



HAL
open science

Numerical and convergence analysis of the stochastic Lagrangian averaged Navier-Stokes equations

Jad Doghman, Ludovic Goudenège

► **To cite this version:**

Jad Doghman, Ludovic Goudenège. Numerical and convergence analysis of the stochastic Lagrangian averaged Navier-Stokes equations. *Journal of Computational and Applied Mathematics*, 2022, 10.1016/j.cam.2022.114446 . hal-03406041v3

HAL Id: hal-03406041

<https://hal.science/hal-03406041v3>

Submitted on 23 Dec 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Numerical and convergence analysis of the stochastic Lagrangian averaged Navier-Stokes equations *

Jad Doghman^{1†} and Ludovic Goudenège¹

¹*CNRS, Fédération de Mathématiques de CentraleSupélec FR 3487, Univ. Paris-Saclay, CentraleSupélec, F-91190 Gif-sur-Yvette, France*

Abstract

The primary emphasis of this work is the development of a finite element based space-time discretization for solving the stochastic Lagrangian averaged Navier-Stokes (LANS- α) equations of incompressible fluid turbulence with multiplicative random forcing, under nonperiodic boundary conditions within a bounded polygonal (or polyhedral) domain of \mathbb{R}^d , $d \in \{2, 3\}$. The convergence analysis of a fully discretized numerical scheme is investigated and split into two cases according to the spacial scale α , namely we first assume α to be controlled by the step size of the space discretization so that it vanishes when passing to the limit, then we provide an alternative study when α is fixed. A preparatory analysis of uniform estimates in both α and discretization parameters is carried out. Starting out from the stochastic LANS- α model, we achieve convergence toward the continuous strong solutions of the stochastic Navier-Stokes equations in 2D when α vanishes at the limit. Additionally, convergence toward the continuous strong solutions of the stochastic LANS- α model is accomplished if α is fixed. Neither of the mentioned convergences involves the Skorokhod theorem.

Keywords: stochastic Lagrangian averaged Navier-Stokes, stochastic Navier-Stokes, finite element, Euler method

PACS: 02.70.Dh, 02.70.Bf, 02.60.Lj, 02.60.Jh, 02.60.Gf, 02.60.Cb, 02.50.Ey, 02.50.Cw

2010 MSC: 76D05, 65M12, 35Q35, 35Q30, 60H15, 60H35

*Jad Doghman is supported by a public grant as part of the Investissement d'avenir project [ANR-11-LABX-0056-LMH, LabEx LMH], and both authors are part of the SIMALIN project [ANR-19-CE40-0016] of the French National Research Agency.

[†]Corresponding author

Email addresses: jad.doghman@centralesupelec.fr (J. Doghman),
goudenege@math.cnrs.fr (L. Goudenège)

1 Introduction

Over the last few decades, many regularization models of the Navier-Stokes equations (NSEs) have arisen, especially the α -regularizations, for the sake of better understanding the closure problem of averaged quantities in turbulent flows. Such turbulent modeling schemes (e.g. Leray- α , Navier-Stokes- α , Clark- α , Modified Leray- α) were introduced as effective subgrid-scale models of the NSEs which require massive grid points or Fourier modes, allowing for approximation to capture all the spatial scales down to the Kolmogorov scale (see for instance [9] and the references therein), as well as their suitability with the empirical and experimental data for a thorough range of Reynolds numbers.

In the present paper, we consider the stochastic version of the LANS- α equations [29] (also known as the viscous Camassa-Holm equations [4], or the Navier-Stokes- α model [13, 25])

$$\begin{cases} \partial_t (\bar{u} - \alpha^2 \Delta \bar{u}) - \nu \Delta (\bar{u} - \alpha^2 \Delta \bar{u}) - \bar{u} \times (\nabla \times (\bar{u} - \alpha^2 \Delta \bar{u})) + \nabla p = f(\cdot, \bar{u}) + g(\cdot, \bar{u}) \dot{W}, \\ \operatorname{div} \bar{u} = 0, \\ \bar{u}(0, \cdot) = \bar{u}_0, \end{cases} \quad (1.1)$$

for internal flow i.e. for a bounded domain in \mathbb{R}^d , $d \in \{2, 3\}$. The unknown vector field \bar{u} is called the filtered fluid velocity, and it depends on time and space variables, ν is the fluid kinematic viscosity, and α is a small spatial scale at which fluid motion is filtered. Note that both ν and α are positive constants. $f = f(t, \bar{u})$ is an external force, the scalar quantity $p = p(t, x)$ represents the pressure and \bar{u}_0 is the corresponding initial datum. The last term of equations (1.1)₁ describes a state-dependent random noise, and it is defined by $g(\cdot, \bar{u}) \dot{W} := g(t, \bar{u}) \partial_t W(t, x)$, where g is a diffusion coefficient. One of the aims herein is to approach the two-dimensional solutions of the stochastic NSEs via the LANS- α model, numerically. Whence the need to evoke the former equations with similar configurations:

$$\begin{cases} \partial_t v - \nu \Delta v + [v \cdot \nabla] v + \nabla \lambda = f(\cdot, v) + g(\cdot, v) \dot{W}, \\ \operatorname{div} v = 0, \\ v(0, \cdot) = v_0, \end{cases} \quad (1.2)$$

where v (resp. λ) is the corresponding fluid velocity (resp. pressure), and v_0 embodies its initial datum.

Equations (1.1) and (1.2) are usually employed as a complementary model to their deterministic versions to better understand the situation of tiny variations or perturbations present in fluid flows. The former represents a modification of the latter by performing Lagrangian means, asymptotic expansions, and an assumption of isotropy of fluctuations in the Hamilton principle, which grant further physical properties (e.g. conservation laws for energy and momentum). More specifically, the convective nonlinearity $[v \cdot \nabla] v$ in the NSEs is adjusted so that the cascading of turbulence at scales under specific length stops. The latter adjustment is called a nonlinearly dispersive modification.

The existence and uniqueness of a variational solution to the problem (1.1) were investigated in [10] under Lipschitz-continuous conditions in a three-dimensional bounded domain. A similar study is proposed in [15], but this time with a genuine finite-dimensional Wiener process depending only on time. LANS- α model driven by an additive space-time noise of trace class was considered in [22], where the authors proved the existence and uniqueness of an invariant measure, and a probabilistic strong solution.

Speaking of the numerical approach, the convergence analysis of suitable numerical methods for the stochastic LANS- α equations is less well developed. In connection with the deterministic version, both convergence rate and convergence analysis of an algorithm consisting of a finite element method were investigated in [14] where the spatial scale α is considered in terms of the space discretization's step. The author in [8] conducted a similar study, with α being independent of the discretization parameters. On the other hand, numerical schemes for stochastic nonlinear equations admitting local Lipschitz nonlinearities related to the Navier-Stokes systems had been already investigated. For instance, authors in [7] studied a finite element-based space-time discretization of the incompressible NSEs driven by a multiplicative noise. An enhancement of [7] in dimension 2 was carried out in [3].

This paper aims to provide a fully discrete finite element-based discretization of equations (1.1) in a bounded convex polygonal or polyhedral domain. Notice that the underlying model consists of a fourth order problem, nevertheless we avoid the use of C^1 piecewise polynomials-based finite element methods by introducing a notion of differential filters that transform equations (1.1) into a coupled problem of second order. The employed time-discretization herein is an Euler scheme. One highly valued characteristic of the finite element method is the prospect of meticulous interpretation provided by the functional analysis framework. In contrast to the linear stochastic partial differential equations, since we are dealing here with a nonlinear model, one cannot make use of the semigroup method or Green's function. Those techniques are effectively replaced by monotone or Lipschitz-continuous drift functions. It is worth highlighting the importance of constructing practical numerical schemes provided with exact divergence-free finite element functions (e.g. see [5, 16, 31, 32]). However, due to their computational complexity, one may notice the usage of a weak divergence-free condition that compensates for the strong sense's absence.

The associated spatial scale α will be considered hereafter either in terms of the space discretization's step (case 1) or independently of all discretization's parameters (case 2). Therefore, our main results consist of the convergence in both 2D and 3D of Algorithm 1 toward the continuous solution of the 2D stochastic NSEs for the case 1, together with the convergence toward the continuous solution of the stochastic LANS- α model for the case 2. Speaking of the followed approach, we begin by performing a priori estimates characterized by their uniformity in α (for case 1) and the discretizations' parameters (for cases 1 and 2), allowing us to extract convergent subsequences of the approximate solution. As mentioned in the abstract, Skorokhod's theorem is not employed to achieve solutions' existence, which means that the probability basis that was defined in Section 2 is maintained throughout the demonstration. This avoidance took place due to two different identification methods, according to the imposed condition on the spatial scale α . In other words, if the target was the LANS- α solution, the identification method would rely on its solution's regularity, which is known to be high compared with the NSEs'. If one was aiming for the NSEs solution, the followed technique exploits a local monotonicity property attributed to the nonlinear term of equations (1.1). Each of these two approaches were first introduced in [10, 30] to illustrate solutions' existence through an abstract Faedo-Galerkin method. However, since we are dealing with discrete settings herein, Itô's formula is no longer applicable. Fortunately, discrete derivations solve this issue despite the appearance of a few associated extra terms.

The paper is organized as follows. We introduce in Section 2 a few notions and preliminaries, including the spatial framework, the needed assumptions, the time and space discretizations alongside their properties, definition of solutions to problems (1.1) and (1.2), the definition of continuous and discrete differential filters along with the investigated algorithm. Section 3 is tailored for the main results of this paper. We dedicate Section 4 to all possible a priori estimates within standard regularities, together with the local monotonicity property related to the nonlinear term. In Section 5, we study the convergence analysis of the proposed numerical scheme. Accordingly, we identify both deterministic and stochastic integrals, as the discretization steps tend to 0, with their corresponding counterparts. We terminate this paper (Section 6) with a conclusion concerning the obtained limiting functions and how one can relate them to the stochastic NSEs and LANS- α model. We equip this section with a computational experiment to visualize the outcomes and to evaluate the performance of the proposed numerical scheme.

2 Notations and preliminaries

We state, in this section, preliminary background material following the usual notation employed in the context of the mathematical theory of Navier-Stokes equations.

Given $T > 0$, we denote by $D \subset \mathbb{R}^d$, $d \in \{2, 3\}$ a bounded convex polygonal or polyhedral domain with boundary ∂D , in which we seek a solution, namely a stochastic process $(\bar{u}(t), p(t))$, $t \in [0, T]$ satisfying equations (1.1) in a certain sense. Define almost everywhere on ∂D the unit outward normal vector field $\bar{n}: \partial D \rightarrow \mathbb{R}^d$. The following function spaces are required hereafter:

$$\mathbb{H} := \left\{ z \in (L^2(D))^d \mid \operatorname{div} z = 0 \text{ a.e. in } D, z \cdot \bar{n} = 0 \text{ a.e. on } \partial D \right\},$$

$$\mathbb{V} := \left\{ z \in (H_0^1(D))^d \mid \operatorname{div} z = 0 \text{ a.e. in } D \right\}.$$

From now on, the spaces of vector valued functions will be indicated with blackboard bold letters, for instance $\mathbb{L}^2 := (L^2(D))^d$ denotes the Lebesgue space of vector valued functions defined on D . Denote by $\mathcal{P}: \mathbb{L}^2 \rightarrow \mathbb{H}$ the Leray projector, and by $A: D(A) \rightarrow \mathbb{H}$ the Stokes operator defined by $A := -\mathcal{P}\Delta$ with domain $D(A) = \mathbb{H}^2 \cap \mathbb{V}$. A is a self-adjoint positive operator, and has a compact inverse, see for instance [11]. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ be a complete probability space, Q be a nuclear operator, and K be a separable Hilbert space on which we define the Q -Wiener process $W(t)$, $t \in [0, T]$ such that

$$W(t) = \sum_{k \in \mathbb{N}} \sqrt{q_k} \beta^k(t) w_k, \quad \forall t \in [0, T], \quad (2.1)$$

where $\{\beta^k(\cdot), k \in \mathbb{N}\}$ is a sequence of independent and identically distributed \mathbb{R} -valued Brownian motions on the probability basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, $\{w_k, k \in \mathbb{N}\}$ is a complete orthonormal basis of the Hilbert space K consisting of the eigenfunctions of Q , with eigenvalues $\{q_k\}_{k \in \mathbb{N}^*}$. The following estimate will play an essential role in the sequel, cf. [26].

$$\mathbb{E} \left[\|W(t) - W(s)\|_K^{2r} \right] \leq (2r - 1)!! (t - s)^r (\operatorname{Tr}(Q))^r, \quad \forall r \in \mathbb{N}, \quad (2.2)$$

where $(2r - 1)!! := (2r - 1)(2r - 3) \dots \times 5 \times 3 \times 1$, and $\operatorname{Tr}(Q)$ denotes the trace of Q .

For any arbitrary Hilbert spaces X, Y , the sets $\mathcal{L}_1(X, Y)$ and $\mathcal{L}_2(X, Y)$ denote the nuclear, and Hilbert-Schmidt operators from X to Y , respectively. For brevity's sake, if $X = Y$, we set $\mathcal{L}_i(X, X) = \mathcal{L}_i(X)$, $i \in \{1, 2\}$. Hereafter, $M_{\mathcal{F}_t}^p(0, T; X)$ denotes the space of all \mathcal{F}_t -progressively measurable processes belonging to $L^p(\Omega \times (0, T), d\mathbb{P} \times dt; X)$, for any Banach space X .

Throughout this paper, the nonnegative constant C_D depends only on the domain D , the symbols (\cdot, \cdot) and $\langle \cdot, \cdot \rangle$ stand for the inner product in \mathbb{L}^2 and the duality product between \mathbb{H}^{-1} and \mathbb{H}^1 , respectively. Recall that α is a small spatial scale, thereby we assume that $\alpha \leq 1$. The latter leads to the following norm equivalence

$$\alpha \|\cdot\|_{\mathbb{H}^1} \leq \|\cdot\|_{\alpha} \leq \|\cdot\|_{\mathbb{H}^1}, \quad (2.3)$$

where $\|\cdot\|_{\alpha}$ is defined by $\|\cdot\|_{\alpha}^2 := \|\cdot\|_{\mathbb{L}^2}^2 + \alpha^2 \|\nabla \cdot\|_{\mathbb{L}^2}^2$. We point out that the whole study herein maintains all the stated properties if one chooses $\alpha \leq \tilde{\alpha}$, for some $\tilde{\alpha} \in \mathbb{R}_+^*$. For arbitrary real numbers x, y , the inequality $x \lesssim y$ is a shorthand for $x \leq cy$ for some universal constant $c > 0$. We list below the needed assumptions on the data \bar{u}_0, g, Q , and f .

Assumptions

(S₁) $Q \in \mathcal{L}_1(K)$ is a symmetric, positive definite operator.

(S₂) $f \in L^2(\Omega; C([0, T]; \mathbb{H}^{-1}))$ and $g \in L^2(\Omega; C([0, T]; \mathcal{L}_2(K, \mathbb{L}^2)))$ are sublinear Lipschitz-continuous mappings, i.e. for all $z_1, z_2 \in \mathbb{V}$, $g(\cdot, z_1)$ and $f(\cdot, z_2)$ are \mathcal{F}_t -progressively measurable, and $d\mathbb{P} \times dt$ -a.e. in $\Omega \times (0, T)$,

$$\begin{aligned} \|g(\cdot, z_1) - g(\cdot, z_2)\|_{\mathcal{L}_2(K, \mathbb{L}^2)} &\leq L_g \|z_1 - z_2\|_{\alpha}, \quad \forall z_1, z_2 \in \mathbb{V}, \\ \|g(\cdot, z)\|_{\mathcal{L}_2(K, \mathbb{L}^2)} &\leq K_1 + K_2 \|z\|_{\alpha}, \quad \forall z \in \mathbb{V}, \\ \|f(\cdot, z_1) - f(\cdot, z_2)\|_{\mathbb{H}^{-1}} &\leq L_f \|z_1 - z_2\|_{\alpha}, \quad \forall z_1, z_2 \in \mathbb{V}, \\ \|f(\cdot, z)\|_{\mathbb{H}^{-1}} &\leq K_3 + K_4 \|z\|_{\alpha}, \quad \forall z \in \mathbb{V}, \end{aligned}$$

for some time-independent nonnegative constants $K_1, K_2, K_3, K_4, L_f, L_g$ such that $L_f \leq \nu/\sqrt{2}C_P^2$ and $L_g \leq \operatorname{Tr}(Q)^{-1/2} (\nu^2 - 2L_f^2 C_P^4)^{1/2} / C_P \sqrt{2\nu}$, with $C_P > 0$ being the Poincaré constant.

(S₃) $\bar{u}_0 \in L^{2p}(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{V})$, for some $p \in [1, +\infty)$.

Remark 2.1 *The time-continuity of both f and g in assumption (S₂) can be readily relaxed and turned into $L^2(0, T)$. This emerges from the approximation choice of f and g occurring in Algorithm 1. For instance, $f(t_{m-1}, \cdot)$ could have been approximated by $f^m(\cdot) := \frac{1}{k} \int_{(m-1)k}^{mk} f(t, \cdot) dt$, where k and m are introduced in subsection 2.2.*

To avoid repetitions later on, we state the following identity

$$(a - b, a) = \frac{1}{2} \left(\|a\|_{\mathbb{L}^2}^2 - \|b\|_{\mathbb{L}^2}^2 + \|a - b\|_{\mathbb{L}^2}^2 \right) \quad \text{for all } a, b \in \mathbb{L}^2. \quad (2.4)$$

$$|a + b|^p \leq 2^{p-1} (|a|^p + |b|^p), \quad \text{for all } a, b \in \mathbb{R} \text{ and } p \geq 1. \quad (2.5)$$

The trilinear form We define the trilinear form \tilde{b} , associated with the LANS- α equations, by

$$\tilde{b}(z_1, z_2, w) = -\langle z_1 \times (\nabla \times z_2), w \rangle, \quad \forall z_1, z_2, w \in \mathbb{H}_0^1.$$

The following proposition contains a few corresponding properties.

Proposition 2.1

- (i) $(z_1 \times (\nabla \times z_2), w) = -(w \times (\nabla \times z_2), z_1)$ for all $z_1, z_2, w \in \mathbb{H}^1$. Particularly, $(z_1 \times (\nabla \times z_2), z_1) = 0$.
- (ii) $-(z_1 \times (\nabla \times z_2), w) = ([z_1 \cdot \nabla]z_2, w) + ((\nabla z_1)^T \cdot z_2, w) - (\nabla(z_1 \cdot z_2), w)$, for all $z_1, z_2, w \in \mathbb{H}_0^1$. In particular, $\tilde{b}(z_1, z_2, w) = \langle [z_1 \cdot \nabla]z_2, w \rangle + \langle (\nabla z_1)^T \cdot z_2, w \rangle$ if $z_1, z_2 \in \mathbb{H}_0^1$ and $w \in \mathbb{V}$.
- (iii) $-(z_1 \times (\nabla \times z_2), w) = ([z_1 \cdot \nabla]z_2, w) - ([w \cdot \nabla]z_2, z_1)$, for all $z_1, z_2, w \in \mathbb{H}_0^1$.
- (iv) $\forall z_1, z_2, w \in \mathbb{H}_0^1$, $|(z_1 \times (\nabla \times z_2), w)| \leq \begin{cases} C_D \|z_1\|_{\mathbb{L}^4} \|\nabla z_2\|_{\mathbb{L}^2} \|w\|_{\mathbb{L}^2}^{1/2} \|\nabla w\|_{\mathbb{L}^2}^{1/2}, & \text{if } d = 2, \\ C_D \|z_1\|_{\mathbb{L}^2}^{1/2} \|\nabla z_1\|_{\mathbb{L}^2}^{1/2} \|\nabla z_2\|_{\mathbb{L}^2} \|\nabla w\|_{\mathbb{L}^2}, & \text{if } d = 3. \end{cases}$
- (v) For $d = 3$, $\|z_1 \times (\nabla \times z_2)\|_{\mathbb{H}^{-1}} \leq C_D \|z_1\|_{\mathbb{H}^1}^{1/2} \|z_1\|_{\mathbb{H}^2}^{1/2} \|z_2\|_{\mathbb{L}^2}$, for all $z_1 \in \mathbb{L}^2$ and $z_2 \in \mathbb{H}^2 \cap \mathbb{H}_0^1$.

Proof: Identities (i) and (ii) follow straightforwardly from the triple product property $(a \times b) \cdot c = -(c \times b) \cdot a$, and the equality $[z_1 \cdot \nabla]z_2 + (\nabla z_1)^T \cdot z_2 - \nabla(z_1 \cdot z_2) = -z_1 \times (\nabla \times z_2)$, respectively. To justify assertion (iii), we make use of (ii); more precisely we apply two integrations by parts to $((\nabla z_1)^T \cdot z_2, w) = \sum_{i,j=1}^d \int_D \partial_i z_1^j z_2^j w^i dx$ to obtain $-([w \cdot \nabla]z_2, z_1) + (\nabla(z_1 \cdot z_2), w)$. Plugging it back in (ii) yields the result. Estimate (iv) can therefore be concluded from assertion (iii) after employing the Hölder and Gagliardo-Nirenberg inequalities for both dimensions 2 and 3. To demonstrate (v), we shall use assertion (iii). Indeed, let $z_1, z_2, \varphi \in \mathbb{H}^2$ such that $\|\varphi\|_{\mathbb{H}^1} \leq 1$. Thus, $\langle z_1 \times (\nabla \times z_2), \varphi \rangle = ([\varphi \cdot \nabla]z_2, z_1) - ([z_1 \cdot \nabla]z_2, \varphi) =: I - J$. Performing an integration by parts on I yields $I = -([\varphi \cdot \nabla]z_1, z_2) - (z_1 \operatorname{div}(\varphi), z_2)$. Therefore, by the Hölder, Agmon and Gagliardo-Nirenberg inequalities in \mathbb{R}^3 , we get

$$|I| \leq \|\varphi\|_{\mathbb{L}^6} \|\nabla z_1\|_{\mathbb{L}^3} \|z_2\|_{\mathbb{L}^2} + \|z_1\|_{\mathbb{L}^\infty} \|\operatorname{div} \varphi\|_{\mathbb{L}^2} \|z_2\|_{\mathbb{L}^2} \leq C_D \|z_1\|_{\mathbb{H}^1}^{1/2} \|z_1\|_{\mathbb{H}^2}^{1/2} \|z_2\|_{\mathbb{L}^2}.$$

Integrating by parts J , it follows $J = -([z_1 \cdot \nabla]\varphi, z_2) - (z_2 \operatorname{div} z_1, \varphi)$. Applying once again the same inequalities that were employed to illustrate the bound of I , we obtain

$$|J| \leq \|z_1\|_{\mathbb{L}^\infty} \|\nabla \varphi\|_{\mathbb{L}^2} \|z_2\|_{\mathbb{L}^2} + \|z_2\|_{\mathbb{L}^2} \|\operatorname{div} z_1\|_{\mathbb{L}^3} \|\varphi\|_{\mathbb{L}^6} \leq C_D \|z_1\|_{\mathbb{H}^1}^{1/2} \|z_1\|_{\mathbb{H}^2}^{1/2} \|z_2\|_{\mathbb{L}^2}.$$

Putting it all together and arguing by density complete the proof. \square

It is well-known that finite element methods based on C^1 piecewise polynomials are not easily implementable. Therefore, our fourth-order partial differential equation (1.1) must undergo a modification so that it turns into a second-order problem. To this end, we shall propose a differential filter that deals with a Stokes problem. Such an idea emerges from [18] within a slight adjustment for the sake of fitting the current framework.

Definition 2.1 (Continuous differential filter)

Given a vector field $v \in \mathbb{L}^2$, its continuous differential filter, denoted \bar{u} , is part of the unique solution $(\bar{u}, \bar{p}) \in \mathbb{V} \times L_0^2(D)$ to

$$\begin{cases} -\alpha^2 \Delta \bar{u} + \bar{u} + \nabla \bar{p} = v, & \text{in } D, \\ \operatorname{div}(\bar{u}) = 0, & \text{in } D, \\ \bar{u} = 0, & \text{on } \partial D. \end{cases} \quad (2.6)$$

Note that the differential filter of a function v is usually denoted by \bar{v} . Nevertheless, the employed notation herein will be \bar{u} to obtain a clear vision of the relationship between the differential filter and equations (1.1). For a given $v \in \mathbb{L}^2$, problem (2.6) yields a unique $\bar{u} \in \mathbb{H}^2 \cap \mathbb{V}$ provided that $D \subset \mathbb{R}^d$, is a bounded convex two-dimensional polygonal (three-dimensional polyhedral) domain. Moreover, the solution \bar{u} satisfies $\|\bar{u}\|_{\mathbb{H}^2} \leq C_D \alpha^{-2} \|v\|_{\mathbb{L}^2}$. The former and the latter properties are provided in [23, Subsection 8.2].

2.1 Definition of solutions

Relying on paper [10], a solution to equations (1.1) can be defined as follows:

Definition 2.2 Let $T > 0$ and assume that (S_1) - (S_3) are valid. A \mathbb{V} -valued stochastic process \bar{u} is said to be a variational solution to problem (1.1) if it belongs to $M_{\mathbb{F}_t}^2(0, T; D(A)) \cap L^2(\Omega; L^\infty(0, T; \mathbb{V}))$, is weakly continuous with values in \mathbb{V} , and it satisfies \mathbb{P} -a.s. and for all $t \in [0, T]$ the following:

$$\begin{aligned} & (\bar{u}(t), \phi) + \alpha^2 (\nabla \bar{u}(t), \nabla \phi) + \nu \int_0^t (\bar{u}(s) + \alpha^2 A \bar{u}(s), A \phi) ds \\ & + \int_0^t \tilde{b}(\bar{u}(s), \bar{u}(s) - \alpha^2 \Delta \bar{u}(s), \phi) ds = (\bar{u}_0, \phi) + \alpha^2 (\nabla \bar{u}_0, \nabla \phi) \\ & + \int_0^t \langle f(s, \bar{u}(s)), \phi \rangle ds + \left(\int_0^t g(s, \bar{u}(s)) dW(s), \phi \right), \quad \forall \phi \in D(A). \end{aligned} \quad (2.7)$$

If \bar{u} is a solution to problem (1.1) in the sense of Definition 2.2, then it has the sufficient regularity to define the right-hand side v of system (2.6). Therefore, considering $v = v(t)$ as in problem (2.6) grants a new (equivalent) formula for equation (2.7), namely for all $t \in [0, T]$, it holds \mathbb{P} -almost surely

$$\begin{aligned} & (v(t), \phi) + \nu \int_0^t (\nabla v(s), \nabla \phi) ds + \int_0^t \tilde{b}(\bar{u}(s), v(s), \phi) ds = (v_0, \phi) \\ & + \int_0^t \langle f(s, \bar{u}(s)), \phi \rangle ds + \left(\int_0^t g(s, \bar{u}(s)) dW(s), \phi \right), \quad \forall \phi \in \mathbb{V}, \end{aligned} \quad (2.8)$$

where $v_0 \in \mathbb{L}^2$ is given by equation (2.6) when $\bar{u} = \bar{u}_0$.

