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► **To cite this version:**

Jad Doghman, Ludovic Goudenège. Numerical and convergence analysis of the stochastic Lagrangian averaged Navier-Stokes equations. 2021. hal-03406041v1

HAL Id: hal-03406041

<https://hal.science/hal-03406041v1>

Preprint submitted on 28 Oct 2021 (v1), last revised 23 Dec 2022 (v3)

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Numerical and convergence analysis of the stochastic Lagrangian averaged Navier-Stokes equations *

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Abstract

The primary emphasis of this work is the development of a finite element based space-time discretization for solving the stochastic Lagrangian averaged Navier-Stokes (LANS- α) equations of incompressible fluid turbulence with multiplicative random forcing, under nonperiodic boundary conditions within a bounded polygonal (or polyhedral) domain of \mathbb{R}^d , $d \in \{2, 3\}$. The convergence analysis of a fully discretized numerical scheme is investigated and split into two cases according to the spacial scale α , namely we first assume α to be controlled by the step size of the space discretization so that it vanishes when passing to the limit, then we provide an alternative study when α is fixed. A preparatory analysis of uniform estimates in both α and discretization parameters is carried out. Starting out from the stochastic LANS- α model, we achieve convergence toward the continuous strong solutions of the stochastic Navier-Stokes equations in 2D when α vanishes at the limit. Additionally, convergence toward the continuous strong solutions of the stochastic LANS- α model is accomplished if α is fixed.

Keywords: stochastic Lagrangian averaged Navier-Stokes, stochastic Navier-Stokes, finite element, Euler method

PACS: 02.70.Dh, 02.70.Bf, 02.60.Lj, 02.60.Jh, 02.60.Gf, 02.60.Cb, 02.50.Ey, 02.50.Cw

2010 MSC: 76D05, 65M12, 35Q35, 35Q30, 60H15, 60H35

*Jad Doghman is supported by a public grant as part of the Investissement d'avenir project [ANR-11-LABX-0056-LMH, LabEx LMH], and both authors are part of the SIMALIN project [ANR-19-CE40-0016] of the French National Research Agency.

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1 Introduction

Over the last few decades, many regularization models of the Navier-Stokes equations (NSEs) have arisen, especially the α -regularizations, for the sake of better understanding the closure problem of averaged quantities in turbulent flows. Such turbulent modeling schemes (e.g. Leray- α , Navier-Stokes- α , Clark- α , Modified Leray- α) were introduced as effective subgrid-scale models of the NSEs which require massive grid points or Fourier modes, allowing for approximation to capture all the spatial scales down to the Kolmogorov scale (see for instance [8] and the references therein), as well as their suitability with the empirical and experimental data for a thorough range of Reynolds numbers.

In the present paper, we consider the stochastic version of the LANS- α equations [26] (also known as the viscous Camassa-Holm equations [3], or the Navier-Stokes- α model [11, 22])

$$\begin{cases} \partial_t (\bar{u} - \alpha^2 \Delta \bar{u}) - \nu \Delta (\bar{u} - \alpha^2 \Delta \bar{u}) + [\bar{u} \cdot \nabla] (\bar{u} - \alpha^2 \Delta \bar{u}) + (\nabla \bar{u})^T \cdot (\bar{u} - \alpha^2 \Delta \bar{u}) + \nabla p \\ \quad = f(\cdot, \bar{u}) + g(\cdot, \bar{u}) \dot{W}, \\ \operatorname{div} \bar{u} = 0, \\ \bar{u}(0, \cdot) = \bar{u}_0, \end{cases} \quad (1.1)$$

for internal flow i.e. for a bounded domain in \mathbb{R}^d , $d \in \{2, 3\}$. The unknown vector field \bar{u} is called the filtered fluid velocity, and it depends on time and space variables, ν is the fluid kinematic viscosity, and α is a small spatial scale at which fluid motion is filtered. Note that both ν and α are positive constants. $f = f(t, \bar{u})$ is an external force, the scalar quantity $p = p(t, x)$ represents the pressure and \bar{u}_0 is the corresponding initial datum. The last term of equations (1.1)₁ describes a state-dependent random noise, and it is defined by $g(\cdot, \bar{u}) \dot{W} := g(t, \bar{u}) \partial_t W(t, x)$, where g is a diffusion coefficient. One of the aims herein is to approach the two-dimensional solutions of the stochastic NSEs via the LANS- α model, numerically. Whence the need to evoke the former equations with similar configurations:

$$\begin{cases} \partial_t v - \nu \Delta v + [v \cdot \nabla] v + \nabla \lambda = f(\cdot, v) + g(\cdot, v) \dot{W}, \\ \operatorname{div} v = 0, \\ v(0, \cdot) = v_0, \end{cases} \quad (1.2)$$

where v (resp. λ) is the corresponding fluid velocity (resp. pressure), and v_0 embodies its initial datum.

Equations (1.1) and (1.2) are usually employed as a complementary model to their deterministic versions to better understand the situation of tiny variations or perturbations present in fluid flows. The former represents a modification of the latter by performing Lagrangian means, asymptotic expansions, and an assumption of isotropy of fluctuations in the Hamilton principle, which grant further physical properties (e.g. conservation laws for energy and momentum). More specifically, the convective nonlinearity $[v \cdot \nabla] v$ in the NSEs is adjusted so that the cascading of turbulence at scales under specific length stops. The latter adjustment is called a nonlinearly dispersive modification.

For stochastic hydrodynamical systems, a general theory was developed in [13], involving well-posedness and large deviations. It covers, for instance, 3D Leray- α model, but not completely the LANS- α equations which, in parallel, have been a subject of several papers. The existence and uniqueness of a variational solution to the problem (1.1) were investigated in [9] under Lipschitz-continuous conditions in a three-dimensional bounded domain. A similar study is proposed in [15], but this time with a genuine finite-dimensional Wiener process depending only on time. LANS- α

model driven by an additive space-time noise of trace class was considered in [19], where the authors proved the existence and uniqueness of an invariant measure, and a probabilistic strong solution.

Speaking of the numerical approach side, the convergence analysis of suitable numerical methods for the stochastic LANS- α equations is less well developed. In connection with the deterministic version, both convergence rate and convergence analysis of an algorithm consisting of a finite element method were investigated in [14] where the spatial scale α is considered in terms of the space discretization's step. The author in [7] conducted a similar study, with α being independent of the discretization parameters. On the other hand, numerical schemes for stochastic nonlinear equations admitting local Lipschitz nonlinearities related to the Navier-Stokes systems had been already investigated. For instance, authors in [6] studied a finite element-based space-time discretization of the incompressible NSEs driven by a multiplicative noise. An enhancement of [6] in dimension 2 was carried out in [2].

This paper aims to provide a fully discrete finite element-based discretization of equations (1.1) in a bounded convex polygonal or polyhedral domain. Notice that the underlying model consists of a fourth order problem, nevertheless we avoid the use of C^1 piecewise polynomials-based finite element methods by introducing a notion of differential filters that transform equations (1.1) into a coupled problem of second order. The employed time-discretization herein is an Euler scheme. One highly valued characteristic of the finite element method is the prospect of meticulous interpretation provided by the functional analysis framework. In contrast to the linear stochastic partial differential equations, since we are dealing here with a nonlinear model, one cannot make use of the semigroup method or Green's function. Those techniques are effectively replaced by monotone or Lipschitz-continuous drift functions. It is worth highlighting the importance of constructing practical numerical schemes provided with exact divergence-free finite element functions. However, due to their computational complexity, one may notice the usage of a weak divergence-free condition that compensates for the strong sense's absence.

The associated spatial scale α will be considered hereafter either in terms of the space discretization's step (case 1) or independently of all discretization's parameters (case 2). Therefore, our main results consist of the convergence in both 2D and 3D of Algorithm 1 toward the continuous solution of the 2D stochastic NSEs for the case 1, together with the convergence toward the continuous solution of the stochastic LANS- α model for the case 2. Speaking of the followed approach, we begin by performing a priori estimates characterized by their uniformity in α and the discretizations' parameters, allowing us to extract convergent subsequences of the approximate solution. We also demonstrate iterates' uniqueness, thanks to the scheme's linearity. The second step consists in applying two compactness lemmas (Lemmas 2.2 and 2.3) to achieve strong convergence of the iterates in the same probability space, toward the NSEs and the LANS- α model. In other words, we do not use the Skorokhod theorem. Although the mentioned compactness lemmas do not include spaces with randomness, we accomplish a preliminary convergence through a suggested sample subset $\Omega_{k,h}^\varepsilon \subset \Omega$, whose measure is bigger than $1 - \varepsilon$, with ε being independent of α and the discretizations' parameters. The latter allows us to demonstrate the strong convergence in the whole probability set Ω . Afterwards, we identify the limit of each term of Algorithm 1 with its continuous (in time) counterpart.

The paper is organized as follows. We introduce in section 2 a few notions and preliminaries, including the spatial framework, the needed assumptions, the time and space discretizations alongside their properties, definition of solutions to problems (1.1) and (1.2), the definition of continuous and discrete differential filters, the investigated algorithm together with the needed compactness lemmas. Section 3 is tailored for the main results of this paper. We exhibit part of the proofs in section 4, where we focus on the a priori estimates and iterates' uniqueness. In

section 5, we study the convergence analysis of the corresponding numerical scheme. Accordingly, we identify both deterministic and stochastic integrals, as the discretization steps tend to 0, with the corresponding exact solution. We terminate this paper (section 6) with a conclusion concerning the obtained limiting functions and how one can relate them to the stochastic NSEs and LANS- α model. We equip this section with a computational experiment to visualize the outcomes and to evaluate the performance of the proposed numerical scheme.

2 Notations and preliminaries

We state, in this section, preliminary background material following the usual notation employed in the context of the mathematical theory of Navier-Stokes equations.

Given $T > 0$, we denote by $D \subset \mathbb{R}^d$, $d \in \{2, 3\}$ a bounded convex polygonal or polyhedral domain with boundary ∂D , in which we seek a solution, namely a stochastic process $(\bar{u}(t), p(t))$, $t \in [0, T]$ satisfying equations (1.1) in a certain sense. Define almost everywhere on ∂D the unit outward normal vector field $\vec{n}: \partial D \rightarrow \mathbb{R}^d$. The following function spaces are required hereafter:

$$\begin{aligned} \mathcal{V} &:= \left\{ z \in [C_c^\infty(D)]^d \mid \operatorname{div} z = 0 \right\}, \\ \mathbb{H} &:= \left\{ z \in (L^2(D))^d \mid \operatorname{div} z = 0 \text{ a.e. in } D, z \cdot \vec{n} = 0 \text{ a.e. on } \partial D \right\}, \\ \mathbb{V} &:= \left\{ z \in (H_0^1(D))^d \mid \operatorname{div} z = 0 \text{ a.e. in } D \right\}. \end{aligned}$$

From now on, the spaces of vector valued functions will be indicated with blackboard bold letters, for instance $\mathbb{L}^2 := (L^2(D))^d$ denotes the Lebesgue space of vector valued functions defined on D . Denote by $\mathcal{P}: \mathbb{L}^2 \rightarrow \mathbb{H}$ the Leray projector, and by $A: D(A) \rightarrow \mathbb{H}$ the Stokes operator defined by $A := -\mathcal{P}\Delta$ with domain $D(A) = \mathbb{H}^2 \cap \mathbb{V}$. A is a self-adjoint positive operator, and has a compact inverse, see for instance [10]. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ be a complete probability space, Q a nuclear operator, and denoted by K a separable Hilbert space on which we define the Q -Wiener process $W(t)$, $t \in [0, T]$ such that

$$W(t) = \sum_{k \in \mathbb{N}} \sqrt{q_k} \beta^k(t) w_k, \quad \forall t \in [0, T], \quad (2.1)$$

where $\{\beta^k(\cdot), k \in \mathbb{N}\}$ is a sequence of independent and identically distributed \mathbb{R} -valued Brownian motions on the probability basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, $\{w_k, k \in \mathbb{N}\}$ is a complete orthonormal basis of the Hilbert space K consisting of the eigenfunctions of Q , with eigenvalues $\{q_k\}_{k \in \mathbb{N}^*}$. The following estimate will play an essential role in the sequel, cf. [23].

$$\mathbb{E} \left[\|W(t) - W(s)\|_K^{2r} \right] \leq (2r - 1)!! (t - s)^r (\operatorname{Tr}(Q))^r, \quad \forall r \in \mathbb{N}, \quad (2.2)$$

where $(2r - 1)!! := (2r - 1)(2r - 3) \dots \times 5 \times 3 \times 1$.

For any arbitrary Hilbert spaces X, Y , the sets $\mathcal{L}_1(X, Y)$ and $\mathcal{L}_2(X, Y)$ denote the nuclear, and Hilbert-Schmidt operators from X to Y , respectively. For brevity's sake, if $X = Y$, we set $\mathcal{L}_i(X, X) = \mathcal{L}_i(X)$, $i \in \{1, 2\}$. Hereafter, $M_{\mathcal{F}_t}^p(0, T; X)$ denotes the space of all \mathcal{F}_t -progressively measurable processes belonging to $L^p(\Omega \times (0, T), d\mathbb{P} \times dt; X)$, for any Banach space X .

Throughout this paper, the nonnegative constant C_D depends only on the domain D , the symbols (\cdot, \cdot) and $\langle \cdot, \cdot \rangle$ stand for the inner product in \mathbb{L}^2 and the duality product between \mathbb{H}^{-1} and

\mathbb{H}^1 , respectively. Recall that α is a small spatial scale, thereby we assume that $\alpha \leq 1$. The latter leads to the following norm equivalence

$$\alpha \|\cdot\|_{\mathbb{H}^1} \leq \|\cdot\|_{\alpha} \leq \|\cdot\|_{\mathbb{H}^1}, \quad (2.3)$$

where $\|\cdot\|_{\alpha}$ is defined by $\|\cdot\|_{\alpha}^2 := \|\cdot\|_{\mathbb{L}^2}^2 + \alpha^2 \|\nabla \cdot\|_{\mathbb{L}^2}^2$. We point out that the whole study herein maintains all the stated properties if one chooses $\alpha \leq \alpha_0$, for some $\alpha_0 \in \mathbb{R}_+^*$. For arbitrary real numbers x, y , the inequality $x \lesssim y$ is a shorthand for $x \leq cy$ for some universal constant $c > 0$.

We list the assumptions needed below for the data \bar{u}_0, g, Q , and f .

Assumptions

(S₁) $Q \in \mathcal{L}_1(K)$ is a symmetric, positive definite operator,

(S₂) $f \in L^2(\Omega; C([0, T]; \mathbb{H}^{-1}))$ and $g \in L^2(\Omega; C([0, T]; \mathcal{L}_2(K, \mathbb{L}^2)))$ are sublinear Lipschitz-continuous mappings, i.e. for all $z_1, z_2 \in \mathbb{V}$, $g(\cdot, z_1)$ and $f(\cdot, z_2)$ are \mathcal{F}_t -progressively measurable, and $d\mathbb{P} \times dt$ -a.e. in $\Omega \times (0, T)$,

$$\begin{aligned} \|g(\cdot, z_1) - g(\cdot, z_2)\|_{\mathcal{L}_2(K, \mathbb{L}^2)} &\leq L_g \|z_1 - z_2\|_{\alpha}, \quad \forall z_1, z_2 \in \mathbb{V}, \\ \|g(\cdot, z)\|_{\mathcal{L}_2(K, \mathbb{L}^2)} &\leq K_1 + K_2 \|z\|_{\alpha}, \quad \forall z \in \mathbb{V}, \\ \|f(\cdot, z_1) - f(\cdot, z_2)\|_{\mathbb{H}^{-1}} &\leq L_f \|z_1 - z_2\|_{\alpha}, \quad \forall z_1, z_2 \in \mathbb{V}, \\ \|f(\cdot, z)\|_{\mathbb{H}^{-1}} &\leq K_3 + K_4 \|z\|_{\alpha}, \quad \forall z \in \mathbb{V}, \end{aligned}$$

for some time-independent nonnegative constants $K_1, K_2, K_3, K_4, L_f, L_g$.

(S₃) $\bar{u}_0 \in L^4(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{V})$.

To avoid repetitions later on, we state the following assertions

$$(a - b, a) = \frac{1}{2} \left(\|a\|_{\mathbb{L}^2}^2 - \|b\|_{\mathbb{L}^2}^2 + \|a - b\|_{\mathbb{L}^2}^2 \right) \quad \text{for all } a, b \in \mathbb{L}^2, \quad (2.4)$$

$$\left(\sum_{m=1}^M |a_m| \right)^2 \leq 3 \sum_{m=1}^M |a_m|^2, \quad \forall M \in \mathbb{N} \setminus \{0\}. \quad (2.5)$$

The trilinear form We define the trilinear form \tilde{b} , associated with the LANS- α equations, by

$$\tilde{b}(z_1, z_2, w) = \left\langle [z_1 \cdot \nabla] z_2, w \right\rangle + \left\langle (\nabla z_1)^T \cdot z_2, w \right\rangle, \quad \forall z_1, z_2, w \in \mathbb{H}_0^1.$$

We list in the following proposition a few corresponding properties.

Proposition 2.1

- (i) $\tilde{b}(z_1, z_2, w) = -\tilde{b}(w, z_2, z_1)$, for all $z_1, w \in \mathbb{V}$, $z_2 \in \mathbb{H}_0^1$,
- (ii) $\tilde{b}(z_1, z_2, z_1) = 0$, for all $(z_1, z_2) \in \mathbb{V} \times \mathbb{H}_0^1$,
- (iii) $|\tilde{b}(z_1, z_2, w)| \leq C_D \|z_1\|_{\mathbb{H}^1} \|z_2\|_{\mathbb{H}^1} \|w\|_{\mathbb{H}^1}$, for all $z_1, z_2, w \in \mathbb{H}_0^1$,
- (iv) $\int_D (\nabla z_1)^T \cdot z_2 w dx = - \int_D [w \cdot \nabla] z_2 z_1 dx$, for all $z_1, z_2 \in \mathbb{H}_0^1$ and $w \in \mathbb{V}$.

