Inventory control of a class of logistic networks

Rosa Abbou, Berna Bou Farraa, J.-J. Loiseau

Inventory Control of a Class of Logistic Networks

Berna BouFarraa[a,*,1], Rosa Abbou[a,1] and Jean Jacques Loiseau[a,1]

[a]*Laboratory of Digital Science of Nantes - LS2N*
Université Bretagne Loire, Nantes, FRANCE.
1 Rue de la Noë, 44321 Nantes, FRANCE.

ARTICLE INFO

Keywords:
Time-delays systems
Continuous-time systems
Convolution Systems
Invariant polytopes
Polyhedral constraints
$\mathcal{D}$-invariance property
Artstein reduction
Production network

ABSTRACT

This paper focuses on a prediction-based control for a class of convolution systems subject to constant input delays, also known as the reduction approach. We propose here a characterization of the $\mathcal{D}$-invariance property of polytope sets for continuous-time systems. Our motivations come from production management, where a general model for a production network is considered. The system is subject to product losses and multiple delays, with bounded demand. This model is transformed into an equivalent free-delay system using Artstein reduction. The question of regulating the inventory levels of the nodes of this production network is then reduced to a pair of sub-problems, that are the regulation of the equivalent system without delay, and the relationship between the output of the system and that of the reduced system. A pair of conditions is therefore obtained, for the verification that a large class of control solutions allow to satisfy the external demand, while meeting the constraints that are imposed to the system. At the end, explicit conditions are found for an example of a supply chain with two nodes.

1. Introduction

In this paper, we are interested to the control of a class of delayed systems, that correspond effectively to the management of dynamic networks. Such systems represent a wide range of applications such as logistic systems, production and distribution systems, communication networks, manufacturing processes, and transportation systems. They can be addressed by network models in which nodes represent warehouses and arcs represent the process flows.

Such dynamic systems face different problems. Forrester in [10] has observed that a distribution system of cascaded inventories and ordering procedures amplifies small disturbances that occur at the retail level. The same effect is observed in shops with many workstations. The flows between workstations, and the levels of buffer stores, are subject to variations, that are amplified along the production chain. This phenomenon is called Bullwhip Effect, also known as the Forrester effect. It was named for the way the amplitude of a whip increases down its length. The further from the originating signal, the greater the distortion of the wave pattern. It has generated a continuous flow of research work for the last fifty years. Plenty of clever strategies already exist to avoid the Bullwhip Effect as well as minimizing the costs. We can cite stochastic optimization, echelon policies (MRP), lot sizing, Kanban and its variants, CONWIP, and also control approaches. A classical study of the Bullwhip effect is provided by Lee in [13], and a recent survey by Wang and Disney in [25].

The first author who formulates the problem of inventory management as a control theory problem is Simon (Nobel prize winner, 1978) [21]. He introduced a fundamental model, and proposed to use the theory of servomechanisms to analyze the supply chain stability. Then, Forrester (Nobel prize, 1961) introduced the system dynamics to study the logistic systems and more generally the economical systems in [10]. Furthermore, Towill used in [24] the Laplace transform to study the control of inventory and order for a based production system. More recently, some authors, Blanchini [2], Hennet [11], and Ignaciuk and Bartoszewicz [12] have used the invariance principle for the design of control laws in this context. Moreover, the problem of controlling production systems is formulated in terms of polyhedral invariance in Tarbouriech and Hennet in [23], and Blanchini *et al.* in [4]. A similar controller was derived in [18] using a formulation in terms of invariant sets for an elementary logistic systems.

In this paper, we are particularly interested into the description of a production network as an interconnected system. From the point of view of control, each node is a system, for which the inventory level is the output, the production order is the control input and the demand is a perturbation. Indeed, the production orders and corresponding supplies orders are calculated on-line so as to satisfactorily answer the external demand, regulating the inventory level. The instantaneous production order at a given place is directly related to the consumption of goods that are ordered to the upstream nodes of the network, so that it plays the role of the demand for these ones. The characteristic features of such a system is the presence of delays and constraints. At a first level, the production rate, inventory levels, and demand are non-negative and bounded. At a second level, their range are intervals, which size depends on the demand variations. Our aim is to propose a control law that takes into account both delays and constraints, and permits to regulate the inventory level and limit the Forrester effect.

In our work, we carry out another study on this problem. The control methodology is formulated in terms of $\mathcal{D}$-invariance of polytopic sets. An interesting application of the concept of $\mathcal{D}$-invariance is to investigate the possibil-

Berna.Boufarraa@ls2n.fr (B. BouFarraa); Rosa.Abbou@ls2n.fr (R. Abbou); Jean-Jacques.Loiseau@ls2n.fr (J.J. Loiseau)
ORCID(s):

ity of controlling a constrained system. Dorea and Hennet have obtained in [9] a first characterization of the controlled invariance property of polyhedral sets with respect to linear continuous-time systems, where the conditions are established in the form of linear matrix relations. Their work is then extended to control the constrained disturbed systems. Furthermore, the work of Blanchini and Miani [3] provided another characterization, in terms of a convex representation of a polytope and using the dual analysis for the (A, B)-invariance of polyhedral region. Our paper provides an third characterization of the $\mathcal{D}$-invariance property using a double description of the polytope sets for continuous-time systems. This new formulation of the conditions for the controlled invariance has the advantage to define explicitly an admissible control law. In addition, this approach reduces the algorithmic complexity with respect to that of Blanchini and Miani. Moreover, our characterization presents a simpler test than that of Dorea and Hennet for the implementation of the control law in practical applications.

The paper is organized as follows. We introduce the basic concepts of convex sets, polytopes and bounds for convolution systems in section 2. In section 3, new conditions are defined for the $\mathcal{D}$-invariance of a polytope for a system without delay subject to a disturbance evolving in a given polytope $\mathcal{D}$. In section 4, the production system is stated in terms of constrained orders and inventories, and unknown customer demands variations. The controller design has been investigated in section 5. A pair of conditions for the existence of an admissible control is found, for a input time-delay system with positivity constraints and bounded disturbances. We end the study in section 6 with some important remarks. In section 7, we show that the conditions lead to an explicit solution, in the case of a supply chain with two elements. Finally, some conclusions are pointed out in section 8.

Notations and definitions: $\mathbb{N}$ and $\mathbb{R}$ denote, respectively, the set of natural and real numbers including the 0. n, p and $q \in \mathbb{N}$. We denote the real vectors z, v, w and π such that $z \in \mathbb{R}^n$, $v \in \mathbb{R}^p$, $w \in \mathbb{R}^p$ and $\pi \in \mathbb{R}^q$. w^T denotes the transpose of the vector w. M and P denote real matrices such that $M \in \mathbb{R}^{n\times p}$ and $P \in \mathbb{R}^{q\times n}$. In addition, M_{c_j} denotes the j^{th} column of the matrix M, and P_{r_i} the i^{th} row of the matrix P. The identity matrix is represented by I. Moreover, the notations Conv, min, max and Pol are, respectively, diminutive of *convex*, *minimum*, *maximum* and *polytope*. $\mathcal{P}$ denotes a convex set such that $\mathcal{P} \subset \mathbb{R}^n$. Given an input-output system of the form $y(t) = (h * u)(t)$, u, y and h are, respectively, the input system, the output system and the convolution kernel. The operator $*$ denotes the convolution product. Finally, given two vectors v and $v' \in \mathbb{R}^n$, the vector inequality $v \leq v'$ is component-wise, which means that every component $i = 1...n$ verifies $v_i \leq v'_i$.

2. Background concepts

we start by introducing the preliminary properties of polyhedra, convex sets and bounds calculation, which will be used subsequently in the control study of a class of constrained delayed systems.