The trilinear term involving the pressure $\int_0^t \tilde{b}(\bar{u}(s), \nabla \bar{p}(s), \phi) ds$ does not appear in equation (2.8) because

$$\tilde{b}(\bar{u}, \nabla \bar{p}, \phi) = \sum_{i,j=1}^d \int_D \bar{u}^i \partial_i \partial_j \bar{p} \phi^j dx + \sum_{i,j=1}^d \int_D \partial_i \bar{u}^j \partial_j \bar{p} \phi^i dx.$$

The first term on the right-hand side turns into $-\int_D [\phi \cdot \nabla] \bar{u} \nabla \bar{p} dx$ after performing an integration by parts, and the second term can be rewritten as $\int_D [\phi \cdot \nabla] \bar{u} \nabla \bar{p} dx$. We highlight the absence of $(\nabla(\bar{u} \cdot \nabla \bar{p}), \phi)$ in the above calculation which results from Proposition 2.1-(ii). It is worth mentioning that (2.8), coupled with the weak formulation of (2.6), establishes a well-posed problem whose solution satisfies equations (1.1) in the sense of Definition 2.2.

According to Definition 2.1, it is straightforward to infer that assumption (S_3) can be substituted by

(S_3^{alt}) $v_0 \in L^{2p}(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{H})$, for some $p \in [1, +\infty)$.

Indeed, the \mathbb{V} regularity of \bar{u}_0 arises from the variational formulation study of system (2.6) which implies that $\|\bar{u}_0\|_\alpha \leq \|v_0\|_{\mathbb{L}^2}$.

Remark 2.2 Both functions v and \bar{u} of equation (2.8) must vanish at the boundary ∂D , which includes the condition $A\bar{u} = 0$ on ∂D . This goes back to the way in which the LANS- α model is constructed. For additional interpretations, the reader may refer to article [29].

Next, we give a definition of strong solutions to problem (1.2) in 2D.

Definition 2.3 Given $T > 0$, let assumptions (S_1), (S_2) and (S_3^{alt}) be fulfilled. An \mathbb{H} -valued stochastic process $v(t)$, $t \in [0, T]$ is said to be a strong solution to equations (1.2) if it belongs to $M_{\mathbb{F}_t}^2(0, T; \mathbb{V}) \cap L^2(\Omega; C([0, T]; \mathbb{H}))$ and it satisfies \mathbb{P} -a.s., for all $t \in [0, T]$:

$$\begin{aligned} (v(t), \varphi) + \nu \int_0^t (\nabla v(s), \nabla \varphi) ds + \int_0^t \langle [v(s) \cdot \nabla] v(s), \varphi \rangle ds &= (v_0, \varphi) \\ + \int_0^t \langle f(s, v(s)), \varphi \rangle ds + \left(\int_0^t g(s, v(s)) dW(s), \varphi \right), \quad \forall \varphi \in \mathbb{V}. \end{aligned} \quad (2.9)$$

2.2 Discretizations and algorithm

Time Discretization

Let $M \in \mathbb{N}^*$ be given, and $I_k = \{t_\ell\}_{\ell=0}^M$ be an equidistant partition of the interval $[0, T]$, where $t_0 := 0$, $t_M := T$ and $k := T/M$ is the time-step size. The nodes' equidistance is not mandatory in the sequel; it is imposed, however, for simplicity. One can generalize the presented method by associating a time-step k_m with each sub-interval $[t_{m-1}, t_m]$, for all $m \in \{1, \dots, M\}$.

Space discretization For simplicity's sake, we let \mathcal{T}_h be a quasi-uniform triangulation of the domain $D \subset \mathbb{R}^2$ into simplexes of maximal diameter $h > 0$, and $\bar{D} = \bigcup_{K \in \mathcal{T}_h} \bar{K}$. The space of polynomial vector fields on an arbitrary set O with degree less than or equal to $n \in \mathbb{N}$ is denoted by $\mathbb{P}_n(O) := (P_n(O))^2$. For $n_1, n_2 \in \mathbb{N} \setminus \{0\}$, we let

$$\begin{aligned} \mathbb{H}_h &:= \left\{ z_h \in \mathbb{H}_0^1 \cap [C^0(\bar{D})]^2 \mid z_h|_K \in \mathbb{P}_{n_1}(K), \quad \forall K \in \mathcal{T}_h \right\}, \\ L_h &:= \left\{ q_h \in L_0^2(D) \mid q_h|_K \in P_{n_2}(K), \quad \forall K \in \mathcal{T}_h \right\}, \\ \mathbb{V}_h &:= \left\{ z_h \in \mathbb{H}_h \mid (\operatorname{div}(z_h), q_h) = 0, \quad \forall q_h \in L_h \right\}, \end{aligned}$$

be the finite element function spaces. For fixed $n_1, n_2 \in \mathbb{N} \setminus \{0\}$, we assume that (\mathbb{H}_h, L_h) satisfies the discrete inf-sup condition; namely there is a constant $\beta > 0$ independent of the mesh size h such that

$$\sup_{z_h \in \mathbb{H}_h \setminus \{0\}} \frac{(\operatorname{div}(z_h), q_h)}{\|\nabla z_h\|_{\mathbb{L}^2}} \geq \beta \|q_h\|_{L^2}, \quad \forall q_h \in L_h. \quad (2.10)$$

Examples of finite-dimensional spaces that satisfy inequality (2.10) are provided in [1, 19, 24].

Given $z \in \mathbb{L}^2$, we denote by $\Pi_h : \mathbb{L}^2 \rightarrow \mathbb{V}_h$ the \mathbb{L}^2 -orthogonal projections, defined as the unique solution of the identity

$$(z - \Pi_h z, \varphi_h) = 0, \quad \forall \varphi_h \in \mathbb{V}_h. \quad (2.11)$$

For $z \in \mathbb{H}_0^1$, $\Delta^h : \mathbb{H}_0^1 \rightarrow \mathbb{V}_h$ denotes the discrete Laplace operator, defined as the unique solution of

$$\left(\Delta^h z, \varphi_h \right) = -(\nabla z, \nabla \varphi_h), \quad \forall \varphi_h \in \mathbb{V}_h. \quad (2.12)$$

Operator Δ^h can also be seen as a discrete Stokes operator since its image is \mathbb{V}_h . We point out that one must not blindly interchange \mathbb{H}_h and \mathbb{V}_h in the definition of Δ^h because a discrete pressure shall be eliminated from Algorithm 1 later on. This also appears in Lemma 2.1.

Estimate (2.13) and the inverse inequality (2.14) below need to be satisfied by the recently defined approximate function spaces. Let \mathbb{S}_h be a finite-dimensional subspace of \mathbb{H}_0^1 equipped with an \mathbb{L}^2 -projector $\Pi_{\mathbb{S}_h} : \mathbb{L}^2 \rightarrow \mathbb{S}_h$, satisfying the following property:

Let $n \in \mathbb{N}$ be the polynomials' degree in \mathbb{S}_h . For all $s \in [2, n + 1]$, and $z \in \mathbb{H}^s \cap \mathbb{H}_0^1$, there is a positive constant C independent of h such that

$$\sum_{j=0}^1 h^j \left\| D^j (z - \Pi_{\mathbb{S}_h} z) \right\|_{\mathbb{L}^2} \leq Ch^s \|z\|_{\mathbb{H}^s}. \quad (2.13)$$

Furthermore, assume that \mathbb{S}_h fulfills the following inverse inequality:

For $\ell \in \mathbb{N}$, $1 \leq p, q \leq +\infty$ and $0 \leq m \leq \ell$, there exists a constant C independent of h such that

$$\|z_h\|_{\mathbb{W}^{\ell,p}} \leq Ch^{m-\ell+2\min(\frac{1}{p}-\frac{1}{q},0)} \|z_h\|_{\mathbb{W}^{m,q}}, \quad \forall z_h \in \mathbb{S}_h. \quad (2.14)$$

Provided the triangulation of the domain D is quasi-uniform, one can easily check that the space \mathbb{H}_h satisfies both estimates (2.13) and (2.14). The reader may refer to [6] for adequate proofs. Subsequently, we take $\mathbb{S}_h = \mathbb{H}_h$.

Identity (2.12) together with the inverse inequality (2.14) ensure the following estimate:

$$\left\| \Delta^h z_h \right\|_{\mathbb{L}^2} \leq Ch^{-1} \|\nabla z_h\|_{\mathbb{L}^2}, \quad \forall z_h \in \mathbb{V}_h. \quad (2.15)$$

The discrete differential filter is somewhat defined as its continuous counterpart, but this time by involving the weak formulation of problem (2.6).

Definition 2.4 (Discrete differential filter)

Let v be the vector field of Definition 2.1. Its discrete differential filter, denoted by $\bar{u}_h \in \mathbb{V}_h$, is given by the unique solution of

$$\alpha^2 (\nabla \bar{u}_h, \nabla \varphi_h) + (\bar{u}_h, \varphi_h) = (v, \varphi_h), \quad \forall \varphi_h \in \mathbb{V}_h.$$

Additional information are stated in article [27, Section 4]. We list some of its properties in the following lemma.

Lemma 2.1 *Let $v = v_h \in \mathbb{V}_h$ and $\bar{u}_h \in \mathbb{V}_h$ be its discrete differential filter. Then,*

(i) $v_h = \bar{u}_h - \alpha^2 \Delta^h \bar{u}_h$ and $\nabla v_h = \nabla \bar{u}_h - \alpha^2 \nabla \Delta^h \bar{u}_h$ a.e. in D .

(ii) $(\nabla v_h, \nabla \bar{u}_h) = \|\nabla \bar{u}_h\|_{\mathbb{L}^2}^2 + \alpha^2 \left\| \Delta^h \bar{u}_h \right\|_{\mathbb{L}^2}^2$.

Proof: Assertions (i) and (ii) are covered in [14, Lemma 2.1]. □

Before exhibiting the algorithm, we will define new notations for the approximate functions. The subscript h of the utilized test functions will be dropped throughout the rest of this paper for the sake of clarity. For $t \in [0, T]$, we set $V(t) := v_h(t)$ for $v_h \in \mathbb{V}_h$, and denote by $U(t)$ its discrete differential filter, i.e. $U(t) := \bar{u}_h(t)$. Besides, let $\Pi(t) := p_h(t)$ and $\tilde{\Pi}(t) := \bar{p}_h(t)$ be the (space) approximate pressures. We point out that Algorithm 1 is derived from equation (2.8), which contains both terms \bar{u} and v .

Algorithm 1

Let $U^0 \in \mathbb{H}_h$ be given.

Project v_0 through $(V^0, \psi) = (U^0, \psi) + \alpha^2 (\nabla U^0, \nabla \psi)$, for all $\psi \in \mathbb{H}_h$.

If for $m \in \{1, \dots, M\}$, iterates U^{m-1} , V^{m-1} , Π^{m-1} , $\tilde{\Pi}^{m-1}$ are known, find a 4-tuple stochastic process

$(U^m, V^m, \Pi^m, \tilde{\Pi}^m) \in \mathbb{H}_h \times \mathbb{H}_h \times L_h \times L_h$ such that for all $(\varphi, \psi, \Lambda_1, \Lambda_2) \in \mathbb{H}_h \times \mathbb{H}_h \times L_h \times L_h$, there holds \mathbb{P} -a.s.

$$\begin{cases} (V^m - V^{m-1}, \varphi) + k\nu(\nabla V^m, \nabla \varphi) - k(U^m \times (\nabla \times V^m), \varphi) - k(\Pi^m, \operatorname{div}(\varphi)) \\ \quad = k\langle f(t_{m-1}, U^{m-1}), \varphi \rangle + (g(t_{m-1}, U^{m-1})\Delta_m W, \varphi), \\ (V^m, \psi) = (U^m, \psi) + \alpha^2(\nabla U^m, \nabla \psi) - (\tilde{\Pi}^m, \operatorname{div}(\psi)), \\ (\operatorname{div}(U^m), \Lambda_1) = (\operatorname{div}(V^m), \Lambda_2) = 0, \end{cases}$$

where $\Delta_m W = W(t_m) - W(t_{m-1})$ for all $m \in \{1, \dots, M\}$.

The choice of the starting point U^0 of Algorithm 1 is provided in Section 3. Although the trilinear term's second variable could have been chosen to be explicit i.e. $\nabla \times V^{m-1}$, this choice may force an extra smoothness assumption on v_0 when illustrating the a priori estimates of $\{V^m\}_{m=1}^M$; namely one needs v_0 to be in \mathbb{H}_0^1 , which is not really practical as v_0 is just a transfer tool that should not play an important role within the study. For each $m \in \{0, \dots, M\}$, we may conclude from the second and third equations of Algorithm 1 along with Definition 2.4 two facts:

- (i) U^m is the discrete differential filter of V^m and thereby, all the associated properties are valid.
- (ii) The Algorithm's starting point U^0 could be exchanged with V^0 .

3 Main results

In the light of the preceding preliminaries, we are now able to state the main results of this paper. Theorem 3.1 concerns the stochastic LANS- α model and Theorem 3.2 is devoted to the stochastic Navier-Stokes equations.

Theorem 3.1 *Let $T > 0$, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ be a filtered probability space, and $D \subset \mathbb{R}^3$ be a bounded convex polyhedral domain. Suppose that assumptions (S_1) , (S_2) and (S_3^{alt}) are fulfilled. Given an $M \in \mathbb{N} \setminus \{0\}$, define the discretization step size $k := \frac{T}{M}$ such that $k \in (0, 1)$ and I_k makes up a uniform partition of the time interval $[0, T]$. Let $h \in (0, 1)$ be the space discretization step size such that the triangulation \mathcal{T}_h is quasi-uniform, and let $\alpha > 0$ be a fixed non-vanishing parameter that does not depend on k and h . Define the finite element triple $(\mathbb{H}_h, L_h, U_h^0)$ such that the pair (\mathbb{H}_h, L_h) satisfies the discrete LBB condition (2.10) and the initial datum U_h^0 ($:= U^0$) belongs to \mathbb{H}_h . The following results are valid:*

- For a given $(k, h, \alpha) \in (0, 1) \times (0, 1) \times (0, +\infty)$, there is a solution $\left\{ (U^m, V^m, \Pi^m, \tilde{\Pi}^m) \right\}_{m=1}^M$ to Algorithm 1 that fulfills Lemmas 4.1 and 4.3.
- For a family $\{k, h\}$ of discretization parameters satisfying $k, h \rightarrow 0$ instantly such that $U^0 \rightarrow \bar{u}_0$ as $h \rightarrow 0$ in the space $L^4(\Omega; \mathbb{H}^1)$ (particularly $\|U^0\|_{L^4(\Omega; \mathbb{H}^1)}$ is uniformly bounded in h), the solution $\left\{ (U^m, V^m, \Pi^m, \tilde{\Pi}^m) \right\}_{m=1}^M$ of Algorithm 1 converges toward the unique solution of the Lagrangian averaged Navier-Stokes equations (1.1) in the sense of Definition 2.2.

Theorem 3.2 *Let $T > 0$, $D \subset \mathbb{R}^2$ be a bounded convex polygonal domain and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ be a filtered probability space. Assume assumptions (S_1) , (S_2) and (S_3^{alt}) . Given an integer $M \in \mathbb{N} \setminus \{0\}$, define the time discretization step size $k := \frac{T}{M}$ such that $0 < k < 1$, and I_k forms a uniform partition of the time interval $[0, T]$. Let $h \in (0, 1)$ be the space discretization step size such that the triangulation \mathcal{T}_h is quasi-uniform. Define the finite element triple $(\mathbb{H}_h, L_h, U_h^0)$ such that the pair (\mathbb{H}_h, L_h) satisfies the LBB condition (2.10) and U_h^0 ($:= U^0$) belongs to \mathbb{H}_h . The following results hold:*

- For a given $(k, h, \alpha) \in (0, 1)^3$, there exists a solution $\left\{ (U^m, V^m, \Pi^m, \tilde{\Pi}^m) \right\}_{m=1}^M$ to Algorithm 1 that satisfies Lemma 4.1.

- For a family $\{k, h, \alpha\}$ of parameters satisfying $k, h, \alpha \rightarrow 0$ instantly such that the following supplementary assumptions are imposed:

- ★ the spatial scale α lies in the range $[\alpha_{min}h, \alpha_{max}h]$ for some $\alpha_{min}, \alpha_{max} > 0$ independent of k and h ,

- ★ $U^0 \rightarrow \bar{u}_0$ in the space $L^4(\Omega; \mathbb{H}^1)$ (particularly, $\|U^0\|_{L^4(\Omega; \mathbb{H}^1)}$ is uniformly bounded in h),

the solution $\left\{ \left(U^m, V^m, \Pi^m, \tilde{\Pi}^m \right) \right\}_{m=1}^M$ of Algorithm 1 fulfills Lemma 4.5 and it converges toward the unique solution of the stochastic Navier-Stokes equations (1.2) in the sense of Definition 2.3.

The initial datum U^0 of Algorithm 1 can be prescribed through the Ritz operator which grants a uniform bound in \mathbb{H}^1 with respect to h . That is, we evoke the Ritz operator \mathcal{R}_h which is stable in \mathbb{H}^1 i.e. there is a positive non-decreasing function ζ , uniform in h such that $\|\mathcal{R}_h v\|_{\mathbb{H}^1} \leq \zeta \|v\|_{\mathbb{H}^1}$ for all $v \in \mathbb{H}^1$, such that for a given $v \in \mathbb{H}^1$, $\mathcal{R}_h: \mathbb{H}^1 \rightarrow \mathbb{V}_h$ is defined as the unique solution of

$$(\nabla \mathcal{R}_h v, \nabla v_h) = (\nabla v, \nabla v_h), \quad \forall v_h \in \mathbb{V}_h.$$

Therefore, we define U^0 by $U^0 = \mathcal{R}_h \bar{u}_0$ where \bar{u}_0 is the initial datum of equations (1.1), which also represents the continuous differential filter of v_0 . Besides, the second equation in Algorithm 1 together with Lemma 2.1-(i), inequality (2.15), $\alpha \leq 1$ and $\alpha \leq \alpha_{max}h$ lead to $\|V^0\|_{\mathbb{L}^2} \leq \sqrt{2} \max(1, C\alpha_{max}) \|U^0\|_{\mathbb{H}^1}$ which means that $\|V^0\|_{\mathbb{L}^2}$ is also uniformly bounded in h .

4 Solvability, stability and a priori estimates

Notice that the system of equations proposed in Algorithm 1 can be reformulated after taking the test functions φ and ψ in \mathbb{V}_h :

$$\begin{cases} (V^m - V^{m-1}, \varphi) + k\nu(\nabla V^m, \nabla \varphi) - k(U^m \times (\nabla \times V^m), \varphi) \\ = k\langle f(t_{m-1}, U^{m-1}), \varphi \rangle + (g(t_{m-1}, U^{m-1})\Delta_m W, \varphi), \quad \forall \varphi \in \mathbb{V}_h. \\ (V^m, \psi) = (U^m, \psi) + \alpha^2(\nabla U^m, \nabla \psi), \quad \forall \psi \in \mathbb{V}_h. \end{cases} \quad (4.1)$$

In the lemma down below, we illustrate the solvability of Algorithm 1, the iterates' measurability, and some a priori estimates whose role is to afford the proposed numerical scheme with stability.

Lemma 4.1 *Assume that assumptions (S_1) - (S_3) are valid and let $p \in [2, \infty) \cap \mathbb{N}$. Then, there exists a $\mathbb{V}_h \times \mathbb{V}_h \times L_h \times L_h$ -valued sequence of random variables $\{(U^m, V^m, \Pi^m, \tilde{\Pi}^m)\}_{m=1}^M$ that solves \mathbb{P} -a.s. Algorithm 1, and fulfills the following assertions:*

(i) for any $m \in \{1, \dots, M\}$, the maps $U^m, V^m: \Omega \rightarrow \mathbb{H}_h$ are \mathcal{F}_{t_m} -measurable.

(ii)
$$\mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_\alpha^2 + \frac{k\nu}{2} \sum_{m=1}^M \left(\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right) + \frac{1}{4} \sum_{m=1}^M \|U^m - U^{m-1}\|_\alpha^2 \right] \leq C_T,$$

(iii)
$$\mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_\alpha^{2p} + \sum_{m=1}^M \|U^m\|_\alpha^{2p-1} \|U^m - U^{m-1}\|_\alpha^2 + k\nu \sum_{m=1}^M \|U^m\|_\alpha^{2p-1} \left(\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right) \right] \leq C_{T,p},$$

where $C_{T,p} > 0$ depends on $\|U^0\|_{L^{2p}(\Omega; \mathbb{H}^1)}$, T , $\{K_i\}_{i=1}^4$, $Tr(Q)$, ν , and D , and is independent of α , k and h . Note that $C_T := C_{T,1}$.

Proof: *Solvability*

To prove the Algorithm's solvability, we will follow a technique similar to that in [2, Lemma 4.1] while relying on equations (4.1). Since $V^m \in \mathbb{V}_h$ for all $m \in \{1, \dots, M\}$ then, by Lemma 2.1-(i), we get $V^m = U^m - \alpha^2 \Delta^h U^m$, \mathbb{P} -a.s. and a.e. in D . This means that the existence of U^m implies that of V^m . Assume that, for some $2 \leq \ell \leq M$ and for almost every $\omega \in \Omega$, a sequence $\{(U^m(\omega), V^m(\omega))\}_{m=1}^{\ell-1}$ has been found by induction. For $\omega \in \Omega$, define \mathbb{P} -a.s. the mapping $\mathcal{F}_{\ell-1}^\omega : \mathbb{V}_h \rightarrow \mathbb{V}'_h$ by

$$\begin{aligned} \mathcal{F}_{\ell-1}^\omega(\varphi) := & \varphi - \alpha^2 \Delta^h \varphi - V^{\ell-1}(\omega) - k\nu \Delta \left(\varphi - \alpha^2 \Delta^h \varphi \right) - k\varphi \times \left(\nabla \times (\varphi - \alpha^2 \Delta^h \varphi) \right) \\ & - kf(t_{\ell-1}, U^{\ell-1}(\omega)) - g(t_{\ell-1}, U^{\ell-1}(\omega)) \Delta_\ell W(\omega), \end{aligned}$$

for all $\varphi \in \mathbb{V}_h$. The continuity of $\mathcal{F}_{\ell-1}^\omega$ can be shown by a straightforward argument. Since, \mathbb{V}_h equipped with the inner product (\cdot, \cdot) , is a Hilbert space, then by Riesz representation theorem, functional $\mathcal{F}_{\ell-1}^\omega$ can be defined through the L^2 -inner product, namely for $\varphi \in \mathbb{V}_h$,

$$(\mathcal{F}_{\ell-1}^\omega(\varphi))(\psi) = (\mathcal{F}_{\ell-1}^\omega(\varphi), \psi), \quad \text{for all } \psi \in \mathbb{V}_h.$$

Therefore, for $\psi = \varphi \in \mathbb{V}_h$ and by Proposition 2.1-(i), the discrete Laplace operator (2.12), assumption (S_2) , the Cauchy-Schwarz and Young inequalities,

$$\begin{aligned} (\mathcal{F}_{\ell-1}^\omega(\varphi), \varphi) & \geq \|\varphi\|_{\mathbb{L}^2}^2 + (\alpha^2 + k\nu) \|\nabla \varphi\|_{\mathbb{L}^2}^2 - \|V^{\ell-1}(\omega)\|_{\mathbb{L}^2} \|\varphi\|_{\mathbb{L}^2} + k\nu \alpha^2 \|\Delta^h \varphi\|_{\mathbb{L}^2}^2 \\ & - k \left(K_3 + K_4 \|U^{\ell-1}(\omega)\|_\alpha \right) \|\varphi\|_{\mathbb{H}^1} - \left(K_1 + K_2 \|U^{\ell-1}(\omega)\|_\alpha \right) \|\Delta_\ell W(\omega)\|_K \|\varphi\|_{\mathbb{L}^2} \\ & \geq \frac{1}{2} \|\varphi\|_{\mathbb{L}^2}^2 + \left(\alpha^2 + \frac{k\nu}{2} \right) \|\nabla \varphi\|_{\mathbb{L}^2}^2 - \|V^{\ell-1}(\omega)\|_{\mathbb{L}^2} \|\varphi\|_{\mathbb{L}^2} - \frac{kC_D^2}{2\nu} \left(K_3 + K_4 \|U^{\ell-1}(\omega)\|_\alpha \right)^2 \\ & - \left(K_1 + K_2 \|U^{\ell-1}(\omega)\|_\alpha \right)^2 \|\Delta_\ell W(\omega)\|_K^2 \geq \frac{1}{2} \|\varphi\|_{\mathbb{L}^2}^2 - L_{\ell-1}(\omega), \end{aligned}$$

where

$$L_{\ell-1} := 2K_1^2 \|\Delta_\ell W\|_K^2 + \frac{kC_D^2 K_3^2}{\nu} + \|V^{\ell-1}\|_{\mathbb{L}^2}^2 + \left(\frac{kC_D^2 K_4^2}{\nu} + 2K_2^2 \|\Delta_\ell W\|_K^2 \right) \|U^{\ell-1}\|_\alpha^2.$$

By inequality (2.2) and the induction's hypothesis, it holds \mathbb{P} -a.s. $L_{\ell-1}(\omega) < +\infty$. Therefore, taking $\varphi \in \mathbb{V}_h$ such that $\|\varphi\|_{\mathbb{L}^2} = \sqrt{2L_{\ell-1}(\omega)}$ yields $(\mathcal{F}_{\ell-1}^\omega(\varphi), \varphi) \geq 0$. Subsequently, Brouwer's fixed point theorem (see [20, Corollary 1.1, p. 279]) ensures the existence (but not uniqueness) of a function $\phi = \phi(\omega) \in \mathbb{V}_h$ such that $\mathcal{F}_{\ell-1}^\omega(\phi) = 0$. Hence, $(U^\ell, V^\ell) \in \mathbb{V}_h \times \mathbb{V}_h$ exists \mathbb{P} -a.s. . The discrete LBB condition (2.10) yields the existence of an $L_h \times L_h$ -valued process $\{(\Pi^m, \tilde{\Pi}^m)\}_{m=1}^M$ satisfying Algorithm 1.

