Bubble and Profile Decompositions Methods: a common Thread through many Works in Geometry, Physics and Fluid Mechanics

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BUBBLE AND PROFILE DECOMPOSITIONS METHODS : A COMMON THREAD THROUGH MANY WORKS IN GEOMETRY, PHYSICS AND FLUID MECHANICS

HAJER BAHOURI, ZEYU LYU, AND FRANÇOIS VIGNERON

Abstract. Bubble and profile decompositions methods are a powerful tool, which is deeply connected with compactness issues in function spaces. The spectrum of their applications is quite broad and encompass in particular questions in geometry, mathematical physics and fluid mechanics. In this text, we survey the fundamental principles of this theory and showcase its various applications by presenting a few landmark articles.

Contents

1. Introduction

1.1. **Brief history of the bubble and profile decompositions.** Bubble decompositions first appeared in the eighties, in the studies by Brézis-Coron in [21] and Struwe in [90], in the context of geometric problems. A decade later, they were also used in the framework of evolution equations (under the name of profile decompositions) in the works of Bahouri-Gérard [11] and Merle-Vega [76]. These later works are in the same vein as the result of P. Gérard [49] that characterized (by means of profiles) the lack of compactness in the critical Sobolev embedding[1]:

$$\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \quad \text{with} \quad 0 < s < d/2 \quad \text{and} \quad p = 2d/(d-2s)\cdot \tag{1.1}$$

The result of P. Gérard remains a cornerstone of the applications of profile decompositions to evolution equations. It states that a sequence $(u_n)_{n\geq 0}$ bounded in $\dot{H}^s(\mathbb{R}^d)$ can be

2000 *Mathematics Subject Classification.* 35R03, 35Q40.
Key words and phrases. Profile decompositions, Partial differential equations.

[1] This embedding can be established using several different methods; see *e.g.* [20, § IX.3] or [2]. Among others, one can mention the proof of Chemin-Xu [29].

decomposed, up to the extraction of a subsequence, in the following way:

$$u_n = \sum_{\ell=1}^{L} h_{\ell,n}^{-d/p} \phi^\ell\Big(\frac{\cdot - x_{\ell,n}}{h_{\ell,n}}\Big) + r_{n,L} \tag{1.2}$$

where $(\phi^\ell)_{\ell\geq 1}$ is a family of functions (called profiles) in $\dot{H}^s(\mathbb{R}^d)$, $(\underline{x}^\ell)_{\ell\geq 1}$ is a family of cores[2] and $(\underline{h}^\ell)_{\ell\geq 1}$ is a family of scales. The remainders satisfy

$$\lim_{L\to+\infty} \big(\limsup_{n\to+\infty} \|r_{n,L}\|_{L^p(\mathbb{R}^d)}\big) = 0.$$

The profiles are "asymptotically orthogonal" in the sense that for $\ell \neq \ell'$

$$|\log(h_{\ell,n}/h_{\ell',n})| \to +\infty \ \text{ or } h_{\ell,n} = h_{\ell',n} \text{ and } |x_{\ell,n} - x_{\ell',n}|/h_{\ell,n} \to +\infty, \text{ as } n \to +\infty. \tag{1.3}$$

The following energy balance expresses this asymptotic orthogonality:

$$\|u_n\|^2_{\dot{H}^s(\mathbb{R}^d)} = \sum_{\ell=1}^{L} \|\phi^\ell\|^2_{\dot{H}^s(\mathbb{R}^d)} + \|r_{n,L}\|^2_{\dot{H}^s(\mathbb{R}^d)} + \circ(1), \quad n \to \infty. \tag{1.4}$$

Since then many developments based on profile decompositions have been achieved by several authors. Some broaden the *functional framework of Sobolev embeddings*: see for instance the article of Jaffard [55] concerning Riesz potential spaces; the articles of Bahouri-Majdoub-Masmoudi [12, 13] and Bahouri-Perelman [14] about Orlicz spaces[3]; the works of Adimurthi-Tintarev [4], Bahouri-Cohen-Koch [9], Fieseler-Tintarev [39], Solimini [88] and Schindler-Tintarev [86] devoted to abstract functional frameworks including Sobolev, Besov, Triebel-Lizorkin, Lorentz, Hölder, BMO and BV spaces; the profile decompositions derived in anisotropic settings by Bahouri-Gallagher [10] and Bahouri-Chemin-Gallagher [8]; the result of Ben Ameur [17] describing the lack of compactness of the Sobolev embedding in Lebesgue spaces for the Heisenberg group and the articles of Christ-Shao [30], Frank-Lieb-Sabin [41] and Shao [87] concerning profile decompositions by means of Knapp examples, which are linked to the Fourier restriction inequalities initiated by Tomas-Stein [89, 94].

Other works investigate *nonlinear PDEs* that arise in geometry, physics and fluid mechanics. It is actually not possible to mention all the remarkable and interesting applications of profile decompositions in the study of nonlinear PDEs. Instead, we will limit ourselves to presenting a few of them in more details. We will pay a special attention to the founding article of Brézis-Coron [21], to the profile decomposition of P. Gérard [49], to the article of Gallagher [43] where the method of profile decompositions was applied in fluid mechanics for the first time and, finally, to the article of Hmidi-Keraani [53] that initiated the use of profiles in the study of the formation of singularities of solutions of L^2-critical nonlinear Schrödinger equations[4].

Even though we will not be able to review it thoroughly, we would also like to underline, in this introduction, the role of the outstanding article of Kenig-Merle [61] as it marked a great turning point in this theory and inspired many other results in the study of nonlinear evolution PDEs, either for global well-posedness problems or to solve various issues related to questions of blow-up, scattering or the soliton resolution conjecture,...

Overall, the profile decompositions are *a very versatile tool* that can serve many different purposes, ranging from the proof of the existence of extremals of some functionals or the computation of Sobolev constants to the construction of traveling waves for nonlinear evolution PDEs (such as for instance in the article of Gassot [47], where the author

[2] We follow here the terminology of P. Gérard [49] and call a *core* any real sequence $\underline{x}^\ell = (x_{\ell,n})_{n\in\mathbb{N}}$ of points in $\mathbb{R}^d$. Similarly, by *scale* $\underline{h}^\ell$, we denote any sequence $(h_{\ell,n})_{n\in\mathbb{N}}$ of positive real numbers.

[3] For an introduction to Orlicz spaces, we refer the reader to the monograph of Rao-Ren [83].

[4] Note that in the radial framework, Merle-Tsutsumi [75] obtained before the same result by another approach.

takes advantage of the profile decomposition of Ben Ameur [17] to construct families of traveling waves for the cubic Schrödinger equation on the Heisenberg group) or the determination of the quantized levels of energy where sequences of solutions of some elliptic PDEs concentrate at infinity, etc. Among others, one can mention the articles of Adimurthi-Druet [3], Druet-Hebey-Robert [37], Fieseler-Tintarev [39], Hutchings-Morgan-Ritoré-Ros [54], Struwe [90] and the references therein.

Let us point out, and we will be discussed it further below, that the profile decompositions that are fit for *geometric* problems (actually rather called bubbles in this context) present some major differences with those involved in the study of evolution PDEs. In geometric questions, the decompositions of type (1.2) usually only include a finite number of bubbles, *i.e.* a finite number of profiles ϕ^ℓ. In most cases, these bubbles are explicitly known; they solve an elliptic PDE and, actually, are the translated and rescaled version of some fundamental function. In such a case, all the bubbles have the same energy; thanks to the orthogonality in the balance of energy (1.4), this ensures for example that only a finite number may arise in the decomposition and that the energy will concentrate at well-defined quantized levels. Such a geometric rigidity is quite exceptional in the general framework of PDEs.

The study of the *lack of compactness* in critical Sobolev embedding in Lebesgue spaces as well as in Orlicz spaces was initiated by P.-L. Lions [72, 73] in the eighties by means of defect measures[5]. In these pioneering works, the author highlighted that the Sobolev embeddings are not compact for two reasons. The first reason is the lack of compactness at infinity: a typical example is given by a traveling bump sequence $(u_n)_{n\in\mathbb{N}}$ defined by

$$u_n = \tau_{x_n}\varphi = \varphi(\cdot - x_n) \quad \text{with} \quad \varphi \in \mathcal{D}\setminus\{0\} \quad \text{and} \quad \|x_n\| \to \infty\,. \tag{1.5}$$

The second reason is related to concentration phenomena. In the case of homogeneous spaces like *e.g.* for the embedding $\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, it is illustrated by a sequence of rescaled functions of the following type

$$u_n = \delta_{h_n} u = \frac{1}{h_n^{d/p}}\varphi\left(\frac{\cdot}{h_n}\right), \tag{1.6}$$

where $\varphi \in \mathcal{D}\setminus\{0\}$ is a bump function and $(h_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers tending either to 0 or to infinity. In the case of inhomogeneous spaces, *e.g.* in the Sobolev embedding into the Orlicz space

$$H^1(\mathbb{R}^2) \hookrightarrow \mathcal{L}(\mathbb{R}^2) \tag{1.7}$$

where

$$\|u\|_{\mathcal{L}(\mathbb{R}^2)} = \inf\left\{\lambda > 0, \quad \int_{\mathbb{R}^2}\left(\mathrm{e}^{\frac{|u(x)|^2}{\lambda^2}} - 1\right)dx \le 1\right\},$$

the lack of compactness by concentration is an inhomogeneous phenomenon, which is illustrated by Moser's sequence $(f_{\alpha_n})_{n\in\mathbb{N}}$ defined by:

$$f_{\alpha_n}(x) \;=\; \begin{cases} 0 & \text{if} \quad |x| \ge 1, \\ -\frac{\log|x|}{\sqrt{2\alpha_n\pi}} & \text{if} \quad \mathrm{e}^{-\alpha_n} \le |x| \le 1, \\ \sqrt{\frac{\alpha_n}{2\pi}} & \text{if} \quad |x| \le \mathrm{e}^{-\alpha_n}, \end{cases} \tag{1.8}$$

