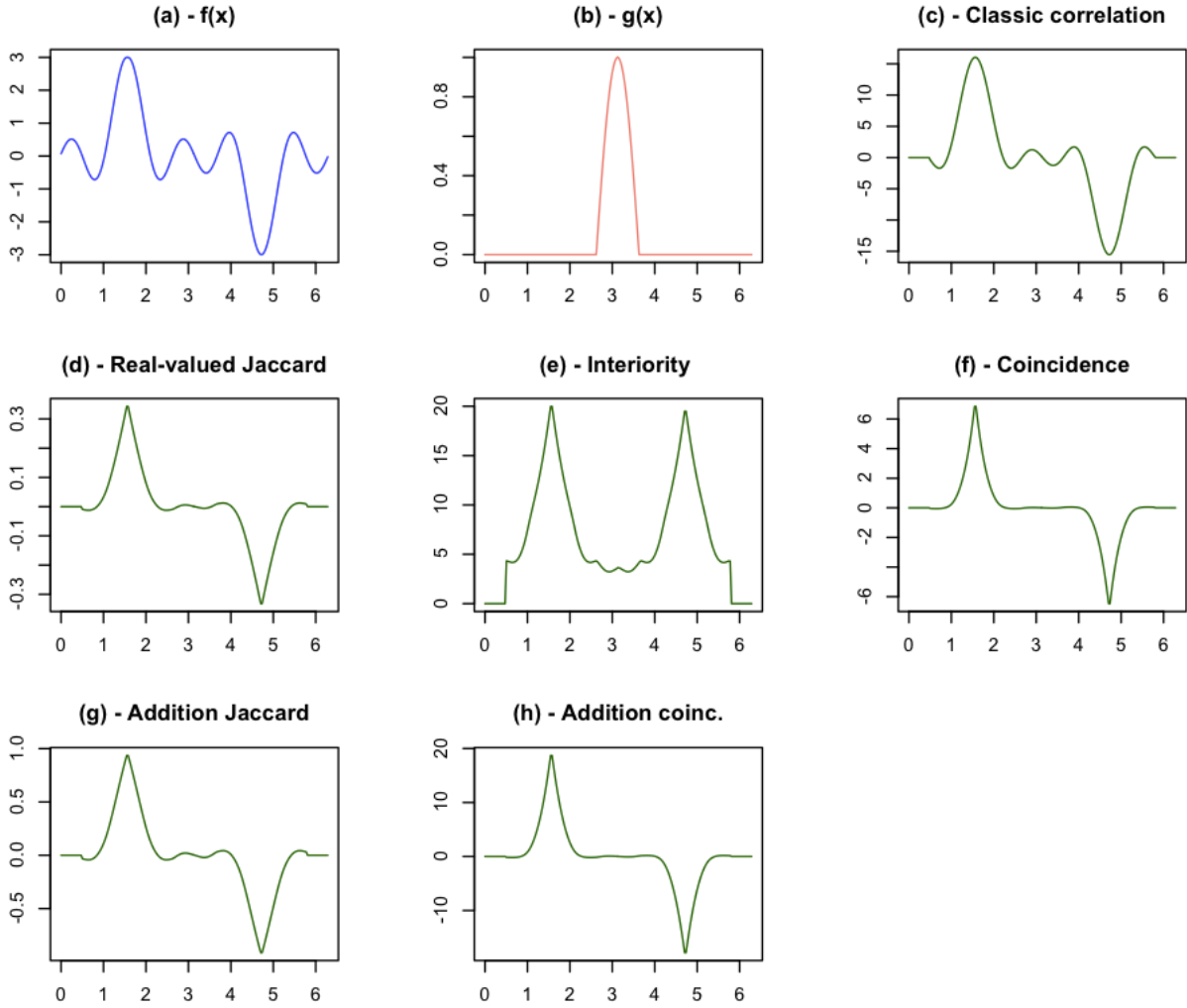


Figure 1: Comparison of the several multifunction convolution methods with respect to two noiseless functions (a) and (b).

## 5 The Comparison Framework

In order to compare the performance of the considered correlation methods in an objective and systematic manner, it is first necessary to develop a formal and comprehensive respective framework taking into account all the parameters of the object and template function, as well as defining several suitable quantifications of the several performance aspects of interest.

The preliminary performance investigation reported in the previous section contributed to obtaining a more effective and objective systematic approach. For instance, since the real-valued addition-based Jaccard and respective coincidence indices were found to yield performance very close to the real-valued Jaccard and coincidence indices regarding the addressed operation of template matching, only the two latter indices are taken into account in a more systematic manner. Also, given that the interiority index is focused on providing a partial indication of the similarity between the template and object

functions, related mainly to the relative interiority between these two functions, the interiority index will also not be considered further in our subsequent performance evaluation.

However, despite its almost hopeless performance of the classic cross-correlation approach, it will be considered as a reference because of its extensive application in the most diverse areas.

Figure 3 depicts the basic framework proposed in this work for systematic comparison of the performance of the application of the considered correlation methods to the task of template matching.