Measurability

After proving the algorithm's solvability through the functional $\mathcal{F}_{\ell-1}^\omega$, the measurability of iterates U^m , $m \in \{1, \dots, M\}$ follows by induction (see [2, Lemma 4.1]). Moreover, by Lemma 2.1-(i), one infers the measurability of $\{V^m\}_{m=1}^M$.

A priori energy estimate

Let us denote by $\|\cdot\|_{h,\alpha}^2$ the quantity $\|\nabla \cdot\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h \cdot\|_{\mathbb{L}^2}^2$. In equation (4.1), we take $\varphi = \psi = U^m$ and employ identity (2.4), Lemma 2.1-(ii) together with Proposition 2.1-(i):

$$\begin{aligned} \frac{1}{2} \left(\|U^m\|_\alpha^2 - \|U^{m-1}\|_\alpha^2 + \|U^m - U^{m-1}\|_\alpha^2 \right) + k\nu \|U^m\|_{h,\alpha}^2 & = k \langle f(t_{m-1}, U^{m-1}), U^m \rangle \\ & + (g(t_{m-1}, U^{m-1}) \Delta_m W, U^m - U^{m-1}) + (g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1}). \end{aligned} \quad (4.2)$$

After employing the Cauchy-Schwarz and Young inequalities along with assumption (S_2) , we take the sum over m from 1 to M :

$$\begin{aligned}
 & \frac{1}{2} \|U^M\|_\alpha^2 - \frac{1}{2} \|U^0\|_\alpha^2 + \frac{1}{4} \sum_{m=1}^M \|U^m - U^{m-1}\|_\alpha^2 + \frac{k\nu}{2} \sum_{m=1}^M \|U^m\|_{h,\alpha}^2 \\
 & \leq \frac{C_D^2 T K_3^2}{\nu} + \frac{C_D^2 K_4^2}{\nu} k \sum_{m=1}^M \|U^{m-1}\|_\alpha^2 + \sum_{m=1}^M \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 \\
 & \quad + \sum_{m=1}^M (g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1}).
 \end{aligned} \tag{4.3}$$

Due to the measurability of U^m , the last term on the right-hand side vanishes when taking its expectation. The penultimate term is controlled as follows:

$$\begin{aligned}
 & \mathbb{E} \left[\|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \right] \\
 & = \mathbb{E} \left[\|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \right] \mathbb{E} \left[\|\Delta_m W\|_K^2 \middle| \mathcal{F}_{t_{m-1}} \right] \\
 & \leq 2Tr(Q) K_1^2 k + 2K_2^2 k Tr(Q) \mathbb{E} [\|U^{m-1}\|_\alpha^2],
 \end{aligned} \tag{4.4}$$

thanks to the tower property of the conditional expectation, the increments independence of the Wiener process, property (2.2), and assumption (S_2) . Plugging estimate (4.4) in equation (4.3) returns

$$\begin{aligned}
 & \frac{1}{2} \mathbb{E} [\|U^M\|_\alpha^2] + \frac{1}{4} \sum_{m=1}^M \mathbb{E} [\|U^m - U^{m-1}\|_\alpha^2] + \frac{k\nu}{2} \sum_{m=1}^M \mathbb{E} [\|U^m\|_{h,\alpha}^2] \leq \frac{1}{2} \mathbb{E} [\|U^0\|_\alpha^2] \\
 & \quad + \left(\frac{C_D^2 K_3^2}{\nu} + 2Tr(Q) K_1^2 \right) T + \left(\frac{C_D^2 K_4^2}{\nu} + 2K_2^2 Tr(Q) \right) k \sum_{m=0}^{M-1} \mathbb{E} [\|U^m\|_\alpha^2].
 \end{aligned} \tag{4.5}$$

Now, we employ the discrete Grönwall inequality (see for instance [35, Lemma 10.5]) in order to prove the sought estimate. We replace M in equation (4.5) by any other index $\ell \geq 1$. We get

$$\mathbb{E} [\|U^\ell\|_\alpha^2] \leq \left[\mathbb{E} [\|U^0\|_{\mathbb{H}^1}^2] + 2 \left(\frac{C_D^2 K_3^2}{\nu} + 2Tr(Q) K_1^2 \right) T \right] e^{T \left(\frac{C_D^2 K_4^2}{\nu} + 2K_2^2 Tr(Q) \right)} =: K_T$$

for all $\ell \in \{1, \dots, M\}$, where $\|U^0\|_\alpha \leq \|U^0\|_{\mathbb{H}^1}$ thanks to (2.3). Consequently,

$$\max_{1 \leq m \leq M} \mathbb{E} [\|U^m\|_\alpha^2] \leq K_T. \tag{4.6}$$

By virtue of estimate (4.5) and the discrete Grönwall lemma, one also obtains the following two estimates:

$$\frac{k\nu}{2} \sum_{m=1}^M \mathbb{E} [\|U^m\|_{h,\alpha}^2] \leq K_T \quad \text{and} \quad \frac{1}{4} \sum_{m=1}^M \mathbb{E} [\|U^m - U^{m-1}\|_\alpha^2] \leq K_T.$$

We still need to prove $\mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_\alpha^2 \right] \leq C_T$, for a certain positive constant C_T independent of α , k and h . To this end, we make use of estimate (4.3), but this time by summing from $m = 1$ to $m = \ell$ where $\ell \geq 1$ is an integer. Then, we take the maximum over ℓ and apply the mathematical expectation on both sides to get

$$\begin{aligned}
 & \frac{1}{2} \mathbb{E} \left[\max_{1 \leq \ell \leq M} \|U^\ell\|_\alpha^2 \right] \leq \frac{1}{2} \mathbb{E} [\|U^0\|_\alpha^2] + \frac{C_D^2 T K_3^2}{\nu} + \frac{C_D^2 K_4^2}{\nu} k \sum_{m=1}^M \mathbb{E} [\|U^{m-1}\|_\alpha^2] \\
 & \quad + \sum_{m=1}^M \mathbb{E} [\|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2] + \mathbb{E} \left[\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} (g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1}) \right].
 \end{aligned} \tag{4.7}$$

To bound the last term on the right-hand side, we use assumption (S_2) , the Burkholder-Davis-Gundy and Young inequalities, after considering the sum as the stochastic integral of a piecewise constant integrand:

$$\begin{aligned}
& \mathbb{E} \left[\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} (g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1}) \right] \\
& \lesssim \mathbb{E} \left[\left(k \sum_{m=1}^M \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|U^{m-1}\|_{\mathbb{L}^2}^2 \right)^{1/2} \right] \\
& \leq \frac{1}{4} \mathbb{E} [\|U^0\|_{\mathbb{L}^2}^2] + 2K_1^2 T + \mathbb{E} \left[\frac{1}{4} \max_{1 \leq \ell \leq M} \|U^\ell\|_{\mathbb{L}^2}^2 + 2K_2^2 k \sum_{m=1}^M \|U^{m-1}\|_{\alpha}^2 \right].
\end{aligned} \tag{4.8}$$

Returning to estimate (4.7), we avail ourselves of (4.4), (4.6) and (4.8) to conclude

$$\mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_{\alpha}^2 \right] \leq C_T,$$

where $C_T > 0$ depends only on the parameters of K_T .

Bounds for higher velocity moments

We will demonstrate below the case $p = 2$. The reader may refer to [7] for additional hints. We start by multiplying equation (4.2) by the norm $\|U^m\|_{\alpha}^2$.

$$\begin{aligned}
& \frac{1}{2} \|U^m\|_{\alpha}^4 - \frac{1}{2} \|U^{m-1}\|_{\alpha}^2 \|U^m\|_{\alpha}^2 + \frac{1}{2} \|U^m - U^{m-1}\|_{\alpha}^2 \|U^m\|_{\alpha}^2 + k\nu \|U^m\|_{h,\alpha}^2 \|U^m\|_{\alpha}^2 \\
& = k \langle f(t_{m-1}, U^{m-1}), U^m \rangle \|U^m\|_{\alpha}^2 + (g(t_{m-1}, U^{m-1}) \Delta_m W, U^m - U^{m-1}) \|U^m\|_{\alpha}^2 \\
& \quad + (g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1}) \|U^m\|_{\alpha}^2 = I + II + III.
\end{aligned} \tag{4.9}$$

For I , we apply the norm equivalence (2.3), the Young inequality and estimate $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ for $p = 4$:

$$\begin{aligned}
I & \leq kC_D (K_3 + K_4 \|U^{m-1}\|_{\alpha}) \|\nabla U^m\|_{\mathbb{L}^2}^{\frac{3}{2}} \|U^m\|_{\alpha}^{\frac{3}{2}} \leq \frac{kC_D^4}{4\nu^3} (K_3 + K_4 \|U^{m-1}\|_{\alpha})^4 \\
& \quad + \frac{3k\nu}{4} \|U^m\|_{h,\alpha}^2 \|U^m\|_{\alpha}^2 \leq \frac{2kC_D^4 K_3^4}{\nu^3} + \frac{2kC_D^4 K_4^4}{\nu^3} \|U^{m-1}\|_{\alpha}^4 + \frac{3k\nu}{4} \|U^m\|_{h,\alpha}^2 \|U^m\|_{\alpha}^2.
\end{aligned}$$

For II ,

$$\begin{aligned}
II & \leq \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 (\|U^m\|_{\alpha}^2 - \|U^{m-1}\|_{\alpha}^2 + \|U^{m-1}\|_{\alpha}^2) \\
& \quad + \frac{1}{4} \|U^m - U^{m-1}\|_{\mathbb{L}^2}^2 \|U^m\|_{\alpha}^2 \\
& \leq \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \|U^{m-1}\|_{\alpha}^2 + \frac{1}{16} \left| \|U^m\|_{\alpha}^2 - \|U^{m-1}\|_{\alpha}^2 \right|^2 \\
& \quad + 4 \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^4 \|\Delta_m W\|_K^4 + \frac{1}{4} \|U^m - U^{m-1}\|_{\mathbb{L}^2}^2 \|U^m\|_{\alpha}^2.
\end{aligned}$$

For III ,

$$\begin{aligned}
III & := (g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1}) (\|U^m\|_{\alpha}^2 - \|U^{m-1}\|_{\alpha}^2 + \|U^{m-1}\|_{\alpha}^2) \\
& \leq (g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1}) \|U^{m-1}\|_{\alpha}^2 + \frac{1}{16} \left| \|U^m\|_{\alpha}^2 - \|U^{m-1}\|_{\alpha}^2 \right|^2 \\
& \quad + 4 \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \|U^{m-1}\|_{\alpha}^2.
\end{aligned}$$

Equation (4.9) becomes

$$\frac{1}{2} \|U^m\|_{\alpha}^4 - \frac{1}{2} \|U^{m-1}\|_{\alpha}^2 \|U^m\|_{\alpha}^2 + \frac{1}{4} \|U^m - U^{m-1}\|_{\alpha}^2 \|U^m\|_{\alpha}^2 + \frac{k\nu}{4} \|U^m\|_{h,\alpha}^2 \|U^m\|_{\alpha}^2$$

$$\begin{aligned}
 &\leq \frac{2kC_D^4 K_3^4}{\nu^3} + \frac{2kC_D^4 K_4^4}{\nu^3} \|U^{m-1}\|_\alpha^4 + \frac{1}{8} \left| \|U^m\|_\alpha^2 - \|U^{m-1}\|_\alpha^2 \right|^2 \\
 &\quad + (g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1}) \|U^{m-1}\|_\alpha^2 + 4 \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^4 \|\Delta_m W\|_K^4 \\
 &\quad + 5 \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \|U^{m-1}\|_\alpha^2.
 \end{aligned}$$

Note that $\|U^m\|_\alpha^4 - \|U^{m-1}\|_\alpha^4 \|U^m\|_\alpha^2 = \frac{1}{2} (\|U^m\|_\alpha^4 - \|U^{m-1}\|_\alpha^4 + \left| \|U^m\|_\alpha^2 - \|U^{m-1}\|_\alpha^2 \right|^2)$, therefore

$$\begin{aligned}
 &\frac{1}{4} \left(\|U^m\|_\alpha^4 - \|U^{m-1}\|_\alpha^4 + \frac{1}{2} \left| \|U^m\|_\alpha^2 - \|U^{m-1}\|_\alpha^2 \right|^2 + \|U^m - U^{m-1}\|_\alpha^2 \|U^m\|_\alpha^2 \right. \\
 &\quad \left. + k\nu \|U^m\|_{h,\alpha}^2 \|U^m\|_\alpha^2 \right) \leq \frac{2kC_D^4 K_3^4}{\nu^3} + \frac{2kC_D^4 K_4^4}{\nu^3} \|U^{m-1}\|_\alpha^4 \\
 &\quad + (g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1}) \|U^{m-1}\|_\alpha^2 + 4 \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^4 \|\Delta_m W\|_K^4 \\
 &\quad + 5 \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \|U^{m-1}\|_\alpha^2,
 \end{aligned} \tag{4.10}$$

Proceeding as (4.4), the penultimate term can be estimated as follows

$$\mathbb{E} \left[\|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^4 \|\Delta_m W\|_K^4 \right] \lesssim K_1^4 Tr(Q)^2 k^2 + K_2^4 Tr(Q)^2 k^2 \mathbb{E} [\|U^{m-1}\|_\alpha^4]. \tag{4.11}$$

Next, we bound the last term on the right-hand side of (4.10)

$$\begin{aligned}
 &\mathbb{E} \left[\|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \|U^{m-1}\|_\alpha^2 \right] \\
 &\quad \lesssim K_1^2 k Tr(Q) \mathbb{E} [\|U^{m-1}\|_\alpha^2] + K_2^2 Tr(Q) k \mathbb{E} [\|U^{m-1}\|_\alpha^4].
 \end{aligned} \tag{4.12}$$

The third term on the right-hand side of (4.10) vanishes after taking its expectation, thanks to the measurability of the iterates U^m , $m \in \{1, \dots, M\}$. We collect and plug the above estimates back in (4.10), and we sum it up over m from $m = 1$ to $m = M$. Then, we apply the mathematical expectation, and employ the discrete Grönwall lemma to get

$$\max_{1 \leq m \leq M} \mathbb{E} [\|U^m\|_\alpha^4] \leq C_{T,2}, \tag{4.13}$$

where $C_{T,2} > 0$ does not depend on α , k and h . We also get by Grönwall lemma the following two estimates:

$$\frac{1}{4} \mathbb{E} \left[\sum_{m=1}^M \|U^m - U^{m-1}\|_\alpha^2 \|U^m\|_\alpha^2 \right] \leq C_{T,2} \text{ and } \frac{k\nu}{4} \mathbb{E} \left[\sum_{m=1}^M \|U^m\|_{h,\alpha}^2 \|U^m\|_\alpha^2 \right] \leq C_{T,2}.$$

It remains to show that $\mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_\alpha^4 \right] \leq C_{T,2}$. To do so, we follow the technique which was employed in the previous step (A priori energy estimate) by summing up inequality (4.10) over m from 1 to $\ell \geq 1$. We will only need to control the following stochastic term:

$$\begin{aligned}
 &\mathbb{E} \left[\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} (g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1}) \|U^{m-1}\|_\alpha^2 \right] \\
 &\quad \lesssim \mathbb{E} \left[\left(k \sum_{m=1}^M \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|U^{m-1}\|_\alpha^6 \right)^{\frac{1}{2}} \right] \\
 &\quad \leq \mathbb{E} \left[\frac{1}{8} \|U^0\|_{\mathbb{H}^1}^4 + \frac{1}{8} \max_{1 \leq m \leq M} \|U^m\|_\alpha^4 + 4K_1^2 k \sum_{m=1}^M \|U^{m-1}\|_\alpha^2 + 4K_2^2 k \sum_{m=1}^M \|U^{m-1}\|_\alpha^4 \right].
 \end{aligned}$$

Collecting all estimates together and using (4.13) complete the proof of estimate (iii). \square

Remark 4.1 *The iterates' uniqueness can be shown to hold in a subspace Ω_ϵ of Ω such that $\epsilon = \epsilon(k, h) > 0$ and $\mathbb{P}(\Omega_\epsilon) \rightarrow 1$ as $\epsilon \rightarrow 0$. Uniqueness in the whole probability space does not occur because of the nonlinearity of the proposed numerical scheme. The reader may refer to [7, Lemma A.1] for a similar approach.*

Lemma 4.2 *Assume the hypothesis of Lemma 4.1. Iterates $\{U^m\}_{m=1}^M$ satisfy the following estimate:*

$$\mathbb{E} \left[\left(k\nu \sum_{m=1}^M \left(\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right) \right)^{2^{p-1}} \right] \leq C_{T,p},$$

where $C_{T,p} > 0$ has same ingredients as that of Lemma 4.1.

Proof: The demonstration is straightforward; it can be illustrated as follows: we pick the adequate inequality from equation (4.3):

$$\begin{aligned} \frac{k\nu}{2} \sum_{m=1}^M \|U^m\|_{h,\alpha}^2 &\leq \frac{1}{2} \|U^0\|_\alpha^2 + \frac{C_D^2 T K_3^2}{\nu} + \frac{C_D^2 K_4^2 T}{\nu} \max_{1 \leq m \leq M} \|U^{m-1}\|_\alpha^2 \\ &+ \sum_{m=1}^M \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 + \max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} (g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1}) \end{aligned}$$

Next, we raise the above inequality to the power 2^{p-1} and use the estimate $|a + b|^q \leq 2^{q-1} (|a|^q + |b|^q)$, $\forall a, b \in \mathbb{R}, \forall q \geq 1$. It holds that

$$\begin{aligned} \left(\frac{k\nu}{2} \sum_{m=1}^M \|U^m\|_{h,\alpha}^2 \right)^{2^{p-1}} &\lesssim \|U^0\|_\alpha^{2^p} + \left(\frac{C_D^2 T K_3^2}{\nu} \right)^{2^{p-1}} + \left(\frac{C_D^2 K_4^2 T}{\nu} \right)^{2^{p-1}} \max_{1 \leq m \leq M} \|U^{m-1}\|_\alpha^{2^p} \\ &+ \left(\sum_{m=1}^M \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 \right)^{2^{p-1}} + \left(\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} (g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1}) \right)^{2^{p-1}}. \end{aligned}$$

We will only focus on bounding the mathematical expectation of the last two terms. By the Burkholder-Davis-Gundy inequality, assumption (S_2) , and Lemma (4.1)-(iii), it follows

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{m=1}^M \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 \right)^{2^{p-1}} \right] &\leq M^{2^{p-1}} \mathbb{E} \left[\max_{1 \leq m \leq M} \left\| \int_{t_{m-1}}^{t_m} g(t_{m-1}, U^{m-1}) dW(t) \right\|_{\mathbb{L}^2}^{2^p} \right] \\ &\lesssim M^{2^{p-1}} \mathbb{E} \left[k^{2^{p-1}} \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^{2^p} \right] \lesssim T^{2^{p-1}} K_1^{2^p} + T^{2^{p-1}} K_2^{2^p} \mathbb{E} \left[\max_{1 \leq m \leq M} \|U^{m-1}\|_\alpha^{2^p} \right] \\ &\leq T^{2^{p-1}} K_1^{2^p} + T^{2^{p-1}} K_2^{2^p} C_{T,p} \end{aligned}$$

On the other hand,

$$\begin{aligned} &\mathbb{E} \left[\left(\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} (g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1}) \right)^{2^{p-1}} \right] \\ &\lesssim \mathbb{E} \left[\left(k \sum_{m=1}^M \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|U^{m-1}\|_{\mathbb{L}^2}^2 \right)^{2^{p-2}} \right] \\ &\leq \mathbb{E} \left[\left(k \sum_{m=1}^M 2K_1^2 + 2(K_1^2 + K_2^2) \|U^{m-1}\|_\alpha^4 \right)^{2^{p-2}} \right] \end{aligned}$$

$$\begin{aligned} &\leq \mathbb{E} \left[\left(2TK_1^2 + 2T(K_1^2 + K_2^2) \max_{1 \leq m \leq M} \|U^{m-1}\|_\alpha^4 \right)^{2^{p-2}} \right] \\ &\lesssim (2TK_1^2)^{2^{p-2}} + [2T(K_1^2 + K_2^2)]^{2^{p-2}} C_{T,p}, \end{aligned}$$

thanks to the Burkholder-Davis-Gundy inequality, assumption (S_2) , Lemma 4.1-(iii), inequalities $x^2 \leq 1 + x^4, \forall x \in \mathbb{R}$ and (2.5). This completes the proof of this lemma. \square

The provided a priori estimates in Lemma 4.1 are not sufficient to derive the existence of a (continuous) solution when α is fixed. Whence the need of the following lemma which consists of further stability properties for Algorithm 1.

Lemma 4.3 *Assume that α is fixed away from h and k and that (S_1) - (S_2) are valid. Let $\{V^m\}_{m=1}^M$ be the iterates of Algorithm 1. For $p \in [2, +\infty) \cap \mathbb{N}$, it holds*

$$\begin{aligned} (i) \quad &\mathbb{E} \left[\max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^2 + k\nu \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2 + \sum_{m=1}^M \|V^m - V^{m-1}\|_{\mathbb{L}^2}^2 \right] \leq C(\alpha), \\ (ii) \quad &\mathbb{E} \left[\max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^{2p} + \left(k\nu \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2 \right)^{2^{p-1}} + \left(\sum_{m=1}^M \|V^m - V^{m-1}\|_{\mathbb{L}^2}^2 \right)^{2^{p-1}} \right] \leq C_p(\alpha), \end{aligned}$$

for some constant $C_p(\alpha) > 0$ depending on $\alpha, \|V^0\|_{L^{2p}(\Omega; \mathbb{L}^2)}, \|U^0\|_{L^{2p+2}(\Omega; \mathbb{H}_0^1)}$, but not on k and h . Note that $C(\alpha) := C_1(\alpha)$.

