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ALGEBRAIC INTERSECTION IN REGULAR POLYGONS

ERWAN LANNEAU AND DANIEL MASSART

ABSTRACT. We study the function $\text{KVVol} : (X, \omega) \mapsto \text{Vol}(X, \omega) \sup_{\alpha, \beta} \frac{\text{Int}(\alpha, \beta)}{l_g(\alpha)l_g(\beta)}$ defined on the moduli spaces of translation surfaces. More precisely, let \mathcal{T}_n be the Teichmüller discs of the original Veech surface (X_n, ω_n) arising from right-angled triangle with angles $(\pi/2, \pi/n, (n-2)\pi/2n)$ by the unfolding construction for $n \geq 5$. For $n \not\equiv 2 \pmod{4}$ and any $(X, \omega) \in \mathcal{T}_n$, we establish the (sharp) bounds

$$\frac{a(n)}{4} \cot \frac{\pi}{n} \leq \text{KVVol}(X, \omega) \leq \frac{a(n)}{4} \cot \frac{\pi}{n} \cdot \frac{1}{\sin \frac{2\pi}{a(n)}},$$

where $a(n) = 2n$ if n is odd, and $a(n) = n$ otherwise. The lower bound is uniquely realized at (X_n, ω_n) .

1. INTRODUCTION

1.1. Motivation and context. Let X be a closed oriented surface, that is, a compact, connected manifold of dimension 2, without boundary. The algebraic intersection endows the first homology group $H_1(X, \mathbb{R})$ with a symplectic bilinear form denoted $\text{Int}(\cdot, \cdot)$. When X is endowed with a Riemannian metric g , one can ask the following question: how much can two curves of a given length intersect? We denote by $\text{Vol}(X, g)$ the Riemannian volume of X with respect to the metric g , and for any piecewise smooth closed curve α in X , we denote $l_g(\alpha)$ the length of α with respect to g . When there is no ambiguity we omit the reference to g . The natural quantity related to this question is the following:

$$\text{KVVol}(X, g) = \text{Vol}(X, g) \sup_{\alpha, \beta} \frac{\text{Int}(\alpha, \beta)}{l_g(\alpha)l_g(\beta)},$$

where the supremum ranges over all piecewise smooth closed curves α and β in X . The $\text{Vol}(X, g)$ factor is there to make KVVol invariant to re-scaling of the metric g . This quantity is well defined and finite (see [7]).

Very little is known on the function KVVol . For any Riemannian surface (X, g) , we have $\text{KVVol}(X, g) \geq 1$, and equality holds if and only if (X, g) is a flat torus [7]. Almost all of the obvious questions about KVVol on hyperbolic surfaces are currently open.

1.2. KVVol on arithmetic Teichmüller discs. In this paper we propose to continue the study of KVVol as a function on the moduli space of translation surfaces, originally initiated by the second named author in [1, 2]. In these papers, KVVol is studied for translation surfaces in the stratum $\mathcal{H}(2)$: the surface X has genus two, the metric is flat everywhere, except at one conical point, and the holonomy is trivial (see [11] for a nice survey on translation surfaces and moduli spaces). More specifically, in [2],

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KVol is studied for ramified covers of the torus (or arithmetic Teichmüller disc). It is proved in [2] that KVol, defined on the Teichmüller disc of the surface tiled with three squares, is unbounded, but it does have a minimum, achieved at a surface, unique modulo symmetries, and otherwise fairly undistinguished. The interesting surfaces, *i.e.* the three square surfaces, and the surface tiled with six equilateral triangles, are local maxima, with $\text{KVol} = 3$, where 3 should be thought of as the ratio of the total area of the surface, to the area of the smallest cylinder of closed geodesics. The local maxima are not locally unique, they come in hyperbolic geodesics, in the Teichmüller curve.

1.3. KVol on non arithmetic Teichmüller discs. In the current paper, we extend the study to other non arithmetic Teichmüller discs. Recall that Teichmüller curves are isometrically immersed algebraic curves in the moduli space of Riemann surfaces. These arise as $\text{SL}(2, \mathbb{R})$ -orbit (or Teichmüller disc) of special flat surfaces that are called Veech surfaces. We will consider the Teichmüller discs \mathcal{T}_n of the original Veech surfaces [10], namely the surfaces X_n arising from a right-angled triangle with angles $(\pi/2, \pi/n, (n-2) \cdot \pi/2n)$ by the unfolding construction described in [5] when $n \geq 4$ (see Section 2.3 for a precise definition). The main result we will show is the following.

Theorem 1.1. *For any $n \geq 4$, we set $a(n) = 2n$ if n is odd, and $a(n) = n$ otherwise. If $n \not\equiv 2 \pmod{4}$ then for any $(X, \omega) \in \mathcal{T}_n$ the following holds*

$$\frac{a(n)}{4} \cot \frac{\pi}{n} \leq \text{KVol}(X, \omega) \leq \frac{a(n)}{4} \cot \frac{\pi}{n} \cdot \frac{1}{\sin \frac{2\pi}{a(n)}}.$$

Moreover the bounds are sharp and:

- (1) *The maximum of the function KVol on \mathcal{T}_n is achieved, precisely, by the image of the right-angled staircases under the Teichmüller geodesic flow. In particular it is finite.*
- (2) *The minimum of the function KVol on \mathcal{T}_n is achieved, uniquely, at X_n .*

Finally, in the definition of KVol, the supremum is achieved: if n is odd, the maximizing pairs of curves are (images of) pairs of sides of the double regular n -gon, while if n is even, it is the (images of) pairs formed by a side and a short diagonal of the regular n -gon.

1.4. Comments. When $n \equiv 2 \pmod{4}$, the surface X_n belongs to a stratum with two conical points. This case is geometrically more complicated and we set it aside for future work.

Unlike to the three-square surface case [2], KVol is bounded on the Teichmüller discs of the odd regular double- n -gon, and the even regular n -gon. The reason for this is that in the three-square surface there are parallel geodesics which intersect (at the singular point), while in the regular polygons, there are not.

Another difference is that the minimum of KVol on the Teichmüller disc of a regular polygon is achieved, uniquely, by the most interesting surface in the disc, the regular polygon itself. On the other hand, similarly to the three-square case, the local maxima, which are also global maxima in the regular polygon case, are achieved along hyperbolic geodesics in the Teichmüller disc, which correspond to surfaces with a right-angled template, see Figure 3.

1.5. Proof outline. We end this section with a sketch of the proof of Theorem 1.1. Let α and β be two saddle connections in $X \in \mathcal{T}_n$, and let d and d' be their directions, making an angle θ . Swapping the places of α and β if we have to, we assume $\sin \theta \geq 0$.

- (1) **Intersections:** If α and β intersect transversally at a point $p \in X$, we set $\text{Int}_p(\alpha, \beta) = 1$ if β crosses α from right to left, and $\text{Int}_p(\alpha, \beta) = -1$ otherwise. The algebraic intersection $\text{Int}(\alpha, \beta)$ is the sum over all intersection points.
- (2) Define $K(X) = \sup_{\alpha, \beta} \frac{\text{Int}(\alpha, \beta)}{l_g(\alpha)l_g(\beta)}$. It is invariant by rotations, so the study of $K(\cdot)$ on the $\text{SL}(2, \mathbb{R})$ -orbit of X_n reduces to its study on the quotient space $\text{SO}(2, \mathbb{R}) \backslash \text{SL}(2, \mathbb{R}) = \mathbb{H}^2$. We view d as a point in $\mathbb{R} \cup \infty = \partial \mathbb{H}^2$. Thus the pair (d, d') determines a hyperbolic geodesic in \mathbb{H}^2 .
- (3) Since X_n is a Veech surface, it decomposes into cylinders of periodic geodesics in both directions d and d' . Let C_1, \dots, C_m be the cylinders of the direction d . We write $\beta = \bigcup_{i=1}^m C_i \cap \beta$, and for each piece $C_i \cap \beta$, we estimate its length by

$$l(C_i \cap \beta) \sin \theta = \text{height}(C_i) \text{Int}(C_i, \beta),$$

where $\text{Int}(C_i, \beta)$ denotes the intersection between the core curve of C_i , and β . From this, we easily deduce (§ 2.7 and § 2.8) a formula involving the intersections of β with all the saddle connections $\alpha_1, \dots, \alpha_m$ of direction d :

$$\sin \theta = \sum_{i=1}^m \text{Area}(C_i) \frac{\text{Int}(\alpha_i, \beta)}{l(\alpha_i)l(\beta)}.$$

We distinguish two cases depending on the signs of $\text{Int}(\alpha_i, \beta)$. This prompts us to define $K^+(X)$, respectively $K^-(X)$, where the sup is taken over all α, β satisfying $\text{Int}(\alpha, \beta) > 0$, respectively there is an intersection satisfying $\text{Int}(\alpha, \beta) = -1$.

- (4) If all intersections $\text{Int}(\alpha_i, \beta)$ are nonnegative then for all i , the equation in Step (3) gives:

$$\frac{\text{Int}(\alpha_i, \beta)}{l(\alpha_i)l(\beta)} \leq \frac{\sin \theta}{\text{Area}(C_i)}.$$

- (a) The right-hand side is maximal when $C_i = C_{i_0}$ has minimal area among the cylinders of d , and the bound is sharp when $\text{Int}(\alpha_i, \beta) = 0$ for all $i \neq i_0$. We then identify \mathcal{D}^+ (§ 3.1, § 4.1), the set of pairs of directions (d, d') such that for some saddle connection β of direction d' , $\text{Int}(\alpha_i, \beta) = 0$ for all $i \neq i_0$.
- (b) Let $\mathcal{Z}^+ \subset \mathbb{H}^2$ be the set geodesics determined by \mathcal{D}^+ as explained in Step (2). The distance $d^+ = d(X, \mathcal{Z}^+)$ is realized by a geodesic $\gamma_{u,v} \in \mathcal{Z}^+$ where the directions u, v make an angle θ^+ in X (Lemma 2.2). Thus (Corollary 4.3)

$$K^+(X) = \frac{\sin \theta^+}{\min \text{Vol}(C_i)}.$$

- (5) If there is one intersection $\text{Int}(\alpha_i, \beta) = -1$ then it imposes strong restrictions on the directions d and d' , allowing us to treat them separately. We define similarly \mathcal{D}^- , \mathcal{Z}^- and show (Proposition 3.2)

$$K^-(X) = \frac{\sin \theta^-}{\min l(\alpha_i)^2}.$$

- (6) Obviously $K(X) = \max\{K^+(X), K^-(X)\}$. Thus computing $K(X)$ amounts to maximizing $\sin \theta^-$ and $\sin \theta^+$ in Equations in Steps (4b) and (5), that is, finding geodesics in \mathcal{Z}^+ and \mathcal{Z}^- which is closest to X . For the maximum of $K(X)$, we prove that it is achieved at \mathcal{Z}^- (§ 3.2) when $\sin \theta^- = 1$. The minimum of $K\text{Vol}$ on \mathcal{T}_n is achieved by points at maximal distance from \mathcal{Z}^+ and \mathcal{Z}^- (see § 3.3).

