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CHARLOTTE HARDOUIN AND MICHAEL F. SINGER

ABSTRACT. We refine necessary and sufficient conditions for the generating series of a weighted model of a quarter plane walk to be differentially algebraic. In addition, we give algorithms based on the theory of Mordell-Weil lattices, that, for each weighted model, yield polynomial conditions on the weights determining this property of the associated generating series.

CONTENTS

1. INTRODUCTION

The enumeration of planar lattice walks confined to the north-east quadrant has attracted a considerable amount of interest over the past fifteen years. For the lattice $\mathbb{Z}^2$, a lattice path model is comprised of a finite set $\mathcal{D}$ of lattice vectors called the *step set* together with a starting point $P \in \mathbb{Z}^2$. The combinatorial question boils down to the count $q_{i,j}(n)$ of n-step walks, i.e., of polygonal chains, that remain in the first quadrant, starting from P, ending at (i, j) and consisting of n oriented line segments whose associated translation vectors belong to $\mathcal{D}$. This question is ubiquitous since lattice walks encode several classes of mathematical objects, in discrete mathematics (permutations, trees, planar maps), in probability theory (lucky games, sums of discrete random variables), statistics (non-parametric

tests). We refer to the introduction of [BF02] for more details on these applications as well as [Hum10] for applications in other scientific areas.

Many algebraic and analytic properties of the combinatorial sequence of a lattice walk are embodied in the algebraic nature of the associated generating function. For instance, for the lattice $\mathbb{Z}^2$, the linear recurrences satisfied by the sequence $(q_{i,j}(n))_{i,j,n}$ corresponds to the fact that the generating function

$$(1.1) \qquad Q(x,y,t) = \sum_{i,j,n \geq 0} q_{i,j}(n) x^i y^j t^n$$

is D-finite, that is, satisfies non-trivial linear differential equations in each of the derivation with respect to x, y and t. This correspondence yields a classification of the generating series as to: algebraic functions over $\mathbb{Q}(x,y,t)$, D-finite functions, differentially algebraic functions (those satisfying a non-trivial polynomial relation with their derivatives) and differentially transcendental functions, that is, functions that are not differentially algebraic. Recently, the works of many authors led to a complete classification of generating series associated to lattice walks with small steps, that is, with step set $\mathcal{D} \subset \{-1,0,1\}^2 \backslash \{(0,0)\}$. These works combine a wide variety of technics: singularity analysis via the Kernel Method, probabilistic method, guess and proof strategies and Galois theory of functional equations. Many researchers have contributed answers to these questions and our brief exposition below does not do justice to these contributions. Nonetheless, since detailed descriptions of these various contributions exist elsewhere (see for example [BBMR17, DHRS18, DHRS19]) we will limit ourselves to a brief summary.

Of the $2^8 - 1$ possible choices of step sets it is shown in [BMM10] that taking symetries into account and eliminating trivial sets, one need only consider 79 of these models. Doing this, one only eliminates models whose generating series are algebraic. Of these, 23 models have D-finite (in all variables) generating series ([BMM10, BvHK10]) of which 4 are algebraic. The remaining 56 models were shown to have non-D-finite generating series with respect to various variables in [KR12, MR09, MM14, BRS14]. In [BBMR17, DHRS18, DHRS19, DR19], the more general question of differential transcendence is addressed. In [BBMR17], the authors give new uniform proofs of the 4 algebraic cases and also show that 9 (see Figure 1) of the 56 non-D-finite models in fact have differentially algebraic generating functions. Using criteria from the Galois theory of difference equations, the authors of [DHRS18, DHRS19] show that 47 of the 56 non-D-finite models have differentially transcendental generating functions with respect to x and y and reproved the fact from [BBMR17] that the remaining 9 are differentially algebraic with respect to x and y. (Figure 1 below reproduces Figure 2 of [DHRS18] with a table comparing the notations of [BBMR17, Table 2] and [DHRS18, Figure 2]).

At the core of all these works, one finds two geometric objects: an algebraic curve defined over $\mathbb{Q}(t)$ called the *kernel curve* of genus 0 or 1 and a group of automorphisms of the curve called *the group of the walk*. Though the finiteness of the group had been clearly related to the D-finiteness of the generating function, no combinatorial as well as geometric criteria had been proposed to characterize the differential algebraicity of the generating function. [DHRS18] proposed a criteria based on the computation of residues of elliptic functions and [BBMR17] discovered the more algebraic notion of *decoupled model* by a case-by-case analysis of the nine models of Figure 1.The notion of decoupled model allowed the authors of [BBMR17] to give an explicit expression of the generating function, which led to an explicit differential algebraic equation.

The study of weighted walks (that is, lattice walks whose steps have been endowed with weights) yielded a more fecund understanding of these criteria. When all these weights are equal, a rescaling allows one to consider them all equal to 1

[BBMR17, Tab 2]	[DHRS18, Fig. 2]	values of parameters
1	$w_{\text{IIB.1}}$ (after $x \leftrightarrow y$)	all
2	$w_{\text{IIB.2}}$ (after $x \leftrightarrow y$)	all
3	$w_{\text{IIC.1}}$	all
4	$w_{\text{IIB.3}}$	all
5	$w_{\text{IIC.4}}$	$d_{-1,-1}d_{1,1} - d_{1,0}d_{-1,0}$
6	$w_{\text{IIC.2}}$	$d_{0,1}d_{0,-1} - d_{1,1}d_{-1,-1}$
7	$w_{\text{IIB.6}}$ (after $x \leftrightarrow y$)	all
8	$w_{\text{IIC.5}}$	all
9	$w_{\text{IIB.7}}$	$d_{-1,1}d_{1,-1} - d_{0,-1}d_{0,1}$

FIGURE 1. The 9 non-D-finite models that have D-algebraic generating series when unweighted together with a table comparing notations of [BBMR17] and [DHRS18]. The last column indicates for which weights the generating series of the corresponding weighted models are D-algebraic with respect to x and y. The determination of these values is a consequence of the main results of this paper. See Section 5 Example 5.2 and Remark 5.3.

whence the terminology *unweighted model* to denote now the $2^8 - 1$ models introduced in the above paragraphs. The need for a classification of weighted walks confined in the quadrant arose in the classification of three dimensional walks confined in the octant. As shown in [BBMKM16], some of these three dimensional models can be reduced by projection to two-dimensional models with weights. Similarly to unweighted models, one attaches to a weighted model a kernel curve of genus zero or one and a group of automorphisms of this curve. When the group is finite, [DR19, Cor.43] proves that the generating function is D-finite in x and y. When the kernel curve is of genus zero, the generating function is differentially transcendental with respect to x and y by [DHRS19] and with respect to t by [DH19]. The case of a kernel curve of genus one remained open until now and only some partial cases were treated. In [DR19], the authors adapt some arguments of [DHRS18] and proved the differential transcendence with respect to x and y of the generating function for some classes of weighted walks. In [DH19], the authors proved that the differential transcendence with respect to x and y implies the differential transcendence with respect to t. In [KY15] and [CMMR17], the authors study families of weighted models with finite group and the algebraicity of their generating functions.

In this paper, we focus on weighted models with small steps and on the differential algebraicity with respect to the variables x and y. For these models, we unify the approaches of [DHRS18] and [BBMR17] and show that a weighted model is decoupled if and only if its generating function is differentially algebraic with respect to the variables x and y. A recent work by Dreyfus inspired by [BBMR17] proved that in the weighted case, a decoupled model has a differentially algebraic generating series with respect to t (see [D21]). Moreover we translate the combinatorial question of the differential algebraicity of the generating function in the purely arithmetic question of the linear dependence of two given points of the Mordell-Weil group of the kernel curve. Previous works had considered the kernel curve as a fixed elliptic curve by choosing a value of t, even transcendental over $\mathbb{Q}$. The

novelty of our strategy is that we allow t to vary so that we work with a pencil of elliptic curves or equivalently with a rational surface whose general fiber is the kernel curve. Relying on the theory of Mordell-Weil lattices and their classification for rational elliptic surfaces (see for instance [SS19]), we construct an algorithm which given a weighted model determines the polynomial relations between the weights that correspond to a differentially algebraic generating function. For instance, let us consider for the weighted model

with nonzero weights $d_{1,1}, d_{0,-1}, d_{-1,-1}, d_{-1,0}, d_{0,1}, d_{0,0}$, where $d_{i,j}$ is the probabilistic weight attached to the direction (i, j). We then have that the associated generating series is differentially algebraic if and only if

$$d_{0,1}d_{0,-1} - d_{1,1}d_{-1,-1} = 0.$$

This relation is automatically satisfied when all the weights are equal to one so that the corresponding model w_{IIC2} is one of the nine differentially algebraic models of Figure 1. This shows that these nine cases are coincidences; they are just the only weighted models for which the weights equal to one satisfy the polynomial equations guaranteeing the differential algebraicity.

This geometric strategy has therefore a combinatorial interest since it builds a bridge between the combinatorics of the walks and the combinatorics of the Mordell-Weil lattices. For walks in the first quadrant, the nature of the Mordell-Weil lattice is controlled by the relative position of the base points of the pencil of kernel curves. This arithmetic point of view might be also well suited to attack the question of the specialization of the variable t since it might be translated in terms of specialization of independent points of the general fiber of an elliptic surface to linearly dependent points in a specialized fiber.

The rest of the paper is organized as follows. In Section 2 we review the notions of the kernel of a walk and the group of a walk. In Section 3 we show that the criteria of [BBMR17] and [DHRS18] are equivalent. In Section 4 we simplify the latter criteria of [DHRS18] by showing they are equivalent to showing that two poles of a certain function lie in the same orbit under an action already considered in [DHRS18]. Combining this with ideas from the theory of elliptic surfaces, we give, in Section 5 an algorithm and some refinements which allow one to characterize in terms of polynomial relations those weights for which the generating series are differentially algebraic. In Section 5.1, we present some basic facts concerning the Kodaira-Néron Model of our family of elliptic curves, its Mordell-Weill lattice and Néron-Tate heights and present an algorithm which, once these are facts are accepted, reduces the computation of these polynomial conditions to the calculation of an associated Weierstass equation and simple arithmetic. In Section 5.2, we give a more detailed description of these objects and concepts, yielding a significant refinement of the algorithm. In Appendix A we recall some facts concerning local parameters, poles and the notion of orbit residue introduced in [DHRS18].

2. Kernel curve and group of the walk

From now on, we will fix a set of steps $\mathcal{D}$ and weights $\{d_{i,j}\}$. We also fix once and for all a value of t, transcendental over $\mathbb{Q}$ and occasionally suppress the symbol t in our notation. All studies concerning the behavior of the generating series (1.1) begin with the functional equation it satisfies (c.f., [BMM10]). One first defines a

Laurent polynomial called the *inventory* of the step set $\mathcal{D}$

$$(2.1) \qquad S(x, y) := \sum_{(i,j) \in \mathcal{D}} d_{i,j} x^i y^j$$

and a polynomial called the *kernel* of the walk

$$(2.2) \qquad K(x, y, t) := xy(1 - tS(x, y)).$$

One then has that $Q(x, y, t)$ satisfies

$$(2.3) \qquad K(x, y, t)Q(x, y, t) = xy - F^1(x, t) - F^2(y, t) + td_{-1,-1}Q(0, 0, t)$$

where

$$(2.4) \quad F^1(x, t) := -K(x, 0, t)Q(x, 0, t) \quad \text{and} \quad F^2(y, t) := -K(0, y, t)Q(0, y, t).$$

2.1. **The Curve.** The equation $K(x, y) = 0$ defines an affine curve E_t in $\mathbb{C} \times \mathbb{C}$. As in [DHRS18, DHRS19], it is useful to consider a compactification $\overline{E_t}$ of this curve in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. This curve is defined by homogenizing each variable separately in $K(x, y)$, that is,

Definition 2.1. *The* kernel curve *associated to a quadrant model is the curve*

$$\overline{E_t} = \{([x_0 : x_1], [y_0 : y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \mid \overline{K}(x_0, x_1, y_0, y_1, t) = 0\}$$

where $\overline{K}(x_0, x_1, y_0, y_1, t)$ is the following bihomogeneous polynomial
(2.5)
$$\overline{K}(x_0, x_1, y_0, y_1, t) = x_1^2 y_1^2 K\left(\frac{x_0}{x_1}, \frac{y_0}{y_1}, t\right) = x_0 x_1 y_0 y_1 - t \sum_{i,j=0}^{2} d_{i-1,j-1} x_0^i x_1^{2-i} y_0^j y_1^{2-j}.$$

The reducibility of $K(x, y, t)$ as an element of $\mathbb{C}[x, y]$ can be expressed as a condition on the set of steps of the model (see [FIM17, Lemma 2.3.2] for $t = 1$ and [DHRS20, Proposition 1.2]). The walks having reducible kernel polynomials or degree in x or degree in y less than or equal to 1 are called degenerate and their generating series is algebraic. Thus, we will discard these cases and will assume that $K(x, y, t)$ is irreducible and has degree 2 in each of its variables x and y. If $\overline{E_t}$ is nonsingular it is of genus 1, otherwise it has genus 0. The genus zero curves correspond to the 28 sets of steps that are included in a half plane [DHRS20, Cor. 2.6] (note that each of the 4 families in (G0) corresponds to 7 models). Using the observations in [DHRS20, Remark 2.9] and taking symmetry into account, one need only focus on the following five sets of steps

