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Numerical method for the computation of first arrival time of seismic waves with general anisotropy

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SUMMARY

We present a numerical method for computing the first arrival time of seismic waves. The scheme can tackle 3D media with elastic properties defined by a fully anisotropic Hooke tensor. Our numerical method proceeds in a single pass over the domain, similar to the Fast Marching method, and achieves third order accuracy in smooth test cases, with quasi-linear computation time. We also validate our solver by comparing it with the solution to the elastic wave equation on a medium with general anisotropy.

INTRODUCTION

The first arrival time of a seismic wave is characterized by the eikonal equation. Although the elastic wave equation gives more information on the behaviour of a seismic wave, the solution to the eikonal equation is still of interest as it can typically be computed for a lower cost. Besides, numerical solvers for the eikonal equation can be used for several applications, such as earthquake hypocenter relocation (Moser et al. (1992)) or tomographic inversions (Nolet (2008); Billette and Lambaré (1998); Taillandier et al. (2009)).

Efficient algorithms for the eikonal equation have been developed thanks to the level-set framework (Sethian (1996)), such as the FSM (Fast Sweeping method) and the FMM (Fast Marching Method), which respectively generalize the algorithms of Bellman-Ford and of Djikstra for graph distance computation, at first for isotropic settings. The FMM is usually prefered as it is a single-pass method, whereas the FSM is an iterative method. However, there are more technical challenges for the FMM, which makes the FSM easier to implement, especially for generalizations in anisotropic settings.

Geological media are typically regarded as anisotropic, with the anisotropy described by a Hooke tensor. Specific forms of anisotropy have been considered, which translates to additional symmetries in the Hooke tensor, such as (from less complex to more complex): VTI (vertical transverse isotropy), TTI (tilted transverse isotropy), TOR (tilted orthorhombic), or a fully general Hooke tensor.

Algorithms based on the FSM have been developed to deal with anisotropic settings up to the TOR symmetry (Waheed et al. (2015); Le Bouteiller et al. (2019)). However, up to now, the FMM has only been generalized to the TTI symmetry in bin Waheed (2020) and with no formal proof of convergence. With work from Mirebeau (2014a, 2019) on the FMM with elliptic anisotropy, we now have a generalization of the FMM which works for a fully general Hooke tensor (Desquilbet et al. (2020)). The method can be implemented up to third order

accuracy, illustrated on smooth test cases, with a quasi-linear complexity in computation time.

In this abstract, we first review the eikonal equation with anisotropy defined by a Hooke tensor, then we present our numerical solver using the FMM and its features. Finally, we illustrate the performances of the solver on a 3D medium defined with a fully anisotropic Hooke tensor, and compare our solution with the solution to the elastic wave equation.

EIKONAL EQUATION

Notations: In this abstract, the position is denoted by the variable $x \in \mathbb{R}^3$, the velocity by the variable $v \in \mathbb{R}^3$ and the time by the variable $t \in \mathbb{R}^+$. All the indices i, j, k, l belong to the set $\{1, 2, 3\}$, and we use Einsten convention of summation (from 1 to 3) over repeated indices in a same formula.

The elastic properties of a medium are usually characterized by a Hooke tensor, denoted by $c_{ijkl}(x)$, as well as the density $\rho(x)$. The amplitude displacement vector of a seismic wave is denoted $U_i(x,t)$. The *elastic wave equation* is an equation for the displacement vector $U_i(x,t)$, of the form

$$\rho \partial_{tt}^2 U_i - \partial_i (c_{ijkl} \partial_l U_k) = F_i, \tag{1}$$

where F_i is the source field.

The first arrival time of the seismic wave is denoted u(x). The *Christoffel equation* can be obtained from a high-frequency approximation of the elastic wave equation (see Slawinski (2003)), and is an equation for the first arrival time u(x), of the form

$$\det(\rho \,\delta_{ik} - c_{i\,ikl} \,\partial_i u \partial_l u) = 0, \tag{2}$$

with $u(x_0) = 0$ for a source point x_0 , where δ_{ik} denotes Kronecker's symbol and the determinant is taken over the matrix with indices (i,k).