The henceforth adopted object function in (a), corresponding to the addition of two gaussians, incorporates all the aspects of interest in our comparison approach. Here, we have a *principal* peak with height  $h_p$  and width  $w_p$  together with a *secondary* peak with respective  $h_s$  and  $w_s$ . Each of these two gaussians are placed at respective

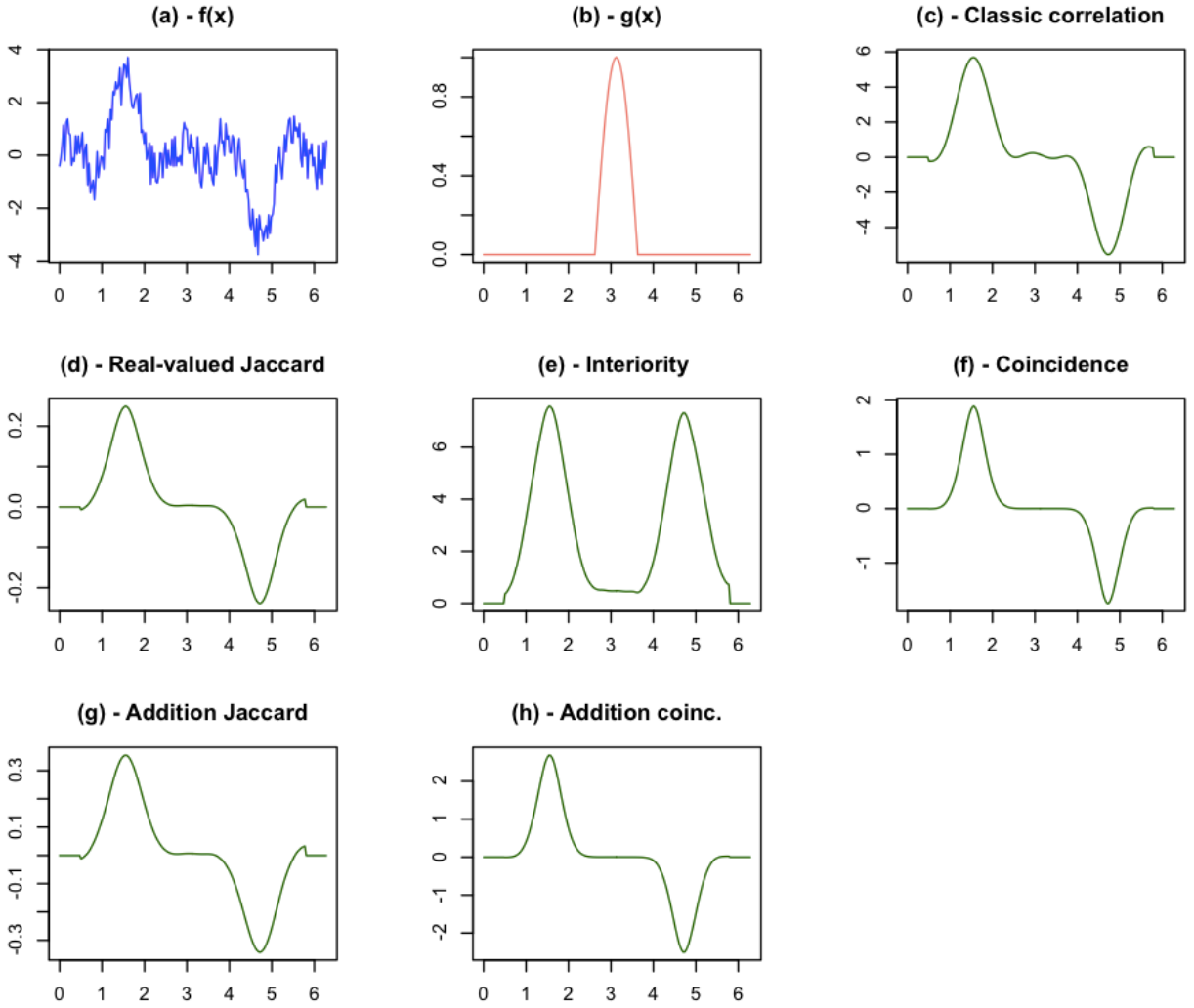


Figure 2: Results of the combination of cross-correlation with multifunction convolutions. The cross correlation was first obtained between the object signal with level noise  $L = 2$  and respective template function, then followed by the several considered methods. It can be readily observed that this respective methodology allowed the combination of the good characteristics of the classic cross correlation (higher robustness to noise) with the enhanced peaks provided by the multifunction convolutions considered in the present work.

abscissae values  $x_p$  and  $x_s$ .

The expected cross-correlation result is to emphasize as much as possible the dominant peak while the secondary match is attenuated. More precisely, the ideal result would correspond to a single Dirac's delta peak at position  $x_p$ . However, the typical expected result will be as indicated in Figure 3(b). The main resulting peak will be positioned at  $x_1$ , having height  $h_1$  and width  $w_1$ . A secondary peak is likely to be obtained at respective position  $x_2$ , with height  $h_2$  and width  $h_2$ . In addition, an overlap region extending from  $x_2$  to  $x_1$  is also likely to be obtained as result of the cross-correlation methods. The shapes of the detected peaks will typically be different from those of the respective original counterparts.