Proof: For (i)-(iii), see for instance [16, Lemma 1], and the last assertion was covered in [9, Proposition 2.1] with a slight modification here regarding the utilized spaces. \square

It is well-known that finite element methods based on C^1 piecewise polynomials are not easily implementable. This means that our fourth-order partial differential equation (1.1) must undergo a modification so that it turns into a second-order problem. To this end, we shall propose a differential filter that deals with a Stokes problem. Such an idea emerges from [17] within a slight adjustment for the sake of fitting the current framework. The divergence-free condition in the definition below is not mandatory as one can always use the Helmholtz decomposition to subsume the resulting gradient term within $\nabla \tilde{p}$.

Definition 2.1 (Continuous differential filter)

Given a (divergence-free) vector field $v \in \mathbb{L}^2$ vanishing on ∂D , its continuous differential filter, denoted by \bar{u} , is part of the unique solution $(\bar{u}, \tilde{p}) \in \mathbb{V} \times L_0^2(D)$ to

$$\begin{cases} -\alpha^2 \Delta \bar{u} + \bar{u} + \nabla \tilde{p} = v, & \text{in } D, \\ \operatorname{div} \bar{u} = 0, & \text{in } D, \\ \bar{u} = 0, & \text{on } \partial D. \end{cases} \quad (2.6)$$

Note that the differential filter of a function v is usually denoted by \bar{v} . Nevertheless, the employed notation herein will be \bar{u} to obtain a clear vision of the relationship between the differential filter and equations (1.1). For a given $v \in \mathbb{L}^2$, problem (2.6) yields a unique $\bar{u} \in \mathbb{H}^2 \cap \mathbb{V}$ provided that $D \subset \mathbb{R}^d$, is a bounded convex two-dimensional polygonal (three-dimensional polyhedral) domain. Moreover, the solution \bar{u} satisfies $\|\bar{u}\|_{\mathbb{H}^2} \leq C_D \alpha^{-2} \|v\|_{\mathbb{L}^2}$. The former and the latter property are provided in [20, Subsection 8.2]. Observe that v in equations (2.6) is assumed to be null on ∂D due to the occurring equality $\bar{u} = v$ when one passes to the limit in α after projecting (2.6)₁ using the Leray projector \mathcal{P} .

2.1 Definition of solutions

Relying on paper [9], a solution to equations (1.1) can be defined as follows:

Definition 2.2

Let $T > 0$ and assume that (S₁)-(S₃) are valid. A \mathbb{V} -valued stochastic process $\bar{u}(t)$, $t \in [0, T]$ is said to be a variational solution to problem (1.1) if it fulfills the following conditions:

- (i) $\bar{u} \in M_{\mathcal{F}_t}^2(0, T; D(A)) \cap L^2(\Omega; L^\infty(0, T; \mathbb{V}))$,
- (ii) \bar{u} is weakly continuous with values in \mathbb{V} , \mathbb{P} -almost surely,
- (iii) for all $t \in [0, T]$, \bar{u} satisfies the following equation \mathbb{P} -almost surely

$$\begin{aligned} & (\bar{u}(t), \phi) + \alpha^2 (\nabla \bar{u}(t), \nabla \phi) + \nu \int_0^t (\bar{u}(s) + \alpha^2 A \bar{u}(s), A \phi) ds \\ & + \int_0^t \tilde{b}(\bar{u}(s), \bar{u}(s) - \alpha^2 \Delta \bar{u}(s), \phi) ds = (\bar{u}_0, \phi) + \alpha^2 (\nabla \bar{u}_0, \nabla \phi) \\ & + \int_0^t \langle f(s, \bar{u}(s)), \phi \rangle ds + \left(\int_0^t g(s, \bar{u}(s)) dW(s), \phi \right), \quad \forall \phi \in D(A). \end{aligned} \quad (2.7)$$

If \bar{u} is a solution to problem (1.1) in the sense of Definition 2.2, then considering $v = v(t)$ as in problem (2.6) grants a new (equivalent) formula for equation (2.7), namely for all $t \in [0, T]$, there holds \mathbb{P} -almost surely

$$\begin{aligned} (v(t), \phi) + \nu \int_0^t (\nabla v(s), \nabla \phi) ds + \int_0^t \tilde{b}(\bar{u}(s), v(s), \phi) ds &= (v_0, \phi) \\ + \int_0^t \langle f(s, \bar{u}(s)), \phi \rangle ds + \left(\int_0^t g(s, \bar{u}(s)) dW(s), \phi \right), \quad \forall \phi \in D(A), \end{aligned} \quad (2.8)$$

where $v_0 \in \mathbb{L}^2$ is given by equation (2.6) when $\bar{u} = \bar{u}_0$. The trilinear term involving the pressure $\int_0^t \tilde{b}(\bar{u}(s), \nabla \tilde{p}(s), \phi) ds$ does not appear in equation (2.8) because

$$\tilde{b}(\bar{u}, \nabla \tilde{p}, \phi) = \sum_{i,j=1}^d \int_D \bar{u}^i \partial_i \partial_j \tilde{p} \phi^j dx + \sum_{i,j=1}^d \int_D \partial_i \bar{u}^j \partial_j \tilde{p} \phi^i dx.$$

The first term on the right-hand side turns into $-\int_D [\phi \cdot \nabla] \bar{u} \nabla \tilde{p} dx$ after performing an integration by parts, and the second term can be rewritten as $\int_D [\phi \cdot \nabla] \bar{u} \nabla \tilde{p} dx$. It is worth mentioning that (2.8), coupled with the weak formulation of (2.6), establishes a well-posed problem whose solution satisfies equations (1.1) in the sense of Definition 2.2.

Next, we give a definition of strong solutions to problem (1.2) in 2D.

Definition 2.3

Given $T > 0$, let assumptions (S_1) and (S_2) be fulfilled, $d = 2$ and $v_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{H})$ be the initial datum. An \mathbb{H} -valued stochastic process $v(t), t \in [0, T]$ is said to be a strong solution to equations (1.2) if it satisfies:

- (i) $v \in M_{\mathcal{F}_t}^2(0, T; \mathbb{V}) \cap L^2(\Omega; C([0, T]; \mathbb{H}))$,
- (ii) for all $t \in [0, T]$, there holds \mathbb{P} -a.s.

$$\begin{aligned} (v(t), \varphi) + \nu \int_0^t (\nabla v(s), \nabla \varphi) ds + \int_0^t \langle [v(s) \cdot \nabla] v(s), \varphi \rangle ds &= (v_0, \varphi) \\ + \int_0^t \langle f(s, v(s)), \varphi \rangle ds + \left(\int_0^t g(s, v(s)) dW(s), \varphi \right), \quad \forall \varphi \in D(A). \end{aligned} \quad (2.9)$$

2.2 Discretizations and algorithm

Time Discretization Let $M \in \mathbb{N}^*$ be given, and set $I_k = \{t_\ell\}_{\ell=0}^M$ an equidistant partition of the interval $[0, T]$, where $t_0 := 0$, $t_M := T$ and $k := T/M$ is the time-step size. The equidistance condition is not mandatory in the sequel, but it is imposed for simplicity. One can generalize the presented method by associating a time-step k_m with each sub-interval $[t_{m-1}, t_m]$, for all $m \in \{1, \dots, M\}$.

Space discretization For simplicity's sake, we let \mathcal{T}_h be a quasi-uniform triangulation of the domain $D \subset \mathbb{R}^d$, $d \in \{2, 3\}$ into simplexes of maximal diameter $h > 0$, and $\bar{D} = \bigcup_{K \in \mathcal{T}_h} \bar{K}$. The space of polynomial vector fields on an arbitrary set O with degree less than or equal to $n \in \mathbb{N}$ is denoted by $\mathbb{P}_n(O) := (P_n(O))^d$. For $n_1, n_2 \in \mathbb{N} \setminus \{0\}$, we let

$$\mathbb{H}_h := \left\{ z_h \in \mathbb{H}_0^1 \cap [C^0(\bar{D})]^d \mid z_h|_K \in \mathbb{P}_{n_1}(K), \quad \forall K \in \mathcal{T}_h \right\},$$

$$\begin{aligned} L_h &:= \left\{ q_h \in L_0^2(D) \mid q_h|_K \in P_{n_2}(K), \quad \forall K \in \mathcal{T}_h \right\}, \\ \mathbb{V}_h &:= \left\{ z_h \in \mathbb{H}_h \mid (\operatorname{div} z_h, q_h) = 0, \quad \forall q_h \in L_h \right\}, \end{aligned}$$

be the finite element function spaces. For fixed $n_1, n_2 \in \mathbb{N} \setminus \{0\}$, we assume that (\mathbb{H}_h, L_h) satisfies the discrete inf-sup condition; namely there is a constant $\beta > 0$ independent of the mesh size h such that

$$\sup_{z_h \in \mathbb{H}_h \setminus \{0\}} \frac{(\operatorname{div} z_h, q_h)}{\|\nabla z_h\|_{\mathbb{L}^2}} \geq \beta \|q_h\|_{L^2}, \quad \forall q_h \in L_h. \quad (2.10)$$

Given $z \in \mathbb{L}^2$, we denote by $\Pi_h: \mathbb{L}^2 \rightarrow \mathbb{V}_h$ the \mathbb{L}^2 -orthogonal projections, defined as the unique solution of the identity

$$(z - \Pi_h z, \varphi_h) = 0, \quad \forall \varphi_h \in \mathbb{V}_h. \quad (2.11)$$

For $z \in \mathbb{H}_0^1$, $\Delta^h: \mathbb{H}_0^1 \rightarrow \mathbb{V}_h$ denotes the discrete Laplace operator, defined as the unique solution of

$$\left(\Delta^h z, \varphi_h \right) = - (\nabla z, \nabla \varphi_h), \quad \forall \varphi_h \in \mathbb{V}_h. \quad (2.12)$$

Estimate (2.13) and the inverse inequality (2.14) below need to be satisfied by the recently defined approximate function spaces. Let \mathbb{S}_h be a finite dimensional subspace of \mathbb{H}_0^1 equipped with an \mathbb{L}^2 -projector $\Pi_{\mathbb{S}_h}: \mathbb{L}^2 \rightarrow \mathbb{S}_h$, satisfying the following property:

For $z \in \mathbb{H}_0^1 \cap \mathbb{W}^{s,2}$, there is a positive constant C independent of h such that

$$\sum_{j=0}^1 h^j \|D^j (z - \Pi_{\mathbb{S}_h} z)\|_{\mathbb{L}^2} \leq Ch^s \|z\|_{\mathbb{W}^{s,2}}, \quad 2 \leq s \leq n+1, \quad (2.13)$$

where n is the polynomials' degree in \mathbb{S}_h .

Furthermore, assume that \mathbb{S}_h fulfills the following inverse inequality:

For $\ell \in \mathbb{N}$, $1 \leq p, q \leq +\infty$ and $0 \leq m \leq \ell$, there exists a constant C independent of h such that

$$\|z_h\|_{\mathbb{W}^{\ell,p}} \leq Ch^{m-\ell+d \min(\frac{1}{p}-\frac{1}{q}, 0)} \|z_h\|_{\mathbb{W}^{m,q}}, \quad \forall z_h \in \mathbb{S}_h. \quad (2.14)$$

Provided the triangulation of the domain D is quasi-uniform, one can easily check that the space \mathbb{H}_h satisfies both estimates (2.13) and (2.14). The reader may refer to [5] for adequate proofs. Subsequently, we take $\mathbb{S}_h = \mathbb{H}_h$.

The discrete differential filter is somewhat defined as its continuous counterpart, but this time by involving the weak formulation of problem (2.6).

Definition 2.4 (Discrete differential filter)

Let v be the vector field of Definition 2.1. Its discrete differential filter, denoted by $\bar{u}_h \in \mathbb{V}_h$, is given by

$$\alpha^2 (\nabla \bar{u}_h, \nabla \varphi_h) + (\bar{u}_h, \varphi_h) = (v, \varphi_h), \quad \forall \varphi_h \in \mathbb{V}_h.$$

Additional information are stated in article [24, Section 4]. We list some of its properties in the following lemma.

Lemma 2.1

Let $v = v_h \in \mathbb{V}^h$ and $\bar{u}_h \in \mathbb{V}_h$ be its discrete differential filter. Then,

$$(i) \quad v_h = \bar{u}_h - \alpha^2 \Delta^h \bar{u}_h \text{ and } \nabla v_h = \nabla \bar{u}_h - \alpha^2 \nabla \Delta^h \bar{u}_h \text{ a.e. in } D.$$

$$(ii) \quad (\nabla v_h, \nabla \bar{u}_h) = \|\nabla \bar{u}_h\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h \bar{u}_h\|_{\mathbb{L}^2}^2.$$

Proof: Assertions (i) and (ii) are covered by [14, Lemma 2.1]. \square

Before exhibiting the algorithm, we will define new notations for the approximate functions. The subscript h of the utilized test functions will be dropped throughout the rest of this paper for the sake of clarity. For $t \in [0, T]$, we set $V(t) := v_h(t)$ for $v_h \in \mathbb{V}_h$, and denote by $U(t)$ its discrete differential filter, i.e. $U(t) := \bar{u}_h(t)$. Besides, let $\Pi(t) := p_h(t)$ and $\tilde{\Pi}(t) := \tilde{p}_h(t)$ be the (space) approximate pressures. We point out that Algorithm 1 is derived from equation (2.8), which contains both variables \bar{u} and v .

Algorithm 1

Given a starting point $U^0 \in \mathbb{H}_h$, find for every $m \in \{1, \dots, M\}$, a 4-tuple stochastic process $(U^m, V^m, \Pi^m, \tilde{\Pi}^m) \in \mathbb{H}_h \times \mathbb{H}_h \times L_h \times L_h$ such that for all $(\varphi, \psi, \Lambda_1, \Lambda_2) \in \mathbb{H}_h \times \mathbb{H}_h \times L_h \times L_h$, there holds \mathbb{P} -a.s.

$$\begin{cases} \bullet (V^m - V^{m-1}, \varphi) + k\nu(\nabla V^m, \nabla \varphi) + k\tilde{b}(U^m, V^{m-1}, \varphi) - k(\Pi^m, \operatorname{div} \varphi) \\ = k \left\langle f(t_{m-1}, U^{m-1}), \varphi \right\rangle + \left\langle g(t_{m-1}, U^{m-1}) \Delta_m W, \varphi \right\rangle, \\ \bullet (V^m, \psi) = (U^m, \psi) + \alpha^2 (\nabla U^m, \nabla \psi) - (\tilde{\Pi}^m, \operatorname{div} \psi), \\ \bullet (\operatorname{div} U^m, \Lambda_1) = (\operatorname{div} V^m, \Lambda_2) = 0, \end{cases}$$

where $\Delta_m W = W(t_m) - W(t_{m-1})$ for all $m \in \{1, \dots, M\}$.

For each $m \in \{0, \dots, M\}$, we may conclude from the second and third equations of Algorithm 1 along with Definition 2.4 two facts:

- (i) U^m is the discrete differential filter of V^m and thereby, all the associated properties are valid.
- (ii) The Algorithm's starting point U^0 could be exchanged with V^0 .

We still need to state two mainly important lemmas that will contribute in the convergence of Algorithm 1. Besides, given a function u , the shift operator τ_ℓ is defined by $\tau_\ell u(t, x) := u(t + \ell, x)$, for all $(t, x) \in [0, T - \ell] \times D$.

The following lemma is provided in [12, Lemma 6], and will be employed when α is assumed to be controlled by h .

Lemma 2.2

Let Y be a Banach space and M_+ be a normed vector space in Y . Assume that the embedding $M_+ \hookrightarrow Y$ is compact and that U is a bounded subset of $L^2(0, T; M_+)$. We suppose in addition that $\|\tau_\ell u - u\|_{L^2(0, T - \ell; Y)} \rightarrow 0$ as $\ell \rightarrow 0$, uniformly in $u \in U$. Then, U is relatively compact in $L^2(0, T; Y)$.

Conversely, when α is considered independently of h and k , the below lemma will play an alternative role, and it consists of a discrete version of Lemma 2.2.

Lemma 2.3

Let $n := (k, h) \in (\mathbb{R}_+^*)^2$, $(B, \|\cdot\|_B)$ be a Banach space and $(M_h, \|\cdot\|_{M_h})$ be a normed space in B . Let $(u_n)_n$ be a sequence in $L^2(0, T; B)$. Assume that

- (i) if $(\varphi_h)_h$ is a sequence of B such that $\|\varphi_h\|_{M_h} \leq C$ for all $h > 0$, for some $C > 0$ then, $(\varphi_h)_h$ is relatively compact in B ,
- (ii) $\|u_n\|_{L^2(0, T; M_h)} \leq C_1$ and $\|u_n\|_{L^1_{loc}(0, T; B)} \leq C_2$ for all n , for some $C_1, C_2 > 0$,
- (iii) $\|\tau_\ell u_n - u_n\|_{L^2(0, T - \ell; B)} \rightarrow 0$ as $\ell \rightarrow 0$, uniformly in $n \in \mathbb{R}_+^* \times \mathbb{R}_+^*$.

Then, $(u_n)_n$ is relatively compact in $L^2(0, T; B)$.

Proof: For $h > 0$, define $b_h(v) := \begin{cases} \frac{\|v\|_B}{\|v\|_{M_h}} v & \text{if } v \in M_h \setminus \{0\}, \\ 0 & \text{if } v \in (B \setminus M_h) \cup \{0\} \end{cases}$. Let us show that $b_h: B \rightarrow$

B is a (nonlinear) compact operator. Indeed, assume that $(v_h)_{h>0}$ is a bounded sequence of B i.e. there is $M \geq 0$ such that $\|v_h\|_B \leq M$ for all $h > 0$. We have $b_h(v_h) \in M_h$ and $\|b_h(v_h)\|_{M_h} = \|v_h\|_B \leq M$ for all $h > 0$. Therefore, by assumption (i), $(b_h(v_h))_h$ is relatively compact in B . Whence the compactness of b_h . For $n = (k, h) \in (\mathbb{R}_+^*)^2$ and $t \in [0, T]$, define the sequence $v_n(t) := \begin{cases} \frac{\|u_n(t)\|_{M_h}}{\|u_n(t)\|_B} u_n(t) & \text{on } \{u_n(t) \neq 0\}, \\ 0 & \text{on } \{u_n(t) = 0\} \end{cases}$. We have $\|v_n\|_{L^2(0, T; B)} \leq \|u_n\|_{L^2(0, T; M_h)} \leq C_1$ for all n ,

thanks to assertion (ii). Thus, $(v_n)_n$ is bounded in $L^2(0, T; B)$, particularly in $L^1(0, T; B)$. On the

other hand, $b_h(v_n(t)) = \begin{cases} \frac{\|v_n(t)\|_B}{\|v_n(t)\|_{M_h}} v_n(t) & \text{on } \{v_n(t) \neq 0\}, \\ 0 & \text{on } \{v_n(t) = 0\} \end{cases} = \begin{cases} u_n(t) & \text{on } \{u_n(t) \neq 0\}, \\ 0 & \text{on } \{u_n(t) = 0\} \end{cases} = u_n(t)$

for all $t \in [0, T]$. Thereby, $u_n = b_h(v_n)$, for all $n \in (\mathbb{R}_+^*)^2$. Using the above results together with assertions (ii), (iii) and applying Theorem 1 in [12] yield the relative compactness of $(u_n)_n$ in $L^2(0, T; B)$. \square

3 Main results

In the light of the preceding preliminaries, we are now able to state the main results of this paper. Theorem 3.1 concerns the stochastic LANS- α model and Theorem 3.2 is devoted to the stochastic Navier-Stokes equations.

