Comparing Cross Correlation-Based Similarities

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Abstract

Introduced recently, the common product between two multisets, or functions represented as multisets (multifunctions), can be understood as being analogue to the inner product in real vector or function spaces. In addition to providing resources for quantifying joint variations, as well as the similarity between clusters in pattern recognition, the common product also allows a respective multiset convolution/correlation to be derived which, in addition to its conceptual and computational simplicity, has been verified to be able to provide enhanced results in tasks such as template matching, tending to yield peaks that are sharper and narrower than those typically obtained by standard cross-correlation, while also attenuating substantially secondary matching peaks. Given that the multiset convolution can be adapted to employ other similarity functionals, such as the coincidence and addition-based multiset Jaccard index, it remains an interesting subject to compare the performance of multifunction correlations based on these alternative implementations of the multifunction correlation. The present work addresses this subject, with encouraging results that can have immediate applications not only in pattern recognition and deep learning, but also in scientific modeling in general. As expected, the multifunction convolution/correlation methods presented enhanced performance, characterized by sharper and narrower peaks while secondary peaks were attenuated, which was maintained even in presence of increasing levels of noise. In particular, the two methods derived from the coincidence index led to the sharpest and narrowest peaks, as well as intense attenuation of the secondary peaks. The cross correlation, however, presented the best robustness to symmetric additive noise, which suggested a new combined method in which the cross-correlation is applied prior to the multifunction convolution methods, therefore allying the best characteristics of each of these two families of methods.

'*Deep inside the mirror, a whole universe of similarities.*'

LdaFC

1 Introduction

The Jaccard index (e.g. [1, 2, 3]) has been extensively used in many areas as an interesting and effective means for comparing any two sets. A multiset Jaccard index, in which multisets (e.g. [4, 5, 6, 7, 8, 9]) are taken instead of sets, has also been used in similar applications, with the ability to take into account the multiplicities of multiset elements. More recently [3], several further generalizations of the Jaccard index have been proposed based on several respective motivations.

In another recent work [10], multisets have been extended to cope with negative and real-valued multiplicities, which allowed the identification of the multiset universe as corresponding to the multiset with all multiplicities null. As a consequence of the definition of the multiset universe, the complementation operation becomes possible, allowing results such as the De Morgan theorem to be incorporated into multiset theory. In addition, these two extensions paved the way to extending multisets to represent generic functions and scalar/vector fields [3, 10], leading to the concept of multifunctions.

Multifunctions, however, require an adaptation of the inner product binary operation in order to properly take into account the relative signs of the multifunctions. Reflecting an analogy with the traditional inner product of vectors or functions, which can also be understood as quantifications of similarity in the sense of multiset differences, a respective operator has been proposed, called the *common product* [3, 10].

The common product can be readily applied not only to quantify the similarity between two clusters in pattern recognition, but also be employed as a joint variation measurement similar to the Pearson correlation coefficient. Preliminary results [3] indicate that the common product provides a purportedly quantification of joint variation between two random variables that is more compatible with human perception than product-based measurements such as the Pearson correlation coefficient, especially given the tendency of the latter to saturate as the correlation increases [3].

The common product also allowed the definition of a convolution/correlation between two multifunctions [3, 10]. As with its real function counterpart, the multifunction convolution can be employed for typical signal processing tasks, including but not being limited to filtering and template matching. Indeed, preliminary results [3, 10] have been obtained that are particularly promising and encouraging. More specifically, when used for template matching, the obtained peaks corresponding to the maximum similarity between the target and template functions are not only substantially sharper and narrower, but also the secondary matching peaks result much more attenuated [10].

The combination of these two features suggests that the multifunction convolution has substantial potential for applications in pattern recognition and deep learning, as well as other related areas involving estimations of similarity and/or convolutions. Actually, the linear part of neurons in artificial neuronal network can be made to correspond to common products, instead of the traditionally adopted inner product of inputs.

Since the multifunction convolution is based on the functional of the common product for several relative displacements between the two involved multifunctions, it becomes of particular interest to adopt alternative similarity indices, such as those proposed in [3]. That constitutes the main purpose of the present work. More specifically, we compare several multifunction convolutions derived from the generalized Jaccard indices presented and developed in [3].

We start by reviewing the adopted indices, then proceed to a numeric comparison of the respectively obtained multifunction convolutions with respect to two multifunctions, while also considering progressive levels of additive noise.

2 The Considered Similarity Indices

We start with the traditional cross-correlation between two functions $f(x)$ and $g(x)$:

$$Corr(f,g)[y] = \int_{-\infty}^{\infty} f(x)g(x-y)dx \tag{1}$$

The *basic Jaccard index* between two sets with non-negative multiplicity can be defined as:

$$\mathcal{J}(A,B) = \frac{|A \cap B|}{|A \cup B|} \tag{2}$$

where A and B are any two sets to be compared. It can be verified that $0 \leq \mathcal{J}(A,B) \leq 1$.