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2. FLAT AND HYPERBOLIC GEOMETRIES

2.1. **The set $\mathcal{M}(u, v, \theta)$.** Let u, v be two non-collinear vectors in \mathbb{R}^2 . Let $\bar{u} \neq \bar{v}$ be their equivalence classes in $\mathbb{R}P^1$. If the vector u has coordinates (x, y) , we view the equivalence class \bar{u} as x/y , if $y \neq 0$, or ∞ , if $y = 0$. This allows us to identify $\mathbb{R}P^1$ with the boundary at infinity of the hyperbolic plane \mathbb{H}^2 . For $\theta \in \mathbb{R}/\pi\mathbb{Z}$ we define

$$\mathcal{M}(u, v, \theta) = \{M \in GL_2^+(\mathbb{R}) : \text{angle}(Mu, Mv) = \theta\}.$$

Observe that $\mathcal{M}(u, v, \theta)$ only depends on the equivalence classes \bar{u}, \bar{v} (this is the reason for taking angles mod π).

Lemma 2.1. *For any $G \in GL_2^+(\mathbb{R})$, $\mathcal{M}(u, v, \theta).G = \mathcal{M}(G^{-1}u, G^{-1}v, \theta)$.*

Proof. Take $M \in \mathcal{M}(u, v, \theta)$. Then $\text{angle}(Mu, Mv) = \theta$, that is,

$$\text{angle}(MG.G^{-1}u, MG.G^{-1}v) = \theta,$$

hence $MG \in \mathcal{M}(G^{-1}u, G^{-1}v, \theta)$. Conversely, take $M \in \mathcal{M}(G^{-1}u, G^{-1}v, \theta)$. Then $\text{angle}(MG^{-1}u, MG^{-1}v) = \theta$, that is, $MG^{-1} \in \mathcal{M}(u, v, \theta)$, so $M \in \mathcal{M}(u, v, \theta).G$. \square

We map $SO_2(\mathbb{R}) \backslash GL_2^+(\mathbb{R})$ to the hyperbolic plane \mathbb{H}^2 by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{di + b}{ci + a}.$$

Note that this defines a right action of $GL_2^+(\mathbb{R})$. See [6], Section 7.1, as to why we should quotient by $SO_2(\mathbb{R})$ on the left, and act by $GL_2^+(\mathbb{R})$ on the right.

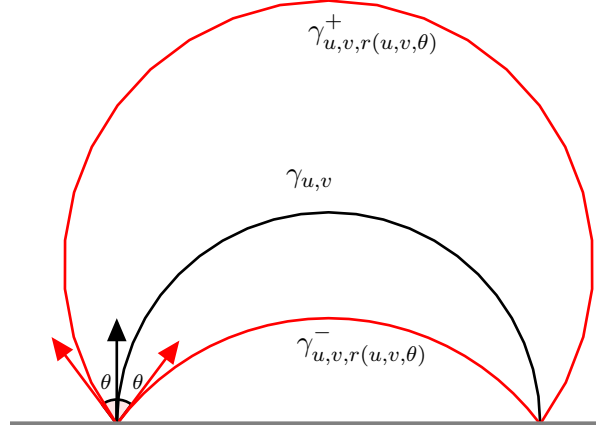
Denote $\bar{\mathcal{M}}(u, v, \theta)$ the projection of $\mathcal{M}(u, v, \theta)$ to \mathbb{H}^2 . Observe that $\mathcal{M}(u, v, \theta)$ is invariant by left multiplication by $SO_2(\mathbb{R})$, so any matrix in $GL_2^+(\mathbb{R})$ that projects to an element of $\bar{\mathcal{M}}(u, v, \theta)$, is actually in $\mathcal{M}(u, v, \theta)$.

Let $\gamma_{u,v}$ be the hyperbolic geodesic with endpoints \bar{u}, \bar{v} . For $r \in \mathbb{R}_+$, denote

$$\gamma_{u,v,r} = \{z \in \mathbb{H}^2 : d(z, \gamma_{u,v}) = r\}$$

where d is the hyperbolic distance. Note that $\gamma_{u,v,r}$ has two connected components, $\gamma_{u,v,r}^+$ and $\gamma_{u,v,r}^-$, such that the angle between $\gamma_{u,v}$ and $\gamma_{u,v,r}^+$ (resp. $\gamma_{u,v,r}^-$) at u is in $[0, \pi/2[$ (resp. $] -\pi/2, 0]$.

Let $\theta(u, v, r) \in [0, \pi/2[$ be the angle at u of $\gamma_{u,v}$ with $\gamma_{u,v,r}^+$. The map $r \mapsto \theta(u, v, r)$ is a homeomorphism from \mathbb{R}_+ to $[0, \pi/2[$. Denote $\theta \mapsto r(u, v, \theta)$ the inverse map.

FIGURE 1. The sets $\gamma_{u,v,r(u,v,\theta)}^+$ and $\gamma_{u,v,r(u,v,\theta)}^-$

Lemma 2.2. *We have, for any $u, v \in \mathbb{R}^2$, and any $\theta \in]0, \pi[$,*

$$\begin{aligned} \bar{\mathcal{M}}(u, v, \theta) &= \gamma_{u,v,r(u,v,\theta-\pi/2)}^+ \text{ if } \theta \in [\pi/2, \pi[\\ &= \gamma_{u,v,r(u,v,\pi/2-\theta)}^- \text{ if } \theta \in]0, \pi/2]. \end{aligned}$$

Proof. First let us look at the case $u = (1, 0), v = (0, 1)$, that is, $\bar{u} = \infty$ and $\bar{v} = 0$. Observe that in that case $\bar{\mathcal{M}}(u, v, \theta)$ is invariant by $z \mapsto \lambda z$, for any $\lambda > 0$. Indeed, take $\lambda > 0$ and $z \in \bar{\mathcal{M}}(u, v, \theta)$, and let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be an element of $\mathcal{M}(u, v, \theta) \subset GL_2^+(\mathbb{R})$ which projects to z . Then the matrix

$$M' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \in GL_2^+(\mathbb{R})$$

projects to λz . But the equivalence class, in $\mathbb{R}P^1$, of $M'u$ (resp. $M'v$), is \bar{u} (resp. \bar{v}), and we have seen that $\mathcal{M}(u, v, \theta)$ only depends on the equivalence classes \bar{u}, \bar{v} , so $M \in \mathcal{M}(u, v, \theta)$ entails $M' \in \mathcal{M}(u, v, \theta)$. Therefore $\lambda z \in \bar{\mathcal{M}}(u, v, \theta)$.

Thus, to determine $\bar{\mathcal{M}}(u, v, \theta)$, it suffices to determine its intersection with the horizontal straight line $\{y = 1\}$, which we parametrize as

$$\{i + \cot \alpha : \alpha \in]0, \pi[\}$$

A corresponding set of matrices in $GL_2^+(\mathbb{R})$ is given by

$$\left\{ \begin{pmatrix} 1 & \cot \alpha \\ 0 & 1 \end{pmatrix} : \alpha \in]0, \pi[\right\}$$

which send u and v to, respectively, u and $(\cot \alpha, 1)$. The angle of the latter vectors is α , so $\bar{\mathcal{M}}(u, v, \theta) \cap \{y = 1\} = \{(\cot \theta, 1)\}$. Therefore, $\bar{\mathcal{M}}(u, v, \theta)$ is the half-line which starts at the origin, with slope $\cot \theta$. This is precisely $\gamma_{u,v,r(u,v,\pi/2-\theta)}^+$ if $\theta \in]0, \pi/2]$, and $\gamma_{u,v,r(u,v,\theta-\pi/2)}^-$ if $\theta \in [-\pi/2, \pi[$.

Now let us consider the general case. Let G be an element of $GL_2^+(\mathbb{R})$ which takes u (resp. v) to $(1, 0)$ (resp. $(0, 1)$). Then, by Lemma 2.1, $\mathcal{M}((1, 0), (0, 1), \theta) = \mathcal{M}(u, v, \theta).G$, so $\bar{\mathcal{M}}(u, v, \theta)$ is the image of $\bar{\mathcal{M}}((1, 0), (0, 1), \theta)$ by an orientation-preserving isometry of \mathbb{H}^2 . This finishes the proof. \square

2.2. Unfolding construction and regular n -gons. For $n \geq 3$, the surfaces X_n (arising from the unfolding of a right triangle with angles $(\pi/2, \pi/n, (n-2)\pi/2n)$) can be described as follows (see [5, 10]). If $n \geq 8$ satisfies $n \equiv 0 \pmod{4}$ then X_n is the quotient of the regular n -gon (with radius 1) by gluing opposite sides by translation (see Figure 2). If $n \geq 5$ is odd, X_n the quotient of the double of the regular n -gon (with radius 1) by gluing opposite sides by translation. The translation surface X_n , belongs to the stratum $\mathcal{H}(n/2 - 2)$ if n is even, and $\mathcal{H}(n - 3)$ if n is odd.