[5] The study of P.-L. Lions [72, 73] had a lot of impact and thereafter microlocal tools called H-measures or microlocal defect measures were introduced by Tartar [93] and P. Gérard [48] to investigate related problems. See also Murat-Tartar [79].

where $(\alpha_n)_{n\in\mathbb{N}}$ denotes any sequence of positive real numbers tending to infinity. Observe that the above sequence $(f_{\alpha_n})_{n\in\mathbb{N}}$ also reads as follows:

$$f_{\alpha_n}(x) = \sqrt{\frac{\alpha_n}{2\pi}}\,\mathbf{L}\left(\frac{-\log|x|}{\alpha_n}\right) \quad \text{with} \quad \mathbf{L}(s) \;=\; \begin{cases} 0 & \text{if} \quad s \leq 0 \\ s & \text{if} \quad 0 \leq s \leq 1 \\ 1 & \text{if} \quad s \geq 1\,. \end{cases}$$

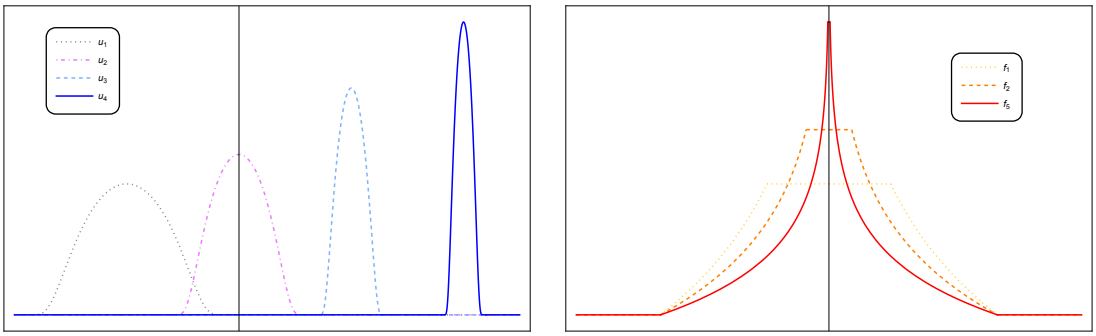

FIGURE 1. *Left: a typical sequence (u_n) illustrating the default of compactness due to the simultaneous effect of translations (1.5) and concentration (1.6). Right: a few terms of Moser's (f_α) sequence (1.8).*

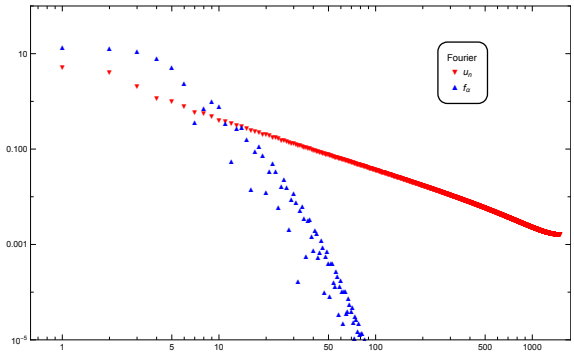

FIGURE 2. *Typical frequency profile (modulus of the Fourier transform against the frequency) of a function u_n and f_α from Figure 1. An FFT was performed on a discretized version of each function. The units are arbitrary and in log-log scale.*

The Sobolev embedding (1.7) follows from the well-known Trudinger-Moser inequality due to Ruf [84] (see also [1]):

$$\sup_{\|u\|_{H^1}\leq 1} \int_{\mathbb{R}^2} \left(\mathrm{e}^{4\pi|u|^2} - 1\right) dx < \infty\,.$$

It was characterized by means of profile decomposition in the spirit of (1.2) by Bahouri-Majdoub-Masmoudi [12, 13] and Bahouri-Perelman [14] for higher dimensions. The profiles involved in this setting are of the same type as the example by Moser and take the following form:

$$g_{\alpha_n}(x) \overset{\text{def}}{=} \sqrt{\frac{\alpha_n}{2\pi}}\,\psi\left(\frac{-\log|x-x_n|}{\alpha_n}\right),$$

where the fundamental profile ψ is defined on $\mathbb{R}$ and satisfies $\psi_{|]-\infty,0]} = 0$ and $\psi' \in L^2(\mathbb{R})$, where $\alpha_n \to \infty$ and where, of course, the remainder term tends to 0 in $\mathcal{L}(\mathbb{R}^2)$. Observe that the sequences $(f_{\alpha_n})_{n\in\mathbb{N}}$ and $(g_{\alpha_n})_{n\in\mathbb{N}}$ are strikingly different from the scaling

sequences (1.6) arising in (1.2) to describe the lack of compactness of the homogeneous embedding (1.1). Note however that in the two cases, the obstruction to compactness is expressed in the same manner in terms of defect measures, according to [72, 73]. Actually, contrary to the profiles $h_{\ell,n}^{-d/p}\phi^\ell\Big(\frac{\cdot - x_{\ell,n}}{h_{\ell,n}}\Big)$, which turn out to be asymptotically spectrally localized in rings, the sequence f_{α_n} remains spectrally spread out (see Figure 2). This example reveals the importance of *microlocal analysis* in the choice of a pertinent profile decomposition. In what follows, we will not pursue the analysis of this subtle phenomenon and refer the interested reader to [6].

In the rest of this text, we will focus on the *homogeneous* profile decomposition (1.2) because it can be generalized in many settings, including for instance the framework of critical embeddings Besov spaces[6], which arises naturally in the study of evolution equations. Note however that the proof of P. Gérard [49] that we will sketch below is mainly based on Fourier analysis. The description of the defect of compactness among more general function spaces requires other tools such as for instance wavelet decompositions (see for instance the book of Meyer [78] and the references therein for an introduction to wavelet theory) and ideas from the nonlinear approximation theory, which has been studied *e.g.* by DeVore [35]. For further details, one can consult [9, 55].

1.2. **Layout.** The sketch of proof of the profile decomposition of P. Gérard [49] is presented in Section 2. An introduction to the article of Brézis-Coron [21] is provided in Section 3 with a brief overview on the efficiency of bubble decompositions for geometric questions. The last section is dedicated to presenting targeted applications of profile decompositions in the analysis of nonlinear evolution PDEs. As mentioned above, despite the vast literature on the subject[7], we shall limit ourselves to a presentation of some ideas from the articles of Gallagher [43] and Hmidi-Keraani [53] with a few comments about related issues.

1.3. **Notations.** To avoid heaviness, C will denote a positive constant which may vary from line to line; we also use $A \lesssim B$ to denote an estimate of the form $A \leq CB$. For simplicity, we shall also still denote by (u_n) any subsequence of a sequence (u_n) and designate by $\circ(1)$ any sequence which tends to 0 as n goes to infinity.

2. Profile decompositions for critical Sobolev embeddings

In this section, we focus on the description of the lack of compactness of the critical embedding (1.1) of homogeneous Sobolev spaces into Lebesgue spaces. As mentioned above, such a description, initially due to P. Gérard [49], was generalized to many other pairs of function spaces including, for instance, Besov spaces $\dot{B}^s_{r,q}(\mathbb{R}^d)$, which come to play in several results related to fluid mechanics and mathematical physics. For instance, if we consider the critical Sobolev embedding $X \hookrightarrow Y$ where $X = \dot{B}^s_{r,q}(\mathbb{R}^d)$, $0 < s < d/r$, and where $Y = L^p(\mathbb{R}^d)$ with $p = rd/(d-rs)$ or $Y = \dot{B}^t_{a,b}(\mathbb{R}^d)$, $0 < t < d/a$, with $t < s$ and $d/r - s = d/a - t$, then any bounded sequence $(u_n)_{n\in\mathbb{N}}$ in X admits, up to the extraction of a subsequence, an asymptotically orthogonal decomposition of the type (1.2) where the remainder term $r_{n,L}$ satisfies

$$\lim_{L\to+\infty}\big(\limsup_{n\to+\infty}\|r_{n,L}\|_Y\big) = 0.$$

Even though all such profile decompositions start with a diagonal extraction procedure, the original strategy adopted by P. Gérard in [49], which uses Fourier analysis in a crucial

[6] For an introduction to Besov spaces, one can for instance consult the monographs [7, 95] and the references therein.

[7] Among others, one can mention [11, 36, 38, 44, 45, 56, 58, 61, 62, 63, 66, 69, 76, 80, 82, 92] and the references therein.

way, cannot be generalized mutatis mutandis to Besov spaces $\dot{B}^s_{r,q}(\mathbb{R}^d)$ when $(r,q)\neq(2,2)$. Recall that $\dot{B}^s_{2,2}(\mathbb{R}^d)$ coincides with the Sobolev space $\dot{H}^s(\mathbb{R}^d)$. The characterization of critical Sobolev embeddings for general Besov spaces, as well as for Triebel-Lizorkin, Lorentz or Hölder spaces relies on wavelet decompositions and on nonlinear approximation theories (see Appendix A for a succinct introduction to this topic). In this section, we will not seek the maximal generality possible and we invite the interested reader to explore [8, 9, 35, 55, 64, 78] and the references therein. Neither will we discuss the results of inhomogeneous type that where mentioned briefly in the introduction and that revolve around Moser and Knapp examples; see for instance [12, 13, 30, 41].