The *homogeneity* or *interiority index* can be expressed as:

$$\mathcal{I}(A,B) = \frac{|A \cap B|}{\min\{|A|, |B|\}} \tag{3}$$

with $0 \leq \mathcal{I}(A,B) \leq 1$ and $0 \leq \mathcal{J}(A,B) \leq \mathcal{H}(A,B) \leq 1$.

The combination of the basic Jaccard and the interiority indices yields the *coincidence index*, proposed in [3] as:

$$\mathcal{C}(A,B) = \mathcal{J}(A,B)\, \mathcal{I}(A,B) \tag{4}$$

which can also be expressed in expanded form as:

$$\mathcal{C}(A,B) = \frac{|A \cap B|^2}{|A \cup B| \ \min\{|A|, |B|\}} \tag{5}$$

again with $0 \leq \mathcal{C}(A,B) \leq 1$.

The Jaccard index extended to multisets with non-negative multiplicities can be expressed as:

$$\mathcal{J}_M(A,B) = \frac{\sum_{i=1}^{N} \min(a_i, b_i)}{\sum_{i=1}^{N} \max(a_i, b_i)} \tag{6}$$

with $0 \leq \mathcal{J}_M(A,B) \leq$. Therefore, this index is capable of taking into account the multiplicity of the elements in the involved multisets.

The extension of the Jaccard index to negative, real-valued multiplicities, involves the following developments [3, 10]. In particular, the possibility to have positive and/or negative multiplicity values now requires the application of a binary operator analogous to the inner product in function spaces, in which the multiplicities are properly mirrored among the four quadrants depending on their signs [3, 10]. This operator, which will be referenced here as *common product*, can be expressed as:

$$\ll f(x), g(x) \gg = \int_{-\infty}^{\infty} s_f s_g \min(s_f f(x), s_g g(x))dx \tag{7}$$

This binary operator corresponds to the direct counterpart of inner product between multisets with non-negative multiplicities.

From the above results, the *multifunction convolution* [3, 10] (*mconvolution*) of two functions can now be derived:

$$f(x) \blacksquare g(x)[y] = \int_{-\infty}^{\infty} \ll f(x)g(y-x) \gg dx \tag{8}$$

Similarly, we derive the *multifunction correlation*, or *mcorrelation* as:

$$f(x) \square g(x)[y] = \int_{-\infty}^{\infty} \ll f(x)g(x-y) \gg dx \tag{9}$$

It is also possible [10] to define:

$$f(x) \circledast g(x)[y] = \int_{-\infty}^{\infty} \max(f(x), g(x-y))dx \qquad (10)$$

Now, the multifunction convolution, when normalized by the above function, yields:

$$f(x)\Box g(x)[y] = \int_{-\infty}^{\infty} \frac{\ll f(x), g(x-y) \gg}{f(x) \circledast g(x-y)} dx \qquad (11)$$

observe that the integrand corresponds to the traditional Jaccard index adapted to cope also with negative multiplicities, here called *negative multiset Jaccard index*.

The interiority index also needs to be adapted to possibly negative multiplicities as;

$$\mathcal{I}(f(x), g(x)) = sign(\ll f(x), g(x) \gg) \frac{\ll f(x), g(x) \gg}{\min\{|f(x)|, |g(x)|\}} \qquad (12)$$

The respective combination with the interiority index yields the *negative multiset coincidence*, written as:

$$\mathcal{C}_N(f(x), g(x)) = \mathcal{J}_N(f(x), g(x))\, \mathcal{I}(f(x), g(x)) \qquad (13)$$

The multiset Jaccard index generalized to negative multiplicities can be adapted for taking into account the *sum* of the two sets A and B instead of their respective union, which leads to the *addition-based multiset Jaccard index*:

$$\mathcal{J}_A(f(x), g(x)) = \frac{2\sum_{i=1}^{N} \min(f(x), g(x))}{\sum_{i=1}^{N} (f(x) + g(x))} \qquad (14)$$

with $0 \leq \mathcal{J}_S(f(x), g(x)) \leq 1$.

As with the original Jaccard index, the addition-based multiset Jaccard index can be combined with the interiority index, leading to the respective *addition-based coincidence index*:

$$\mathcal{C}_A(f(x), g(x)) = \mathcal{J}_A(f(x), g(x))\, \mathcal{I}(f(x), g(x)) \qquad (15)$$

Each of these indices lead to respective multifunction convolutions and correlations, which involve sliding one function with respect to the other while calculating the respective index, followed by the respective integration.

Table 1 summarizes the several similarity index underlying the correlation/convolution methods that will be compared in the present work.

3 Results

Figure 1 presents the results of applying the several types of correlations/convolutions considered in the present work to the matching between a target ($f(x)$) and a template ($g(x)$) function.

method	definition
Traditional cross-correlation	Eq. 1
multifunction correlation	Eq. 9
Multiset Jaccard, neg. mult.	Eq. 11
Multiset coincidence, neg. mult.	Eq. 13
Addition-based multiset Jaccard	Eq. 14
Addition-based multiset coinc.	Eq. 15

Table 1: A summary of the similarity correlation/convolution methods adopted in this work respectively to their underlying indices, also including their abbreviations and respective equations.