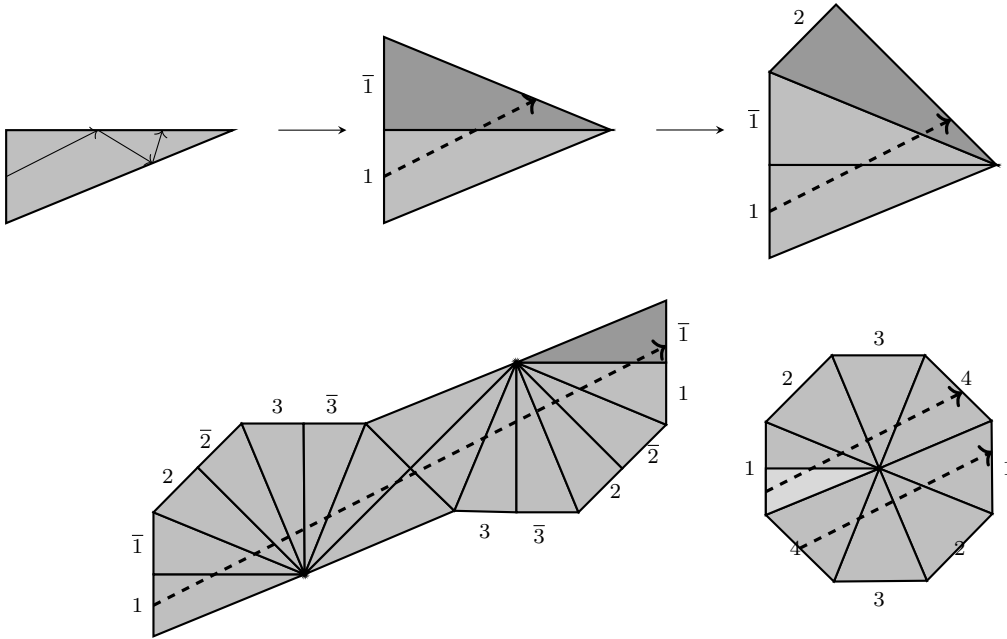


FIGURE 2. Unfolding a billiard trajectory on X_n , where $n = 8$.

2.3. The staircase models for the regular n -gons. We provide a staircase model, denoted by S_n , for X_n , with parameters $(\alpha_i, \beta_i)_{i=1, \dots, m}$. This is mostly contained in [8] with a few modifications.

2.3.1. Staircase model S_n for $n \in 4\mathbb{N}$. Set $n = 4m$. We label the vertices of the regular n -gon counterclockwise by s_i for $i = 0, \dots, n - 1$ (see Figure 4). For $i = 1, \dots, m$ the horizontal saddle connection connecting s_{m+i} to s_{m-i} has holonomy

$$\left(\cos \frac{2(m-i)\pi}{4m} - \cos \frac{2(m+i)\pi}{4m}, 0 \right) = \left(2 \sin \frac{2i\pi}{n}, 0 \right).$$

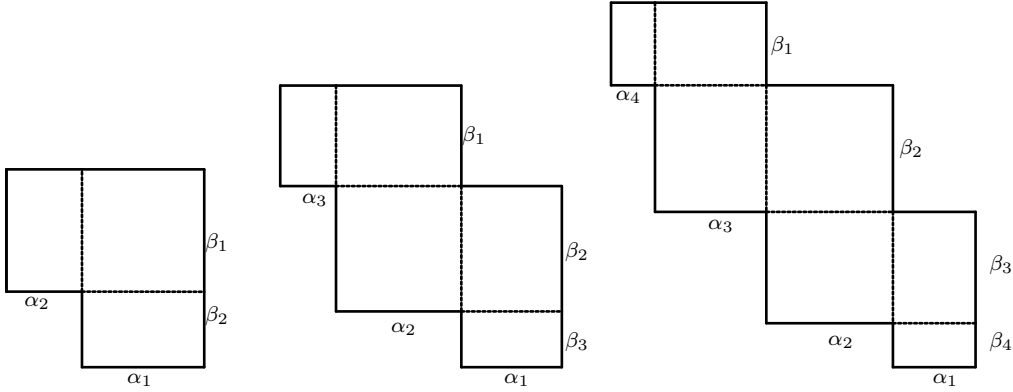


FIGURE 3. A staircase model S_n for X_n ($n = 2m + 1$), for $m = 2, 3, 4$. The surface S_5 for $m = 2$ is usually shown rotated by 180 degrees, as the golden L (see [3], [4]). The staircase model for X_n ($n = 4m$) is laid out similarly but the relative sizes of α_i and β_i are different.

Similarly, the y -coordinate of the holonomy vector $s_{i-1}s_{2m-i}$ is

$$\sin \frac{2(2m-i)\pi}{4m} - \sin \frac{2(i-1)\pi}{4m} = 2 \sin \frac{\pi}{n} \sin \frac{(2m-2i+1)\pi}{n}.$$

Furthermore the image of the saddle connection from s_{i-1} to s_{2m-i} by the shear $\begin{pmatrix} 1 & \cot \frac{\pi}{n} \\ 0 & 1 \end{pmatrix}$ is a vertical saddle connection. Thus, up to renormalizing by $\begin{pmatrix} 2 & 0 \\ 0 & 2 \sin \frac{\pi}{n} \end{pmatrix}^{-1}$, the matrix

$$A_n := \begin{pmatrix} 2 & 0 \\ 0 & 2 \sin \frac{\pi}{n} \end{pmatrix}^{-1} \begin{pmatrix} 1 & \cot \frac{\pi}{n} \\ 0 & 1 \end{pmatrix} = \frac{1}{2 \sin \frac{\pi}{n}} \begin{pmatrix} \sin \frac{\pi}{n} & \cos \frac{\pi}{n} \\ 0 & 1 \end{pmatrix}$$

maps S_n to X_n as depicted in Figure 4. We define the saddle connection α_i in S_n as the image by A_n of the horizontal saddle connection from s_{m+i} to s_{m-i} , and β_i as the image by A_n of the saddle connection from s_{i-1} to s_{2m-i} . From the discussion above we get

$$(1) \quad l(\alpha_i) = \sin \frac{2i\pi}{n}, \quad l(\beta_i) = \sin \frac{(2m-2i+1)\pi}{n}.$$

It follows that the area of the i -th horizontal cylinder is

$$\begin{aligned} (l(\alpha_{i-1}) + l(\alpha_i))l(\beta_{m-i+1}) &= \left(\sin \frac{2(i-1)\pi}{n} + \sin \frac{2i\pi}{n} \right) \sin \frac{(2i-1)\pi}{n} \\ &= 2 \cos \frac{\pi}{n} \sin^2 \frac{(2i-1)\pi}{n} = \Phi_n \sin^2 \frac{(2i-1)\pi}{n} \end{aligned}$$

where we have set $\Phi_n := 2 \cos \frac{\pi}{n}$. The modulus of the i -th horizontal cylinder is

$$(2) \quad \frac{l(\beta_{m-i+1})}{l(\alpha_{i-1}) + l(\alpha_i)} = \frac{1}{\Phi_n}.$$

Likewise, for $i \geq 2$, the area of the i -th vertical cylinder is

$$\begin{aligned} (l(\beta_{i-1}) + l(\beta_i))l(\alpha_{m-i+1}) &= \left(\sin \frac{(2m-2i+3)\pi}{n} + \sin \frac{(2m-2i+1)\pi}{n} \right) \sin \frac{(2m-2i+2)\pi}{n} \\ &= 2 \cos \frac{\pi}{n} \sin^2 \frac{2(m-i+1)\pi}{n} = \Phi_n \sin^2 \frac{2(m-i+1)\pi}{n} \end{aligned}$$

and its modulus is

$$(3) \quad \frac{l(\alpha_{m-i+1})}{l(\beta_{i-1}) + l(\beta_i)} = \frac{1}{\Phi_n}.$$

However, the first vertical cylinder is special: recalling that $n = 4m$, its area is

$$l(\beta_1)l(\alpha_m) = \sin \frac{(2m-1)\pi}{n} \sin \frac{2m\pi}{n} = \frac{\Phi_n}{2},$$

and its modulus is $\frac{2}{\Phi_n}$.

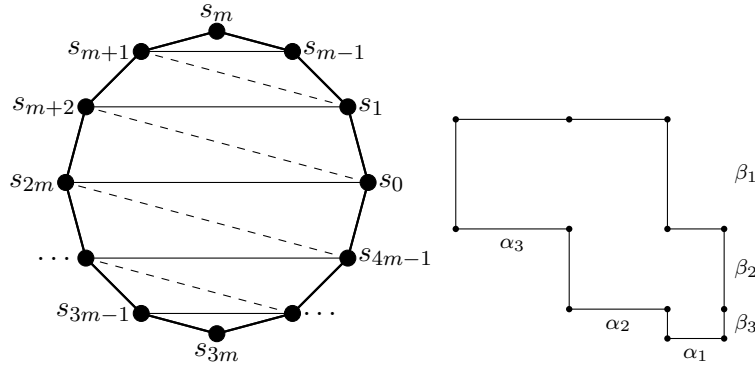


FIGURE 4. The regular $4m$ -gon X_n and its staircase model S_n (here $m = 3$).

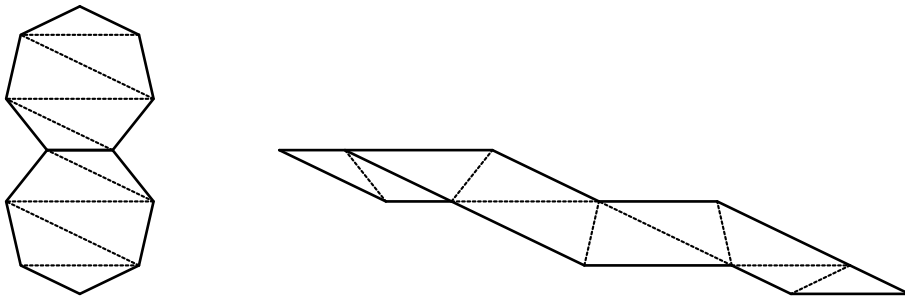


FIGURE 5. The double heptagon X_7 is cut into ten triangles, which are rearranged into a slanted stair-shape, whose slanted sides are then rotated to create the right-angled stair-shape S_7 in the middle of Figure 3.