The main result of [DHRS19] is to show that the generating series of any weighted model attached to one of the above set of steps is differentially transcendental. Thus in the whole paper, we will always assume that the model of our walk corresponds to a genus one curve, that is according to [DHRS20, Cor.2.6], we will focus on the

weighted models whose set of steps is not included in any half plane.
(G1)
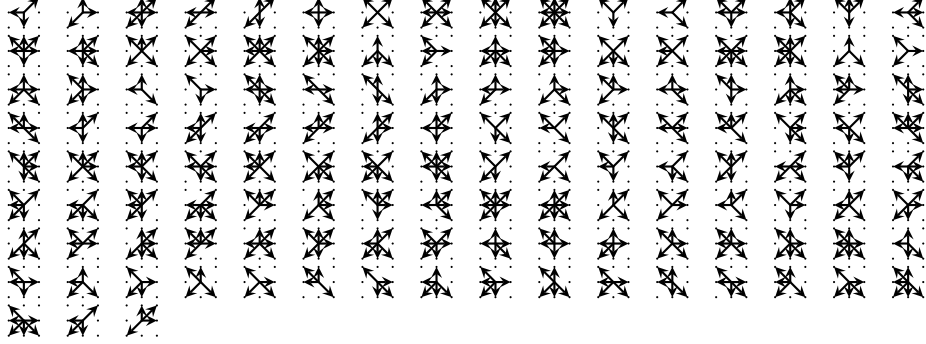
The ring $\mathbb{C}[x,y]/(K(x,y,t))$ is an integral domain and we will denote its quotient field by $\mathbb{C}(\overline{E_t})$. This is the field of rational functions from $\overline{E_t}$ to $\mathbb{P}^1$. To see this, any element $f(x,y) \in \mathbb{C}[x,y]/(K(x,y,t))$ can be homogenized in x and y separately by setting $F([x_0:x_1],[y_0:y_1]) = x_1^{\deg_x f} y_1^{\deg_y f} f\left(\frac{x_0}{x_1}, \frac{y_0}{y_1}\right)$. For an element $f(x,y)/g(x,y)$ in this quotient field, we homogenize $f(x,y)$ and $g(x,y)$ separately to get F and G as in the previous sentence. One then multiplies each F and G (if necessary) by a suitable power of x_1 and y_1 to get new polynomials $\bar{F}$ and $\bar{G}$ so that the total degree in the x variables are the same for $\bar{F}$ and $\bar{G}$ the same holds for the y variables. This latter operation ensures that the resulting map $\overline{E_t} \to \mathbb{P}^1$ defined by

$$([x_0:x_1],[y_0:y_1]) \mapsto [\bar{F}([x_0:x_1],[y_0:y_1]), \bar{G}([x_0:x_1],[y_0:y_1])]$$

is well defined. For example the rational function

$$\frac{x^2 + y^4}{x^3 + y^2}$$

corresponds to

$$[x_1(x_0^2 y_1^4 + x_1^2 y_0^4) : y_1^2(x_0^3 y_1^2 + x_1^3 y_0^2)].$$

Conversely any algebraic map from $\overline{E_t}$ to $\mathbb{P}^1$ is of the previous form. We will abuse notation and use x and y to denote the image of these variables in this field as well. From the context it will be clear which sense is being used.

2.2. **The Group.** Since the polynomial $K(x,y,t)$ has degree 2 in each variable, we can define two involutive automorphisms of $\overline{E_t}$. Let $P = (a,b) = ([a_0:a_1],[b_0:b_1]) \in \overline{E_t}$. Then, $\overline{K}(a_0,a_1,b_0,b_1,t) = 0$. The polynomial $\overline{K}(a_0,a_1,y_0,y_1,t)$ is homogeneous of degree 2 in $[y_0:y_1]$ and has therefore two roots $b, \tilde{b}$ in $\mathbb{P}^1$ (possibly $b = \tilde{b}$). We define $\iota_1(P) = (a, \tilde{b})$. Similarly, one can define $\iota_2(P) = (\tilde{a}, b)$ where $a, \tilde{a}$ are the roots of $\overline{K}(x,b,t) = 0$. The maps ι_1, ι_2 are involutions which are induced by rational maps on $\mathbb{C} \times \mathbb{C}$ (formulas are given in [BMM10] and [DHRS18]) and so can be extended to involutions of $\overline{E_t}$ i.e., for any $P = (x,y) \in \overline{E_t}$ we have

$$\{P, \iota_1(P)\} = \overline{E_t} \cap (\{x\} \times \mathbb{P}^1(\mathbb{C})) \text{ and } \{P, \iota_2(P)\} = \overline{E_t} \cap (\mathbb{P}^1(\mathbb{C}) \times \{y\}).$$

We furthermore define an an automorphism $\tau : \overline{E_t} \to \overline{E_t}$ by the formula

$$\tau := \iota_2 \circ \iota_1.$$

Definition 2.2. *The* group of the walk *is the group generated by ι_1, ι_2.*

Remark 2.3. *The map ι_1 induces an automorphism of $\mathbb{C}(\overline{E_t})$ via $\iota_1(f(Q)) = f(\iota_1(Q))$ for $Q \in \overline{E_t}$ (we are abusing notation and using the same symbol for the map on $\overline{E_t}$ and $\mathbb{C}(\overline{E_t})$). Similarly, ι_2 and τ induce automorphisms of $\mathbb{C}(\overline{E_t})$. One*

needs to be careful of the context when using these symbols. In particular, $\tau = \iota_2 \circ \iota_1$ on $\overline{E_t}$ but $\tau = \iota_1 \circ \iota_2$ on $\mathbb{C}(\overline{E_t})$. Indeed, any automorphism σ of $\overline{E_t}$ acts on $\mathbb{C}(\overline{E_t})$, the field of rational functions from $\overline{E_t}$ to $\mathbb{P}^1$ by inner composition, that is, $\sigma(f) = f \circ \sigma$. Therefore

$$\tau(f) = (\iota_2 \circ \iota_1)(f) = f \circ \iota_2 \circ \iota_1 = \iota_1 (f \circ \iota_2) = \iota_1(\iota_2(f)).$$

The subfields $\mathbb{C}(x)$ and $\mathbb{C}(y)$ of $\mathbb{C}(\overline{E_t})$ are pure transcendental extensions and are the fixed fields of ι_1 and ι_2 respectively.

In [BBMR17], the authors show that the group of the walk is finite if and only if there exists a nonconstant $g \in (\mathbb{C}(x) \cap \mathbb{C}(y)) \subset \mathbb{C}(\overline{E_t})$. When such a g exists one says that the walk admits *invariants*. We give an equivalent property.

Lemma 2.4. *1. The group G of the walk is finite if and only if τ has finite order.*
2. The element τ has finite order if and only if there exists $f \in \mathbb{C}(\overline{E_t}) \backslash \mathbb{C}$ such that $\tau(f) = f$.
3. There exists a nonconstant $g \in (\mathbb{C}(x) \cap \mathbb{C}(y)) \subset \mathbb{C}(\overline{E_t})$ if and only if there exists $f \in \mathbb{C}(\overline{E_t}) \backslash \mathbb{C}$ such that $\tau(f) = f$

Proof. 1. This follows from the fact that the group generated by τ has index 2 in the group of the walk.
2. Assume τ has finite order n. Let $\mathbb{C}(\overline{E_t})^\tau$ be the field of invariants of τ. For any $f \in \mathbb{C}(\overline{E_t})$, the polynomial

$$P_f(X) = \prod_{i=0}^{n-1} (X - \tau^i(f))$$

has coefficients in $\mathbb{C}(\overline{E_t})^\tau$ and therefore any element of $\mathbb{C}(\overline{E_t})$ is algebraic over $\mathbb{C}(\overline{E_t})^\tau$. Since $\mathbb{C}(\overline{E_t})$ has transcendence degree 1 over $\mathbb{C}$, there must be an element in $\mathbb{C}(\overline{E_t})^\tau \backslash \mathbb{C}$.

Now assume that there exists an $f \in \mathbb{C}(\overline{E_t}) \backslash \mathbb{C}$ such that $\tau(f) = f$. Since $\mathbb{C}$ is algebraically closed, we have that $\mathbb{C}(f)$ has transcendence degree one over $\mathbb{C}$. Furthermore, since $\mathbb{C}(\overline{E_t})$ has transcendence degree 1 over $\mathbb{C}$, x and y must each be algebraic over $\mathbb{C}(f)$. Let $P_x(X) \in \mathbb{C}(f)[X]$ (resp. $P_y(X) \in \mathbb{C}(f)[X]$) be the monic minimal polynomial of x (resp. y) over $\mathbb{C}(f)$ and let $S_x = \{\alpha \in \mathbb{C}(\overline{E_t}) \mid P_x(\alpha) = 0\}$ and $S_y = \{\alpha \in \mathbb{C}(\overline{E_t}) \mid P_y(\alpha) = 0\}$. The automorphism τ permutes the elements of S_x and the elements of S_y. Since these sets are finite sets, there is some positive integer n such that τ^n leaves all the elements of these sets fixed. In particular, τ^n leaves x and y fixed and so must be the identity.
3. Of course, [BBMR17, Theorem 7] and 2. above yield this equivalence but we give a direct proof. If $g \in (\mathbb{C}(x) \cap \mathbb{C}(y)) \backslash \mathbb{C}$ then $\iota_1(g) = g$ and $\iota_2(g) = g$, so $\tau(g) = \iota_1 \iota_2(g) = \iota_1(g) = g$. Conversely assume that $f \in \mathbb{C}(\overline{E_t}) \backslash \mathbb{C}$ such that $\tau(f) = f$. Since $\mathbb{C}$ is algebraically closed, we then have that f is transcendental over $\mathbb{C}$ so x is algebraic over $\mathbb{C}(f) \subset \mathbb{C}(\overline{E_t})^\tau$. Let $P(X) \in \mathbb{C}(\overline{E_t})^\tau[X]$ be the minimal unitary polynomial of x and denote by $P^{\iota_1}(X)$ the polynomial resulting from applying ι_1 to the coefficients of $P(X)$. One sees that the coefficients of $P^{\iota_1}(X)$ again lie in $\mathbb{C}(\overline{E_t})^\tau$ so we must have that $P^{\iota_1}(X) = P(X)$ since they both have x as a root. Therefore the coefficients of $P(x)$ are left fixed by ι_1 (as well as by τ) and thus lie in $\mathbb{C}(x)$. Not all of these coefficients lie in $\mathbb{C}$ since x is not algebraic over $\mathbb{C}$ so there exists $g \in \mathbb{C}(x)$ such that $\tau(g) = g$. We then have that $g = \iota_1(g) = \iota_1(\tau(g)) = \iota_2(g)$ so $g \in (\mathbb{C}(x) \cap \mathbb{C}(y)) \backslash \mathbb{C}$. $\square$

Example 2.5. For the weighted cross model, that is, the model such that $d_{1,1} = d_{-1,1} = d_{-1,1} = d_{-1,-1} = 0$, it is easily seen that $\tau^2((\infty, 0)) = (\infty, 0)$. Therefore, the automorphism of the walk is finite of order 2. One can easily show that $d_{1,0}(x + \iota_2(x)) = d_{1,0}x + \frac{d_{-1,0}}{x}$ is fixed by ι_2 and by ι_1. It is therefore a constant with respect to τ but also an element of $\mathbb{C}(x) \cap \mathbb{C}(y)$. Indeed, one has

$$d_{1,0}x + \frac{d_{-1,0}}{x} = \frac{1}{t} - \left(\frac{d_{0,-1}}{y} + d_{0,0} + d_{0,1}y\right) \text{ modulo } K(x,y).$$

We have the following additional facts concerning the group of the walk and its relation to the kernel curve.

- For a dense set of values of $t \in [0, 1]$, this group is finite for 23 unweighted models (as well as some of these models with weights). These have been shown to have generating series that are holonomic (or even algebraic). [BMM10, BvHK10, BBMR17].
- For a dense set of values of $t \in [0, 1]$, this group is infinite for the remaining 56 unweighted models. Furthermore,
  - for the 51 unweighted models with associated curve of genus 1, there exists a point $P \in \overline{E_t}$ such that the element τ of the group is given by

    $$\tau(Q) = Q \oplus P$$

    where $\oplus$ denotes addition on the elliptic cuve $\overline{E_t}$ [Proposition 2.5.2 in [Dui10]]. If $\tau^n(Q) = Q$ for some point $Q \in \overline{E_t}$ and some integer $n \in \mathbb{Z}$, the automorphism τ^n is the identity. The fact that the group is infinite is also equivalent to the point P having infinite order in the group structure on $\overline{E_t}$.
  - for the 5 weighted models with associated curve of genus 0, there exists a rational map $\phi : \mathbb{P}^1(\mathbb{C}) \to \overline{E_t}$ such that the pullback of τ is a q-dilation $z \mapsto qz$ for some $q \in \mathbb{C}, |q| \neq 1$.

A remaining question is: for which values of the weights are the models attached to the set of steps G1 differentially algebraic or D-algebraic for short. If the group of the walk is finite, [DR19, Theorem 42] shows that the generating series is holonomic. When the group is infinite and the models unweighted, the question was solved case by case in [BBMR17] and [DHRS18]. In the next sections of this paper, we will show that the D-algebraicity of weighted models with genus one kernel curve is encoded by the position of the *base points* of a pencil of elliptic curves. This gives a more geometric understanding of the differential behavior of the weighted models and allows one to produce an algorithm to test their D-algebraicity.

3. Decoupling pairs and certificates

In this section we compare the criteria presented in [BBMR17] and [DHRS18] ensuring that the generating series of a quadrant model is D-algebraic. We shall assume that the model is non-degenerate, that is, that the curve $\overline{E_t}$ defined by $K(x, y, t) = 0$ is an irreducible curve and K is of degree 2 in x and y.

3.1. **Decoupling pairs.** In [BBMR17, Definition 8], the authors introduce the notion of a *decoupling*.