As expected, the multifunction convolutions yielded substantially enhanced results regarding the identification of the peaks corresponding to the matches, leading to sharper and narrower peaks and attention of secondary matches, while the standard cross correlation resulted in peaks that are even wider than in the original target.

Of particular interest is to observe the results of the interiority-based convolution, yielding two sharp peaks that are, however, less narrow than those obtained for the other multifunction convolutions. This interesting property contributed to the observed enhanced performance of the two coincidence index-based multifunction convolution methods, which yielded the sharpest and narrowest matching peaks while almost eliminating the secondary matches.

Subsequent results considered the incorporation of progressive noise levels into the target function. More specifically, we added noised points at each of the x values drawn from the symmetric, uniform density:

$$n(x) = L(u(x) - 0.5) \qquad (16)$$

where $u()$ is the uniform random distribution in the interval $[0, 1]$ and L is the *noise level*.

Figures 2 to 2 present the obtained results respectively to noise levels 0.2, 0.5, 1 and 2.

It follows from the analysis of these results that all considered methods are robust to the applied levels of noise. Actually, in an analogous manner to the inner product of a target function with a smooth template, a *low-pass filtering* action can be observed that attenuates the added noise to a good extent.

However, two effects are of particular importance. First, we have that the added noised implied in slight loss of sharpness in the case of all the multifunction methods. At the same time, the cross correlation accounted for the best robustness to noise. These complementary features motivated the combination of these two types of methods as described in the following section.

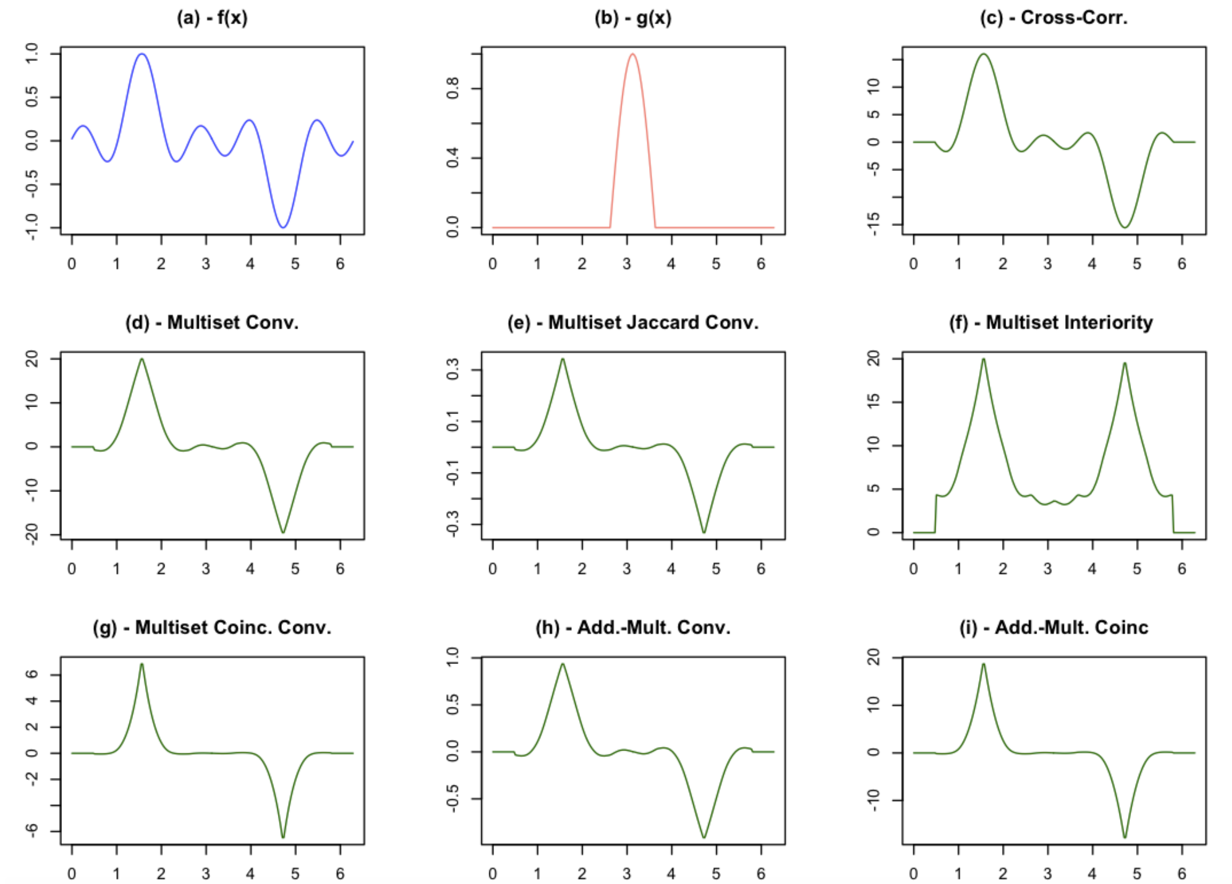

Figure 1: Comparison of the several multifunction convolution methods with respect to two noiseless functions (a) and (b).