2.3.2. *Staircase model S_n for odd n .* In the staircase model S_n for odd n , a similar computation leads to the same parameters (1). Observe that since $n = 2m + 1$ we have the relations

$$l(\alpha_i) = l(\beta_i) = \sin \frac{2i\pi}{n}, \text{ for any } i = 1, \dots, m.$$

The area of the i -th cylinder is

$$\begin{aligned} (l(\alpha_{i-1}) + l(\alpha_i)) l(\beta_{m-i+1}) &= \left(\sin \frac{2(i-1)\pi}{n} + \sin \frac{2i\pi}{n} \right) \sin \frac{(2i-1)\pi}{n} \\ &= 2 \cos \frac{\pi}{n} \sin^2 \frac{(2i-1)\pi}{n} = \Phi_n \sin^2 \frac{(2i-1)\pi}{n}. \end{aligned}$$

The modulus of the i -th cylinder is

$$(4) \quad \frac{l(\beta_{m-i+1})}{l(\alpha_{i-1}) + l(\alpha_i)} = \frac{1}{\Phi_n}.$$

Remark 2.3. *Since the area of X_n is $\frac{n}{2} \sin \frac{2\pi}{n} = n \sin \frac{\pi}{n} \cos \frac{\pi}{n}$ for $n = 4m$, and $n \sin \frac{2\pi}{n}$ for $n = 2m + 1$, and the determinant of A_n is $\frac{1}{4 \sin \frac{\pi}{n}}$ we get*

- (1) $\text{Vol}(S_n) = \frac{n}{4} \cos \frac{\pi}{n}$ for $n = 4n$ and $\text{Vol}(S_n) = \frac{n}{2} \cos \frac{\pi}{n}$ for odd n .
- (2) *In the construction for odd n , we see that α_m, β_m correspond to the sides of X_n while α_1, β_1 correspond to short diagonals. Similarly, for $n = 4m$, β_m corresponds to a side of X_n while α_1 corresponds to a short diagonal.*

2.4. **Veech groups and Fundamental domains.** For $n \geq 3$, we denote by Γ_n the Hecke triangle group of level n (or signature $(2, n, \infty)$) generated by

$$T = \begin{pmatrix} 1 & \Phi_n \\ 0 & 1 \end{pmatrix} \text{ and } R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ recalling that } \Phi_n = 2 \cos \frac{\pi}{n}.$$

The group Γ_n , acting on the hyperbolic plane \mathbb{H}^2 , has a fundamental domain, depicted in Figure 6, comprised between the vertical geodesics with abscissae $-\Phi_n/2$ and $\Phi_n/2$, and the geodesic with endpoints ± 1 .

When $n = 2m + 1$, the Veech group of S_n coincides with Γ_n , whereas when $n = 4m$, the Veech group of S_n is a subgroup of index 2 of Γ_n : it is generated by T and RTR . The regular n -gon X_n , for $n = 4m$, is fixed by the elliptic isometry S^2 , where $S = RT$, while the staircase model S_n lies on the vertical geodesic $(0, \infty)$ (see Figure 6). In the case of the double n -gon, for n odd, in the fundamental domain depicted in Figure 6, the staircase model S_n is represented by the point i , while X_n corresponds to the lower corners of \mathcal{D}_n (the intersection between a vertical boundary and the circular boundary).

2.5. **Intersections in S_n .** If two C^1 closed curves α and β in X intersect transversally at a point $p \in X$, we set $\text{Int}_p(\alpha, \beta) = 1$ if β crosses α from right to left, and $\text{Int}_p(\alpha, \beta) = -1$ otherwise. Then the algebraic intersection $\text{Int}(\alpha, \beta)$ of α and β is the sum over all intersection points p of $\text{Int}_p(\alpha, \beta)$. The algebraic intersection endows the first homology $H_1(X, \mathbb{R})$ with a symplectic bilinear form. In particular $\text{Int}(\alpha, \beta)$ is finite, and only depends on the homology classes of α and β .

Since $\alpha_k + \alpha_{k+1}$ (for $k = 0, 2, \dots, m-1$) are homologous to core curves of horizontal cylinders, which are non-singular, it is visually obvious that $\text{Int}(\alpha_k + \alpha_{k+1}, \beta_l) = \delta_{m-k, l}$.

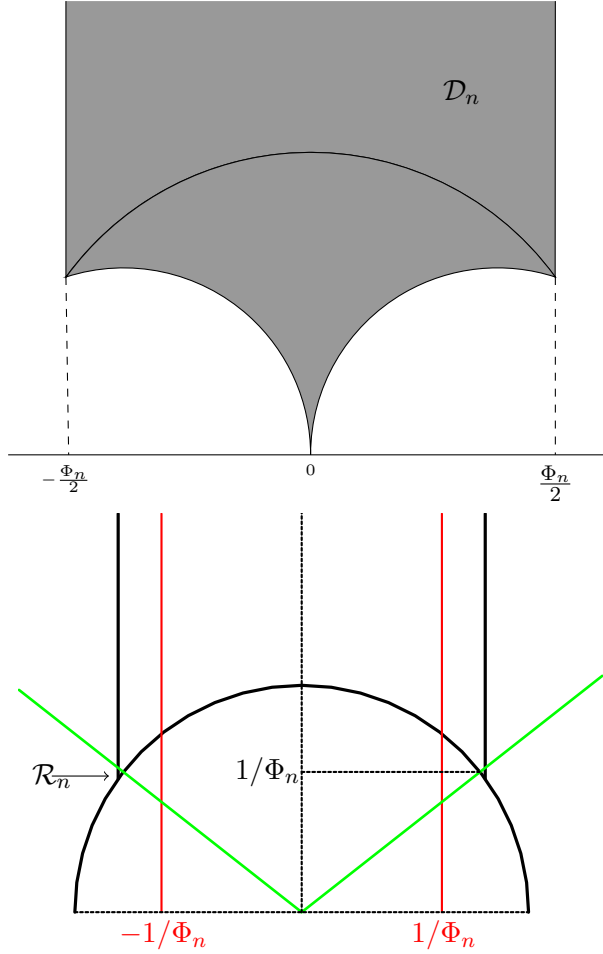


FIGURE 6. A fundamental domain \mathcal{D}_n of the Veech group of the staircase model of the regular n -gon for $n = 4m$ (above) and $n = 2m + 1$ (below). In the lower drawing, \mathcal{D}_n is bounded by the geodesics in black, and the tubular neighborhood of $(0, \infty)$ defined by $\sin \theta \geq 1/\Phi_n$ is bounded by the two green half-lines.

From there we easily deduce

$$(5) \quad \text{Int}(\alpha_k, \beta_l) = \begin{cases} 0 & \text{if } k + l \leq m \\ 1 & \text{if } k + l > m \text{ and } k + l = m - 1 \pmod{2} \\ -1 & \text{if } k + l > m \text{ and } k + l = m \pmod{2}. \end{cases}$$

Observe that $\text{Int}(\alpha_k, \beta_l) = \text{Int}(\alpha_l, \beta_k)$.

2.6. Directions in the Teichmüller disc. For $d \in \mathbb{R}P^1$, we say a geodesic in S_n has direction d if it has direction d in the plane template of Figure 3. This makes sense because the sides of the template are identified by translations. For $A \in \text{GL}_2^+(\mathbb{R})$ we say

that a geodesic γ in $A.S_n$ has direction d if $A^{-1}.\gamma$ has direction d in S_n . This is a bit counter-intuitive because γ may not have direction d in a plane template for $A.S_n$.

The reason for choosing this terminology is that it allows a nice way of looking at the set $\mathcal{M}(u, v, \theta)$ of Lemma 2.2: this is just the set of surfaces X in the $\mathrm{GL}_2^+(\mathbb{R})$ -orbit such that the directions u and v make an angle θ in X . Likewise, since the angle between two directions is preserved by $\mathrm{SO}_2(\mathbb{R})$, $\mathcal{M}(u, v, \theta)$ is just the set of (equivalence classes modulo $\mathrm{SO}_2(\mathbb{R})$ of) surfaces X , viewed as points in the hyperbolic plane such that the directions u and v make an angle θ in X . As a particular case, the set of surfaces X in which the directions u and v are orthogonal is just the hyperbolic geodesic with endpoints u and v .

We say a direction d is periodic if all geodesics with direction d are either periodic, or saddle connections. This makes sense because the property of being periodic or a saddle connection is preserved by the action of $\mathrm{GL}_2^+(\mathbb{R})$.

2.7. Cusp identities in S_n for $n = 2m + 1$. The Teichmüller curve \mathcal{T}_n of S_n is the quotient \mathbb{H}^2/Γ_n . It is a hyperbolic surface with one elliptic point of order $2m + 1$, which is X_n , one elliptic point of order 2, which is S_n , and one cusp. By [10], the fact that there is only one cusp entails that Γ_n acts transitively on the set of periodic directions, that is, for any periodic direction d in S_n , there exists $\gamma \in \Gamma_n$ such that $\gamma(d) = \infty$.

In a periodic direction d on X in \mathcal{T}_n , the surface is decomposed into m cylinders. We denote by $\alpha_1(d, X), \dots, \alpha_m(d, X)$ the saddle connections of the direction d , labeled so that

$$(6) \quad \frac{l(\alpha_k(d, X))}{l(\alpha_m(d, X))} = \frac{l(\alpha_k)}{l(\alpha_m)} = \frac{\sin \frac{2k\pi}{n}}{\sin \frac{\pi}{n}}.$$

Proposition 2.4. *Take X in \mathcal{T}_n , $n = 2m + 1$. Let $\theta \in [0, 2\pi[$ be the angle between two periodic directions d and d' on X , and let β be a saddle connection in direction d' . We have*

$$(7) \quad \sin \theta = \sum_{k=1}^m l(\alpha_k)(l(\alpha_{m-k}) + l(\alpha_{m-k+1})) \frac{\mathrm{Int}(\alpha_k(d, X), \beta)}{l(\alpha_k(d, X))l(\beta)}.$$

with the convention $\alpha_0 = \emptyset$.