Theorem 3.1

Let $T > 0$, $D \subset \mathbb{R}^d$, $d \in \{2, 3\}$ be a bounded convex polygonal or polyhedral domain and $(\Omega, \mathcal{F}, (F_t)_{t \in [0, T]}, \mathbb{P})$ be a filtered probability space. Assume that assumptions (S_1) - (S_3) are fulfilled. For any finite positive pair (k, h) , let \mathcal{T}_h be a quasi-uniform triangulation of D , I_k be an equidistant partition of the time interval $[0, T]$, (\mathbb{H}_h, L_h) be a pair of finite element spaces satisfying the LBB-condition (2.10), and U^0 be in \mathbb{H}_h such that $\|U^0\|_{\mathbb{H}^1}$ is uniformly bounded in $h > 0$. If $\frac{\sqrt{k}}{h} < L \leq \alpha$ for some $L \in (0, 1)$ independent of k and h then, there exists a solution $\{(U^m, V^m, \Pi^m, \tilde{\Pi}^m)\}_{m=1}^M$ of Algorithm 1, and it satisfies Lemma 4.1. Moreover, if $U^0 \rightarrow \bar{u}_0$ in $L^4(\Omega; \mathbb{H}^1)$ as $h \rightarrow 0$, Algorithm 1 converges toward the unique solution of equations (1.1) in the sense of Definition 2.2.

Theorem 3.2

Let $T > 0$, $D \subset \mathbb{R}^2$ be a bounded convex polygonal domain and $(\Omega, \mathcal{F}, (F_t)_{t \in [0, T]}, \mathbb{P})$ be a filtered probability space. Assume assumptions (S_1) and (S_2) and let $v_0 \in L^4(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{H})$ be the initial datum of equations (1.2). For any finite positive pair (k, h) , let \mathcal{T}_h be a quasi-uniform triangulation of D , I_k be an equidistant partition of the time interval $[0, T]$, (\mathbb{H}_h, L_h) be a pair of finite element spaces satisfying the LBB-condition (2.10), and V^0 be in \mathbb{H}_h such that $\|V^0\|_{\mathbb{L}^2}$ is uniformly bounded in $h > 0$. If $\alpha \leq Ch$ for some $C > 0$ independent of k and h then, there exists a solution $\{(U^m, V^m, \Pi^m, \tilde{\Pi}^m)\}_{m=1}^M$ of Algorithm 1, and it satisfies Lemmas 4.1 and 4.4. Further, if $V^0 \rightarrow v_0$ in $L^4(\Omega; \mathbb{L}^2)$ as $h \rightarrow 0$ then, Algorithm 1 converges toward the unique solution of equations (1.2) in the sense of Definition 2.3.

As stated in the hypothesis of both Theorems 3.1 and 3.2, one needs to bound the initial datum U^0 (or V^0) of Algorithm 1 independently of $h > 0$. To do so, we evoke the Ritz operator \mathcal{R}_h

which is stable in \mathbb{H}^1 i.e. there is a positive non-decreasing function ζ , uniform in h such that $\|\mathcal{R}_h v\|_{\mathbb{H}^1} \leq \zeta \|v\|_{\mathbb{H}^1}$ for all $v \in \mathbb{H}^1$. Given $v \in \mathbb{H}^1$, the Ritz operator $\mathcal{R}_h: \mathbb{H}^1 \rightarrow \mathbb{V}_h$ is defined as the unique solution of

$$(\nabla \mathcal{R}_h v, \nabla v_h) = (\nabla v, \nabla v_h), \quad \forall v_h \in \mathbb{V}_h.$$

Therefore, we define U^0 by $U^0 = \mathcal{R}_h \bar{u}_0$ where \bar{u}_0 is the initial datum of equations (1.1). The same operator is suitable if V^0 was chosen to be the starting point of Algorithm 1. In this case, we set $V^0 = \mathcal{R}_h v_0$, where v_0 is given by problem (2.6) when $\bar{u} = \bar{u}_0$. For further properties, the reader may refer to [21, Lemma 4.2].

4 Solvability, stability and a priori estimates

Notice that the system of equations proposed in Algorithm 1 can be reformulated after taking the test functions φ and ψ in \mathbb{V}_h :

$$\begin{cases} \bullet (V^m - V^{m-1}, \varphi) + k\nu(\nabla V^m, \nabla \varphi) + k\tilde{b}(U^m, V^{m-1}, \varphi) \\ \quad = k\langle f(t_{m-1}, U^{m-1}), \varphi \rangle + (g(t_{m-1}, U^{m-1})\Delta_m W, \varphi), \quad \forall \varphi \in \mathbb{V}_h. \\ \bullet (V^m, \psi) = (U^m, \psi) + \alpha^2 (\nabla U^m, \nabla \psi), \quad \forall \psi \in \mathbb{V}_h. \end{cases} \quad (4.1)$$

In the below lemma, we illustrate the solvability of Algorithm 1, the iterates' measurability, and some a priori estimates whose role is to afford the proposed numerical scheme with stability.

Lemma 4.1

Assume that assumptions (S_1) - (S_3) are valid. Then, there exists a $\mathbb{V}_h \times \mathbb{V}_h \times L_h \times L_h$ -valued sequence of random variables $\{(U^m, V^m, \Pi^m, \tilde{\Pi}^m)\}_{m=1}^M$ that solves \mathbb{P} -a.s. Algorithm 1, and fulfills the following assertions:

(i) for any $m \in \{1, \dots, M\}$, the maps $U^m, V^m: \Omega \rightarrow \mathbb{H}_h$ are \mathcal{F}_{t_m} -measurable.

$$(ii) \quad \mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_\alpha^2 + \frac{k\nu}{2} \sum_{m=1}^M \left(\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right) + \frac{1}{4} \sum_{m=1}^M \|U^m - U^{m-1}\|_\alpha^2 \right] \leq C_T,$$

$$(iii) \quad \mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_\alpha^4 + \frac{1}{4} \sum_{m=1}^M \|U^m\|_\alpha^2 \|U^m - U^{m-1}\|_\alpha^2 + \frac{k\nu}{4} \sum_{m=1}^M \|U^m\|_\alpha^2 \left(\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right) \right] \leq C_{T,2},$$

where $C_{T,q} = C_{T,q} \left(\|U^0\|_{L^{2q}(\Omega; \mathbb{H}^1)}, T, (K_i)_{i=1}^4, Tr(Q), \nu, D \right)$, $q \in \{1, 2\}$ is a positive constant, independent of α , k and h . Note that $C_T := C_{T,1}$.

Proof: *Solvability*

To prove the Algorithm's solvability, we will follow a technique similar to that in [1, Lemma 4.1] while relying on equations (4.1). Since $V^m \in \mathbb{V}_h$ for all $m \in \{1, \dots, M\}$ then, by Lemma 2.1-(i), we get $V^m = U^m - \alpha^2 \Delta^h U^m$, \mathbb{P} -a.s. and a.e. in D . This means that the existence of U^m implies that of V^m . Assume that, for some $2 \leq \ell \leq M$ and for almost every $\omega \in \Omega$, a sequence

$\{(U^m(\omega), V^m(\omega))\}_{m=1}^{\ell-1}$ has been found by induction. For $\omega \in \Omega$, define \mathbb{P} -a.s. the mapping $\mathcal{F}_{\ell-1}^\omega: \mathbb{V}_h \rightarrow \mathbb{V}'_h$ by

$$\begin{aligned} \mathcal{F}_{\ell-1}^\omega(\varphi) := & \varphi - \alpha^2 \Delta^h \varphi - V^{\ell-1}(\omega) - k\nu \left(\Delta \varphi - \alpha^2 \Delta \Delta^h \varphi \right) + k[\varphi \cdot \nabla] V^{\ell-1}(\omega) \\ & + k(\nabla \varphi)^T \cdot V^{\ell-1}(\omega) - kf(t_{\ell-1}, U^{\ell-1}(\omega)) - g(t_{\ell-1}, U^{\ell-1}(\omega)) \Delta_\ell W(\omega), \end{aligned}$$

for all $\varphi \in \mathbb{V}_h$. The continuity of $\mathcal{F}_{\ell-1}^\omega$ can be shown by a straightforward argument. Since, \mathbb{V}_h equipped with the inner product (\cdot, \cdot) , is a Hilbert space, then by Riesz representation theorem, functional $\mathcal{F}_{\ell-1}^\omega$ can be defined through the L^2 -inner product, namely for $\varphi \in \mathbb{V}_h$, $(\mathcal{F}_{\ell-1}^\omega(\varphi))(\psi) = (\mathcal{F}_{\ell-1}^\omega(\varphi), \psi)$ for all $\psi \in \mathbb{V}_h$. Therefore, for $\psi = \varphi \in \mathbb{V}_h$ and by Proposition 2.1-(ii), the discrete Laplace operator (2.12), assumption (S_2) , the Cauchy-Schwarz and Young inequalities,

$$\begin{aligned} (\mathcal{F}_{\ell-1}^\omega(\varphi), \varphi) & \geq \|\varphi\|_{\mathbb{L}^2}^2 + (\alpha^2 + k\nu) \|\nabla \varphi\|_{\mathbb{L}^2}^2 - \|V^{\ell-1}(\omega)\|_{\mathbb{L}^2} \|\varphi\|_{\mathbb{L}^2} + k\nu\alpha^2 \|\Delta^h \varphi\|_{\mathbb{L}^2}^2 \\ & - k \left(K_3 + K_4 \|U^{\ell-1}(\omega)\|_\alpha \right) \|\varphi\|_{\mathbb{H}^1} - \left(K_1 + K_2 \|U^{\ell-1}(\omega)\|_\alpha \right) \|\Delta_\ell W(\omega)\|_K \|\varphi\|_{\mathbb{L}^2} \\ & \geq \frac{1}{2} \|\varphi\|_{\mathbb{L}^2}^2 + (\alpha^2 + \frac{k\nu}{2}) \|\nabla \varphi\|_{\mathbb{L}^2}^2 - \|V^{\ell-1}(\omega)\|_{\mathbb{L}^2}^2 - \frac{kC_D^2}{2\nu} \left(K_3 + K_4 \|U^{\ell-1}(\omega)\|_\alpha \right)^2 \\ & - \left(K_1 + K_2 \|U^{\ell-1}(\omega)\|_\alpha \right)^2 \|\Delta_\ell W(\omega)\|_K^2 \geq \frac{1}{2} \|\varphi\|_{\mathbb{L}^2}^2 - L_{\ell-1}(\omega), \end{aligned}$$

where $L_{\ell-1} := 2K_1^2 \|\Delta_\ell W\|_K^2 + \frac{kC_D^2 K_3^2}{\nu} + \|V^{\ell-1}\|_{\mathbb{L}^2}^2 + \left(\frac{kC_D^2 K_4^2}{\nu} + 2K_2^2 \|\Delta_\ell W\|_K^2 \right) \|U^{\ell-1}\|_\alpha^2$. By (2.2) and the induction's hypothesis, there holds \mathbb{P} -a.s. $L_{\ell-1}(\omega) < +\infty$. Therefore, taking $\varphi \in \mathbb{V}_h$ such that $\|\varphi\|_{\mathbb{L}^2} = \sqrt{2L_{\ell-1}(\omega)}$ yields $(\mathcal{F}_{\ell-1}^\omega(\varphi), \varphi) \geq 0$. Subsequently, Brouwer's fixed point theorem (see [18, Corollary 1.1, p. 279]) ensures the existence (but not uniqueness) of a $\phi = \phi(\omega) \in \mathbb{V}_h$ such that $\mathcal{F}_{\ell-1}^\omega(\phi) = 0$. Hence, $(U^\ell, V^\ell) \in \mathbb{V}_h \times \mathbb{V}_h$ exists \mathbb{P} -a.s. . The discrete LBB-condition (2.10) yields the existence of an $L_h \times L_h$ -valued process $\{(\Pi^m, \tilde{\Pi}^m)\}_{m=1}^M$ satisfying Algorithm 1.

Measurability

After proving the algorithm's solvability through the functional $\mathcal{F}_{\ell-1}^\omega$, the measurability of iterates U^m , $m \in \{1, \dots, M\}$ follows by induction (see [1, Lemma 4.1]). Moreover, by Lemma 2.1-(i), one infers the measurability of $\{V^m\}_{m=1}^M$.

A priori energy estimate

Let us denote by $\|\cdot\|_{h,\alpha}^2$ the quantity $\|\nabla \cdot\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h \cdot\|_{\mathbb{L}^2}^2$. In equation (4.1), we take $\varphi = \psi = U^m$ and employ identity (2.4) and Lemma 2.1-(ii):

$$\begin{aligned} \frac{1}{2} \left(\|U^m\|_\alpha^2 - \|U^{m-1}\|_\alpha^2 + \|U^m - U^{m-1}\|_\alpha^2 \right) + k\nu \|U^m\|_{h,\alpha}^2 & = k \langle f(t_{m-1}, U^{m-1}), U^m \rangle \\ & + \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^m - U^{m-1} \right) + \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right). \end{aligned} \quad (4.2)$$

After employing the Cauchy-Schwarz and Young inequalities along with assumption (S_2) , we take the sum over m from 1 to M :

$$\begin{aligned} \frac{1}{2} \|U^M\|_\alpha^2 - \frac{1}{2} \|U^0\|_\alpha^2 + \frac{1}{4} \sum_{m=1}^M \|U^m - U^{m-1}\|_\alpha^2 + \frac{k\nu}{2} \sum_{m=1}^M \|U^m\|_{h,\alpha}^2 \\ \leq \frac{C_D^2 T K_3^2}{\nu} + \frac{C_D^2 K_4^2}{\nu} k \sum_{m=1}^M \|U^{m-1}\|_\alpha^2 + \sum_{m=1}^M \|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 \\ + \sum_{m=1}^M \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right). \end{aligned} \quad (4.3)$$

Due to the measurability of U^m , the last term on the right-hand side vanishes when taking its expectation. The penultimate term is controlled as follows:

$$\begin{aligned} & \mathbb{E} \left[\|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \right] \\ &= \mathbb{E} \left[\|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \right] \mathbb{E} \left[\|\Delta_m W\|_K^2 \middle| \mathcal{F}_{t_{m-1}} \right] \\ &\leq 2Tr(Q)K_1^2 k + 2K_2^2 k Tr(Q) \mathbb{E} [\|U^{m-1}\|_\alpha^2], \end{aligned} \quad (4.4)$$

thanks to the tower property of the conditional expectation, the increments independence of the Wiener process, property (2.2), and assumption (S_2) . Plugging estimate (4.4) in equation (4.3) returns

$$\begin{aligned} & \frac{1}{2} \mathbb{E} [\|U^M\|_\alpha^2] + \frac{1}{4} \sum_{m=1}^M \mathbb{E} [\|U^m - U^{m-1}\|_\alpha^2] + \frac{k\nu}{2} \sum_{m=1}^M \mathbb{E} [\|U^m\|_{h,\alpha}^2] \leq \frac{1}{2} \mathbb{E} [\|U^0\|_\alpha^2] \\ &+ \left(\frac{C_D^2 K_3^2}{\nu} + 2Tr(Q)K_1^2 \right) T + \left(\frac{C_D^2 K_4^2}{\nu} + 2K_2^2 Tr(Q) \right) k \sum_{m=0}^{M-1} \mathbb{E} [\|U^m\|_\alpha^2]. \end{aligned} \quad (4.5)$$

Now, we employ the discrete Gronwall inequality (see for instance [28, Lemma 10.5]) in order to prove the sought estimate. We replace M in equation (4.5) by any other index $\ell \geq 1$. We get

$$\mathbb{E} [\|U^\ell\|_\alpha^2] \leq \left[\mathbb{E} [\|U^0\|_{\mathbb{H}^1}^2] + 2 \left(\frac{C_D^2 K_3^2}{\nu} + 2Tr(Q)K_1^2 \right) T \right] e^{T \left(\frac{C_D^2 K_4^2}{\nu} + 2K_2^2 Tr(Q) \right)} =: K_T$$

for all $\ell \in \{1, \dots, M\}$, where $\|U^0\|_\alpha \leq \|U^0\|_{\mathbb{H}^1}$ thanks to (2.3). Consequently,

$$\max_{1 \leq m \leq M} \mathbb{E} [\|U^m\|_\alpha^2] \leq K_T. \quad (4.6)$$

By virtue of estimate (4.5) and the discrete Gronwall lemma, one also obtains the following two estimates: $\frac{k\nu}{2} \sum_{m=1}^M \mathbb{E} [\|U^m\|_{h,\alpha}^2] \leq K_T$ and $\frac{1}{4} \sum_{m=1}^M \mathbb{E} [\|U^m - U^{m-1}\|_\alpha^2] \leq K_T$. We still need to prove $\mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_\alpha^2 \right] \leq C_T$, for a certain positive constant C_T independent of α , k and h . To this end, we make use of estimate (4.3), but this time by summing from $m = 1$ to $m = \ell$ where $\ell \geq 1$ is an integer. Then, we take the maximum over ℓ and apply the mathematical expectation on both sides to get

$$\begin{aligned} & \frac{1}{2} \mathbb{E} \left[\max_{1 \leq \ell \leq M} \|U^\ell\|_\alpha^2 \right] \leq \frac{1}{2} \mathbb{E} [\|U^0\|_\alpha^2] + \frac{C_D^2 T K_3^2}{\nu} + \frac{C_D^2 K_4^2}{\nu} k \sum_{m=1}^M \mathbb{E} [\|U^{m-1}\|_\alpha^2] \\ &+ \sum_{m=1}^M \mathbb{E} \left[\|g(t_{m-1}, U^{m-1}) \Delta_m W\|_{\mathbb{L}^2}^2 \right] + \mathbb{E} \left[\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} (g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1}) \right]. \end{aligned} \quad (4.7)$$

To bound the last term on the right-hand side, we use assumption (S_2) , the Burkholder-Davis-Gundy and Young inequalities, after considering the sum as the stochastic integral of a piecewise

constant integrand:

$$\begin{aligned}
 & \mathbb{E} \left[\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) \right] \\
 & \lesssim \mathbb{E} \left[\left(k \sum_{m=1}^M \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|U^{m-1}\|_{\mathbb{L}^2}^2 \right)^{1/2} \right] \\
 & \leq \frac{1}{4} \mathbb{E} [\|U^0\|_{\mathbb{L}^2}^2] + 2K_1^2 T + \mathbb{E} \left[\frac{1}{4} \max_{1 \leq \ell \leq M} \|U^\ell\|_{\mathbb{L}^2}^2 + 2K_2^2 k \sum_{m=1}^M \|U^{m-1}\|_{\alpha}^2 \right].
 \end{aligned} \tag{4.8}$$

Returning to estimate (4.7), we avail ourselves of (4.4), (4.6) and (4.8) to conclude

$$\mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_{\alpha}^2 \right] \leq C_T,$$

where $C_T > 0$ depends only on the parameters of K_T .