Proof: We replace φ by V^m in equation (4.1), then apply identity (2.4) to get

$$\begin{aligned} &\frac{1}{2} \|V^m\|_{\mathbb{L}^2}^2 - \frac{1}{2} \|V^{m-1}\|_{\mathbb{L}^2}^2 + \frac{1}{2} \|V^m - V^{m-1}\|_{\mathbb{L}^2}^2 + k\nu \|\nabla V^m\|_{\mathbb{L}^2}^2 \\ &= k (U^m \times (\nabla \times V^m), V^m) + k \langle f(t_{m-1}, U^{m-1}), V^m \rangle + (g(t_{m-1}, U^{m-1}) \Delta_m W, V^m). \end{aligned}$$

Taking the sum over m from 1 to $\ell \in \{1, \dots, M\}$, then the maximum over ℓ yields

$$\begin{aligned} &\frac{1}{2} \max_{1 \leq \ell \leq M} \|V^\ell\|_{\mathbb{L}^2}^2 + k\nu \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2 + \frac{1}{4} \sum_{m=1}^M \|V^m - V^{m-1}\|_{\mathbb{L}^2}^2 \leq \frac{1}{2} \|V^0\|_{\mathbb{L}^2}^2 \\ &+ k \sum_{m=1}^M |(U^m \times (\nabla \times V^m), V^m)| + k \sum_{m=1}^M \|f(t_{m-1}, U^{m-1})\|_{\mathbb{H}^{-1}} \|V^m\|_{\mathbb{H}^1} \\ &+ \sum_{m=1}^M \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 + \max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} (g(t_{m-1}, U^{m-1}) \Delta_m W, V^{m-1}). \end{aligned} \tag{4.14}$$

We exploit the identity $V^m = U^m - \alpha^2 \Delta^h U^m$ and Proposition 2.1-(i) to write

$$(U^m \times (\nabla \times V^m), V^m) = \alpha^2 \left(\Delta^h U^m \times (\nabla \times V^m), U^m \right). \tag{4.15}$$

Moreover, plugging $\psi = U^m$ in equation (4.1)₂ and using the Cauchy-Schwarz, Young and Poincaré inequalities return $\|\nabla U^m\|_{\mathbb{L}^2} \leq C_D \alpha^{-1} \|\nabla V^m\|_{\mathbb{L}^2}$. Further, by Lemma 2.1-(i), it follows

$$\alpha^2 \|\nabla \Delta^h U^m\|_{\mathbb{L}^2} \leq \|\nabla U^m\|_{\mathbb{L}^2} + \|\nabla V^m\|_{\mathbb{L}^2} \leq (C_D \alpha^{-1} + 1) \|\nabla V^m\|_{\mathbb{L}^2}.$$

Therefore, identity (4.15), Proposition 2.1-(iv), the norm equivalence $\alpha \|\cdot\|_{\mathbb{H}^1} \leq \|\cdot\|_\alpha \leq \|\cdot\|_{\mathbb{H}^1}$, the Hölder and Young inequalities imply

$$\begin{aligned} k \sum_{m=1}^M |(U^m \times (\nabla \times V^m), V^m)| &\leq C_D \alpha k \sum_{m=1}^M \|U^m\|_\alpha \|\nabla V^m\|_{\mathbb{L}^2} \|\Delta^h U^m\|_{\mathbb{L}^2}^{\frac{1}{2}} \alpha^{-1} \sqrt{C_D \alpha^{-1} + 1} \|\nabla V^m\|_{\mathbb{L}^2}^{\frac{1}{2}} \\ &\leq C_D \alpha^{-1/2} \sqrt{C_D \alpha^{-1} + 1} \left(k \sum_{m=1}^M \|U^m\|_\alpha^4 \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right)^{1/4} \left(k \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2 \right)^{3/4} \\ &\leq \frac{27C_D^4 (C_D + \alpha)^2}{4\nu^3 \alpha^4} k \sum_{m=1}^M \|U^m\|_\alpha^4 \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 + \frac{k\nu}{4} \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2. \end{aligned}$$

Furthermore, employing assumption (S_2) , the Poincaré and Young inequalities gives

$$k \sum_{m=1}^M \|f(t_{m-1}, U^{m-1})\|_{\mathbb{H}^{-1}} \|V^m\|_{\mathbb{H}^1} \leq \frac{2C_D^2}{\nu} k \sum_{m=1}^M (K_3^2 + K_4^2 \|U^{m-1}\|_\alpha^2) + \frac{k\nu}{4} \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2.$$

Collecting all inequalities together, equation (4.14) becomes

$$\begin{aligned} \frac{1}{2} \max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^2 + \frac{k\nu}{2} \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2 + \frac{1}{4} \sum_{m=1}^M \|V^m - V^{m-1}\|_{\mathbb{L}^2}^2 &\leq \frac{1}{2} \|V^0\|_{\mathbb{L}^2}^2 \\ &+ \frac{27C_D^4 (C_D + \alpha)^2}{4\nu^3 \alpha^4} k \sum_{m=1}^M \|U^m\|_\alpha^4 \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 + \frac{2C_D^2 T K_3^2}{\nu} + \frac{2C_D^2 K_4^2 T}{\nu} \max_{1 \leq m \leq M} \|U^{m-1}\|_\alpha^2 \quad (4.16) \\ &+ \sum_{m=1}^M \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 + \max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} (g(t_{m-1}, U^{m-1}) \Delta_m W, V^{m-1}). \end{aligned}$$

Next, we take the mathematical expectation of equation (4.16) to bound each term on its right-hand size. We have

$$\mathbb{E} \left[k \sum_{m=1}^M \|U^m\|_\alpha^4 \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right] \leq \mathbb{E} \left[k \sum_{m=1}^M \|U^m\|_{\mathbb{L}^2}^4 \left(\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right) \right] \leq C_{T,3},$$

thanks to Lemma 4.1-(iii). Moreover, making use of estimate (4.4) and Lemma 4.1-(ii), we infer that

$$\mathbb{E} \left[\sum_{m=1}^M \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 \right] \leq 2Tr(Q)TK_1^2 + 2Tr(Q)TK_2^2 \mathbb{E} [\|U^0\|_\alpha^2] + 2Tr(Q)K_2^2 TC_T.$$

To control the last term on the right-hand side of equation (4.16), we shall employ the Burkholder-Davis-Gundy and Young inequalities, assumption (S_2) , and Lemma 4.1-(ii)

$$\begin{aligned} \mathbb{E} \left[\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} (g(t_{m-1}, U^{m-1}) \Delta_m W, V^{m-1}) \right] &\lesssim \mathbb{E} \left[\left(k \sum_{m=1}^M \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|V^{m-1}\|_{\mathbb{L}^2}^2 \right)^{\frac{1}{2}} \right] \\ &\leq \frac{1}{4} \mathbb{E} \left[\max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^2 \right] + \mathbb{E} \left[\frac{1}{4} \|V^0\|_{\mathbb{L}^2}^2 + k \sum_{m=1}^M \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \right] \\ &\leq \frac{1}{4} \mathbb{E} \left[\max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^2 \right] + \frac{1}{4} \mathbb{E} [\|V^0\|_{\mathbb{L}^2}^2] + 2K_2^2 T \mathbb{E} [\|U^0\|_\alpha^2] + 2K_1^2 T + 2K_2^2 TC_T. \end{aligned}$$

The first term on the right-hand side of the above equation must be absorbed in the left-hand side of estimate (4.16) after taking its mathematical expectation. This finishes up the demonstration of assertion (i). To illustrate

estimate (ii), we shall raise inequality (4.16) to the power 2^{p-1} , apply estimate (2.5), then take the mathematical expectation:

$$\begin{aligned}
 & \mathbb{E} \left[\max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^{2p} + \left(k\nu \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2 \right)^{2^{p-1}} + \left(\sum_{m=1}^M \|V^m - V^{m-1}\|_{\mathbb{L}^2}^2 \right)^{2^{p-1}} \right] \lesssim \mathbb{E} [\|V^0\|_{\mathbb{L}^2}^{2p}] \\
 & + \frac{C_D^{2p+1} (C_D + \alpha)^{2p}}{\nu^3 \times 2^{p-1} \alpha^{2p+1}} \mathbb{E} \left[\left(k \sum_{m=1}^M \|U^m\|_{\alpha}^4 \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right)^{2^{p-1}} \right] + \frac{C_D^{2p} T^{2p-1} K_3^{2p}}{\nu^{2p-1}} \\
 & + \frac{C_D^{2p} T^{2p-1} K_4^{2p}}{\nu^{2p-1}} \mathbb{E} \left[\max_{1 \leq m \leq M} \|U^{m-1}\|_{\alpha}^{2p} \right] + \mathbb{E} \left[\left(\sum_{m=1}^M \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 \right)^{2^{p-1}} \right] \\
 & + \mathbb{E} \left[\left(\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} (g(t_{m-1}, U^{m-1}) \Delta_m W, V^{m-1}) \right)^{2^{p-1}} \right] \\
 & =: \mathbb{E} [\|V^0\|_{\mathbb{L}^2}^{2p}] + J_1 + \frac{C_D^{2p} T^{2p-1} K_3^{2p}}{\nu^{2p-1}} + J_2 + J_3 + J_4.
 \end{aligned}$$

Using the Cauchy-Schwarz inequality and Lemma 4.1, it holds that

$$\begin{aligned}
 J_1 & \leq \frac{C_D^{2p+1} (C_D + \alpha)^{2p}}{\nu^3 \times 2^{p-1} \alpha^{2p+1}} \mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_{\alpha}^{2p+2} \right]^{\frac{1}{2}} \mathbb{E} \left[\left(k \sum_{m=1}^M (\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2) \right)^{2p} \right]^{\frac{1}{2}} \\
 & \leq \frac{C_D^{2p+1} (C_D + \alpha)^{2p}}{\nu^3 \times 2^{p-1} \alpha^{2p+1}} \sqrt{C_{T,p+2}} \sqrt{C_{T,p+1}}.
 \end{aligned}$$

J_2 can be readily bounded through Lemma 4.1-(ii). On the other hand, we have

$$\begin{aligned}
 J_3 & \leq M^{2p-1} \mathbb{E} \left[\max_{1 \leq m \leq M} \left\| \int_{t_{m-1}}^{t_m} g(t_{m-1}, U^{m-1}) dW(t) \right\|_{\mathbb{L}^2}^{2p} \right] \\
 & \lesssim M^{2p-1} \mathbb{E} \left[\left(\int_{t_{m-1}}^{t_m} \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right)^{2^{p-1}} \right] \leq T^{2p-1} \mathbb{E} \left[\max_{1 \leq m \leq M} (K_1 + K_2 \|U^{m-1}\|_{\alpha})^{2p} \right] \\
 & \lesssim T^{2p-1} K_1^{2p} + T^{2p-1} K_2^{2p} \mathbb{E} [\|U^0\|_{\alpha}^{2p}] + T^{2p-1} K_2^{2p} C_{T,p},
 \end{aligned}$$

thanks to the Burkholder-Davis-Gundy inequality, assumption (S_2), estimate (2.5), and Lemma 4.1-(iii). Similarly, J_4 can be controlled by the use of Burkholder-Davis-Gundy's inequality, assumption (S_2), Young's inequality, estimate (2.5), and Lemma 4.1-(iii):

$$\begin{aligned}
 J_4 & \lesssim \mathbb{E} \left[\left(k \sum_{m=1}^M \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)} \|V^{m-1}\|_{\mathbb{L}^2} \right)^{2^{p-2}} \right] \\
 & \leq \frac{1}{4} \mathbb{E} \left[\max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^{2p} \right] + \mathbb{E} \left[\frac{1}{4} \|V^0\|_{\mathbb{L}^2}^{2p} + \left(T \max_{1 \leq m \leq M} (K_1 + K_2 \|U^{m-1}\|_{\alpha})^2 \right)^{2^{p-1}} \right] \\
 & \leq \frac{1}{4} \mathbb{E} \left[\max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^{2p} \right] + \frac{1}{4} \mathbb{E} [\|V^0\|_{\mathbb{L}^2}^{2p}] + T^{2p-1} K_2^{2p} \mathbb{E} [\|U^0\|_{\alpha}^{2p}] + T^{2p-1} (K_1^{2p} + K_2^{2p} C_{T,p}).
 \end{aligned}$$

Putting it all together and absorbing the first term on the right-hand side of the above equation terminate the proof. \square

Remark 4.2 According to the proof of Lemma 4.3, one can assume the relation $\nu^3\alpha^4 \geq 1$ to tackle the non-uniformness in α of the obtained estimate.

In order to obtain a priori estimates for $\{V^m\}_{m=1}^M$ in Sobolev spaces, uniformly in α , we shall assume that $\alpha \leq \alpha_{max}h$ for some $\alpha_{max} > 0$ independent of h and k . We will present in Lemma 4.4 some preliminary estimates.

Lemma 4.4 Let $\{(U^m, V^m)\}_{m=1}^M$ be the iterates of Algorithm 1 and $0 < \alpha \leq \alpha_{max}h$, where $\alpha_{max} > 0$ independent of α , h and k . Then, for all $m \in \{1, \dots, M\}$ and \mathbb{P} -a.s.

$$(i) \quad \|V^m\|_{\mathbb{L}^2} \leq C_1 \|U^m\|_{\alpha},$$

$$(ii) \quad \|\nabla V^m\|_{\mathbb{L}^2}^2 \leq C_1 \left(\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right),$$

$$(iii) \quad \|V^{m+\ell} - V^m\|_{\mathbb{L}^2} \leq C_1 \|U^{m+\ell} - U^m\|_{\alpha}, \text{ for all } \ell \in \{1, \dots, M - m\},$$

where $C_1 > 0$ depends only on α_{max} and the constant C of the inverse inequality (2.14).

Proof: Let $m \in \{1, \dots, M\}$. From equation (4.1)₂, taking $\psi = V^m$ and applying the Cauchy-Schwarz and Young inequalities yield

$$\|V^m\|_{\mathbb{L}^2}^2 \leq \|U^m\|_{\mathbb{L}^2}^2 + \frac{1}{4}\|V^m\|_{\mathbb{L}^2}^2 + \frac{\alpha^2}{\epsilon}\|\nabla U^m\|_{\mathbb{L}^2}^2 + \frac{\epsilon\alpha^2}{4}\|\nabla V^m\|_{\mathbb{L}^2}^2,$$

where $\epsilon > 0$. Taking $\epsilon = \frac{1}{\alpha_{max}^2 C^2}$ and applying the inverse inequality (2.14) complete the proof of assertion

(i). On the other hand, by Lemma 2.1-(i), $\nabla V^m = \nabla U^m - \alpha^2 \nabla \Delta^h U^m$, \mathbb{P} -a.s. and a.e. in D . Thus,

$$\|\nabla V^m\|_{\mathbb{L}^2}^2 \leq 2\|\nabla U^m\|_{\mathbb{L}^2}^2 + 2\alpha_{max}^2 C^2 \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2,$$

thanks to the inverse inequality (2.14). Estimate (iii) has similar proof to that of assertion (i). \square

Clearly, one must incorporate Lemmas 4.1 and 4.4 to obtain:

Lemma 4.5 Let $\{V^m\}_{m=1}^M$ be the iterates of Algorithm 1. Assume that assumptions (S_1) - (S_3) are fulfilled and that $0 < \alpha \leq \alpha_{max}h$, for some $\alpha_{max} > 0$ independent of k and h . Then,

$$\mathbb{E} \left[\max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^2 + \frac{k\nu}{2} \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2 + \frac{1}{4} \sum_{m=1}^M \|V^m - V^{m-1}\|_{\mathbb{L}^2}^2 \right] \leq C'_T,$$

where $C'_T > 0$ does not depend on α , k and h .

We terminate this section with a local monotonicity property associated with the trilinear term of the underlying equations, as stated in the following proposition.

Proposition 4.1 Assume that $\alpha \leq \alpha_{max}h$ for some $\alpha_{max} > 0$ independent of k and h , and that $L_f \leq \frac{\nu}{\sqrt{2C_P^2}}$

and $L_g \leq \frac{1}{\sqrt{\text{Tr}(Q)}} \left(\frac{\nu}{2C_P^2} - \frac{L_f^2 C_P^2}{\nu} \right)^{1/2}$, where C_P represents here the Poincaré constant. For $v_h^1, v_h^2 \in \mathbb{V}_h$,

let \bar{u}_h^1 and \bar{u}_h^2 be their discrete differential filters, respectively. Denote $w_h = \bar{u}_h^1 - \bar{u}_h^2$. There is a constant $\mathcal{K} > 0$ depending only on D , α_{max} and the inverse inequality's constant C such that

$$\begin{aligned} & \left\langle \nu \Delta(v_h^1 - v_h^2) + \bar{u}_h^1 \times (\nabla \times v_h^1) - \bar{u}_h^2 \times (\nabla \times v_h^2) + f(\cdot, \bar{u}_h^1) - f(\cdot, \bar{u}_h^2) - \frac{\mathcal{K}}{\nu^3} \|\bar{u}_h^2\|_{\mathbb{L}^4}^4 w_h, w_h \right\rangle \\ & + \text{Tr}(Q) \|g(\cdot, \bar{u}_h^1) - g(\cdot, \bar{u}_h^2)\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 \leq 0. \end{aligned} \quad (4.17)$$

Proof: The first target in this proof will be the estimate

$$\|\nabla(v_h^1 - v_h^2)\|_{\mathbb{L}^2} \leq (1 + C^2 \alpha_{max}^2) \|\nabla(\bar{u}_h^1 - \bar{u}_h^2)\|_{\mathbb{L}^2}. \quad (4.18)$$

Indeed, from Lemma 2.1-(i), we get $\nabla(v_h^1 - v_h^2) = \nabla(\bar{u}_h^1 - \bar{u}_h^2) - \alpha^2 \nabla \Delta^h (\bar{u}_h^1 - \bar{u}_h^2)$ a.e. in D . Therefore, a simple application of the inverse inequalities (2.14), (2.15) and the hypothesis $\alpha \leq \alpha_{max} h$ justifies (4.18). On the other hand,

$$\begin{aligned} & |(\bar{u}_h^1 \times (\nabla \times v_h^1) - \bar{u}_h^2 \times (\nabla \times v_h^2), w_h)| = |(\bar{u}_h^2 \times (\nabla \times (v_h^1 - v_h^2)), w_h)| \\ & \leq C_D \|\bar{u}_h^2\|_{\mathbb{L}^4} \|\nabla(v_h^1 - v_h^2)\|_{\mathbb{L}^2} \|w_h\|_{\mathbb{L}^2}^{1/2} \|\nabla w_h\|_{\mathbb{L}^2}^{1/2} \leq C_D (1 + C^2 \alpha_{max}^2) \|\bar{u}_h^2\|_{\mathbb{L}^4} \|\nabla w_h\|_{\mathbb{L}^2}^{3/2} \|w_h\|_{\mathbb{L}^2}^{1/2} \\ & \leq \frac{\nu}{4} \|\nabla w_h\|_{\mathbb{L}^2}^2 + \frac{\mathcal{K}}{\nu^3} \|\bar{u}_h^2\|_{\mathbb{L}^4}^4 \|w_h\|_{\mathbb{L}^2}^2, \end{aligned} \quad (4.19)$$

for some constant $\mathcal{K} > 0$ depending on C , α_{max} and C_D , where Proposition 2.1-(i), (iv), estimate (4.18) and Young's inequality were employed. In addition, assumption (S_2) implies

$$\begin{aligned} & |\langle f(\cdot, \bar{u}_h^1) - f(\cdot, \bar{u}_h^2), w_h \rangle| \leq L_f C_P \|w_h\|_{\alpha} \|\nabla w_h\|_{\mathbb{L}^2} \leq \frac{L_f^2 C_P^2}{\nu} \|w_h\|_{\alpha}^2 + \frac{\nu}{4} \|\nabla w_h\|_{\mathbb{L}^2}^2 \\ & \leq \frac{L_f^2 C_P^4}{\nu} (\|\nabla w_h\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h w_h\|_{\mathbb{L}^2}^2) + \frac{\nu}{4} \|\nabla w_h\|_{\mathbb{L}^2}^2, \end{aligned} \quad (4.20)$$

where the Poincaré inequality and $\|\nabla \varphi_h\|_{\mathbb{L}^2} \leq C_P \|\Delta^h \varphi_h\|_{\mathbb{L}^2}$, $\forall \varphi_h \in \mathbb{V}_h$, were used in the last inequality. Similarly, by virtue of assumptions (S_2) , the norm equivalence (2.3), and the Poincaré inequality, one gets

$$\|g(\cdot, \bar{u}_h^1) - g(\cdot, \bar{u}_h^2)\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 \leq L_g^2 C_P^2 \|\nabla w_h\|_{\mathbb{L}^2}^2.$$

By Lemma 2.1-(i) and identity 2.12, it holds that

$$\nu (\nabla(v_h^1 - v_h^2), \nabla w_h) = \nu \|\nabla w_h\|_{\mathbb{L}^2}^2 + \nu \alpha^2 \|\Delta^h w_h\|_{\mathbb{L}^2}^2.$$

The sum of the former and the latter identities along with inequalities (4.19) and (4.20) yields estimate (4.17). \square

Remark 4.3 *The assumed conditions on L_f and L_g in Proposition 4.1 are mainly imposed to maintain the monotonicity. They appear in this context due to the dependence of both f and g on the solution. In other words, if $f = f(t)$ and $g = g(t)$, these conditions would no longer make sense. Observe, in addition, that one could have omitted the factor $\frac{1}{\sqrt{\text{Tr}(Q)}}$ by adjusting the Lipschitz-continuity of the diffusion coefficient g to $\|g(\cdot, z_1) - g(\cdot, z_2)\|_{\mathcal{L}_2(Q^{1/2}(K); \mathbb{L}^2)} \leq L_g \|z_1 - z_2\|_{\alpha}$.*

5 Convergence

All the previous analysis relied on $\{(U^m, V^m)\}_{m=1}^M$, which does not depend explicitly on the time variable. To investigate the convergence in continuous-time spaces, e.g. $L^2(\Omega; L^2(0, T; \mathbb{H}_0^1))$, we need to define the following processes

$$\left(\mathcal{U}_{k,h}^-(t, x), \mathcal{V}_{k,h}^-(t, x) \right) := (U^{m-1}(x), V^{m-1}(x)), \quad \forall (t, x) \in [t_{m-1}, t_m] \times D, \quad (5.1)$$

$$\left(\mathcal{U}_{k,h}^+(t, x), \mathcal{V}_{k,h}^+(t, x) \right) := (U^m(x), V^m(x)), \quad \forall (t, x) \in (t_{m-1}, t_m] \times D, \quad (5.2)$$

$$(f^-(t, \cdot), g^-(t, \cdot)) = (f(t_{m-1}, \cdot), g(t_{m-1}, \cdot)), \quad \forall t \in [t_{m-1}, t_m]. \quad (5.3)$$

Discrete derivation with respect to time will be required later on. For this purpose, we list a few rules in the proposition below.

Proposition 5.1 Denote by d_t the discrete derivation defined by $d_t z^m = \frac{z^m - z^{m-1}}{k}$, for all $m \in \{1, \dots, M\}$. Let $z^+, z^- : [0, T] \rightarrow \mathbb{R}$ be the piecewise constant functions defined by $z^+(t) := z^m$ for all $t \in (t_{m-1}, t_m]$, and $z^-(t) := z^{m-1}$ for all $t \in [t_{m-1}, t_m)$. The following properties hold true:

- (i) $d_t(\zeta^+ \xi^+) = \zeta^+ d_t \xi^+ + \xi^- d_t \zeta^+$.
- (ii) $\int_0^T \zeta^+ d_t \xi^+(t) dt = \zeta^+(T) \xi^+(T) - \zeta^-(0) \xi^-(0) - \int_0^T (d_t \zeta^+(t)) \xi^- dt$.
- (iii) $d_t e^{\zeta^+} = e^{\zeta^-} d_t \zeta^+ + e^\eta \frac{(\zeta^+ - \zeta^-)^2}{2k}$, for some $\eta \in (\zeta^-, \zeta^+)$.

Proof: See [7, Appendix B]. □

The remaining two subsections of this section are solely devoted to giving adequate proof for solutions' existence. For further analysis, the reader may refer to Section 6.

5.1 Convergence when $\alpha \leq \alpha_{max} h$

We assume within this part that $d = 2$. We point out that the demonstration technique which is followed for a non-vanishing α (see the next subsection), cannot be employed here due to the lack of solution's regularity. Nevertheless, Skorokhod's theorem will be kept away in the steps down below. This avoidance is valid by virtue of Proposition 4.1. We need to go through a few steps to illustrate the convergence of Algorithm 1.

Step 1: Boundedness

The following sequences $\{\mathbf{u}_{k,h}^+\}_{k,h}$, $\{\mathcal{V}_{k,h}^+\}_{k,h}$, $\{f^-(\cdot, \mathbf{u}_{k,h}^-)\}_{k,h}$ and $\{g^-(\cdot, \mathbf{u}_{k,h}^-)\}_{k,h}$ are respectively bounded in $L^2(\Omega; L^\infty(0, T; \mathbb{L}^2)) \cap L^2(\Omega; L^2(0, T; \mathbb{H}_0^1))$, $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$ and $L^2(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$ by virtue of Lemmas 4.1, 4.5 and assumption (S_2) . Therefore, there are $u, v \in L^2(\Omega; L^\infty(0, T; \mathbb{L}^2) \cap L^2(0, T; \mathbb{H}_0^1))$, $F_0 \in L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$, $G_0 \in L^2(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$ and two subsequences denoted by $\{\mathbf{u}_{k',h'}^+\}_{k',h'}$, $\{\mathcal{V}_{k',h'}^+\}_{k',h'}$ such that

$$\mathbf{u}_{k',h'}^+ \rightharpoonup u \ \& \ \mathcal{V}_{k',h'}^+ \rightharpoonup v \quad \text{in } L^2(\Omega; L^2(0, T; \mathbb{H}_0^1)), \quad (5.4)$$

$$\mathbf{u}_{k',h'}^+ \overset{*}{\rightharpoonup} u \ \& \ \mathcal{V}_{k',h'}^+ \overset{*}{\rightharpoonup} v \quad \text{in } L^2(\Omega; L^\infty(0, T; \mathbb{L}^2)), \quad (5.5)$$

$$f^-(\cdot, \mathbf{u}_{k',h'}^-) \rightharpoonup F_0 \quad \text{in } L^2(\Omega; L^2(0, T; \mathbb{H}^{-1})), \quad (5.6)$$

$$g^-(\cdot, \mathbf{u}_{k',h'}^-) \rightharpoonup G_0 \quad \text{in } L^2(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2))). \quad (5.7)$$

Let $\varphi \in D(A)$. Set $\varphi_h = \Pi_h \varphi$ and $R(\mathbf{u}_{k,h}^+) := \nu \Delta \mathcal{V}_{k,h}^+ + \mathbf{u}_{k,h}^+ \times (\nabla \times \mathcal{V}_{k,h}^+)$. By summing equation (4.1) over m from 1 to M , we achieve:

$$\begin{aligned} \int_0^T \langle R(\mathbf{u}_{k',h'}^+), \varphi_{h'} \rangle dt &= \left(\mathcal{V}_{k',h'}^+(T) - \mathcal{V}_{k',h'}^-(0), \varphi_{h'} \right) - \int_0^T \langle f^-(t, \mathbf{u}_{k',h'}^-), \varphi_{h'} \rangle dt \\ &\quad - \left(\int_0^T g^-(t, \mathbf{u}_{k',h'}^-) dW(t), \varphi_{h'} \right). \end{aligned}$$

Since all terms on the right-hand side converge after applying the mathematical expectation, thanks to (5.4)-(5.7), we define the operator R_0 as

$$\mathbb{E} \left[\int_0^T \langle R_0(t), \varphi \rangle dt \right] = \lim_{k',h' \rightarrow 0} \mathbb{E} \left[\int_0^T \langle R(\mathbf{u}_{k',h'}^+), \Pi_{h'} \varphi \rangle dt \right], \ \forall \varphi \in D(A).$$

Subsequently, the limiting function v fulfills, for all $\varphi \in D(A)$, for all $t \in [0, T]$ and \mathbb{P} -a.s. the following equation:

$$(v(t), \varphi) - \int_0^t \langle R_0(s), \varphi \rangle ds = (v_0, \varphi) + \int_0^t \langle F_0(s), \varphi \rangle ds + \left(\int_0^t G_0(s) dW(s), \varphi \right). \quad (5.8)$$

Besides the convergence results (5.4)-(5.7), we will also need the following:

$$f^-(\cdot, \mathbf{u}_{k',h'}^+) \rightharpoonup F_0 \quad \text{in } L^2(\Omega; L^2(0, T; \mathbb{H}^{-1})), \quad (5.9)$$

$$g^-(\cdot, \mathbf{u}_{k',h'}^+) \rightharpoonup G_0 \quad \text{in } L^2(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2))). \quad (5.10)$$

Convergence (5.9) can be illustrated as follows: $\{f^-(\cdot, \mathbf{u}_{k',h'}^+)\}_{k',h'}$ is bounded in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$, thanks to assumption (S_2) and Lemma 4.1. Thus, there are \bar{F}_0 and a subsequence of $\{f^-(\cdot, \mathbf{u}_{k',h'}^+)\}_{k',h'}$ (still denoted $f^-(\cdot, \mathbf{u}_{k',h'}^+)$) such that $f^-(\cdot, \mathbf{u}_{k',h'}^+) \rightharpoonup \bar{F}_0$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$. To unify the limiting functions F_0 and \bar{F}_0 , we let $\varphi \in L^2(\Omega; L^2(0, T; \mathbb{H}_0^1))$. Thus,

$$\begin{aligned} \langle \bar{F}_0(t) - F_0(t), \varphi(t) \rangle &\leq \langle \bar{F}_0(t) - f^-(t, \mathbf{u}_{k',h'}^+), \varphi(t) \rangle + L_f \|\mathbf{u}_{k',h'}^+ - \mathbf{u}_{k',h'}^-\|_\alpha \|\varphi(t)\|_{\mathbb{H}^1} \\ &\quad + \langle f^-(t, \mathbf{u}_{k',h'}^-) - F_0(t), \varphi(t) \rangle. \end{aligned}$$

Integrating with respect to t and applying the mathematical expectation while taking into account the strong convergence toward 0 of $\mathbb{E} \left[\int_0^T \|\mathbf{u}_{k',h'}^+ - \mathbf{u}_{k',h'}^-\|_\alpha^2 dt \right]$ (thanks to Lemma 4.1) yield $F_0 = \bar{F}_0$ in the space $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$. Convergence (5.10) follows similarly.