Definition 3.1. *A quadrant model is* decoupled *if there exist $f(x) \in \mathbb{Q}(t)(x)$ and $g(x) \in \mathbb{Q}(t)(y)$ such that $xy = f(x) + g(y)$ in $\mathbb{C}(\overline{E_t})$. The functions $f(x)$ and $g(y)$ are said to form a* decoupling pair *for $h(x, y)$.*

A main result of [BBMR17] is that, of the 79 relevant unweighted quadrant models, precisely 13 are decoupled. Of these, 9, as in Figure 1, correspond to models with infinite group and an additional 4 have finite group. The authors further show that those models admitting an invariant and having a decoupling pair are precisely the models having algebraic generating series. For the 9 decoupled unweighted models with infinite group, the authors give explicit expressions for the generating series and show that these series are D-algebraic.

The strategy of [BBMR17] is to give an explicit expression of the generating series in terms of a certain *weak invariant*, which is written in terms of the elliptic functions. This explicit expression allows one to find explicit differential algebraic equation for the generating series. The approach of [BBMR17] should also work for decoupled weighted model.

Without being as explicit as [BBMR17], we can indicate why these expressions exist. Note that when the kernel curve has genus one, the elliptic curve $\overline{E_t}$ admits an uniformization of the form $\{(x(\omega), y(\omega)) \text{ with } \omega \in \mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)\}$ where ω_1 is a non-zero real number and ω_2 a purely imaginary number [FIM17, Lemma 3.3.2]. The functions $x(\omega), y(\omega)$ are rational functions of the Weierstrass functions $\wp_{1,2}, \wp'_{1,2}$ attached to the elliptic curve $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$. The automorphisms ι_1, ι_2 and τ then lift to $\mathbb{C}$ as $\iota_1(\omega) = -\omega, \iota_2(\omega) = -\omega + \omega_3$ and $\tau(\omega) = \omega + \omega_3$[1], which is a non-zero real number. By [DR19, Proposition 2.8], the generating series $F^1(x, t)$ and $F^2(x, t)$ can be lifted to the universal cover of $\overline{E_t}$ as meromorphic functions denoted by $r_x(\omega)$ and $r_y(\omega)$

- $r_y(\omega)$ coincides with $F^2(y(\omega), t)$ on a nonempty open subset $\mathcal{D}_{x,y}$ ([DR19, Lemma 24]);
- $r_y(\omega + \omega_1) = r_y(\omega)$;
- $r_x(\omega + \omega_1) = r_x(\omega)$;
- $r_x(\omega + \omega_) = r_x(\omega) + y(-\omega)(x(\omega + \omega_3) - x(\omega))$;
- $r_y(\omega + \omega_3) = r_y(\omega) + b \circ (x(\omega), y(\omega))$.

When the model is decoupled, one can express $r_x(\omega)$ in terms of elliptic functions as follows.

Lemma 3.2. *Assume that the weighted model is decoupled and has a genus one kernel curve and infinite group of the walk. Let $f(x)$ and $g(y)$ be a decoupling pair for xy. Then, there exist a unique rational function $G(X, Y) \in \mathbb{C}(X, Y)$ such that*

$$r_x(\omega) = f(x(\omega)) + G(\wp_{1,3}(\omega), \wp'_{1,3}(\omega)),$$

where $\wp_{1,3}$ is the Weierstrass function attached to the elliptic curve $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_3)$.

Proof. Since the group of the walk is infinite, the automorphism τ had infinite order and the complex number ω_3 is $\mathbb{Z}$-linearly independent with ω_1 so that they both form a $\mathbb{Z}$-lattice in $\mathbb{C}$. Noting that $y(\omega)$ is fixed by ι_2, one finds that

$$\iota_2 \circ \iota_1(y(\omega)) = \iota_1(\iota_2(y(\omega))) = \iota_1(y(\omega)) = y(-\omega)$$

by Remark 2.3. Similarly, one finds $x(-\omega + \omega_3) = x(\omega + \omega_3)$. Since the model is decoupled, one has

$$x(\omega)y(\omega) = f(x(\omega)) + g(y(\omega)),$$

which gives by applying ι_1 the following equation

(3.1) $$x(\omega)y(-\omega) = f(x(\omega)) + g(y(-\omega)),$$

and then ι_2

(3.2) $$x(-\omega + \omega_3)y(-\omega) = f(x(-\omega + \omega_3)) + g(y(-\omega)).$$

---

[1]There is a discrepancy in signs between [DR19] and this paper. We choose $F^1(x, t) = -Q(x, 0, t)K(x, 0, t)$ and the opposite is chosen in [DR19].

Substracting (3.1) to (3.3), one finds

$$(3.3) \qquad y(-\omega)\,(x(-\omega + \omega_3) - x(\omega)) \quad = \quad f(x(-\omega + \omega_3)) - f(x(\omega))$$

which gives $y(-\omega)(x(\omega + \omega_3) - x(\omega)) = f(x(\omega + \omega_3)) - f(x(\omega))$. Since $f(x(\omega))$ is ω_1-periodic, we deduce from the functional equation above satisfied by $r_x(\omega)$ that $r_x(\omega) - f(x(\omega))$ is a meromorphic functions that is ω_1, ω_3-periodic. It is therefore an elliptic function with respect to the elliptic curve $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_3)$. We conclude the proof via the characterization of elliptic functions in terms of Weierstrass functions. $\qquad\qquad\square$

3.2. **Certificates.**

Definition 3.3. *Let K be a field, τ an automorphism of K and $f \in K$. We say that g is a* certificate *for f if*

$$f = \tau(g) - g.$$

This terminology comes from a similar term used in the theory of telescopers and certificates for deriving and verifying combinatorial identities [BCCL10, WZ90]. In [DHRS19, Section 2.2] the authors, using a result of Ishizaki [Ish98], show

Proposition 3.4. *Assume that the kernel curve of a weighted quadrant model $\overline{E_t}$ has genus 0 and has infinite group. The series $F^1(x,t) = -K(x,0,t)Q(x,0,t)$ and $F^2(y,t) = -K(0,y,t)Q(0,y,t)$ are differentially algebraic in x and in y, which we shall call D-algebraic for short [2] if and only if the element $b = x(\iota_1(y) - y) \in \mathbb{C}(\overline{E_t})$ has a certificate in $\mathbb{C}(\overline{E_t})$, i.e., there exists $g \in \mathbb{C}(\overline{E_t})$ such that*

$$(3.4) \qquad\qquad\qquad b = \tau(g) - g.$$

In the genus 1 case and for unweighted models, the authors of [DHRS18] proved a slightly weaker result The following proposition shows how this latter result can be reproduced word for word for weighted models. We will just sketch the proof since its only new ingredient relies on the uniformization results of [DR19] for weighted models, which allows the direct use of the Galois theoretic tools of [DHRS18]. The existence of a certificate depends on the position of the poles of b and the vanishing of certain *orbit residues*. Once these criteria are fulfilled, it is quite easy to build the certificate g by adjusting the poles and residues of g so that they fulfill the condition $b = \tau(g) - g$. We work through a complete example in *Example 5.2 revisited: The weighted model $w_{IIC.2}$.* and make explicit the certificate in §5.2.3.

If D is a divisor of $\overline{E_t}$, we will denote by $\mathcal{L}(D)$ the finite dimensional $\mathbb{C}$-space $\{f \mid (f) + D \geq 0\}$ where (f) is the divisor of f. Recall that there exists a point $P \in \overline{E_t}$ such that $\tau(Q) = Q \oplus P$ for all $Q \in \overline{E_t}$.

Proposition 3.5. *Assume that the kernel curve $\overline{E_t}$ of a weighted quadrant model is of genus 1 and has infinite group. We then have that $F^1(x,t) = -K(x,0,t)Q(x,0,t)$ and $F^2(y,t) = -K(0,y,t)Q(0,y,t)$ are D-algebraic if and only if there exist $g \in \mathbb{C}(\overline{E_t})$, a $Q \in \overline{E_t}$ and an $h \in \mathcal{L}(Q + \tau(Q)) = \{f \mid (f) + Q + \tau(Q) \geq 0\}$ such that*

$$(3.5) \qquad\qquad\qquad b = \tau(g) - g + h,$$

where $b = x(\iota_1(y) - y) \in \mathbb{C}(\overline{E_t})$ (see 3.1).

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[2]In this paper, we do not investigate the differential dependencies with respect to the variable t.

Proof. Let us assume that $F^1(x,t)$ and $F^2(y,t)$ are D-algebraic over $\mathbb{C}$. By [DHRS18, Lemma 6.3], the function $r_y(\omega)$, which coincides with $F^2 \circ y(\omega)$ on some open set is ω-D-algebraic and satisfies $r_y(\omega + \omega_3) = r_y(\omega) + b(x(\omega), y(\omega))$ (see 3.1). By [DHRS18, Proposition 3.6 and Proposition B.5], there exits $g \in \mathbb{C}(\overline{E_t})$, a $Q \in \overline{E_t}$ and an $h \in \mathcal{L}(Q + \tau(Q))$ such that $b = \tau(g) - g + h$. Conversely, if $b = \tau(g) - g + h$ then [DHRS18, Proposition B.5] implies the existence of $L \in \mathbb{C}[\frac{d}{d\omega}]$ such that $L(b \circ (x(\omega), y(\omega))) = g(\omega + \omega_3) - g(\omega)$ for some $g \in \mathbb{C}(\overline{E_t})$, the latter field being identified with the field of meromorphic functions that are (ω_1, ω_2)-periodic. From the functional equations satisfied by r_y, one obtains that the function $L(r_y) - g$ is (ω_1, ω_3)-periodic. Since elliptic functions are differentially algebraic over $\mathbb{C}$, the functions $L(r_y) - g$ and g are differentially algebraic over $\mathbb{C}$ and so is r_y. [DHRS18, Lemma 6.4] allows one to conclude that, since $F^2(y,t) = r_y(y^{-1}(\omega))$ on some open set, the function $F^2(y,t)$ is y-D-algebraic over $\mathbb{C}$. By [DHRS18, Proposition 3.10], the function $F^1(x,t)$ is also x-D-algebraic over $\mathbb{C}$. $\qquad\square$

Remark 3.6. *In* [DH19], *the authors show that if a weighted quadrant model has a generating series that is neither x- nor y-D-algebraic, then the generating series is also t-D-transcendental.*

In fact, one can further improve Proposition 3.5 so that the condition (3.5) is replaced with the simpler *b has a certificate in* $\mathbb{C}(\overline{E_t})$, making the condition uniform for genus 0 and 1.

Note that $\iota_1(x) = x$ so for $b = x(\iota_1(y) - y)$, one has $\iota_1(b) = -b$. We refer to Appendix A for the required facts concerning poles and residues.

Lemma 3.7. *Let $\overline{E_t}$ be of genus 1 and $b \in \mathbb{C}(\overline{E_t})$ such that $\iota_1(b) = -b$. Assume that the group of the walk is infinite. If there exist a $g \in \mathbb{C}(\overline{E_t})$, a $Q \in \overline{E_t}$ and an $h \in \mathcal{L}(Q + \tau(Q))$ such that*

$$(3.6) \qquad\qquad b = \tau(g) - g + h,$$

then there exists a $\tilde{g} \in \mathbb{C}(\overline{E_t})$ such that

$$(3.7) \qquad\qquad b = \tau(\tilde{g}) - \tilde{g}.$$

Proof. Note that $\tau = \iota_1 \iota_2, \iota_1 \tau = \iota_2$, and $\tau \iota_2 = \iota_1$ on $\mathbb{C}(\overline{E_t})$ so

$$(3.8) \qquad 2b = b - \iota_1(b) = \tau(g + \iota_2(g)) - (g + \iota_2(g)) + (h - \iota_1(h)).$$

If $h \in \mathbb{C}$, we have that $b = \tau(\tilde{g}) - \tilde{g}$ where $\tilde{g} = \frac{g + \iota_1(g)}{2}$.

If $h \notin \mathbb{C}$, then it will be sufficient to prove that there exists an $\tilde{h} \in \mathbb{C}(\overline{E_t})$ such that $h - \iota_1(h) = \tau(\tilde{h}) - \tilde{h}$. Lemma A.7, which is a consequence of the fact that the sum of the residues of an elliptic function is zero, implies that the configuration of poles and residues of h is the following

Divisor	Q	$\tau(Q)$
Residues of order 1	α	$-\alpha$

for some $\alpha \in \mathbb{C}^*$. Since ι_1 is an involution of the curve, Lemma A.9.1 implies that the configuration of poles and residues of $-\iota_1(h)$ is

Divisor	$\tau^{-1}(\iota_1(Q))$	$\iota_1(Q)$
Residues of order 1	$-\alpha$	α

If $\iota_1(Q) = \tau(Q)$, then the function $\hat{h} = h - \iota_1(h)$ has no poles and is therefore constant. Note that a may not equal zero but the poles of h and $\iota_1(h)$ cancel. Since $\hat{h} = -\iota_1(\hat{h})$, the constant $\hat{h}$ must be zero and so from (3.8) we can conclude that $b = \tau(\tilde{g}) - \tilde{g}$ where $\tilde{g} = \frac{g + \iota_2(g)}{2}$.