Proof of Proposition 2.4. Let C_1, \dots, C_m be the cylinders of (X, d) , labeled so that the core curve of C_i is homologous to $[\alpha_{i-1}] + [\alpha_i]$, for $i = 1, \dots, m$, with the convention $\alpha_0 := \emptyset$. The height of C_i is $h_i(d) = l(\alpha_{m-i+1}(d, X))$. Again we make the convention that $h_{m+1}(d) = 0$. By cutting a saddle connection β of d' into its intersections with the cylinders of d , we get

$$\begin{aligned} l(\beta) \sin \theta &= \sum_{k=1}^m h_k(d) \cdot \mathrm{Int}(\alpha_k(d) + \alpha_{k-1}(d), \beta) \\ &= \sum_{k=1}^m (h_k(d) + h_{k+1}(d)) \cdot \mathrm{Int}(\alpha_k(d), \beta). \end{aligned}$$

This reads

$$\sin \theta = \sum_{k=1}^m c_k(d) \frac{\mathrm{Int}(\alpha_k(d), \beta)}{l(\alpha_k(d))l(\beta)}$$

with $c_k(d) = l(\alpha_k(d))(h_k(d) + h_{k+1}(d))$. Acting by the Veech group, which is a subgroup of $\mathrm{SL}_2(\mathbb{R})$, does not change the area $c_k(\infty)$ of the k -th horizontal cylinder, so we can send d to the horizontal direction: $c_k(d) = c_k(\infty) = l(\alpha_k)(l(\alpha_{m-k}) + l(\alpha_{m-k+1}))$ as desired. \square

2.8. Cusp identities in S_n , $n = 4m$. Now if $n = 4m$ then the Teichmüller disc has two cusps, represented by the horizontal and the vertical directions in S_n . We label a periodic direction d by d_h or d_v , depending on which cusp it represents. In any periodic direction d_h , X in \mathcal{T}_n decomposes into m cylinders. Let $\alpha_1(d, X), \dots, \alpha_m(d, X)$ be the saddle connections in the direction d , labeled so that

$$(8) \quad \frac{l(\alpha_k(d, X))}{l(\alpha_m(d, X))} = \frac{l(\alpha_k)}{l(\alpha_m)} = \frac{\sin \frac{2k\pi}{n}}{\sin \frac{2m\pi}{n}} = \sin \frac{2k\pi}{n}.$$

Let C_1, \dots, C_m be the cylinders of (X, d) , labeled so that the core of C_i is homologous to $[\alpha_{i-1}] + [\alpha_i]$, for $i = 1, \dots, m$, with the convention $\alpha_0 := \emptyset$. The height of C_i is $l(\beta_{m-i+1}(d, X))$.

For the other cusp, we get a slightly different formula. Let C_1, \dots, C_m be the cylinders of (X, d_v) , labeled so that the core of C_i is homologous to $[\beta_i] + [\beta_{i-1}]$, for $i = 1, \dots, m$, with the convention $\beta_0 := \emptyset$. The height of C_i is $l(\alpha_{m-i+1}(d, X))$.

Following the lines of the proof of Proposition 2.4 we get

Proposition 2.5. *Let $\theta \in [0, 2\pi[$ be the angle between two periodic direction d and d' on X . If d represents the vertical cusp and α is a saddle connection in direction d' then*

$$(9) \quad \sin \theta = \sum_{k=1}^m l(\beta_k)(l(\alpha_{m-k}) + l(\alpha_{m-k+1})) \frac{\mathrm{Int}(\beta_k(d_v, X), \alpha)}{l(\beta_k(d_v, X))l(\alpha)}.$$

otherwise, if d represents the horizontal cusp and β is a saddle connection in direction d'

$$(10) \quad \sin \theta = \sum_{k=1}^m l(\alpha_k)(l(\beta_{m-k}) + l(\beta_{m-k+1})) \frac{\mathrm{Int}(\alpha_k(d_h, X), \beta)}{l(\alpha_k(d_h, X))l(\beta)}.$$

Proof. If d represents the vertical cusp, let θ be the angle (in $[0, 2\pi[$) between d and d' . Recall that the cylinder C_k , for $k = 1, \dots, m$, in the direction d has core curve $\beta_k + \beta_{k-1}$ and height $h_k = l(\alpha_{m-k+1})$. By cutting a saddle connection α of d' into its intersections with the cylinders of d , we get

$$\begin{aligned} l(\alpha) \sin \theta &= \sum_{k=1}^m h_k(d) \cdot \mathrm{Int}(\beta_k(d) + \beta_{k-1}(d), \alpha) \\ &= \sum_{k=1}^m (h_k(d) + h_{k+1}(d)) \cdot \mathrm{Int}(\beta_k(d), \alpha). \end{aligned}$$

This reads

$$\sin \theta = \sum_{k=1}^m c_k(d) \frac{\mathrm{Int}(\beta_k(d), \alpha)}{l(\beta_k(d))l(\alpha)}$$

where $c_k(d) = l(\beta_k(d))(h_k(d) + h_{k+1}(d))$ is the area of the k -th vertical cylinder. By the action of the Veech group, which is a subgroup of $\mathrm{SL}_2(\mathbb{R})$, we get $c_k(d) = c_k(0) =$

$l(\beta_k)(l(\alpha_{m-k} + l(\alpha_{m-k+1}))$ as desired. For the horizontal direction, we proceed as above proving the proposition. \square

We end this section with the following useful elementary result.

Proposition 2.6. *Let \mathcal{D} be the set of periodic directions in X , where X is any translation surface. For any $d \in \mathcal{D}$, let $\alpha_1(d, X), \dots, \alpha_{m(d)}(d, X)$ be the saddle connections of X in the direction d . We have*

$$(11) \quad KVol(X) = Vol(X) \sup_{\substack{d, d' \in \mathcal{D} \\ i, j = 1, \dots, m(d)}} \frac{Int(\alpha_i(d, X), \alpha_j(d', X))}{l(\alpha_i(d, X))l(\alpha_j(d', X))}$$

3. PROOF OF THEOREM 1.1 FOR ODD n

3.1. Positive and negative algebraic intersections. We make a case-by-case analysis, depending on the sign of the intersections of $\alpha_i(d)$ with $\alpha_j(d)$ in Equation (11). Namely we partition the set of pairs of periodic directions on $X \in \mathcal{T}_n$ into the subset $\mathcal{D}^+(X)$, of pairs of directions (d, d') such that all intersections between saddle connections of d and that of d' have the same sign, and the complement $\mathcal{D}^-(X)$.

So we may naturally rewrite Equation (11) as

$$(12) \quad KVol(X) = Vol(X) \max \{K^+(X), K^-(X)\}$$

where

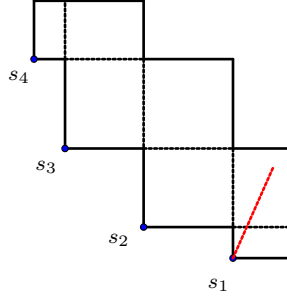
$$K^\varepsilon(X) = \sup_{\substack{(d, d') \in \mathcal{D}^\varepsilon(X) \\ i, j = 1, \dots, m}} \frac{Int(\alpha_i(d, X), \alpha_j(d', X))}{l(\alpha_i(d, X))l(\alpha_j(d', X))}, \text{ for } \varepsilon \in \{+, -\}.$$

Lemma 3.1. *The set $\mathcal{D}^-(X_n)$ consists of the orbit of $(\infty, 0)$ under the Veech group acting diagonally on $\mathbb{R}P^1 \times \mathbb{R}P^1$.*

Proof. Note that all intersections between $\alpha_i(d, X)$ and $\alpha_j(d', X)$, outside the singular point, have the same sign, that of $\sin \theta$, where θ is the angle, in X , of d and d' . So if, for some i and j , the intersection of $\alpha_i(d, X)$ and $\alpha_j(d', X)$ has the same sign as $-\sin \theta$, then $\alpha_i(d, X)$ and $\alpha_j(d', X)$ intersect only once, at the singular point.

Let us look for pairs of directions (d, d') in the staircase model S_n such that for some indices i and j , $\alpha_i(d)$ and $\alpha_j(d')$ intersect only at the saddle point of S_n .

This property is preserved by diffeomorphisms. In addition, \mathcal{T}_n has only one cusp: the Veech group acts transitively on periodic directions. So we may assume d is horizontal (that is, $d = \infty$). Furthermore, acting by some power of the horizontal shear $T = \begin{pmatrix} 1 & \Phi_n \\ 0 & 1 \end{pmatrix}$ we may assume that $d \in [0, \Phi_n[$. It is clear (see Figure 7, where $m = 4$) that any saddle connection, starting from s_1, s_2, \dots, s_m in the stair-shaped template, with co-slope $< \Phi_n$ (or equivalently, slope $> 1/\Phi_n$) must intersect α_i , for some index i , outside the singularity, unless it is vertical (*i.e.* its co-slope is zero). Thus, the set \mathcal{D}^- of pairs of directions (d, d') in S_n , such that not all intersections between a saddle connection of d , and a saddle connection of d' , have the same sign, consists of $(\infty, 0)$, and its orbit under the Veech group acting on $\mathbb{R}P^1 \times \mathbb{R}P^1$. \square

FIGURE 7. Saddle connections of slope $> \frac{1}{\Phi_n}$

We denote by $\mathcal{Z}^-(S_n) \subset \mathbb{H}^2$ the orbit of the geodesic $(\infty, 0)$ under the Veech group. For $X \in \mathcal{T}_n$, define $d^- = d(X, \mathcal{Z}^-)$ and $\theta^- = \theta(d^-)$. The distance d^- is realized by a geodesic $\gamma_{u,v} \in \mathcal{Z}^-$. By Lemma 2.2 the directions u, v make an angle θ^- in X .

Proposition 3.2. *For $X \in \mathcal{T}_n$, $n = 2m + 1$, we have*

$$K^-(X) = \frac{\sin \theta^-}{l(\alpha_m)^2}.$$

Proof of Proposition 3.2. For any $(d, d') \in \mathcal{D}^-(X)$, we have $\text{Int}(\alpha_i(d, X), \alpha_j(d', X)) = \text{Int}(\alpha_i, \beta_j)$, which is given by (5), and is always $-1, 0$, or 1 . By (6),

$$\max_{i,j=1,\dots,m} \frac{1}{l(\alpha_i(d, X))l(\alpha_j(d', X))} = \frac{1}{l(\alpha_m(d, X))l(\alpha_m(d', X))}$$

and by (5) again, $\text{Int}(\alpha_m(d, X), \alpha_m(d', X)) \neq 0$. Since $l(\alpha_m(d, X))l(\alpha_m(d', X)) \sin \theta = l(\alpha_m)l(\beta_m)$, where θ is the angle between d, d' , we draw

$$K^-(X) = \sup_{(d,d') \in \mathcal{D}^-(X)} \frac{1}{l(\alpha_m(d, X))l(\alpha_m(d', X))} = \frac{\sin \theta}{l(\alpha_m)^2}.$$

Proposition 3.2 is proved. \square

Lemma 3.3. *For any periodic directions d, d' making an angle θ and satisfying $(d, d') \in \mathcal{D}^+(X)$, and any saddle connection β in direction d' , the following holds*

$$\frac{\text{Int}(\alpha_i(d), \beta)}{l(\alpha_i(d))l(\beta)} \leq \frac{|\sin \theta|}{l(\alpha_1)l(\alpha_m)},$$

with equality if and only if $\text{Int}(\alpha_i(d), \beta) = 0$ for all $i = 1, \dots, m - 1$.