Step 2: \mathbf{u} and \mathbf{v} are equal

From equation (4.1), there holds $\mathcal{V}_{k',h'}^+ = \mathbf{u}_{k',h'}^+ - \alpha^2 \Delta^{h'} \mathbf{u}_{k',h'}^+$, \mathbb{P} -a.s. and a.e. in $(0, T) \times D$. Moreover, for all $\varphi \in L^2(\Omega; L^2(0, T; \mathbb{L}^2))$, we have

$$\begin{aligned} \alpha^2 \mathbb{E} \left[\int_0^T \left(\Delta^{h'} \mathbf{u}_{k',h'}^+, \varphi(t) \right) dt \right] &\leq \alpha \|\varphi\|_{L^2(\Omega; L^2(0, T; \mathbb{L}^2))} \mathbb{E} \left[\int_0^T \alpha^2 \|\Delta^{h'} \mathbf{u}_{k',h'}^+\|_{\mathbb{L}^2}^2 dt \right]^{\frac{1}{2}} \\ &\leq \alpha_{max} h' \|\varphi\|_{L^2(\Omega; L^2(0, T; \mathbb{L}^2))} C_T \rightarrow 0 \text{ as } k', h' \rightarrow 0, \end{aligned}$$

where Lemma 4.1-(ii) is exploited along with the hypothesis $\alpha \leq \alpha_{max} h$. As a result, $\{\Delta^{h'} \mathbf{u}_{k',h'}^+\}_{k',h'}$ is weakly convergent to 0 in $L^2(\Omega; L^2(0, T; \mathbb{L}^2))$. Consequently, it follows from convergence (5.4) and the relationship which was stated in the beginning of this step that $u = v$ \mathbb{P} -a.s. and a.e. in $(0, T) \times D$.

Step 3: Identification of \mathbf{R}_0 , \mathbf{F}_0 and \mathbf{G}_0

We shall denote, from this step onwards, $(k', h') = (k, h)$ for the sake of clarity. For a given $z \in C([0, T]; D(A))$, we define $z_h^+(t) = \Pi_h z(t_m)$ for all $t \in (t_{m-1}, t_m]$ and all $m \in \{0, 1, \dots, M\}$. We also denote by \bar{z}_h^+ its discrete differential filter. For all $m \in \{1, \dots, M\}$, set

$$r_m := \frac{2\mathcal{K}}{\nu^3} k \sum_{i=1}^m \|\bar{z}_h(t_i)\|_{\mathbb{L}^4}^4,$$

and we associate with it the piecewise constant function $r^+(t) = r_m$ when $t \in (t_{m-1}, t_m]$. The constant \mathcal{K} in r_m emerges from Proposition 4.1 and will play a relevant role in the upcoming analysis. We finally define a non-increasing function $\rho: [0, T] \rightarrow \mathbb{R}_+$ verifying $\rho(0) = 1$ and such that its discrete version reads $\rho_m := e^{-r_m}$, for all $m \in \{1, \dots, M\}$ and we assign ρ^+ and ρ^- the usual piecewise constant definition. We replace φ with U^m in equation (4.1), employ the Cauchy-Schwarz and Young inequalities, then apply the mathematical expectation to achieve:

$$\begin{aligned} \mathbb{E} [\|U^m\|_\alpha^2 - \|U^{m-1}\|_\alpha^2 - 2k \langle R(U^m) + f(t_{m-1}, U^{m-1}), U^m \rangle] \\ \leq \mathbb{E} [\|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2]. \end{aligned} \quad (5.11)$$

Afterwards, we multiply (5.11) by ρ_{m-1} and sum it over m from 1 to M . It follows:

$$\begin{aligned} \mathbb{E} \left[\rho^+(T) \|\mathbf{u}_{k,h}^+(T)\|_\alpha^2 - \|\mathbf{u}_{k,h}^-(0)\|_\alpha^2 \right] &\leq \mathbb{E} \left[2 \int_0^T \rho^-(t) \langle R(\mathbf{u}_{k,h}^+) + f^-(t, \mathbf{u}_{k,h}^+), \mathbf{u}_{k,h}^+ \rangle dt \right] \\ &\quad + \mathbb{E} \left[2 \int_0^T \rho^-(t) \langle f^-(t, \mathbf{u}_{k,h}^-) - f^-(t, \mathbf{u}_{k,h}^+), \mathbf{u}_{k,h}^+ \rangle dt \right] \\ &\quad + \mathbb{E} \left[\int_0^T \rho^-(t) \|g^-(t, \mathbf{u}_{k,h}^-)\|_{\mathcal{L}_2(Q^{1/2}(K); \mathbb{L}^2)}^2 dt \right] + \mathbb{E} \left[\int_0^T \|\mathbf{u}_{k,h}^+\|_\alpha^2 d_t \rho^+ dt \right], \end{aligned} \quad (5.12)$$

where the first two terms on the left-hand side in inequality (5.11) were handled through Proposition 5.1 as follows:

$$\begin{aligned} \sum_{m=1}^M \rho_{m-1} (\|U^m\|_\alpha^2 - \|U^{m-1}\|_\alpha^2) &= \int_0^T \rho^-(t) dt \|u_{k,h}^+\|_\alpha^2 dt \\ &= \rho^+(T) \|u_{k,h}^+(T)\|_\alpha^2 - \int_0^T \|u_{k,h}^+\|_\alpha^2 d_t \rho^+(t) dt, \end{aligned}$$

and the right-hand side of (5.11) is treated with Itô's isometry. Taking into account the discrete derivation (see Proposition 5.1) and adjusting a few terms in equation (5.12), we obtain

$$\begin{aligned} \mathbb{E} \left[\rho^+(T) \|u_{k,h}^+(T)\|_{\mathbb{L}^2}^2 - \|u_{k,h}^-(0)\|_{\mathbb{L}^2}^2 \right] &\leq \alpha^2 \mathbb{E} \left[\|\nabla u_{k,h}^-(0)\|_{\mathbb{L}^2}^2 \right] + \mathbb{E} \left[\int_0^T \|u_{k,h}^+ - \bar{z}_h^+\|_\alpha^2 d_t \rho^+ dt \right] \\ &+ \mathbb{E} \left[\int_0^T \left\{ 2 \left((u_{k,h}^+, \bar{z}_h^+) \right)_\alpha - \|\bar{z}_h^+\|_\alpha^2 \right\} d_t \rho^+ dt \right] + \mathbb{E} \left[2 \int_0^T \rho^-(t) \langle f^-(t, u_{k,h}^-) - f^-(t, u_{k,h}^+), u_{k,h}^+ \rangle dt \right] \\ &+ \mathbb{E} \left[2 \int_0^T \rho^-(t) \langle R(u_{k,h}^+) - R(\bar{z}_h^+) + f^-(t, u_{k,h}^+) - f^-(t, \bar{z}_h^+), u_{k,h}^+ - \bar{z}_h^+ \rangle dt \right] \\ &+ \mathbb{E} \left[2 \int_0^T \rho^-(t) \langle R(u_{k,h}^+) - R(\bar{z}_h^+) + f^-(t, u_{k,h}^+) - f^-(t, \bar{z}_h^+), \bar{z}_h^+ \rangle dt \right] \\ &+ \mathbb{E} \left[2 \int_0^T \rho^-(t) \langle R(\bar{z}_h^+) + f^-(t, \bar{z}_h^+), u_{k,h}^+ \rangle dt \right] + \mathbb{E} \left[\int_0^T \rho^-(t) \left\{ \|g^-(t, u_{k,h}^-) - g^-(t, u_{k,h}^+)\|_{\mathcal{L}_2^Q}^2 \right. \right. \\ &+ \|g^-(t, u_{k,h}^+) - g^-(t, \bar{z}_h^+)\|_{\mathcal{L}_2^Q}^2 - \|g^-(t, \bar{z}_h^+)\|_{\mathcal{L}_2^Q}^2 + 2 \left(g^-(t, u_{k,h}^+), g^-(t, \bar{z}_h^+) \right)_{\mathcal{L}_2^Q} \\ &\left. \left. + 2 \left(g^-(t, u_{k,h}^-) - g^-(t, u_{k,h}^+), g^-(t, u_{k,h}^+) \right)_{\mathcal{L}_2^Q} \right\} dt \right] =: I_1 + \dots + I_7 + I_8^Q + \dots + I_{12}^Q. \end{aligned}$$

where we recall that $((\cdot, \cdot))_\alpha := (\cdot, \cdot) + \alpha^2 (\nabla \cdot, \nabla \cdot)$, $R(\bar{z}_h^+) = \nu \Delta z_h^+ + \bar{z}_h^+ \times (\nabla \times z_h^+)$, and \mathcal{L}_2^Q is a shorthand for $\mathcal{L}_2(Q^{1/2}(K); \mathbb{L}^2)$. Since $z \in C([0, T]; D(A))$, it follows straightforwardly that z_h^+ and \bar{z}_h^+ converge toward z in $L^p(0, T; \mathbb{H}^1)$ as $k, h \rightarrow 0$, for all $p \geq 1$ (e.g. [27, Theorem 4.0.1]). It suffices now to study the limit of all I_1, \dots, I_{12}^Q . It is easy to see that $I_1 \rightarrow 0$ as $k, h \rightarrow 0$ because $u_{k,h}^-(0) = U^0$ and $\|U^0\|_{\mathbb{H}^1}$ is uniformly bounded in h . By Proposition 5.1,

$$I_2 \leq \mathbb{E} \left[\int_0^T -\frac{2\mathcal{K}}{\nu^3} \rho^- \|\bar{z}_h^+\|_{\mathbb{L}^4}^4 \|u_{k,h}^+ - \bar{z}_h^+\|_{\mathbb{L}^2}^2 dt \right] + \mathbb{E} \left[\int_0^T \|u_{k,h}^+ - \bar{z}_h^+\|_\alpha^2 e^\eta \frac{(r^+ - r^-)^2}{2k} dt \right] =: I_{2,1} + I_{2,2},$$

for some $\eta \in (-r^+, -r^-)$. By a simple application of Proposition 4.1, it follows that $I_5 + I_{2,1} + I_9^Q \leq 0$. We also mention that $I_{2,2} \rightarrow 0$ as $k, h \rightarrow 0$ by utilizing Lemma 4.1 and the fact that

$$\frac{(r^+ - r^-)^2}{2k} = \frac{2\mathcal{K}^2}{\nu^6} k \|\bar{z}_h^+\|_{\mathbb{L}^4}^8.$$

Furthermore, we know by Lemma 4.1-(ii) that $\mathbb{E} \left[\int_0^T \|u_{k,h}^+ - u_{k,h}^-\|_\alpha^2 dt \right]$ goes to 0 as $k, h \rightarrow 0$, therefore I_4, I_8^Q and I_{12}^Q converge to 0, thanks to assumption (S_2) . Collecting the recently derived limits and using convergences (5.4)-(5.10), we acquire:

$$\begin{aligned} \lim_{k,h \rightarrow 0} \mathbb{E} \left[\rho^+(T) \|u_{k,h}^+(T)\|_{\mathbb{L}^2}^2 - \|u_{k,h}^-(0)\|_{\mathbb{L}^2}^2 \right] &\leq \mathbb{E} \left[\int_0^T \left\{ 2(v, z) - \|z\|_{\mathbb{L}^2}^2 \right\} \partial_t \rho(t) dt \right] \\ &+ \mathbb{E} \left[2 \int_0^T \rho(t) \langle R_0(t) - R(z) + F_0(t) - f(t, z), z \rangle dt \right] + \mathbb{E} \left[2 \int_0^T \rho(t) \langle R(z) + f(t, z), v \rangle dt \right] \quad (5.13) \\ &+ \mathbb{E} \left[\int_0^T \rho(t) \left\{ -\|g(t, z)\|_{\mathcal{L}_2^Q}^2 + 2(G_0(t), g(t, z))_{\mathcal{L}_2^Q} \right\} dt \right] \end{aligned}$$

Next, Itô's formula employed to the stochastic process $(t, v) \mapsto \rho(t) \|v(t)\|_{\mathbb{L}^2}^2$ (where v fulfills equation (5.8)) together with inequality (5.13), condition $\alpha \leq \alpha_{max} h$, convergence $U^0 \rightarrow \bar{u}_0 = v_0$ as $h \rightarrow 0$, and the fact that $\mathbb{E} [\rho(T) \|v(T)\|_{\mathbb{L}^2}^2] \leq \liminf \mathbb{E} [\rho^+(T) \|u_{k,h}^+(T)\|_{\mathbb{L}^2}^2]$ grant:

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \partial_t \rho(t) \|v(t) - z(t)\|_{\mathbb{L}^2}^2 dt \right] + \mathbb{E} \left[\int_0^T \rho(t) \|G_0(t) - g(t, z(t))\|_{\mathcal{L}_2(Q^{1/2}(K); \mathbb{L}^2)}^2 dt \right] \\ & \leq \mathbb{E} \left[2 \int_0^T \rho(t) \langle R_0(t) - R(z(t)) + F_0(t) - f(t, z(t)), z(t) - v(t) \rangle dt \right], \end{aligned} \quad (5.14)$$

for all $z \in C([0, T]; D(A))$. Particularly, inequality (5.14) holds true for all $z \in L^4(\Omega; L^\infty(0, T; \mathbb{H})) \cap L^2(\Omega; L^2(0, T; \mathbb{V}))$ due to the density of $C([0, T]; D(A))$ in $L^4(\Omega; L^\infty(0, T; \mathbb{H})) \cap L^2(\Omega; L^2(0, T; \mathbb{V}))$. Hence, taking $z = v$ yields $G_0 = g(\cdot, v)$ in $L^2(\Omega; L^2(0, T; \mathcal{L}_2(Q^{1/2}(K); \mathbb{L}^2)))$. Therewith, plugging $z = v + \lambda w$, for $w \in L^2(\Omega; L^\infty(0, T; \mathbb{H})) \cap L^2(\Omega; L^2(0, T; \mathbb{V}))$ and $\lambda > 0$, into inequality (5.14) implies the following

$$\begin{aligned} & \lambda \mathbb{E} \left[\int_0^T \partial_t \rho(t) \|w(t)\|_{\mathbb{L}^2}^2 dt \right] \\ & \leq \mathbb{E} \left[2 \int_0^T \rho(t) \langle R_0(t) - R(v + \lambda w) + F_0(t) - f(t, v + \lambda w), w(t) \rangle dt \right]. \end{aligned}$$

Taking into account the hemi-continuity of operator R and the fact the f is Lipschitz-continuous with respect to its space-variable, one may take $\lambda \rightarrow 0$ to obtain

$$R_0 + F_0 = R(v) + f(\cdot, v) \text{ in } L^2(\Omega; L^2(0, T; \mathbb{H}^{-1})).$$

5.2 Convergence when α is fixed

Step 1: Boundedness

We aim here at bounding each term of equation (4.1) in a reflexive Banach space. By virtue of Lemmas 4.1 and 4.3, the sequences $\{u_{k,h}^+\}_{k,h}$ and $\{\mathcal{V}_{k,h}^+\}_{k,h}$ are bounded in $L^{2p}(\Omega; L^\infty(0, T; \mathbb{L}^2) \cap L^2(0, T; \mathbb{H}_0^1))$ and $L^{2p}(\Omega; L^\infty(0, T; \mathbb{H}_0^1) \cap L^2(0, T; \mathbb{H}_0^1))$, respectively. In addition, one may bound the sequence $\{u_{k,h}^+ \times (\nabla \times \mathcal{V}_{k,h}^+)\}_{k,h}$ in $L^{2p}(\Omega; L^2(0, T; \mathbb{H}^{-1}))$ as follows:

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^T \|u_{k,h}^+ \times (\nabla \times \mathcal{V}_{k,h}^+)\|_{\mathbb{H}^{-1}}^2 dt \right)^{2p-1} \right] \leq C_D \mathbb{E} \left[\left(\int_0^T \|\nabla u_{k,h}^+\|_{\mathbb{L}^2}^2 \|\nabla \mathcal{V}_{k,h}^+\|_{\mathbb{L}^2}^2 dt \right)^{2p-1} \right] \\ & \leq C_D \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\nabla u_{k,h}^+\|_{\mathbb{L}^2}^{2p+1} \right]^{1/2} \mathbb{E} \left[\left(\int_0^T \|\nabla \mathcal{V}_{k,h}^+\|_{\mathbb{L}^2}^2 dt \right)^{2p-1} \right]^{1/2}, \end{aligned}$$

thanks to Proposition 2.1-(iv). The right-hand side is bounded by a constant due to Lemmas 4.1-(iii) and 4.3-(ii). Moreover, for $p \geq 1$, $\{f^-(\cdot, u_{k,h}^-)\}_{k,h}$ is bounded in $L^{2p}(\Omega; L^2(0, T; \mathbb{H}^{-1}))$; indeed, making use of assumption (S_2) , inequality (2.5), and Lemma 4.1-(iii), it follows

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^T \|f^-(t, u_{k,h}^-)\|_{\mathbb{H}^{-1}}^2 dt \right)^{2p-1} \right] \leq \mathbb{E} \left[\left(2K_3^2 T + 2K_4^2 \int_0^T \|u_{k,h}^-\|_{\mathbb{L}^2}^2 dt \right)^{2p-1} \right] \\ & \leq 2^{2p-1-1} \mathbb{E} \left[2^{2p-1} K_3^{2p} T^{2p-1} + 2^{2p-1} K_4^{2p} T^{2p-1} \max_{1 \leq m \leq M} \|U^{m-1}\|_{\mathbb{L}^2}^{2p} \right] \leq 2^{2p-1} T^{2p-1} (K_3^{2p} + K_4^{2p} C_{T,p}). \end{aligned}$$

By applying the same technique, one obtained the boundedness of $\{g^-(\cdot, u_{k,h}^-)\}_{k,h}$ in the reflexive Banach space $L^{2p}(\Omega; L^2(0, T; \mathcal{L}_2(K, \mathbb{L}^2)))$. Therewith, by virtue of Lemmas 4.1-(iii)-(iv), 4.3-(ii), there are two sub-sequences $\{\mathcal{V}_{k',h'}^+\}_{k',h'}$ and $\{u_{k',h'}^+\}_{k',h'}$ of $\{\mathcal{V}_{k,h}^+\}_{k,h}$ and $\{u_{k,h}^+\}_{k,h}$, respectively, permitting the following

convergences

$$\mathcal{V}_{k',h'}^+ \rightharpoonup v_\alpha \quad \& \quad \mathcal{U}_{k',h'}^+ \rightharpoonup u_\alpha \quad \text{in } L^{2p}(\Omega; L^2(0, T; \mathbb{H}_0^1)), \quad (5.15)$$

$$\mathcal{V}_{k',h'}^+ \xrightarrow{*} v_\alpha \quad \text{in } L^{2p}(\Omega; L^\infty(0, T; \mathbb{L}^2)), \quad (5.16)$$

$$\mathcal{U}_{k',h'}^+ \xrightarrow{*} u_\alpha \quad \text{in } L^{2p}(\Omega; L^\infty(0, T; \mathbb{H}_0^1)), \quad (5.17)$$

$$\mathcal{U}_{k',h'}^+ \times (\nabla \times \mathcal{V}_{k',h'}^+) \rightharpoonup B_0 \quad \text{in } L^{2p}(\Omega; L^2(0, T; \mathbb{H}^{-1})), \quad (5.18)$$

$$f^-(\cdot, \mathcal{U}_{k',h'}^-) \rightharpoonup F_0 \quad \text{in } L^{2p}(\Omega; L^2(0, T; \mathbb{H}^{-1})), \quad (5.19)$$

$$g^-(\cdot, \mathcal{U}_{k',h'}^-) \rightharpoonup G_0 \quad \text{in } L^{2p}(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2))), \quad (5.20)$$

for some functions $v_\alpha \in L^{2p}(\Omega; L^\infty(0, T; \mathbb{L}^2) \cap L^2(0, T; \mathbb{H}_0^1))$, $u_\alpha \in L^{2p}(\Omega; L^\infty(0, T; \mathbb{H}_0^1) \cap L^2(0, T; \mathbb{H}_0^1))$, $B_0 \in L^{2p}(\Omega; L^2(0, T; \mathbb{H}^{-1}))$, $F_0 \in L^{2p}(\Omega; L^2(0, T; \mathbb{H}^{-1}))$ and $G_0 \in L^{2p}(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$.

Remark 5.1 For $p \in [1, +\infty)$, since $L^{2p}(\Omega) \hookrightarrow L^2(\Omega)$ continuously, all the above obtained convergence results hold particularly in $L^2(\Omega; \mathcal{X})$, where \mathcal{X} corresponds to the space of each convergence.

Step 2: $v_\alpha = u_\alpha + \alpha^2 Au_\alpha$ and properties of u_α

By convergence (5.15) and Lemmas-4.1-(iv), 4.3-(ii), it holds

$$\mathbb{E} \left[\left(\int_0^T \|\nabla v_\alpha(t)\|_{\mathbb{L}^2}^2 dt \right)^{2p-1} \right] \leq C_p(\alpha) \quad \text{and} \quad \mathbb{E} \left[\left(\int_0^T \|\nabla u_\alpha(t)\|_{\mathbb{L}^2}^2 dt \right)^{2p-1} \right] \leq C_{T,p}.$$

Moreover, by convergences (5.16)-(5.17) and Lemmas 4.1-(iii), 4.3-(ii), we get

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|v_\alpha(t)\|_{\mathbb{L}^2}^{2p} \right] \leq C_p(\alpha) \quad \text{and} \quad \mathbb{E} \left[\sup_{0 \leq t \leq T} \|u_\alpha(t)\|_{\mathbb{H}^1}^{2p} \right] \leq C_{T,p}.$$

Let $w \in L^2(\Omega; L^2(0, T; D(A)))$ be arbitrary. From equation (4.1), it follows that

$$\mathbb{E} \left[\int_0^T \left(\mathcal{V}_{k',h'}^+, \Pi_{h'} w(t) \right) dt \right] = \mathbb{E} \left[\int_0^T \left(\mathcal{U}_{k',h'}^+, \Pi_{h'} w(t) \right) dt \right] + \alpha^2 \mathbb{E} \left[\int_0^T \left(\nabla \mathcal{U}_{k',h'}^+, \nabla \Pi_{h'} w(t) \right) dt \right].$$

Taking into account that $\Pi_{h'} w \rightarrow w$ as $h' \rightarrow 0$, strongly in $L^2(\Omega; L^2(0, T; \mathbb{H}^1))$ together with convergence (5.15) and the embedding $L^{2p}(\Omega) \hookrightarrow L^2(\Omega)$, one gets

$$\mathbb{E} \left[\int_0^T (v_\alpha(t), w(t)) dt \right] = \mathbb{E} \left[\int_0^T (u_\alpha(t), w(t)) dt \right] + \alpha^2 \mathbb{E} \left[\int_0^T (\nabla u_\alpha(t), \nabla w(t)) dt \right],$$

for all $w \in L^2(\Omega; L^2(0, T; D(A)))$, which implies $v_\alpha = u_\alpha + \alpha^2 Au_\alpha$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$. Owing to convergences (5.16) and (5.17), $\alpha^2 Au_\alpha = v_\alpha - u_\alpha \in L^2(\Omega; L^\infty(0, T; \mathbb{L}^2))$. Subsequently, $v_\alpha = u_\alpha + \alpha^2 Au_\alpha$, \mathbb{P} -a.s. and a.e. in $(0, T) \times D$. This implies

$$\begin{aligned} \mathbb{E} \left[\left(\alpha^4 \int_0^T \|Au_\alpha(t)\|_{\mathbb{L}^2}^2 dt \right)^{2p-1} \right] &\leq 2^{2p-1} T^{2p-1} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|v_\alpha(t)\|_{\mathbb{L}^2}^{2p} + \sup_{0 \leq t \leq T} \|u_\alpha(t)\|_{\mathbb{L}^2}^{2p} \right] \\ &\leq 2^{2p-1} T^{2p-1} (C_p(\alpha) + C_{T,p}), \end{aligned}$$

thanks to inequality (2.5).

The weak time-continuity of the process u_α (i.e. $u_\alpha \in L^2(\Omega; C_w([0, T]; \mathbb{H}^1))$) can be illustrated through the time-continuity of v_α (see step 3).