2.1. Polytopes and convex sets

This section is dedicated to the set invariance characterization of dynamical systems affected by time-delays. Starting from a continuous-time dynamical system described by a delay-differential equation and based on convolution kernel formula, the construction of a positive invariant set is searched. In fact, the positive invariance is involved in different problems in control theory [5], and is used in many topics such as predictive control [16], fault tolerant control [20] and reference governor design [22], to ensure system stability and to guarantee constraint satisfaction [14].

A *polytope* is a bounded polyhedron. In this way, we will use the celebrated Minkowski-Weyl theorem (see for instance Schrijver [19] for a detailed presentation). This result states that every polytope can be represented in two different ways, either by a finite set of linear inequalities, or as a set of convex combinations of a finite number of generators. The Theorem 2.1 resumes those properties.

Theorem 2.1. *(Minkowski-Weyl Theorem) Consider a set $\mathcal{P} \subset \mathbb{R}^n$. The following statements are equivalent.*

(i) $\mathcal{P}$ is bounded and there exists an integer q, a real matrix $P \in \mathbb{R}^{q\times n}$ and a real vector $\pi \in \mathbb{R}^q$ such that the following equality holds true:

$$\mathcal{P} = \mathrm{Pol}\,(P, \pi) := \{z \in \mathbb{R}^n \mid Pz \leq \pi\} \tag{1}$$

(ii) There exists an integer p and a matrix $M \in \mathbb{R}^{n\times p}$ such that $\mathcal{P}$ is the convex set defined by:

$$\mathcal{P} = \mathrm{Conv}\, M := \{z \in \mathbb{R}^n \mid \exists v \in \Gamma(p), z = Mv\} \tag{2}$$

where $\Gamma(p)$ is the set of p-dimensional convex combinations, defined by:

$$\Gamma(p) = \{v \in \mathbb{R}^p \mid v \geq 0, \sum_{i=1}^{p} v_i = 1\}$$

This theorem shows that any polytope $\mathcal{P}$ can be described equivalently by its extreme points using the convex representation Conv M, or by its faces through the definition of a polyhedron Pol (P, π). The columns of the matrix M are the vertices of the polytope, and its faces are associated to the rows of the matrix P and the corresponding components of the vector π. Furthermore, we introduce in the following a well-known result about convex set properties, that is useful in the sequel (for more details, see for e.g. [17]).

Lemma 2.1. *Consider the notation introduced in Theorem 2.1. For every vector $w \in \mathbb{R}^p$, we have the equality:*

$$\max_{v\in\Gamma(p)} w^T v = \max_i w_i \tag{3}$$

As a consequence, the following expression holds true:

$$\max_{z\in\mathrm{Conv}\, M} w^T z = \max_i (w^T M)_i \tag{4}$$

PROOF. We first observe that $\max_{v\in\Gamma(p)}(w^T v)$ is equal to $\max_{v\in\Gamma(p)}(\sum_{i=1}^{p} w_i v_i)$, using the notation introduced in Theorem 2.1. By definition, the vectors v_i are non negative and the sum of their components is equal to one. As a consequence, we deduce the inequality:

$$\max_{v\in\Gamma(p)}(w^T v) \leq \max_{v\in\Gamma(p)} \sum_{i=1}^{p}(\max_{i=1}^{p} w_i)v_i \ = \max_{i=1}^{p}(w_i)$$

The reverse inequality holds true since $w_i = w^T e^i$, with $e^i \in \Gamma(p)$ verifying:

$$e^i_j = \begin{cases} 1, & \text{if } j = i \\ 0, & \text{if } j \neq i \end{cases}$$

Hence, the equality (4) is obtained by applying the definition of a convex set expressed by (2) on the expression (3). □

This latter result allows to easily calculate the maximal value taken by a linear functional over a polytope. The value of z that maximize $w^T z$ is not unique in general. We shall use this result in section 2.2 in order to identify the bounds for input-output operators.

2.2. Bounds for input-output convolution systems

Let us consider the input-output system of the form:

$$y(t) = (h * u)(t) \tag{5}$$

where u the input, y is the system output and h the convolution kernel. The operator $*$ denotes the convolution product. Such a system is well defined, provided that the kernel h, and the input u are measurable. This class covers finite dimensional systems and infinite dimensional ones as well, such as linear time delayed systems. In the multivariable case, input-output systems are characterized by a matrix kernel H and defined by:

$$y(t) = \int_0^t H(\tau)u(t-\tau)d\tau \tag{6}$$

Such a system is said to be BIBO stable (Bounded Input-Bounded Output) if every bounded input u results in an output that is bounded. As it is well known (see for instance Desoer and Vidyasagar [8]), the system (6) is BIBO stable if and only if the entries of its kernel H are integrable, over the whole positive real line. We use in the sequel a variant of this result that is useful to investigate bounds for linear systems. This latter is useful to investigate some bound properties for linear systems, and to solve fundamental problems of constraints meeting for systems with delays. Hence, the authors in [15] and [17] formulated the following result on polyhedral bounds for convolution systems.

Theorem 2.2. *Consider the system (6) together with two matrices $M \in \mathbb{R}^{n\times p}$ and $P \in \mathbb{R}^{q\times n}$, and a vector $\pi \in \mathbb{R}^q$. The output $y(t)$ belongs to* $\mathrm{Pol}\,(P,\pi)$*, for $t \geq 0$, for every input $u(t)$ evolving in* $\mathrm{Conv}\,M$*, for $t \geq 0$, if and only if the following inequalities hold true, for $i = 1$ to q, and $t \geq 0$:*

$$\int_0^t \max_j (PH(\tau)M)_{ij}\, d\tau \leq \pi_i$$

PROOF. We proceed by equivalences successively obtained, using, first, the definition (1) of $\mathrm{Pol}\,(P,\pi)$, then the definition (6) of the system output $y(t)$. After that, we use the definition (2) of the convex set $\mathrm{Conv}\ M$, and apply finally Lemma 2.1:

$$\begin{aligned} y(t) \in \mathrm{Pol}\,(P,\pi) &\Leftrightarrow Py \leq \pi \\ &\Leftrightarrow \int_0^t PH(\tau)u(t-\tau)d\tau \leq \pi \\ &\Leftrightarrow \int_0^t PH(\tau)Mv(t-\tau)d\tau \leq \pi \\ &\Leftrightarrow \int_0^t \max_{j=1}^{q}(PH(\tau)M)_{ij}\, d\tau \leq \pi_i \end{aligned}$$

The integrals in these equivalences are well defined since the matrix H is a measurable kernel, with integrable inputs over $[0,t]$. This ends the proof. □

We move on to define the way we express a multivariable interval in the form of a polytope and convex set. Thus, the following section is very useful in order to apply the previous result on polyhedral bounds for delayed systems, that will be introduce in the sequel.

2.3. Polyhedral bounds for delayed systems

We define a multivariable interval that is represented in the form $\mathcal{P} = \{z \in \mathbb{R}^n \mid z^{\min} \leq z \leq z^{\max}\}$, for some vectors $z^{\min}, z^{\max} \in \mathbb{R}^n$, and satisfying $z^{\min} \leq z^{\max}$ component by component. It is denoted by $\mathcal{P} = [z^{\min}, z^{\max}]$.
The faces of a multivariable interval are in number of $2\times n$. They correspond to the subsets of those vectors that satisfy either $z_i = z_i^{\min}$ or $z_i = z_i^{\max}$, for some index i. The vertices of the multivariable interval are the vectors that, for each i, satisfy either $z_i = z_i^{\min}$ or $z_i = z_i^{\max}$. This set of vertices forms a matrix M that has 2^n elements. These remarks are used to get polyhedral and convex representations of a multivariable interval, as introduced in the following proposition.