The width of the four peaks is determined by measuring the extension of the respective slice at 75% of the respective height. We have not adopted the standard devia-

tion (other for initial specification of the peaks) because, though it would be viable regarding the object function in (a), it is impossible to be properly characterized from the resulting matching profile  $s(x)$  in (b), as this would require some criterion and respective methodology for separating the two typically overlapping peaks.

We can take the following interesting additional parameters regarding the object function that can be used while comparing and discussing the correlation methods:

$$\Delta x = x_p - x_s \quad (20)$$

$$r_h = \frac{h_p}{h_s} \quad (21)$$

$$r_w = \frac{w_s}{w_p} \quad (22)$$

The first parameter,  $\Delta x$ , specifies the separation between the two peaks. The ratio  $r_h$  indicates how sec-

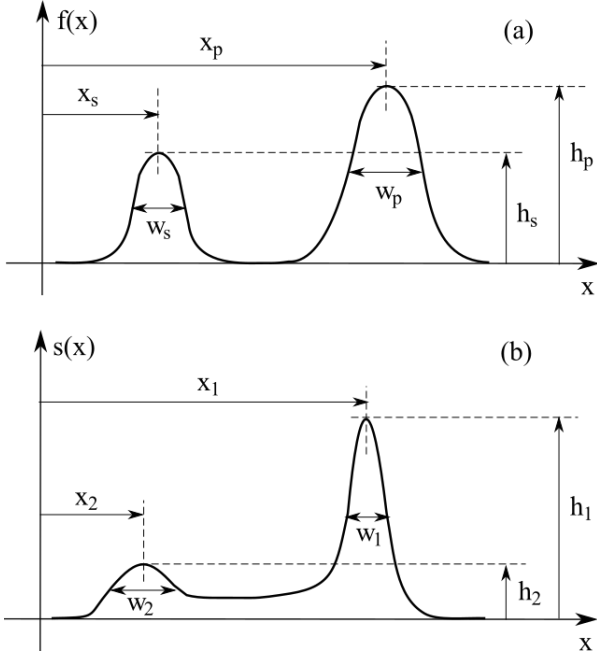


Figure 3: The basic framework for comparison of correlation methods. The object function has two peaks, one principal ( $p$ ) and another secondary ( $s$ ). The result of the correlation yields two peaks, one corresponding to the principal original peak (1), and the other to the secondary original peak (2).

ondary the smaller peak is, while the ratio  $r_w$  expresses the relationship between the two widths. These parameters correspond to those that are most likely to influence the performance of the correlation approaches. Good results will be favored by relatively large values of  $\Delta x$  and  $r_h$ . The influence of the ratio  $r_w$  is not so straightforward.

Now that we have identified all relevant parameters, we are in position to objectively define figures of merit regarding the respective performance. Consider the prototypical matching function depicted in Figure 3(b). The adopted performance indicators are as follows:

$$\begin{aligned}
 r_{x_p} &= \frac{x_p - x_1}{x_p} && \text{desirable low} \\
 r_{x_s} &= \frac{x_s - x_2}{x_s} && \text{desirable low} \\
 r_h &= \frac{h_1/h_2}{h_p/h_s} && \text{desirable high} \\
 r_{w,p} &= \frac{w_1}{h_1} && \text{desirable low} \\
 r_{w,s} &= \frac{w_2}{h_2} && \text{desirable high} \\
 \alpha &= \int_{x_2}^{x_1} s(x) dx && \text{desirable low}
 \end{aligned}$$

The ratio  $r_{x_p}$  quantifies how much the main resulting peak displaces itself from the original principal peak. As such, this corresponds to a relative error and should be as small as possible in order to ensure accurate template de-

tection. The ratio  $r_{x_s}$  plays an analogue role respectively to the secondary peaks, and therefore also should be as small as possible.

The relationship between the relative heights of the main and secondary original and detected peaks is quantified by the relative index  $r_h$ . The higher this value, the most the main peak will differentiate in height from the secondary peak. This immediately implies that the secondary peak will result relatively more attenuated. As such, this index should be as high as possible.

The ratio  $r_{w,p}$  quantifies how narrow the obtained detected peak is and, as such, should be as small as possible. The index  $r_{w,s}$  plays a similar role regarding the secondary peak, and as such should be as high as possible.

The index  $\alpha$  corresponds to the integral of the obtained matching function  $s(t)$ , therefore quantifying how intense the resulting overlap is. Ideally, we should have  $\alpha = 0$ , so that the lower this value, the better the performance will be.

## 6 Performance in Presence of Additive Noise

The comprehensive systematic framework developed in the previous section is now applied as a means for comparing the similarity-based correlation approaches. More specifically, we will incorporate progressive levels of additive noise to the object function as in Figure 3, with the following parameters:

$$\begin{aligned}
 \sigma_p &= 0.3 \\
 \sigma_s &= 0.15 \\
 h_p &= 2 \\
 h_s &= 1 \\
 x_p &= 4.5 \\
 x_s &= 1.8
 \end{aligned}$$

The 21 noise levels are as follows:

$$ns[v] = \frac{v}{20} [u(x) - 0.5], \text{ with } v = 0, 1, \dots, 20 \quad (23)$$

While the first level correspond to noiseless object function, the noise level implied by  $v = 20$  is markedly intense on purpose.