2.1. **Description of the default of compactness in critical homogeneous Sobolev embeddings.** The result of P. Gérard [49] concerning the critical Sobolev embedding (1.1) can be stated precisely as follows:

Theorem 1 ([49]). *Let $0<s<d/2$ and $p=2d/(d-2s)$ and consider $(u_n)_{n\in\mathbb{N}}$ a sequence of functions that is bounded in $\dot{H}^s(\mathbb{R}^d)$. Then up to extracting a subsequence (which we denote in the same way), there is a family of functions $(\phi^\ell)_{\ell\geq1}$ in $\dot{H}^s(\mathbb{R}^d)$, and a family of cores $(\underline{x}^\ell)_{\ell\geq1}$, as well as a family of sequences of positive real numbers $((h_{\ell,n})_{n\in\mathbb{N}})_{\ell\geq1}$, orthogonal in the sense of (1.3), such that for all integers $L\geq1$*

$$u_n=\sum_{\ell=1}^{L}h_{\ell,n}^{-d/p}\phi^\ell\Big(\frac{\cdot-x_{\ell,n}}{h_{\ell,n}}\Big)+r_{n,L}\quad\text{with}\quad\lim_{L\to+\infty}\big(\limsup_{n\to+\infty}\|r_{n,L}\|_{L^p(\mathbb{R}^d)}\big)=0. \tag{2.1}$$

In addition, one has the following orthogonality equality

$$\|u_n\|^2_{\dot{H}^s(\mathbb{R}^d)}=\sum_{\ell=1}^{L}\|\phi^\ell\|^2_{\dot{H}^s(\mathbb{R}^d)}+\|r_{n,L}\|^2_{\dot{H}^s(\mathbb{R}^d)}+\circ(1),\quad n\to\infty. \tag{2.2}$$

This result highlights the fact that translational and scaling invariance are the only features responsible for the lack of compactness in the homogeneous Sobolev embedding (1.1). Note that the decomposition (2.1) is not unique and can either be finite or infinite.

Remark 2.1. *Theorem 1 calls for some natural complementary statements.*

- *The scale invariance of the statement can be checked by straightforward computations of the norms:*

$$\|h_{\ell,n}^{-d/p}\phi^\ell\Big(\frac{\cdot-x_{\ell,n}}{h_{\ell,n}}\Big)\|_{\dot{H}^s}=\|\phi^\ell\|_{\dot{H}^s}\quad\text{and}\quad\|h_{\ell,n}^{-d/p}\phi^\ell\Big(\frac{\cdot-x_{\ell,n}}{h_{\ell,n}}\Big)\|_{L^p}=\|\phi^\ell\|_{L^p}.$$

- *The orthogonality condition (1.3) ensures both that the interaction between any pair of different elements in the profile decomposition (2.1) is asymptotically negligible in $\dot{H}^s(\mathbb{R}^d)$ as n goes to infinity and that the profiles ϕ^ℓ satisfy :*

$$\lim_{\ell\to\infty}\|\phi^\ell\|_{\dot{H}^s(\mathbb{R}^d)}=0. \tag{2.3}$$

  As we shall see in Section 4, such properties play a crucial role in the applications.
- *The profiles ϕ^ℓ can be obtained in the following way:*

$$h_{\ell,n}^{d/p}u_n(h_{\ell,n}(\cdot+x_{\ell,n}))\rightharpoonup\phi^\ell\,,\quad\text{as } n\to\infty\,. \tag{2.4}$$

  i.e. as weak limits of translations and scaling transforms of the sequence $(u_n)_{n\in\mathbb{N}}$.

Theorem 1 allows, for example, to recover the fact that the best Sobolev constant

$$C=\inf\Big\{\|u\|_{\dot{H}^s(\mathbb{R}^d)}\,;\,u\in\dot{H}^s(\mathbb{R}^d)\text{ and }\|u\|_{L^p(\mathbb{R}^d)}=1\Big\}$$

is achieved as well as on the structure of the corresponding minimizing sequences. Indeed, taking advantage of the elementary inequality

$$\Big|\big|\sum_{\ell=1}^{L}\alpha_\ell\big|^p-\sum_{\ell=1}^{L}|\alpha_\ell|^p\Big|\le C_L\sum_{\ell\neq k}|\alpha_\ell||\alpha_k|^{p-1}\,, \tag{2.5}$$

one may deduce, under the assumptions of Theorem 1, that

$$\|u_n\|^p_{L^p(\mathbb{R}^d)}\overset{n\to\infty}{\longrightarrow}\sum_{\ell=1}^{\infty}\|\phi^\ell\|^p_{L^p(\mathbb{R}^d)}\,.$$

The energy balance (2.2) obviously implies also that

$$\limsup_{n\to+\infty}\|u_n\|^2_{\dot H^s(\mathbb{R}^d)}\ge\sum_{\ell=1}^{\infty}\|\phi^\ell\|^2_{\dot H^s(\mathbb{R}^d)}\,.$$

This ensures easily the fact that C is achieved. For further details, one can consult the article of P.-L. Lions [73].

2.2. **Sketch of proof of the profile decomposition by P. Gérard.** The proof of Theorem 1 uses Fourier analysis in a crucial way. It comports three main steps:

(i). The first step is devoted to the extraction of the scales $(\underline{h}^\ell)_{\ell\ge1}$ and consists (up to the extraction of subsequences) in spliting u_n as follows:

$$u_n=\sum_{\ell=1}^{L}u_n^\ell+r_n^L\,, \tag{2.6}$$

where, in the phrasing of P. Gérard in [49], each component $(|D|^s u_n^\ell)_{n\in\mathbb{N}}$ is $(h_{\ell,n})$-oscillating, *i.e.* is asymptotically spectrally localized in a ring of size $1/h_{\ell,n}$, with orthogonality between the scales in the sense that for $\ell\neq\ell'$

$$|\log(h_{\ell,n}/h_{\ell',n})|\overset{n\to\infty}{\longrightarrow}+\infty.$$

The purpose at this stage is to ensure that the remainder term r_n^L satisfies

$$\limsup_{n\to\infty}\|r_n^L\|_{\dot B^s_{2,\infty}(\mathbb{R}^d)}\to0,\quad\text{as } L\to\infty\,.$$

This property allows one to get rid of the remainder r_n^L thanks to the refined Sobolev inequality due to Gérard-Meyer-Oru [50] (see also [7] for another proof):

$$\|f\|_{L^p(\mathbb{R}^d)}\lesssim\|f\|^{2/p}_{\dot H^s(\mathbb{R}^d)}\|f\|^{1-2/p}_{\dot B^s_{2,\infty}(\mathbb{R}^d)}\,.$$

To be more precise, let us start by introducing the concept of a sequence, which is oscillatory with respect to a scale $(h_n)_{n\in\mathbb{N}}$ and the concept of being unrelated to a scale.

Definition 2.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of functions of $L^2(\mathbb{R}^d)$. The sequence $(f_n)_{n\in\mathbb{N}}$ is said (h_n)-oscillating if*

$$\limsup_{n\to\infty}\Big(\int_{h_n|\xi|\le\frac{1}{R}}|\widehat{f_n}(\xi)|^2d\xi+\int_{h_n|\xi|\ge R}|\widehat{f_n}(\xi)|^2d\xi\Big)\overset{R\to\infty}{\longrightarrow}0.$$

The sequence $(f_n)_{n\in\mathbb{N}}$ is said unrelated to the scale (h_n) if, for all $0<a<b$

$$\lim_{n\to\infty}\int_{a\le h_n|\xi|\le b}|\widehat{f_n}(\xi)|^2d\xi=0.$$

Remark 2.3. *Observe that*

- *With the notations of Theorem 1, the sequences*[8] $\Big(h_{\ell,n}^{-(s+d/p)}(|D|^s\phi^\ell)\Big(\frac{\cdot-x_{\ell,n}}{h_{\ell,n}}\Big)\Big)_{n\in\mathbb{N}}$ *are* $(h_{\ell,n})$*-oscillating.*
- *One can show that an* L^2*-bounded sequence* $(f_n)_{n\in\mathbb{N}}$ *is unrelated to any scale if and only if* $\|f_n\|_{\dot{B}^0_{2,\infty}} \overset{n\to\infty}{\longrightarrow} 0$ *(see [49] for a detailed proof).*
- *One can easily check that if* $(f_n)_{n\in\mathbb{N}}$ *is* (h_n)*-oscillating and* $(g_n)_{n\in\mathbb{N}}$ *is* $(\widetilde{h}_n)$*-oscillating with*
$$|\log(h_n/\widetilde{h}_n)| \overset{n\to\infty}{\longrightarrow} +\infty,$$
then
$$\|f_n+g_n\|^2_{L^2(\mathbb{R}^d)} = \|f_n\|^2_{L^2(\mathbb{R}^d)} + \|g_n\|^2_{L^2(\mathbb{R}^d)} + \circ(1), \quad n\to\infty.$$

The key argument in the proof of Decomposition (2.6) consists to apply (recursively) the following proposition to the L^2-bounded sequence $(|D|^s u_n)_{n\in\mathbb{N}}$.