Bounds for higher velocity moments

We start by multiplying equation (4.2) by the norm $\|U^m\|_{\alpha}^2$.

$$\begin{aligned}
 & \frac{1}{2} \|U^m\|_{\alpha}^4 - \frac{1}{2} \|U^{m-1}\|_{\alpha}^2 \|U^m\|_{\alpha}^2 + \frac{1}{2} \|U^m - U^{m-1}\|_{\alpha}^2 \|U^m\|_{\alpha}^2 + k\nu \|U^m\|_{h, \alpha}^2 \|U^m\|_{\alpha}^2 \\
 & = k \langle f(t_{m-1}, U^{m-1}), U^m \rangle \|U^m\|_{\alpha}^2 + \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^m - U^{m-1} \right) \|U^m\|_{\alpha}^2 \\
 & \quad + \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) \|U^m\|_{\alpha}^2 = I + II + III.
 \end{aligned} \tag{4.9}$$

For I , we apply the norm equivalence (2.3), the Young inequality and estimate $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ for $p = 4$:

$$\begin{aligned}
 I & \leq kC_D \left(K_3 + K_4 \|U^{m-1}\|_{\alpha} \right) \|\nabla U^m\|_{\mathbb{L}^2}^{\frac{3}{2}} \|U^m\|_{\alpha}^{\frac{3}{2}} \leq \frac{kC_D^4}{4\nu^3} \left(K_3 + K_4 \|U^{m-1}\|_{\alpha} \right)^4 \\
 & \quad + \frac{3k\nu}{4} \|U^m\|_{h, \alpha}^2 \|U^m\|_{\alpha}^2 \leq \frac{2kC_D^4 K_3^4}{\nu^3} + \frac{2kC_D^4 K_4^4}{\nu^3} \|U^{m-1}\|_{\alpha}^4 + \frac{3k\nu}{4} \|U^m\|_{h, \alpha}^2 \|U^m\|_{\alpha}^2.
 \end{aligned}$$

For II ,

$$\begin{aligned}
 II & \leq \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \left(\|U^m\|_{\alpha}^2 - \|U^{m-1}\|_{\alpha}^2 + \|U^{m-1}\|_{\alpha}^2 \right) \\
 & \quad + \frac{1}{4} \|U^m - U^{m-1}\|_{\mathbb{L}^2}^2 \|U^m\|_{\alpha}^2 \\
 & \leq \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \|U^{m-1}\|_{\alpha}^2 + \frac{1}{16} \left| \|U^m\|_{\alpha}^2 - \|U^{m-1}\|_{\alpha}^2 \right|^2 \\
 & \quad + 4 \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^4 \|\Delta_m W\|_K^4 + \frac{1}{4} \|U^m - U^{m-1}\|_{\mathbb{L}^2}^2 \|U^m\|_{\alpha}^2.
 \end{aligned}$$

For III ,

$$\begin{aligned}
 III & := \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) \left(\|U^m\|_{\alpha}^2 - \|U^{m-1}\|_{\alpha}^2 + \|U^{m-1}\|_{\alpha}^2 \right) \\
 & \leq \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) \|U^{m-1}\|_{\alpha}^2 + \frac{1}{16} \left| \|U^m\|_{\alpha}^2 - \|U^{m-1}\|_{\alpha}^2 \right|^2 \\
 & \quad + 4 \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \|U^{m-1}\|_{\alpha}^2.
 \end{aligned}$$

Equation (4.9) becomes

$$\begin{aligned}
& \frac{1}{2} \|U^m\|_\alpha^4 - \frac{1}{2} \|U^{m-1}\|_\alpha^2 \|U^m\|_\alpha^2 + \frac{1}{4} \|U^m - U^{m-1}\|_\alpha^2 \|U^m\|_\alpha^2 + \frac{k\nu}{4} \|U^m\|_{h,\alpha}^2 \|U^m\|_\alpha^2 \\
& \leq \frac{2kC_D^4 K_3^4}{\nu^3} + \frac{2kC_D^4 K_4^4}{\nu^3} \|U^{m-1}\|_\alpha^4 + \frac{1}{8} \left| \|U^m\|_\alpha^2 - \|U^{m-1}\|_\alpha^2 \right|^2 \\
& \quad + \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) \|U^{m-1}\|_\alpha^2 + 4 \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^4 \|\Delta_m W\|_K^4 \\
& \quad + 5 \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \|U^{m-1}\|_\alpha^2.
\end{aligned}$$

Note that $\|U^m\|_\alpha^4 - \|U^{m-1}\|_\alpha^2 \|U^m\|_\alpha^2 = \frac{1}{2} (\|U^m\|_\alpha^4 - \|U^{m-1}\|_\alpha^4 + \|U^m\|_\alpha^2 - \|U^{m-1}\|_\alpha^2)^2$, therefore

$$\begin{aligned}
& \frac{1}{4} \left(\|U^m\|_\alpha^4 - \|U^{m-1}\|_\alpha^4 + \frac{1}{2} \left| \|U^m\|_\alpha^2 - \|U^{m-1}\|_\alpha^2 \right|^2 + \|U^m - U^{m-1}\|_\alpha^2 \|U^m\|_\alpha^2 \right. \\
& \quad \left. + k\nu \|U^m\|_{h,\alpha}^2 \|U^m\|_\alpha^2 \right) \leq \frac{2kC_D^4 K_3^4}{\nu^3} + \frac{2kC_D^4 K_4^4}{\nu^3} \|U^{m-1}\|_\alpha^4 \\
& \quad + \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) \|U^{m-1}\|_\alpha^2 + 4 \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^4 \|\Delta_m W\|_K^4 \\
& \quad + 5 \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \|U^{m-1}\|_\alpha^2,
\end{aligned} \tag{4.10}$$

Proceeding as (4.4), the penultimate term can be estimated as follows

$$\mathbb{E} \left[\|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^4 \|\Delta_m W\|_K^4 \right] \lesssim K_1^4 \text{Tr}(Q)^2 k^2 + K_2^4 \text{Tr}(Q)^2 k^2 \mathbb{E} \left[\|U^{m-1}\|_\alpha^4 \right]. \tag{4.11}$$

Next, we bound the last term on the right-hand side of (4.10)

$$\begin{aligned}
& \mathbb{E} \left[\|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|\Delta_m W\|_K^2 \|U^{m-1}\|_\alpha^2 \right] \\
& \lesssim K_1^2 k \text{Tr}(Q) \mathbb{E} \left[\|U^{m-1}\|_\alpha^2 \right] + K_2^2 \text{Tr}(Q) k \mathbb{E} \left[\|U^{m-1}\|_\alpha^4 \right].
\end{aligned} \tag{4.12}$$

The third term on the right-hand side of (4.10) vanishes after taking its expectation, thanks to the measurability of the iterates U^m , $m \in \{1, \dots, M\}$. We collect and plug the above estimates back in (4.10), and we sum it up over m from $m = 1$ to $m = M$. Then, we apply the mathematical expectation, and employ the discrete Gronwall lemma to get

$$\max_{1 \leq m \leq M} \mathbb{E} \left[\|U^m\|_\alpha^4 \right] \leq C_{T,2}, \tag{4.13}$$

where $C_{T,2} > 0$ does not depend on α , k and h . We also get by Gronwall lemma the following two estimates:

$$\frac{1}{4} \mathbb{E} \left[\sum_{m=1}^M \|U^m - U^{m-1}\|_\alpha^2 \|U^m\|_\alpha^2 \right] \leq C_{T,2} \text{ and } \frac{k\nu}{4} \mathbb{E} \left[\sum_{m=1}^M \|U^m\|_{h,\alpha}^2 \|U^m\|_\alpha^2 \right] \leq C_{T,2}.$$

It remains to show that $\mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_\alpha^4 \right] \leq C_{T,2}$. To do so, we follow the technique which was employed in the previous step (A priori energy estimate) by summing up inequality (4.10) over m from 1 to $\ell \geq 1$. We will only need to control the following stochastic term:

$$\mathbb{E} \left[\max_{1 \leq \ell \leq M} \sum_{m=1}^{\ell} \left(g(t_{m-1}, U^{m-1}) \Delta_m W, U^{m-1} \right) \|U^{m-1}\|_\alpha^2 \right]$$

$$\begin{aligned} &\lesssim \mathbb{E} \left[\left(k \sum_{m=1}^M \|g(t_{m-1}, U^{m-1})\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 \|U^{m-1}\|_{\alpha}^6 \right)^{\frac{1}{2}} \right] \\ &\leq \mathbb{E} \left[\frac{1}{8} \|U^0\|_{\mathbb{H}^1}^4 + \frac{1}{8} \max_{1 \leq m \leq M} \|U^m\|_{\alpha}^4 + 4K_1^2 k \sum_{m=1}^M \|U^{m-1}\|_{\alpha}^2 + 4K_2^2 k \sum_{m=1}^M \|U^{m-1}\|_{\alpha}^4 \right]. \end{aligned}$$

Collecting all estimates together and using (4.13) complete the proof. \square

While proving the solvability of Algorithm 1, we found out that the iterates $\{(U^m, V^m)\}_{m=1}^M$ might not be unique. We will discuss this property in the upcoming lemma.

Lemma 4.2

Iterates $\{(U^m, V^m)\}_{m=1}^M$ of Algorithm 1, are unique \mathbb{P} -almost surely and almost everywhere in D .

Proof: Assume that the $\mathbb{V}_h \times \mathbb{V}_h$ -valued processes $\{(U_1^m, V_1^m)\}_{m=1}^M$ and $\{(U_2^m, V_2^m)\}_{m=1}^M$ solve the projected version (4.1) of Algorithm 1, starting from the same initial datum U^0 . For all $m \in \{0, 1, \dots, M\}$, denote by $\mathcal{U}^m := U_1^m - U_2^m$ and by $\mathcal{V}^m := V_1^m - V_2^m$. Clearly, $\mathcal{U}^0 = \mathcal{V}^0 = 0$ \mathbb{P} -almost surely. Replace both solutions in equation (4.1) and subtract them to get for all $(\varphi, \psi) \in \mathbb{V}_h \times \mathbb{V}_h$:

$$\begin{cases} \bullet \left(\mathcal{V}^m - \mathcal{V}^{m-1}, \varphi \right) + k\nu (\nabla \mathcal{V}^m, \nabla \varphi) + k [\tilde{b}(U_1^m, V_1^{m-1}, \varphi) - \tilde{b}(U_2^m, V_2^{m-1}, \varphi)] \\ \bullet = k \langle f(t_{m-1}, U_1^{m-1}) - f(t_{m-1}, U_2^{m-1}), \varphi \rangle + \left(g(t_{m-1}, U_1^{m-1}) - g(t_{m-1}, U_2^{m-1}), \varphi \right), \\ \bullet (\mathcal{V}^m, \psi) = (\mathcal{U}^m, \psi) + \alpha^2 (\nabla \mathcal{U}^m, \nabla \psi). \end{cases} \quad (4.14)$$

We proceed by induction. For $m = 1$, the trilinear term becomes $\tilde{b}(\mathcal{U}^1, 0, \varphi) = 0$ and equation (4.14)₁ leads to $(\mathcal{V}^1, \varphi) + k\nu (\nabla \mathcal{V}^1, \nabla \varphi) = 0$. Taking $\varphi = \psi = \mathcal{U}^1$ in both (4.14)₂ and the latter equation and subtracting them yield $\|\mathcal{U}^1\|_{\alpha}^2 = -k\nu (\nabla \mathcal{V}^1, \nabla \mathcal{U}^1)$. From Lemma 2.1-(i), one gets $\nabla \mathcal{V}^1 = \nabla \mathcal{U}^1 - \alpha^2 \nabla \Delta^h \mathcal{U}^1$ \mathbb{P} -a.s. and a.e. in D . Thereby, $\|\mathcal{U}^1\|_{\alpha}^2 = -k\nu \|\nabla \mathcal{U}^1\|_{\mathbb{L}^2}^2 + k\nu \alpha^2 (\nabla \Delta^h \mathcal{U}^1, \nabla \mathcal{U}^1)$. Using identity (2.12), we infer that

$$\|\mathcal{U}^1\|_{\alpha}^2 + k\nu \|\nabla \mathcal{U}^1\|_{\mathbb{L}^2}^2 + k\nu \alpha^2 \|\Delta^h \mathcal{U}^1\|_{\mathbb{L}^2}^2 = 0, \quad \mathbb{P}\text{-a.s.}$$

Hence, $\mathcal{U}^1 = \mathcal{V}^1 = 0$, \mathbb{P} -a.s. and a.e. in D . By following the same technique while assuming $(U_1^{m-1}, V_1^{m-1}) = (U_2^{m-1}, V_2^{m-1})$, one gets eventually the uniqueness \mathbb{P} -a.s. and a.e. in D for all $m \in \mathbb{N}$. \square

In order to obtain a priori estimates for $\{V^m\}_{m=1}^M$ in Sobolev spaces, independently of the discretization's parameters, we shall assume that there is a positive constant C , independent of α , h and k such that $0 < \alpha \leq Ch$. We will present in Lemma 4.3 some preliminary estimates.

Lemma 4.3

Let $\{(U^m, V^m)\}_{m=1}^M$ be the iterates of Algorithm 1 and $0 < \alpha \leq Ch$, where $C > 0$ independent of α , h and k . Then, for all $m \in \{1, \dots, M\}$ and \mathbb{P} -a.s.

- (i) $\|V^m\|_{\mathbb{L}^2} \leq C_1 \|U^m\|_{\alpha}$,
- (ii) $\|\nabla V^m\|_{\mathbb{L}^2}^2 \leq C_1 \left(\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right)$,
- (iii) $\|V^{m+\ell} - V^m\|_{\mathbb{L}^2} \leq C_1 \|U^{m+\ell} - U^m\|_{\alpha}$, for all $\ell \in \{1, \dots, M - m\}$,

where $C_1 > 0$ depends only on C and the constant C of the inverse inequality (2.14).

Proof: Let $m \in \{1, \dots, M\}$. From equation (4.1)₂, taking $\psi = V^m$ and applying the Cauchy-Schwarz and Young inequalities yield $\|V^m\|_{\mathbb{L}^2}^2 \leq \|U^m\|_{\mathbb{L}^2}^2 + \frac{1}{4}\|V^m\|_{\mathbb{L}^2}^2 + \frac{\alpha^2}{\epsilon}\|\nabla U^m\|_{\mathbb{L}^2}^2 + \frac{\epsilon\alpha^2}{4}\|\nabla V^m\|_{\mathbb{L}^2}^2$, where $\epsilon > 0$. Taking $\epsilon = \frac{1}{c^2C^2}$ and applying the inverse inequality (2.14) complete the proof of assertion (i). On the other hand, by Lemma 2.1-(i), $\nabla V^m = \nabla U^m - \alpha^2 \nabla \Delta^h U^m$, \mathbb{P} -a.s. and a.e. in D . Thus, $\|\nabla V^m\|_{\mathbb{L}^2}^2 \leq 2\|\nabla U^m\|_{\mathbb{L}^2}^2 + 2C^2C^2\alpha^2\|\Delta^h U^m\|_{\mathbb{L}^2}^2$, thanks to the inverse inequality (2.14). Estimate (iii) has similar proof to that of assertion (i). \square

Clearly, one must incorporate Lemmas 4.1 and 4.3 to obtain:

Lemma 4.4

Let $\{V^m\}_{m=1}^M$ be the iterates of Algorithm 1. Assume that assumptions (S₁)-(S₃) are fulfilled and that $0 < \alpha \leq Ch$, for a certain $C > 0$ independent of α , k and h . Then,

$$\mathbb{E} \left[\max_{1 \leq m \leq M} \|V^m\|_{\mathbb{L}^2}^2 + \frac{k\nu}{2} \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2 + \frac{1}{4} \sum_{m=1}^M \|V^m - V^{m-1}\|_{\mathbb{L}^2}^2 \right] \leq C'_T,$$

where $C'_T > 0$ does not depend on α , k and h .

We end up this section with the following a priori estimate for $\{V^m\}_{m=1}^M$, where the scale α is not necessarily assumed to be controlled by h .