A total of 300 realizations are obtained for each of the noise levels above. The methods to be compared are the classic cross-correlation, real-valued Jacard and coincidence indices, as well as the combination of cross-correlation and coincidence indices-based cross-correlations. In all cases, the template signal is a half sine identical to that shown in Figure 1.



Figure 4 presents the average  $\pm$  standard deviation results obtained respectively to the real-valued Jaccard (results shown in salmon) and coincidence method (in blue). Several interesting and important results can be verified.

First, we have that both these methods yielded quite similar performance regarding the accuracy for identification of the original position of the peaks (Fig. 4(a,b)) which, in both cases, are nearly zero, which is a quite good result. However, when we move to the ratio  $r_h$  shown in (c), which should correspond to a high value, the coincidence method outperforms the real-valued Jaccard by a large margin.

The performance ratio  $r_{wp}$ , which should have low values, again revealed the enhanced relative performance of the coincidence index. The relative performance for the index  $r_{ws}$  resulted even more striking, further corroborating the special characteristics of the coincidence method.

The index  $\alpha$  obtained for the coincidence method, shown in Figure 4(f), resulted nearly half of that observed for the real-valued Jaccard approach, again revealing enhanced performance of the former methodology.

Though it could be expected from our preliminary analysis that the coincidence correlation method would outperformed the real-valued Jaccard correlation approach, the obtained results indicate that the relative advantage is surprisingly significant. This is especially so in the case of the ratio  $r_h$ , which indicates that the coincidence method attenuates substantially the secondary peaks.

Still regarding the results in Figure 4, it is interesting to observe that the real-valued Jaccard and coincidence methods tend to have more similar performance as the added noise level increases. This is expected because too high noise levels tend to completely undermine the patterns in the object function, as illustrated in the object function in Figure 5, which is respective to the highest considered noise level.

Figure 6 presents the performance figures obtained for the classic cross-correlation method. As already hinted by the preliminary results described in Section 3, this method is nearly hopeless for identification of the patterns in terms of narrow, sharp peaks, while the secondary peaks are duly attenuated. Except for the  $r_h$  index in Figure 6(c), all other indices resulted almost constant with the noise levels. Though a slight performance improvement can be observed in (c), its values are far away from those allowed by the real-valued Jaccard and coincidence methods.

Figure 7 presents the results obtained for the combination of correlation as preliminary processing before the ap-

plication of the real-valued Jaccard or coincidence methods, as proposed in Section 4.

The obtained results confirm the previous expectation that, in the average in case of object functions with intense level of noise it can be advantageous to apply the real-valued Jaccard or coincidence correlations after the object function is filtered by the classic cross-correlation. Indeed, though the obtained performance figures are all worse in the case of low level noise, they become nevertheless better than would be otherwise obtained by not using a priori correlation in the case of the highest considered levels of noise. However, larger error bars are observed for more intense noise levels.

Another interesting result has been obtained in which the unfolding of the several merit figures became much more straight or linear along the levels of noise. At the same time, the combined coincidence correlation approach resulted substantially better than the combined real-valued Jaccard method for all levels of noise.

It should be kept in mind that these results concern situations where the template function is smooth, being therefore capable of incorporating respective low-pass filtering action.

In order to complete our comparative performance analysis, we shown in Figure 8 the two main axis obtained by principal component analysis (PCA [14, 15]) of the performance figures for the real-valued Jaccard, coincidence, and classic cross-correlation methods. Three PCA projections are shown, corresponding to the second lowest (a), middle (b), and highest (c) considered noise levels.

Several interesting results are revealed from the PCA projections in Figure 8. First, we have the fact that a substantial deal of the original data variance (about 70% as indicated in parentheses in the axes labels) has been explained by the two main PCA axes, which indicates that the PCA projection is particularly representative of the original data distribution. Second, it becomes clear that, in all noise levels, the performance of the classic correlation method is well separated from that of the two other multiset-based methods.

Particularly important is to observe the relative dispersions of each of the three groups, with the coincidence index yielding the largest dispersion, followed by the real-valued Jaccard, and then the classic correlation approach. These dispersions of the performance figures reflect directly the sensitivity of each method to the specificities of the object function, being also related to the accuracy of the methods, in the sense that a method that always produce a same meaningless result independently of the shape of the object function would have null dispersion.