4 Combining Cross Correlation and Multifunction Convolutions

The complementary advantages of the cross correlation and multifunction convolutions respectively to template matching and filtering can be combined so as to harness their respective best features. This can be immediately achieved by applying the multifunction methods *after* the two signals have been cross-correlated, with the result being taken as the next target function. It is also possible to apply low-pass filtering on the noise functions, but this may not be necessary in case the template function is itself smooth as in the case of the current examples.

Figure 6 illustrates the results obtained by application of the above proposed methodology on the functions in Figure 5, a situation which corresponds to the highest levels of noise considered in this work, and the obtained results corroborate the effectiveness of the approach.

5 Concluding Remarks

We have presented a comparison among several multifunction correlations applied to detecting the similarity between two functions $f(x)$ and $g(x)$ according to increasing levels of noise. This operation relates directly to template matching as well as filtering. The considered methods consisted of the multifunction correlation as well as its adaptation with respect to the several similarity index described in [3], as well as the traditional cross-correlation, which is directly related to the cosine similarity.

The results can be understood to be markedly encouraging and promising. First, we have that the multifunction convolutions yield sharper and narrower peaks, while attenuating secondary peaks. Therefore, the local similarity between the two functions can be effectively quantified by using the multifunction correlations.

Among the latter type of methods, we have that the multifunction convolution tended to have similar performance when compared to its normalized version and addition-based convolution. However, the two approaches incorporating the interiority index were capable of empha-

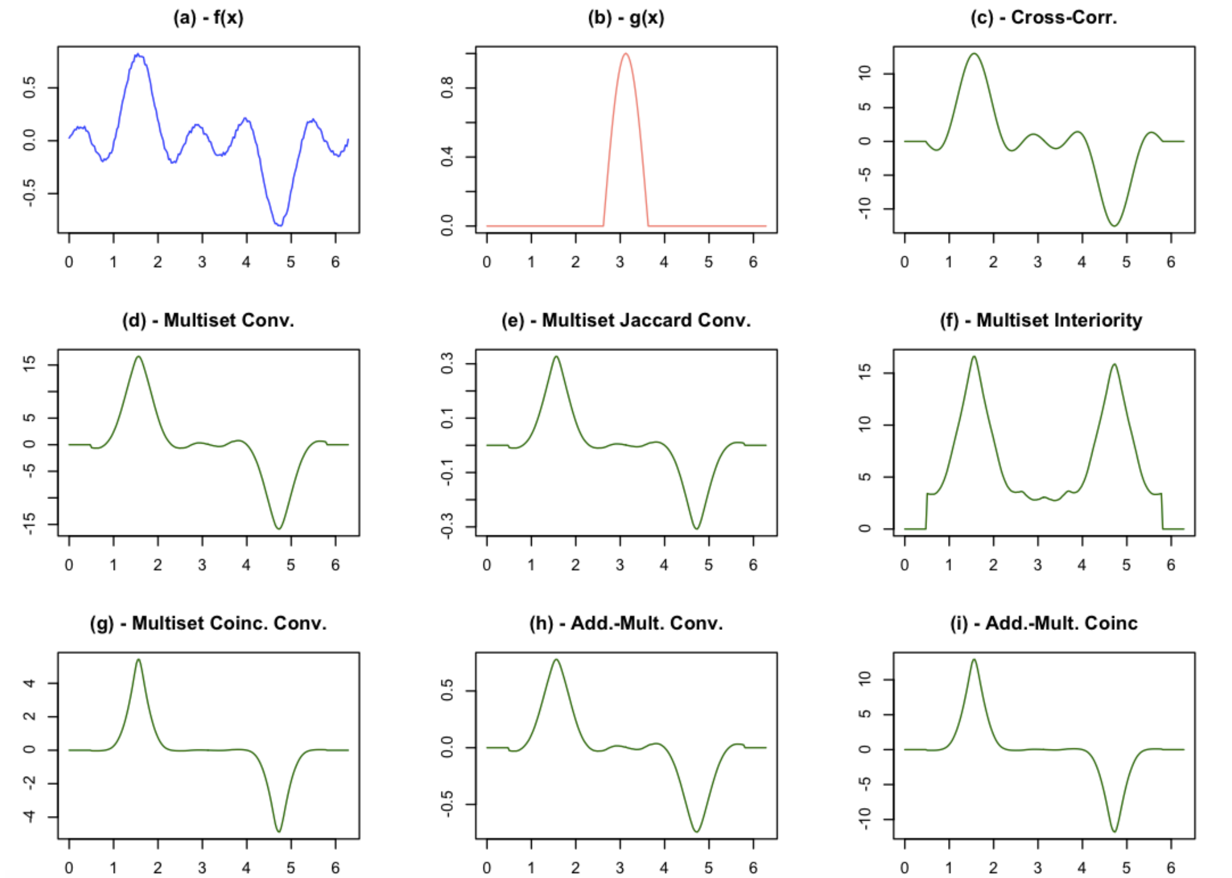

Figure 2: Comparison of the several multifunction convolution methods with respect to two noise level 0.2.

sizing even further the matching peaks while attenuating secondary matches.