Proof of Lemma 3.3. Let d, d' be two periodic directions on $X \in \mathcal{T}_n$ and β be a saddle connection in direction d' . If all the intersections between β and $\alpha_i(d)$ are nonnegative (for $i = 1, \dots, m$) then Equations (7) and (6) give

$$\frac{\text{Int}(\alpha_i(d), \beta)}{l(\alpha_i(d))l(\beta)} \leq \frac{\sin \theta}{l(\alpha_i)(l(\alpha_{m-i}) + l(\alpha_{m-i+1}))} \leq \frac{\sin \theta}{l(\alpha_1)l(\alpha_m)}.$$

To justify the last inequality, we remark that for every $i = 1, \dots, m$ one has (see (4))

$$\frac{1}{l(\alpha_i)(l(\alpha_{m-i}) + l(\alpha_{m-i+1}))} = \frac{\Phi_n^{-1}}{l(\alpha_i)^2},$$

that is maximal for $i = m$ (Φ_n^{-1} is the common modulus of the horizontal cylinders). Clearly this upper bound is achieved if and only if β intersects only $\alpha_m(d)$ and no other $\alpha_i(d)$. \square

We now define the set \mathcal{Z}^+ . Since the equality of Lemma 3.3 is achieved when β intersects only $\alpha_m(d)$, let us assume $\text{Int}(\alpha_i(d), \beta) = 0$ for all $i = 1, \dots, m-1$. We want to determine the corresponding set of directions on S_n . As in the proof of Lemma 3.1, up to action of the Veech group, we may assume that $d = \infty$ (d is horizontal) and $d' \in]-\frac{1}{2\Phi_n}, \frac{1}{2\Phi_n}[$. Furthermore, by the symmetry of the staircase model with respect to the vertical axis, we may assume $d' \in]0, \frac{1}{2\Phi_n}[$. Then, since $\alpha_i(d) = \alpha_i$, we have $\text{Int}(\alpha_i, \beta) = 0$ for all $i = 1, \dots, m-1$.

Now look again at Figure 7: the saddle connections starting, in the stair-shaped template, from S_1, \dots, S_{m-1} , do intersect α_i transversally, for some $i \leq m-1$; and so does the saddle connection starting from S_m , unless it closes up at the upper right corner of the small cylinder on the highest step of the staircase, before leaving the small cylinder, which only happens if the slope is a multiple of Φ_n , that is, $d' = (k\Phi_n)^{-1}$, for some $k \in \mathbb{N}^*$.

Hence we will define the set $\mathcal{Z}^+(S_n) \subset \mathbb{H}^2$ to be the orbit of the geodesics $(\infty, (k\Phi_n)^{-1})$, $k \in \mathbb{Z}^*$, under the Veech group. For $X \in \mathcal{T}_n$, we also define d^+, θ^+ in the same way as for d^- and θ^- . The above discussion leads to the following result.

Corollary 3.4. *For $X \in \mathcal{T}_n$, $n = 2m + 1$, we have*

$$K^+(X) = \frac{\sin \theta^+}{l(\alpha_1)l(\alpha_m)}.$$

3.2. The maximum.

Proof of the upper bound in Theorem 1.1 for odd n . Let $X \in \mathcal{T}_n$ be a maximum for KVVol and let (d, d') be two directions giving the maximizing pair of saddle connections. By Proposition 3.2, $K^-(X) = \frac{\sin \theta}{l(\alpha_m)^2}$ is maximal when the angle between d and d' is $\theta = \pi/2$. Thus by Lemma 2.2, $X = M.S_n$ where M projects to $\mathcal{Z}^- \subset \mathbb{H}^2$, the orbit of the geodesic $(\infty, 0)$ by the Veech group.

On the other hand Lemma 3.3 implies that $K^+(X) \leq \frac{\sin \theta}{l(\alpha_1)l(\alpha_m)}$. Since $\frac{1}{l(\alpha_1)l(\alpha_m)} < \frac{1}{l(\alpha_m)^2}$ we draw that the maximum of KVVol is achieved at \mathcal{Z}^- and for those surfaces one has

$$\text{KVVol}(X) = \frac{n \cos \frac{\pi}{n}}{2 \sin^2 \frac{\pi}{n}}$$

recalling that the area of X is $\frac{n}{2} \cos \frac{\pi}{n}$ (see Remark 2.3). \square

3.3. The minimum. Now let us investigate what happens outside of \mathcal{Z}^- . Pick $X \in \mathcal{T}_n$. For $\varepsilon = \pm$, let $\gamma_{u^\varepsilon, v^\varepsilon}(X)$ be a geodesic in \mathcal{Z}^ε which comes closest to X . We also set $d^\varepsilon = d(X, \gamma^\varepsilon)$, $\theta^\varepsilon = \theta(d^\varepsilon)$, and $\gamma^\varepsilon(X) = \gamma_{u^\varepsilon, v^\varepsilon, \theta^\varepsilon}(X)$, so

$$X \in \gamma^+ \cap \gamma^-.$$

We refer to Figure 10. By Lemma 2.2 the directions u^ε and v^ε make an angle θ^ε in X , so that

$$K^+(X) = \frac{\sin \theta^+}{l(\alpha_1)l(\alpha_m)} \quad \text{and} \quad K^-(X) = \frac{\sin \theta^-}{l(\alpha_m)^2}.$$

Observe that for any two saddle connections α, β in X , by the identities of Subsection 2.7, we have

$$\left| \frac{\text{Int}(\alpha, \beta)}{l(\alpha)l(\beta)} \right| \leq \max \left\{ \frac{\sin \theta^-}{l(\alpha_m)^2}, \frac{\sin \theta^+}{l(\alpha_1)l(\alpha_m)}, \frac{\Phi_n^{-1}}{l(\alpha_k)^2}, \text{ for } k \neq 1, m \right\}.$$

Theorem 1.1 for odd n will follow from inequalities

$$(13) \quad \frac{\sin \theta^-}{l(\alpha_m)^2} \geq \frac{\sin \theta^+}{l(\alpha_1)l(\alpha_m)} > \frac{\Phi_n^{-1}}{l(\alpha_k)^2}, \text{ for } k \neq 1, m,$$

which in turn will imply

$$\text{KVVol}(X) = \text{Vol}(X) \frac{\sin \theta^-}{l(\alpha_m)^2}.$$

Recall that the action of the Veech group Γ_n on \mathbb{H}^2 has a fundamental domain \mathcal{D}_n which is the geodesic triangle whose sides are the $(\infty, -\Phi_n/2)$, $(\infty, \Phi_n/2)$, $(-1, 1)$ geodesics (see Figure 9 and Figure 8). It is enough to prove (13) for $X \in \mathcal{D}_n$.

Lemma 3.5 (Geodesic γ^-). *For any $X \in \mathcal{D}_n$, the closest (to X) geodesic in \mathcal{Z}^- is $(\infty, 0)$. It is unique except when X lies on a vertical boundary of \mathcal{D}_n .*

Proof of Lemma 3.5. Let $S = RT = \begin{pmatrix} 0 & -1 \\ 1 & \Phi_n \end{pmatrix}$ be the elliptic element of the Veech group Γ , of order $2m+1$, fixing the double regular $2m+1$ -gon P_n at coordinate $-\frac{1}{2}\Phi_n + \frac{i}{2}\sqrt{4 - \Phi_n^2}$. For $i = 1, \dots, 2m$, let $\gamma_0 = (0, \infty)$, and $\gamma_i = S^i(\gamma_0)$. The geodesics γ_i , $i = 0, \dots, 2m$, are contained in \mathcal{Z}^- . Observe that $\gamma_i \subset S^i(\mathcal{D}_n) \cup S^i R(\mathcal{D}_n)$. These $2m+1$ geodesics, and their endpoints at infinity, define an ideal regular $2m+1$ -gon Q_n in the compactification $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ (see Figure 8). By construction Q_n contains the double regular $2m+1$ -gon (the fixed point of the isometry RS). Moreover, by definition of a fundamental domain, $\mathcal{Z}^- \cap \mathcal{D}_n = (0, \infty)$, so we see that the interior of Q_n does not meet any element of \mathcal{Z}^- .

So, pick some $x \in \mathcal{D}_n$ with $\text{Re}(x) \leq 0$. Any geodesic with initial point x must intersect one of $\gamma_0, \dots, \gamma_{2m+1}$ before it reaches any other element of \mathcal{Z}^- . Therefore,

$$\forall \gamma \in \mathcal{Z}^-, d(x, \gamma) \geq \min\{d(x, \gamma_i), i = 0, \dots, 2m\}.$$

So all we have to show is

$$d(x, \gamma_0) \leq \min\{d(x, \gamma_i), i = 1, \dots, 2m\}.$$

Note that the center of the ideal regular polygon Q_n is the double regular n -gon X_n . So for any $i = 0, \dots, 2m$, the equidistant locus (angular bisector) between γ_i and γ_{i+1} (with the convention $\gamma_{2m+1} = \gamma_0$) is the geodesic g_i between X_n and the common endpoint to γ_i and γ_{i+1} . So, for any point $x \in Q_n$, if i is such that x lies in the interior of the

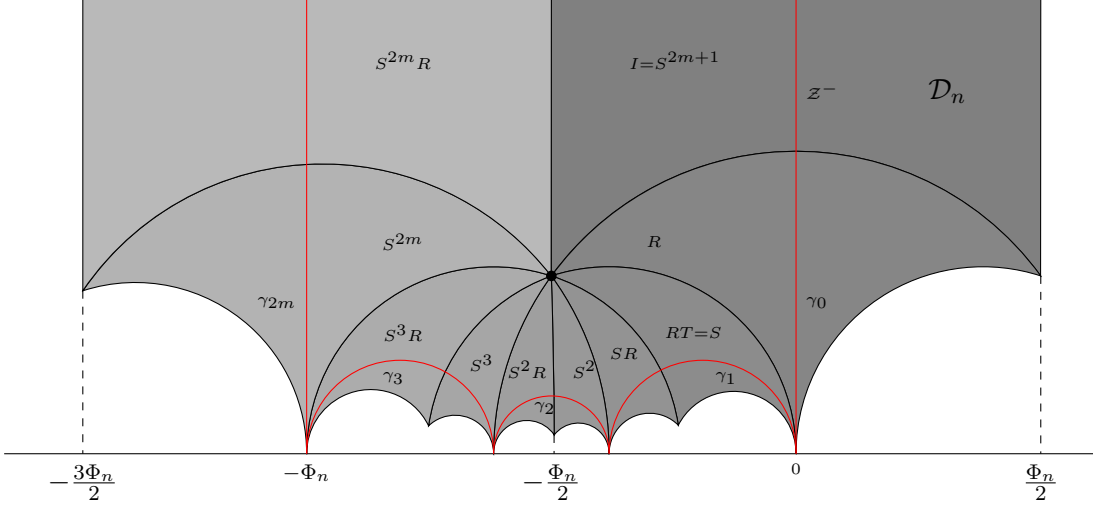


FIGURE 8. The fundamental domain \mathcal{D}_n , in grey, in the case $m = 2$, and the geodesics $\gamma_0, \dots, \gamma_{2m}$ (in red). The double $2m + 1$ -gon X_n is the black point fixed by the isometry RT .

triangle with boundary c_i, c_{i-1} , and γ_i , then the closest (to x) boundary point of Q_n lies on γ_i . In particular, since $x \in \mathcal{D}_n$, the closest γ_i to x is γ_0 .