Proposition 2.1. *The multivariable interval defined by $z^{\min}$ and $z^{\max}$ satisfies:*

$$[z^{\min}, z^{\max}] = \mathrm{Conv}\ M = \mathrm{Pol}\,(P,\pi)$$

with the matrices M and P, and the vector π defined as follows:

$$M_{ij} = \begin{cases} z_i^{\min}, \text{ if } \beta_i(j-1) = 0 \\ z_i^{\max}, \text{if } \beta_i(j-1) = 1 \end{cases}$$

for $j = 1$ to 2^n, where $\beta_i(k)$ denotes the i^{th} bit of a binary representation of any positive integer k, and:

$$P = \begin{pmatrix} I_n \\ -I_n \end{pmatrix} \qquad \text{and} \qquad \pi = \begin{pmatrix} z^{max} \\ -z^{min} \end{pmatrix}$$

where I_n is the identity matrix of size $n \times n$.

PROOF. The interval $[z^{\min}, z^{\max}]$ is the set of vectors $z \in \mathbb{R}^n$ that satisfies both conditions $(z \leq z^{\max})$ and $(z^{\min} \leq z)$. The definition of the matrix P and the vector π as in Proposition

2.1 corresponds to these inequalities. The columns of the matrix M are the vertices of the interval $[z^{\min}, z^{\max}]$. They form the family $\{z \in \mathbb{R}^n \mid \text{for } i = 1 \text{ to } n,\ z_i = z_i^{\min} \text{ or } z_i = z_i^{\max}\}$. The matrix M is obtained by ordering this family as indicated in the Proposition 2.1. □

In this section, we introduced the preliminary results about polytopes, multivariable intervals and bounds for convolutions systems being introduced. In section 3, we move to define the $\mathcal{D}$-invariance property of polytopes sets, in order to determine a new characterization for continuous systems.

3. Invariance and $\mathcal{D}$-invariance of polytopes

In this section, we first introduce the following well-known definitions in the literature about the invariance, the controlled invariance and the $\mathcal{D}$-invariance of polytopes.

Definition 3.1. A set $\mathcal{P}$ is said to be invariant for the system $\dot{z}(t) = -Az(t)$ if, for every initial condition $z(0) \in \mathcal{P}$, the trajectory $z(t)$ remains in $\mathcal{P}$, for $t \geq 0$.

Definition 3.2. A set $\mathcal{P}$ is said to be controlled invariant for the system $\dot{z}(t) = -Az(t)+Bu(t)$, with an admissible control evolving in a set $\mathcal{U}$ if, for every initial condition $z(0) \in \mathcal{P}$, there exists a control law $u(t) \in \mathcal{U}$ such that the trajectory $z(t)$ remains in $\mathcal{P}$ for $t \geq 0$.

Definition 3.3. A set $\mathcal{P}$ is said to be controlled $\mathcal{D}$-invariant for the system:

$$\dot{z}(t) = -Az(t) + Bu(t) - Ed(t) \tag{7}$$

with admissible control in a set $\mathcal{U}$ if, for every initial condition $z(0) \in \mathcal{P}$, there exists a control law $u(t)$ such that the trajectory $z(t)$ remains in $\mathcal{P}$, for $t \geq 0$, whatever be the disturbance $d(t) \in \mathcal{D}$.

In the following, we shall denote P_{r_i} the i^{th} row of a matrix P, and M_{c_j} the j^{th} column of a matrix M.

3.1. Characterization of the invariant polytopes

Dorea and Hennet in [9] have obtained an explicit characterization of the controlled invariance property of polyhedral sets for linear continuous-time systems. The necessary and sufficient conditions for a polyhedron to be (A, B)-invariant are established in the form of linear matrix relations, and presented in the Theorem 3.1.

Theorem 3.1. *A polytope $\mathcal{P} = \mathrm{Pol}(P, \pi)$ is controlled invariant if and only if, for every point z on the face corresponding to the i^{th} row of P, such that $P_{r_i} z = \pi_i$, and $P_{r_j} z \leq \pi_j$, for $j \neq i$, there exists a vector $u \in \mathcal{U}$ such that $P_{r_i}(-Az + Bu) \leq 0$.*

The study of controlled invariance is then extended in their work to control the constrained disturbed systems. So, the Theorem 3.2 presents the conditions of $\mathcal{D}$-invariance for general convex polyhedra.

Theorem 3.2. *A polytope $\mathcal{P} = \mathrm{Pol}(P, \pi)$ is $\mathcal{D}$-invariant if and only if, for every point z on the face corresponding to the i^{th} row of P, such that $P_{r_i} z = \pi_i$, and $P_{r_j} z \leq \pi_j$ for $j \neq i$, there exists a vector $u \in \mathcal{U}$ satisfying $P_{r_i}(-Az+Bu-Ed) \leq 0$, for every $d \in \mathcal{D}$.*

Dorea and Hennet in [9] reformulated the conditions of the controlled invariance of general convex polyhedral sets in terms of the existence of constrained matrices, that can be checked using linear programming and differential analysis. Blanchini and Miani in [3] provided another characterization, in terms of a convex representation of a polytope and using the dual analysis for the (A, B)-invariance of polyhedral region.

In this study, we provide another slightly different characterization defined in Theorem 3.3, where a double description of the polytope is considered.

Theorem 3.3. *Let D be any matrix such that $\mathcal{D} = \mathrm{Conv}\, D$, and*

$$\delta(i) = \min_j (PED)_j \ , \ \textit{for } i = 1 \textit{ to } q.$$

The polytope $\mathcal{P} = \mathrm{Conv}\, M = \mathrm{Pol}(P, \pi)$ is $\mathcal{D}$-invariant for the system (7) if and only if, for each vertex of $\mathcal{P}$, say M_{c_j}, there exists a vector, say $U_{c_j} \in \mathcal{U}$, such that:

$$P_{r_i}(-AM_{c_j} + BU_{c_j}) \leq \delta(i) \tag{8}$$

for each face containing the vertex M_{c_j}, such that $P_{r_i} M_{c_j} = \pi_i$. In addition, if $\mathcal{U}$ is a convex set, then an admissible control law is given in the form:

$$u(t) = Uv(t) \tag{9}$$

where, for $t \geq 0$, $v(t) \in \Gamma(p)$ is any convex combination vector satisfying:

$$z(t) = Mv(t) \tag{10}$$

Proof. According to Theorem 3.2, the conditions expressed in (8) are necessarily satisfied by each vertex if the polytope is $\mathcal{D}$-invariant. The vectors U_{c_j} that are successively determined for each vertex form a matrix U that permits to define the control law (9) for every vector $z(t)$ in $\mathcal{P}$. Notice that, by definition of the matrix M, a vector $v(t) \in \Gamma(p)$ satisfying (10) can be defined for each vector $z(t) \in \mathcal{P}$. The sufficiency of the conditions comes from the fact that this control law can be used to check the conditions of Theorem 3.2. Indeed, let us consider a vector z that is on the face of $\mathcal{P}$ that corresponds to the i^{th} row of matrix P. This vector is a convex combination of the vertices that satisfy $P_{r_i} M_{c_j} = \pi_i$. If J_i denotes the set of indexes of these vertices, we have $z = \sum_{j \in J_i} \alpha_j M_{c_j}$, with $\alpha_j \geq 0$, for $j \in J_i$, and $\sum_{j \in J_i} \alpha_j = 1$. Defining v by $v_j = \alpha_j$, for $j \in J_i$, and $v_j = 0$, if $j \notin J_i$, one can verify that $u = Uv$ is an admissible control that verifies $P_{r_i}(-Az + Bu) \leq 0$. Indeed, u is admissible since it is a convex combination of admissible control vectors, and $\mathcal{U}$ is convex. The condition of Theorem 3.2 holds true since, for every $d \in \mathcal{D}$, we have $\sum_{j \in J_i} \alpha_j \times d = d$. □

This new formulation of the conditions for the controlled invariance has the advantage to control directly the system by defining a construction of the matrix U. This matrix allows the calculation of a set of admissible control solutions. This latter is explained in details in section 3.2. In addition, this approach reduces the algorithmic complexity with respect to the approach of Blanchini and Miani, which will be explained in section 3.3.