In case the secondary peaks need to be also identified, it is always possible to employ a recursive procedure in which the largest peaks are removed from the target signal, and new detections involving the multifunction convolutions are employed.

The application of the correlation methods in presence of noise also yielded remarkable results, revealing a substantial robustness of all considered methods up to relatively high levels of additive uniform noise. Nevertheless, the noise has been observed to imply wider matching peaks, and also to affect the multifunction correlations slightly more than the traditional cross-correlation, which presented the best resilience to the considered type of noise.

The superior robustness of the cross-correlation motivated a new method in which the multifunction convolutions are applied not directly onto the target and template signals, but on their respective cross correlation. This method therefore combines both advantages of the cross-correlation (or cosine similarity-based methods) with the enhanced performance of the multifunction convolutions for peak detection.

The results reported in this work corroborate the potential of the multifunction convolutional methods for enhanced pattern recognition, filtering, as well as other related tasks. Sharper peak identification has frequently been pursued while considering non-linear methods. Interestingly, the proposed methodology is also non-linear (the use of minimum and maximum functions), but demanding minimal computational expenses when compared to more sophisticated methodologies involving complex variables, differentiation, etc.

Indeed, in addition to the observed good performance, we also have that the multifunction methods are conceptually, mathematically, and computationally simple, requiring only comparisons between values and additions instead of the products involved in the traditional cross-correlation and cosine distance-based methods.

In addition, the identified enhanced performance of the multifunction correlation methods is believed to provide insights also about the performance of the directly associated approach of using the common product as a measurement of the similarity between two clusters [3, 10], or as a quantification of joint variation of two random

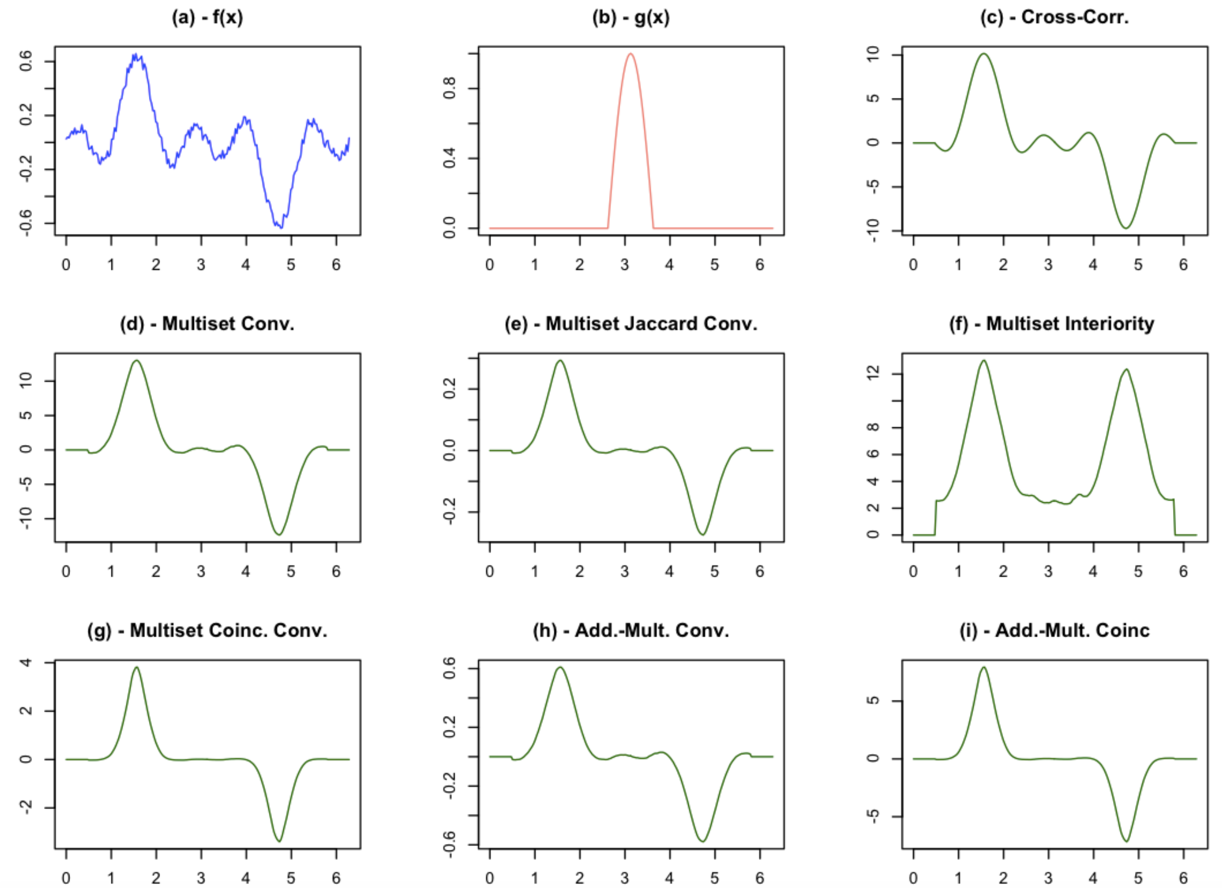

Figure 3: Comparison of the several multifunction convolution methods with respect to two noise level 0.5.

variables [3].