Proposition 2.4. *Let* $(f_n)_{n\in\mathbb{N}}$ *be a bounded sequence of functions of* $L^2(\mathbb{R}^d)$ *and* (h_n) *a scale. Then up to extracting a subsequence (which we denote in the same way), there exists an* L^2*-bounded sequence* $(g_n)_{n\in\mathbb{N}}$ *such that*

- *The sequence* $(g_n)_{n\in\mathbb{N}}$ *is* (h_n)*-oscillating.*
- *The sequence* $(f_n-g_n)_{n\in\mathbb{N}}$ *is unrelated to* (h_n)*.*

Proof. Consider the sequence of functions $(L_n)_{n\in\mathbb{N}}$ defined by
$$L_n\left\{\begin{array}{ccc}]1,\infty[& \longrightarrow & [0,\infty[\\ R & \longmapsto & \displaystyle\int_{\frac{1}{R}\le h_n|\xi|\le R} |\widehat{f}_n(\xi)|^2 d\xi\,. \end{array}\right.$$
For any integer n, it is straightforward that the function L_n is a non decreasing bounded function. Helly's lemma (see for instance [40] or Appendix B in this paper) implies that there exists an increasing sequence of positive integers $\phi(n)$ such that
$$\forall R>1, \quad L_{\phi(n)}(R) \overset{n\to\infty}{\longrightarrow} L(R)\,,$$
and
$$\forall n\in\mathbb{N}^*, \quad L_{\phi(n)}(n) - \frac{1}{n} \le L_{\phi(n)}(n) \le L_{\phi(n)}(n) + \frac{1}{n}\,.$$
Since obviously the limit L is also a non decreasing bounded function, we infer that there exists a positive real number ℓ such that $L(R) \overset{R\to\infty}{\longrightarrow} \ell$, and thus $L(n) \overset{n\to\infty}{\longrightarrow} \ell$. Now for any integer n, define g_n so that
$$\widehat{g}_n(\xi) = \widehat{f}_{\phi(n)}(\xi)\mathbf{1}_{\frac{\mathbf{1}}{\mathbf{n}}\le \mathbf{h}_{\phi(\mathbf{n})}|\xi|\le \mathbf{n}}\,.$$
Obviously $(g_n)_{n\in\mathbb{N}}$ is a bounded sequence of functions of $L^2(\mathbb{R}^d)$ which satisfies for $n\ge R\ge 1$
$$\begin{aligned}\int_{h_{\phi(n)}|\xi|\le\frac{1}{R}} |\widehat{g}_n(\xi)|^2 d\xi + \int_{h_{\phi(n)}|\xi|\ge R} |\widehat{g}_n(\xi)|^2 d\xi \\ = L_{\phi(n)}(n) - L_{\phi(n)}(R) \overset{n\to\infty}{\longrightarrow} \ell - L_{\phi(n)}(R)\,,\end{aligned}$$
which according to Definition 2.2 ensures that the sequence $(g_n)_{n\in\mathbb{N}}$ is (h_n)-oscillating. Finally, since for all $b>a>0$ and n sufficiently large, we have
$$\widehat{g}_n(\xi) = \widehat{f}_{\phi(n)}(\xi)\,, \text{ for } a\le h_{\phi(n)}|\xi|\le b\,,$$
we deduce that (up to a subsequence) the sequence $(f_n-g_n)_{n\in\mathbb{N}}$ is unrelated to (h_n). This completes the proof of the proposition. □

[8] which, according to the fact that $0<s<d/2$ and $p=2d/(d-2s)$, are bounded sequence of functions of $L^2(\mathbb{R}^d)$.

By iterating the above extraction and taking advantage of the exhaustion method of Metivier-Schochet [77] on can complete the first step.

(ii). The second step is devoted to the extraction of the profiles and the cores in each $(h_{\ell,n})$-oscillating component. First, scale invariance allows for a reduction of the study to the case of a **1**-oscillating sequence $(w_n)_{n\in\mathbb{N}}$, *i.e.* a sequence for which $h_{\ell,n}\equiv 1$. Next, one extracts a non-trivial weak limit in $\dot{H}^s(\mathbb{R}^d)$, among all possible translations:

$$w_n(\cdot + x_n^1) \overset{n\to\infty}{\rightharpoonup} \varphi^1\,.$$

Then, one considers the sequence $(r_n^1)_{n\in\mathbb{N}}$ defined by

$$r_n^1 = w_n - \varphi^1(\cdot - x_n^1)\,,$$

which is obviously **1**-oscillating and satisfies $r_n^1(\cdot + x_n^1) \overset{n\to\infty}{\rightharpoonup} 0$. This readily implies that

$$\|w_n\|^2_{\dot{H}^s(\mathbb{R}^d)} = \|\varphi^1\|^2_{\dot{H}^s(\mathbb{R}^d)} + \|r_n^1\|^2_{\dot{H}^s(\mathbb{R}^d)} + \circ(1), \quad n\to\infty\,.$$

Arguing similarly, one extracts a weak limit of $(r_n^1)_{n\in\mathbb{N}}$ among its translations, namely

$$r_n^1(\cdot + x_n^2) \overset{n\to\infty}{\rightharpoonup} \varphi^2 \quad \text{with} \quad |x_n^1 - x_n^2| \overset{n\to\infty}{\to} \infty\,,$$

which now ensures that

$$w_n = \varphi^1(\cdot - x_n^1) + \varphi^2(\cdot - x_n^2) + r_n^2$$

and with

$$\|w_n\|^2_{\dot{H}^s(\mathbb{R}^d)} = \|\varphi^1\|^2_{\dot{H}^s(\mathbb{R}^d)} + \|\varphi^2\|^2_{\dot{H}^s(\mathbb{R}^d)} + \|r_n^2\|^2_{\dot{H}^s(\mathbb{R}^d)} + \circ(1), \quad n\to\infty\,.$$

As long as all the $\dot{H}^s(\mathbb{R}^d)$ energy of the sequence is not spent, one can reiterate this construction and define the next cores and profiles.

(iii). The last step of the proof consists in showing that the process is indeed convergent, which we will not detail here.

3. Some insight on profiles in geometric issues

This section is primarily devoted to the introduction of the result of Brézis-Coron [21] where a profile (bubble) decomposition of sequences of solutions to an elliptic system was derived for the first time. This result was the starting point of a very active field of research tackling both geometric and variational problems. Another line of research consists in constructing solutions to nonlinear elliptic PDEs with a prescribed number of bubbles. However, we will not go further into all those interesting aspects that are at the crossroad of PDEs and Geometry. For recent advances, see *e.g.* [54], [74].

The founding result of Brézis-Coron [21] concerns the problem of finding surfaces $\Sigma\subset\mathbb{R}^3$ of constant mean curvatures spanned by a Jordan curve Γ. For that purpose, they consider surfaces parametrized by the unit disc D of $\mathbb{R}^2$, *i.e.* $\Sigma = u(\overline{D})$ where $u:\overline{D}\to\mathbb{R}^3$ solves the following system:

$$\begin{cases} \Delta u = 2u_x\wedge u_y & \text{on } D\\ |u_x|^2 - |u_y|^2 = u_x\cdot u_y = 0 & \text{on } D\\ u(\partial D) = \Gamma. \end{cases} \tag{3.1}$$

The vectors (u_x, u_y) constitute an orthogonal moving frame along Σ (see Figure 3). The first equation expresses the constancy of the mean curvature of Σ. The last one is the geometrical anchor that constraints the restriction of u to the circle ∂D to be a parametrization of Γ. This problem has a deep connection with physical models of soap bubbles leaning on a wire Γ, hence the name "bubbles" for the profiles exhibited in [21].

In [21], the authors investigate an asymptotic regime of (3.1) in the case when $\Gamma\to 0$. Namely, given a sequence of Jordan curves $(\Gamma_n)_{n\in\mathbb{N}}$ where $\Gamma_n\subset B(0,R_n)$ with $R_n\to 0$,

they show that if the total area of the associated surfaces Σ_n remain bounded, then, up to a subsequence, the surfaces Σ_n converge either to the origin (collapse) or to a finite connected union of spheres of radius one, such that at least one of them contains the origin.