3.2. Parameterization of the control law

This section deals with the parametrization of the control solution $u(t)$. As expressed in (9), the vector v is not uniquely defined in general. This lets a first degree of freedom in the definition of the matrix U. Another degree of freedom comes from the fact that the expression (8) is an inequality. Therefore, the control matrix $U \in \mathbb{R}^{q*p}$ is constructed column by column, such that each column U_{c_j} is defined as follows:

$$U_{c_j} = \left\{ \begin{array}{l} u \in \mathbb{R}^q \mid u \in \mathcal{U}, \text{ and } \forall i \ , \\ P_{r_i} M_{c_j} = \pi_i \ , \Rightarrow \ P_{r_i} Bu \leq P_{r_i} A M_{c_j} + \delta(i) \end{array} \right\}. \tag{11}$$

In other terms, for every vertex M_{c_j} of the polytope $\mathcal{P}$ is associated a control vector U_{c_j}, which satisfies the controlled $\mathcal{D}$-invariance property defined in Theorem 3.3. In this way, the matrix U defines not just a control solution but a set of control laws, because this parametrization is implicit which provides different choices for control laws implementations. This parameterization permits to easily check the existence of a solution.

In addition, we define for each index j, the set $I(j)$ verifying

$$I(j) = \left\{ i \in \mathbb{N} \mid 1 \leq i \leq q, \ P_{r_i} M_{c_j} = (PM)_{ij} = \pi_i \right\},$$

and the matrix $P(j)$ whose rows are selected from that of the matrix P with index i verifying $i \in I(j)$. Then, the condition (11) can be reformulated as the intersection of two sets as follows:

$$U_{c_j} = \mathcal{U} \cap \mathrm{Pol}\,(Q, \kappa),$$

where

$$\mathrm{Pol}\,(Q, \kappa) := \{u \in \mathbb{R}^q \mid P(j)Bu \leq P(j)AM_{c_j} + \delta(i)\}.$$

As consequence, we can say that the set of control vectors U_{c_j} is a polyhedron if the set $\mathcal{U}$ is itself a polyhedron.

3.3. Algorithmic complexity of the verification

Following the double representation of the polytope $\mathcal{P}$, four algorithmic formulations of controlled $\mathcal{D}$-invariance principle are defined. Thus, the complexity problem is related to the number of matrix calculation whose size depends on the dimensions n, p, and q. We notice that the number $p = 2^n$ of vertices of a polyhedron grows exponentially with respect to the size $q = 2 \times n$ of faces.

By looking at the matrices dimensions, we find that the formulation of Dorea and Hennet in [9] allows to verify the existence of an economical solution from the algorithmic complexity point of view. While in the approach of Blanchini and Miani, a non-negative matrix with $p \times p$ dimension must be calculated, whose columns are elements of set of convex combinations $\Gamma(p)$. Therefore, our approach leads to a complexity reduction of order $\frac{n}{2^n}$, based on the calculation of the matrix M of reduced dimension.

n	$q = 2 \times n$	$p = 2^n$	$U \in \mathbb{R}^{q \times p}$	$M \in \mathbb{R}^{n \times p}$
2	4	4	$\mathbb{R}^{4\times4}$	$\mathbb{R}^{2\times4}$
3	6	8	$\mathbb{R}^{6\times8}$	$\mathbb{R}^{3\times8}$
4	8	16	$\mathbb{R}^{8\times16}$	$\mathbb{R}^{4\times16}$
10	20	1024	$\mathbb{R}^{20\times1024}$	$\mathbb{R}^{10\times1024}$
20	40	1048576	$\mathbb{R}^{40\times1048576}$	$\mathbb{R}^{20\times1048576}$

Table 1
Dimensions of matrices U and M

Moreover, as far as the construction of the control law is concerned, Dorea and Hennet also used implicitly the polyhedron representation in their characterization. The method of Theorem 3.3 actually follows the same line of though. One can estimate that the limit number of vertices that can be handled in practice for the construction of matrix U is of the order of 10^6. So, when it comes to verify the sufficiency and to construct the control law u, they come back to our characterization for practical implementation of the control.

4. Delayed production network

4.1. Problem formulation

An elementary logistic system which is composed of a supplying unit and a storage unit was first introduced by Simon in [21], where the inventory dynamic level $x(t)$ is described by the following first order delayed equation expressed by (12):

$$\dot{x}(t) = u(t - \theta) - d(t), \quad \text{for } \ t \geq \theta \tag{12}$$

The three variables $x(t)$, $u(t)$, and $d(t)$ are real variable, well defined for $t \geq 0$. The delay θ corresponds to the time needed to complete the task. This model can actually be used in many situations, to model an elementary production system, a buffer, or a simple transportation device, for instance. We consider in the sequel a production network described by the interconnection of n elementary logistic systems on different nodes, each one presents a elementary production system. Hence, the problem formulation is defined by a continuous time-delay system which is described by the delayed differential equation:

$$\dot{x}(t) = -Ax(t) + \sum_{k=1}^{h} B_k u(t - \theta_k) - B_0 u(t) - Ed(t) \tag{13}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $d(t) \in \mathbb{R}^r$ are, respectively, the state vector, the control input vector and the disturbance vector, while A, B_k, for $k = 0$ to h, and E are constant matrices of appropriate dimensions. The components of the system output variable $x(t)$ represent the resources in the inventory levels of the different nodes. These resources are raw materials, intermediate and finished products, as well as any other

resource used in production processes. The components of the control variable $u(t)$ represent the instantaneous production orders between the different edges. These controlled flows are acquired with delays θ_k from multiple production stages, while $B_0u(t)$ presents the external production orders that can feed the different nodes of production. The products are used to respond to external requests from customers at each node. Therefore the disturbance $d(t)$ is an unknown external signal representing the demand or, more generally, uncontrollable flows. In addition, the storage levels on each node are subject to a continuous decrease due to losses on perishable products modeled by the matrix A. Its components correspond to the loss factors, which are real positive values, namely $\sigma \geq 0$.

4.2. System constraints and objectives

The considered system is subject to different constraints that are frequently used in practical applications, being generally associated to physical limitations. Thus, the system variables are positive and bounded, such that the constraints are formulated in the following form:

$$x^{\min} \leq x(t) \leq x^{\max} \tag{14}$$

$$u^{\min} \leq u(t) \leq u^{\max} \tag{15}$$

$$d^{\min} \leq d(t) \leq d^{\max} \tag{16}$$

where the inequalities are meant component by component, with $x^{\min}, x^{\max} \in \mathbb{R}^n$, $u^{\min}, u^{\max} \in \mathbb{R}^m$, and $d^{\min}, d^{\max} \in \mathbb{R}^r$. In the case of a production system, $u^{\min}$ and $u^{\max}$ define, respectively, the minimum and the maximum of the production capacity. With the same manner, $x^{\min}$ and $x^{\max}$ correspond to, respectively, the minimum and the maximum of the storage capacity of the system. The bounds $d^{\min}$ and $d^{\max}$ for the demand are determined by, respectively, the minimum and the maximum of the market specifically to the considered production system. These constraints define three interval polytopes, that are denoted by $\mathcal{P}_x$, $\mathcal{P}_u$, and $\mathcal{P}_d$ respectively.

In terms of control theory, the fundamental problem we face is to design online an admissible control law $u(t)$ that stabilize the delayed system while ensuring that the constraints on the output $x(t)$ are met for any bounded perturbation $d(t)$. The control problem is formulated in terms of $\mathcal{D}$-invariance, in such a way that the conditions for the existence of an admissible control law are determined.

5. Control design

The control design procedure starts with the elimination of the delays using Artstein's reduction. The problem is then twofold. The first sub-problem is to determine conditions for the resulting reduced model, while the second sub-problem is to express the relationship between the variables of the initial model and the reduced one, in order to obtain finally the conditions for the original delayed system.