That is so because the multifunction convolution methods consists in the repeated application of indices such as the common product. Therefore, the sharper and narrower peaks obtained, as well as the attenuation of secondary peaks, provides a direct indication that the use of the common product for similarity quantification and joint variation will provide more discrimination between the possible relative locations of the compared clusters or correlation. In other words, convolutional quantification of the similarity between two multifunctions actually corresponds to repeated estimations of similarity, representing a more systematic and demanding task that measuring the similarity between two multisets. Therefore, the performance of similarity index will be closely related to the performance of respectively convolution approaches. If a similarity index is good when applied through convolution, it should be expected to be good also for the direct similarity quantification between two multisets.

Two of the considered indices, namely the coincidence and addition-based coincidence, correspond to the product of two other indices, namely the interiority and Jaccard. The fact that these two indices led to the best performance for peak identification motivates further combination of indices, as it is indeed the case with the combination of cross correlation and multifunction convolutions. This possibility expands combinatorially among the several adopted similarity indices, expanding further in case other similarity indices (e.g. [11]) are also taken into account.

One point of particular interest is to contemplate to which extent similar methods could be employed in natural recognition systems involving neuronal cells, as motivated by their relative simplicity and low computational cost. In addition, the results reported in this work have direct implications in pattern recognition and deep learning, motivating new neuronal architectures and respective convolutional methods.

As future works, It would be interesting to consider other types of functions, as well as to study the effect of the relative function magnitudes and other types of noise on the results. Related research is being conducted and results are to be published opportunely.

Acknowledgments.

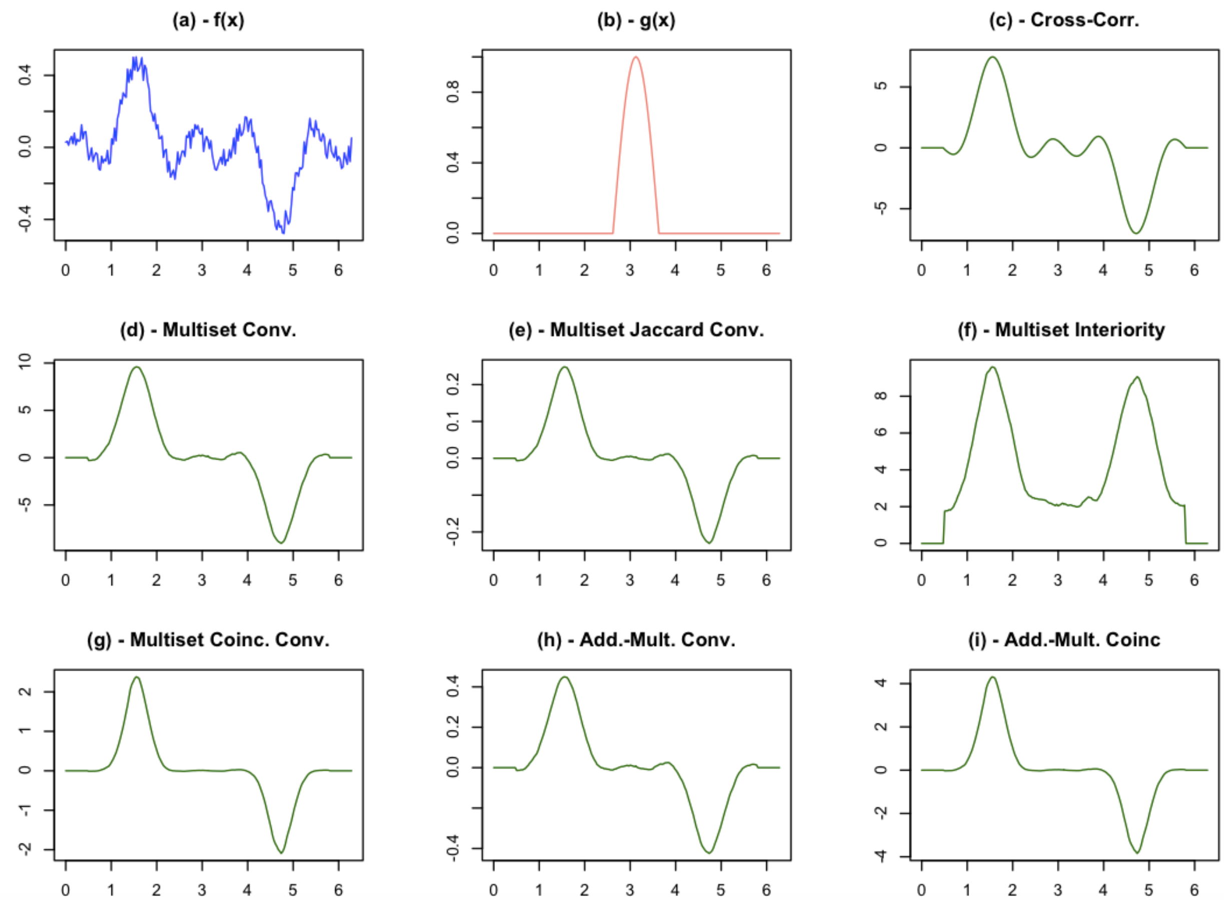

Figure 4: Comparison of the several multifunction convolution methods with respect to two noise level 1.