Lemma 4.5

Assume (S₁)-(S₃) and let $\{V^m\}_{m=1}^M$ be the iterates of Algorithm 1. Then,

$$\mathbb{E} \left[k \sum_{m=1}^M \|V^m\|_{\mathbb{L}^2}^2 \right] \leq C_T.$$

Proof: From equation (4.1), replacing ψ by V^m and employing Lemma 2.1-(ii) lead to $\|V^m\|_{\mathbb{L}^2}^2 = \|U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \left(\|\nabla U^m\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h U^m\|_{\mathbb{L}^2}^2 \right)$. Multiplying by k , summing both sides over m from 1 to M , using the condition $\alpha \leq 1$, then applying the mathematical expectation and Lemma 4.1-(ii) complete the proof. \square

5 Convergence

All the previous interpretation relied on $\{(U^m, V^m)\}_{m=1}^M$, which does not depend explicitly on the time variable. To investigate the convergence in continuous-time spaces, e.g. $L^2(\Omega; L^2(0, T; \mathbb{H}^1))$, we need to define the following processes

$$\mathcal{V}_{k,h}(t, x) := \frac{t - t_{m-1}}{k} V^m(x) + \frac{t_m - t}{k} V^{m-1}(x), \quad \forall (t, x) \in [t_{m-1}, t_m] \times D, \quad (5.1)$$

$$\left(\mathcal{U}_{k,h}^-(t, x), \mathcal{V}_{k,h}^-(t, x) \right) := \left(U^{m-1}(x), V^{m-1}(x) \right), \quad \forall (t, x) \in [t_{m-1}, t_m] \times D, \quad (5.2)$$

$$\left(\mathcal{U}_{k,h}^+(t, x), \mathcal{V}_{k,h}^+(t, x) \right) := \left(U^m(x), V^m(x) \right), \quad \forall (t, x) \in (t_{m-1}, t_m] \times D. \quad (5.3)$$

Note that $\mathcal{V}_{k,h}$ is continuous on $[0, T]$,

$$\mathcal{V}_{k,h}(t, \cdot) - \mathcal{V}_{k,h}^-(t, \cdot) = \frac{t - t_{m-1}}{k} \left(V^m - V^{m-1} \right) \text{ for all } t \in [t_{m-1}, t_m), \quad (5.4)$$

and

$$\mathcal{V}_{k,h}(t, \cdot) - \mathcal{V}_{k,h}^+(t, \cdot) = \frac{t - t_m}{k} \left(V^m - V^{m-1} \right) \text{ for all } t \in (t_{m-1}, t_m]. \quad (5.5)$$

Plugging both new processes in equation (4.1), we get for every $t \in [t_{m-1}, t_m)$, $(\varphi, \psi) \in \mathbb{V}_h \times \mathbb{V}_h$ and \mathbb{P} -a.s. the following:

$$\begin{cases} \bullet \left(\mathcal{V}_{k,h} - \mathcal{V}_{k,h}^-, \varphi \right) + \nu \int_{t_{m-1}}^t \left(\nabla \mathcal{V}_{k,h}^+, \nabla \varphi \right) ds + \int_{t_{m-1}}^t \tilde{b}(\mathcal{U}_{k,h}^+, \mathcal{V}_{k,h}^-, \varphi) ds \\ = \int_{t_{m-1}}^t \left\langle f^-(t, \mathcal{U}_{k,h}^-), \varphi \right\rangle ds + \frac{t - t_{m-1}}{k} \left(\int_{t_{m-1}}^t g^-(s, \mathcal{U}_{k,h}^-) dW(s), \varphi \right), \\ \bullet \left(\mathcal{V}_{k,h}^+, \psi \right) = \left(\mathcal{U}_{k,h}^+, \psi \right) + \alpha^2 \left(\nabla \mathcal{U}_{k,h}^+, \nabla \psi \right). \end{cases}$$

where $f^-(t, \cdot) = f(t_{m-1}, \cdot)$ and $g^-(t, \cdot) = g(t_{m-1}, \cdot)$ for all $t \in [t_{m-1}, t_m)$ and $m \in \{1, \dots, M\}$.

Since Lemmas 2.2 and 2.3 are essential to retrieve a strong convergence in spaces that do not depend on randomness, we must consider applying them in a probability subset. We shall give first a dedicated configuration for the case $0 < \alpha \leq Ch$. It will be adjusted afterward to fit the converse case.

Denote by $\mathcal{U}^{k,h} := k \sum_{m=1}^M \|\nabla U^m\|_{\mathbb{L}^2}^2$ and by $\mathcal{V}^{k,h} := k \sum_{m=1}^M \|\nabla V^m\|_{\mathbb{L}^2}^2$. Both law families $\{\mathcal{L}(\mathcal{U}^{k,h})\}_{k,h}$ and $\{\mathcal{L}(\mathcal{V}^{k,h})\}_{k,h}$ are tight. Indeed, for $\varepsilon > 0$, we consider the closed \mathbb{R} -ball $K_\varepsilon := \bar{B}_{\mathbb{R}}(0, 1/\varepsilon)$. Clearly, K_ε is a compact set of \mathbb{R} , and

$$\mathcal{L}(\mathcal{U}^{k,h})(K_\varepsilon) = \mathbb{P}(\mathcal{U}^{k,h} \in K_\varepsilon) = \mathbb{P}(\{\omega \in \Omega \mid k \sum_{m=1}^M \|\nabla U^m\|_{\mathbb{L}^2}^2 \leq 1/\varepsilon\}) \geq 1 - C_T \varepsilon,$$

for all $k, h > 0$, thanks to the Markov inequality and Lemma 4.1-(ii). Similarly, by Lemma 4.4 and the Markov inequality, the tightness of $\{\mathcal{L}(\mathcal{V}^{k,h})\}_{k,h}$ follows. Such an argument is summoned just to emphasize the non-dependence of ε on k and h . Concerning the Wiener increments' tightness, we define $W_{k,h} := \frac{1}{k} \max_{1 \leq m \leq M} \|\Delta_m W\|_K^2$, and consider the same compact set K_ε . Therefore, by virtue of estimate (2.2),

$$\mathcal{L}(W_{k,h})(K_\varepsilon^W) = \mathbb{P}(W_{k,h} \leq 1/\varepsilon) \geq 1 - Tr(Q)\varepsilon,$$

for all $k, h > 0$, thanks again to the Markov inequality. With that being said, we fix $\varepsilon > 0$, and define the sample subset $\Omega_{k,h}^\varepsilon := \Omega_{k,h}^1 \cap \Omega_{k,h}^2 \subset \Omega$, where $\Omega_{k,h}^1 := \{\omega \in \Omega \mid W_{k,h} \leq 1/\varepsilon\}$, and $\Omega_{k,h}^2 := \left\{ \omega \in \Omega \mid \int_0^T \left(\|\nabla \mathcal{U}_{k,h}^+\|_{\mathbb{L}^2}^2 + \|\nabla \mathcal{V}_{k,h}^+\|_{\mathbb{L}^2}^2 \right) dt \leq 1/\varepsilon \right\}$. In the light of the preceding analysis, there is a $c > 0$ depending only on $Tr(Q)$, C_T and C_T' such that

$$\mathbb{P}(\Omega_{k,h}^\varepsilon) \geq 1 - c\varepsilon. \quad (5.6)$$

The next lemma fulfills one of the hypotheses of Lemmas 2.2 and 2.3 in the sample subset $\Omega_{k,h}^\varepsilon$. Note that assumption $\alpha \leq Ch$ can be replaced by an alternative hypothesis $\sqrt{k}/h < L \leq \alpha$, for some $L \in (0, 1)$ along with a minor modification in $\Omega_{k,h}^\varepsilon$, as explained in Remark 5.1. Such conditions are mandatory if we are aiming at converging Algorithm 1 toward the LANS- α equations instead of the NSEs.

Lemma 5.1

Let $0 < \alpha \leq Ch$, for a certain $C > 0$ independent of α, h and k . For almost every $\omega \in \Omega_{k,h}^\varepsilon$, and $\ell \in \{1, \dots, M-1\}$,

$$k \sum_{m=1}^{M-\ell} \|U^{m+\ell}(\omega) - U^m(\omega)\|_\alpha^2 \leq Ct_\ell^{\frac{1}{2}},$$

where $C = C(T, \nu, D, K_1, K_2, K_3, K_4, \varepsilon) > 0$ independent of α, k and h .

Proof: Let $\ell \in \{1, \dots, M-1\}$, and $\omega \in \Omega_{k,h}^\varepsilon$. We replace the index m by i in (4.1), and sum it over i from $m+1$ to $m+\ell$. We get for all $\varphi \in \mathbb{V}_h$ and \mathbb{P} -a.s.

$$\begin{aligned} & (V^{m+\ell} - V^m, \varphi) + k\nu \sum_{i=1}^{\ell} (\nabla V^{m+i}, \nabla \varphi) + k \sum_{i=1}^{\ell} \tilde{b}(U^{m+i}, V^{m+i-1}, \varphi) \\ &= k \sum_{i=1}^{\ell} \langle f(t_{m+i-1}, U^{m+i-1}), \varphi \rangle + \sum_{i=1}^{\ell} \left(g(t_{m+i-1}, U^{m+i-1}) \Delta_{m+i} W, \varphi \right). \end{aligned} \quad (5.7)$$

Set $\varphi = U^{m+\ell} - U^m$, sum (5.7) up over m from 1 to $M-\ell$, then multiply by k :

$$\begin{aligned} & k \sum_{m=1}^{M-\ell} \|U^{m+\ell} - U^m\|_\alpha^2 = -k^2 \nu \sum_{m=1}^{M-\ell} \sum_{i=1}^{\ell} (\nabla V^{m+i}, \nabla(U^{m+\ell} - U^m)) \\ & - k^2 \sum_{m=1}^{M-\ell} \sum_{i=1}^{\ell} \tilde{b}(U^{m+i}, V^{m+i-1}, U^{m+\ell} - U^m) \\ & + k^2 \sum_{m=1}^{M-\ell} \sum_{i=1}^{\ell} \langle f(t_{m+i-1}, U^{m+i-1}), U^{m+\ell} - U^m \rangle \\ & + k \sum_{m=1}^{M-\ell} \sum_{i=1}^{\ell} \left(g(t_{m+i-1}, U^{m+i-1}) \Delta_{m+i} W, U^{m+\ell} - U^m \right) = I + II + III + IV. \end{aligned}$$

Our aim is to bound each term by $t_\ell^{1/2}$. For the term I , we use the Cauchy-Schwarz inequality,

$|I| \leq 2\nu t_\ell^{1/2} k^{3/2} \sum_{m=1}^M \|\nabla U^m\|_{\mathbb{L}^2} \left(\sum_{i=1}^M \|\nabla V^i\|_{\mathbb{L}^2}^2 \right)^{1/2} \leq \frac{2\nu T^{1/2}}{\varepsilon} t_\ell^{1/2}$. On the other hand,

$$\begin{aligned} |II| & \leq C_D k^{\frac{1}{2}} t_\ell^{\frac{1}{2}} k \sum_{m=1}^{M-\ell} \|\nabla(U^{m+\ell} - U^m)\|_{\mathbb{L}^2} \left(\sum_{i=1}^{\ell} \|\nabla U^{m+i}\|_{\mathbb{L}^2}^2 \|\nabla V^{m+i-1}\|_{\mathbb{L}^2}^2 \right)^{1/2} \\ & \leq 2C_D k^{\frac{1}{2}} t_\ell^{\frac{1}{2}} k \sum_{m=1}^M \|\nabla U^m\|_{\mathbb{L}^2} \left(\sum_{i=1}^M \|\nabla U^i\|_{\mathbb{L}^2}^2 \right)^{1/2} \left(\sum_{i=1}^M \|\nabla V^i\|_{\mathbb{L}^2}^2 \right)^{1/2} \\ & \leq 6C_D t_\ell^{\frac{1}{2}} k \sum_{m=1}^M \|\nabla U^m\|_{\mathbb{L}^2}^2 \left(k \sum_{i=1}^M \|\nabla V^i\|_{\mathbb{L}^2}^2 \right)^{1/2} \leq \frac{6C_D}{\varepsilon \sqrt{\varepsilon}} t_\ell^{\frac{1}{2}}, \end{aligned}$$

where Proposition 2.1-(iii), the Cauchy-Schwarz inequality for sums, $\ell^1(\mathbb{N}) \hookrightarrow \ell^2(\mathbb{N})$ and estimate (2.5) were employed. For the term III , we need assumption (S_2) , the norm equivalence (2.3), the Poincaré and Cauchy-Schwarz inequalities, estimate (2.5) and $\ell^1(\mathbb{N}) \hookrightarrow \ell^2(\mathbb{N})$:

$$|III| \leq C_D k \sum_{m=1}^{M-\ell} \|\nabla(U^{m+\ell} - U^m)\|_{\mathbb{L}^2} k \sum_{i=1}^{\ell} \left(K_3 + C_D K_4 \|\nabla U^{m+i-1}\|_{\mathbb{L}^2} \right)$$

$$\begin{aligned} &\leq 2C_D t_\ell^{\frac{1}{2}} K_3 T \left(k \sum_{m=1}^M \|\nabla U^m\|_{\mathbb{L}^2}^2 \right)^{1/2} + 6C_D t_\ell^{\frac{1}{2}} K_4 k^{\frac{1}{2}} k \sum_{m=1}^M \|\nabla U^m\|_{\mathbb{L}^2}^2 \\ &\leq 2C_D (K_3 T / \sqrt{\varepsilon} + 3K_4 T^{\frac{1}{2}} / \varepsilon) t_\ell^{\frac{1}{2}}. \end{aligned}$$

For the term IV , we need the Cauchy-Schwarz inequality, assumption (S_2) , $\ell^1(\mathbb{N}) \hookrightarrow \ell^2(\mathbb{N})$ and inequality (2.5):

$$\begin{aligned} |IV| &\leq C_D K_2 k^{\frac{1}{2}} \sum_{m=1}^{M-\ell} \|U^{m-\ell} - U^m\|_{\mathbb{L}^2} \left(\sum_{i=1}^{\ell} \|\nabla U^{m+i-1}\|_{\mathbb{L}^2}^2 \|\Delta_{m+i} W\|_K^2 \right)^{1/2} t_\ell^{\frac{1}{2}} \\ &\quad + 2K_1 t_\ell \sum_{m=1}^M \|U^m\|_{\mathbb{L}^2} \max_{1 \leq m \leq M} \|\Delta_m W\|_K \leq \frac{2K_1 C_D}{\varepsilon} t_\ell + \frac{2C_D K_2}{\varepsilon \sqrt{\varepsilon}} t_\ell^{\frac{1}{2}}. \end{aligned}$$

Collecting all the above estimates together yields the final result. \square

Remark 5.1 *Observe that only terms I and II in the above proof required the use of the condition $\alpha \leq Ch$. We avert this assumption as follows: firstly, we assume the relation $\frac{\sqrt{k}}{h} < L \leq \alpha$, for a certain $L \in (0, 1)$ independent of k and h . Secondly, since Lemma 4.4 is no longer valid, we adjust the sample subset $\Omega_{k,h}^2$ by replacing $\|\nabla \mathcal{V}_{k,h}^+\|_{\mathbb{L}^2}^2$ with $\|\mathcal{V}_{k,h}^+\|_{\mathbb{L}^2}^2$. We also need to intersect $\Omega_{k,h}^\varepsilon$ with a supplementary sample subset $\Omega_{k,h}^3 := \{\omega \in \Omega \mid \max_{1 \leq m \leq M} \|U^m\|_\alpha^2 \leq 1/\varepsilon\}$.*

Indeed, by virtue of Lemmas 4.1-(ii), 4.5 and the Markov inequality, there holds $\mathbb{P}(\Omega_{k,h}^\varepsilon) \geq 1 - c\varepsilon$, for some $c > 0$ depending only on $\text{Tr}(Q)$ and C_T . We recall that ε does not depend on k and h . For the term I, we use inequalities (2.14), (2.5) and some elementary calculations: $|I| \leq \frac{2C_D k}{h} t_\ell^{1/2} \sum_{m=1}^M \|\nabla U^m\|_{\mathbb{L}^2} \left(k \sum_{i=1}^M \|V^i\|_{\mathbb{L}^2}^2 \right)^{1/2} < \frac{2\sqrt{3}C_D L \nu}{\varepsilon} t_\ell^{1/2}$. On the other hand, from Proposition 2.1-(iii), the Poincaré and Cauchy-Schwarz inequalities, estimate (2.5) and the inverse inequality (2.14),

$$\begin{aligned} |II| &\leq \frac{2CC_D k}{h} t_\ell^{1/2} \sum_{m=1}^M \|\nabla U^m\|_{\mathbb{L}^2} \left(k \sum_{i=2}^M \|V^{i-1}\|_{\mathbb{L}^2}^2 \|\nabla U^i\|_{\mathbb{L}^2}^2 \right)^{1/2} \\ &\leq 2CC_D \frac{1}{\sqrt{\varepsilon}} t_\ell^{1/2} \max_{1 \leq i \leq M} \alpha \|\nabla U^i\|_{\mathbb{L}^2} \sqrt{k} \sum_{m=1}^M \|\nabla U^m\|_{\mathbb{L}^2} \leq \frac{2\sqrt{3}CC_D}{\varepsilon \sqrt{\varepsilon}} t_\ell^{1/2}. \end{aligned}$$

We summarize, in the upcoming proposition, all the convergence results emerging from the condition $\alpha \leq Ch$. We give afterwards, in Proposition 5.2, convergence outcomes concerning the case $\sqrt{k}/h < L \leq \alpha$.