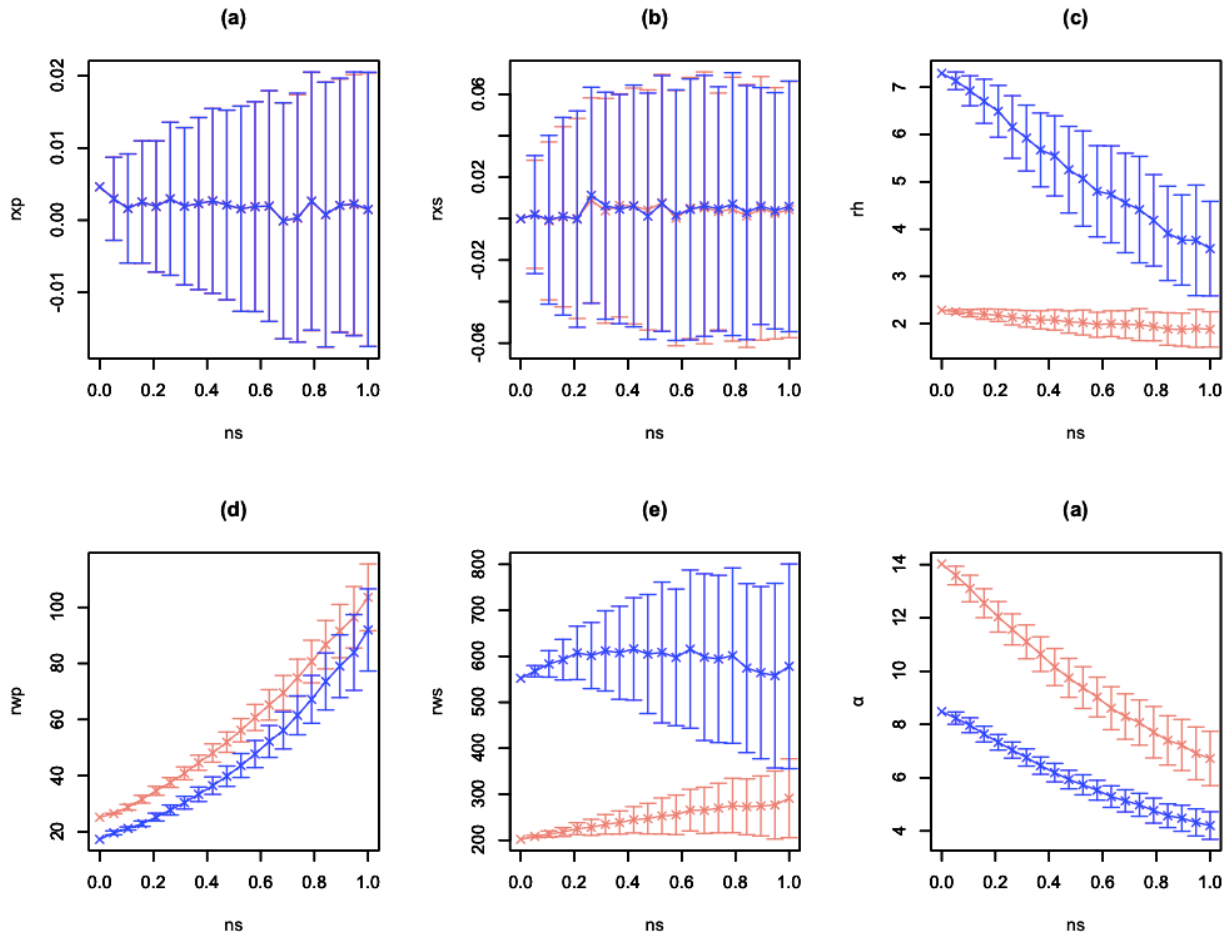


Figure 4: The performance results in terms of the considered level noise obtained while comparing the real-valued Jaccard (shown in salmon) and coincidence (in blue) methods. Except for the localization of the peaks, which resulted in similar indices (b,c) in both methods, the coincidence correlation approach revealed substantially superior relative performance. The plots correspond to the average values for 300 realizations,  $\pm$  the respective standard deviations.

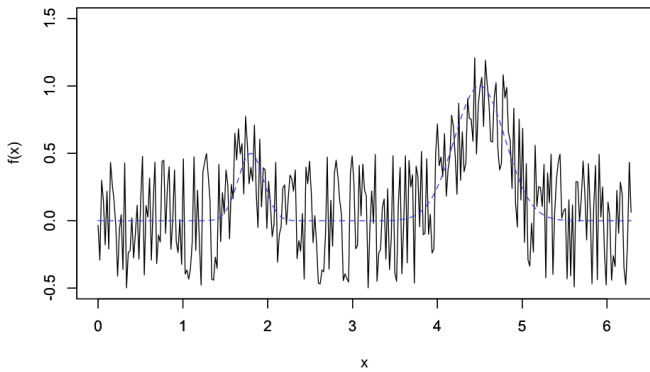


Figure 5: Example of object function incorporating the maximum considered level of noise (i.e. = 20). The original, noiseless version of the object function is also shown in dashed blue for comparison purposes.

As expected, as the noise effectively changes the shape of the object function, higher dispersions will be obtained by more accurate and sensitive methods respectively to higher levels of noise. It is also interesting to realize that it is the relative insensitivity of the classic correlation method that paves the way to its respective combination with the real-valued Jaccard and coincidence correlations in the case of particularly high levels of noise.

Yet, despite the larger fluctuations of the coincidence method, the respective performance values are largely higher than those obtained by the real-valued Jaccard approach and much less so by the extensively used classic cross-correlation. Observe also the confirmation of the already verified tendency of the performances of the coincidence and real-valued Jaccard correlations to merge for the highest levels of noise.

Yet, relatively smaller dispersions resulted in the cases of more typical lower noise levels considered in this work.