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References

[1] P. Jaccard. Étude comparative de la distribution florale dans une portion des alpes et des jura. *Bulletin de la Société vaudoise des sciences naturelles*, 37:547–549, 1901.

[2] Wikipedia. Jaccard index. https://en.wikipedia.org/wiki/Jaccard_index. [Online; accessed 10-Oct-2021].

[3] L. da F. Costa. Further generalizations of the jaccard index. https://www.researchgate.net/publication/355381945_Further_Generalizations_of_the_Jaccard_Index, 2021. [Online; accessed 21-Aug-2021].

[4] J. Hein. *Discrete Mathematics*. Jones & Bartlett Pub., 2003.

[5] D. E. Knuth. *The Art of Computing*. Addison Wesley, 1998.

[6] W. D. Blizard. Multiset theory. *Notre Dame Journal of Formal Logic.*, 30:36—66, 1989.

[7] W. D. Blizard. The development of multiset theory. *Modern Logic*, 4:319?352, 1991.

[8] P. M. Mahalakshmi and P. Thangavelu. Properties of multisets. *International Journal of Innovative Technology and Exploring Engineering*, 8:1–4, 2019.

[9] D. Singh, M. Ibrahim, T. Yohana, and J. N. Singh. Complementation in multiset theory. *International Mathematical Forum*, 38:1877–1884, 2011.

[10] L. da F. Costa. Multisets. https://www.researchgate.net/publication/355437006_Multisets, 2021. [Online; accessed 20-Oct-2021].

[11] M. Brusco, J. D. Cradit, and D. Steinley. A comparison of 71 binary similarity coefficients: The effect of base rates. *PLOS One*, 16(4):e0247751, 2021.