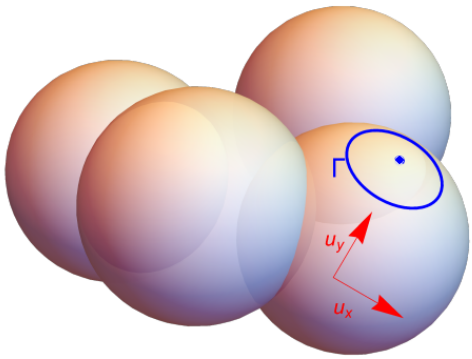

FIGURE 3. *Solutions of (3.1) are parametrizations of surfaces of constant curvatures spanned by* Γ. *According to [21], if* Γ *shrinks to a point, then the solutions are, asymptotically, a finite union of spheric bubbles that contain this point.*

Their approach relies on the following "bubble" decomposition:

Theorem 2 ([21]). *Let* $(u^n)_{n\in\mathbb{N}}$ *be a bounded sequence in* $H^1(D;\mathbb{R}^3)$ *of solutions of the following system*

$$\begin{cases} \Delta u^n = 2u^n_x \wedge u^n_y & \text{on } D \\ u^n = \gamma^n & \text{on } \partial D, \end{cases} \tag{3.2}$$

where $\gamma^n \to 0$ *in* $H^{1/2}(\partial D)$. *Then, there exists a finite family* $(\omega^{(i)})_{i=1\dots p}$ *in* $H^1(\mathbb{R}^2)$, *a family of cores* $(\underline{a}^{(i)})_{i=1\dots p}$ *in* D, *and a family of scales* $((\epsilon_n^{(i)})_{n\in\mathbb{N}})_{i=1\dots p}$ *in* $]0,\infty[$ *that converge to* 0 *and are orthogonal in the sense of (1.3), and such that, for some subsequence, still denoted by* u_n *for simplicity:*

$$u^n(x) = \sum_{i=1}^{p} \omega^{(i)}\left(\frac{x - a_n^{(i)}}{\epsilon_n^{(i)}}\right) + \mathrm{r}_n(x)\,, \qquad \|\mathrm{r}_n\|_{H^1(D)} \overset{n\to\infty}{\longrightarrow} 0. \tag{3.3}$$

Moreover, one has the following orthogonal balance of energy:

$$\|\nabla u^n\|^2_{L^2(D)} = \sum_{i=1}^{p} \|\nabla \omega^{(i)}\|^2_{L^2(\mathbb{R}^2)} + \circ(1), \quad n \to \infty. \tag{3.4}$$

In contrast to the result of P. Gérard concerning any bounded sequence in $\dot{H}^s(\mathbb{R}^d)$, here the "bubble" decomposition is applied to a bounded sequence of solutions to an elliptic nonlinear system, which provides more rigidity.

More precisely, let us underline that the functions ω^i solve an analogous PDE on the whole plane, namely

$$\Delta\omega = 2\omega_x \wedge \omega_y \quad \text{on } \mathbb{R}^2. \tag{3.5}$$

As this equation is invariant by translations and scaling transforms, each bubble

$$\omega^{(i)}\left(\frac{x-a_n^{(i)}}{\epsilon_n^{(i)}}\right)$$

is therefore also a solution of (3.5). All the solutions of (3.5) are known explicitly and their energy $\int_{\mathbb{R}^2}|\nabla\omega|^2$ is quantized as an integer multiple of 8π (see [21, Lemma 0.1]). Consequently, thanks to the orthogonality of the bubbles (3.4), the decomposition (3.3) only includes a finite number p of bubbles, contrary to the decomposition (2.1), which can be infinite. As n goes to infinity, the energy (3.4) of the sequence $(u_n)_{n\in\mathbb{N}}$ therefore concentrates on quantized levels, which are multiple of 8π.

Remark 3.1. *Up to the extraction of a subsequence, the functions u^n concentrate around a finite number of points, which, with the previous notations, are given by*

$$a^{(i)}=\lim_{n\to\infty}a_n^{(i)}.$$

See Figure 3.

We refer the interested reader to [21] for a detailed proof of Theorem 2. The result is achieved by a diagonal extraction process, but in a different way from the proof of profile decomposition of P. Gérard. As we have already seen in Section 2.2, the proof of Theorem 1 is based on Fourier analysis, while the proof of Brézis-Coron [21] is direct and relies on a refined analysis of the sequence $(u_n)_{n\in\mathbb{N}}$ depending on the size of its energy. Roughly speaking, the proof of Theorem 2 comports two steps ((assuming in addition that the sequence $(u_n)_{n\in\mathbb{N}}$ is also bounded in L^∞):

(i). The first step consists in extracting the first bubble, whenever it is possible to do so. Assuming that the energy of the sequence $(u^n)_{n\in\mathbb{N}}$ is bounded from below, namely $\inf_n\|\nabla u^n\|_{L^2}^2=\alpha>0$, a preliminary family of interesting scales $\epsilon_n>0$ and cores $a_n\in\overline{D}$ are constructed by applying the intermediate value theorem to the following functions that quantify the concentration:

$$Q_n(t)=\sup_{z\in\mathbb{R}^2}\int_{z+tD}|\nabla u^n|^2,\ \text{for } t\geq 0,$$

where u^n are extended by zero outside of D. Obviously, each function Q_n is continuous and non-decreasing in t, and satisfies

$$Q_n(0)=0\quad\text{and}\quad Q_n(1)=Q_n(\infty)=\int_D|\nabla u^n|^2\geq\alpha>0\,.$$

So, if one choses an intermediary value $v\in]0,\alpha[$, there exists $0<\epsilon_n<1$ and $a_n\in\bar{D}$ such that

$$Q_n(\epsilon_n)=\int_{a_n+\epsilon_nD}|\nabla u^n|^2=v\,.$$

Alaoglu's theorem then provides a subsequence of scales and cores, denoted respectively by $\epsilon_n^{(1)}$ and $a_n^{(1)}$, such that

$$u^n(\epsilon_n^{(1)}z+a_n^{(1)})\overset{n\to\infty}{\longrightarrow}\omega^{(1)}(z)\quad a.e.\ z\in\mathbb{R}^2,$$
$$\nabla_z u^n(\epsilon_n^{(1)}z+a_n^{(1)})\overset{n\to\infty}{\rightharpoonup}\nabla_z\omega^{(1)}(z)\quad\text{in } L^2(\mathbb{R}^2).$$

One of the key points at this stage is the general identity [21, Lemma 2]

$$\left|\int_\Omega u\cdot v_x\wedge v_y\right|\leq c_0\|\nabla u\|_{L^2}\|\nabla v\|_{L^2},$$

which allows one to use (locally) $u^n(\epsilon_n^{(1)}z+a_n^{(1)})$ itself as a test function of the equation (see [21, eq. 31]). In turn, this allows to convert the weak H^1 convergence of the sequence into a locally strong convergence (see [21, Lemma 3]). One can thus reason by contradiction

and use the non-vanishing energy of (u^n) to ensure that $d(a_n^{(1)}, \partial D)/\epsilon_n^{(1)} \overset{n\to\infty}{\longrightarrow} \infty$. This means that $\omega^{(1)}$ is a solution of (3.5) on $\mathbb{R}^2$ and has therefore a quantized $\dot{H}^1$ energy.

If $\|u^n\|_{H^1(D)} < 8\pi$ then, necessarily, $\omega^{(1)} = 0$ and $\|u^n\|_{H^1(D)} \overset{n\to\infty}{\longrightarrow} 0$. In this case, the final statement holds directly, with no bubble *i.e.* $p = 0$. On the contrary, if $\|u^n\|_{H^1} \geq 8\pi$, then the profile $\omega^{(1)}$ is non trivial and the first bubble takes the form:

$$\omega^{(1)}\left(\frac{\cdot - a_n^{(1)}}{\epsilon_n^{(1)}}\right).$$

(ii). The second step consists in successive iterations of the above construction, each extraction providing a new non-trivial profile, until the remaining energy passes bellow the threshold of 8π.

4. Profile decompositions and nonlinear evolution equations

The method of profile decompositions is now a prerequisite in the analysis of nonlinear evolution PDEs. It can be used to establish global wellposedness and study blow up issues. It is also helpful for the investigation of the qualitative behavior of the solutions and for the description of their asymptotic dynamics, as the time variable goes to infinity (in the case of global wellposedness) or as the time variable goes to the lifespan $T^* < \infty$ (in the case of blow-up at finite time). In the study of long-time behavior of global solutions, it is essential to decide wether they scatter or not, namely if these global solutions behave asymptotically like a solution of the corresponding linear solution or not. When the scattering fails, the next challenge[9] is to show that, asymptotically, the global solutions are a sum of decoupled solitons[10] and a scattering solution. There is an extensive literature on the subject which gained momentum from the remarkable work of Kenig-Merle [61]. Among others, one can mention [8, 10, 11, 13, 15, 36, 38, 43, 44, 45, 53, 56, 58, 60, 61, 62, 63, 66, 76, 80, 82, 85, 92] and the references therein.

In this review, we shall limit ourselves to a presentation of the article of Gallagher [43] and the article of Hmidi-Keraani [53]. Both articles where a historical first as they each opened a new realm of applications to the method of profile decompositions: in [43], this strategy was used to investigate the global wellposedness issue in fluid mechanics, namely for the Navier-Stokes system while, in [53], the authors employed this technique to the problem of formation of singularities for the L^2-critical nonlinear Schrödinger equation.

4.1. Profile decompositions and fluid mechanics.

Before stating the result of Gallagher [43], let us start by recalling briefly some basic facts about the three dimensional[11] incompressible Navier-Stokes equations

$$\begin{cases} \partial_t v + v\cdot\nabla v - \Delta v = -\nabla p \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} v = 0 \\ v_{|t=0} = v_0\,, \end{cases} \tag{NS}$$

where v_0 is a divergence free vector field; $v(t,x)$ and $p(t,x)$ are respectively the velocity and the pressure of the fluid at time $t \geq 0$ and position $x \in \mathbb{R}^3$. The most fundamental result regarding the Cauchy problem was obtained by Leray in [70], who proved that for divergence free Cauchy data $v_0 \in L^2(\mathbb{R}^3)$, there is a global weak solution of (NS) such that

$$v \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^3))$$

[9] known as the soliton resolution conjecture.

[10] That is to say, well-localized solutions traveling at a fixed speed c, which can vanish or not.