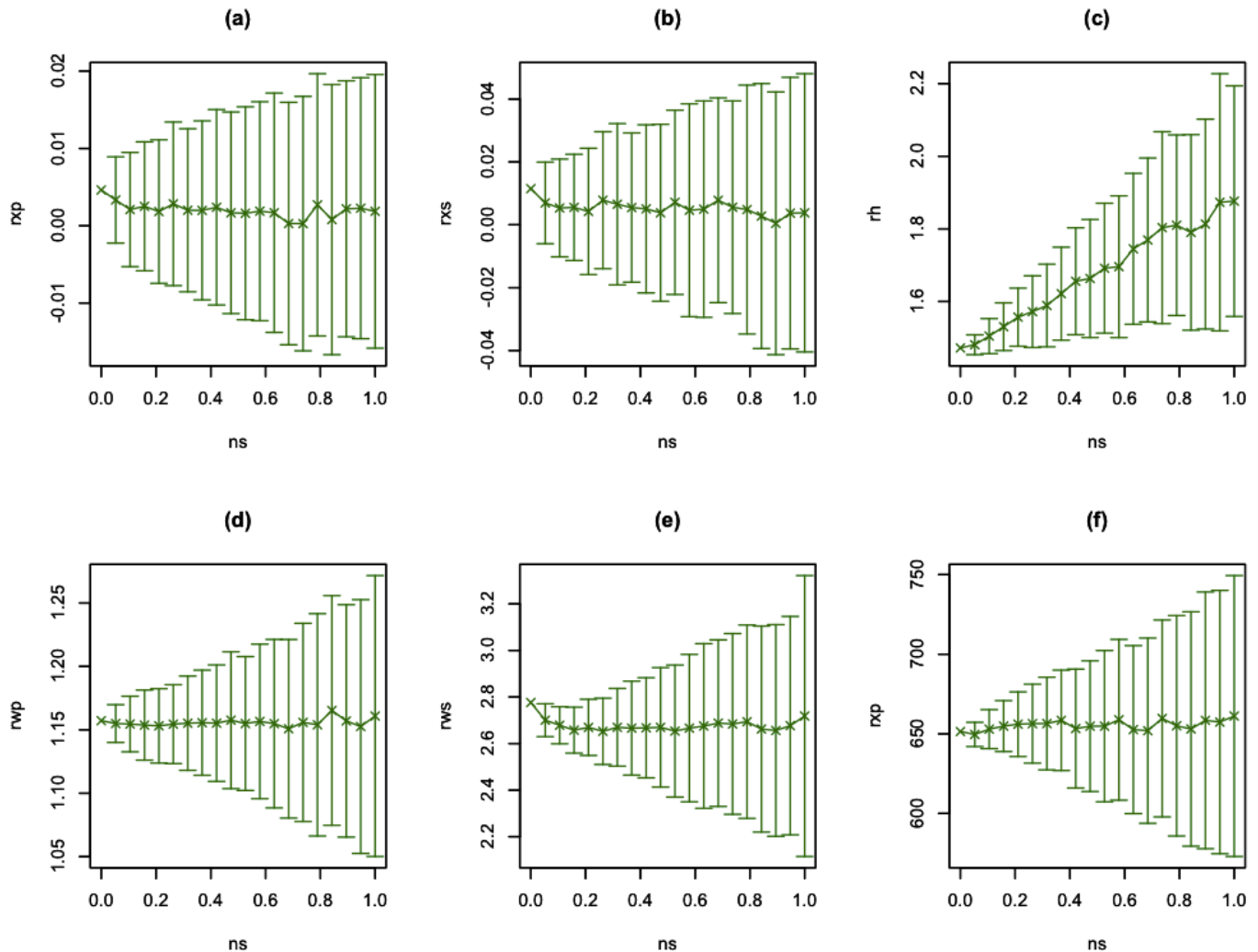


Figure 6: Performance figures obtained for the classic correlation method. Except for the localization of the peaks in (a) and (b), this method resulted almost hopeless regarding all other performance indices. The plots correspond to the average values for 300 realizations,  $\pm$  the respective standard deviations.

## 7 Concluding Remarks

The operation of cross-correlating two functions or signals constitutes an important mathematical resource that has been extensively applied for the most diverse scientific and technological purposes.

Though the classic cross-correlation approach relies on successive inner products between one function displaced relatively to the other, it is also possible to apply multiset-based approaches [3, 10, 11] to derived respective cross-correlation methodologies that have several intrinsic advantages, including being mainly based on the non-linear binary operations (in the mathematical sense of taking two arguments) of minimum and maximum, both of which being intuitive and requiring low computational costs.

A set of multiset-based cross correlation approaches has been recently introduced [3, 10, 11] that includes the real-

valued Jaccard, interiority, coincidence, addition-based real-valued Jaccard, as well as the addition-based coincidence correlations. Preliminary results [10, 11] indicated that these operations, except the interiority index which is used as an ancillary resource in the definition of the coincidence indices have, when compared with the classic cross-correlation approach, substantially enhanced potential for accurate and detailed similarity quantification, including in tasks such as filtering and template matching.

In the present work, we set out at developing a comprehensive and systematic comparison approach between the performance allowed by the above mentioned cross-correlation methods with respect to the important pattern recognition task of template matching, which can be readily understood as a kind of filtering.