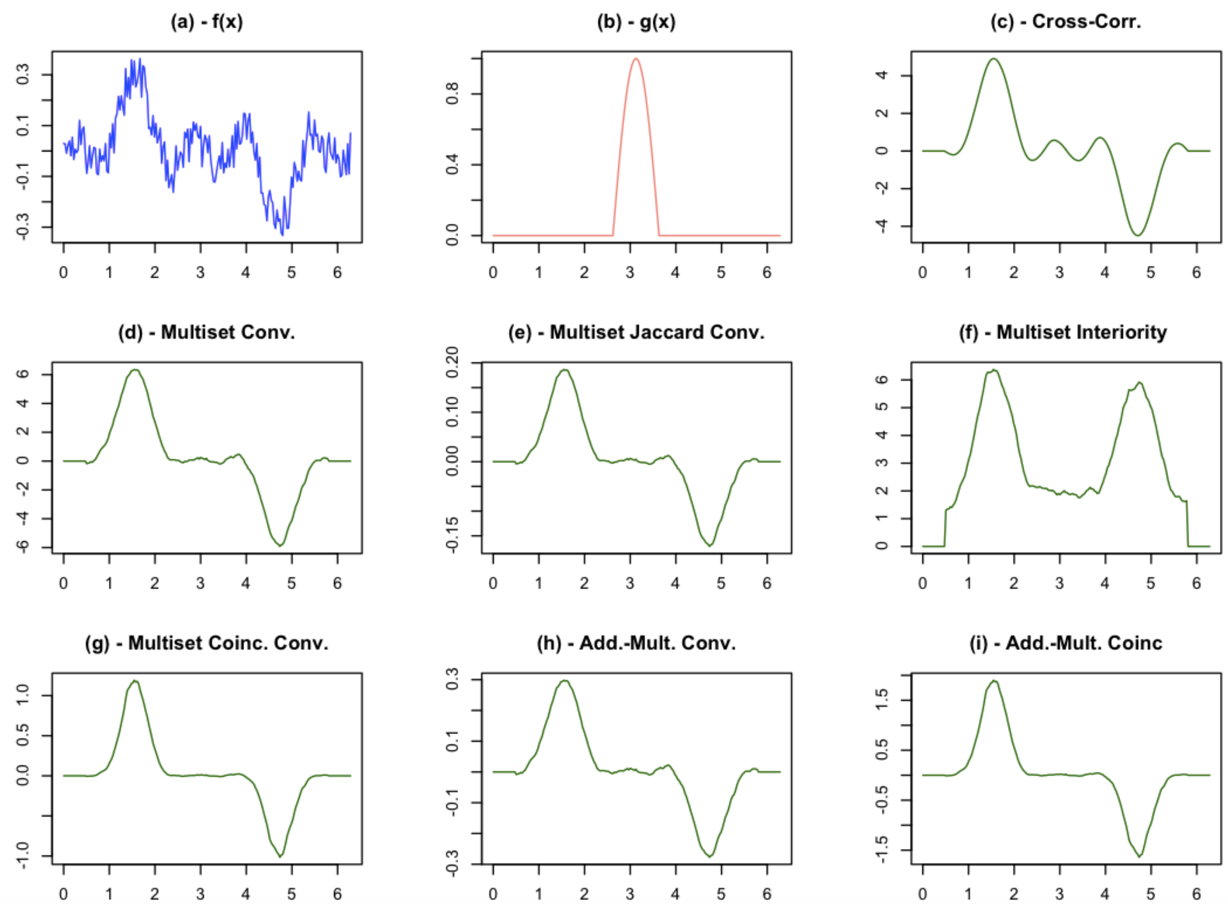

Figure 5: Comparison of the several multifunction convolution methods with respect to two noise level 2.

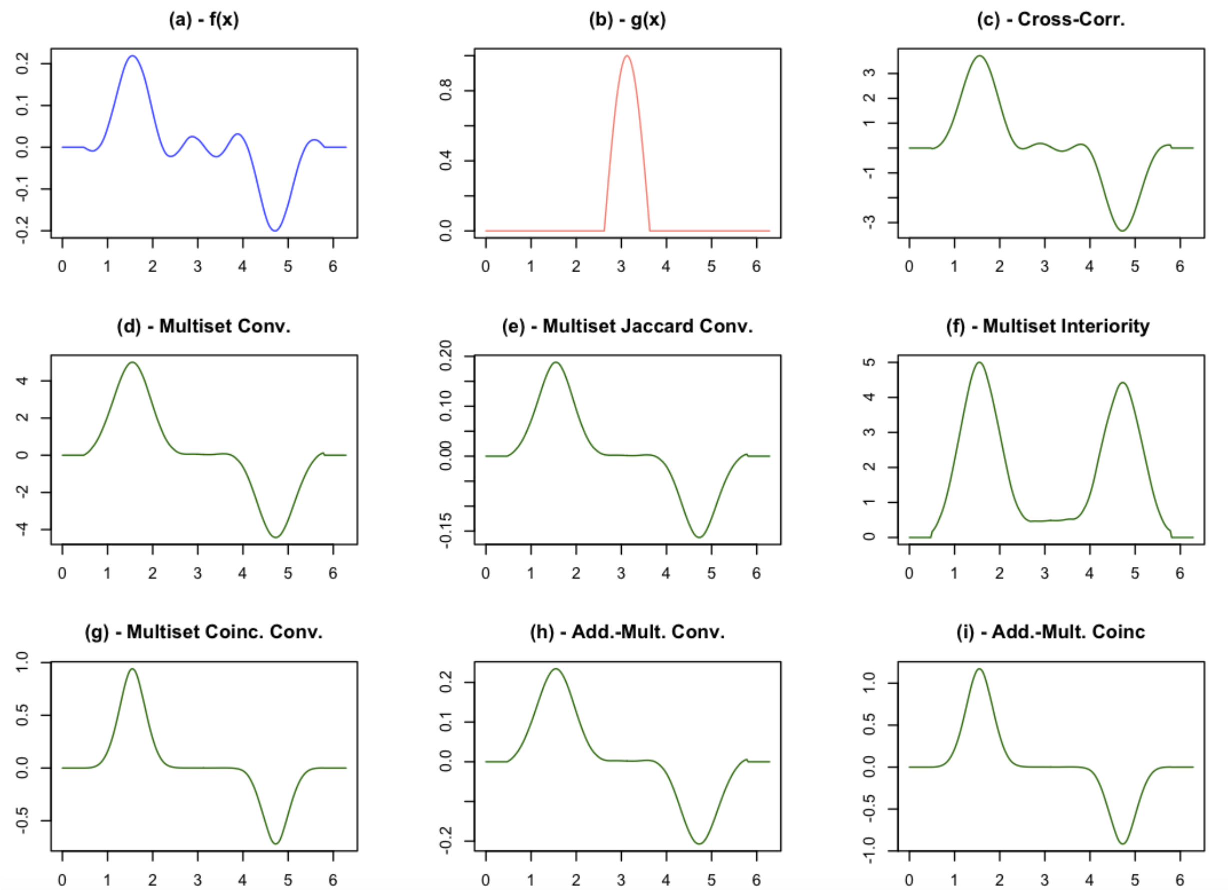

Figure 6: Results of the combination of cross-correlation with multifunction convolutions. The cross correlation was first obtained between the target signal with level noise $L = 2$ and respective template function in Figure 5, then followed by the several considered methods. It can be readily observed that this respective methodology allowed the combination of the best characteristics of the traditional cross correlation (higher robustness to noise) with the enhanced peaks provided by the multifunction convolutions considered in the present work.