5.1. Artstein's reduction

Using Artstein reduction developed in [1], delayed systems are transformed into systems without delay. Thus, the new system is an ordinary differential control equation, so that various control problems can be analysed using the reduced model. Given first the model of the production network, say (13), we define a reduced variable $z(t)$ by the following equality:

$$z(t) = x(t) + \sum_{k=1}^{h} \int_{t-\theta_k}^{t} \mathrm{e}^{-A(t-\tau)} \mathrm{e}^{A\theta_k} B_k u(\tau)\, d\tau \ . \tag{17}$$

Proposition 5.1. *Using the definition (17) of the reduced variable $z(t)$, and verifying the system model (13), we can deduce the following differential equation, called the reduced system:*

$$\dot{z}(t) = -Az(t) + Bu(t) - Ed(t) \tag{18}$$

with $B = \sum_{k=1}^{h} \mathrm{e}^{A\theta_k} B_k - B_0$.

Proof. One first remark that the integral in the definition (17) can be rewritten as $\mathrm{e}^{-At} \int_{t-\theta_k}^{t} \mathrm{e}^{A\tau}\ \mathrm{e}^{A\theta_k} B_k u(\tau)\, d\tau$. Its derivative is then calculated using the usual formulas for the product rule of derivation, and of an integral which integrator does not depend on the integration bounds. We obtain the following result:

$$\frac{d}{dt} \int_{t-\theta_k}^{t} \mathrm{e}^{-A(t-\tau)} \mathrm{e}^{A\theta_k} B_k u(\tau)\, d\tau \ =$$

$$-A \int_{t-\theta_k}^{t} \mathrm{e}^{-A(t-\tau)} \mathrm{e}^{A\theta_k} B_k u(\tau)\, d\tau + \mathrm{e}^{A\theta_k} B_k u(t) - B_k u(t-\theta_k).$$

The conclusion is then obtained by the time derivation of the identity (17), and using the definition (13) of the system. □

We use the Artstein reduction in order to eliminate the delays and to obtain a free-delay dynamic system expressed by (18). Then, we notice that the delayed system (13) depends on a constrained control input $u(t)$ and a constrained output vector $x(t)$, verifying (15) and (14) respectively. In this context, the property of BIBO-stability is interpreted in terms of invariance for constrained systems. We can conclude that the principle of invariance is one of the basic notions in automatic. For this reason, we address in the following section, the control of the reduced model, by determining the controlled $\mathcal{D}$-invariance conditions for the reduced system using Theorem 3.3.

5.2. Controlled $\mathcal{D}$-invariance conditions for the reduced system

In our study, we define the dynamics of the reduced vector $z(t)$, in a polytope named $\mathcal{P}$. This polyhedron is based on two equivalent representations, of half-spaces $\mathrm{Pol}\,(P, \pi)$ and convex set $\mathrm{Conv}\,M$. In addition, let us denote $\mathcal{D}$ the polytope $\mathcal{P}_d = [d^{\min}, d^{\max}]$, and let D be a matrix such that $\mathcal{D} = \mathrm{Conv}\,D$. Then, according to Theorem 3.3, a given polyhedron of $\mathbb{R}^n$, say $\mathcal{P} = \mathrm{Pol}\,(P, \pi) = \mathrm{Conv}\,M$, is $\mathcal{D}$-invariant

for the reduced system if and only if there exists a matrix $U \in \mathbb{R}^{m\times p}$ satisfying the conditions (8) of Theorem 3.3.

Moreover, the matrix U is not uniquely defined. According to Section 3.2, the j^{th} column of this matrix can be chosen in the polyhedron expressed by:

$$\mathcal{P}_u \cap \{u \in \mathbb{R}^m \mid P_{r_i} B u \leq \delta_i + P_{r_i} A M_{c_j} \text{ if } P_{r_i} M_{c_j} = \pi_i\} \quad (19)$$

This characterization allows to effectively check the existence of a solution. The fact that the polyhedra corresponding to the p vertices of $\mathcal{P}$ are non-empty is a necessary and sufficient condition for the $\mathcal{D}$-invariance of the polytope $\mathcal{P}$, and for the existence of an admissible matrix U.

In addition, an admissible control law is given by $u(t) = U v(t)$, for every vector $v(t)$ satisfying $z(t) = M v(t)$, which necessarily exists, provided that $z(t) \in \mathcal{P}$. Thus, remark that we have the equality:

$$\begin{pmatrix} z(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} M \\ U \end{pmatrix} v(t) \quad (20)$$

To summarize the results, we have developed a algorithm to verify that a control solution is stable. First, it consists on the elimination of delays from the control input vector. Then, we define the double representation of the polytope $\mathcal{P}$ for the reduced system $z(t)$, and we test the $\mathcal{D}$-invariance property of this polytope. If this latter holds true, we deduced a parametrization of the set of matrices U, that determine stable control solutions. Hence, the computation of the matrix U is implicit and general, which gives multiple ways to implement the control laws. We can cite for example an implementation by a linear feedback or by a hybrid control law. Those two types of control laws are applied in [6] for solving similar problems in the case of a single input-delay system. In addition, the control solution makes the closed-loop system meet the constraints $z(t) \in \mathcal{P}$ and $u(t) \in \mathcal{P}_u$, for every disturbance verifying $d(t) \in \mathcal{P}_d$. For this reason, we should verify for each matrix U, if the output constraint $x(t) \in \mathcal{P}_x$, is satisfied for the delayed system (13). This feature is developed in the following section, in order to determine the $\mathcal{D}$-invariance conditions of time-delay system.

5.3. $\mathcal{D}$-invariance of the time-delay system

Given the system (13) with multiple input delays, we have determined in Theorem 3.3 a new characterization of the $\mathcal{D}$-invariance of the reduced system $z(t)$ without delays. Moreover, the construction of the set of implicit control solutions U verifying (19)-(20), allows to check the input polyhedral constraints $u(t) \in \mathcal{P}_u$. Thus, the first sub-problem of designing a stable and constrained control solution is solved. We move to develop the second sub-problem, that focuses on the admissiblity property of the controlled system. It consists of finding the conditions of the output constraints verification $x(t)$ of the delayed system (13). In this direction, we start this section by finding the relation between the reduced model expressed by (18) and the original delayed system expressed by (13). From the definition of $z(t)$ given in (17), and using the control parametrization expressed by (20), we obtain:

$$x(t) = M v(t) - \sum_{k=1}^{h} \int_{t-\theta_k}^{t} \mathrm{e}^{-A(t-\tau)} \mathrm{e}^{A\theta_k} B_k U v(\tau)\, d\tau \quad (21)$$

where $v(t)$ is a vector of convex combination, that lies in $\Gamma(q)$ at each instant t. One can remark that (21) is the definition of a convolution system, which kernel is equal to:

$$H(t) = M\delta(t) - \sum_{k=1}^{h} H_k(t)\,,$$

where the terms $H_k(t)$ are $n \times q$ matrices, function of the time, and defined by:

$$H_k(t) = \begin{cases} \mathrm{e}^{-At}\mathrm{e}^{A\theta_k} B_k U & \text{if } t \in [0, \theta_k] \\ 0 & \text{if } t \geq \theta_k \end{cases}$$

Analysing the expression (21), this convolution system has $x(t)$ as output and $v(t)$ as input, which is constrained by the condition $v(t) \in \Gamma(q)$. We can also remark that $\Gamma(q) = \mathrm{Conv} I_q$. The kernel $H(t)$ is integrable, so that the system (21) is BIBO-stable. These preliminary remarks are useful to apply *Theorem 2.2* in order to obtain the conditions under which the variable $x(t)$ meets a given set of constraints.