If $\iota_1(Q) \neq \tau(Q)$ the configuration of poles and residues of $h - \iota_1(h)$ is

Divisor	$\tau^{-1}(\iota_1(Q))$	$\iota_1(Q)$	Q	$\tau(Q)$
Residues of order 1	$-\alpha$	α	α	$-\alpha$

The point Q may coincide with $\iota_1(Q)$ and so the residue there may be 2α but this will not change the reasoning below. Since $\iota_1(Q) \neq \tau(Q)$, the Riemann-Roch Theorem implies that there exists an $f \in \mathbb{C}(\overline{E_t})$ with simple poles at these points and whose configuration of poles and residues is

Divisor	$\iota_1(Q)$	$\tau(Q)$
Residues of order 1	$-\alpha$	α

The configuration of poles and residues of $\tau(f) - f$ and of $h - \iota_1(h)$ are the same. Therefore $\hat{h} := h - \iota_1(h) = \tau(f) - f + d$ for some $d \in \mathbb{C}$. We will now use the facts that $\tau = \iota_1\iota_2, \iota_1\tau = \iota_2, \tau\iota_2 = \iota_1$ on $\mathbb{C}(\overline{E_t})$ and $\iota_1(d) = d$ since $d \in \mathbb{C}$. Since $\iota_1(\hat{h}) = -\hat{h}$, we have that

$$(3.9) \quad 2\hat{h} = \hat{h} - \iota_1(\hat{h}) = \tau(f + \iota_2(f)) - (f + \iota_2(f)) + (d - \iota_1(d)) = \tau(2\tilde{h}) - 2\tilde{h},$$

where $\tilde{h} = \frac{f+\iota_2(f)}{2}$. Thus, $\hat{h} = h - \iota_1(h) = \tau(\tilde{h}) - \tilde{h}$. $\qquad\square$

Combining Corollaries 3.4 and 3.7, we therefore can give a uniform statement for the generating series of weighted quadrant models

Theorem 3.8. *Assume that the kernel curve of a non-degenerate weighted quadrant model $\overline{E_t}$ has infinite group. The series $F^1(x,t) = -K(x,0,t)Q(x,0,t)$ and $F^2(y,t) = -K(0,y,t)Q(0,y,t)$ are D-algebraic if and only if the element $b = x(\iota_1(y) - y) \in \mathbb{C}(\overline{E_t})$ has a certificate in $\mathbb{C}(\overline{E_t})$, i.e., there exists $g \in \mathbb{C}(\overline{E_t})$ such that*

$$(3.10) \qquad\qquad b = \tau(g) - g.$$

3.3. **The relation between decoupling pairs and certificates.** We now turn to showing that, for quadrant models with infinite group, being decoupled is equivalent to the existence of $g \in \mathbb{C}(\overline{E_t})$ such that $x(\iota_1(y) - y) = \tau(g) - g$. The following handles both the genus 0 and genus 1 cases in a uniform way.

Proposition 3.9. *Assume that the quadrant model is non-degenerate and has an infinite group. The following are equivalent*

(1) The model is decoupled.
(2) The element $b = x(\iota_1(y) - y)$ has a certificate in $\mathbb{C}(\overline{E_t})$.

In fact, if $(f(x), g(y))$ is a decoupling pair for xy then $g(y)$ is a certificate for b. Conversely, if g is a certificate for b, then $(f = xy - g, g)$ is a decoupling pair for xy.

Proof. Recall that the fixed field of ι_1 is $\mathbb{C}(x) \subset \mathbb{C}(\overline{E_t})$ and the fixed field of ι_2 is $\mathbb{C}(y) \subset \mathbb{C}(\overline{E_t})$.

Assume *(1)*, that the quadrant model is decoupled. We then have that

$$(3.11) \qquad\qquad xy = f(x) + g(y)$$

for some $f(x) \in k(x)$ and $g(x) \in k(y)$. Applying ι_1 to this equation, we have that

$$(3.12) \qquad\qquad x\iota_1(y) = f(x) + \iota_1(g(y)).$$

Subtracting (3.11) from (3.12) we have $x\iota_1(y) - xy = x(\iota_1(y) - y) = \iota_1(g(y)) - g(y)$. Since $\iota_2(g(y)) = g(y)$, we have

$$(3.13) \qquad\qquad x(\iota_1(y) - y) = \tau(g(y)) - g(y)$$

yielding *(2)*.

Now assume *(2)*, that there exists $g \in \mathbb{C}(\overline{E_t})$ such that $x(\iota_1(y) - y) = \tau(g) - g$. We let $b_1 := x(\iota_1(y) - y) = x(\tau(y) - y)$ and $b_2 := \tau(y)(\tau(x) - x)$. We then have

$$(3.14) \qquad b_1 + b_2 = \tau(y)(\tau(x) - x) + x(\tau(y) - y) = \tau(xy) - xy.$$

We therefore have $b_2 = \tau(f) - f$ where $f = xy - g$. We shall show that $f \in \mathbb{C}(x)$ and $g \in \mathbb{C}(y)$, which implies that *(1)* holds.

To see that $f \in \mathbb{C}(x)$, note that $\iota_2\iota_1(b_2) = y(x - \iota_2\iota_1(x)) = y(x - \iota_2(x))$ which yields $\iota_1\iota_2\iota_1(b_2) = \iota_1(y)(x - \iota_1\iota_2(x)) = -b_2$ since $\iota_1\iota_2(y) = \iota_1(y)$. Combining this with $b_2 = \tau(f) - f = \iota_1\iota_2(f) - f$ yields

$$\iota_1(f) - \iota_1\iota_2\iota_1(f) = f - \iota_1\iota_2(f).$$

This implies that $\tau(\iota_1(f) - f) = \iota_1(f) - f$. Lemma 2.4.2 implies that $\iota_1(f) - f = c \in \mathbb{C}$. Applying ι_1 to this last equation implies that $f - \iota_1(f) = c$ so $c = 0$. Therefore f is left fixed by ι_1 and so must belong to $\mathbb{C}(x)$.

To see that $g \in \mathbb{C}(y)$, note that $\iota_1(b_1) = -b_1$. Combining this with $b_1 = \tau(g) - g$, we have

$$g - \iota_1\iota_2(g) = \iota_2(g) - \iota_1(g).$$

This implies that $\tau(\iota_2(g) - g) = \iota_2(g) - g$ so, as before $\iota_2(g) - g = c \in \mathbb{C}$. Applying ι_2 to this last equation implies $g - \iota_2(g) = c$ so $c = 0$. Therefore g is left fixed by ι_2 and so must belong to $\mathbb{C}(y)$. Therefore, $xy = f(x) + g(y) \in \mathbb{C}(\overline{E_t})$ with $f \in \mathbb{C}(x)$ and $g \in \mathbb{C}(y)$.

Let us now prove that there exist $F \in \mathbb{Q}(t)(x)$ and $G \in \mathbb{Q}(t)(y)$ such that $xy = F(x) + G(y) \in \mathbb{C}(\overline{E_t})$. Since $xy = f(x) + g(y) \in \mathbb{C}(\overline{E_t})$, there exists $r(x, y) \in \mathbb{C}(x, y)$ such that

$$(3.15) \qquad xy = f(x) + g(y) + K(x, y, t)r(x, y).$$

The ring extension $\mathbb{Q}(t) \subset R \subset \mathbb{C}$ generated over $\mathbb{Q}(t)$ by the coefficients of f, g and r as rational functions in x, y is finitely generated. Applying [Lang, Theorem 1.1 and Corollary 1.3, Ch.IX, §1], one can specialize these coefficients to elements in $\overline{\mathbb{Q}(t)}$, the algebraic closure of $\mathbb{Q}(t)$ such that (3.15) holds, that is, there exist $\overline{F}(x), \overline{G}(y), \overline{R}(x, y) \in \overline{\mathbb{Q}(t)}[x, y]$ such that

$$(3.16) \qquad xy = \overline{F}(x) + \overline{G}(y) + K(x, y, t)\overline{R}(x, y).$$

The coefficients of $\overline{F}(x), \overline{G}(y), \overline{R}(x, y)$ lie in a finite normal extension k of $\mathbb{Q}(t)$, so using the trace map and dividing by $[k : \mathbb{Q}(t)]$ one finds

$$(3.17) \qquad xy = F(x) + G(y) + K(x, y, t)R(x, y),$$

for some $F, G, R \in \mathbb{Q}(t)(x, y)$. This proves that the model decouples. $\qquad \square$

4. THE ORBIT RESIDUE CRITERION

In Section 3.2 we reviewed and refined results from [DHRS18] and [DHRS19], to conclude that to determine if a generating series of a quadrant model with infinite group is x- and y-D-algebraic it is enough to determine if the element $b = x(\iota_1(y) - y)$ has a certificate $g \in \mathbb{C}(\overline{E_t})$. This condition is equivalent to the cancellation of the orbit residues of the function b (see Proposition A.4). The definition of the orbit residues of b involves the computation of the poles of b and their orbits with respect to τ as well as various residues at these points. Nevertheless, we show below that there are *a priori* criteria that allow us to avoid these calculations. In Proposition 4.3 we show that if the poles of b behave in a certain way with respect to the involutions ι_1, ι_2 then the orbit residues are never zero. In Proposition 4.4 and 4.6 we show that for the remaining cases b has a certificate if and only if two distinguished poles lie in the same τ-orbit. This simplifies the application of Proposition A.4 and is exploited in our considerations of weighted quadrant walks having

D-algebraic generating series.

The potential poles of $b = x(\iota_1(y) - y)$ are the poles of x, y, and $\iota_1(y)$ in $\mathbb{P}^1 \times \mathbb{P}^1$:

- $P_i = (\infty, b_i)$ where $\infty = [1 : 0]$ and $b_i = [b_{i,0}, b_{i,1}]$, $i = 0, 1$,
- $Q_i = (a_i, \infty)$ where $a_i = [a_{i,0}, a_{i,1}]$, $i = 0, 1$,
- $\iota_1(Q_i) = (a_i, c_i)$ where $c_i = [c_{i,0}, c_{i,1}]$, $i = 0, 1$.

In the rest of the paper, we make the following convention: the indexes of the points $P_i, Q_k, \iota_1(Q_l)$ have to be considered modulo 2. For instance, if $Q_l = Q_1$ the point Q_{l+1} corresponds to Q_0.

4.1. **Symmetries and positions of the poles.** Note that $P_i = \iota_1(P_j)$ and $Q_i = \iota_2(Q_j)$ for $i \neq j$. We collect some useful facts concerning these points in the following Lemma. The notation $R \sim S$ for R, S points of $\overline{E_t}$ is used to denote the fact that there exists an $n \in \mathbb{Z}$ such that $R = \tau^n(S)$.

Lemma 4.1.   (1) $\iota_1(Q_i) = \tau^{-1}(Q_j)$ for $i \neq j$.
   (2) If $Q_i \sim P_j$ then $Q_{i+1} \sim P_{j+1}$.
   (3) If the point Q_i is fixed by ι_1 then $Q_i = P_j = (\infty, \infty)$ for some j or $Q_i = (0, \infty) := ([0 : 1], [1 : 0])$.
   (4) If the point P_i is fixed by ι_2 then $P_i = Q_j$ for some j or $P_i = (\infty, 0) := ([1 : 0], [0 : 1])$.

Proof. 1. The result follows from the facts that $\tau = \iota_2\iota_1$ and $Q_i = \iota_2(Q_j)$ with $i \neq j$.
2. Note that $\iota_1\tau^n = \tau^{-n}\iota_1$. For simplicity, assume $i = j = 1$. If $P_1 = \tau^n(Q_1)$, then $P_0 = \iota_1(P_1) = \tau^{-n}(\iota_1(Q_1)) = \tau^{-n-1}(Q_0)$ since $\iota_1(Q_1) = \tau^{-1}(Q_0)$.
3. Since $\overline{K}(a_{i,0}, a_{i,1}, y_0, y_1) = 0$ has $y_1 = 0$ as a solution, we see that $\overline{K}(a_{i,0}, a_{i,1}, y_0, y_1)$ has no y_0^2 term, that is,

$$\overline{K}(a_{i,0}, a_{i,1}, y_0, y_1) = (a_{i,0}a_{i,1} - t\sum_{\ell=0}^{2} d_{\ell-1,0}a_{i,0}^{\ell}a_{i,1}^{2-\ell})y_0y_1$$

$$+ t(\sum_{\ell=0}^{2} d_{\ell-1,-1}a_{i,0}^{\ell}a_{i,1}^{2-\ell})y_1^2.$$

If Q_i is fixed by ι_1 this expression must have no y_0y_1 term so $a_{i,0}a_{i,1} - t\sum d_{\ell-1,0}a_{i,0}^{\ell}a_{i,1}^{2-\ell} = 0$. Since t is transcendental over $\mathbb{Q}$, we have

$$a_{i,0}a_{i,1} = 0 = \sum d_{\ell-1,0}a_{i,0}^{\ell}a_{i,1}^{2-\ell}$$

which implies that either $Q_i = P_j = (\infty, \infty)$ or $Q_i = (0, \infty)$.
Claim 4. is entirely symmetric to 3. $\square$

We will use the following alternative expression for b (c.f., [DHRS18, Lemma 4.11]):

(4.1) $$b^2 = \frac{x_0^2 \Delta_{[x_0:x_1]}^{x}}{x_1^2(\sum_{i=1}^{2} x_0^i x_1^{2-i} t d_{i-1,1})^2}$$

where $\Delta_{[x_0:x_1]}^{x}$ is the discriminant of the polynomial $y \mapsto K(x_0, x_1, y, t)$,

(4.2)
$$\Delta_{[x_0:x_1]}^{x} = t^2\Big[(d_{-1,0}x_1^2 - \frac{1}{t}x_0x_1 + d_{1,0}x_0^2)^2$$
$$- 4(d_{-1,1}x_1^2 + d_{0,1}x_0x_1 + d_{1,1}x_0^2)(d_{-1,-1}x_1^2 + d_{0,-1}x_0x_1 + d_{1,-1}x_0^2)\Big].$$

Let us first give a symmetry argument which will allow us to simplify the enumeration of the distinct poles configurations. Let $d_{i,j}$ be a set of weights and let us denote by $K(x,y)$ the associated kernel polynomial and by $\overline{E_t}$ the kernel curve. Let us consider now the polynomial $\widetilde{K}(\widetilde{x}, \widetilde{y}) = \widetilde{x}\widetilde{y} - t \sum_{i,j} d_{j,i} \widetilde{x}^i \widetilde{y}^j$ and the corresponding projective curve $\widetilde{E_t}$. These objects are obtained by exchanging the roles of x and y. Let us denote by $\widetilde{\iota_1}, \widetilde{\iota_2}, \widetilde{\tau}$ the horizontal, vertical switches and the automorphism of the walk on $\widetilde{E_t}$. Moreover, we denote by $\widetilde{b}$ the element of $\mathbb{C}(\widetilde{E_t}) = \mathbb{C}(\widetilde{x}, \widetilde{y})$ defined by $\widetilde{x}(\widetilde{\iota_1}(\widetilde{y}) - \widetilde{y})$. The following holds.