[11] All the results presented here hold in the more general case of $\mathbb{R}^d$, with obvious adaptations in the functional spaces considered.

and that satisfies the energy inequality

$$\|v(t)\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^t \|\nabla v(t')\|^2_{L^2(\mathbb{R}^3)} dt' \leq \|v_0\|^2_{L^2(\mathbb{R}^3)}\,.$$

The solutions constructed by Leray are not known to be unique, except in the case of a bidimensional ambient space (see [71]). Many studies exist on that problem of uniqueness: one may consult for instance the monographs of Bahouri-Chemin-Danchin [7], Chemin [25], Lemarié-Rieusset [68] and the references therein for an overview on this topic and in particular the founding article of Fujita-Kato [42], where the authors established that for $v_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, there exists a maximal time T^* and a unique solution v of (NS) stemming from v_0 such that

$$\forall T < T^*, \quad v \in E_T \stackrel{\text{def}}{=} C^0([0,T];\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \cap L^2([0,T];\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)) \tag{4.1}$$

and that satisfies the energy inequality

$$\forall 0 \leq t < T^*, \quad \|v(t)\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} + 2\int_0^t \|v(t')\|^2_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} dt' \leq \|v_0\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}.$$

Below, one will denote this unique solution by $v = \mathrm{NS}(v_0)$. Furthermore, there exists a universal positive constant c such that if

$$\|v_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leq c\,, \tag{4.2}$$

then the associated solution v is global in time[12].

Many results of this type are known to hold. For instance it is possible to replace $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ by the larger Lebesgue space $L^3(\mathbb{R}^3)$ (see [24, 26, 51, 59, 97]) or by the Besov spaces $B^{-1+\frac{3}{p}}_{p,\infty}$ for finite p (see [22, 81]). To this day, the best known result on the uniqueness of solutions to (NS) is due to H. Koch and D. Tataru [65]. It is proved, as most results of this type, by a fixed point theorem in an appropriate Banach space. The smallness condition is the following:

$$\begin{aligned}\|u_0\|_{\mathrm{BMO}^{-1}(\mathbb{R}^3)} \stackrel{\text{def}}{=} & \sup_{t>0} t^{\frac{1}{2}} \|e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^3)} \\ & + \sup_{\substack{x\in\mathbb{R}^3 \\ R>0}} \frac{1}{R^{\frac{3}{2}}} \Big(\int_{[0,R^2]\times B(x,R)} |(e^{t\Delta}u_0)(t,y)|^2\, dydt \Big)^{\frac{1}{2}} \leq c\,.\end{aligned}$$

At this stage, one should emphasize that the Navier-Stokes system (NS) enjoys a scaling invariance property: defining the scaling operators, for any positive real number λ and any point x_0 of $\mathbb{R}^3$,

$$\Lambda_{\lambda,x_0} v(t,x) \stackrel{\text{def}}{=} \lambda v(\lambda^2 t, \lambda(x-x_0)), \tag{4.3}$$

if v solves (NS) with data v_0, then $\Lambda_{\lambda,x_0} v$ solves (NS) with data $\Lambda_{\lambda,x_0} v_0$.

The main result in [43] concerns the family of solutions of the Navier-Stokes system associated with a bounded sequence $(\varphi_n)_{n\in\mathbb{N}}$ of data in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. According to the result of P. Gérard [49] (see Theorem 1 in this article), the sequence $(\varphi_n)_{n\in\mathbb{N}}$ admits, up to the extraction of a subsequence, a decomposition of the form (2.1), where of course $p = 3$. The goal of the following theorem, established by Gallagher in [43], is to investigate how the profile decomposition for (φ_n) is propagated by the Navier-Stokes equation.

[12] Note for instance that in Chemin-Gallagher [27] and Chemin-Gallagher-Zhang [28], global wellposedness for the (NS) system was established for several examples of large data.

Theorem 3 ([43]). *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a family of divergence free vector fields, bounded in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. One applies Theorem 1 to (φ_n) and denote by $(\phi^\ell)_{\ell\in\mathbb{N}}$ the corresponding sequence of profiles (after extracting a subsequence, which we denote in the same way). One adopts the other notations of this Theorem. Let us also introduce*

$$v_n = \mathrm{NS}(\varphi_n) \quad \text{and} \quad V^\ell = \mathrm{NS}(\phi^\ell).$$

The following results hold[13].

(1) There exists a family $(T^\ell)_{\ell\in\mathbb{N}}$ of elements of $\mathbb{R}_+^*\cup\{+\infty\}$ and a finite subset $J\in\mathbb{N}$ such that

$$\forall \ell\in\mathbb{N},\ V^\ell\in E_{T^\ell} \quad \text{and} \quad \forall \ell\in\mathbb{N}\setminus J,\ T^\ell=+\infty. \tag{4.4}$$

Let us define $\tau_n \overset{def}{=} \min\limits_{\ell\in J} h_{\ell,n}^2 T^\ell$.

(2) Then $\|v_n\|_{E_{\tau_n}}$ is bounded, and we have for all $0\le t\le\tau_n$ and all integers $L\ge 1$:

$$v_n(t,x) = V^0(t,x) + \sum_{\ell=1}^{L} \frac{1}{h_{\ell,n}} V^\ell\left(\frac{t}{h_{\ell,n}^2}, \frac{x-x_{\ell,n}}{h_{\ell,n}}\right) + w_n^L(t,x) + R_n^L(t,x)\,, \tag{4.5}$$

where w_n^L solves the heat equation (with data $r_{n,L}$ given by Theorem 1)

$$\begin{cases} \partial_t w_n^L - \Delta w_n^L = 0 \quad \text{in} \quad \mathbb{R}_+\times\mathbb{R}^3 \\ (w_n^L)_{|t=0} = r_{n,L}\,, \end{cases} \tag{4.6}$$

and where

$$\begin{cases} \lim\limits_{L\to+\infty} \big(\limsup\limits_{n\to+\infty} \|w_n^L\|_{L^\infty(\mathbb{R}_+,L^3(\mathbb{R}^3))}\big) = 0, \\ \lim\limits_{L\to+\infty} \big(\limsup\limits_{n\to+\infty} \|R_n^L\|_{E_{\tau_n}}\big) = 0\,. \end{cases}$$

This theorem was at the origin of many studies on the Navier-Stokes system and more generally on several equations arising in fluid mechanics; among others, one can mention [5, 8, 10, 32, 45, 46, 60, 82, 85] and the references therein.

Remark 4.1. *As a by product of the decomposition (4.5), one can easily deduce that the life span T_n of v_n is bounded from below by the smallest of the life spans of each involved profile (up to scaling), namely*

$$T_n \ge \min_{\ell\in J} h_{\ell,n}^2 T^\ell = \tau_n\,.$$

Proof. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a bounded sequence of divergence free vector fields in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, and let $\phi^0\in\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ be any weak limit of (φ_n), then it follows from Theorem 1 that (up to the extraction of a subsequence) the following asymptotic orthogonal decomposition holds

$$\varphi_n = \phi^0 + \sum_{\ell=1}^{L} h_{\ell,n}^{-d/p}\phi^\ell\Big(\frac{\cdot - x_{\ell,n}}{h_{\ell,n}}\Big) + r_{n,L} \quad \text{with} \quad \lim_{L\to+\infty}\big(\limsup_{n\to+\infty}\|r_{n,L}\|_{L^3(\mathbb{R}^3)}\big) = 0\,. \tag{4.7}$$

First observe that, in light of (2.3), one has $\|\phi^\ell\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}\le c$ for ℓ sufficiently large, where c is the universal constant arising in (4.2). This readily ensures the first statement of Theorem 3.

Granted with the above decomposition for $(\varphi_n)_{n\in\mathbb{N}}$, the second statement then results from the scaling properties of the Navier-Stokes system, the orthogonality condition (1.3) and the common properties of the heat equation. More precisely, with the notations of

[13] with E_T defined by (4.1).

Theorem 3, applying a perturbative argument, one can show that a good approximation for v_n is given by

$$v_n^{\rm ap}(t,\cdot) \stackrel{\rm def}{=} V^0(t,\cdot) + \sum_{\ell=1}^{L} h_{\ell,n} V^\ell\Big(\frac{t}{h_{\ell,n}^2}, \frac{\cdot - x_{\ell,n}}{h_{\ell,n}}\Big) + e^{t\Delta} r_{n,L},$$

which is well-defined for all $0 \leq t \leq \tau_n$. □

Let us end this section by presenting a brief proof (using profile decompositions) of a result of Rusin-Şverák [85] whose original proof did not use profile decompositions. This result can be stated as follows.