Step 3: Auxiliary scheme

Owing to equation (4.1) and the convergence results (5.15)-(5.20), the stochastic process $v_\alpha(t)$, $t \in [0, T]$ belongs to $L^2(\Omega; C([0, T]; \mathbb{L}^2))$ (e.g. [33]), and fulfills \mathbb{P} -a.s., for all $t \in [0, T]$ and $\varphi \in \mathbb{V}$ the equation

$$\begin{aligned} & (v_\alpha(t), \varphi) + \nu \int_0^t (\nabla v_\alpha(s), \nabla \varphi) ds - \int_0^t \langle B_0(s), \varphi \rangle ds \\ &= (v_0, \varphi) + \int_0^t \langle F_0(s), \varphi \rangle ds + \int_0^t (G_0(s) dW(s), \varphi). \end{aligned} \quad (5.21)$$

In order to identify the obtained limiting functions in Step 1 with their counterparts, we need to do a subtraction in one way or another allowing us to appear the difference between the abstract and the solution-dependent functions, for instance $\|F_0 - f(\cdot, u_\alpha)\|$ for some norm $\|\cdot\|$ to be determined later. Since neither the employed finite element method involves strong divergence-free vector fields as test functions nor the space of strongly divergence-free vector fields is included in the space of weakly divergence-free vectors fields, the subtraction of equation (5.21) from the scheme (4.1) does not seem to make any sense. Instead, we propose a fully discrete auxiliary scheme arising from equation (5.21) and permitting the subtraction we mentioned shortly before. We will make use of the time and space discretizations that were introduced in Section 2. We define a starting point of the auxiliary scheme $(V_\alpha^0, U_\alpha^0) = (V^0, U^0)$ and the discrete versions of B_0, F_0 and G_0 as follows: for all $m \in \{1, \dots, M\}$,

$$\begin{aligned} B_0^m &= \frac{1}{k} \int_{t_{m-1}}^{t_m} B_0(t) dt, \\ (F_0^m, G_0^m) &= (F_0(0), G_0(0)), \text{ and } (F_0^m, G_0^m) = \left(\frac{1}{k} \int_{t_{m-1}}^{t_m} F_0(t) dt, \frac{1}{k} \int_{t_{m-1}}^{t_m} G_0(t) dt \right). \end{aligned} \quad (5.22)$$

Such approximations will be required within the last step for the sake of obtaining strong convergence in time toward their non-discretized counterparts. For all $(\varphi, \psi) \in \mathbb{V}_h \times \mathbb{V}_h$ and for every $m \in \{1, \dots, M\}$, the auxiliary scheme reads

$$\begin{cases} (V_\alpha^m - V_\alpha^{m-1}, \varphi) + k\nu(\nabla V_\alpha^m, \nabla \varphi) = k \langle B_0^m, \varphi \rangle + k \langle F_0^{m-1}, \varphi \rangle + (G_0^{m-1} \Delta_m W, \varphi), \\ (V_\alpha^m, \psi) = (U_\alpha^m, \psi) + \alpha^2 (\nabla U_\alpha^m, \nabla \psi). \end{cases} \quad (5.23)$$

Equation (5.21) can be considered as a stochastic Stokes problem driven by an additive noise with diffusion coefficient $G_0 \in L^2(\Omega; L^\infty(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$, initial datum $v_0 \in L^4(\Omega; \mathbb{L}^2)$ and a source term $B_0 + F_0 \in L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$. For all $m \in \{1, \dots, M\}$, define the following piecewise constant processes:

$$\begin{aligned} (\mathcal{V}_\alpha^+(t), \mathcal{U}_\alpha^+(t), B_0^+(t)) &= (V_\alpha^m, U_\alpha^m, B_0^m), \quad \forall t \in (t_{m-1}, t_m], \\ (\mathcal{V}_\alpha^-(t), \mathcal{U}_\alpha^-(t), F_0^-(t), G_0^-(t)) &= (V_\alpha^{m-1}, U_\alpha^{m-1}, F_0^{m-1}, G_0^{m-1}), \quad \forall t \in [t_{m-1}, t_m). \end{aligned}$$

For a complete investigation of scheme (5.23), the reader may refer to [17], from which we pick the following convergence results as $k, h \rightarrow 0$ (see Theorem 6 therein):

$$\mathcal{V}_\alpha^+ \rightarrow v_\alpha \text{ in } L^2(\Omega; L^\infty(0, T; \mathbb{L}^2)) \quad \text{and} \quad \mathcal{V}_\alpha^+ \rightarrow v_\alpha \text{ in } L^2(\Omega; L^2(0, T; \mathbb{H}_0^1)). \quad (5.24)$$

The above convergences are in the strong sense. Taking now the second equation of (5.23) and plugging it back into its first one, we get

$$((U_\alpha^m - U_\alpha^{m-1}, \varphi))_\alpha + k\nu((U_\alpha^m, \varphi))_{\alpha, h} = k \langle B_0^m, \varphi \rangle + k \langle F_0^{m-1}, \varphi \rangle + (G_0^{m-1} \Delta_m W, \varphi),$$

for all $\varphi \in \mathbb{V}_h$, where $((\cdot, \cdot))_\alpha := (\cdot, \cdot) + \alpha^2 (\nabla \cdot, \nabla \cdot)$ and $((\cdot, \cdot))_{\alpha, h} := (\nabla \cdot, \nabla \cdot) + \alpha^2 (\Delta^h \cdot, \Delta^h \cdot)$ can be treated in the same way as the \mathbb{L}^2 and the \mathbb{H}_0^1 -inner products, respectively. Thereby, applying once again [17, Theorem 6] yields the following strong convergence as $k, h \rightarrow 0$:

$$\mathcal{U}_\alpha^+ \rightarrow u_\alpha \text{ in } L^2(\Omega; L^\infty(0, T; \mathbb{H}^1)). \quad (5.25)$$

Notice that the limiting function in (5.25) is u_α which turns out to be true after adjusting the \mathbb{L}^2 and the \mathbb{H}_0^1 -inner products in equation (5.21) to fit the framework of that of the scheme, through the identity $v_\alpha = u_\alpha + \alpha^2 A u_\alpha$ that was illustrated in Step 2. We point out that $((\cdot, \cdot))_{\alpha, h}$ corresponds in this case to $(\nabla \cdot, \nabla \cdot) + \alpha^2 (\Delta \cdot, \Delta \cdot)$, which makes sense because $u_\alpha \in L^2(\Omega; L^2(0, T; D(A)))$. We still need to exhibit the a priori estimates satisfied by $\{V_\alpha^m\}_m$ and $\{U_\alpha^m\}_m$ before moving on to the next step. To this end, we replace φ by V_α^m in equation (5.23), then we follow the same demonstration technique of Lemma 4.3 to obtain eventually for all $p \in [1, +\infty)$,

$$\mathbb{E} \left[\max_{1 \leq m \leq M} \|V_\alpha^m\|_{\mathbb{L}^2}^{2p} + \left(k\nu \sum_{m=1}^M \|\nabla V_\alpha^m\|_{\mathbb{L}^2}^2 \right)^{2p-1} + \left(\sum_{m=1}^M \|V_\alpha^m - V_\alpha^{m-1}\|_{\mathbb{L}^2}^2 \right)^{2p-1} \right] \leq C_{\alpha, v}, \quad (5.26)$$

where $C_{\alpha, v} > 0$ does not depend on k and h . On the other hand, replacing φ in scheme (5.23) with U_α^m , and following the proof steps of Lemma 4.1, we acquire for all $p \in [1, +\infty)$:

$$\mathbb{E} \left[\max_{1 \leq m \leq M} \|U_\alpha^m\|_{\alpha}^{2p} + \sum_{m=1}^M \|U_\alpha^m - U_\alpha^{m-1}\|_{\alpha}^2 + \left(k\nu \sum_{m=1}^M \|U_\alpha^m\|_{\alpha, h}^2 \right)^{2p-1} \right] \leq C_{\alpha, u}, \quad (5.27)$$

for some $C_{\alpha, u} > 0$ independent of k and h , where $\|\cdot\|_{\alpha, h}^2 := \|\nabla \cdot\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta \cdot\|_{\mathbb{L}^2}^2$.

Step 4: Identification of B_0, F_0 and G_0

From now on, the indices k' and h' that were derived in Step 1, will be denoted k and h for the sake of clarity. For each $n \in \mathbb{N} \setminus \{0\}$ and for all $m \in \{0, 1, \dots, M\}$, we define the discrete stopping time

$$\tau_m^n := \min \left(t_m, \min_{0 \leq \ell \leq m} \left\{ t_\ell \mid \|u_\alpha(t_\ell)\|_{\alpha}^2 + \int_0^{t_\ell} \|u_\alpha(s)\|_{D(A)}^2 ds > n \right\} \right).$$

Obviously, τ_m^n belongs to the time discretization grid points $\{t_0, \dots, t_M\}$. It is worth mentioning that when $k \rightarrow 0$ (i.e. $M \rightarrow +\infty$), τ_m^n converges \mathbb{P} -a.s. toward the following \mathcal{F}_s -stopping time

$$\tau^n := \min \left(t, \inf \left\{ s \in [0, t] \mid \|u_\alpha(s)\|_{\alpha}^2 + \int_0^s \|u_\alpha(\lambda)\|_{D(A)}^2 d\lambda > n \right\} \right).$$

For all $m \in \{1, \dots, M\}$, we define the discrete weight

$$\rho_m := \exp \left(-\eta_1 t_m - \eta_2 \int_0^{t_m} \|u_\alpha(s)\|_{\mathbb{H}^1} \|u_\alpha(s)\|_{\mathbb{H}^2} ds \right) =: \rho(t_m),$$

where $\eta_1, \eta_2 > 0$ are to be fixed later. Clearly, ρ_m is \mathcal{F}_{t_m} -measurable and is non-increasing. In addition to that, we need to deal with a piecewise-constant version of ρ_m , which is why we introduce first the notations

$$i^+(t) = t_m, \quad \forall t \in (t_{m-1}, t_m] \quad \text{and} \quad i^-(t) = t_{m-1}, \quad \forall t \in [t_{m-1}, t_m).$$

We therefore set

$$\rho^+(t) = \rho(i^+(t)), \quad \forall t \in (t_{m-1}, t_m] \quad \text{and} \quad \rho^-(t) = \rho(i^-(t)), \quad \forall t \in [t_{m-1}, t_m).$$

We subtract both equations (4.1) and (5.23), then apply identity (2.4):

$$\begin{aligned} & \frac{1}{2} \|U_\alpha^m - U^m\|_{\alpha}^2 - \frac{1}{2} \|U_\alpha^{m-1} - U^{m-1}\|_{\alpha}^2 + k\nu \|U_\alpha^m - U^m\|_{\alpha, h}^2 \\ & \leq k \langle B_0^m - U^m \times (\nabla \times V^m), U_\alpha^m - U^m \rangle + k \langle F_0^{m-1} - f(t_{m-1}, U^{m-1}), U_\alpha^m - U^m \rangle \\ & \quad + ([G_0^{m-1} - g(t_{m-1}, U^{m-1})] \Delta_m W, U_\alpha^{m-1} - U^{m-1}) \\ & \quad + \frac{1}{2} \|[G_0^{m-1} - g(t_{m-1}, U^{m-1})] \Delta_m W\|_{\mathbb{L}^2}^2. \end{aligned} \quad (5.28)$$

The next step would be to multiply equation (5.28) by ρ_{m-1} and sum it over m . However, we must clarify a few identities before. Let $\ell \in \{1, \dots, M\}$ be arbitrary. By Proposition 5.1, it holds that

$$\begin{aligned} \sum_{m=1}^{\ell} \rho_{m-1} (\|U_{\alpha}^m - U^m\|_{\alpha}^2 - \|U_{\alpha}^{m-1} - U^{m-1}\|_{\alpha}^2) &= \int_0^{t_{\ell}} \rho^{-}(t) dt \|u_{\alpha}^{+} - u_{k,h}^{+}\|_{\alpha}^2 dt \\ &= \rho^{+}(t_{\ell}) \|u_{\alpha}^{+}(t_{\ell}) - u_{k,h}^{+}(t_{\ell})\|_{\alpha}^2 - \int_0^{t_{\ell}} \|u_{\alpha}^{+} - u_{k,h}^{+}\|_{\alpha}^2 dt \rho^{+}(t) dt. \end{aligned} \quad (5.29)$$

Moreover,

$$\begin{aligned} \sum_{m=1}^{\ell} \rho_{m-1} ([G_0^{m-1} - g(t_{m-1}, U^{m-1})] \Delta_m W, U_{\alpha}^{m-1} - U^{m-1}) \\ = \int_D \int_0^{t_{\ell}} \rho^{-}(t) (u_{\alpha}^{-} - u_{k,h}^{-}) \cdot [G_0^{-}(t) - g^{-}(t, u_{k,h}^{-})] dW(t) dx =: M_1(t_{\ell}). \end{aligned} \quad (5.30)$$

And,

$$\begin{aligned} \sum_{m=1}^{\ell} \rho_{m-1} \|[G_0^{m-1} - g(t_{m-1}, U^{m-1})] \Delta_m W\|_{\mathbb{L}^2}^2 \\ = \sum_{m=1}^{\ell} \left\| \int_{t_{m-1}}^{t_m} \sqrt{\rho^{-}(t)} [G_0^{-}(t) - g^{-}(t, u_{k,h}^{-})] dW(t) \right\|_{\mathbb{L}^2}^2 =: M_2(t_{\ell}). \end{aligned} \quad (5.31)$$

Subsequently, we multiply equation (5.28) by ρ_{m-1} , sum it over m from 1 to $\ell \in \{1, \dots, M\}$, make use of (5.29)-(5.31), replace afterwards the node t_{ℓ} by the discrete stopping time τ_{m-1}^n , and then apply the mathematical expectation to get

$$\begin{aligned} \frac{1}{2} \mathbb{E} \left[\rho^{+}(\tau_{m-1}^n) \|u_{\alpha}^{+}(\tau_{m-1}^n) - u_{k,h}^{+}(\tau_{m-1}^n)\|_{\alpha}^2 \right] + \nu \mathbb{E} \left[\int_0^{\tau_{m-1}^n} \rho^{-}(t) \|u_{\alpha}^{+} - u_{k,h}^{+}\|_{\alpha,h}^2 dt \right] \\ \leq \mathbb{E} \left[\int_0^{\tau_{m-1}^n} \rho^{-}(t) \left\langle B_0^{+}(t) - u_{k,h}^{+} \times (\nabla \times v_{k,h}^{+}) + F_0^{-}(t) - f^{-}(t, u_{k,h}^{-}), u_{\alpha}^{+} - u_{k,h}^{+} \right\rangle dt \right] \\ + \frac{1}{2} M_2(\tau_{m-1}^n) + \frac{1}{2} \int_0^{\tau_{m-1}^n} \|u_{\alpha}^{+} - u_{k,h}^{+}\|_{\alpha}^2 dt \rho^{+}(t) dt, \end{aligned} \quad (5.32)$$

where $\mathbb{E} [M_1(\tau_{m-1}^n)] = 0$ due to assumption (S_2) which implies that the integrand of M_1 is a martingale, together with the fact that τ_{m-1}^n is a stopping time. Next, we handle $M_2(\tau_{m-1}^n)$ as follows:

$$\begin{aligned} \mathbb{E} [M_2(\tau_{m-1}^n)] &= \sum_{i=1}^{m-1} \mathbb{E} \left[\left\| \int_{\tau_{i-1}^n}^{\tau_i^n} \sqrt{\rho^{-}(t)} [G_0^{-}(t) - g^{-}(t, u_{k,h}^{-})] dW(t) \right\|_{\mathbb{L}^2}^2 \right] \\ &= \sum_{i=1}^{m-1} \mathbb{E} \left[\int_{\tau_{i-1}^n}^{\tau_i^n} \rho^{-}(t) \left\| G_0^{-}(t) - g^{-}(t, u_{k,h}^{-}) \right\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right] \\ &\leq Tr(Q) \mathbb{E} \left[\int_0^{\tau_{m-1}^n} \rho^{-}(t) \left\| G_0^{-}(t) - g^{-}(t, u_{k,h}^{-}) \right\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 dt \right], \end{aligned} \quad (5.33)$$

thanks to the Itô isometry. On the other hand,

$$\begin{aligned} \|G_0^{-}(t) - g^{-}(t, u_{k,h}^{-})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 &= \|g^{-}(t, u_{\alpha}(t)) - g^{-}(t, u_{k,h}^{-})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \\ &+ 2 \left(G_0^{-}(t) - g^{-}(t, u_{k,h}^{-}), G_0^{-}(t) - g^{-}(t, u_{\alpha}(t)) \right)_{\mathcal{L}_2(K, \mathbb{L}^2)} - \|g^{-}(t, u_{\alpha}(t)) - G_0^{-}(t)\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \\ &\leq 2L_g^2 \|u_{\alpha}(t) - u_{\alpha}^{-}\|_{\alpha}^2 + 2L_g^2 \|u_{\alpha}^{-} - u_{k,h}^{-}\|_{\alpha}^2 \\ &+ 2 \left(G_0^{-}(t) - g^{-}(t, u_{k,h}^{-}), G_0^{-}(t) - g^{-}(t, u_{\alpha}(t)) \right)_{\mathcal{L}_2(K, \mathbb{L}^2)} - \|g^{-}(t, u_{\alpha}(t)) - G_0^{-}(t)\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2, \end{aligned} \quad (5.34)$$

where the Lipschitz-continuity of g has been utilized. Before heading toward the calculation of $d_t\rho^+(t)$, we define

$$\gamma^+(t) := -\eta_1 t_m - \eta_2 \int_0^{t_m} \|u_\alpha(s)\|_{\mathbb{H}^1} \|u_\alpha(s)\|_{\mathbb{H}^2} ds, \quad \text{for all } t \in (t_{m-1}, t_m], \quad \text{for all } m \in \{1, \dots, M\}.$$

The function γ^- can be defined accordingly. We point out that $\rho^+ = e^{\gamma^+}$. By Proposition 5.1, it holds for $t \in (t_{m-1}, t_m)$,

$$d_t\rho^+(t) = -\eta_1\rho^-(t) - \eta_2\rho^-(t) \frac{1}{k} \int_{i^-(t)}^{i^+(t)} \|u_\alpha(s)\|_{\mathbb{H}^1} \|u_\alpha(s)\|_{\mathbb{H}^2} ds + e^{\xi(t)} \frac{(\gamma^+ - \gamma^-)^2}{2k},$$

for all $m \in \{1, \dots, M\}$, for some $\xi \in (\gamma^-, \gamma^+)$. Now, for each $m \in \{1, \dots, M\}$, there is a $\zeta_m \in (t_{m-1}, t_m)$ such that

$$\int_{t_{m-1}}^{t_m} \|u_\alpha(s)\|_{\mathbb{H}^1} \|u_\alpha(s)\|_{\mathbb{H}^2} ds = k \|u_\alpha(\zeta_m)\|_{\mathbb{H}^1} \|u_\alpha(\zeta_m)\|_{\mathbb{H}^2},$$

thanks to the mean value theorem. Thus, for all $m \in \{1, \dots, M\}$ and $t \in (t_{m-1}, t_m)$,

$$d_t\rho^+(t) = -\eta_1\rho^-(t) - \eta_2\rho^-(t) \|u_\alpha(\zeta_m)\|_{\mathbb{H}^1} \|u_\alpha(\zeta_m)\|_{\mathbb{H}^2} + e^{\xi(t)} \frac{(\gamma^+ - \gamma^-)^2}{2k}. \quad (5.35)$$

Furthermore, we take advantage of all (i), (iv) and (v) of Proposition 2.1 to write for all $t \in (t_{m-1}, t_m)$:

$$\begin{aligned} & \langle B_0^+(t) - \mathbf{u}_{k,h}^+ \times (\nabla \times \mathcal{V}_{k,h}^+), \mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+ \rangle \leq \langle B_0^+(t) - \mathbf{u}_\alpha^+ \times (\nabla \times \mathcal{V}_\alpha^+), \mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+ \rangle \\ & + C_D \|\nabla(\mathbf{u}_\alpha^+ - u_\alpha(t))\|_{\mathbb{L}^2} \|\nabla(\mathcal{V}_\alpha^+ - \mathcal{V}_{k,h}^+)\|_{\mathbb{L}^2} \|\nabla(\mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+)\|_{\mathbb{L}^2} \\ & + C_D \|\nabla(u_\alpha(t) - u_\alpha(\zeta_m))\|_{\mathbb{L}^2} \|\nabla(\mathcal{V}_\alpha^+ - \mathcal{V}_{k,h}^+)\|_{\mathbb{L}^2} \|\nabla(\mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+)\|_{\mathbb{L}^2} \\ & + C_D \|u_\alpha(\zeta_m)\|_{\mathbb{H}^1}^{\frac{1}{2}} \|u_\alpha(\zeta_m)\|_{\mathbb{H}^2}^{\frac{1}{2}} \|\mathcal{V}_\alpha^+ - \mathcal{V}_{k,h}^+\|_{\mathbb{L}^2} \|\nabla(\mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+)\|_{\mathbb{L}^2}. \end{aligned} \quad (5.36)$$

The last term can be bounded through Young's inequality by

$$\frac{2C_D^2 \max(C_D, \alpha^2)^2}{\nu\alpha^4} \|u_\alpha(\zeta_m)\|_{\mathbb{H}^1} \|u_\alpha(\zeta_m)\|_{\mathbb{H}^2} \|\mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+\|_\alpha^2 + \frac{\nu}{4} \|\mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+\|_{\alpha,h}^2,$$

where the estimate $\|\mathcal{V}_\alpha^+ - \mathcal{V}_{k,h}^+\|_{\mathbb{L}^2} \leq \max(C_D, \alpha^2) \|\mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+\|_{\alpha,h}$ was employed. Moreover, by assumption (S_2) , Young's inequality and $\|\cdot\|_{\mathbb{H}^1} \leq C_D \|\cdot\|_{\alpha,h}$, it follows

$$\begin{aligned} & \langle F_0^-(t) - f^-(t, \mathbf{u}_{k,h}^-), \mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+ \rangle \leq \langle F_0^-(t) - f^-(t, \mathbf{u}_\alpha^-), \mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+ \rangle \\ & + \frac{L_f^2 C_D^2}{\nu} \|\mathbf{u}_\alpha^- - \mathbf{u}_{k,h}^-\|_\alpha^2 + \frac{\nu}{4} \|\mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+\|_{\alpha,h}^2. \end{aligned} \quad (5.37)$$

On the other hand, since $U_\alpha^0 = U^0$ and $(\rho_m)_m$ is non-increasing, the following holds:

$$\int_0^{\tau_{m-1}^n} \rho^- \|\mathbf{u}_\alpha^- - \mathbf{u}_{k,h}^-\|_\alpha^2 dt \leq \int_0^{\tau_{m-1}^n} \rho^+ \|\mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+\|_\alpha^2 dt \leq \int_0^{\tau_{m-1}^n} \rho^- \|\mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+\|_\alpha^2 dt. \quad (5.38)$$

By setting $\eta_1 = 2 \frac{L_f^2 C_D^2}{\nu} + 2Tr(Q) L_g^2$ and $\eta_2 = \frac{4C_D^2 \max(C_D, \alpha^2)^2}{\nu\alpha^4}$ and after assembling the obtained

estimates (5.33)-(5.38) together, equation (5.32) becomes

$$\begin{aligned}
 & \frac{1}{2} \mathbb{E} \left[\rho^+(\tau_{m-1}^n) \| \mathbf{u}_\alpha^+(\tau_{m-1}^n) - \mathbf{u}_{k,h}^+(\tau_{m-1}^n) \|_\alpha^2 + \nu \int_0^{\tau_{m-1}^n} \rho^- \| \mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+ \|_{\alpha,h}^2 dt \right. \\
 & \left. + Tr(Q) \int_0^{\tau_{m-1}^n} \rho^- \| g^-(t, u_\alpha(t)) - G_0^-(t) \|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 dt \right] \\
 & \leq \mathbb{E} \left[\int_0^{\tau_{m-1}^n} e^{\xi} \frac{(\gamma^+ - \gamma^-)^2}{4k} \| \mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+ \|_\alpha^2 dt + \int_0^{\tau_{m-1}^n} \rho^- \langle B_0^+ - \mathbf{u}_\alpha^+ \times (\nabla \times \mathcal{V}_\alpha^+), \mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+ \rangle dt \right. \\
 & \left. + C_D \int_0^{\tau_{m-1}^n} \rho^- \| \nabla(\mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+) \|_{\mathbb{L}^2} \| \nabla(\mathcal{V}_\alpha^+ - \mathcal{V}_{k,h}^+) \|_{\mathbb{L}^2} \| \nabla(\mathbf{u}_\alpha^+ - u_\alpha(t)) \|_{\mathbb{L}^2} dt \right. \\
 & \left. + C_D \int_0^{\tau_{m-1}^n} \rho^- \| \nabla(u_\alpha(t) - u_\alpha(\zeta_m)) \|_{\mathbb{L}^2} \| \nabla(\mathcal{V}_\alpha^+ - \mathcal{V}_{k,h}^+) \|_{\mathbb{L}^2} \| \nabla(\mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+) \|_{\mathbb{L}^2} dt \right. \\
 & \left. + \int_0^{\tau_{m-1}^n} \rho^- \langle F_0^-(t) - f^-(t, \mathbf{u}_\alpha^-), \mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+ \rangle dt + Tr(Q) L_g^2 \int_0^{\tau_{m-1}^n} \rho^- \| u_\alpha(t) - \mathbf{u}_\alpha^- \|_\alpha^2 dt \right. \\
 & \left. + Tr(Q) \int_0^{\tau_{m-1}^n} \rho^- \left(G_0^-(t) - g^-(t, \mathbf{u}_{k,h}^-), G_0^-(t) - g^-(t, u_\alpha(t)) \right)_{\mathcal{L}_2(K; \mathbb{L}^2)} dt \right] =: I + \dots + VII. \tag{5.39}
 \end{aligned}$$

Step 4.1: $(\rho_m)_m$ is strongly convergent in $L^4(\Omega; L^2(0, T))$

We recall the notation $\rho_m = \rho^+(t) = \rho(i^+(t))$ for $t \in (t_{m-1}, t_m]$. We have

$$\begin{aligned}
 & \mathbb{E} \left[\left(\int_0^T |\rho(t) - \rho(i^+(t))|^2 dt \right)^2 \right] \leq T \mathbb{E} \left[\int_0^T \left| e^{-\eta_1(t-i^+(t)) - \eta_2 \int_{i^+(t)}^t \|u_\alpha(s)\|_{\mathbb{H}^1} \|u_\alpha(s)\|_{\mathbb{H}^2} ds} - 1 \right|^4 dt \right] \\
 & \leq T \eta_1 \int_0^T |t - i^+(t)| dt + T \eta_2 \mathbb{E} \left[\int_0^T \int_t^{i^+(t)} \|u_\alpha(s)\|_{\mathbb{H}^1} \|u_\alpha(s)\|_{\mathbb{H}^2} ds dt \right],
 \end{aligned}$$

where Jensen's inequality and $|e^{-|x|} - 1| \leq |x|$ were employed in the first and second inequalities, respectively. Since for each $t \in (t_{m-1}, t_m)$, we have $|t - i^+(t)| \leq k$, the first term goes to 0 as $k \rightarrow 0$. Similarly, the second term converges to 0 by a simple application of the dominated convergence theorem.