Lemma 4.2. *The morphism $\phi : \overline{E_t} \to \widetilde{E_t}, (a,b) \mapsto (b,a)$ is an isomorphism such that*

- $\widetilde{\iota_2} \circ \phi = \phi \circ \iota_1$,
- $\widetilde{\iota_1} \circ \phi = \phi \circ \iota_2$,
- $\widetilde{\tau}^{-1} \circ \phi = \phi \circ \tau$.

In particular $\overline{E_t}$ is a curve of genus one and τ has infinite order if and only if $\widetilde{E_t}$ is a curve of genus one and $\widetilde{\tau}$ has infinite order. Moreover, b has a certificate g if and only if $\widetilde{b}$ has a certificate $\widetilde{g}$.

Proof. The first part of the Lemma is obvious since the inverse of ϕ is given by $\phi^{-1}((c,d)) = (d,c)$. The equivalence is entirely symmetric so that one just has to prove one direction. Let us assume that $\widetilde{b}$ has a certificate $\widetilde{g}$, that is,

$$(4.3) \qquad\qquad \widetilde{b} = \widetilde{\tau}(\widetilde{g}) - \widetilde{g}.$$

The isomorphism ϕ induces an isomorphism $\psi : \mathbb{C}(\widetilde{E_t}) \to \mathbb{C}(\overline{E_t}), f \mapsto f \circ \phi$. Noting that $\widetilde{\iota_2} \circ \phi = \phi \circ \iota_1$ as morphism from $\overline{E_t}$ to $\widetilde{E_t}$, one obtains by duality that $\iota_1 \circ \psi = \psi \circ \widetilde{\iota_2}$ on the function fields. Similarly, one has $\iota_2 \circ \psi = \psi \circ \widetilde{\iota_1}$ and $\tau^{-1} \circ \psi = \psi \circ \widetilde{\tau}$. Applying ψ to (4.3) and noting that $\psi(\widetilde{x}) = y$ and $\psi(\widetilde{y}) = x$ yields

$$\psi(\widetilde{b}) = \psi(\widetilde{x})(\psi\widetilde{\iota_1}(\widetilde{y}) - \psi(\widetilde{y})) = \psi\widetilde{\tau}(\widetilde{g}) - \widetilde{g}$$
$$= y(\iota_2(x) - x) = \tau^{-1}(\psi(\widetilde{g})) - \psi(\widetilde{g}).$$

Setting $g = -\tau^{-1}(\psi(\widetilde{g}))$, one finds $\tau(g) = -\psi(\widetilde{g})$ because the action of τ on $\mathbb{C}(\overline{E_t})$ is $\mathbb{C}$-linear and thereby $\psi(\widetilde{b}) = y(\iota_2(x) - x) = \tau(g) - g$. We apply ι_1 to the latter equation and, noting that $\tau = \iota_1\iota_2$ by Remark 2.3, find

$$\iota_1(y)(\tau(x) - x) = \iota_1\tau(g) - \iota_1(g) = \iota_2(g) - \iota_1(g) = -h + \tau(h),$$

where $h = -\iota_2(g)$ (Noting that $\tau(h) = \iota_1\iota_2(h) = -\iota_1(g)$). Thus the function $c = \iota_1(y)(\tau(x) - x)$ has a certificate. Since $b + c = \tau(xy) - xy$, we conclude that b has also a certificate. This ends the proof. $\qquad\square$

4.2. The involutions.

In this section, we study the behavior of the orbit residues of b when its poles are fixed by involutions. Our proof proceeds by considering the various configurations and orders of the poles. To do this one determines the order of vanishing of the numerators and denominators of the expression on the right hand side of (4.1). Useful facts for carrying out this task are:

- As noted in the proof of Lemma 4.1(2), at the points $Q_i = (a_i, \infty)$ where $y_0 = 0$, we have that $\sum_{i=1}^2 x_0^i x_1^{2-i} t d_{i-1,1}$ vanishes. If $Q_0 = Q_1$, we have that his latter expression has a double zero.
- If we have a point R where $\iota_1(R) = R$, then $\Delta^x_{[x_0:x_1]} = 0$ at this point. In particular this happens when $P_1 = P_0$ or $Q_i = \iota_1(Q_i)$. Furthermore, at this point one has ramification and the order of $x = [x_0 : x_1]$ is 2.

In what follows we will state the polar divisor $(b)_\infty$ and residue configurations and rely on the reader to do the simple verification using the facts.

Proposition 4.3. *Assume that $\overline{E_t}$ is a curve of genus one and that the automorphism of the walk is not of finite order. If one of the P_i's and one of the Q_j's is fixed by an involution then the function b has no certificate.*

Proof. By Proposition A.4, the function b has a certificate if and only if its orbit residues are zero. We shall frequently use the fact that since τ has infinite order, if $\tau^n(Q) = Q$ for some point Q then $n = 0$. This follows from the fact that $\tau(Q) = Q \oplus P$ where P has infinite order in the groups structure on $\overline{E_t}$ (see the remarks following Lemma 2.4).

We now use a case-by-case argument to prove this proposition.

Case a: P_j is fixed by ι_1 and Q_i is fixed by ι_1.

By Lemma 4.1, we find that either $Q_i = P_0 = P_1$ or $Q_i = (0, \infty)$. Moreover, $Q_i \neq Q_{i+1}$ since otherwise $\tau(Q_i) = Q_i$ and τ would be the identity.

- *Case a.1: $Q_i = P_0 = P_1$.* Then, the polar divisor $(b)_\infty$ of b is $3P_1 + \epsilon Q_{i+1} + \epsilon\tau^{-1}(Q_i)$ where ϵ is zero if $Q_{i+1} = (0, \infty)$ and otherwise $\epsilon = 1$. It is easily seen that the orbit residue of order 3 of P_1 is never zero.

- *Case a.2: $P_0 = P_1$ and $Q_i = (0, \infty) \neq Q_{i+1}$.* In that situation, $Q_{i+1} = \tau(Q_i)$ and $\iota_1(Q_{i+1}) = \tau^{-1}(Q_i)$, Lemma A.9 allows one to show that the residues of b are as follows

Points	P_0	$\tau(Q_i)$	$\tau^{-1}(Q_i)$
Residues of order 1	α	β	β

  with $\alpha + 2\beta = 0$ and $\alpha, \beta \neq 0$. Then, the orbit residues of b are all zero if and only if $P_0 \sim Q_i$. This last condition will never happen. Suppose to the contrary that $P_0 = \tau^n(Q_i)$ then $\iota_1(P_0) = P_0 = \tau^{-n}(\iota_1(Q_i)) = \tau^{-n}(Q_i)$. Thus $\tau^{2n}(Q_i) = Q_i$ which implies $n = 0$. This is absurd since $Q_i = (0, \infty)$ and $P_0 = (\infty, [b_{0,0} : b_{0,1}])$.

Case b: P_j is fixed by ι_2 and Q_i is fixed by ι_2.
This case is symmetric with Case a by exchanging x and y. Lemma 4.2 allows to conclude that b has no certificate in that case either.

Case c: Q_j fixed by ι_2 and P_i fixed by ι_1
In that case, note that $P_0 = P_1$ and $Q_0 = Q_1$. Moreover, since τ is not the identity, one has $Q_0 \neq P_0$. Lemma A.9 allows to show that $(b)_\infty = P_0 + \epsilon Q_0 + \epsilon\tau^{-1}(Q_0)$ with $\epsilon = 1$ if $Q_0 = (0, \infty)$ and $\epsilon = 2$ if $Q_0 \neq (0, \infty)$. Thus, the residues of b are as follows

Points	P_0	Q_0	$\tau^{-1}(Q_0)$
Residues of order 1	α	β	β
Residues of order 2	0	γ	$-\gamma$

with $\alpha + 2\beta = 0$ and $\alpha \neq 0$, $\beta \neq 0$ if $Q_0 = (0, \infty)$ and $\gamma \neq 0$ if and only if $Q_0 \neq (0, \infty)$. Thus the orbit residues are zero if and only if $P_0 \sim Q_0$. The latter condition is never true. Indeed, if $P_0 = \tau^n(Q_0)$ then

$$\iota_1(P_0) = P_0 = \tau^{-n}(\iota_1(Q_0)) = \tau^{-n}(\iota_1(\iota_2(Q_0))) = \tau^{-n-1}(Q_0) = \tau^n(Q_0).$$

Since τ is of infinite order, we must have $n = -n - 1$ which is absurd since $n \in \mathbb{Z}$.

Case d: Q_j fixed by ι_1 and P_i fixed by ι_2.
Using Lemma 4.1, we see that if Q_j is fixed by ι_1 then $Q_j = (0, \infty)$ or (∞, ∞). Moreover, if P_i is fixed by ι_2 then $P_i = (\infty, 0)$ or (∞, ∞). Some of these possibilities will never occur:

- if $P_i = (\infty, \infty)$ is fixed by ι_2 then $P_i = Q_0 = Q_1$. Thus, none of the Q_j's can be fixed by ι_1. Otherwise, P_i would be fixed by τ.
- if $P_i = (\infty, 0)$ is fixed by ι_2 then $P_{i+1} \notin \{Q_j, Q_{j+1}\}$. Indeed if $P_{i+1} = Q_j$ then $P_{i+1} = P_i = Q_j$ because Q_j is fixed by ι_1. This is absurd since $P_i = (\infty, 0)$ and $Q_j = (a, \infty)$. If $P_{i+1} = Q_{j+1}$ then $\tau^3(Q_j) = Q_j$ which is absurd since τ has infinite order.

Thus the only possibility is $Q_j = (0, \infty)$ fixed by ι_1, $Q_{j+1} \notin \{P_i, P_{i+1}\}$ and $P_i = (\infty, 0)$ fixed by ι_2. The polar divisor of $(b)_\infty$ is $P_0 + P_1 + \tau(Q_j) + \tau^{-1}(Q_j)$ and using Lemma A.9, one gets

Points	P_0	P_1	$\tau(Q_j)$	$\tau^{-1}(Q_j)$
Residues of order 1	α	α	β	β

where $2\alpha + 2\beta = 0$ and $\alpha, \beta \neq 0$. Noting that $P_0 \sim Q_j$ if and only if $P_1 \sim Q_j$, one sees that b has orbit residues zero (in one or two orbits) if and only if $P_i \sim Q_j$. The latter condition is never true. Indeed, if $Q_j = \tau^n(P_i)$ then $\iota_1(Q_j) = Q_j = \tau^{-n}(\iota_1(P_i)) = \tau^{-n-1}(P_i) = \tau^n(P_i)$. Since τ is not of finite order, we must have $n = -n - 1$. Absurd since $n \in \mathbb{Z}$. $\qquad\square$

4.3. **Remaining cases.** In this section, we shall consider the cases where one of the P_i's and one the Q_j's are not simultaneously fixed by an involution. We shall prove that b has orbit residues zero if and only if two precise points of the polar divisor are in the same orbit.

We distinguish two cases: $d_{1,1} = 0$ and $d_{1,1} \neq 0$. They corresponds to the fact that the point (∞, ∞) belongs to the curve or not.

Proposition 4.4. *Assume that $d_{1,1} = 0$, $\overline{E_t}$ is a genus one curve and τ is of infinite order. Assume moreover that one of the P_i's and one of the Q_j's are not simultaneously fixed by an involution. Then, b has a certificate if and only if $P_0 \sim P_1$.*

Proof. Note that $P_j = Q_k = (\infty, \infty)$ for some j, k. Moreover since we assume that one of the P_i's and one the Q_j's are not simultaneously fixed by an involution, we have $P_0 \neq P_1$ and $Q_0 \neq Q_1$. Indeed, if for instance $P_0 = P_1 = Q_k$ then P_0 and Q_k are fixed by the involution contradicting our hypothesis. We shall prove the statement case by case according to the configuration of poles of b.

Case a. *There are no other equalities among the base points:* Then the polar divisor $(b)_\infty$ of b equals $2P_j + 2P_{j+1} + \tau(P_{j+1}) + \tau^{-1}(P_j)$. Lemma A.9 shows that the residues of b are as follows

Points	P_j	P_{j+1}	$\tau(P_{j+1})$	$\tau^{-1}(P_j)$
Residues of order 1	α	α	β	β
Residues of order 2	γ	$-\gamma$	0	0

with $2\alpha + 2\beta = 0$, $\beta, \gamma \neq 0$. Then the orbit residues are zero if and only if $P_j \sim P_{j+1}$.

Case b. $P_j = Q_k$ *and* $Q_{k+1} = (0, \infty)$
- *Case b.1: and there are no other equalities among the base points:* Then the polar divisor $(b)_\infty$ of b equals $2P_j + 2P_{j+1}$. Lemma A.9 shows that the residues of b are as follows

| Points	P_j	P_{j+1}
Residues of order 1	α	α
Residues of order 2	γ	$-\gamma$

with $2\alpha = 0$, $\gamma \neq 0$. Then the orbit residues are zero if and only if $P_j \sim P_{j+1}$.

- *Case b.2: and Q_{k+1} is fixed by ι_1* Note that $P_0 \neq P_1$ as noted in the first paragraph of the proof. Moreover, since Q_{k+1} is fixed by ι_1, we get that $P_j = \tau^2(P_{j+1})$ so that $P_0 \sim P_1$. The divisor is the same than in Case b.1. and and since P_0 is in the same orbit than P_1, the orbit residues are always zero.