Theorem 4 ([85])**.** *Let ρ_{max} be the supremum of the set of $\rho > 0$ such that the Navier-Stokes system is globally wellposed for all v_0 in $\dot{H}^{1/2}(\mathbb{R}^3)$ satisfying $\|v_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} < \rho$. Then, if $\rho_{max} < \infty$, there exists a particular data $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$, which generates a solution to (NS) that blows up at a finite time. Moreover $\|u_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} = \rho_{max}$.*

Proof. Assume that $\rho_{max} < \infty$. Then by definition, there exists a sequence $(v_0^n) \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ such that

$$T^*(v_0^n) < \infty \quad \text{and} \quad \|v_0^n\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \stackrel{n\to\infty}{\longrightarrow} \rho_{max}\,.$$

Invoking the profile decomposition for the sequence (v_0^n), one deduces from the orthogonal energy estimate (1.4) that the involved profiles (ϕ^ℓ) satisfy

$$\sum_{\ell\in\mathbb{N}} \|\phi^\ell\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leq \rho_{max}^2\,.$$

The control of the life span given by the theorem of Gallagher stated above implies that, since one assumes that $T^*(v_0^n) < \infty$, there must be at least one profile ϕ^{ℓ_0} in (4.7) with a finite life span and so such that

$$\|\varphi^{\ell_0}\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \geq \rho_{max}^2\,.$$

The two previous estimates are only compatible if and only if the profile decomposition of (v_0^n) includes exactly only one such profile φ^{ℓ_0} and if the remainder term tends to 0 strongly in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, namely

$$\|\varphi^{\ell_0}\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = \rho_{max}\,,$$

which completes the proof of the theorem. □

4.2. **The blow-up result of Hmidi-Keraani.** At first, the profile decompositions of P. Gérard in [49] found there way into the study of nonlinear evolution equations through the question of wether a given profile decomposition for a sequence of bounded data is propagated by the equation at hand, as was the case in the previous section (see for example Bahouri-Gérard [11], Gallagher [43], Gallagher-Gérard [44], Keraani [62],...). To the best of our knowledge, the first instance where these profile techniques were used for another purpose (at least in the framework of evolution equations) happened in the work of Hmidi-Keraani [53].

In [53], the authors were interested by the L^2-critical nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t u + \Delta u + |u|^{\frac{4}{d}} u = 0 \text{ on } \mathbb{R}^d \\ u_{|t=0} = u_0\,. \end{cases} \tag{4.8}$$

Many results have been devoted to the study of this equation (see for instance Bégout [16], Bourgain [19], Cazenave [23], Colliander-Raynor-Sulem-Wright [33], Ginibre-Velo [52], Merle-Vega [76], and Sulem-Sulem [91]). In particular, it was established by Ginibre-Velo [52] that the Cauchy problem (4.8) is locally well-posed in[14] $H^1(\mathbb{R}^d)$ (see also the

[14] Here $H^1(\mathbb{R}^d)$ stands for the inhomogeneous Sobolev space.

monograph of Cazenave [23] for further details), that is to say for all $u_0 \in H^1(\mathbb{R}^d)$, there exist $T > 0$ and a unique solution $u \in C([0,T]; H^1(\mathbb{R}^d))$ of the Cauchy problem (4.8) associated to u_0. Moreover defining the lifespan T^* of the solution u of (4.8) as the supremum of positive times T such that the Cauchy problem (4.8) has a solution $u \in C([0,T]; H^1(\mathbb{R}^d))$, we have the following blow-up criterion

$$T^* < \infty \Rightarrow \|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^d)} \overset{t\to T^*}{\longrightarrow} \infty\,. \tag{4.9}$$

Finally, this unique solution enjoys the following conservation laws regarding the total mass:

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} = \|u_0\|^2_{L^2(\mathbb{R}^d)} \tag{4.10}$$

and the balance of energy:

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(t,x)|^2 dx - \frac{d}{4+2d}\int_{\mathbb{R}^d} |u(t,x)|^{\frac{4}{d}+2} dx = E(0)\,. \tag{4.11}$$

Nowaday, it is well-established that the global well-posedness for the Cauchy problem (4.8) is connected to the so-called ground-state, namely the unique positive radial solution of

$$\Delta Q - Q + |Q|^{\frac{4}{d}} Q = 0\,.$$

In particular, it turns out that the quantity $\|Q\|^2_{L^2(\mathbb{R}^d)}$ is the critical mass for the formation of singularities. Indeed on the one hand, it is well-known that there exists a solution u of (4.8) whose mass is equal to $\|Q\|^2_{L^2}$ and that blows up at finite time (see [23, 91]). On the other hand, if one combines the energy estimate (4.11) with the following inequality of Gagliardo-Niremberg (due to Weinstein [96])

$$\|f\|^{\frac{4}{d}+2}_{L^{\frac{4}{d}+2}(\mathbb{R}^d)} \leq C_d \|f\|^{\frac{4}{d}}_{L^2(\mathbb{R}^d)} \|\nabla f\|^2_{L^2(\mathbb{R}^d)} \quad \text{with} \quad C_d = \frac{d+2}{d}\|Q\|^{\frac{4}{d}}_{L^2(\mathbb{R}^d)}\,, \tag{4.12}$$

one can easily prove that for any data $u_0 \in H^1(\mathbb{R}^d)$ such that

$$\|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)}\,, \tag{4.13}$$

the associated Cauchy problem (4.8) is globally well-posed. Indeed, in that case taking advantage of the conservation laws (4.10)-(4.11) and the Gagliardo-Niremberg inequality (4.12), we readily gather that the solution at hand satisfies

$$\frac{1}{2}\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^d)} \leq E(0) + \frac{1}{2}\frac{\|u_0\|^{\frac{4}{d}}_{L^2(\mathbb{R}^d)}}{\|Q\|^{\frac{4}{d}}_{L^2(\mathbb{R}^d)}}\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^d)}\,,$$

which shows that under the assumption (4.13), $\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^d)}$ remains uniformly bounded and thus, by virtue of the blow-up criterion (4.9), the solution is global in time.

In the radial framework, it was shown by Merle-Tsutsumi ([75]) that in case of blow-up, there is a minimal amount of concentration of the L^2-norm at the origin, namely that if u denotes a radial solution of the Cauchy problem (4.8), which blows-up at a finite time T^*, then for all $R > 0$,

$$\liminf_{t\to T^*} \int_{|x|\leq R} |u(t,x)|^2 dx \geq \int_{\mathbb{R}^d} Q^2 dx\,.$$

In [53], Hmidi and Keraani established the following generalization:

Theorem 5 ([53]). *Let u be a solution to the Cauchy problem (4.8) and assume that it blows-up at a finite time T^*. Then given $\lambda(t)$ such that $\frac{\sqrt{T^*-t}}{\lambda(t)} \to 0$, as $t \nearrow T^*$, there exists $x(t)$ such that*

$$\int_{|x-x(t)|\leq\lambda(t)} |u(t,x)|^2 dx \geq \int_{\mathbb{R}^d} Q^2 dx\,.$$

Proof. The proof comports three steps that we will briefly sketch here. According to the blow-up criterion (4.9), if we assume that u is a solution of (4.8) that blows-up at a finite time T^*, then $\|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^d)} \overset{t\to T^*}{\longrightarrow} \infty$. In particular, there exists a sequence of times $(t_n)_{n\in\mathbb{N}}$ converging to T^* such that $\|\nabla u(t_n,\cdot)\|_{L^2(\mathbb{R}^d)} \overset{t\to T^*}{\longrightarrow} \infty$.

(i). In the first step, one rescales the sequence $(u(t_n,\cdot))_{n\in\mathbb{N}}$ as follows[15]

$$v_n = \rho_n^{\frac{d}{2}} u(t_n, \rho_n x) \quad \text{with} \quad \rho_n = \frac{\|\nabla Q\|_{L^2(\mathbb{R}^d)}}{\|\nabla u(t_n,\cdot)\|_{L^2(\mathbb{R}^d)}}. \tag{4.14}$$

This transform ensures that $(v_n)_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\mathbb{R}^d)$. Obviously the sequence $(\rho_n)_{n\in\mathbb{N}}$ tends to 0 as n goes to infinity. Then, according to the energy conservation law (4.11), one has

$$E(v_n) = \rho_n^2 E(0) \overset{n\to\infty}{\longrightarrow} 0\,,$$

which implies that

$$\frac{1}{2}\|\nabla v_n\|_{L^2(\mathbb{R}^d)} \overset{n\to\infty}{\longrightarrow} \frac{d}{4+2d}\|v_n\|^{\frac{4}{d}+2}_{L^{\frac{4}{d}+2}(\mathbb{R}^d)}\,.$$

Invoking (4.14), we infer that

$$\|v_n\|^{\frac{4}{d}+2}_{L^{\frac{4}{d}+2}(\mathbb{R}^d)} \overset{n\to\infty}{\longrightarrow} \frac{d+2}{d}\|\nabla Q\|^2_{L^2(\mathbb{R}^d)}\,.$$

(ii). In the second step, taking advantage of the profile decomposition method, one shows the existence of a sequence $(x_n)_{n\in\mathbb{N}}$ of points in $\mathbb{R}^d$ such that

$$v_n(\cdot + x_n) \rightharpoonup V \quad \text{with} \quad \|V\|_{L^2(\mathbb{R}^d)} \geq \|Q\|_{L^2(\mathbb{R}^d)}\,. \tag{4.15}$$

To achieve that goal, one starts by applying the profile decomposition of P. Gérard [49], which (up to the extraction of a subsequence) provides the following orthogonal decomposition:

$$v_n = \sum_{j=0}^{L} V^j(\cdot - x_n^j) + \mathcal{R}_n^L\,,$$

with, for all $2 < p < 2^*$, $\lim_{L\to+\infty}(\limsup_{n\to\infty}\|\mathcal{R}_n^L\|_{L^p(\mathbb{R}^d)}) = 0$ and such that

$$\|v_n\|^2_{L^2(\mathbb{R}^d)} = \sum_{j=0}^{L}\|V^j\|^2_{L^2(\mathbb{R}^d)} + \|\mathcal{R}_n^L\|^2_{L^2(\mathbb{R}^d)} + \circ(1), \quad n\to\infty\,, \tag{4.16}$$

as well as

$$\|\nabla Q\|^2_{L^2(\mathbb{R}^d)} = \|\nabla v_n\|^2_{L^2(\mathbb{R}^d)} = \sum_{j=0}^{L}\|\nabla V^j\|^2_{L^2(\mathbb{R}^d)} + \|\nabla\mathcal{R}_n^L\|^2_{L^2(\mathbb{R}^d)} + \circ(1), \quad n\to\infty\,. \tag{4.17}$$