After presenting the adopted multiset concepts and respectively derived cross-correlation approaches, we per-

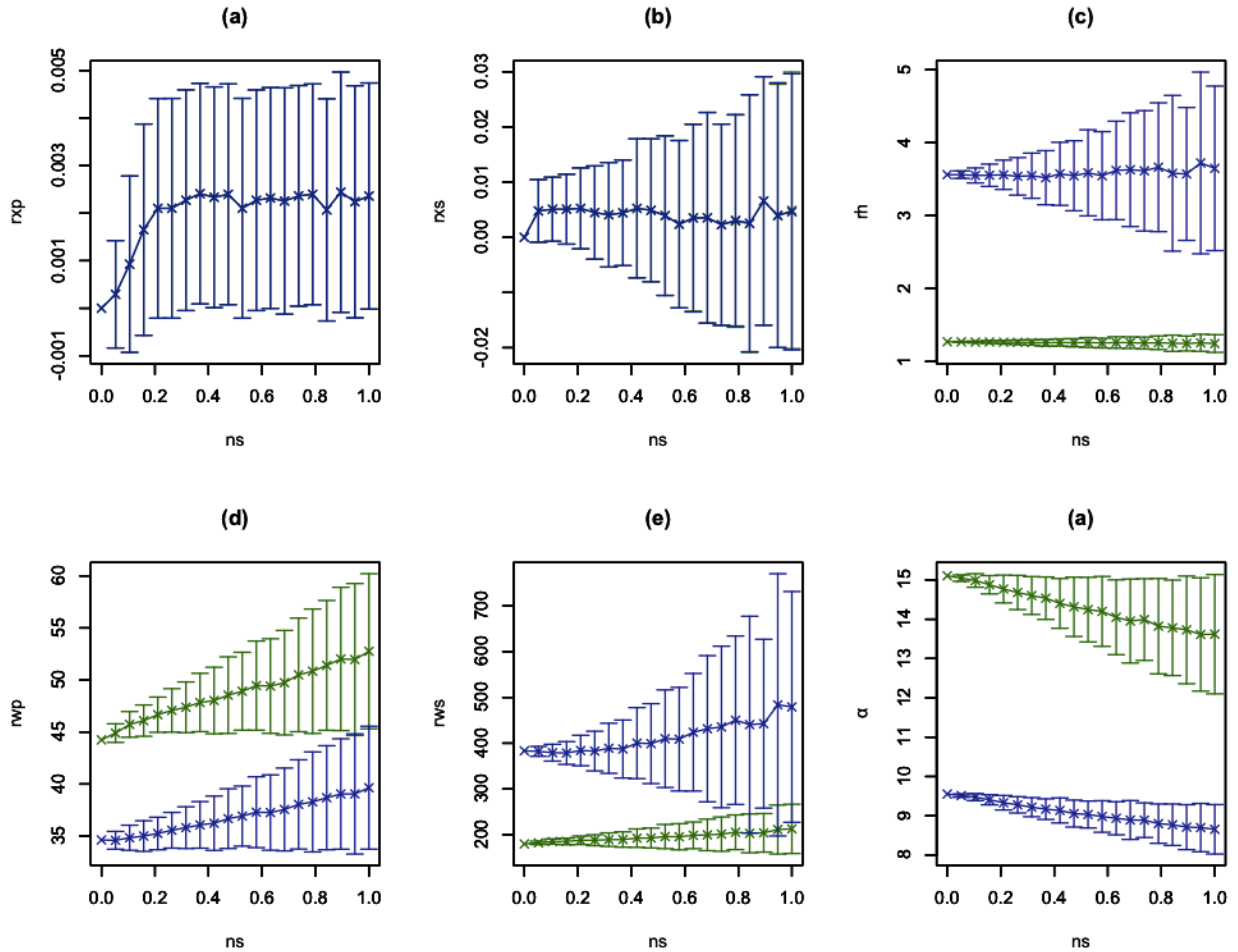


Figure 7: Performance indices obtained for the combination of correlation prior to the real-valued Jaccard (dark green) and coincidence (dark blue) methods. As expected, though this approach is less effective than the respective methods without the a priori correlation for low noise levels, relatively better results are observed for the largest levels of noise. These results corroborate the advantage of combining correlation and the real-valued Jaccard or coincidence methods in the case of very noisy object functions. Also of particular interest is that the several of the obtained curves present a markedly more linear variation with the noise level than for the other correlation configurations considered in this work. The plots correspond to the average values for 300 realizations,  $\pm$  the respective standard deviations.

formed a preliminary comparison aimed at providing an overall appreciation of the relative potential of the several considered methods. The results clearly corroborated the enhanced potential of the multiset methods respectively to the classic cross-correlation. In addition, it was verified that the addition-based methods led to similar performances to those obtained by the real-valued Jaccard and coincidence approaches. These preliminary results also indicated that the interiority approach is not particularly powerful when considered without being combined with the Jaccard index in order to obtain the coincidence method. These findings allowed us to narrow the focus of our comparison on three main alternative cross-correlation methods: the real-valued Jaccard, coincidence, and classic cross-correlation approaches.

In addition, the identification of the enhanced robustness of the classic cross-correlation to intense noise lev-

els observe in the preliminary comparison also motivated the proposal of a method in which the cross-correlation is performed prior to the application of the multiset-based methods as means to reduce the noise as a preparation for the matching detection.

In order to characterize and understand better the respective performance of the three chosen methods, we proposed a formal and comprehensive framework to be used for systematic performance comparison. This framework, which can also be understood as an additional contribution of the present work, takes into account several important quantitative indices expressing the performance of the cross-correlation with respect to several relevant features characterizing good performance.