There are no other cases since the remaining configurations will correspond to the situations where one of the P_i's and one the Q_j's are simultaneously fixed by an involution. □

Remark 4.5. *In the proof of Proposition 4.4, we prove that if $P_j = Q_k$ and $Q_{k+1} = (0, \infty)$ is fixed by ι_1 the function b always has a certificate. This corresponds to walks where the directions North East, North West and West do not belong to the steps set. The models of such walks are as follows*

$$w_{IIB.1} \qquad w_{IIB.2} \qquad w_{IIB.6}$$
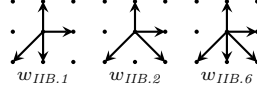
That is we prove that among the 9 models of walks that were differentially algebraic when unweighted, the three models above remain differentially algebraic with weights.

Proposition 4.6. *Assume that $d_{1,1} \neq 0$, $\overline{E_t}$ is a genus one curve and τ is of infinite order. Assume moreover that the P_i's and the Q_j's are not simultaneously fixed by an involution. Then, b has a certificate if and only if there exist j and k such that $P_j \sim Q_k$.*

Proof. Since $d_{1,1} \neq 0$, the sets $\{P_0, P_1\}$ and $\{Q_0, Q_1\}$ have empty intersection.

Case a.: Assume the six points $P_i, Q_i, \iota_1(Q_i), \ i = 0, 1$ are all distinct

- *Case a.1: and $Q_i \neq (0, \infty)$:* Then, $(b)_\infty = P_0 + P_1 + Q_0 + \tau^{-1}(Q_1) + Q_1 + \tau^{-1}(Q_0)$.

  Since $\iota_1(b) = -b$, Lemma A.9 implies that the residues are given by

Points	P_0	P_1	Q_0	$\tau^{-1}(Q_1)$	Q_1	$\tau^{-1}(Q_0)$
Residues of order 1	α	α	β	β	γ	γ

  with $2\alpha + 2\beta + 2\gamma = 0$ and $\alpha, \beta, \gamma \neq 0$. Assume that all the orbit residue are zero. Since $\alpha \neq 0$ the set $\{P_0, P_1\}$ cannot form a single τ-orbit. Therefore $P_i \sim Q_j$ for some i, j. Conversely assume that $P_i \sim Q_j$. Then, by Lemma 4.12.), we have $P_{i+1} \sim Q_{j+1}$. We then have that either there are two τ-orbits $\{P_{i+\epsilon}, Q_{j+\epsilon}, \tau^{-1}(Q_{j+\epsilon})\}, \epsilon = 0, 1$, each of whose orbit residues are $\alpha + \beta + \gamma = 0$ or there is one τ-orbit $\{P_0, P_1, Q_0, \tau^{-1}(Q_1), Q_1, \tau^{-1}(Q_0)\}$ whose orbit residue is $2\alpha + 2\beta + 2\gamma = 0$. Thus the orbit residues are all zero.
- *Case a.2: $Q_i = (0, \infty)$:* For simplicity assume $Q_1 = (0, \infty)$. In this case, $[0 : 1]$ is a double zero of both the numerator and denominator of (4.1) so Q_1 and $\iota_1(Q_1)$ are not poles. Therefore $(b)_\infty = P_0 + P_1 + Q_0 + \tau^{-1}(Q_1)$. Since $\iota_1(b) = -b$, Lemma A.9.2 implies that the residues are given by

Points	P_0	P_1	Q_0	$\tau^{-1}(Q_1)$
Residues of order 1	α	α	β	β

One easily modifies the argument above to prove the Proposition in this case.

We now examine all of the cases when at least two of the putative poles coincide. Notice that we always have that $\tau^{-1}(Q_i) \neq Q_i$ since τ has infinite order (see the remarks following Lemma 2.4).

Case b: $Q_0 = Q_1$

- *Case b.1: and $P_0, P_1, Q_1, \iota_1(Q_1)$ distinct; $Q_1 \neq (0, \infty)$.* In that case, the polar divisor of b is $(b)_\infty = P_0 + P_1 + 2Q_1 + 2\tau^{-1}(Q_1)$. Lemma A.9 implies that the configuration of residues is

Points	P_0	P_1	$2Q_1$	$2\tau^{-1}(Q_1)$
Residues of order 1	α	α	β	β
Residues of order 2			γ	$-\gamma$

  with $2\alpha + 2\beta = 0$ and $\alpha, \gamma \neq 0$. If the orbit sums are zero then $\{P_0, P_1\}$ cannot be an orbit so for some i, j some $P_i \sim Q_j$ ($P_i \neq Q_j$ by assumption). If $P_i \sim Q_j$ then, since $Q_0 = Q_1$, Lemma 4.1 implies that all the poles must lie in the same orbit. Lemma A.9 implies that all orbit sums are zero.
- *Case b.2: and $P_0, P_1, Q_1, \iota_1(Q_1)$ distinct; $Q_1 = (0, \infty)$.* In that case, the polar divisor of b is $(b)_\infty = P_0 + P_1 + Q_1 + \tau^{-1}(Q_1)$.

Points	P_0	P_1	Q_1	$\tau^{-1}(Q_1)$
Residues of order 1	α	α	β	β

  The argument is similar to Case a.1).

Case c : $P_0 = P_1$ This case is obtained by symmetry exchanging x and y from Case b. Lemma 4.2 allows to conclude. Note that the condition $P_i \sim Q_j$ becomes $Q_i \sim P_j$. That is, this condition remains unchanged by symmetry.

Finally, it remains to study the situation when the poles of y and the zero of x coincide, that is, when $Q_i = (0, \infty)$ for some i. In that case, the points Q_i and $\iota_1(Q_i)$ are no longer poles of b (see Formula 4.16 in [DHRS18] and the Case a.2)

Case d : $Q_i = (0, \infty)$

- *Case d.1: and there are no other equalities among the base points:* This is a.2.
- *Case d.2: and Q_i fixed by ι_1* The divisor is the same than in a.2.
- *Case d.3: and Q_{i+1} is fixed by ι_1* This case can not occur. Indeed, Lemma 4.1 implies that $Q_{i+1} = P_l$ or $Q_{i+1} = (0, \infty) = Q_i$. The first case contradicts the assumption $d_{1,1} \neq 0$ whereas the second implies $\tau(Q_i) = \iota_2(\iota_1(Q_i)) = \iota_2(Q_i) = Q_{i+1} = Q_i$ which is in contradiction with the fact that the curve has genus 1.
- *Case d.4: and P_0 is fixed by ι_1:* Assume $Q_1 = (0, \infty)$. In this case $d_{-1,1} = 0$ so b does not have a pole at Q_1. $(b)_\infty = P_0 + Q_0 + \iota_1(Q_0)$.

Points	P_0	Q_0	$\iota_1(Q_0)$
Residues of order 1	α	β	β

  with $\alpha\beta \neq 0$. Lemma A.9 implies that $\alpha + 2\beta = 0$. If the orbit residues are zero, then we must have all the poles in the same orbit, so $P_0 \sim Q_0$. If $P_0 \sim Q_0$, then $P_0 = P_1 \sim Q_1 \sim \tau^{-1}(Q_1) = \iota_1(Q_0)$, so all the poles are in

the same orbit. If $P_0 \sim Q_1$, then $P_0 \sim \tau^{-1}(Q_1) = \iota_1(Q_0)$ so all the poles are in the same orbit and the orbit sum is zero.

$\square$

Note that many cases disappear because we avoid having a Q_i and a P_j fixed simultaneously by an involution and also avoid one of the Q_i equaling one of the P_j. We summarize the results of this section and combine them with Theorem 3.8 to find the following Theorem:

Theorem 4.7. *Assume that $\overline{E_t}$ is a curve of genus one and that the automorphism of the walk is not of finite order. Then,*

- *if one of the P_i's and one of the Q_i's is fixed by an involution then the generating series $Q(x, 0, t)$ and $Q(0, y, t)$ are x and y-differentially transcendental over $\mathbb{Q}(x, y)$,*
- *if one of the P_i's and one of the Q_j's are not simultaneously fixed by an involution, the following holds*
  - *Case $d_{1,1} \neq 0$: the generating series $Q(x, 0, t)$ and $Q(0, y, t)$ are x and y-differentially algebraic over $\mathbb{Q}(x, y)$ if and only if there exist j, k such that $P_j \sim Q_k$;*
  - *Case $d_{1,1} = 0$: the generating series $Q(x, 0, t)$ and $Q(0, y, t)$ are x and y-differentially algebraic over $\mathbb{Q}(x, y)$ if and only if there exist j, k such that $P_0 \sim P_1$. Moreover, this last condition is automatically fulfilled if $(0, \infty)$ belongs to the curve $\overline{E_t}$ and is fixed by ι_1.*

5. DETERMINING WEIGHTS FOR WHICH THE GENERATING SERIES ARE D-ALGEBRAIC.

In Section 4, we show that either b has no certificate or that the existence of a certificate is equivalent to two special points being in the same τ-orbit. In this section we will describe an algorithm and its refinements to decide the question of two such points being in the same orbit.

The algorithm and its refinement are based on well known tools developed in arithmetic algebraic geometry to study elliptic surfaces, that is, families of elliptic curves. In particular the Neron-Tate height $\hat{h}$ on elliptic curves E over function fields k[3], is the crucial ingredient. This is a function $\hat{h} : E(k) \to \mathbb{R}$ one of whose properties is that if $P, Q \in E(k)$ and $Q = nP, n \in \mathbb{Z}$ (which means that Q is the n-multiple of P with respect to the group law defined on $E(k)$) then $\hat{h}(Q) = n^2 \hat{h}(P)$. In Section 5.1, we describe how the question of determining if points P and Q lie in the same τ-orbit can be reduced to deciding if some point is a multiple of another point.

For fixed values of the weights, the Sage Package `comb_walks` (see [BCJPL20]) allows one to calculate nP for fixed integers n and points $P \in E$ as well as the necessary ancillary objects. In addition an implemented algorithm in MAGMA computes exactly the height of a point P and so, for fixed weights, one can calculate if P and Q lie in the same orbit. However, our goal is to characterize the D-algebraicity of a weighted model in terms of a set of polynomial equations on the weights. Therefore, we need to unravel the height computation. The height of a point P is given by a formula (5.1) involving certain numerical data associated with E, P and Q. In Section 5.1 we show how one can determine $\hat{h}(Q), \hat{h}(P)$ up to a finite number of possibilities by estimating these numerical data using the celebrated Tate algorithm (calculating the Weierstrass equation equation for $\overline{E_t}$ and deducing certain properties from tables produced by Tate) as well as estimating the other

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[3] for instance $k = \mathbb{C}(t)$

numerical data by further consulting tables produced by Kodaira, Néron, Oguiso and Shioda. From the possible values of $\hat{h}(Q), \hat{h}(P)$, we can determine a finite set of possible n with $\hat{h}(Q) = n^2\hat{h}(P)$. A computation then allows one to determine which values of n (if any) imply $Q = \tau^n(P)$ for some integer n. We emphasize that thanks to the deep work of those authors, once the Weierstrass equation is determined, only simple arithmetic is required to carry out this algorithm.

The key object lying behind these calculations is an elliptic surface associated with E. In Section 5.2, we construct this elliptic surface by blowing up the base points of the pencil of elliptic curves attached to $\overline{E_t}$ that is, the points in $\mathbb{P}^1 \times \mathbb{P}^1$ which belong to each member of the pencil. We use it to refine the algorithm of Section 5.1. This point of view emphasizes the importance of the relative position of the base points in the study of the D-algebraicity of the weighted model and also allows one one to reduce drastically the number of possible values of n as well as other information related to the mapping τ.

5.1. **An algorithm.** So far, we have considered the kernel of the walk as defining, for a fixed $t \in \mathbb{C}$, transcendental over $\mathbb{Q}$ a curve $\overline{E_t} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. The algorithm described in this and the next section depends on another object associated with the kernel. We now consider t as a variable and consider $\overline{E_t}$ as an elliptic curve defined over the field $\mathbb{C}(t)$. The group law of this elliptic curve is defined over $\mathbb{C}(t)$ and and we consider the maps ι_1, ι_2, τ as automorphisms of $\overline{E_t}$. We will make use of the *Kodaira-Néron model* $\mathcal{S}$ associated to $\overline{E_t}$ (see [SS19, Def. 5.18 and 5.2 and Proposition 5.4] for the most recent reference on the subject but also [Dui10, OS91, Shi90, Sil94] as general references). In Section 5.2 we will give a description of the construction of $\mathcal{S}$ as well as a more precise explanation of its properties but for this algorithm we will only need the following properties:

(1) $\mathcal{S}$ is a smooth projective rational surface defined over $\mathbb{C}$ with a surjective morphism $\pi : \mathcal{S} \to \mathbb{P}^1(\mathbb{C})$;
(2) Almost all fibers are isomorphic to $\overline{E_t}$, that is, they are nonsingular elliptic curves.
(3) The remaining fibers (finite in number) are called singular fibers and are singular (reduced) curves. The fiber over 0 is singular.
(4) There exists a section $\sigma_0 : \mathbb{P}^1(\mathbb{C}) \to \mathcal{S}$ $(\pi \circ \sigma_0 = \mathrm{id}_\mathcal{S})$ and there is a bijection between $\mathbb{C}(t)$-points P of $\overline{E_t}$ and sections $\sigma_P : \mathbb{P}^1 \to \mathcal{S}$ $(\pi \circ \sigma_P = \mathrm{id}_\mathcal{S})$ so that σ_0 corresponds to the origin of the elliptic curve $\overline{E_t}$.

We wish to emphasize that although we allow t to vary when we consider the surface $\mathcal{S}$ and specialize it to 0, we do not specialize t in the generating series. For all statements in this paper concerning the generating series we are assuming t is transcendental over $\mathbb{Q}$ (or, in general, the field of definition of the weights).