Proposition 5.1

Assume $0 < \alpha \leq Ch$ for some $C > 0$ independent of α , k and h . There exist $u, v \in L^2(\Omega; L^2(0, T; \mathbb{H}_0^1))$ such that the convergences below occur, up to extractions, as $k, h \rightarrow 0$:

$$\begin{aligned} &\mathcal{U}_{k,h}^+ \rightharpoonup u \quad \& \quad \mathcal{V}_{k,h}^+ \rightharpoonup v \quad \text{in } L^2(\Omega; L^2(0, T; \mathbb{H}_0^1)), \\ &\mathcal{U}_{k,h}^+ \rightarrow u \quad \& \quad \mathcal{V}_{k,h}^+ \rightarrow v \quad \text{in } L^2(\Omega; L^2(0, T; \mathbb{L}^2)), \\ &\mathcal{V}_{k,h} \rightarrow v \quad \text{in } L^2(\Omega; L^2(0, T; \mathbb{L}^2)), \\ &\mathcal{V}_{k,h}^- \rightarrow v \quad \text{in } L^2(\Omega; L^2(0, T; \mathbb{L}^2)). \end{aligned} \tag{5.8}$$

Proof: We point out that all subsequences in this proof will be denoted as their original sequences for the sake of clarity. By virtue of Lemmas 4.1-(ii) and 4.4, $\{\mathcal{U}_{k,h}^+\}_{k,h}$ and $\{\mathcal{V}_{k,h}^+\}_{k,h}$ are bounded in the Hilbert space $L^2(\Omega; L^2(0, T; \mathbb{H}_0^1))$. This implies the existence of two subsequences of $\{\mathcal{U}_{k,h}^+\}_{k,h}$ and $\{\mathcal{V}_{k,h}^+\}_{k,h}$ such that $\mathcal{V}_{k,h}^+ \rightharpoonup v$ and $\mathcal{U}_{k,h}^+ \rightharpoonup u$ in $L^2(\Omega; L^2(0, T; \mathbb{H}_0^1))$ for some functions v and u belonging to the same space of convergence. To justify (5.8)₂, we need to apply Lemma 2.2. Indeed, we have $\mathbb{H}_0^1 \hookrightarrow \mathbb{L}^2$ compactly, $\{\mathbb{1}_{\Omega_{k,h}^\varepsilon} \mathcal{U}_{k,h}^+\}_{k,h}$ is \mathbb{P} -a.s. bounded in $L^2(0, T; \mathbb{H}_0^1)$, and for all $\ell \in \{1, \dots, M-1\}$, there holds \mathbb{P} -a.s.

$$\left\| \tau_{t_\ell} \mathbb{1}_{\Omega_{k,h}^\varepsilon} \mathcal{U}_{k,h}^+ - \mathbb{1}_{\Omega_{k,h}^\varepsilon} \mathcal{U}_{k,h}^+ \right\|_{L^2(0, T-t_\ell; \mathbb{L}^2)}^2 = \mathbb{1}_{\Omega_{k,h}^\varepsilon} k \sum_{m=1}^{M-\ell} \|U^{m+\ell} - U^m\|_{\mathbb{L}^2}^2 \leq C t_\ell^{\frac{1}{2}} \xrightarrow{t_\ell \rightarrow 0} 0,$$

uniformly in $\mathcal{U}_{k,h}^+$, thanks to Lemma 5.1. As a result, $\{\mathbb{1}_{\Omega_{k,h}^\varepsilon} \mathcal{U}_{k,h}^+\}_{k,h}$ is \mathbb{P} -a.s. relatively compact in $L^2(0, T; \mathbb{L}^2)$. In other words, $\{\|\mathbb{1}_{\Omega_{k,h}^\varepsilon} \mathcal{U}_{k,h}^+\|_{L^2(0, T; \mathbb{L}^2)}^2\}_{k,h}$ is \mathbb{P} -a.s. convergent. Moreover, $\|\mathbb{1}_{\Omega_{k,h}^\varepsilon} \mathcal{U}_{k,h}^+\|_{L^2(0, T; \mathbb{L}^2)}^2 \leq 1/\varepsilon$, \mathbb{P} -a.s. where the function $\omega \mapsto \frac{1}{\varepsilon} \in L^1(\Omega)$. Therefore, by the dominated convergence theorem, $\{\|\mathbb{1}_{\Omega_{k,h}^\varepsilon} \mathcal{U}_{k,h}^+\|_{L^2(\Omega; L^2(0, T; \mathbb{L}^2))}\}_{k,h}$ is convergent. We need to prove in addition to the latter that $\{\mathbb{1}_{\Omega_{k,h}^\varepsilon} \mathcal{U}_{k,h}^+\}_{k,h}$ is weakly convergent in $L^2(\Omega; L^2(0, T; \mathbb{L}^2))$ to achieve strong convergence. We have $\mathbb{E} \left[\int_0^T \|\mathbb{1}_{(\Omega_{k,h}^\varepsilon)^c} \mathcal{U}_{k,h}^+\|_{\mathbb{L}^2}^2 dt \right] = |D|T\mathbb{P}((\Omega_{k,h}^\varepsilon)^c) \lesssim |D|T\varepsilon$, for all $\varepsilon > 0$, thanks to estimate (5.6). By (5.8)₁, one gets $\mathcal{U}_{k,h}^+ \rightharpoonup u$ in $L^2(\Omega; L^2(0, T; \mathbb{L}^2))$. Thus, $\mathbb{E} \left[\int_0^T (\mathbb{1}_{\Omega_{k,h}^\varepsilon} \mathcal{U}_{k,h}^+, \varphi) dt \right] = -\mathbb{E} \left[\int_0^T (\mathcal{U}_{k,h}^+, \mathbb{1}_{(\Omega_{k,h}^\varepsilon)^c} \varphi) dt \right] + \mathbb{E} \left[\int_0^T (\mathcal{U}_{k,h}^+, \varphi) dt \right]$, for all $\varphi \in L^2(\Omega; L^2(0, T; \mathbb{L}^2))$, which converges toward the term $\mathbb{E} \left[\int_0^T (u(t), \varphi) dt \right]$. Note that the test function φ can be considered in a dense set of $L^2(\Omega; L^2(0, T; \mathbb{L}^2))$ since $\{\mathbb{1}_{\Omega_{k,h}^\varepsilon} \mathcal{U}_{k,h}^+\}_{k,h}$ is bounded in the latter space. Subsequently, $\mathbb{1}_{\Omega_{k,h}^\varepsilon} \mathcal{U}_{k,h}^+ \rightarrow u$ in $L^2(\Omega; L^2(0, T; \mathbb{L}^2))$. As a result,

$$\mathbb{E} \left[\int_0^T \|\mathcal{U}_{k,h}^+ - u\|_{\mathbb{L}^2}^2 dt \right] \lesssim \mathbb{E} \left[\int_0^T \left(\|\mathbb{1}_{(\Omega_{k,h}^\varepsilon)^c} \mathcal{U}_{k,h}^+\|_{\mathbb{L}^2}^2 + \|\mathbb{1}_{\Omega_{k,h}^\varepsilon} \mathcal{U}_{k,h}^+ - u\|_{\mathbb{L}^2}^2 \right) dt \right] \xrightarrow{k, h \rightarrow 0} 0.$$

This convergence takes place due to what we have shown so far in this proof and the boundedness of $\{\mathcal{U}_{k,h}^+\}_{k,h}$ in $L^4(\Omega; L^\infty(0, T; \mathbb{L}^2))$, thanks to Lemma 4.1-(iii). The convergence of $\{\mathcal{V}_{k,h}^+\}_{k,h}$ is done similarly through Lemmas 4.3-(iii) and 4.4. Moving on to (5.8)₃, we have

$$\mathbb{E} \left[\int_0^T \|\mathcal{V}_{k,h} - v\|_{\mathbb{L}^2}^2 dt \right] \lesssim \mathbb{E} \left[\int_0^T \left(\|\mathcal{V}_{k,h} - \mathcal{V}_{k,h}^+\|_{\mathbb{L}^2}^2 + \|\mathcal{V}_{k,h}^+ - v\|_{\mathbb{L}^2}^2 \right) dt \right] = I_1 + I_2.$$

From (5.8)₂, we get $I_2 \xrightarrow{k, h \rightarrow 0} 0$. On the other hand, identity (5.5) yields

$$I_1 = \sum_{m=1}^M \mathbb{E} \left[\|\mathbb{V}^m - \mathbb{V}^{m-1}\|_{\mathbb{L}^2}^2 \right] \int_{t_{m-1}}^{t_m} \frac{(t - t_{m-1})^2}{k^2} dt \leq \frac{k}{3} C_T \xrightarrow{k, h \rightarrow 0} 0,$$

thanks to Lemma 4.4. The proof of (5.8)₄ follows from (5.4) and the proving technique of (5.8)₃. \square

Remark 5.2 *In the three-dimensional stochastic NSEs framework, the obtained convergence results in proposition 5.1 remain up to extractions due to the nonuniqueness of the corresponding solution, conversely to the two-dimensional case whose solution is unique.*

The limiting function u_α in the next proposition does not coincide with u that was found in Proposition 5.1. It is worth mentioning that one can demonstrate the convergence of whole

sequences $\{\mathcal{U}_{k,h}^+\}_{k,h}$, $\{\mathcal{V}_{k,h}^+\}_{k,h}$ and $\{\mathcal{V}_{k,h}^-\}_{k,h}$ once we verify that the limiting functions satisfy equation (2.8), \mathbb{P} -a.s. and for $t \in [0, T]$. Such an idea is true due to the solution's uniqueness of equations (1.1) (see for instance [9, Theorem 4.4]).

Proposition 5.2

Assume $0 < \sqrt{k}/h < L \leq \alpha$, for some $L \in (0, 1)$ independent of k and h . Then, there exist two functions $u_\alpha \in L^2(\Omega; L^2(0, T; \mathbb{H}^1_0))$ and $v_\alpha \in L^2(\Omega; L^2(0, T; \mathbb{L}^2))$ such that, up to extractions and as $k, h \rightarrow 0$, one gets

$$\begin{aligned} \mathcal{U}_{k,h}^+ &\rightarrow u_\alpha \text{ in } L^2(\Omega; L^2(0, T; \mathbb{H}^1_0)), \\ \mathcal{V}_{k,h}^+ &\rightarrow v_\alpha \text{ in } L^2(\Omega; L^2(0, T; \mathbb{L}^2)), \\ \mathcal{V}_{k,h}^- &\rightarrow v_\alpha \text{ in } L^2(\Omega; L^2(0, T; \mathbb{L}^2)). \end{aligned} \tag{5.9}$$

and $v_\alpha = u_\alpha - \alpha^2 \Delta u_\alpha$, \mathbb{P} -almost surely and almost everywhere in $(0, T) \times D$.

Proof: Once again, all subsequences in this proof will be denoted as their original sequences. We define $M_h^\alpha := \{u \in \mathbb{V}_h \mid \|u\|_{h,\alpha} < +\infty\}$, where $\|\cdot\|_{h,\alpha}^2 := \|\nabla \cdot\|_{\mathbb{L}^2}^2 + \alpha^2 \|\Delta^h \cdot\|_{\mathbb{L}^2}^2$. Obviously, M_h^α is a subspace of \mathbb{H}^1_0 , and $(M_h^\alpha, \|\cdot\|_{h,\alpha})$ forms a normed space. Note that Lemma 2.2 is no longer applicable since M_h^α depends on h . However, to come out with a strong convergence in \mathbb{H}^1_0 , we shall apply Lemma 2.3 within the sample subset $\Omega_{k,h}^\varepsilon$ that was exclusively modified for the case $\sqrt{k}/h < L$ (see Remark 5.1). We begin by proving the relative compactness of M_h^α in \mathbb{H}^1_0 : let $(\varphi_h)_h$ be a bounded sequence in $(M_h^\alpha, \|\cdot\|_{h,\alpha})$. Therefore, $(\varphi_h)_h$ (resp. $(\Delta^h \varphi_h)_h$) converges weakly as $h \rightarrow 0$, in \mathbb{H}^1_0 (resp. \mathbb{L}^2) toward a function φ (resp. z), up to an extraction. Let $\psi \in [C_c^\infty(D)]^d$. By identities 2.11 and 2.12, $(\Delta^h \varphi_h, \psi) = (\Delta^h \varphi_h, \Pi_h \psi) = -(\nabla \varphi_h, \nabla \Pi_h \psi) \rightarrow -(\nabla \varphi, \nabla \psi) = (\Delta \varphi, \psi)$, thanks to estimate (2.13). Therefore, $z = \Delta \varphi$ a.e. in D . Note that $\varphi \in \mathbb{H}^2 \cap \mathbb{H}^1_0$. The strong convergence of $(\varphi_h)_h$ in \mathbb{H}^1_0 follows from its weak convergence along with the property: $\|\nabla \varphi_h\|_{\mathbb{L}^2}^2 = -(\Delta^h \varphi_h, \varphi_h) \rightarrow -(\Delta \varphi, \varphi) = \|\nabla \varphi\|_{\mathbb{L}^2}^2$, where we used identity 2.12 and the weak and strong (which arises from the compact embedding $\mathbb{H}^1_0 \hookrightarrow \mathbb{L}^2$) convergences of $(\Delta^h \varphi_h)_h$ and $(\varphi_h)_h$ in \mathbb{L}^2 , respectively. On the other hand, $\{\mathbb{1}_{\Omega_{k,h}^\varepsilon} \mathcal{U}_{k,h}^+\}_{k,h}$ is \mathbb{P} -a.s. bounded in $L^2(0, T; M_h^\alpha)$ and $L^2(0, T; \mathbb{H}^1_0)$, thanks to the definition of $\Omega_{k,h}^\varepsilon$. Moreover, by Lemma 5.1, $\mathbb{1}_{\Omega_{k,h}^\varepsilon} \|\tau_{t_\ell} \mathcal{U}_{k,h}^+ - \mathcal{U}_{k,h}^+\|_{L^2(0, T-t_\ell; \mathbb{H}^1_0)} \rightarrow 0$, as $t_\ell \rightarrow 0$, uniformly in k and h . The latter convergence holds in \mathbb{H}^1_0 due to the norm equivalence (2.3). Subsequently, all conditions of Lemma 2.3 are met, we infer that $\{\mathbb{1}_{\Omega_{k,h}^\varepsilon} \mathcal{U}_{k,h}^+\}_{k,h}$ is \mathbb{P} -a.s. relatively compact in $L^2(0, T; \mathbb{H}^1_0)$. Hence, convergence (5.9)₁ can be justified as in the proof of Proposition 5.1, where we recall that $\|\cdot\|_\alpha$ is equivalent to $\|\cdot\|_{\mathbb{H}^1_0}$ because α does not depend on k and h . Let u_α denote the limit of $\{\mathcal{U}_{k,h}^+\}_{k,h}$ in $L^2(\Omega; L^2(0, T; \mathbb{H}^1_0))$. By Lemma 4.1-(ii), $\{\Delta^h \mathcal{U}_{k,h}^+\}_{k,h}$ converges weakly, up to a subsequence, to a function z in $L^2(\Omega; L^2(0, T; \mathbb{L}^2))$. As done earlier in this proof, one can prove that $z = \Delta u_\alpha$. Thus, $\mathbb{E} \left[\int_0^T \|\Delta u_\alpha(t)\|_{\mathbb{L}^2}^2 dt \right] \leq \liminf \mathbb{E} \left[\int_0^T \|\Delta^h \mathcal{U}_{k,h}^+\|_{\mathbb{L}^2}^2 dt \right] \leq \alpha^{-2} C_T$, which implies $u_\alpha \in L^2(\Omega; L^2(0, T; \mathbb{H}^2 \cap \mathbb{H}^1_0))$, thanks to the domain's properties. Moving on to convergence (5.9)₂, by Lemma 4.5, $\{\mathcal{V}_{k,h}^+\}_{k,h}$ converges weakly, up to a subsequence, to a function v_α , in $L^2(\Omega; L^2(0, T; \mathbb{L}^2))$. Additionally, by Lemma 2.1-(i), we have \mathbb{P} -a.s. and a.e. in $(0, T) \times D$ that $\mathcal{V}_{k,h}^+ = \mathcal{U}_{k,h}^+ - \alpha^2 \Delta^h \mathcal{U}_{k,h}^+$. By relying on what we have proven so far in this proof, one gets the identification $v_\alpha = u_\alpha - \alpha^2 \Delta u_\alpha$, \mathbb{P} -a.s. and a.e. in $(0, T) \times D$. For the last convergence, we need the following estimate

$$\|\Delta^h \varphi_h\|_{\mathbb{L}^2} \lesssim h^{-1} \|\nabla \varphi_h\|_{\mathbb{L}^2}, \quad \forall \varphi_h \in \mathbb{V}_h. \tag{5.10}$$

This can be illustrated through identity (2.12), the inverse estimate (2.14) and the Cauchy-Schwarz inequality. Since $\{\mathcal{V}_{k,h}^-\}_{k,h}$ is bounded in the Hilbert space $L^2(\Omega; L^2(0, T; \mathbb{L}^2))$, we let $\varphi \in [C_c^\infty(D)]^d$. Subsequently,

$$\mathbb{E} \left[\int_0^T (\mathcal{V}_{k,h}^-, \varphi) dt \right] = \mathbb{E} \left[\int_0^T (\mathcal{V}_{k,h}^- - \mathcal{V}_{k,h}^+, \varphi) dt \right] + \mathbb{E} \left[\int_0^T (\mathcal{V}_{k,h}^+, \varphi) dt \right] := \zeta_1 + \zeta_2.$$

Using (5.9)₂, we get the convergence of ζ_2 . It remains to show that ζ_1 vanishes on the limits. Indeed, by Lemmas 2.1-(i), 4.1-(ii), the Cauchy-Schwarz inequality and estimate (5.10), we get $\zeta_1 \leq \|\varphi\|_{\mathbb{L}^2} \mathbb{E} \left[k \sum_{m=1}^M (\|U^m - U^{m-1}\|_{\mathbb{L}^2} + \alpha^2 \|\Delta^h(U^m - U^{m-1})\|_{\mathbb{L}^2}) \right] \lesssim \|\varphi\|_{\mathbb{L}^2} C_T k + \|\varphi\|_{\mathbb{L}^2} \alpha h^{-1} k C_T \leq \|\varphi\|_{\mathbb{L}^2} C_T (k + \alpha^2 k^{1/2}) \xrightarrow[k, h \rightarrow 0]{} 0$. \square

Propositions 5.1 and 5.2 give an insight into the limiting functions spaces. Yet, they do not provide the divergence-free property and the relationship between u and v . The following proposition treats this issue.

Proposition 5.3

Let u, v and u_α be the limiting functions provided in Propositions 5.1 and 5.2. Then, u and u_α are divergence-free, and $v = u$ almost everywhere in $(0, T) \times D$ and \mathbb{P} -almost surely.