However, not only the first arrival time is solution of (2), but also later arrival times which correspond to first arrival times of S-waves. In order to select only the first arrival time (of P-wave), we write its associated *eikonal equation*: the first arrival time u(x) is the unique viscosity solution (see Bardi and Capuzzo-Dolcetta (2008)), of an equation of the form

$$N_{\mathbf{x}}^*(\nabla u(\mathbf{x})) = 1,\tag{3}$$

with $u(x_0) = 0$ for a source point x_0 , where $N_x^*(v) := \|\frac{c_{ikl}}{\rho}v_jv_l\|$ and $\|.\|$ denotes the spectral norm (here, of the matrix with indices (i,k)). N_x^* is the Legendre-Fenchel dual of the norm N_x , and N_x can be deduced from N_x^* by the duality relation: $N_x(x) = \sup\{\langle v, w \rangle, N_x^*(w) \le 1\}.$

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This norm N_x can be used to determine the distance between two positions x, y, as

$$d(x,y) := \min\{\int_0^1 N_{\gamma(t)}(\gamma'(t)) \,\mathrm{d}t, \ \gamma \text{ path from } x \text{ to } y\}, \quad (4)$$

and the first arrival time u(x) is related to the distance as

$$u(x) = d(x_0, x),\tag{5}$$

with source point x_0 .

From this characterization, one can derive *Bellman's optimality principle*, which will be useful for the discretization method: for any position x and any neighborhood V of x not containing the source point x_0 , one has

$$u(x) = \min_{y \in \partial V} (u(y) + d(y, x)).$$
(6)

NUMERICAL METHOD

Our discretization of the eikonal equation (3) mimics and discretizes Bellman's optimality principle (6) (following Kimmel and Sethian (1998); Sethian and Vladimirsky (2001); Bornemann and Rasch (2006); Mirebeau (2014a,b)).

For that purpose, we introduce for each position *x* a polygonal neighborhood V_h^x whose vertices lie on the Cartesian grid $h\mathbb{Z}^3$ (for a grid step h > 0). V_h^x is referred to as the *stencil* at position *x*. We also define I_h^x as the piecewise linear interpolation operator on the facets of V_h^x . Given the arrival time *u*, we approximate the right-hand side of (6) by interpolating the arrival times between the vertices of the stencil, and approximating the distance function with the local norm N_x as

$$\Delta_h u(x) := \min_{y \in \partial V_h^x} (I_h^x u(y) + N_x(x-y)).$$
(7)

The numerical approximation of the arrival time is defined as the unique solution u to the discrete system (see Bornemann and Rasch (2006))

$$u(x) = \Lambda_h u(x), \tag{8}$$

and $u(x_0) = 0$ at the source point.

Equation (8) is a finite-difference scheme for the eikonal equation (3). The points used for the finite-difference at position xare exactly the vertices of the stencil V_h^x : the arrival time at xis calculated with (7) from arrival times at neighbours of x.

The solution to (8) may already be computed using iterative methods, such as the Fast Sweeping method. As stated in the introduction, we are more interested in the Fast Marching method, presented in Algorithm 1, which guarantees a computation time of complexity $\mathcal{O}(n\ln(n))$. However, for the Fast Marching method, a careful choice of the stencil V_h^x is needed, as described in the next subsection. Examples of stencils used in our scheme are shown in Figure 1.

Acute stencil and causality property

In this subsection, we establish a property of the numerical

Algorithm 1 The Fast Marching algorithm	
Initialize: $u(x_0) = 0$, and $u(x) = +\infty$ for all $x \neq x_0$. Tag	g all
points as non-accepted.	
While a non-accepted point remains:	1.
Denote by <i>y</i> the non-accepted point minimizing $u(y)$.	2.
Tag y as accepted.	3.
For each non-accepted point x such that $y \in V_h^x$:	4.
$u(x) \leftarrow \Lambda_h u(x)$	5.



Figure 1: Example of stencils (cut-cube, cube and spiky-cube)

scheme, known as *causality*, which guarantees that the Fast Marching method correctly solves the system (8). Following Sethian and Vladimirsky (2001), it is derived from a geometrical *acuteness* property of the norms and discretization stencils.