Let us denote by $\mathcal{P}$ the image in $\mathcal{S}$, of the section σ_P corresponding to a $\mathbb{C}(t)$-point P of $\overline{E_t}$. The image $\mathcal{P}$ is then a curve in the surface $\mathcal{S}$. Abusing terminology, we shall call $\mathcal{P}$ the section associated to P. The Néron-Tate height of a point P is defined in terms of a numerical invariant of $\mathcal{S}$, how the section $\mathcal{P}$ intersects $\mathcal{O}$, the section corresponding to the origin O of $\overline{E_t}$, and how $\mathcal{P}$ intersects some of the singular fibers. The (at first intimidating) formula defining the Néron-Tate height is

$$(5.1) \qquad \hat{h}(P) = 2\chi(\mathcal{S}) + 2(\mathcal{P}.\mathcal{O}) - \sum_{v \in R} \mathrm{contr}_v(P)$$

The term $\chi(\mathcal{S})$ is the arithmetic genus of $\mathcal{S}$. By Lemma 5.5, the surface $\mathcal{S}$ is rational so that its arithmetic genus is 1 ([SS19, Proposition 7.1]). The term $(\mathcal{P}.\mathcal{O})$ is the intersection number of $\mathcal{P}$ and $\mathcal{O}$, where $\mathcal{O}$ is the section corresponding to the origin of $\overline{E_t}$. In our applications, these sections are disjoint so, for us, $(\mathcal{P}.\mathcal{O}) = 0$.

For the remaining sum, R is the finite set of singular fibers v and $\mathrm{contr}_v(P)$ is a rational number determined by how $\mathcal{P}$ intersects the components of v. Much is known about R and the numbers $\mathrm{contr}_v(P)$.

Kodaira [Kod64, Kod66] and Néron [N64] classified the types of fibers which can occur in such a fibration (see also [Sil94, Ch.IV,§9, Table 4.1]). Based on the configuration of the intersections of the components of such a fiber v, one associates a root lattice T_v of type $A, D,$ or E. Up to a finite number of possibilities, $\mathrm{contr}_v(\mathcal{P})$ is determined by the root lattice of the fiber T_v. This information is summarized in Table 5.1 (see [SS19, Table 6.1], [Shi90, (8.16)],[Dui10, Lemma 7.5.3]).

Kodaira Fiber Type	III	III^*	IV	IV^*	$I_n(n > 1)$	I_n^*
Root Lattice T_v of Fiber	A_1	E_7	A_2	E_6	A_{n-1}	D_{n+4}
Possible $\mathrm{contr}_v(P)$	$1/2$	$3/2$	$2/3$	$4/3$	$\begin{aligned}&i(n-i)/n\\&0 \leq i \leq n-1\end{aligned}$	$\begin{cases}1, & i = 1 \\ 1 + n/4, & i > 1\end{cases}$

TABLE 5.1. This table gives the range of possibilities for $\mathrm{contr}_v(P)$. In Section 5.2 we show how i can be determined exactly based on the explicit construction of $\mathcal{S}$ and the specific $\mathcal{P}$ but for now we are only concerned with knowing the finite set of possibilities[4].

The direct sum $T = \oplus_{v \in R} T_v$ is defined to be the *root lattice* associated with the the singular fibers. In [OS91], Oguiso and Shiota give a finite list of the possible root lattices which can occur (there are 74). This implies that if one can determine T_v for at least one fiber, then seeing which root lattices contain T_v allows one to determine the term $\sum_{v \in R} \mathrm{contr}_v(P)$ in (5.1) up to a finite set of possibilities.

Remark 5.1. *By* [SS19, Theorem 6.20], *a point* $P \in \overline{E_t}(\mathbb{C}(t))$ *has height zero if and only* P *is a torsion point. Choosing some point* O *to be the origin of* $\overline{E_t}$, *one remarks that* $\tau^n(O) = O$ *if and only if* $n\tau(O) = O$ *if and only if* $\tau(O)$ *is a torsion point. Therefore the group of the walk is finite if and only* $\hat{h}(\tau(O)) = 0$. *In that situation, the order of the group is* $2n$ *where* n *is the order of torsion of* $\tau(O)$. *If one knows the root lattice of the singular fibers,* [SS19, Table 8.2] *gives the torsion subgroup and thereby an upper-bound for the order of the group of the walk. By* [SS19, Cor. 8.21], *the order of the torsion is bounded by* 6 *and therefore the order of the group is bounded by* 12. *Note that we are considering the group of the walk acting on a generic fiber. If one considers its action on an arbitrary fiber, its order might be bigger than* 12 *but less than* 24 *by Mazur Theorem (assuming that the fiber is defined over* $\mathbb{Q}$*; see* [JTZ17]*).*

An algorithm due to Tate [Tat75] allows us to determine the type of any fiber. We shall use it only to determine the type of the fiber above 0. The Tate algorithm relies on the Weierstrass model of $\overline{E_t}$ (see [SS19, Sections 5.7 and 5.8] and also [Sil94, Ch. IV, §9], [Dui10, Lemma 6.3.1]). This leads to the following algorithm.

---

[4]Kodaira's classification of fiber types included an additional fiber referred to as type II^*. It is not included in this table since in this situation any point $P \in \overline{E_t}(\mathbb{C}(t))$ has finite order and the group of the walk is finite (see [SS19, Table 8.2]).

Algorithm. As noted in Remark 5.1, if the automorphism τ has finite order, then its order is bounded by 6. Calculating τ^n for $1 \leq n \leq 6$ will give polynomial conditions on the $d_{i,j}$ equivalent to τ being of finite order, c.f. [KY15] (in Section 5.2 we will see that a more careful examination of $\mathcal{S}$ and its Mordell-Weil Lattice will yield such equations directly). We can therefore assume that τ is of infinite order and that we are given a kernel K whose associated curve satisfies the conditions of Proposition 4.4 or Proposition 4.6. These propositions say that b has a certificate if and only if two distinct $\mathbb{C}$-points (which we will denote by N and M) of the curve are in the same τ-orbit. By Lemma 5.7, the curves $\mathcal{M}$ and $\tau(\mathcal{N})$ do not intersect $\mathcal{N}$ in $\mathcal{S}$. We will show how to decide if $\tau^n(N) = M$ for some $n \in \mathbb{Z}$. We have freedom to select the point of $\overline{E_t}$ that will be the origin O of the associated group and so we will let $O = N$. Recall that $\tau(P) = P \oplus \tau(N)$ for any point P, so we have that if $\tau^n(N) = M$, then $M = n\tau(N)$. In particular, $\hat{h}(M) = n^2 \hat{h}(N)$. We will first find a finite set H of rational numbers, depending on K, such that if Q is any point of $\overline{E_t}$ such that the corresponding curve $\mathcal{Q}$ does not intersect $\mathcal{O}$, then $\widehat{h}(Q) \in H$. Since this hypothesis holds for M and $\tau(N)$, we can compare all pairs of values r_1, r_2 in H and determine all integers n such that $n^2 = r_1/r_2$. For these integers, a computation will check if $\tau^n(N) = M$.

We shall now give a more detailed description of this algorithm together with the associated computations for the weighted model $w_{IIC.2}$.

Input: The homogenized kernel $K(t, x_0, x_1, y_0, y_1)$ of a model with indeterminate weights $\{d_{i,j}\}$. We assume that $K(t, x_0, x_1, y_0, y_1)$ is irreducible in $\mathbb{C}[t, x_0, x_1, y_0, y_1]$ and that the associated curve $\overline{E_t}$ has no singularities. We furthermore assume that the associated group is infinite.

Output: Polynomial conditions on the $\{d_{i,j}\}$ that are equivalent to the associated generating series being differentially algebraic.

Note that the conditions on K can be checked algorithmically and imply that $\overline{E_t}$ has genus 1. As noted in Remark 5.1, the finiteness of the group can be checked algorithmically. It may happen that all or none of the specializations of the weights lead to a differentially algebraic generating series and this will be indicated in the output as well.

Step 1: Determine the polar divisor $(b)_\infty$ of b and which of the cases of Propositions 4.3, 4.4, and 4.6 holds.

- 1.1 Calculate the potential poles of $b = x(\iota_1(y) - y)$
    - $P_i = (\infty, b_i)$ where $\infty = [1:0]$ and $b_i = [b_{i,0}, b_{i,1}]$, $i = 0, 1$,
    - $Q_i = (a_i, \infty)$ where $a_i = [a_{i,0}, a_{i,1}]$, $i = 0, 1$,
    - $\iota_1(Q_i) = (a_i, c_i)$ where $c_i = [c_{i,0}, c_{i,1}]$, $i = 0, 1$.

    and determine which, if any, coincide and which is a pole.
- 1.2 Determine which of the cases in Propositions 4.3, 4.4, and 4.6 holds. Determine for which two poles being in the same τ-orbit is a necessary and sufficient condition for the generating series to be differentially algebraic.

Equations (4.1) and (4.2) are useful in 1.1. Note that if Proposition 4.3 holds, then one can stop and conclude that there are no values of the weights leading to a differentially algebraic generating series. If case b.2 of Proposition 4.4 applies then one can stop and conclude that all nonzero values of the designated weights lead

to differential algebraic generating series.

Example 5.2. Consider the weighted model:

with nonzero weights $d_{1,1}, d_{0,-1}, d_{-1,-1}, d_{-1,0}, d_{0,1}, d_{0,0}$. When unweighted, this model was called $w_{IIC.2}$ and we shall keep this notation for the weighted model. The associated kernel is

$$K(x_0, x_1, y_0, y_1, t_0, t_1) = x_0 x_1 y_0 y_1 -$$
$$t \left(d_{-1,-1} x_1^2 y_1^2 + d_{-1,0} x_1^2 y_0 y_1 + d_{0,-1} x_0 x_1 y_1^2 + d_{0,0} x_0 x_1 y_0 y_1 + d_{0,1} x_0 x_1 y_0^2 + d_{1,1} x_0^2 y_0^2 \right).$$

Steps 1.1 and 1.2: The polar divisor of b is $(b)_\infty = P_1 + Q_0 + \iota_1(Q_0)$, where

- $P_1 = P_0 = ([1 : 0], [0 : 1])$
- $Q_0 = ([-d_{0,1} : d_{1,1}], [1 : 0])$
- $\iota_1(Q_0) = ([-d_{0,1} : d_{1,1}], [t(d_{-1,-1}d_{1,1} - d_{0,-1}d_{0,1}) : -(d_{0,1} + t(d_{-1.0}d_{1,1} - d_{0,0}d_{0,1})])$

Furthermore, $Q_1 = ([0 : 1] : [1 : 0])$. This means we are in Case d.4 of Proposition 4.6 and we must decide if P_1 and Q_0 are in the same τ-orbit.

<u>*Step 2: Find the Kodaira type of the fiber above 0 and its associated root lattice T_0.*</u>

2.1 Calculate the Weierstrass equation $y^2 = 4x^3 - g_2 x - g_3$ associated to $K(t, x_0, x_1, y_0, y_1) = 0$.

2.2 Calculate the order of vanishing (as functions of t) of the discriminant Δ, and the invariants g_2, g_3.

2.3 Use Table 5.2 to determine the type of the fiber above 0.

2.4 Use Table 5.1 to determine the associated root lattice.

The first three steps are essentially the algorithm of Tate mentioned above. Tate's algorithm determines, in all characteristics, the Kodaira type of a singular fiber (assumed to be above 0) of an elliptic surface whose generic fiber is given by a Weierstrass equation $y^2 = 4x^3 - g_2 x - g_3$ with $g_2, g_3 \in \mathbb{C}(t)$. In characteristic 0, the algorithm shows that the type is determined by the order of vanishing of the discriminant Δ and the invariants g_2 and g_3 at 0. Formulas to express Δ, g_2, g_3 in terms of the coefficients of K are given in [Dui10, Section 2.3.5, Proposition 2.4.3, Corollary 2.5.10]. Restricting to the fiber types in Table 5.1, Table 5.2 gives the type of the fiber in terms of the order of vanishing of Δ, g_2, and g_3 (See also [SS10, Table 1], [Dui10, Lemma 6.3.1]). Note that the table does not deal with the cases when the valuations of g_2 and g_3 with respect to t are respectively greater than 4 and 6. When this is the case successive changes of variables of the form $x \mapsto t^2 x, y \mapsto t^3 y$ will ensure that this condition is met since with this transformation, the order of Δ drops by 12 and this can happen only a finite number of times. One now uses Table 5.1 to find the associated root lattice T_0.

Example 5.2(bis): Steps 2.1 and 2.2: A MAPLE calculation shows that the orders of g_2 and g_3 are 0 and the order of Δ is 7 (see [HS20]).

Steps 2.3 and 2.4: Table 5.2 implies that the associated fiber is I_7 and Table 5.1 implies that the root lattice T_0 is A_6.

<u>*Step 3: Determine T.*</u>

Type	g_2	g_3	Δ
$I_n, n \geq 1$	0	0	n
I_0^*	≥ 2	≥ 3	6
$I_n^*, n \geq 1$	2	3	$n+6$
III	1	≥ 2	3
III^*	3	≥ 5	9
IV	≥ 2	2	4
IV^*	≥ 3	4	8

TABLE 5.2. Local contributions of the singular fibers

3.1 Using the value of T_0, consult the table of all possible root lattices in [SS19, Table 8.2] or the table in [OS91] to find all possible T of which T_0 is a summand.

Example 5.2(bis): 3.1: Since $T_0 = A_6$, the possibilities for T listed in these tables are A_6 and $A_6 \oplus A_1$. This implies that there are one or two singular fibers.

Step 4: Determine possible $\mathrm{contr}_v(Q)$ *and possible* $\hat{h}(Q)$ *for the designated poles of* b.
With our assumptions this will be independent of which pole we consider.