If $x \in \mathcal{D}_n$ and $\operatorname{Re}(x) \geq 0$, the same argument works by the symmetry of \mathcal{D}_n . Lemma 3.5 is proved. \square

Corollary 3.6. *Let X be a translation surface in the tubular neighborhood $\mathcal{N}(\infty, 0) = \{\rho e^{i\theta} : \sin \theta \geq 1/\Phi_n\}$ of $(\infty, 0)$, see Figure 6. Then*

$$KV\text{ol}(X) = \text{Vol}(X) \frac{\sin \theta^-}{l(\alpha_m)^2},$$

where $d^- = d(X, \gamma_{0,\infty})$ and $\theta^- = \theta(d^-)$.

Proof of Corollary 3.6. By definition, $\sin(\theta^-) \geq 1/\Phi_n = l(\alpha_m)/l(\alpha_1)$. Thus

$$\frac{\sin \theta^-}{l(\alpha_m)^2} \geq \frac{l(\alpha_m)}{l(\alpha_1)} \frac{1}{l(\alpha_m)^2} \geq \frac{\sin \theta^+}{l(\alpha_1)l(\alpha_m)}$$

establishing Equation (13) when X belongs to $\mathcal{N}(\infty, 0)$. In particular one has $KV\text{ol}(X) = \text{Vol}(X) \frac{\sin \theta^-}{l(\alpha_m)^2}$ as desired. \square

We denote by \mathcal{R}_n the connected component of $\mathcal{D}_n \setminus \mathcal{N}(\infty, 0)$ containing X_n (see Figure 6). It is bounded by the geodesics $(-1, 1)$ and $(\Phi_n/2, \infty)$, and the (non-geodesic) half-line $\{\rho e^{i\theta} : \sin \theta = 1/\Phi_n\}$. One needs to determine $\delta^+(X)$, for $X \in \mathcal{R}_n$.

Lemma 3.7. *For any X in \mathcal{R}_n , the closest (to X) geodesic in \mathcal{Z}^+ is $(-\Phi_n, \infty)$. It is unique except when X lies on the lower boundary of \mathcal{D}_n .*

Proof of Lemma 3.7. Let $\delta_0 = (-1/\Phi_n, \infty)$, and $\delta_i = S^i(\delta_0)$ for $i = 1, \dots, 2m$. The $2m + 1$ geodesics $\delta_0, \dots, \delta_{2m}$ (see Figure 9) make up a regular $2m + 1$ -gon Q_n in \mathbb{H}^2 , with P_n as its center. Note that $(-\Phi_n, 0)$ is δ_{2m} .

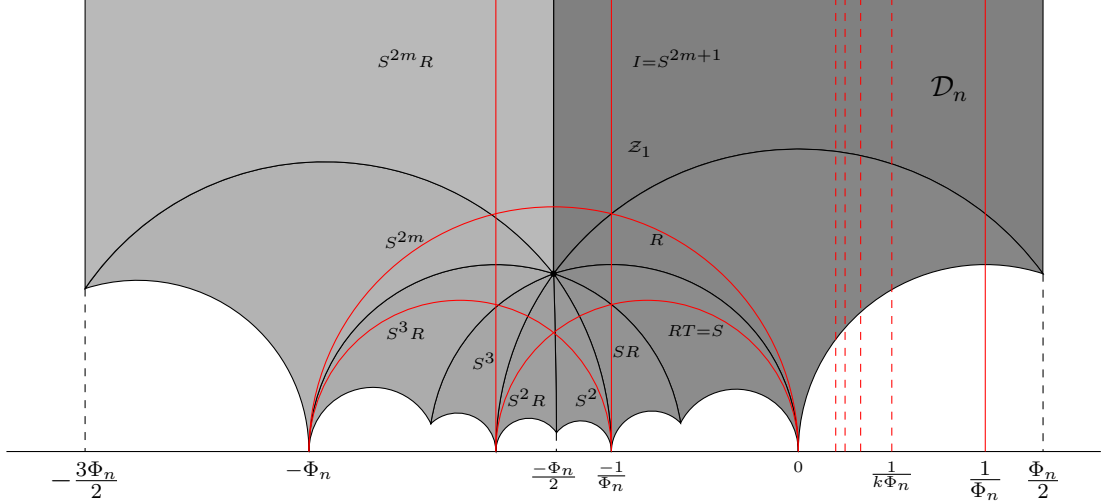


FIGURE 9. The double $2m + 1$ -gon at the center of a hyperbolic regular $2m + 1$ -gon, and the tubular neighborhood of $(0, \infty)$ defined by $\sin \theta \geq 1/\Phi_n$ (bounded by the two green half-lines).

Similarly to the proof of Lemma 3.5, the connected component C of $\mathbb{H}^2 \setminus Q_n$ which contains the region \mathcal{R}_n , does not meet any element of \mathcal{Z}^+ .

So for any $x \in \mathcal{R}_n$, any geodesic with initial point x , must intersect one of $\delta_0, \dots, \delta_{2m}$ before it reaches any other element of \mathcal{Z}^+ . Therefore,

$$\forall x \in \mathcal{R}_n, \forall \delta \in \mathcal{Z}^+, d(x, \delta) \geq \min\{d(x, \delta_i), i = 0, \dots, 2m\}$$

so all we have to do is to prove that

$$\forall x \in \mathcal{R}_n, d(x, \gamma_{2m}) \leq \min\{d(x, \gamma_i), i = 1, \dots, 2m\}.$$

The images, under $S^i, i = 0, \dots, 2m$, of the boundary of \mathcal{D}_n , divide the regular n -gon Q_n into $2n$ isometric triangles. The region \mathcal{R}_n is contained in the upper right triangle, bounded by the geodesics $(-1, 1)$, $(-\Phi_n/2, \infty)$, and $(0, -\Phi_n)$ (see Figure 8). Then the closest red line (i.e. the closest side of the pentagon Q_n) is δ_{2m} . Lemma 3.7 is proved. \square

Corollary 3.8. *For any $X \in \mathcal{R}_n$ one has*

$$KVol(X) = Vol(X) \frac{\sin \theta^-}{l(\alpha_m)^2}.$$

Proof of Corollary 3.8. Pick $X \in \mathcal{R}_n$ and let d^+ be the distance between X and \mathcal{Z}^+ . By Lemma 3.7 $d^+ = d(X, \gamma_{-\Phi_n, 0})$. Setting $\theta^+ = \theta(d^+)$, we only need to establish Inequality (13), which is equivalent to

$$\Phi_n \sin \theta^- \geq \sin \theta^+.$$

To make the situation easier to visualize, we apply the inversion R . The geodesics $(-1, 1)$ and $(0, \infty)$ are left invariant, and so are the two green half-lines in Figure 6.

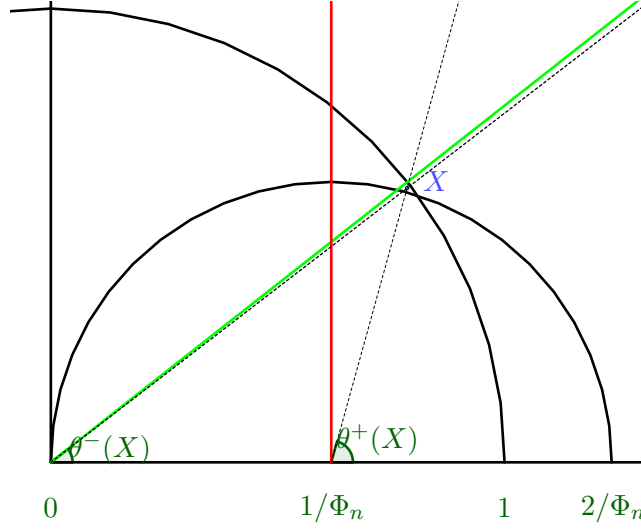


FIGURE 10.

The vertical boundary $(-\Phi_n/2, \infty)$ of the fundamental domain \mathcal{D}_n becomes the half-circle $(0, 2/\Phi_n)$. The geodesic $\delta_{2m} = (0, -\Phi_n)$ becomes the vertical half-line $(1/\Phi_n, \infty)$ (in red in Figures 9 and 10). The angles do not change, so what we want to do is to prove that $\Phi_n \sin \theta^-(X) \geq \sin \theta^+(X)$, with θ^- and θ^+ as in Figure 10.