First, we define the angular width of the facets of a stencil V, measured with respect to a norm N:

$$\Theta(N,V) := \max\{ \sphericalangle(\nabla N(v), w), v, w \text{ in a common facet of } \partial V \}$$
(9)

where $\triangleleft(u, v)$ denotes the unoriented angle between two vectors u, v.

 $\Theta(N,V)$ is a measure of the adequation between the stencil and the anisotropy of the norm: to achieve a lower value, the stencil needs to be more refined in directions in which the wave is faster. A stencil V is called *strictly acute* with respect to a norm N if we have: $\Theta(N,V) < \pi/2$.

The applicability of the Fast Marching method is linked to the acuteness of the stencils: if the stencils V_h^x are strictly acute with respect to N_x for all positions x, then the system (8) is strictly causal, in the sense that each of the solution values $u(x) = \Lambda u(x)$ can be computed in terms of strictly smaller values u(y) at some neighbors y of x in the computation of the minimum in (7). As a result, the system can be solved in a single pass using the Fast Marching method. Numerous variants of this result can be found in Tsitsiklis (1995); Kimmel and Sethian (1998); Sethian and Vladimirsky (2001).

Therefore, the stencils need to be chosen before solving the eikonal equation, in such a way that they all verify the acuteness property with regard to the local norm. Once the stencils are set, they do not need to be updated anymore during the Fast Marching algorithm, or in case of a change of source point with the same medium. As illustrated in the next subsection, the cut-cube stencil (see Figure 1) has been found to be sufficient for most geological media, and the cube or spiky-cube stencils can handle anisotropy of even greater intensities.

Illustration: acute stencils for TTI anisotropy

One form of anisotropy for the Hooke tensor is the TTI (tilted transversal isotropic) symmetry, which is usually parameterized by the Thomson parameters: $(V_p, V_s, \varepsilon, \delta)$ as well as a rotation, see Thomsen (1986). For illustrative purposes, we can determine whether our fixed stencils (cut-cube, cube and spiky-cube) provide a causal scheme for each TTI metric.

We considered only the worst case in terms of the rotation chosen for the TTI metric. Besides, the parameter V_s can be chosen as 0 with very small impact on the value of the first arrival time, and V_p can be taken equal to 1 with no loss of generality. Therefore, it is sufficient to consider only (ε , δ) as parameters. Results are shown in Figure 2. We also plotted the 58 examples of TTI metrics presented in Thomsen (1986). Out of the 58 media, only 4 of them are such that the cut-cube stencil does not guarantee a causal scheme, and correspond to crystallographic media with anisotropy of high intensity. We can conclude that the cut-cube stencil is often good enough for most practical cases of seismic anisotropy, and that the cube or spiky-cube can even cover most of the extreme cases.



Figure 2: TTI metrics and acuteness property. The figure shows whether a given stencil is acute with regard to a TTI norm determined by parameters (ε , δ), with $V_s = 0$ and $V_p = 1$, while taking into account the worst possible rotation. The domain labelled for the spiky-cube also contains the domain labelled for the cube, which also contains the domain labelled for the cut-cube.

Source factorization and convergence orders

The numerical discretization (7) is not very precise near the source point, but we can add a source factorization to improve the accuracy, in the spirit of Luo and Qian (2012); Treister and Haber (2016). We also consider higher order upwind finite differences for (7), in the spirit of Sethian (1999).

In order to validate the convergence order of the proposed method, we generate non-trivial test cases with explicit solution, obtained as the conformal transformation of constant materials (Desquilbet et al. (2020)). With these settings, our scheme achieves a quasi-linear complexity in time, which is the specificity of the Fast Marching algorithm, and third-order accuracy in the test cases. Details on the computations and illustrations can be seen in Desquilbet et al. (2020).

APPLICATION ON A REALISTIC MEDIUM

We consider a 3D model with a fully anisotropic Hooke tensor, obtained through the homogenization (equivalent medium theory) of a fine scale isotropic model known as the SEG/EAGE overthrust model (Aminzadeh et al. (1997)). As an illustration of non-periodic two-scale homogenization for elastic media, the fully anisotropic version of the SEG/EAGE overthrust model has been presented in Cupillard et al. (2020).