Moreover, thanks to the orthogonality condition between the cores, one has

$$\sum_{j=0}^{\infty}\|V^j\|^{\frac{4}{d}+2}_{L^{\frac{4}{d}+2}(\mathbb{R}^d)} = \lim_{n\to\infty}\|v_n\|^{\frac{4}{d}+2}_{L^{\frac{4}{d}+2}(\mathbb{R}^d)} = \frac{d+2}{d}\|\nabla Q\|^2_{L^2(\mathbb{R}^d)}\,. \tag{4.18}$$

Then, Gagliardo Niremberg inequality (4.12) allows one to infer that

$$\sum_{j=0}^{\infty}\|V^j\|^{\frac{4}{d}+2}_{L^{\frac{4}{d}+2}(\mathbb{R}^d)} \leq C_d\left(\sup\|V^j\|^{\frac{4}{d}}_{L^2(\mathbb{R}^d)}\right)\sum_{j=0}^{\infty}\|\nabla V^j\|^2_{L^2(\mathbb{R}^d)}\,,$$

[15] As it will be seen further, the choice of the scaling ρ_n is crucial in the proof of the result.

with $C_d = \dfrac{d+2}{d}\|Q\|_{L^2(\mathbb{R}^d)}^{\frac{4}{d}}$, which in view of (4.17) implies that

$$\sum_{j=0}^{\infty} \|V^j\|_{L^{\frac{4}{d}+2}(\mathbb{R}^d)}^{\frac{4}{d}+2} \leq C_d \left(\sup \|V^j\|_{L^2(\mathbb{R}^d)}^{\frac{4}{d}}\right) \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2 .$$

Since the series $\sum \|V^j\|_{L^2}^2$ is convergent as a consequence of the orthogonality equality (4.16), the supremum on the right-hand side is achieved for some index j_0. The latter estimate thus becomes

$$\|V^{j_0}\|_{L^2(\mathbb{R}^d)}^{\frac{4}{d}} \geq \frac{\sum_{j=0}^{\infty} \|V^j\|_{L^{\frac{4}{d}+2}(\mathbb{R}^d)}^{\frac{4}{d}+2}}{C_d \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2} .$$

Combining this result with (4.18) and with the fact that $C_d = \dfrac{d+2}{d}\|Q\|_{L^2(\mathbb{R}^d)}^{\frac{4}{d}}$, one readily gathers that

$$\|V^{j_0}\|_{L^2(\mathbb{R}^2)} \geq \|Q\|_{L^2(\mathbb{R}^2)} ,$$

which completes the proof of the claim (4.15).

(iii). In the last step, one obtains the result by invoking the scale invariance of the equation and expressing v_n by means of u_n. □

Appendix A. Wavelet decompositions and nonlinear approximation theory

In this appendix, we introduce briefly some basic facts about wavelet decompositions and nonlinear approximation theory; for further details, one can consult [31, 34, 35, 67, 78] and the references therein.

Recall that wavelet decompositions (for instance Haar's system) have the form

$$f = \sum_{\lambda \in \Lambda} d_\lambda \psi_\lambda . \tag{A.1}$$

The set Λ comports both a scale index $j = j(\lambda)$ and a space index $k = k(\lambda)$. The wavelets can be normalized (for example) in the following way:

$$\psi_\lambda = \psi_{j,k} = 2^{rj}\psi(2^j \cdot -k) ,$$

where r is related to the scale invariance of the space at hand (see examples below). In the wavelet decomposition (A.1), the spatial index k provides an additional level of discretization compared to the Littlewood-Paley decomposition (for an introduction to Littlewood-Paley theory, see for instance [7, 18, 25, 68] and the references therein).

It is possible to characterize wether a function belongs to almost any classical functional space in terms of conditions pertaining only to $|d_\lambda|$. For instance, one has

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} \sim \|(d_\lambda)_{\lambda\in\Lambda}\|_{l^2} \quad \text{and} \quad \|f\|_{\dot{B}^t_{q,q}(\mathbb{R}^d)} \sim \|(d_\lambda)_{\lambda\in\Lambda}\|_{l^q}$$

if the previous decomposition is respectively normalized with $r = \dfrac{d}{2} - s$ for $\dot{H}^s(\mathbb{R}^d)$ or with $r = \dfrac{d}{q} - t$ for $\dot{B}^t_{q,q}(\mathbb{R}^d)$.

Nonlinear approximation theory, which is crucial in signal and image processing theories, consists in approximating a given function f by its N most significant components in the wavelet decomposition (A.1), namely

$$Q_N(f) = \sum_{\lambda \in E_N} d_\lambda \psi_\lambda , \tag{A.2}$$

where $E_N = E_N(f)$ is a subset of Λ with cardinal N. In many cases, E_N corresponds to the N largest wavelet coefficients. For instance, for the critical Sobolev embedding

$$\dot{B}^s_{p,p}(\mathbb{R}^d) \hookrightarrow \dot{B}^t_{q,q}(\mathbb{R}^d) \quad \text{with} \quad 0 < \frac{1}{p} - \frac{1}{q} = \frac{s-t}{d}\,, \tag{A.3}$$

it is straightforward that

$$\sup_{\|f\|_{\dot{B}^s_{p,p}(\mathbb{R}^d)}\leq 1} \|f - Q_N f\|_{\dot{B}^t_{q,q}(\mathbb{R}^d)} \leq C N^{-\frac{s-t}{d}}\,. \tag{A.4}$$

Indeed, if one denotes by $(d_m)_{m\geq 0}$ the decreasing rearrangement of $|d_\lambda|$, using the previous equivalence of norms and the fact that $p < q$, one can easily check that

$$\begin{aligned}
\|f - Q_N f\|_{\dot{B}^t_{q,q}(\mathbb{R}^d)} &\sim \Big(\sum_{\lambda\notin E_N} |d_\lambda|^q\Big)^{1/q} = \Big(\sum_{m>N} |d_m|^q\Big)^{1/q} \\
&\leq \Big(N^{-1}\sum_{m=1}^{N} |d_m|^p\Big)^{1/p-1/q}\Big(\sum_{m>N} |d_m|^p\Big)^{1/q} \\
&\lesssim N^{-\frac{s-t}{d}}\|f\|_{\dot{B}^s_{p,p}(\mathbb{R}^d)}.
\end{aligned}$$

This completes the proof of (A.4).

Appendix B. Helly's lemma

For the convenience of the reader, let us briefly recall Helly's extraction lemma, which plays a role in the proof of the profile decomposition by P. Gérard (see §2.2). Further details can be found, for example in [40].

Proposition B.1. *For any sequence of increasing functions $f_n : \mathbb{R} \to [-1, 1]$, there exists a subsequence which converges pointwise on $\mathbb{R}$.*

Proof. First, one considers the restrictions $f_n|_{\mathbb{Q}}$. The compactness of $[-1, 1]$ ensures, by Bolzano-Weierstrass's theorem the existence of a first subsequence that converges at one given point in $\mathbb{Q}$. Iterating this idea, one can successively extract subsequences that converge simultaneously at a finite number of points in $\mathbb{Q}$. As $\mathbb{Q}$ is countable, Cantor's diagonal argument then provides one extraction $\psi : \mathbb{N} \to \mathbb{N}$ such that $(f_{\psi(n)})$ converges at every point of $\mathbb{Q}$. The map

$$f : \left|\begin{array}{ccc} \mathbb{Q} & \to & [-1, 1] \\ q & \mapsto & \lim\limits_{n\to\infty} f_{\psi(n)}(q) \end{array}\right.$$

is obviously increasing and thus the sided limits

$$f(x-0) = \sup\{f(q)\,;\, q \in \mathbb{Q},\ \ q < x\} \quad \text{and} \quad f(x+0) = \inf\{f(q)\,;\, q \in \mathbb{Q},\ \ q > x\}$$

are well defined for any $x \in \mathbb{R}$ and satisfy $f(x-0) \leq f(x+0)$. At all the points $x \in \mathbb{R}$ such that $f(x-0) = f(x+0)$ one defines $f(x)$ as this common value. For any $\varepsilon > 0$, the function $f|_{\mathbb{Q}}$ does not varies by more then ε on a small rational neighborhood whose adherence contains x. Convergence at two such rational points flanking x then ensures that for n large enough,

$$f(x) - 2\varepsilon \leq f_{\psi(n)}(x) \leq f(x) + 2\varepsilon$$

and thus $(f_{\psi(n)}(x))$ converges towards $f(x)$. The remaining set of "bad" points

$$B = \{x \in \mathbb{R}\,;\, f(x-0) < f(x+0)\}$$

is at most countable. Indeed, monotonicity implies that any choice of a rational ϑ_x such that $f(x-0) \leq \vartheta_x \leq f(x+0)$ induces an injective map $\vartheta : B \to \mathbb{Q}$. Then, using the compactness of $[f(x-0), f(x+0)]$ and Cantor's diagonal extraction once more, it is

possible to extract a subsequence $(f_{\psi\circ\varphi(n)})$ that converges pointwise at any point of B and therefore, by construction, at any point of $\mathbb{R}$. □

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