The application of the proposed comparison framework led to several interesting results.

First, we have that the real-valued Jaccard and coinci-

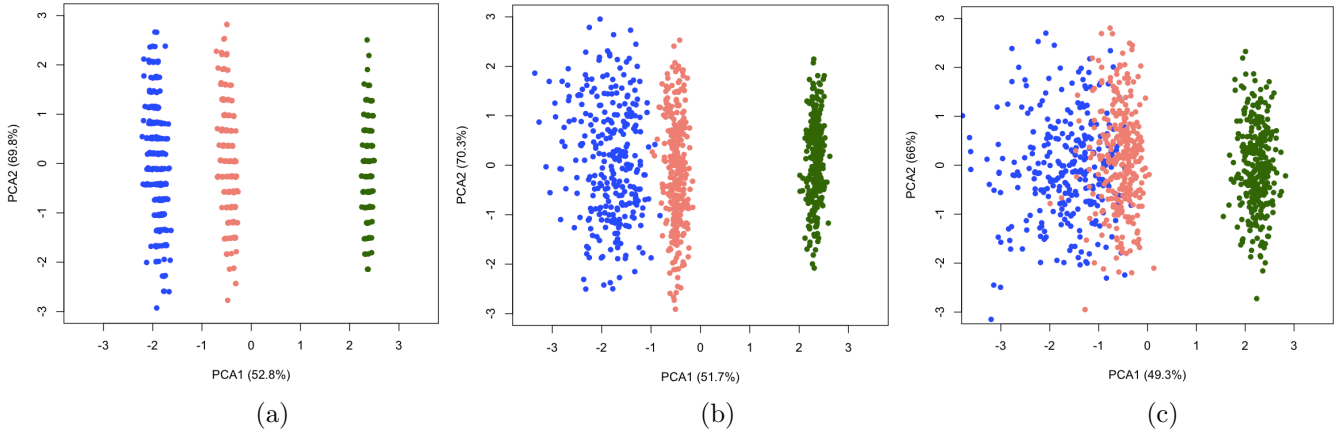


Figure 8: Principal component analysis (PCA) of the performance figures obtained for the real-valued Jaccard (shown in salmon), coincidence (blue) and classic cross-correlation (green) for the second lowest (a), middle (b) and highest considered level noises. Observe the good variance explanation indicated in parentheses on the respective axes. The fact that larger dispersions have been obtained for the real-valued Jaccard and coincidence indices respectively to the classic correlation reflects the greater ability of the two former methods, and especially the coincidence cross-correlation, to capture information about the original object function specificities. These three PCA projections also clearly indicate the substantial relative between the multiset-based correlations and the classic correlation, which is far away. Observe also the confirmation that the real-valued Jaccard and coincidence methods tend to allow similar performance for the highest noise levels.

dence methods allowed the best overall template matching performance respectively to the classic cross-correlation, therefore corroborating further the preliminary expectations. The performance was also observed not to be significantly undermined even by intense levels of additive noise.

Among the two multiset-based approaches, the coincidence method resulted significantly more effective, leading to best performance with respect to virtually every considered index. The proposed cross-correlation method combining the classic cross-correlation before the application of the multiset-based methods was also found to constitute an interesting alternative in cases of intense noise.

Additional characterization of the obtained performance indices by using principal component analysis provided further understanding the potential of each investigated method. It was therefore confirmed that the real-valued Jaccard and coincidence cross-correlation methods have performance markedly different from that of the classic cross-correlation. In addition, the sensitivity of the two multiset-based methods was confirmed by the observation of larger respective dispersions in the PCA space that are consequence of their enhanced sensitivity to the modifications implied by the noise on the specific shape of the object function. These complementary results also substantiated that the coincidence method provided the best overall performance at least for the type of data and problem addressed in our comparative analysis.

The reported concepts, methods, and results have important and wide implications for many areas dependent of the cross-correlation or the related convolution op-

erations, including signal processing (e.g. [16, 17, 18]), pattern recognition (e.g. [19, 20, 21]), shape analysis (e.g. [13]), complex networks (e.g. [22, 23, 24]) as well as deep learning (e.g. [25, 26]), among many other fields.

One point of particular interest is to contemplate to which extent similar methods could be employed in natural recognition systems involving neuronal cells, as motivated by their relative simplicity and low computational cost. An interesting related issue regarding the possible eventual neuronal operations that would be correlate to the multiset-based methods addressed in the present work.

Given the importance of similarity concepts and methods in the physical sciences, not to mention virtually every other scientific and technological fields, the implications of the methods results reported in the present work are particularly ample, paving the way to a large number of further developments. Indeed, every current concept or method based on similarity, inner products, and/or convolution and correlation can be re-assessed in the light of the presented results indicating the enhanced performance of the Jaccard and multiset-derived methods relatively to the extensively applied classic cross-correlation, as well as related concepts including the cosine similarity and inner product.

As more immediate future developments more close and specifically related to the issues addressed in the present work, it would be interesting to consider other types of functions and cross-correlation based methods, higher dimensional vector and function spaces, as well as to study the effect of the relative function magnitudes and other types of noise on the results. Related research is being

conducted and results are to be published opportunely.

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