The model is described on a Cartesian grid of $534 \times 534 \times 107$ points, which represents an area of 20 km \times 20 km \times 4 km. We consider a source located in the middle of the medium at the surface, at x = 10 km, y = 10 km, z = 0 km. We solve the eikonal equation with our numerical solver, and also solve the elastic wave equation with the same setting. By comparing the two, we can illustrate the performances of our solver, as the solution to the eikonal equation is expected to match with the first arrival time of the solution to the elastic wave equation.

The elastic wave equation is solved using the spectral-element based modeling and inversion code SEM46 (Trinh et al. (2019); Cao et al. (2020)). The simulation is performed using a 10 Hz Ricker vertical force source, on a Cartesian-based mesh using $560 \times 560 \times 110$ elements with P^4 Lagrange polynomial. The final time for simulation is set to 2.5s leading to 10000 time steps with a time sampling $\Delta t = 0.00025$ s. The computation has been performed on the Univ. Grenoble Alpes HPC Dahu platform on 6 nodes of 32 cores (192 cores in total) benefiting from the domain-decomposition algorithm implemented in SEM46. Each node is equipped with two xeon Skylake Gold Intel processors, each featuring 16 cores clocked at 2.1 GHz, and 192 GO of RAM. With these settings, the computation of the solution to the elastic wave equation took 3.5 hours.

On the other hand, for the eikonal solver, we can show that the cut-cube stencil provides a causal scheme everywhere in the medium. With this setting, the computation of the solution has been done on a single core of a laptop (with Intel architecture comparable to the one from the Univ. Grenoble Alpes cluster) and took approximately 1600s (less than half an hour).

For illustrations, we present a seismogram in Figure 3. The seismogram is extracted on a receiver line at the surface, in the place of the source, from x=0 to x = 20 km. On this seismogram, we superpose the first arrival time computed through our eikonal solver. We observe a qualitative match between the solution from our eikonal solver and the first arrival time deduced from the solution to the elastic wave equation.

We also present in Figure 4 a 3D view of the superposition of the isochrones for the first arrival time computed through our eikonal solver with the solution to the elastic wave equation computed using SEM46, at time t = 1.5s, t = 2s, and t = 2.5s. The isochrone contours (in red) accurately follow the first arrival time of the wavefront (in black and white) for the different snapshots. The correspondence is not expected to be perfect due to finite frequency effects for the solution to the elastic wave equation, however the qualitative match we observe is a validation of our approach on a realistic setting with fully general anisotropy.



Figure 3: Seismogram recorded along a receiver line located on the surface (z = 0 km), in the source plane (y = 10 km) along the *x*-axis (from x = 0 km to x = 20 km). For the elastic wave equation, the vertical displacement is represented in black and white. The first arrival time computed through our eikonal solver for each receiver position is superposed in red. The resulting red contour matches the first arrival time deduced from the solution to the elastic wave equation.

CONCLUSION

In this study, we present a numerical solver for the eikonal equation in 3D, with anisotropy coming from a fully general Hooke tensor. It uses a single pass method based on the Fast Marching method, and achieves quasi-linear computation time and third-order accuracy on smooth test cases.

This numerical scheme is causal only if the discretization stencils verify an acuteness property with regard to the metric. For the majority of media encountered in seismology, the cutcube stencil is enough to provide a causal scheme. If strongly anisotropic media are considered and the cut-cube fails the acuteness requirements, then more complex stencils can be considered. Otherwise, the Fast Sweeping method can also be directly applied with this same finite-difference scheme.

For future research, another numerical scheme for the eikonal equation is also in consideration, specifically for the TTI symmetry. In that case, the new scheme no longer requires any constraint regarding the intensity of anisotropy. The metric with TTI symmetry is treated as an envelope of metrics with elliptic anisotropy, for which we can adapt the already existing numerical schemes and generate stencils adapted to the metric, instead of the fixed stencils we consider for the current scheme.

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Figure 4: In black and white, vertical displacement of the solution to the elastic wave equation, computed in the 3D fully anisotropic medium coming from the homogenization of the SEG/EAGE overthrust model. The red contour corresponds to the isochrone computing through our eikonal solver. The background corresponds to the P-wave velocity of this model. The different snapshots are obtained at t = 1.5s (top), t = 2s (middle) and t = 2.5s (bottom).

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