Proposition 5.2. *The pair $(G, \gamma) \in \mathbb{R}^{s\times n} \times \mathbb{R}^n$ being given, the variable $x(t)$ of system (13), looped by the control law (20), evolves in the polyhedron $\mathrm{Pol}(G, \gamma)$ for any reduced vector verifying $z(t) \in \mathcal{P}$, if and only if the following conditions hold true for $i = 1$ to s:*

$$\max_{j=1}^{q} (GM)_{ij} - \sum_{k=1}^{h} \int_0^{\theta_k} \min_{j=1}^{q} (G\mathrm{e}^{-A\tau} B_k U)_{ij} d\tau \leq \gamma_i \quad (22)$$

After applying the result of *Theorem 2.2*, we reformulate the inequalities (22) in order to determine the $\mathcal{D}$-invariance conditions for the input-delayed system in the following theorem.

Theorem 5.1. *The variables $x(t)$ and $u(t)$ resulting from the delayed system (13) and looped by a control law of the form (19)-(20), evolve in the polyhedra $\mathcal{P}_x = [x^{\min}, x^{\max}]$ and $\mathcal{P}_u = [u^{\min}, u^{\max}]$ respectively, for every perturbation verifying $d(t) \in \mathcal{P}_d = [d^{\min}, d^{\max}]$, if and only if the following inequalities hold true for $i = 1$ to n:*

$$\max_{j=1}^{q} M_{ij} - \sum_{k=1}^{h} \int_0^{\theta_k} \min_{j=1}^{q} (\mathrm{e}^{-A\tau} B_k U)_{ij} d\tau \leq x_i^{\max} \quad (23)$$

$$x_i^{\min} \leq \min_{j=1}^{q} M_{ij} - \sum_{k=1}^{h} \int_0^{\theta_k} \max_{j=1}^{q} (\mathrm{e}^{-A\tau} B_k U)_{ij} d\tau \quad (24)$$

PROOF. The interval $[x^{\min}, x^{\max}]$ is the set of vectors $x \in \mathbb{R}^n$ that satisfy both inequalities $x \leq x^{\max}$ and $-x \leq -x^{\min}$. As a consequence, the inequalities of the theorem are obtained using twice Proposition 5.2, the first time taking $G = I_n$ and $\gamma = x^{\max}$, and the second time taking $G = -I_n$ and $\gamma = -x^{\min}$. This ends the proof. □

This result allows to check whether or not a given polytope $\mathcal{P}$ provides a solution to our control problem. A way to implement a control law is chosen and expressed by (20), in such a way that the closed-loop system meets the constraints $z(t) \in \mathcal{P}$, $u(t) \in \mathcal{P}_u$, for every demand verifying $d(t) \in \mathcal{P}_d$. Furthermore, using the expression (21) that describes the relation between the variables of the reduced model and the delayed model, we have found the $\mathcal{D}$-invariance conditions for the delayed system using Proposition 5.2 and Theorem 5.1.

6. Study analysis

We identify in the following discussion, the most important issues that remain unsolved, in order to complete this general control approach for the resolution of logistics network.

6.1. Remark on the initialization

We have remarked in Section 4.1 that the evolution of an elementary production system may depend from an initial fonction $\phi(t)$ during the first instant, for $t \leq \theta$, that can be interpreted in terms of initial work-in-progress. More specifically, since $u(t)$ is defined for $t \geq 0$, we have $\dot{x}(t) = \phi(t) - d(t)$, for $t \in [0, \theta]$.

The interpretation of the work-in progress depends on the kind of systems that is modelled and on the interpretation of the initial time. For a production system, the initial time may be the start-up of the process after a long stop, or the time where a new controller starts to manage the system after a release of the management decision process. In the first case, the work-in-progress is initially zero, while in the second case, it could be nonzero, for example if the new management procedures are launched online, without interruption of the production. Notice that such a study was done in the mono-variable case [18].

The same actually occurs in the case of a multivariable production system. As a consequence, the equations (17) and (18) may be satisfied only for t greater than or equal to the maximal element of the set of delays θ_k, where $k = 1$ to h. We can always assume that the sequence θ_k is arranged in increasing order, so that the maximal element is θ_h. During the initialization period, $t \in [0, \theta_h]$, the reduced variable $z(t)$ also depends on some initial functions $\phi_k(t)$, that correspond to the initial work-in-progress along the edges of the production network that is considered. As a consequence, the invariance of $\mathcal{P}$ is actually obtained for $t \geq \theta_h$, if the conditions of Theorem 3.3 are satisfied.

6.2. Robust control implementation

Beyond focusing on the controlled system stability and constraints verification, we can remark that the performance of logistic network dynamics was addressed in this study, by taking into account exact system parameters. In this direction, an important topic concerns the sensitivity of the control strategies against the intrinsic parameters uncertainties, specifically the delays and the loss factors. For this reason, different types of control laws can be proposed and applied for explicit control implementations to logistic networks. Then, important approaches can be developed in order to study the robust control against numerous system uncertainties. In this way, a first robust methodology was made for the elementary logistic system in [7], where the algorithmic complexity is easier to manipulate, than that we face in logistic networks studies.

6.3. Construction of the polytope $\mathcal{P}$

Our study is presented by a production network composed of n elementary systems expressed by (13), where the intrinsic parameters are the delays θ_k for $k = 1$ to h, and the loss factors σ on the storage level of each node. The system is modeled by the state vector $x(t)$, the control vector $u(t)$, and the disturbance vector $d(t)$ that are constrained and evolve, respectively, in the polytopes $\mathcal{P}_x = [x^{\min}, x^{\max}]$, $\mathcal{P}_u = [u^{\min}, u^{\max}]$ and $\mathcal{P}_d = [d^{\min}, d^{\max}]$.
From a theoretical point of view, the obtained sufficient conditions are used to define properly the polytope $\mathcal{P}$. This latter allows to check the admissibility of the proposed control $u(t)$ which stabilizes the delayed system while ensuring that the constraints on the state $x(t)$ are met for any bounded perturbation $d(t)$. But in fact, the research of a convenient polytope $\mathcal{P}$ is a difficult problem, of high complexity in general. The computation is anyway possible in many practical situations, as illustrated in the section 7 for the case of a supply chain with two nodes, equal delays and equal loss factors.

7. Application on a supply chain with two elements

We consider a supply chain with two nodes, the first one feeding the second by an internal demand $d_1(t)$, that in turn satisfies an external demand $d_2(t)$. We suppose that the production delays of both elementary systems are identical, say $\theta_1 = \theta_2 = \theta$, as well as their loss factor σ also equal. The interconnection of the two elementary logistic systems is illustrated in the following schema.

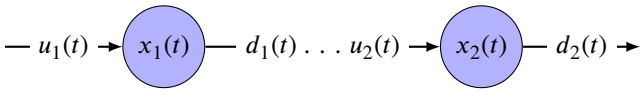

Figure 1: Supply chain illustration with two nodes

The corresponding matrices defining the delayed system (13) for the supply chain example, are:

$$A = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \; B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; E = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; B_0 = 0.$$

We apply the Artstein Reduction in order to fully compensate the delay effects on both nodes of the supply chain. The resulting reduced system is defined by (18) with:

$$B = \begin{pmatrix} e^{\sigma\theta} & 0 \\ 0 & e^{\sigma\theta} \end{pmatrix}.$$

In this case, the reduced model provides an exact prediction of the future storage levels x_1 and x_2 over a time horizon between instants t and $t+\theta$, because the input delays are equal. In addition, we notice that B is a square and invertible matrix. This property allows to apply a second static transformation in order to decouple the supply chain into two elementary systems. In this way, each node of the supply chain corresponds to an independent logistic system having its proper production unit and control system. For this reason, this example is particularly significant and presents an important class of systems in production management. Thus, the transformation $w(t) = B^{-1}z(t)$ is then performed on the reduced system (18), in order to define the new decoupled system that reads:

$$\dot{w}(t) = -B^{-1}AB\, w(t) + u(t) - B^{-1}E\, d(t), \tag{25}$$

with

$$B^{-1} = \begin{pmatrix} \mathrm{e}^{-\sigma\theta} & 0 \\ 0 & \mathrm{e}^{-\sigma\theta} \end{pmatrix}, \text{ and } B^{-1}E = \begin{pmatrix} \mathrm{e}^{-\sigma\theta} \\ \mathrm{e}^{-\sigma\theta} \end{pmatrix}.$$