Step 4.2: Convergence of I, \dots, VII to 0

We have

$$\gamma^+ - \gamma^- = -\eta_1 k - \eta_2 \int_{i^-(t)}^{i^+(t)} \|u_\alpha(s)\|_{\mathbb{H}^1} \|u_\alpha(s)\|_{\mathbb{H}^2} ds, \quad \text{for all } t \in (t_{m-1}, t_m).$$

By Jensen's inequality, one gets

$$\frac{(\gamma^+ - \gamma^-)^2}{4k} \leq \frac{\eta_1^2}{2} k + \frac{\eta_2^2}{2} \int_{i^-(t)}^{i^+(t)} \|u_\alpha(s)\|_{\mathbb{H}^1}^2 \|u_\alpha(s)\|_{\mathbb{H}^2}^2 ds.$$

This implies

$$\begin{aligned}
 I & \leq T \eta_1^2 k (C_{\alpha, u} + C_T) + \frac{\eta_2^2}{2} \mathbb{E} \left[\sup_{0 \leq t \leq T} \| \mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+ \|_\alpha^2 \sum_{m=1}^M \int_{t_{m-1}}^{t_m} \int_{t_{m-1}}^{t_m} \| u_\alpha(s) \|_{\mathbb{H}^1}^2 \| u_\alpha(s) \|_{\mathbb{H}^2}^2 ds dt \right] \\
 & \leq T \eta_1^2 k (C_{\alpha, u} + C_T) + \frac{\eta_2^2 k}{2} \mathbb{E} \left[\sup_{0 \leq t \leq T} \| \mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+ \|_\alpha^2 \sup_{0 \leq t \leq T} \| u_\alpha(t) \|_{\mathbb{H}^1}^2 \int_0^T \| u_\alpha(s) \|_{\mathbb{H}^2}^2 ds \right] \rightarrow 0,
 \end{aligned}$$

thanks to Lemmas 4.1, 4.2, estimate (5.27), and Step 2. Moving on to II , we have

$$\begin{aligned}
 II & = \mathbb{E} \left[\int_0^{\tau^n} \rho^- \langle B_0^+ - \mathbf{u}_\alpha^+ \times (\nabla \times \mathcal{V}_\alpha^+), \mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+ \rangle dt \right] \\
 & \quad + \mathbb{E} \left[\int_{\tau^n}^{\tau_{m-1}^n} \rho^- \langle B_0^+ - \mathbf{u}_\alpha^+ \times (\nabla \times \mathcal{V}_\alpha^+), \mathbf{u}_\alpha^+ - \mathbf{u}_{k,h}^+ \rangle dt \right] = II_1 + II_2.
 \end{aligned}$$

We set $II_1 = II_{1,1} + II_{1,2}$ where

$$\begin{aligned} II_{1,1} &:= \mathbb{E} \left[\int_0^{\tau^n} (\rho^- - \rho) \langle B_0^+ - \mathfrak{u}_\alpha^+ \times (\nabla \times \mathfrak{v}_\alpha^+), \mathfrak{u}_\alpha^+ - \mathfrak{u}_{k,h}^+ \rangle dt \right] \\ &\leq \|\rho^- - \rho\|_{L^4(\Omega; L^2(0, T))} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathfrak{u}_\alpha^+ - \mathfrak{u}_{k,h}^+\|_{\mathbb{H}^1}^4 \right]^{\frac{1}{4}} \mathbb{E} \left[\int_0^T \|B_0^+ - \mathfrak{u}_\alpha^+ \times (\nabla \times \mathfrak{v}_\alpha^+)\|_{\mathbb{H}^{-1}}^2 dt \right]^{\frac{1}{2}} \rightarrow 0 \end{aligned}$$

due to the strong convergence of $(\rho_m)_m$ (see Step 4.1) along with the boundedness of the remaining terms. More precisely, $\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathfrak{u}_\alpha^+ - \mathfrak{u}_{k,h}^+\|_{\mathbb{H}^1}^4 \right]$ is controlled through Lemma 4.1 and estimate (5.27), $\mathfrak{u}_\alpha^+ \times (\nabla \times \mathfrak{v}_\alpha^+)$ obeys the same bounding technique as $\mathfrak{u}_{k,h}^+ \times (\nabla \times \mathfrak{v}_{k,h}^+)$ in Step 1 where we associate with it the estimates (5.26) and (5.27). By the definition of B_0^m together with [34, Lemma 4.5], we get $\mathbb{E} \left[\int_0^T \|B_0^+\|_{\mathbb{H}^{-1}}^2 dt \right] \leq \mathbb{E} \left[\int_0^T \|B_0(t)\|_{\mathbb{H}^{-1}}^2 dt \right]$ which is bounded through the convergence (5.18). On the other hand,

$$\begin{aligned} II_{1,2} &:= \mathbb{E} \left[\int_0^{\tau^n} \rho(t) \langle B_0^+ - \mathfrak{u}_\alpha^+ \times (\nabla \times \mathfrak{v}_\alpha^+), \mathfrak{u}_\alpha^+ - \mathfrak{u}_{k,h}^+ \rangle dt \right] \\ &= \mathbb{E} \left[\int_0^{\tau^n} \rho(t) \langle B_0^+ - B_0(t), \mathfrak{u}_\alpha^+ - \mathfrak{u}_{k,h}^+ \rangle dt + \int_0^{\tau^n} \rho(t) \langle B_0(t) - u_\alpha(t) \times (\nabla \times v_\alpha(t)), \mathfrak{u}_\alpha^+ - \mathfrak{u}_{k,h}^+ \rangle dt \right. \\ &\quad \left. + \int_0^{\tau^n} \rho(t) \langle u_\alpha(t) \times (\nabla \times v_\alpha(t)) - \mathfrak{u}_\alpha^+ \times (\nabla \times \mathfrak{v}_\alpha^+), \mathfrak{u}_\alpha^+ - \mathfrak{u}_{k,h}^+ \rangle dt \right] =: J_1 + J_2 + J_3. \end{aligned}$$

Due to the specific construction of B_0^m together with [34, Lemma 4.9], it holds that $B_0^+ \rightarrow B_0$ in the space $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$ as $k, h \rightarrow 0$. Moreover, $\mathfrak{u}_\alpha^+ - \mathfrak{u}_{k,h}^+ \rightarrow 0$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^1))$ thanks to convergences (5.15) and (5.25). Therefore, $J_1 \rightarrow 0$ as $k, h \rightarrow 0$. Similarly, $J_2 \rightarrow 0$ as $k, h \rightarrow 0$ because $\mathbb{1}_{[0, \tau^n]} \rho (B_0 - u_\alpha \times (\nabla \times v_\alpha)) \in L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$ along with the weak convergence toward 0 of $\mathfrak{u}_\alpha^+ - \mathfrak{u}_{k,h}^+$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^1))$. Making use of Proposition 2.1-(iv)-(v), one gets

$$\begin{aligned} &\mathbb{E} \left[\int_0^T \|\mathbb{1}_{[0, \tau^n]}(t) \rho(t) (u_\alpha(t) \times (\nabla \times v_\alpha(t)) - \mathfrak{u}_\alpha^+ \times (\nabla \times \mathfrak{v}_\alpha^+))\|_{\mathbb{H}^{-1}}^2 dt \right] \\ &\leq 2\mathbb{E} \left[\int_0^T \|u_\alpha(t) \times (\nabla \times (v_\alpha(t) - \mathfrak{v}_\alpha^+))\|_{\mathbb{H}^{-1}}^2 dt \right] + 2\mathbb{E} \left[\int_0^T \|(u_\alpha(t) - \mathfrak{u}_\alpha^+) \times (\nabla \times \mathfrak{v}_\alpha^+)\|_{\mathbb{H}^{-1}}^2 dt \right] \\ &\lesssim \mathbb{E} \left[\sup_{0 \leq t \leq T} \|v_\alpha(t) - \mathfrak{v}_\alpha^+\|_{\mathbb{L}^2}^2 \int_0^T \|u_\alpha(t)\|_{\mathbb{H}^2}^2 dt \right] + \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\nabla(u_\alpha(t) - \mathfrak{u}_\alpha^+)\|_{\mathbb{L}^2}^2 \int_0^T \|\nabla \mathfrak{v}_\alpha^+\|_{\mathbb{L}^2}^2 dt \right]. \end{aligned}$$

Using the generalized Hölder inequality, the first term on the right-hand side can be controlled by

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|v_\alpha(t) - \mathfrak{v}_\alpha^+\|_{\mathbb{L}^2}^4 \right]^{\frac{1}{4}} \|v_\alpha - \mathfrak{v}_\alpha^+\|_{L^2(\Omega; L^\infty(0, T; \mathbb{L}^2))} \mathbb{E} \left[\left(\int_0^T \|u_\alpha(t)\|_{\mathbb{H}^2}^2 dt \right)^4 \right]^{\frac{1}{4}},$$

which tends to 0 as $k, h \rightarrow 0$. Indeed, its first term can be bounded through estimate (5.26), convergence (5.16), and Step 2, its second term goes to 0 by (5.24), and its third term is also bounded by Step 2. Similarly, using the same techniques together with convergence (5.25) imply the convergence toward 0 of the second term on the right-hand side. Therefore, one infers that $J_3 \rightarrow 0$ as $k, h \rightarrow 0$. For II_2 , since all terms of the integrand own estimates with high-moments, one can easily show that $II_2 \rightarrow 0$ after a simple application of the

Cauchy-Schwarz inequality as follows:

$$\begin{aligned} |II_2| &\leq \mathbb{E} \left[\int_0^T \mathbb{1}_{[\tau^n, \tau_{m-1}^n]}(t) dt \right]^{\frac{1}{2}} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathfrak{u}_\alpha^+ - \mathfrak{u}_{k,h}^+\|_{\mathbb{H}^1}^2 \int_0^T \|B_0^+(t) - \mathfrak{u}_\alpha^+ \times (\nabla \times \mathfrak{v}_\alpha^+)\|_{\mathbb{H}^{-1}}^2 dt \right]^{\frac{1}{2}} \\ &\leq \mathbb{E} [|\tau_{m-1}^n - \tau^n|]^{\frac{1}{2}} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathfrak{u}_\alpha^+ - \mathfrak{u}_{k,h}^+\|_{\mathbb{H}^1}^4 \right]^{\frac{1}{4}} \mathbb{E} \left[\left(\int_0^T \|B_0^+(t) - \mathfrak{u}_\alpha^+ \times (\nabla \times \mathfrak{v}_\alpha^+)\|_{\mathbb{H}^{-1}}^2 dt \right)^2 \right]^{\frac{1}{4}}. \end{aligned}$$

The first term $\mathbb{E} [|\tau_{m-1}^n - \tau^n|]$ converges toward 0 because $\tau_{m-1}^n \rightarrow \tau^n$ \mathbb{P} -a.s. as $k \rightarrow 0$. The two remaining terms can be easily controlled through Lemma 4.1-(iii), equation (5.27), and the convergence result (5.18). In addition to that, $III \rightarrow 0$, thanks to convergence (5.25), Lemma 4.1, estimates (5.27) and (5.26). Similarly, $IV \rightarrow 0$ due to the time-continuity of u_α in \mathbb{H}^1 , as mentioned in Step 2. The term V can be handled in a similar way to II by taking into consideration the construction of F_0^m along with assumption (S_2) . Further,

$$VI \leq 2Tr(Q)L_g^2 \mathbb{E} \left[\int_0^T \|u_\alpha(t) - \mathfrak{u}_\alpha^+\|_\alpha^2 dt \right] + 2Tr(Q)L_g^2 \mathbb{E} \left[\int_0^T \|\mathfrak{u}_\alpha^+ - \mathfrak{u}_\alpha^-\|_\alpha^2 dt \right].$$

The first term converges to 0, thanks to convergence (5.25). The second term can be rewritten as follows:

$$2Tr(Q)L_g^2 \mathbb{E} \left[k \sum_{m=1}^M \|U_\alpha^m - U_\alpha^{m-1}\|_\alpha^2 \right] \leq 2Tr(Q)L_g^2 k C_{\alpha,u} \rightarrow 0,$$

by virtue of estimate (5.27). It remains to treat VII . To this end, it will be split into two terms as follows

$$\begin{aligned} VII &= Tr(Q) \mathbb{E} \left[\int_0^{\tau^n} \rho^-(t) \left(G_0^- - g^-(t, \mathfrak{u}_{k,h}^-), G_0^- - g^-(t, u_\alpha(t)) \right)_{\mathcal{L}_2(K; \mathbb{L}^2)} dt \right. \\ &\quad \left. + \int_{\tau^n}^{\tau_{m-1}^n} \rho^-(t) \left(G_0^- - g^-(t, \mathfrak{u}_{k,h}^-), G_0^- - g^-(t, u_\alpha(t)) \right)_{\mathcal{L}_2(K; \mathbb{L}^2)} dt \right] =: VII_1 + VII_2. \end{aligned}$$

We have

$$\begin{aligned} VII_1 &= Tr(Q) \mathbb{E} \left[\int_0^{\tau^n} (\rho^- - \rho(t)) \left(G_0^- - g^-(t, \mathfrak{u}_{k,h}^-), G_0^- - g^-(t, u_\alpha(t)) \right)_{\mathcal{L}_2(K; \mathbb{L}^2)} dt \right. \\ &\quad \left. + \int_0^{\tau^n} \rho(t) \left(G_0^- - g^-(t, \mathfrak{u}_{k,h}^-), G_0^- - g^-(t, u_\alpha(t)) \right)_{\mathcal{L}_2(K; \mathbb{L}^2)} dt \right] =: VII_{1,1} + VII_{1,2}. \end{aligned}$$

$VII_{1,1}$ can be handled in the same way as that of $II_{1,1}$ to achieve convergence to 0. For $VII_{1,2}$,

$$\begin{aligned} VII_{1,2} &= Tr(Q) \mathbb{E} \left[\int_0^{\tau^n} \rho(t) \left(G_0^- - G_0(t), G_0^- - g^-(t, u_\alpha(t)) \right)_{\mathcal{L}_2(K; \mathbb{L}^2)} dt \right. \\ &\quad + \int_0^{\tau^n} \rho(t) \left(G_0(t) - g^-(t, \mathfrak{u}_{k,h}^-), G_0^- - G_0(t) \right)_{\mathcal{L}_2(K; \mathbb{L}^2)} dt \\ &\quad + \int_0^{\tau^n} \rho(t) \left(G_0(t) - g^-(t, \mathfrak{u}_{k,h}^-), G_0(t) - g(t, u_\alpha(t)) \right)_{\mathcal{L}_2(K; \mathbb{L}^2)} dt \\ &\quad \left. + \int_0^{\tau^n} \rho(t) \left(G_0(t) - g^-(t, \mathfrak{u}_{k,h}^-), g(t, u_\alpha(t)) - g^-(t, u_\alpha(t)) \right)_{\mathcal{L}_2(K; \mathbb{L}^2)} dt \right]. \end{aligned}$$

Owing to the construction of G_0^m together with [34, Lemma 4.9], it holds that $G_0^- \rightarrow G_0$ in the space $L^2(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$. Adding on top of that the boundedness properties of the terms G_0^- , $g^-(\cdot, u_\alpha)$, G_0 and $g^-(\cdot, \mathfrak{u}_{k,h}^-)$ (such as Lemma 4.1, Step 2, and assumption (S_2)), we infer that the first and second terms on the right-hand side go to 0 as $k, h \rightarrow 0$. Third term also goes to 0 due to convergence (5.20) and the fact that $\mathbb{1}_{[0, \tau^n]} \rho (G_0 - g(\cdot, u_\alpha)) \in L^2(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$. Similarly, the fourth term vanishes when

$k, h \rightarrow 0$ by virtue of the weak convergence (5.20) and the strong convergence $g^-(\cdot, u_\alpha) \rightarrow g(\cdot, u_\alpha)$ in $L^2(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$ which emerges from the continuity of g with respect to t (see assumption (S_2)). Finally, $VII_2 \rightarrow 0$ as $k, h \rightarrow 0$ because its integrand is uniformly bounded in $L^2(\Omega; L^\infty(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$ and $\tau_{m-1}^n \rightarrow \tau^n$ \mathbb{P} -a.s. as $k \rightarrow 0$. Putting it all together to conclude from equation (5.39) the following:

$$\begin{aligned} \lim_{k, h \rightarrow 0} \mathbb{E} \left[\int_0^{\tau_{m-1}^n} \rho^-(t) \|\mathcal{U}_\alpha^+ - \mathcal{U}_{k, h}^+\|_{\alpha, h}^2 dt \right] &= \lim_{k, h \rightarrow 0} \mathbb{E} \left[\int_0^{\tau_{m-1}^n} \rho^-(t) \|g^-(t, u_\alpha(t)) - G_0^-(t)\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 dt \right] \\ &= 0. \end{aligned}$$

For $t \in (0, \tau_{m-1}^n)$, we know through the discrete stopping time τ_{m-1}^n that $\rho^-(t) > e^{-\eta_1 T - \eta_2 n}$. Subsequently, by making use of $\|\mathcal{V}_\alpha^+ - \mathcal{V}_{k, h}^+\|_{\mathbb{L}^2} \leq \max(C_D, \alpha^2) \|\mathcal{U}_\alpha^+ - \mathcal{U}_{k, h}^+\|_{\alpha, h}$, it follows

$$\lim_{k, h \rightarrow 0} \mathbb{E} \left[\int_0^{\tau_{m-1}^n} \|\mathcal{V}_\alpha^+ - \mathcal{V}_{k, h}^+\|_{\mathbb{L}^2}^2 dt \right] = \lim_{k, h \rightarrow 0} \mathbb{E} \left[\int_0^{\tau_{m-1}^n} \|g^-(t, u_\alpha(t)) - G_0^-(t)\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 dt \right] = 0,$$

which implies

$$\lim_{k, h \rightarrow 0} \mathbb{E} \left[\int_0^{\tau^n} \|\mathcal{V}_\alpha^+ - \mathcal{V}_{k, h}^+\|_{\mathbb{L}^2}^2 dt \right] = \lim_{k, h \rightarrow 0} \mathbb{E} \left[\int_0^{\tau^n} \|g^-(t, u_\alpha(t)) - G_0^-(t)\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 dt \right] = 0. \quad (5.40)$$

Indeed, it suffices to write $\int_0^{\tau^n} = \int_0^{\tau_{m-1}^n} + \int_{\tau_{m-1}^n}^{\tau^n}$. The first integral on the right converges to 0 as already shown and the second one goes to 0 as well because $\tau_{m-1}^n \rightarrow \tau^n$ \mathbb{P} -a.s. as $k \rightarrow 0$ along with the associated estimates of each integrand, such as Lemma 4.3, inequality (5.26), assumption (S_2) , convergence (5.20), and Step 2. Consequently, it follows that

$$\begin{aligned} \mathbb{E} \left[\int_0^{\tau^n} \|G_0(t) - g(t, u_\alpha(t))\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 dt \right] &\leq 2\mathbb{E} \left[\int_0^{\tau^n} \|G_0(t) - G_0^-(t)\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 dt \right] \\ &+ \int_0^{\tau^n} \|G_0^-(t) - g^-(t, u_\alpha(t))\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 dt + \int_0^{\tau^n} \|g^-(t, u_\alpha(t)) - g(t, u_\alpha(t))\|_{\mathcal{L}_2(K; \mathbb{L}^2)}^2 dt \rightarrow 0, \end{aligned}$$

thanks to the strong convergence in $L^2(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$ of G_0^m to G_0 together with (5.40) and the time-continuity of g by assumption (S_2) . Taking into account that $\{\tau^n\}_n$ is increasing to $t \in [0, T]$ leads to $G_0 = g(\cdot, u_\alpha)$ in $L^2(\Omega; L^2(0, T; \mathcal{L}_2(K; \mathbb{L}^2)))$. On the other hand,

$$\mathbb{E} \left[\int_0^{\tau^n} \|\mathcal{V}_{k, h}^+ - v_\alpha(t)\|_{\mathbb{L}^2}^2 dt \right] \leq 2\mathbb{E} \left[\int_0^{\tau^n} \|\mathcal{V}_{k, h}^+ - \mathcal{V}_\alpha^+\|_{\mathbb{L}^2}^2 dt + \int_0^{\tau^n} \|\mathcal{V}_\alpha^+ - v_\alpha(t)\|_{\mathbb{L}^2}^2 dt \right] \rightarrow 0, \quad (5.41)$$

by convergences (5.40) and (5.24). Similarly, by $\|\mathcal{U}_{k, h}^+ - \mathcal{U}_\alpha^+\|_\alpha \lesssim \|\mathcal{V}_{k, h}^+ - \mathcal{V}_\alpha^+\|_{\mathbb{L}^2}$, (5.40) and (5.25),

$$\mathbb{E} \left[\int_0^{\tau^n} \|\mathcal{U}_{k, h}^+ - u_\alpha(t)\|_\alpha^2 dt \right] \leq 2\mathbb{E} \left[\int_0^{\tau^n} \|\mathcal{U}_{k, h}^+ - \mathcal{U}_\alpha^+\|_\alpha^2 dt + \int_0^{\tau^n} \|\mathcal{U}_\alpha^+ - u_\alpha(t)\|_\alpha^2 dt \right] \rightarrow 0. \quad (5.42)$$

For $z \in M_{\mathcal{F}_t}^\infty(0, T; \mathbb{H}_0^1)$, we have,

$$\begin{aligned} &\left| \mathbb{E} \left[\int_0^{\tau^n} \left\langle u_\alpha(t) \times (\nabla \times v_\alpha(t)) - \mathcal{U}_{k, h}^+ \times (\nabla \times \mathcal{V}_{k, h}^+), z(t) \right\rangle dt \right] \right| \\ &\leq \|z\|_{M_{\mathcal{F}_t}^\infty(0, T; \mathbb{H}_0^1)} \mathbb{E} \left[\int_0^{\tau^n} \left\| u_\alpha(t) \times (\nabla \times (v_\alpha(t) - \mathcal{V}_{k, h}^+)) \right\|_{\mathbb{H}^{-1}} dt \right] \\ &\quad + \|z\|_{M_{\mathcal{F}_t}^\infty(0, T; \mathbb{H}_0^1)} \mathbb{E} \left[\int_0^{\tau^n} \left\| (u_\alpha(t) - \mathcal{U}_{k, h}^+) \times (\nabla \times \mathcal{V}_{k, h}^+) \right\|_{\mathbb{H}^{-1}} dt \right] \\ &\leq C_D \|z\|_{M_{\mathcal{F}_t}^\infty(0, T; \mathbb{H}_0^1)} \mathbb{E} \left[\int_0^T \|u_\alpha(t)\|_{\mathbb{H}^1}^2 dt \right]^{\frac{1}{4}} \mathbb{E} \left[\int_0^T \|u_\alpha(t)\|_{\mathbb{H}^2}^2 dt \right]^{\frac{1}{4}} \mathbb{E} \left[\int_0^{\tau^n} \|\mathcal{V}_{k, h}^+ - v_\alpha(t)\|_{\mathbb{L}^2}^2 dt \right]^{\frac{1}{2}} \\ &\quad + C_D \|z\|_{M_{\mathcal{F}_t}^\infty(0, T; \mathbb{H}_0^1)} \mathbb{E} \left[\int_0^{\tau^n} \|\nabla(\mathcal{U}_{k, h}^+ - u_\alpha(t))\|_{\mathbb{L}^2}^2 dt \right]^{\frac{1}{2}} \mathbb{E} \left[\int_0^T \|\nabla \mathcal{V}_{k, h}^+\|_{\mathbb{L}^2}^2 dt \right]^{\frac{1}{2}} \rightarrow 0, \text{ as } k, h \rightarrow 0, \end{aligned}$$

where Proposition 2.1-(iv)-(v), the Cauchy-Schwarz and the generalized Hölder inequalities, Step 2, Lemma 4.1, convergence (5.41) and (5.42) were applied. As a result, the above convergence together with (5.18) yield

$$\begin{aligned} \mathbb{E} \left[\int_0^{\tau^n} \langle B_0(t) - u_\alpha(t) \times (\nabla \times v_\alpha(t)), z(t) \rangle dt \right] &= \mathbb{E} \left[\int_0^{\tau^n} \langle B_0(t) - \mathfrak{u}_{k,h}^+ \times (\nabla \times \mathfrak{v}_{k,h}^+), z(t) \rangle dt \right. \\ &\left. + \int_0^{\tau^n} \langle \mathfrak{u}_{k,h}^+ \times (\nabla \times \mathfrak{v}_{k,h}^+) - u_\alpha(t) \times (\nabla \times v_\alpha(t)), z(t) \rangle dt \right] \rightarrow 0, \quad \forall z \in M_{\mathcal{F}_t}^\infty(0, T; \mathbb{H}_0^1). \end{aligned}$$

Now, since the space $M_{\mathcal{F}_t}^\infty(0, T; \mathbb{H}_0^1)$ is dense in $L^2(\Omega; L^2(0, T; \mathbb{H}_0^1))$ and $\{\tau^n\}$ is increasing to $t \in [0, T]$, we infer the identity $B_0 = u_\alpha \times (\nabla \times v_\alpha)$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$. It remains to identify F_0 with its counterpart. To this purpose, let $z \in L^2(\Omega; L^2(0, T; \mathbb{H}_0^1))$. We have

$$\begin{aligned} \langle f(t, u_\alpha(t)) - f^-(t, \mathfrak{u}_{k,h}^-), z(t) \rangle &\leq \langle f(t, u_\alpha(t)) - f^-(t, u_\alpha(t)), z(t) \rangle + L_f \|z(t)\|_{\mathbb{H}^1} \|u_\alpha(t) - \mathfrak{u}_{k,h}^+\|_\alpha \\ &\quad + L_f \|z(t)\|_{\mathbb{H}^1} \|\mathfrak{u}_{k,h}^+ - \mathfrak{u}_{k,h}^-\|_\alpha. \end{aligned}$$

Therefore, the time-continuity of f , convergence (5.42) and the fact that

$$\mathbb{E} \left[\int_0^T \|\mathfrak{u}_{k,h}^+ - \mathfrak{u}_{k,h}^-\|_\alpha^2 dt \right] = k \mathbb{E} \left[\sum_{m=1}^M \|U^m - U^{m-1}\|_\alpha^2 \right] \leq k C_T \rightarrow 0, \quad (\text{by Lemma 4.1})$$

ensure the weak convergence $f^-(\cdot, \mathfrak{u}_{k,h}^-) \rightharpoonup f(\cdot, u_\alpha)$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$. The latter together with convergence (5.19) permit the following

$$\begin{aligned} \mathbb{E} \left[\int_0^{\tau^n} \langle f(t, u_\alpha(t)) - F_0(t), z(t) \rangle dt \right] &= \mathbb{E} \left[\int_0^{\tau^n} \langle f(t, u_\alpha(t)) - f^-(t, \mathfrak{u}_{k,h}^-), z(t) \rangle dt \right] \\ &\quad + \mathbb{E} \left[\int_0^{\tau^n} \langle f^-(t, \mathfrak{u}_{k,h}^-) - F_0(t), z(t) \rangle dt \right] \rightarrow 0 \quad \text{as } k, h \rightarrow 0, \quad \forall z \in L^2(\Omega; L^2(0, T; \mathbb{H}_0^1)). \end{aligned}$$

Consequently, $F_0 = f(\cdot, u_\alpha)$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}))$.