Proof: To prove that u and u_α are divergence-free, we show that $\{\text{div } \mathcal{U}_{k,h}^+\}_{k,h}$ converges weakly in $L^2(\Omega; L^2(0, T; \mathbb{L}^2))$ toward 0, thanks to (5.8)₁ and (5.9)₁. To this end, we evoke the Lagrange interpolation $I : C^\ell(D) \rightarrow L_h$, $\ell \in \mathbb{N}$ (c.f. [5, Theorem 4.4.4]). For $\ell \in \mathbb{N}$, let $z \in C^\ell(D)$, then

$$\begin{aligned} \mathbb{E} \left[\int_0^T (\text{div } \mathcal{U}_{k,h}^+, z) dt \right] &= \mathbb{E} \left[\int_0^T (\text{div } \mathcal{U}_{k,h}^+, z - Iz) dt \right] + \mathbb{E} \left[\int_0^T (\text{div } \mathcal{U}_{k,h}^+, Iz) dt \right] \\ &\lesssim \mathbb{E} \left[\int_0^T \|\nabla \mathcal{U}_{k,h}^+\|_{L^2} dt \right] \|z - Iz\|_{L^2} \xrightarrow[k, h \rightarrow 0]{} 0, \end{aligned}$$

where the second term in the first equality vanishes because $\{\mathcal{U}_{k,h}^+\}_{k,h}$ is weakly divergence-free. Consider $\varphi \in \mathbb{L}^2$. By Lemma 4.1-(ii), the embedding $L^2 \hookrightarrow L^1$, and $\alpha \leq Ch$, one gets $\left| \mathbb{E} \left[\int_0^T (\alpha^2 \Delta^h \mathcal{U}_{k,h}^+, \varphi) dt \right] \right| \lesssim C_T Ch \|\varphi\|_{\mathbb{L}^2} \rightarrow 0$. Therefore, $\alpha^2 \Delta^h \mathcal{U}_{k,h}^+ \rightarrow 0$ in $L^2(\Omega; L^2(0, T; \mathbb{L}^2))$ if $\alpha \leq Ch$. By Lemma 2.1-(i) and Proposition 5.1, we infer that $u = v$, \mathbb{P} -almost surely and almost everywhere in $(0, T) \times D$. \square

In the below proposition, we identify the limits of deterministic and stochastic integrals of Algorithm 1.

Proposition 5.4

Let $t \in [0, T]$. For all $\varphi \in \mathbb{H}^2 \cap \mathbb{V}$,

1. if $0 < \alpha \leq Ch$, for some $C > 0$ independent of α, h and k , then

$$\begin{aligned} (i) \quad \lim_{k, h \rightarrow 0} \mathbb{E} \left[\int_0^t (\nabla \mathcal{V}_{k,h}^+, \nabla \Pi_h \varphi) ds \right] &= \mathbb{E} \left[\int_0^t (\nabla v(s), \nabla \varphi) ds \right], \\ (ii) \quad \lim_{k, h \rightarrow 0} \mathbb{E} \left[\int_0^t \tilde{b}(\mathcal{U}_{k,h}^+, \mathcal{V}_{k,h}^-, \Pi_h \varphi) ds \right] &= \mathbb{E} \left[\int_0^t ([v(s) \cdot \nabla] v(s), \varphi) ds \right], \end{aligned}$$

$$(iii) \lim_{k,h \rightarrow 0} \mathbb{E} \left[\int_0^t \langle f^-(s, \mathcal{U}_{k,h}^-, \Pi_h \varphi) \rangle ds \right] = \mathbb{E} \left[\int_0^t \langle f(s, v(s), \varphi) \rangle ds \right],$$

$$(iv) \lim_{k,h \rightarrow 0} \left(\int_0^t g^-(s, \mathcal{U}_{k,h}^-) dW(s), \Pi_h \varphi \right) = \left(\int_0^t g(s, v(s)) dW(s), \varphi \right), \mathbb{P}\text{-a.s.},$$

2. if $\frac{\sqrt{k}}{h} < L \leq \alpha$, for some $L > 0$ independent of α, h and k , then

$$(i) \lim_{k,h \rightarrow 0} \mathbb{E} \left[\int_0^t \tilde{b} \left(\mathcal{U}_{k,h}^+, \mathcal{V}_{k,h}^-, \Pi_h \varphi \right) ds \right] = \mathbb{E} \left[\int_0^t \tilde{b}(u_\alpha(s), v_\alpha(s), \varphi) ds \right],$$

(ii) Assertions 1.(iii)-(iv) remain valid, provided that v is replaced by u_α .

Proof: Fix φ in $\mathbb{H}^2 \cap \mathbb{V}$. Starting out with assertion 1, we have $\{\mathcal{V}_{k,h}^+\}_{k,h}$ (resp. $\{\Pi_h \varphi\}_h$) is weakly (resp. strongly) convergent in $L^2(\Omega; L^2(0, T; \mathbb{H}_0^1))$ (resp. \mathbb{H}_0^1), thanks to Proposition 5.1 and estimate (2.13). Therefore, 1.(i) follows. Moreover,

$$\tilde{b}(\mathcal{U}_{k,h}^+, \mathcal{V}_{k,h}^-, \Pi_h \varphi) = \left([\mathcal{U}_{k,h}^+ \cdot \nabla] \mathcal{V}_{k,h}^-, \Pi_h \varphi \right) + \left((\nabla \mathcal{U}_{k,h}^+)^T \cdot \mathcal{V}_{k,h}^-, \Pi_h \varphi \right) =: J_1 + J_2.$$

We have $J_1 = - \left([\mathcal{U}_{k,h}^+ \cdot \nabla] \Pi_h \varphi, \mathcal{V}_{k,h}^- \right) - \left([div \mathcal{U}_{k,h}^+] \mathcal{V}_{k,h}^-, \Pi_h \varphi \right) =: -J_1^1 - J_1^2$ by integrating by parts. Therefore, by Proposition 2.1-(iii) and estimate (2.13),

$$\left| \mathbb{E} \left[\int_0^t \left(-J_1^1 + ([u \cdot \nabla] \varphi, v) \right) ds \right] \right| \leq C_D C h \|\varphi\|_{\mathbb{H}^2} \mathbb{E} \left[\int_0^t \|\nabla \mathcal{U}_{k,h}^+\|_{\mathbb{L}^2} \|\nabla \mathcal{V}_{k,h}^-\|_{\mathbb{L}^2} ds \right]$$

$$+ \left| \mathbb{E} \left[\int_0^t \left\{ - \left([\mathcal{U}_{k,h}^+ \cdot \nabla] \varphi, \mathcal{V}_{k,h}^- \right) + ([u \cdot \nabla] \varphi, v) \right\} ds \right] \right|,$$

The first term on the right hand side tends to 0 after applying the Cauchy-Schwarz inequality, Lemmas 4.1 and 4.4. Similarly, the second term goes to 0 by virtue of Proposition 5.1. Moreover, $J_1^2 = \left([div \mathcal{U}_{k,h}^+] \mathcal{V}_{k,h}^-, \Pi_h \varphi - \varphi \right) + \left(\mathcal{V}_{k,h}^-, [div \mathcal{U}_{k,h}^+] \varphi \right)$. Thus, $\mathbb{E} \left[\int_0^t |J_1^2| ds \right]$ goes to 0 after applying Hölder's inequality, embedding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^4$, Poincaré's inequality, Lemmas 4.1, 4.4, estimate (2.13), convergence (5.8)₄ and Proposition 5.3. On the other hand, by virtue of Proposition 2.1-(iv), $J_2 = - \int_D [\Pi_h \varphi \cdot \nabla] \mathcal{V}_{k,h}^- \mathcal{U}_{k,h}^+ dx$. Thus,

$$\left| \mathbb{E} \left[\int_0^t J_2 ds \right] \right| \leq C_D C h \|\varphi\|_{\mathbb{H}^2} \mathbb{E} \left[\int_0^t \|\nabla \mathcal{V}_{k,h}^-\|_{\mathbb{L}^2} \|\nabla \mathcal{U}_{k,h}^+\|_{\mathbb{L}^2} ds \right]$$

$$+ \left| \mathbb{E} \left[\int_0^t ([\varphi \cdot \nabla] \mathcal{V}_{k,h}^-, \mathcal{U}_{k,h}^+) ds \right] \right|.$$

The first term on the right converges toward 0, thanks to Lemmas 4.1 and 4.4. By Proposition 5.1, the second term goes to $\left| \mathbb{E} \left[\int_0^t ([\varphi \cdot \nabla] v, u) ds \right] \right|$ which vanishes because $u = v$, \mathbb{P} -a.s. and a.e. in $(0, T) \times D$ (see Proposition 5.3) and φ is divergence-free. Subsequently, the trilinear term \tilde{b} tends to $-\mathbb{E} \left[\int_0^t ([u \cdot \nabla] \varphi, v) ds \right]$, which by virtue of Proposition 5.3 and an integration by parts, lead to the sought term. For assertion 1.(iii), we make use of assumption (S_2):

$$\left| \langle f^-(s, \mathcal{U}_{k,h}^-), \Pi_h \varphi \rangle - \langle f(s, u(s)), \varphi \rangle \right| \leq (K_3 + K_4 \|\mathcal{U}_{k,h}^-\|_\alpha) \|\Pi_h \varphi - \varphi\|_{\mathbb{H}^1}$$

$$+ \|f(s, u(s)) - f^-(s, u(s))\|_{\mathbb{H}^{-1}} \|\varphi\|_{\mathbb{H}^1} + L_f \|u(s) - \mathcal{U}_{k,h}^-\|_\alpha \|\varphi\|_{\mathbb{H}^1} =: \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3.$$

We have $\mathbb{E} \left[\int_0^t \mathcal{L}_1 ds \right] \leq Ch \|\varphi\|_{\mathbb{H}^2} (TK_3 + K_4 C_T) \rightarrow 0$, thanks to estimate (2.13) and Lemma 4.1. Besides, $\mathbb{E} \left[\int_0^t \mathcal{L}_2 ds \right] \rightarrow 0$, thanks to the continuity of f with respect to t . Lastly, we decompose the norm $\|\cdot\|_\alpha$ after using the embedding $\mathbb{L}^2 \hookrightarrow \mathbb{L}^1$:

$$\mathbb{E} \left[\int_0^t \mathcal{L}_3 dt \right] \lesssim L_f \|\varphi\|_{\mathbb{H}^1} \mathbb{E} \left[\int_0^t (\|u(s) - \mathcal{U}_{k,h}^-\|_{\mathbb{L}^2}^2 + \alpha^2 \|\nabla(u(s) - \mathcal{U}_{k,h}^-\|_{\mathbb{L}^2})\|_{\mathbb{L}^2}^2) ds \right]^{1/2}.$$

Convergence of $\mathbb{E} \left[\int_0^t \|u(s) - \mathcal{U}_{k,h}^-\|_{\mathbb{L}^2}^2 ds \right]$ is handled by adding and subtracting $\mathcal{U}_{k,h}^+$, then by employing the triangular inequality along with convergence (5.8)₂ and Lemma 4.1-(ii). Furthermore, $\mathbb{E} \left[\int_0^t \alpha^2 \|\nabla(u(s) - \mathcal{U}_{k,h}^-\|_{\mathbb{L}^2})\|_{\mathbb{L}^2}^2 ds \right]^{1/2} \lesssim C_T Ch$, due to Lemma 4.1-(ii), convergence (5.8)₁ and $\alpha \leq Ch$. We now justify assertion 1.(iv). Let us denote by

$$\begin{aligned} j &:= \left(\int_0^t g^-(s, \mathcal{U}_{k,h}^-) dW(s), \Pi_h \varphi \right) - \left(\int_0^t g(s, u(s)) dW(s), \varphi \right) \\ &= \left(\int_0^t g^-(s, \mathcal{U}_{k,h}^-) dW(s), \Pi_h \varphi - \varphi \right) + \left(\int_0^t (g^-(s, \mathcal{U}_{k,h}^-) - g^-(s, u(s))) dW(s), \varphi \right) \\ &\quad + \left(\int_0^t (g^-(s, u(s)) - g(s, u(s))) dW(s), \varphi \right) =: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3. \end{aligned}$$

After squaring both sides and applying the mathematical expectation, we shall bound each term separately. To this end, assumption (S₂), Itô's isometry, the Cauchy-Schwarz inequality, estimate (2.13), and Lemma 4.1 are all needed in the calculations below.

$$\begin{aligned} \mathbb{E} \left[\int_0^t \mathcal{J}_1^2 ds \right] &\lesssim \text{Tr}(\mathcal{Q}) \|\Pi_h \varphi - \varphi\|_{\mathbb{L}^2}^2 \mathbb{E} \left[\int_0^t (K_1 + K_2 \|\mathcal{U}_{k,h}^-\|_\alpha)^2 ds \right] \lesssim h^4 C_T \rightarrow 0, \\ \mathbb{E} \left[\int_0^t \mathcal{J}_2^2 ds \right] &\lesssim \text{Tr}(\mathcal{Q}) \|\varphi\|_{\mathbb{L}^2}^2 L_g^2 \mathbb{E} \left[\int_0^t \|\mathcal{U}_{k,h}^- - u(s)\|_\alpha^2 ds \right] \rightarrow 0 \quad (\text{similar to } \mathcal{L}_3) \text{ and,} \\ \mathbb{E} \left[\int_0^t \mathcal{J}_3^2 ds \right] &\lesssim \text{Tr}(\mathcal{Q}) \|\varphi\|_{\mathbb{L}^2}^2 \mathbb{E} \left[\sum_{m=1}^M \int_{t_{m-1}}^{t_m} \|g(t_{m-1}, u) - g(s, u)\|_{\mathcal{L}_2(K, \mathbb{L}^2)}^2 ds \right] \rightarrow 0, \end{aligned}$$

by the continuity of g with respect to its first variable. Replacing u by v (Proposition 5.3) completes the proof of (iv). Moving on to assertion 2. We have

$$\begin{aligned} \tilde{b}(\mathcal{U}_{k,h}^+, \mathcal{V}_{k,h}^-, \Pi_h \varphi) - \tilde{b}(u_\alpha, v_\alpha, \varphi) &= \tilde{b}(\mathcal{U}_{k,h}^+, \mathcal{V}_{k,h}^-, \Pi_h \varphi - \varphi) + \left(\tilde{b}(\mathcal{U}_{k,h}^+, \mathcal{V}_{k,h}^-, \varphi) - \tilde{b}(u_\alpha, v_\alpha, \varphi) \right) \\ &=: I_1 + I_2. \end{aligned}$$

By Proposition 2.1-(iii), estimate (2.13), the inverse (2.14) and sum (2.5) inequalities, Young's inequality, Lemmas 4.1, 4.5 and $\sqrt{k}/h < \alpha$, one gets

$$\begin{aligned} \mathbb{E} \left[\int_0^t |I_1| ds \right] &\leq \mathbb{E} \left[k \sum_{m=1}^M |I_1| \right] \lesssim h \|\varphi\|_{\mathbb{H}^2} \mathbb{E} \left[\alpha k^{1/2} \max_{1 \leq m \leq M} \|\nabla U^m\|_{\mathbb{L}^2} \sum_{m=1}^M \|V^{m-1}\|_{\mathbb{L}^2} \right] \\ &\lesssim h \|\varphi\|_{\mathbb{H}^2} \mathbb{E} \left[\max_{1 \leq m \leq M} \|U^m\|_\alpha^2 + 3k \sum_{m=1}^M \|V^{m-1}\|_{\mathbb{L}^2}^2 \right] \leq (C_T + C'_T) h \|\varphi\|_{\mathbb{H}^2} \xrightarrow[k, h \rightarrow 0]{} 0. \end{aligned}$$

On the other hand, by employing Proposition 2.1-(iv) and integrating by parts twice, there holds $\tilde{b}(\mathcal{U}_{k,h}^+, \mathcal{V}_{k,h}^-, \varphi) = \left([\varphi \cdot \nabla] \mathcal{U}_{k,h}^+, \mathcal{V}_{k,h}^- \right) - \left([\mathcal{U}_{k,h}^+ \cdot \nabla] \varphi, \mathcal{V}_{k,h}^- \right)$. Proposition 5.2 ensures the convergence $\left| \mathbb{E} \left[\int_0^t I_2 ds \right] \right| \xrightarrow[k, h \rightarrow 0]{} 0$, while taking into account that $([\varphi \cdot \nabla] u_\alpha, v_\alpha) - ([u_\alpha \cdot \nabla] \varphi, v_\alpha)$

coincides \mathbb{P} -a.s. with $\tilde{b}(u_\alpha, v_\alpha, \varphi)$ after integrating it twice by parts and using the null divergence of φ and u_α (see Proposition 5.3). We point out that the low regularity of v_α (in \mathbb{L}^2) does not prevent $\tilde{b}(u_\alpha, v_\alpha, \varphi)$ from being well-defined, because u_α and φ have high regularities. The proof of convergence for f and g is similar to that of assertion 1. One may only need to replace the employed estimate $\alpha \leq Ch$ (when decomposing the norm $\|\cdot\|_\alpha$ in the above steps) by the strong convergence of $\{u_{k,h}^+\}_{k,h}$ in $L^2(\Omega; L^2(0, T; \mathbb{H}_0^1))$, thanks to Proposition 5.2. \square

6 Conclusion

We devote this section to results and conclusions emerging from what we have carried out so far.

6.0.1 Convergence of LANS- α to NS in 2D

Assume $d = 2$, and $V^0 \rightarrow v_0$ in $L^2(\Omega; \mathbb{L}^2)$ as $h \rightarrow 0$. From Propositions 5.3 and 5.4-(1), we infer that v satisfies equation (2.9), \mathbb{P} -a.s., for all $t \in [0, T]$. Moreover, by a standard technique (e.g. [27]), it is easy to check that $v \in L^2(\Omega; C([0, T]; \mathbb{H}))$. We also have, by Proposition 5.1-(1), that $v \in M_{\mathcal{F}_t}^2(0, T; \mathbb{V})$.

6.0.2 Convergence to the LANS- α

Assume $d \in \{2, 3\}$, and $U^0 \rightarrow \bar{u}_0$ in $L^2(\Omega; \mathbb{H}^1)$ as $h \rightarrow 0$. According to Propositions 5.2 and 5.4-(2), one may notice that we still need to illustrate the convergence of $\mathbb{E} \left[\int_0^T \left(\nabla \mathcal{V}_{k,h}^+, \nabla \varphi_h \right) dt \right]$, $\varphi_h \in \mathbb{V}_h$ toward its continuous counterpart. To this end, we define, for $z \in \mathbb{H}_0^1$, the elliptic projection $E_h: \mathbb{H}_0^1 \rightarrow \mathbb{V}_h$ as the unique solution of

$$(\nabla E_h z, \nabla \varphi_h) = (\nabla z, \nabla \varphi_h), \quad \forall \varphi_h \in \mathbb{V}_h.$$

Operator E_h satisfies $\Delta^h E_h z = \Pi_h \Delta z$ for all $z \in \mathbb{H}^2 \cap \mathbb{V}$ (e.g. [21, Page 593]). Therefore, for all $\varphi \in \mathbb{H}^2 \cap \mathbb{V}$, it follows from identities 2.11, 2.12, Proposition 5.2 and the above relation that

$$\left(\nabla \mathcal{V}_{k,h}^+, \nabla E_h \varphi \right) = - \left(\mathcal{V}_{k,h}^+, \Delta^h E_h \varphi \right) = - \left(\mathcal{V}_{k,h}^+, \Pi_h \Delta \varphi \right) = - \left(\mathcal{V}_{k,h}^+, \Delta \varphi \right).$$

Subsequently, $\mathbb{E} \left[\int_0^T \left(\nabla \mathcal{V}_{k,h}^+, \nabla E_h \varphi \right) dt \right]$ converges toward

$$-\mathbb{E} \left[\int_0^T (v_\alpha(t), \Delta \varphi) dt \right] = -\mathbb{E} \left[\int_0^T (v_\alpha(t), A \varphi) dt \right] = \mathbb{E} \left[\int_0^T \left(u_\alpha(t) + \alpha^2 A u_\alpha(t), A \varphi \right) dt \right].$$

Putting it all together yields the sought result. In other words, u_α satisfies equation (2.7), \mathbb{P} -a.s., for all $t \in [0, T]$ and $u_\alpha \in M_{\mathcal{F}_t}^2(0, T; D(A)) \cap L^2(\Omega; L^\infty(0, T; \mathbb{V}))$ together with the fact that u_α is weakly continuous with values in \mathbb{V} .