Since this system (25) is decoupled, the variable $w(t)$ evolves in a bi-dimensional interval $[w^{\min}, w^{\max}]$, that can be represented as explained in section 2.3, by $\mathcal{P}_w = \mathrm{Conv}\, W$, where

$$W = \begin{pmatrix} w_1^{\min} & w_1^{\max} & w_1^{\min} & w_1^{\max} \\ w_2^{\min} & w_2^{\min} & w_2^{\max} & w_2^{\max} \end{pmatrix}.$$

As suggested in Section 6.1, we construct the invariant polytope $\mathcal{P}$ of the reduced variable $z(t) = Bw(t)$ by a convex representation as expressed in the following form:

$$\mathcal{P} = B[w^{\min}, w^{\max}] = \mathrm{Conv}\, M, \text{ where } M = BW \tag{26}$$

Based on the non limitative assumption that B is a square and invertible matrix, we deduce that the rectangle $ABCD$ corresponds to the biggest polytope which includes all the sets of evolution of $z(t)$. The vertices A, B, C, and D are well defined by the columns of the matrix $M = BW$. Each row in the matrix B corresponds to two parallel lines where the variation of $z(t)$ is limited.

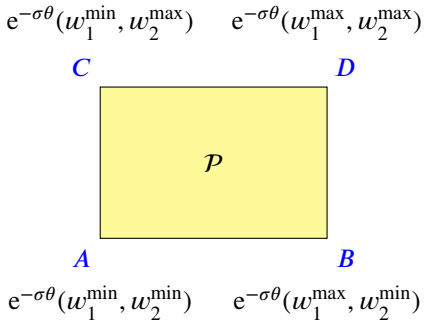

Figure 2: Invariant polytope of $z(t)$

According to the control parametrization (19) and (20), the constrained control solution is defined by a polyhedron $[u^1, u^2]$, where the control vectors u^1 and u^2 should verify the following condition:

$$u^1,\ u^2 \in \mathcal{P}_u = [u^{\min}, u^{\max}]\ . \tag{27}$$

As consequence, the control matrix U is defined in such a way that $\mathrm{Conv}\, U = [u^1, u^2]$, say:

$$U = \begin{pmatrix} u_1^1 & u_1^2 & u_1^1 & u_1^2 \\ \\ u_2^1 & u_2^1 & u_2^2 & u_2^2 \end{pmatrix}. \tag{28}$$

Using the parametrization (27) and the results of Theorem 3.3, we deduce that an interval $\mathcal{P}_w = [w^{\min}, w^{\max}]$ is controlled $\mathcal{D}$-invariant for the decoupled system (25), and the polytope $\mathcal{P} = B[w^{\min}, w^{\max}]$ is controlled $\mathcal{D}$-invariant for (18) by equivalence, using the control law of the form (28), if and only if the following conditions hold true:

$$u^{\min} \le u^1 \le \sigma w^{\max} + \mathrm{e}^{-\sigma\theta} d^{\min}\ , \tag{29}$$

$$\sigma w^{\min} + \mathrm{e}^{-\sigma\theta} d^{\max} \le u^2 \le u^{\max}\ . \tag{30}$$

At this level, we come back to the original time-delay system (13) in order to find the admissibility conditions of the polytope $\mathcal{P}_x = [x^{\min}, x^{\max}]$, using Theorem 5.1. In this example, since the delays are equal, the reduced vector $z(t)$ is the exact prediction of the output vector $x(t)$. In this case, the output vector is defined using the transformation $z(t) = Bw(t)$ by the following expression:

$$x(t+\theta) = Bw(t) - \int_t^{t+\theta} \mathrm{e}^{-A(t+\theta-\tau)} d(\tau)\, d\tau.$$

Hence, the exact and reachable output bounds are determined, and the necessary and sufficient conditions of admissibility are deduced as follows:

$$x^{\min} \le \mathrm{e}^{\sigma\theta} w^{\min} - \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma} d^{\max}, \tag{31}$$

$$\mathrm{e}^{\sigma\theta} w^{\max} - \frac{1-\mathrm{e}^{-\sigma\theta}}{\sigma} d^{\min} \le x^{\max}. \tag{32}$$

As consequence, we have obtained the inequalities (27), (29), (30), (31) and (32), that correspond to the invariance conditions for the polytope $\mathcal{P}_w$ and the admissiblity conditions of the polytope $\mathcal{P}_x$. These inequalities are necessary and sufficient conditions that satisfy the output constraints (14) and the input constraints (15) for every demand $d(t)$ verifying(16), when the system (13) is looped by the control law defined by (27) and (28).

One can remark that these conditions are affine. They define the admissible region in the space of system parameters θ and σ, the space of the specification defined by $d^{\min}$, $d^{\max}$, $u^{\min}$, $u^{\max}$, $x^{\min}$ and $x^{\max}$, and the space of control parameters $w^{\min}$, $w^{\max}$, u^1 and u^2. This region is a non empty polyhedral set provided the following hypothesis is true:

$$w^{\min} \le w^{\max}. \tag{33}$$

This condition is very important to guarantee not only the verification of an admissible control solution but also the existence of such control solution. Then, we eliminate successively the control parameters from the set of inequalities (27), (29), (30), (31), (32) and (33), by projecting the resulting admissible region parallel to the space of control parameters. Thus, the resulting equivalent conditions are described in the following inequalities:

$$\sigma x^{min} + (2 - \mathrm{e}^{-\sigma\theta})d^{\max} \leq \mathrm{e}^{\sigma\theta}u^{max} , \tag{34}$$

$$\mathrm{e}^{\sigma\theta}u^{min} \leq \sigma x^{max} + (2 - \mathrm{e}^{-\sigma\theta})d^{\min} , \tag{35}$$

$$x^{min} + \frac{1 - \mathrm{e}^{-\sigma\theta}}{\sigma}d^{\max} \leq x^{max} + \frac{1 - \mathrm{e}^{-\sigma\theta}}{\sigma}d^{\min} . \tag{36}$$

These conditions determine the sizing of each production and storage unit of one elementary logistic system of the bi-dimensional supply chain. Therefore, the conditions (34), (35) and (36) are completed with the following condition:

$$d_1^{\min} \leq u_2^{\min} < u_2^{\max} \leq d_1^{\max} . \tag{37}$$

This condition describes the interconnection between the two nodes of the supply chain, in order to verify the system constraints ans to guaranty the external customer demand satisfaction at the last node of the supply chain. At the end of the paper, we have obtained a final set of conditions, that depend only on the system parameters ans the imposed specification, and that are necessary and sufficient conditions to prove the existence of an admissible control law defined by (28) and (27). The obtained results are not conservative, since the definition (20)-(26) cover a large class of control laws, that solves the constrained control problem for a supply chain.

7.1. Simulation example

This simulation illustrates the supply chain responses and highlights the effects of the control methodology on the system stability and constraints verification. The delays are $\theta_1 = \theta_2 = 5$ days and the loss factors are equal to $\sigma_1 = \sigma_2 = 0.07$ days^{-1}. In addition, the constraints are given by $u^{\min} = 5.83$, $u^{\max} = 15.11$, $x^{\min} = 0$ and $x^{\max} = 105$. Moreover, the supply chain is initialized by 16 items/day for the production order in progress at each storage node. We assume that the external customer demand $d(t) = d_2(t)$ follows a rectangular signal illustrated in figure (3), which reflects abrupt seasonal changes in a half-year trend.