6 Further properties and conclusion

Section 5 gave an insight into the limiting functions and the existence of solutions. Yet, it has not provided the divergence-free property which must be associated with u_α and v . The following proposition treats this issue.

Proposition 6.1 *The limiting functions v and u_α which were provided in Section 5 are divergence-free almost everywhere in $(0, T) \times D$ and \mathbb{P} -almost surely.*

Proof: To prove that v and u_α are divergence-free, it suffices to show that $\{\text{div } \mathfrak{u}_{k,h}^+\}_{k,h}$ converges weakly in $L^2(\Omega; L^2(0, T; \mathbb{L}^2))$ toward 0, thanks to (5.4) and (5.15). To this end, we evoke the Lagrange interpolation $I_h: C^3(D) \rightarrow L_h$ (c.f. [6, Theorem 4.4.4]). For $z \in C^3(D)$, we have

$$\begin{aligned} \mathbb{E} \left[\int_0^T \left(\text{div } \mathfrak{u}_{k,h}^+, z \right) dt \right] &= \mathbb{E} \left[\int_0^T \left(\text{div } \mathfrak{u}_{k,h}^+, z - I_h z \right) dt \right] + \mathbb{E} \left[\int_0^T \left(\text{div } \mathfrak{u}_{k,h}^+, I_h z \right) dt \right] \\ &\lesssim \mathbb{E} \left[\int_0^T \|\nabla \mathfrak{u}_{k,h}^+\|_{L^2} dt \right] \|z - I_h z\|_{L^2} \xrightarrow[k,h \rightarrow 0]{} 0, \end{aligned}$$

where the second term in the first equality vanishes because $\{\mathfrak{u}_{k,h}^+\}_{k,h}$ is weakly divergence-free. \square

6.1 Convergence of LANS- α to NSE in 2D

Assume $\alpha \leq \alpha_{max}h$ for some $\alpha_{max} > 0$ independent of k and h , and $U^0 \rightarrow \bar{u}_0$ in $L^4(\Omega; \mathbb{H}^1)$ as $h \rightarrow 0$. In Subsection 5.1, we proved that the process v satisfies \mathbb{P} -a.s. and for all $(t, \varphi) \in [0, T] \times \mathbb{V}$ the following equation:

$$\begin{aligned} & (v(t), \varphi) + \nu \int_0^t (\nabla v(s), \nabla \varphi) ds - \int_0^t \langle v(s) \times (\nabla \times v(s)), \varphi \rangle ds \\ & = (v_0, \varphi) + \int_0^t \langle f(s, v(s)), \varphi \rangle ds + \left(\int_0^t g(s, v(s)) dW(s), \varphi \right), \end{aligned}$$

where we recall that $u = v$, \mathbb{P} -a.s. and a.e in $(0, T) \times D$. The above equation does not represent yet the Navier-Stokes problem we are looking for because the trilinear term $\langle v(s) \times (\nabla \times v(s)), \varphi \rangle$ looks unfamiliar. However, by Proposition 2.1-(iii),

$$-\langle v(s) \times (\nabla \times v(s)), \varphi \rangle = \langle [v(s) \cdot \nabla]v(s), \varphi \rangle,$$

where $\langle [\varphi \cdot \nabla]v(s), v(s) \rangle = 0$ because $\varphi \in \mathbb{V}$ and $v \in \mathbb{H}_0^1$ (see for instance [34, Chapter 2, Lemma 1.3]). Moreover, by a standard technique (e.g. [33]), it is easy to check from equation (5.8), that v is continuous in time i.e. $v \in L^2(\Omega; C([0, T]; \mathbb{H}))$. Hence, v is a solution of equations (1.2) in the sense of Definition 2.3. Additionally, owing to [21, Proposition 4.1], the process v is unique and it follows that the whole sequences $\{\mathcal{U}_{k,h}^+\}_{k,h}$ and $\{\mathcal{V}_{k,h}^+\}_{k,h}$ are convergent.

6.1.1 Convergence to the LANS- α model

Assume $d \in \{2, 3\}$, $\alpha > 0$ an non-vanishing parameter, and $U^0 \rightarrow \bar{u}_0$ in $L^4(\Omega; \mathbb{H}^1)$ as $h \rightarrow 0$. According to subsection 5.2, the stochastic process (u_α, v_α) satisfies, \mathbb{P} -a.s. and for all $(t, \varphi, \psi) \in (0, T) \times \mathbb{V} \times \mathbb{V}$, equation (2.8) together with $(v_\alpha(t), \psi) = (u_\alpha(t), \psi) + \alpha^2 (\nabla u_\alpha(t), \nabla \psi)$. We also had $v_\alpha \in L^2(\Omega; C([0, T]; \mathbb{H}))$ according to step 3 in subsection 5.2, which implies that u_α is weakly continuous with values in \mathbb{V} , \mathbb{P} -almost surely. Therewith, u_α makes up a solution of equation (1.1) in the sense of Definition 2.2. Taking advantage of [10, Theorem 4.4], we infer that u_α is unique and that the whole sequence $\{(\mathcal{U}_{k,h}^+, \mathcal{V}_{k,h}^+)\}_{k,h}$ is convergent.

6.2 Numerical experiments

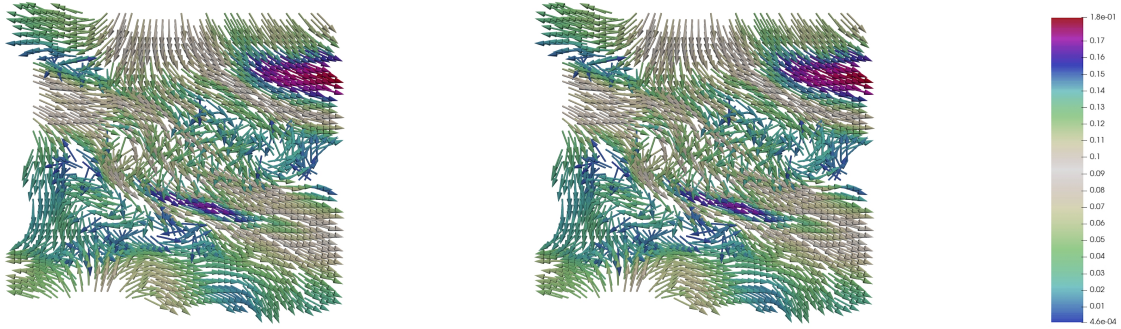
This part is devoted to giving computational experiments in 2D for the stochastic LANS- α model through Algorithm 1 when the spatial scale α fulfills either $\alpha \leq \alpha_{max}h$ or $\alpha > 0$ fixed. Since our primary objective is to compare solutions' behavior of LANS- α to that of Navier-Stokes, we provide simulation of solutions to the latter equations as well through a non-linear scheme covered in [7, Algorithm 1]. The implementation hereafter is performed using the open source finite element software FEniCS [28]. We employ the lower order Taylor-Hood (P_2 - P_1) element for the spatial discretization within a mixed finite element framework. The chosen domain is a unit square $D = (0, 1)^2$ along the time interval $[0, T]$ with $T = 1$. The initial condition $\bar{u}_0 = (\partial_y \psi, -\partial_x \psi)$ is divergence-free and vanishes on the boundary ∂D , where $\psi(x, y) := 10 \sin(100xy^2)x^2(1-x)^2y^2(1-y)^2$, and the viscosity ν is set to 1. On the other hand, the source term $f(\omega, t, u) = e^{-\frac{\omega^2}{2}} \sin(t)u$ is considered in a way to satisfy assumption (S_2) along with the inequality $L_f \leq \nu/\sqrt{2}C_P^2$, where in the present case, $L_f = 1$. It is worth mentioning that C_P , which is the Poincaré constant in this case, is less than $\text{diam}(D)/\pi = \sqrt{2}/\pi$ (see [12]). Finally, the corresponding noise that drives the equations of interest will be considered additive for the sake of simplicity i.e. the drift coefficient g will play the identity operator role.

Q-Wiener process approximation For computational purposes, we must deal with a truncated form of the series (2.1). We consider two independent $H_0^1(D)$ -valued Wiener processes W_1 and W_2 such that $W = (W_1, W_2)$.

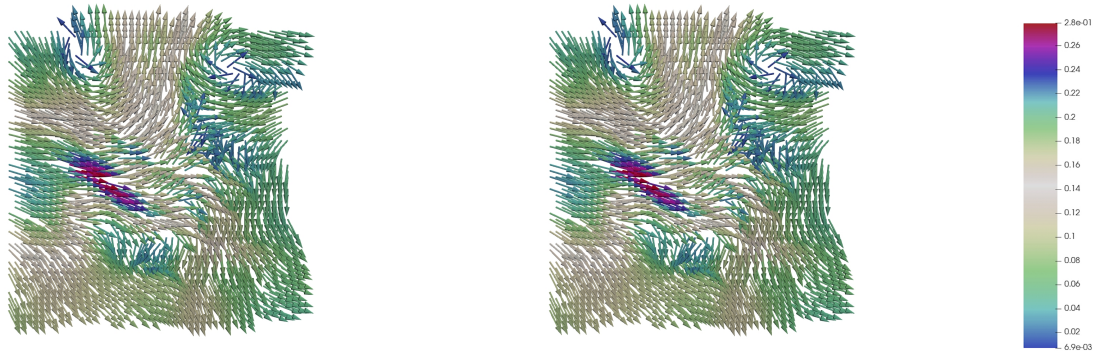
For $J \geq 1$, the used increments are expressed by $\Delta_m W_\ell \approx k^{1/2} \sum_{i,j=1}^J (\lambda_{i,j}^\ell)^{1/2} \xi_{i,j}^{\ell,m} e_{i,j}$, $\ell \in \{1, 2\}$, where J is set to 10 for the simulations down below, and for all $i, j \in \mathbb{N}$ and $(x, y) \in D$, the basis elements

$e_{i,j} := 2 \sin(i\pi x) \sin(j\pi y)$ represent the Laplace eigenfunctions with Dirichlet boundary conditions on D . For $\ell \in \{1, 2\}$, $\{\{\xi_{i,j}^{\ell,m}\}_{i,j}\}_m$ is a family of independent identically distributed standard normal random variables, and $\lambda_{i,j}^\ell := \frac{1}{(i+j)^2}$ for $\ell \in \{1, 2\}$.

Case $\alpha \leq \alpha_{max}h$
 Consider $\alpha = 0.001h$, $h \approx 0.03$ and $k = 0.01$.

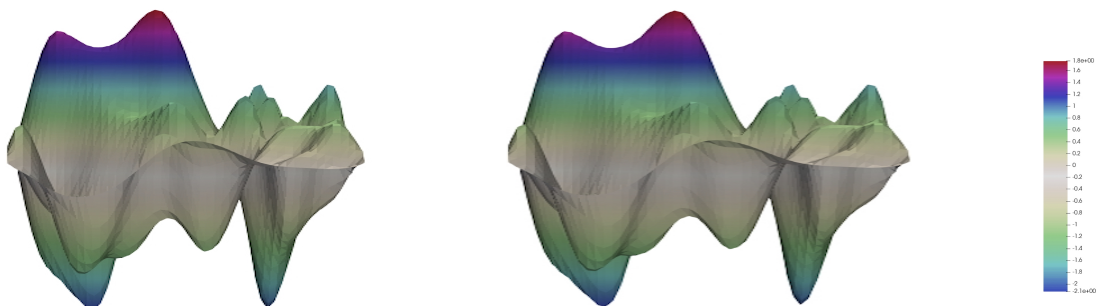


Velocity field of LANS- α at time $t = 0.41$ Velocity field of NS at time $t = 0.41$



Velocity field of LANS- α at time $t = 0.99$ Velocity field of NS at time $t = 0.99$

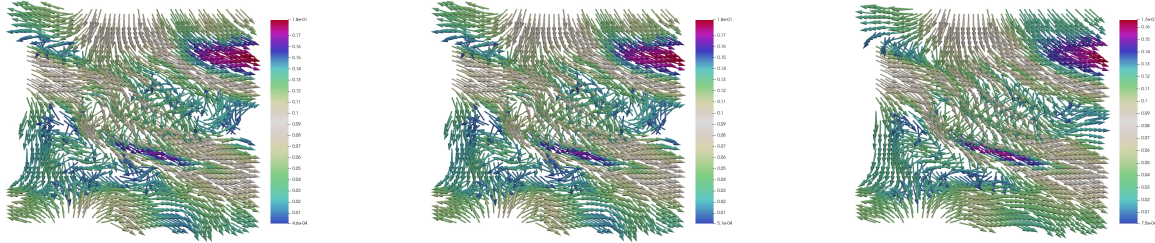
Since this case relates both equations (1.1) and (1.2), we choose two different time values in $[0, T]$, and plot the associated figures side by side. This allows us to compare the solutions' behavior together with the occurring differences. Observe that both LANS- α and NS solutions behave similarly with a tiny variation in values (observable via high resolution monitors). Such a difference was expected since we are dealing here with approximate computations, not to mention the considered space discretization's step h which is not too close to 0, yet its code execution is costly. We also provide the following pressure figures which are barely distinguishable.



LANS- α pressure at $t = 0.41$ NS pressure at $t = 0.41$

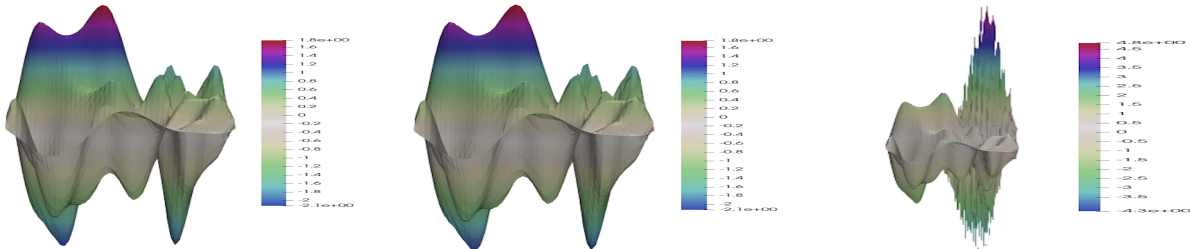
Case $\alpha > 0$ fixed

We set $h \approx 0.03$, $k = 0.01$, and we consider three different values of α : $5 \cdot 10^{-4}$, $5 \cdot 10^{-3}$, and 0.05 . We show down below three figures, each corresponds to a value of α at time $t = 0.41$, and each equipped with a color bar in order to compare their values to that of case $\alpha \leq \alpha_{max}h$.

LANS- $5 \cdot 10^{-4}$ LANS- $5 \cdot 10^{-3}$ LANS- $5 \cdot 10^{-2}$

Velocity fields at time $t = 0.41$ of LANS- α for $\alpha \in \{5 \cdot 10^{-4}, 5 \cdot 10^{-3}, 5 \cdot 10^{-2}\}$

Observe that the velocity fields' behavior when $\alpha = 5 \cdot 10^{-3}$ is tremendously comparable with that of case $\alpha \leq \alpha_{max}h$, although the chosen value of α is not as small. There is however, a remarkable velocity flow variation each time α increases in value. Therewith, the LANS- α equations might not be an alternative model for the NSEs when the spatial scale α is somewhat large. Besides the mentioned variation of the velocity, the pressure field is also heavily impacted by the modification of α as it appears in the upcoming figures. We point out that as α increases, the pressure gains an enormous amplitude which prevent the corresponding simulation outcome to be visible, especially when α exceeds 0.5 .

LANS- $5 \cdot 10^{-4}$ LANS- $5 \cdot 10^{-3}$ LANS- $5 \cdot 10^{-2}$

Pressure fields at time $t = 0.41$ of LANS- α for $\alpha \in \{5 \cdot 10^{-4}, 5 \cdot 10^{-3}, 5 \cdot 10^{-2}\}$

Acknowledgment: The authors would like to thank Professor Guy Vallet (Pau and the Adour Region university) for his suggestions and mainly for identifying a way to improve the previous version of this article.

References

- [1] “A stable finite element for the Stokes equations”. In: *CALCOLO* 21.4 (1984), pp. 337–344.
- [2] L. Bañas et al. “A convergent finite-element-based discretization of the stochastic Landau-Lifshitz-Gilbert equation”. In: *IMAJNA* 34 (Apr. 2014), pp. 502–549. DOI: 10.1093/imanum/drt020.
- [3] Hakima Bessaih and Annie Millet. “Space-time Euler discretization schemes for the stochastic 2D Navier-Stokes equations”. In: *Stochastics and Partial Differential Equations: Analysis and Computations* (2021). DOI: 10.1007/s40072-021-00217-7.

- [4] Clayton Bjorland and Maria E. Schonbek. “On questions of decay and existence for the viscous Camassa–Holm equations”. In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 25.5 (2008), pp. 907–936. ISSN: 0294-1449.
- [5] Francesca Bonizzoni and Guido Kanschat. “H1 -conforming finite element cochain complexes and commuting quasi-interpolation operators on Cartesian meshes”. In: *Calcolo* 58.2 (2021), p. 18. ISSN: 0008-0624. DOI: 10.1007/s10092-021-00409-6.
- [6] Susanne Brenner and Ridgway Scott. *The mathematical theory of finite element methods*. Vol. 15. New York: Springer Sci. & Bus. Media, 2007, p. 400. DOI: 10.1007/978-0-387-75934-0.
- [7] Zdzislaw Brzeźniak, Erich Carelli, and Andreas Prohl. “Finite-element-based discretizations of the incompressible Navier–Stokes equations with multiplicative random forcing”. In: *IMAJNA*. 33.3 (Jan. 2013), pp. 771–824. ISSN: 0272-4979. DOI: 10.1093/imanum/drs032.
- [8] Atife Çağlar. “Convergence analysis of the Navier–Stokes alpha model”. In: *Numer. Methods Partial Differential Equ.* 26.5 (2009), pp. 1154–1167. DOI: 10.1002/num.20481.
- [9] Yanping Cao and Edriss S. Titi. “On the Rate of Convergence of the Two-Dimensional α -Models of Turbulence to the Navier–Stokes Equations”. In: *Numer. Funct. Anal. Optim.* 30.11-12 (2009), pp. 1231–1271. DOI: 10.1080/01630560903439189.
- [10] Tomás Caraballo, José Real, and Takeshi Taniguchi. “On the existence and uniqueness of solutions to stochastic three-dimensional Lagrangian averaged Navier-Stokes equations”. In: *Proc. R. Soc. A: Math., Phys. and Eng. Sci.* 462.2066 (2006), pp. 459–479. DOI: 10.1098/rspa.2005.1574.
- [11] Lamberto Cattabriga. “Su un problema al contorno relativo al sistema di equazioni di Stokes”. In: *Rendiconti del Seminario Matematico della Università di Padova* 31 (1961), pp. 308–340.
- [12] I Chavel and EA Feldman. “An optimal Poincaré inequality for convex domains of non-negative curvature”. In: *Archive for Rational Mechanics and Analysis* 65.3 (1977), pp. 263–273.
- [13] S. Chen et al. “A connection between the Camassa–Holm equations and turbulent flows in channels and pipes”. In: *Physics of Fluids* 11.8 (1999), pp. 2343–2353.
- [14] Jeffrey Connors. “Convergence analysis and computational testing of the finite element discretization of the Navier–Stokes alpha model”. In: *Numerical Methods for Partial Differential Equations* 26.6 (2010), pp. 1328–1350. DOI: <https://doi.org/10.1002/num.20493>.
- [15] Gabriel Deugoue and Mamadou Sango. “On the Stochastic 3D Navier-Stokes- α Model of Fluids Turbulence”. In: *Abstr. Appl. Anal.* 2009.none (2009), pp. 1–27. DOI: 10.1155/2009/723236.
- [16] John A Evans and Thomas JR Hughes. “Isogeometric divergence-conforming B-splines for the unsteady Navier-Stokes equations”. In: *J. Comput. Phys.* 241 (2013), pp. 141–167.
- [17] Xiaobing Feng and Hailong Qiu. “Analysis of Fully Discrete Mixed Finite Element Methods for Time-dependent Stochastic Stokes Equations with Multiplicative Noise”. In: *Journal of Scientific Computing* 88.2 (2021), p. 31. DOI: 10.1007/s10915-021-01546-4.
- [18] M. Germano. “Differential filters for the large eddy numerical simulation of turbulent flows”. In: *The Physics of Fluids* 29.6 (1986), pp. 1755–1757. DOI: 10.1063/1.865649.
- [19] Vivette Girault and Pierre-Arnaud Raviart. *Finite element approximation of the Navier-Stokes equations*. Vol. 749. Springer Berlin, 1979.

- [20] Vivette Girault and Pierre-Arnaud Raviart. *Finite element methods for Navier-Stokes equations: theory and algorithms*. Vol. 5. Berlin Heidelberg: Springer Sci. & Bus. Media, 2012. doi: <https://doi.org/10.1007/978-3-642-61623-5>.
- [21] Nathan Glatt-Holtz and Mohammed Ziane. “Strong pathwise solutions of the stochastic Navier-Stokes system”. In: *Advances in Differential Equations* 14.5/6 (2009), pp. 567–600.
- [22] Ludovic Goudenège and Luigi Manca. “ α -Navier-Stokes equation perturbed by space-time noise of trace class”. working paper or preprint. May 2020.
- [23] Pierre Grisvard. *Elliptic problems in nonsmooth domains*. SIAM, 2011.
- [24] Max D Gunzburger. *Finite element methods for viscous incompressible flows: a guide to theory, practice, and algorithms*. Elsevier, 2012.
- [25] Darryl Holm, Jerrold E Marsden, and Tudor Ratiu. “The Euler-Poincaré Equations and Semidirect Products with Applications to Continuum Theories”. In: *Adv. Math.* 137 (1998), pp. 1–81. doi: 10.1006/aima.1998.1721.
- [26] Akira Ichikawa. “Stability of semilinear stochastic evolution equations”. In: *J. Math. Anal. Appl.* 90.1 (1982), pp. 12–44. ISSN: 0022-247X. doi: [https://doi.org/10.1016/0022-247X\(82\)90041-5](https://doi.org/10.1016/0022-247X(82)90041-5).
- [27] Songul Kaya Merdan and Carolina Manica. “Convergence analysis of the finite element method for a fundamental model in turbulence”. In: *M3AS* 22 (Nov. 2012), 24pp. doi: 10.1142/S0218202512500339.
- [28] Anders Logg, Kent-Andre Mardal, Garth N. Wells, et al. *Automated Solution of Differential Equations by the Finite Element Method*. Springer, 2012. ISBN: 978-3-642-23098-1. doi: 10.1007/978-3-642-23099-8.
- [29] J. E. Marsden and S. Shkoller. “Global well-posedness for the Lagrangian Navier-Stokes (LANS- α) equations on bounded domains”. In: *Philosophical Transactions Of The Royal Society B: Biological Sciences* 359 (2001).
- [30] Jose-Luis Menaldi and Sivaguru S Sritharan. “Stochastic 2-D Navier-Stokes Equation.” In: *Applied Mathematics & Optimization* 46.1 (2002).
- [31] Michael Neilan. “The Stokes complex: A review of exactly divergence-free finite element pairs for incompressible flows”. In: *75 Years of Mathematics of Computation*. Vol. 754. Contemporary Mathematics. American Mathematical Society, 2020.
- [32] Michael Neilan and Duygu Sap. “Stokes elements on cubic meshes yielding divergence-free approximations”. In: *Calcolo* 53 (June 2015), pp. 1–21. doi: 10.1007/s10092-015-0148-x.
- [33] E. Pardoux. “Equations aux dérivées partielles stochastiques non linéaires monotones”. PhD thesis. Université Paris XI, 1975.
- [34] Roger Temam. *Navier-Stokes equations: theory and numerical analysis*. Vol. 343. American Mathematical Soc., 2001.
- [35] Vidar Thomée. *Galerkin finite element methods for parabolic problems*. Vol. 25. Springer Science & Business Media, 2007.