Let X' be the intersection of the half line from the origin to X , with the half-circle $(0, 2/\Phi_n)$. We have $\theta^-(X') = \theta^-(X)$, and $\theta^+(X') \geq \theta^+(X)$, so $\sin \theta^+(X') \geq \sin \theta^+(X)$. Thus, if we can prove

$$(14) \quad \Phi_n \sin \theta^-(X') \geq \sin \theta^+(X'),$$

we have proven $\Phi_n \sin \theta^-(X) \geq \sin \theta^+(X)$. Now X' lies on the half-circle $(0, 2/\Phi_n)$, so by the angle at apex/angle at center theorem, we have $\theta^+(X') = 2\theta^-(X')$, whence, by taking sines,

$$\sin \theta^+(X') = \sin(2\theta^-(X')) = 2 \sin(\theta^-(X')) \cos(\theta^-(X')),$$

so (14) boils down to $2 \cos(\theta^-(X')) \leq \Phi_n$. The latter inequality is true because X' , which lies in $R(\mathcal{R}_n)$, lies to the left of the half-line $(\Phi_n/2, \infty)$. Note that the inequality is strict, unless $X' = X_n$. Thus Corollary 3.8 is proved. \square

Proof of the lower bound in Theorem 1.1 for odd n . By Corollary 3.6 and Corollary 3.8, the minimum of KVVol in the fundamental domain \mathcal{D}_n is achieved by the point which lies furthest from $(0, \infty)$, which is the double $2m + 1$ -gon X_n . Thus

$$\text{KVVol}(X) = \text{Vol}(X) \frac{\sin \theta^-}{l(\alpha_m)^2},$$

where $\text{Vol}(X) = \frac{2m+1}{2} \cos \frac{\pi}{2m+1}$ (see Remark 2.3) and $l(\alpha_m) = \sin(\pi/(2m+1))$. A quick inspection of Figure 10 shows

$$\sin \theta^- = y \text{ coordinate of } X_m = \frac{1}{2} \sqrt{4 - \Phi_n^2}.$$

Since $\Phi_n = 2 \cos \frac{\pi}{2m+1}$, we get $\sin \theta^- = \sin \frac{\pi}{2m+1}$. Therefore

$$\text{KVVol}(X) = \frac{2m+1}{2} \cot \frac{\pi}{2m+1}$$

which concludes the proof of Theorem 1.1 when n is odd. \square

4. PROOF OF THEOREM 1.1 FOR $n = 4m$

4.1. Positive and negative algebraic intersections. Similarly to the case of the odd double- n -gon, we partition the set of pairs of periodic directions on X in the Teichmüller curve \mathcal{T}_n of the $4m$ -gon, $n = 4m$, into the subset $\mathcal{D}^+(X)$, of pairs of directions (d, d') such that all intersections between saddle connections of d and that of d' have the same sign, and the complement $\mathcal{D}^-(X)$.

For $X \in \mathcal{T}_n$, with $n = 4m$, and $\varepsilon = \pm$, define $d^\varepsilon = d(X, \mathcal{Z}^\varepsilon)$ and $\theta^\varepsilon = \theta(d^\varepsilon)$.

Proposition 4.1. *The set $\mathcal{D}^-(S_n)$ consists of the orbit of $(\infty, 0)$ under the Veech group acting diagonally on $\mathbb{R}P^1 \times \mathbb{R}P^1$. We denote by $\mathcal{Z}^-(S_n) \subset \mathbb{H}^2$ the orbit of the geodesic $(\infty, 0)$ under the Veech group. We have*

$$K^-(X) = \frac{\sin \theta^-}{l(\alpha_1)l(\beta_m)} = \frac{\sin \theta^-}{\sin(\frac{2\pi}{n})\sin(\frac{\pi}{n})} = \frac{\sin \theta^-}{\Phi_n \sin^2(\frac{\pi}{n})}.$$

Proof of Proposition 4.1. It follows the same lines as the proof of Proposition 3.2.

For any $(d, d') \in \mathcal{D}^-(X)$, we have $\text{Int}(\alpha_i(d, X), \alpha_j(d', X)) = \text{Int}(\alpha_i, \beta_j)$, which is given by (5), and is always $-1, 0$, or 1 . By (8),

$$\max_{i,j=1,\dots,m} \frac{1}{l(\alpha_i(d, X))l(\alpha_j(d', X))} = \frac{1}{l(\alpha_1(d, X))l(\beta_m(d', X))}$$

and by (5) again, $\text{Int}(\alpha_1(d, X), \beta_m(d', X)) \neq 0$. Since $l(\alpha_1(d, X))l(\beta_m(d', X)) \sin \theta^- = l(\alpha_1)l(\beta_m)$, where θ^- is the angle between d, d' , we draw

$$K^-(X) = \sup_{(d,d') \in \mathcal{D}^-(X)} \frac{1}{l(\alpha_1(d, X))l(\beta_m(d', X))} = \frac{\sin \theta^-}{l(\alpha_1)l(\beta_m)}.$$

Proposition 4.1 is proved. \square

For the same-signed intersections, we have the following result.

Lemma 4.2. *Let $\theta \in [0, 2\pi[$ be the angle between two periodic directions d and d' on X with $(d, d') \in \mathcal{D}^+(X)$.*

(1) *If d represents the vertical cusp and α is a saddle connection in direction d' then*

$$\frac{\text{Int}(\beta_k(d_v, X), \alpha)}{l(\beta_k(d_v, X))l(\alpha)} \leq \frac{\Phi_n^{-1} \sin \theta}{l(\beta_m)^2} = \frac{\sin \theta}{l(\alpha_1)l(\beta_m)},$$

with equality if and only if $\text{Int}(\beta_i(d_v), \alpha) = 0$ for all $i = 1, \dots, m-1$.

(2) If d represents the horizontal cusp and β is a saddle connection in direction d' then

$$\frac{\text{Int}(\alpha_k(d_h, X), \beta)}{l(\alpha_k(d_h, X))l(\beta)} \leq \frac{\Phi_n^{-1} \sin \theta}{l(\alpha_1)^2} = \frac{\sin \theta}{l(\alpha_1)(l(\beta_m) + l(\beta_{m-1}))},$$

with equality if and only if $\text{Int}(\alpha_i(d_h), \beta) = 0$ for all $i = 2, \dots, m$.

Proof of Lemma 4.2. As in the proof of Lemma 4.3, for the vertical cusp, Equation (9) gives

$$\frac{\text{Int}(\beta_k(d_v, X), \alpha)}{l(\beta_k(d_v, X))l(\alpha)} \leq \frac{\sin \theta}{l(\beta_i)(l(\alpha_{m-i}) + l(\alpha_{m-i+1}))} \leq \frac{\sin \theta}{l(\alpha_1)l(\beta_m)}.$$

The last inequality is obtained by noticing that for every $i = 1, \dots, m$

$$\frac{1}{l(\beta_i)(l(\alpha_{m-i}) + l(\alpha_{m-i+1}))} = \frac{\Phi_n^{-1}}{l(\beta_i)^2}$$

which is maximal for $i = m$ (see Equation (3)).

For the horizontal cusp Equation (10) gives

$$\frac{\text{Int}(\alpha_i(d_h), \beta)}{l(\alpha_i(d_h))l(\beta)} \leq \frac{\sin \theta}{l(\alpha_k)(l(\beta_{m-k}) + l(\beta_{m-k+1}))}$$

From (3) we draw

$$\frac{1}{l(\alpha_i)(l(\beta_{m-i}) + l(\beta_{m-i+1}))} = \begin{cases} \frac{\Phi_n^{-1}}{l(\alpha_i)^2} & \text{for any } i = 1, \dots, m-1 \\ \frac{2\Phi_n^{-1}}{l(\alpha_m)^2} & \text{for } i = m \end{cases}$$

The above quantity is maximal for $i = 1$. \square

Since

$$\Phi_n^{-1} \frac{1}{l(\beta_m)^2} > \Phi_n^{-1} \frac{1}{l(\alpha_1)^2},$$

it follows that for any $X \in \mathcal{T}_n$, $K^+(X)$ is realized by a pair of directions (d, d') , with d representing the vertical cusp, so we may assume $d = 0$, and $\text{Int}(\beta_i(d), \alpha) = 0$ for all $i = 1, \dots, m-1$. We want to determine the corresponding set of directions on S_n . As in the proof of Lemma 3.1, $d' = k\Phi_n$, for some $k \in \mathbb{N}^*$.

Hence we define the set $\mathcal{Z}^+(S_n) \subset \mathbb{H}^2$ as the orbit of the geodesics $(0, k\Phi_n)$, $k \in \mathbb{Z}^*$, under the Veech group. The above discussion shows

Corollary 4.3. For $X \in \mathcal{T}_n$, $n = 4m$, we have

$$K^+(X) = \frac{\sin \theta^+}{l(\alpha_1)l(\beta_m)}.$$

4.2. The maximum.

Proof of the upper bound in Theorem 1.1 for $n = 4m$. Let $X \in \mathcal{T}_n^O$ be a maximum for KVOL and let (d, d') be two directions giving the minimizing pair of saddle connection. By Proposition 4.1, $K^-(X) = \frac{\sin \theta^-}{l(\alpha_1)l(\beta_m)}$ is maximal when the angle between d and d' is $\theta^- = \pi/2$. Thus by Lemma 2.2, $X = M.X_m$ where M projects to $\mathcal{Z}^- \subset \mathbb{H}^2$, the orbit of the geodesic $(\infty, 0)$ by the Veech group.

On the other hand Proposition 4.2 implies that $K^+(X) = \frac{\sin \theta^+}{l(\alpha_1)l(\alpha_m)}$ with $\sin \theta^+ \neq 1$ (since the corresponding pair of directions belongs to \mathcal{D}^+). We draw that the maximum of KVol is achieved at \mathcal{Z}^- and for those surfaces one has

$$\text{KVol}(X) = \frac{n}{4} \cos \frac{\pi}{n} \frac{1}{n \sin \frac{2\pi}{n} \sin \frac{\pi}{n}} = \frac{n}{8} \frac{1}{\sin^2 \frac{\pi}{n}}$$

recalling that the area of X is $\frac{n}{4} \cos \frac{\pi}{n}$ (see Remark 2.3). \square

4.3. The minimum.

Proof of the lower bound in Theorem 1.1 for $n = 4m$. Since $K^\pm = \frac{\sin \theta^\pm}{l(\alpha_1)l(\beta_m)}$, the minimum of KVol in the fundamental domain \mathcal{D}_n is achieved by the point which lies furthest from \mathcal{Z}^\pm . Similarly to Lemma 3.5, we immediately see that this is X_n . The geodesics in $\mathcal{Z}^+(S_n)$ which come closest to X_n are $(0, \Phi_n)$ and its image under R , $(\infty, 1/\Phi_n)$. The geodesic in $\mathcal{Z}^-(S_n)$ which comes closest to X_n is $(0, \infty)$. The corresponding angles are $\theta^+ = 2\theta^- = \frac{2\pi}{n}$ (see Figure 10). Hence $\text{KVol}(X_n) = \text{Vol}(X_n) \frac{\sin \theta^+}{l(\alpha_1)l(\beta_m)}$ and the minimum of KVol is

$$\frac{n}{4} \cos \frac{\pi}{n} \cdot \frac{\sin \frac{2\pi}{n}}{l(\alpha_1)l(\beta_m)} = \frac{n}{4} \cot \frac{\pi}{n}.$$

The proof of Theorem 1.1 is now complete. \square

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