Based on the system responses illustrated in figures (3) and (4), we first notice that when the external demand $d(t) = 0$, the controlled structure makes it possible to replenish the inventory levels of the two storage nodes without any undergo beyond $x^{\max}$ and any shortage. Hence, the system constraints are checked for both the output $x(t)$ and the input $u(t)$. Then, the storage levels never fall to zero in the initial phase which implies full demand satisfaction and high service level. In addition, we can see that the input controller at the second node quickly responds to the sudden changes in the demand trend without oscillations or overshoots in the production order u_2. It also establishes bigger storage quantities, compared to the supply source u_1 that leads to smaller storage costs while maintaining the same service level.

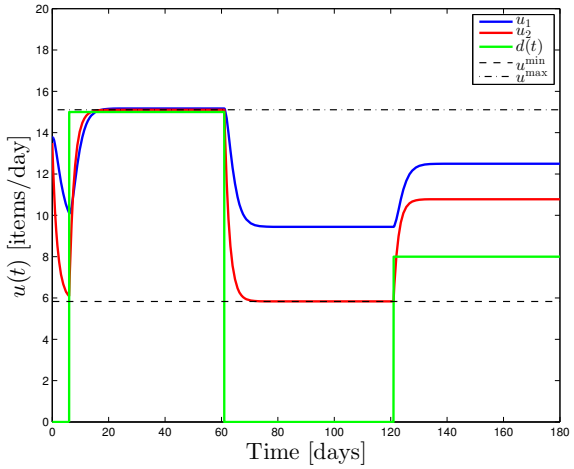

Figure 3: Temporal evolution of the production orders $u(t)$

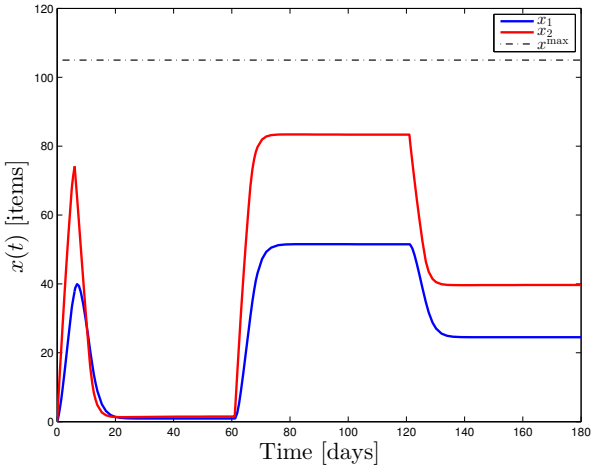

Figure 4: Temporal evolution of storage levels $x(t)$

8. Conclusion

The papers provides a new characterization of the $\mathcal{D}$-invariance property of polytope sets for continuous time-delay systems. An interesting application of the concept of $\mathcal{D}$-invariance is to investigate the possibility of controlling a class of constrained convolution systems. The considered system is the model of an interconnected production network where the intrinsic parameters are the delays and the loss factors. Our motivation is to design management procedures that permit to answer an external demand, that evolves in a specified range, without exceeding the system sizing that is

defined by limitations on the storage level and production order at each node. The controller design problem for supply chain, with constrained orders and inventories, and unknown customer demands variations, has been investigated. The problem is stated in terms of controlled input-delay system, with positivity constraints, subject to bounded disturbances. Sufficient conditions for the existence of an admissible control were determined on a two dimensional example.

References

[1] Z. Artstein, Linear systems with delayed control: A reduction, IEEE Transactions on Automatic Control 27 (4) 869-879, 1982.

[2] F. Blanchini, Feedback control for linear time-invariant systems with state and control bounds in the presence of disturbances, IEEE Transactions on Automatic Control 35 (11) 1231-1234, 1990.

[3] F. Blanchini, S. Miani, C.E.T. Dorea, J.-C. Hennet, Discussion on (A, B)-invariance conditions of polyhedral domains for continuous-time systems by C.E.T. Dorea and J.C. Hennet, European Journal of Control 5 (1) 82-86, 1999.

[4] F. Blanchini, R. Pesenti, F. Rinaldi, W. Ukovich, Feedback control of production-distribution systems with unknown demand and delays, IEEE Transactions on Robotics and Automation 16 (3) 313-317, 2000.

[5] F. Blanchini, S. Miani, Set-theoretic methods in control. Boston: Birkhäuser, 2008.

[6] B. Bou Farraa, R. Abbou, J.J. Loiseau, Necessary and sufficient conditions for the stability of input-delayed systems, 14^{th} IFAC Workshop on Time Delay Systems 51(14), 218–223, 2018.

[7] B. Bou Farraa, R. Abbou, J.J. Loiseau, Robust stabilization of an elementary logistic system with an input delay, 21^{st} IFAC World Congress, 2020.

[8] C.A. Desoer, M. Vidyasagar, Feedback systems: input-output properties, *Academic Press*, 1975.

[9] C.E.T. Dorea, J.-C. Hennet, (A, B)-invariance conditions of polyhedral domains for continuous-time systems, European Journal of Control, 5 (1) 70-81, 1999.

[10] J.W. Forrester, Industrial dynamics, MIT Press, Cambridge, Mass and Wiley, New York, 1961.

[11] J.-C. Hennet, A bimodal scheme for multi-stage production and inventory control, Automatica 39 (3) 793-805, 2003.

[12] P. Ignaciuk, A. Bartoszewicz, Congestion control in data transmission networks, Sliding modes and other designs, Springer, London, 2013.

[13] H.L. Lee, V. Padmanabhan, S. Whang, Information distortion in a supply chain: The bullwhip effect, Management Science 43 (4) 546-558, 1997.

[14] M.T. Laraba, S. Olaru, S.I. Niculescu, F. Blanchini, G. Giordano, D. Casagrande, S. Miani, Set invariance for delay difference equations, IFAC-PapersOnLine, 48 (12) 215-220, 2015.

[15] J.J. Loiseau, Polyhedral invariance for convolution systems over the Callier-Desoer class, In: F. Cacace, L. Farina, R. Setola, A. Germani (eds.), Positive Systems POSTA 2016, Lecture Notes in Control and Information Sciences 471, 151-170, Springer, Cham, 2017.

[16] D.Q. Mayne, J.B. Rawlings, C.V. Rao, P.O. Scokaert, Constrained model predictive control: Stability and optimality, Automatica, 36(6) 789-814, 2000.

[17] C. Moussaoui, R. Abbou, J.J. Loiseau, On bounds of input-output systems, Reachability set determination and polyhedral constraints verification, In: Proc. IFAC World Congress 2014, IFAC Proceedings Volumes 47 (3) 11012-11017, 2014.

[18] C. Moussaou, R. Abbou, J.J. Loiseau, Robust inventory control of production systems subject to uncertainties on demand and lead times, International Journal of Production Research 55 (8) 2177-2196, 2017.

[19] A. Schrijver, Theory of linear and integer programming, John Wiley and sons, 1986.

[20] S. Olaru, J.A. De Doná, M.M. Seron, F. Stoican, Positive invariant sets for fault tolerant multisensor control schemes, International Journal of Control, 83(12), 2622-2640, 2010.

[21] H. Simon, On the application of servomechanism theory in the study of production control, Econometrica 69 (1) 247-268, 1952.

[22] F. Stoican, S. Olaru, M.M. Seron, J.A. De Don, Reference governor design for tracking problems with fault detection guarantees, Journal of Process Control, 22(5), 829-836, 2012.

[23] S. Tarbouriech, J.-C. Hennet, Conditions of stability via positive invariance for delay systems, Journal Européen des Systèmes Automatisés 31 (6) 983-999, 1997.

[24] D.R. Towill, Dynamic analysis of an inventory and order based production control systems with transportation delay, European Journal of Production Economics 92, 3-12, 1982.

[25] X. Wang, S.M. Disney, The bullwhip effect: Progress, trends and directions, European Journal of Operational Research 250 (3) 691-701, 2016.