6.1 Numerical experiments

This part is devoted to giving computational experiments in 2D for the stochastic LANS- α model through Algorithm 1 when the spatial scale α fulfills either $\alpha \leq Ch$ or $\alpha > L \geq \frac{\sqrt{k}}{h}$. Since our primary objective is to compare solutions' behavior of LANS- α to that of Navier-Stokes, we

provide simulation of solutions to the latter equations as well through a linear scheme covered in [6, Algorithm 3]. The implementation hereafter is performed using the open source finite element software FEniCS [25]. We employ the lower order Taylor-Hood (P_2 - P_1) element for the spatial discretization within a mixed finite element framework. The chosen domain is a unit square $D = (0, 1)^2$ along the time interval $[0, T]$ with $T = 1$. The initial condition \bar{u}_0 and viscosity ν are set to 0 and 1, respectively. For the sake of simplicity, the source term f is considered as a deterministic constant and the drift term g plays the identity operator role.

Q-Wiener process approximation For computational purposes, we must deal with a truncated form of the series (2.1). Besides, we consider two independent $H_0^1(D)$ -valued Wiener processes W_1 and W_2 such that $W = (W_1, W_2)$. For $M \in \mathbb{N} \setminus \{0\}$, the utilized increments are expressed by

$$\Delta_m W_\ell \approx k^{1/2} \sum_{i,j=1}^M (\lambda_{i,j}^\ell)^{1/2} \xi_{i,j}^{\ell,m} e_{i,j}, \quad \ell \in \{1, 2\},$$

where for all $i, j \in \mathbb{N}$ and $(x, y) \in D$, the basis elements $e_{i,j} := 2 \sin(i\pi x) \sin(j\pi y)$ represent the Laplace eigenfunctions with Dirichlet boundary conditions on D . For $\ell \in \{1, 2\}$, $\{\{\xi_{i,j}^{\ell,m}\}_{i,j}\}_m$ is a family of independent identically distributed standard normal random variables, $\lambda_{i,j}^\ell := \frac{1}{(i+j)^2}$ for $\ell \in \{1, 2\}$ and $M = 10$.

Case $\alpha \leq Ch$

Consider $\alpha = 10^{-3}h$, $h \approx 0.03$ and $k = 10^{-3}$.

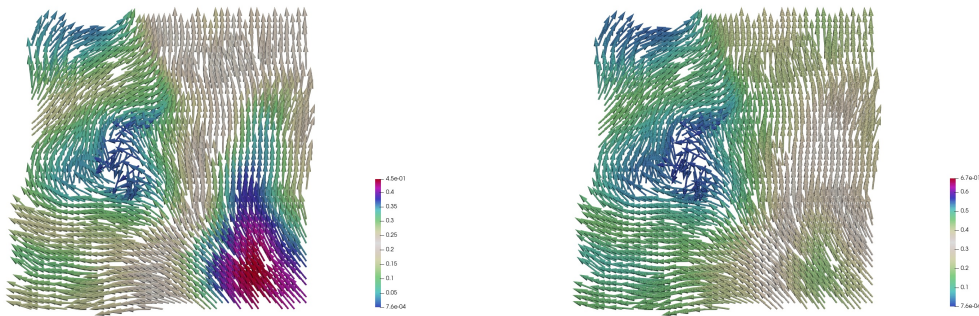
Since this case relates both equations (1.1) and (1.2), we choose two different time values in $[0, T]$, and plot the associated figures side by side. This allows us to compare the solutions' behavior together with the occurring differences. Observe that both LANS- α and NS solutions behave similarly with a slight variation in values. Such a difference was expected since we are dealing here with approximate computations, not to mention the considered space discretization's step h which is not too close to 0, yet its code execution is costly. We also provide the following pressure figures

Case $\alpha \geq L > \sqrt{k}/h$

For this framework, we set $\alpha = 1$, $h \approx 0.03$ and take k in terms of h ; namely $k = 0.9h^2$, so that the condition $\sqrt{k}/h < \alpha$ is met.

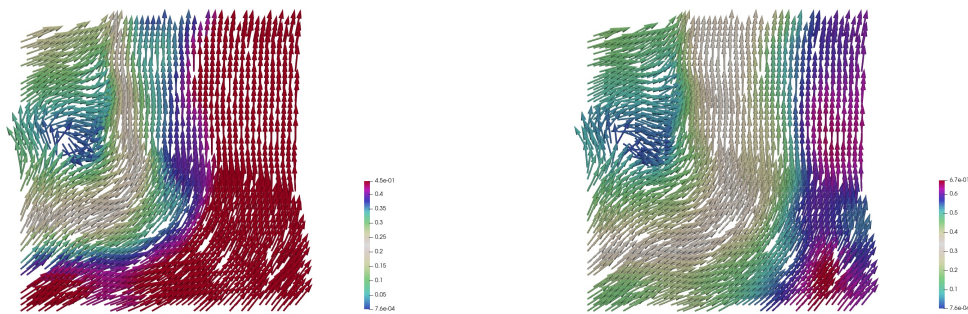
Within one realization, outcomes of the current case are clearly well-behaved, which means a higher regularity in terms of the velocity fields. It is worth mentioning the speed variation stage within the time interval $[0.45 - \varepsilon, 0.65 + \varepsilon]$, $\varepsilon \ll 1$ where the velocity value goes from its lowest to its highest rate. Below $t = 0.45$ and above $t = 0.65$, the velocity field maintains almost a constant value. Here are the associated pressure figures:

We point out that in both cases, the pressure is heavily impacted by the noise. As its curve progresses in time, we notice a random behavior at each time node. This can be thought of as the stochastic pressure decomposition that was evoked in article [4] for the two-dimensional stochastic Navier-Stokes equations, which states that p can be split into a few terms, one of which can be written in terms of the Wiener process W .



(a) Velocity field of LANS- α at time $t = 0.161$

(b) Velocity field of NS at time $t = 0.161$

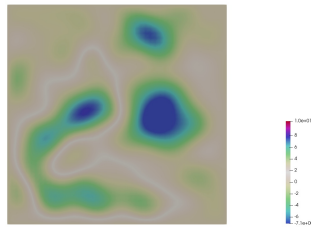
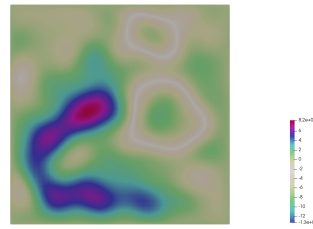
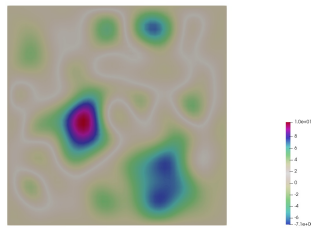
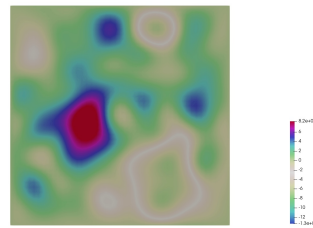
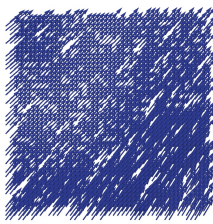
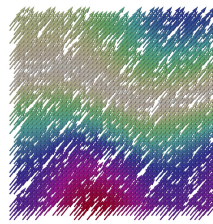
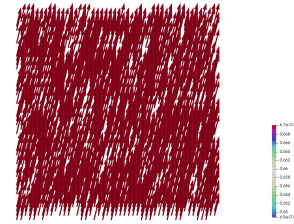


(c) Velocity field of LANS- α at time $t = 0.586$

(d) Velocity field of NS at time $t = 0.586$

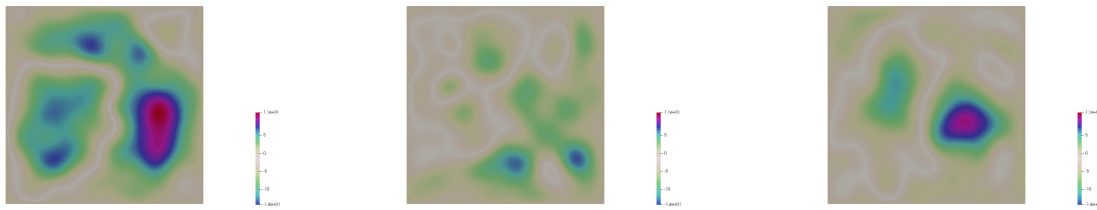
References

- [1] L. Bañas et al. “A convergent finite-element-based discretization of the stochastic Landau-Lifshitz-Gilbert equation”. In: *IMAJNA* 34 (Apr. 2014), pp. 502–549. doi: 10.1093/imanum/drt020.
- [2] Hakima Bessaih and Annie Millet. “Space-time Euler discretization schemes for the stochastic 2D Navier-Stokes equations”. In: *Stochastics and Partial Differential Equations: Analysis and Computations* (2021). doi: 10.1007/s40072-021-00217-7. URL: <https://doi.org/10.1007/s40072-021-00217-7>.
- [3] Clayton Bjorland and Maria E. Schonbek. “On questions of decay and existence for the viscous Camassa–Holm equations”. In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 25.5 (2008), pp. 907–936. ISSN: 0294-1449.
- [4] Dominic Breit and Alan Dodgson. “Convergence rates for the numerical approximation of the 2D stochastic Navier-Stokes equations”. In: *Numerische Mathematik* 147.3 (2021), pp. 553–578. doi: 10.1007/s00211-021-01181-z. URL: <https://doi.org/10.1007/s00211-021-01181-z>.
- [5] Susanne Brenner and Ridgway Scott. *The mathematical theory of finite element methods*. Vol. 15. New York: Springer Sci. & Bus. Media, 2007, p. 400. doi: 10.1007/978-0-387-75934-0.

(a) LANS- α pressure at $t = 0.161$ (b) NS pressure at $t = 0.161$ (c) LANS- α pressure at $t = 0.586$ (d) NS pressure at $t = 0.586$ (a) Time $t = 0.161$ (b) Time $t = 0.586$ (c) Time $t = 1$

Velocity field of LANS- α evaluated at different time values

- [6] Zdzislaw Brzeźniak, Erich Carelli, and Andreas Prohl. “Finite-element-based discretizations of the incompressible Navier–Stokes equations with multiplicative random forcing”. In: *IMAJNA*. 33.3 (Jan. 2013), pp. 771–824. ISSN: 0272-4979. DOI: 10.1093/imanum/drs032. URL: <https://doi.org/10.1093/imanum/drs032>.
- [7] Atife Çağlar. “Convergence analysis of the Navier–Stokes alpha model”. In: *Numer. Methods Partial Differential Equ.* 26.5 (2009), pp. 1154–1167. DOI: 10.1002/num.20481.
- [8] Yanping Cao and Edriss S. Titi. “On the Rate of Convergence of the Two-Dimensional α -Models of Turbulence to the Navier–Stokes Equations”. In: *Numer. Funct. Anal. Optim.* 30.11-12 (2009), pp. 1231–1271. DOI: 10.1080/01630560903439189. eprint: <https://doi.org/10.1080/01630560903439189>. URL: <https://doi.org/10.1080/01630560903439189>.

(a) Time $t = 0.161$ (b) Time $t = 0.586$ (c) Time $t = 1$ Pressure of LANS- α evaluated at different time values

- [9] Tomás Caraballo, José Real, and Takeshi Taniguchi. “On the existence and uniqueness of solutions to stochastic three-dimensional Lagrangian averaged Navier-Stokes equations”. In: *Proc. R. Soc. A: Math., Phys. and Eng. Sci.* 462.2066 (2006), pp. 459–479. DOI: 10.1098/rspa.2005.1574. URL: <https://royalsocietypublishing.org/doi/abs/10.1098/rspa.2005.1574>.
- [10] Lamberto Cattabriga. “Su un problema al contorno relativo al sistema di equazioni di Stokes”. it. In: *Rendiconti del Seminario Matematico della Università di Padova* 31 (1961), pp. 308–340. URL: http://www.numdam.org/item/RSMUP_1961__31__308_0/.
- [11] S. Chen et al. “A connection between the Camassa–Holm equations and turbulent flows in channels and pipes”. In: *Physics of Fluids* 11.8 (1999), pp. 2343–2353.
- [12] Xiuqing Chen and Jian-Guo Liu. “Two nonlinear compactness theorems in $L_p(0, T; B)$ ”. In: *Appl. Math. Lett.* 25.12 (2012), pp. 2252–2257. ISSN: 0893-9659. DOI: <https://doi.org/10.1016/j.aml.2012.06.012>. URL: <https://www.sciencedirect.com/science/article/pii/S0893965912002856>.
- [13] Igor Chueshov and Annie Millet. “Stochastic 2D Hydrodynamical Type Systems: Well Posedness and Large Deviations”. In: *Appl. Math. Optim.* 61.3 (2009), 379–420. ISSN: 1432-0606. DOI: 10.1007/s00245-009-9091-z. URL: <http://dx.doi.org/10.1007/s00245-009-9091-z>.
- [14] Jeffrey Connors. “Convergence analysis and computational testing of the finite element discretization of the Navier–Stokes alpha model”. In: *Numerical Methods for Partial Differential Equations* 26.6 (2010), pp. 1328–1350. DOI: <https://doi.org/10.1002/num.20493>. eprint: <https://onlinelibrary.wiley.com/doi/pdf/10.1002/num.20493>. URL: <https://onlinelibrary.wiley.com/doi/abs/10.1002/num.20493>.
- [15] Gabriel Deugoue and Mamadou Sango. “On the Stochastic 3D Navier-Stokes- α Model of Fluids Turbulence”. In: *Abstr. Appl. Anal.* 2009.none (2009), pp. 1–27. DOI: 10.1155/2009/723236. URL: <https://doi.org/10.1155/2009/723236>.

- [16] Ciprian Foias, Darryl D. Holm, and Edriss S. Titi. “The Three Dimensional Viscous Camassa–Holm Equations, and Their Relation to the Navier–Stokes Equations and Turbulence Theory”. In: *Journal of Dynamics and Differential Equations* 14.1 (2002), pp. 1–35. DOI: 10.1023/A:1012984210582. URL: <https://doi.org/10.1023/A:1012984210582>.
- [17] M. Germano. “Differential filters for the large eddy numerical simulation of turbulent flows”. In: *The Physics of Fluids* 29.6 (1986), pp. 1755–1757. DOI: 10.1063/1.865649. eprint: <https://aip.scitation.org/doi/pdf/10.1063/1.865649>. URL: <https://aip.scitation.org/doi/abs/10.1063/1.865649>.
- [18] Vivette Girault and Pierre-Arnaud Raviart. *Finite element methods for Navier-Stokes equations: theory and algorithms*. Vol. 5. Berlin Heidelberg: Springer Sci. & Bus. Media, 2012. DOI: <https://doi.org/10.1007/978-3-642-61623-5>.
- [19] Ludovic Goudenège and Luigi Manca. “ α -Navier-Stokes equation perturbed by space-time noise of trace class”. working paper or preprint. May 2020. URL: <https://hal.archives-ouvertes.fr/hal-02615640>.
- [20] Pierre Grisvard. *Elliptic problems in nonsmooth domains*. SIAM, 2011.
- [21] Jean-Luc Guermond and Joseph Pasciak. “Stability of Discrete Stokes Operators in Fractional Sobolev Spaces”. In: *J. Math. Fluid Mech.* 10 (Nov. 2008), pp. 588–610. DOI: 10.1007/s00021-007-0244-z.
- [22] Darryl Holm, Jerrold E Marsden, and Tudor Ratiu. “The Euler-Poincaré Equations and Semidirect Products with Applications to Continuum Theories”. In: *Adv. Math.* 137 (1998), pp. 1–81. DOI: 10.1006/aima.1998.1721.
- [23] Akira Ichikawa. “Stability of semilinear stochastic evolution equations”. In: *J. Math. Anal. Appl.* 90.1 (1982), pp. 12–44. ISSN: 0022-247X. DOI: [https://doi.org/10.1016/0022-247X\(82\)90041-5](https://doi.org/10.1016/0022-247X(82)90041-5). URL: <https://www.sciencedirect.com/science/article/pii/0022247X82900415>.
- [24] Songul Kaya Merdan and Carolina Manica. “Convergence analysis of the finite element method for a fundamental model in turbulence”. In: *M3AS* 22 (Nov. 2012), 24pp. DOI: 10.1142/S0218202512500339.
- [25] Anders Logg, Kent-Andre Mardal, Garth N. Wells, et al. *Automated Solution of Differential Equations by the Finite Element Method*. Springer, 2012. ISBN: 978-3-642-23098-1. DOI: 10.1007/978-3-642-23099-8.
- [26] J. E. Marsden and S. Shkoller. “Global well-posedness for the Lagrangian Navier-Stokes (LANS- α) equations on bounded domains”. In: *Philosophical Transactions Of The Royal Society B: Biological Sciences* 359 (2001).
- [27] E. Pardoux. “Equations aux dérivées partielles stochastiques non linéaires monotones”. PhD thesis. Université Paris XI, 1975.
- [28] Vidar Thomée. *Galerkin finite element methods for parabolic problems*. Vol. 25. Springer Science & Business Media, 2007.