4.1 For each of the possible T found in Step 3 and each of the summands T_v, determine the set of possible values of $\mathrm{contr}_v(Q)$ from Table 5.1.
4.2 Determine the set H of possible values of $\hat{h}(Q)$. Our assumption on P and $\mathcal{S}$ imply that (5.1) simplifies to

$$(5.2) \qquad \hat{h}(Q) = 2 - \sum_{v \in R} \mathrm{contr}_v(Q)$$

Note that if one of the T_v is A_n for some n, then we can only determine $\mathrm{contr}_v(Q)$ up to a finite set of possibilities. The techniques of Section 5.2 allows one to determine the exact value.

Example 5.2(bis): Step 4.1: If $T = A_6$, then there is only one reducible fiber v_0 and Table 5.1 implies that $\mathrm{contr}_{v_0}(Q) \in \{0, 6/7, 10/7, 12/7\}$. If $T = A_6 \oplus A_1$ then there are two fibers: v_0 as before and v_1. We have $\mathrm{contr}_{v_1}(Q) \in \{0, 1/2\}$.

Step 4.2: If $T = A_6$, then $\hat{h}(Q) = 2 - \mathrm{contr}_{v_0}(Q) \in \{2, 8/7, 4/7, 2/7\}$. It $T = A_6 \oplus A_1$ then $\hat{h}(Q) = 2 - \mathrm{contr}_{v_0}(Q) - \mathrm{contr}_{v_1}(Q) \in \{2, 8/7, 4/7, 2/7, 3/2, 9/14/, 1/14\} = H$.

Step 5: Determine possible values of n *such that* $\hat{h}(M) = n^2 \hat{h}(N)$ *and test if* $M = \tau^n(N)$ *for these values.*

5.1 For all $r_1, r_2 \in H$, determine if $r_1/r_2 = n^2$ for some integer n. Let $\mathcal{I}$ be the set of such integers.
5.2 For the two designated poles M, N determined in Step 1.2, calculate $\tau^n(N)$ for $n \in \mathcal{I}$ and set this equal to M. This yields a set of polynomial conditions of the weights that are necessary and sufficient for $\tau^n(N) = M$. If these are trivial, then the model has a differentially algebraic generating series for all nonzero values of the designated weights. If these are satisfiable, these will be the output. If these are not satisfiable, then there are no values of the weights yielding a differentially algebraic generating series.

Example 5.2(bis): Step 5.1: One finds that the possible values of n are $-4, -3, -2, -1, 0, 1, 2, 3, 4$.

Step 5.2: The entries in the coordinates of $\tau^n(P_1)$ and Q_0 are polynomials in t and the weights. In all cases, except $n = -1$, we show via a MAPLE calculation (see [HS20]) that $Q_0 \neq \tau^n(P_1)$. For $n = -1$, we have

$$\tau^{-1}(P_1) = \iota_1(\iota_2(([1:0],[[0:1])) = \iota_1(((-d_{-1,-1}:d_{0,-1}],[0:1]))$$
$$= ([-d_{-1,-1}:d_{0,-1}],[y_0:y_1])$$

where

$$y_0 = d_{0,-1}\left(td_{-1,-1}d_{0,0} - td_{-1,0}d_{0,-1} - d_{-1,-1}\right) \text{ and}$$
$$y_1 = td_{-1,-1}\left(d_{-1,-1}d_{1,1} - d_{0,-1}d_{0,1}\right).$$

Since $Q_0 = ([-d_{0,1}:d_{1,1}],[1:0])$ we have that $Q_0 = \tau^{-1}(P_1)$ if and only if

$$d_{-1,-1}d_{1,1} - d_{0,-1}d_{0,1} = 0.$$

This implies that this weighted model has an x- and y-D-algebraic generating series if and only if this latter condition holds. Note that this condition is automatically satisfied if the model is unweighted so we reaffirm that the unweighted model $w_{IIC.2}$ is x-and y-D-algebraic.

Remark 5.3. *1. For each of the nine weighted models of Figure 1, one can show that τ has infinite order. This can be done by verifying that $\tau^n(Q) \neq Q$ for a suitable point Q and $0 < n \leq 6$. Another approach is to write $\tau(Q) = Q \oplus P$ and verify that $\hat{h}(P) \neq 0$. We have verified this latter condition for all values of the parameters for these nine models as well.*
2. The fact that the generating series for $w_{IIB.1}, w_{IIB.2}$ and $w_{IIB.6}$ are differentially algebraic follows from Proposition 4.4 (see remark 4.5). The conditions on the parameters appearing in the other weighted models of Figure 1 are calculated in a similar fashion as for $w_{IIC.2}$. See [HS20] for MAPLE worksheets exhibiting the calculations.

5.2. Refinements.
In this section we shall give a more detailed description of the *Kodaira-Néron model* associated to $\overline{E_t}$ and the computation of the numbers $\text{contr}_v(P)$. This will allow us to refine the algorithm described in the previous section. In particular, we will determine explicitly the type of the fiber above zero of the elliptic fibration and the contribution of this fiber to our height computation. We will assume a familiarity with several concepts from the algebraic geometry of surfaces with a particular emphasis on intersection theory and resolution of singularities via blowups (see for instance [Sha13, Chap 4]).

5.2.1. *The geometric objects.* One attaches to the kernel polynomial some geometric objects. We denote by $S([x_0:x_1],[y_0:y_1])$ the homogeneous biquadratic polynomial defined by $x_1^2 y_1^2 S(\frac{x_0}{x_1}, \frac{y_0}{y_1})$ in the notation of Section 2. First, one can consider the pencil $\mathfrak{C}$ of biquadratic curves $C_{[\lambda:\mu]}$ in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ defined by

$$C_{[\lambda:\mu]} = \{([x_0:x_1],[y_0:y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) | \mu x_0 x_1 y_0 y_1 - \lambda S([x_0:x_1],[y_0:y_1]) = 0\}$$

whose base points, that are the common zeros of $x_0 x_1 y_0 y_1$ and $S([x_0:x_1],[y_0:y_1]) = 0$, are represented in Figure 2.

Any member of the pencil $\mathfrak{C}$ passes through $\{P_0, P_1, Q_0, Q_1, R_0, R_1, S_0, S_1\}$ (see Figure 2). There are 8 of these base points counted with multiplicities.

For any pair of elements $t_1, t_2 \in \mathbb{C}$, each transcendental over $\mathbb{Q}$, the curves $C_{[t_1:1]}$ and $C_{[t_2:1]}$ are isomorphic over $\mathbb{Q}$. These curves are general members of the pencil. They are isomorphic to $\overline{E_t}$ over $\mathbb{C}$. The following lemma shows how to construct a Kodaira-Néron model $\mathcal{S}$ attached to $\overline{E_t}$, that is a relatively minimal fibration over $\mathbb{P}^1(\mathbb{C})$ with a rational section and whose general fiber is $\overline{E_t}$.
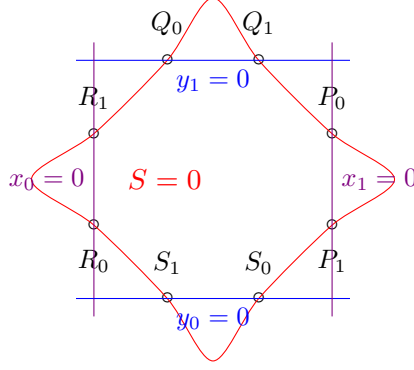
FIGURE 2. Position of the base points

Proposition 5.4 (Cor. 3.3.10 and §3.3.5 in [Dui10]). *Let $\mathcal{S}$ be the surface obtained by successively blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at the eight base points of the pencil $\mathfrak{C}$ counted with multiplicities. Write $\pi = \pi_1 \circ \ldots \circ \pi_8 : \mathcal{S} \mapsto \mathbb{P}^1 \times \mathbb{P}^1$. Then the space W of holomorphic 2-vector fields on $\mathcal{S}$ is two dimensional and $\mathcal{S}$ together with the mapping $\kappa : \mathcal{S} \mapsto \mathbb{P}^1(W), s \mapsto \{w \in W | w(s) = 0\}$ is a Kodaira-Néron model for $\overline{E_t}$. Moreover, the following holds*

- *a member C of the pencil $\mathfrak{C}$ is smooth if and only if its strict transform $\pi'(C)$ is a smooth fiber of κ when $\pi|_{\pi'(C)}$ is an isomorphism from $\pi'(C)$ to C; In particular the general fiber of $\mathcal{S}$ is $\overline{E_t}$.*
- *κ coincides with $\phi \circ \pi$ with $\phi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1, ([x_0 : x_1], [y_0 : y_1]) \mapsto (x_0 x_1 y_0 y_1, S([x_0 : x_1], [y_0 : y_1])$ on the open dense subset of $\mathcal{S}$ where $\phi \circ \pi$ is defined.*

The idea behind the successive blow-ups is to separate the members of the pencil $\mathfrak{C}$ so that they won't meet anymore at the eight base points. Doing this, one constructs a smooth fibration over the projective line. Locally, a blow-up at the point $(0,0) \in \mathbb{C} \times \mathbb{C}$ is the morphism

$$\phi : \{(x,y) \times [z : w] | xz + yw = 0\} \subset \mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{C}^2 \mapsto \mathbb{C}^2, (x,y) \times [z : w] \mapsto (x,y).$$

The preimage of the point $(0,0)$ by ϕ is a projective line $\mathcal{E}$ which is called the *exceptional divisor*. The strict-transform of a curve by the blow-up is the Zariski closure of $\phi^{-1}(\mathcal{C} \setminus (0,0))$. When all the base points are distinct, the blow-up at one base point allows to separate the member of the pencils and the exceptional divisor meets each member of the pencil one time, i.e., it is a section of the fibration. Indeed, the blow-up map separates the lines passing through $(0,0)$ with respect to their slope (see see [Hart77, p.28-29] for a detailed discussion of the blowing-up process as well as an excellent illustration exhibiting how the blow-up map separates these lines).

When some base points are equal, one needs to blow-up several times at the same point because some members of the pencil still intersect after one blow-up. The exceptional divisor of the last blow-up will give rise to a section of the fibration but the intermediate ones will become connected genus zero components of some fibers. Section 5.2.2 shows how one can determine the number of genus zero components of the fiber above zero from the position of the base points. We refer to [Dui10, Chapter 3.3] for more details concerning blowing-up the base points.

Note that the indeterminacy locus of the rational map ϕ is precisely the set of base points. A straightforward corollary of Proposition 5.4 is the following.

Lemma 5.5. *The Kodaira-Néron model $\mathcal{S}$ of $\overline{E_t}$ is rational elliptic surface.*

Proof. Indeed, it is birational to $\mathbb{P}^1 \times \mathbb{P}^1$ via π and $\mathbb{P}^1 \times \mathbb{P}^1$ is birational to $\mathbf{P}^2$. $\square$

Proposition 5.4 of [SS19] describes the correspondence between $\mathbb{C}(t)$-points of E and rational sections of $\kappa : \mathcal{S} \to \mathbb{P}^1$. The following lemma shows how one can make explicit this dictionary in the special cases of base points.

Lemma 5.6. *Let $P = (a, b) \in \mathbb{P}^1 \times \mathbb{P}^1$ be a base point. Then the multiplicity m of P as base point is less than or equal to 3. Moreover, the last exceptional divisor $\mathcal{E}_{(a,b)}$ obtained by blowing up m times at P is the image of the section of $\mathcal{S}$ that corresponds to the point (a, b) in E.*

Proof. The multiplicity is less than or equal to 3 because otherwise the base point would be singular for any member of the pencil contradicting the fact that $\overline{E_t}$ is a genus one curve. The second assertion is [Dui10, Cor.3.3.9]. $\square$

In Section 5.1, we use Propositions 4.4 and 4.6 to implement an algorithm, which allows us to decide if the weighted model was decoupled or not. We use the formula (5.1) defining the Néron-Tate height and claimed that when we apply this formula in our situation, the term representing the intersection multiplicity $(\mathcal{P}.\mathcal{O})$ is zero.

The main purpose of the following lemma is to verify this claim.

Lemma 5.7. *Assume that $\overline{E_t}$ is a genus 1 curve and that there is no P_j's and Q_k's that are simultaneously fixed by an involution. The following holds:*

- *Case 1: $P_j = Q_k$ for some j and k. Then, the section $\mathcal{P}_{j+1}$ has empty intersection with $\mathcal{P}_j$ and $\mathcal{Q}_{k+1}$, which is the section corresponding to $\tau(P_{j+1}) = Q_{k+1}$.*
- *Case 2: $P_j \neq Q_k$ for any j and k. Then, the section $\tau^{-1}(\mathcal{Q}_k)$ corresponding to the point $\tau^{-1}(Q_k)$ does not intersect the sections $\mathcal{Q}_k$ and $\mathcal{P}_j$.*

Proof. In the first case, we have $P_0 \neq P_1$ and $Q_0 \neq Q_1$ by assumption. For simplicity, let us assume that $P_0 = Q_0$. By Lemma 5.6, the section $\mathcal{P}_1$ (resp. $\mathcal{Q}_1$, $\mathcal{P}_0$) is the last exceptional divisor obtained by blowing up at P_1 (resp. Q_1, P_0). Then, $\mathcal{P}_1 \subset \pi^{-1}(P_1)$, $\mathcal{P}_0 \subset \pi^{-1}(P_0)$ and $\mathcal{Q}_1 \subset \pi^{-1}(Q_1)$. Since $P_1 \neq Q_1$ and $P_1 \neq P_0$, we conclude that $\mathcal{P}_1$ has empty intersection with $\mathcal{Q}_1$ and $\mathcal{P}_0$.

In the second case, let $\alpha \in \mathbb{C}$ such that $Q_{k+1} = (\alpha, \infty)$. Then, $\tau^{-1}(Q_k)$ is the point $(\alpha, [-t(\sum d_{i,-1}\alpha^{i+1}) : \alpha - t(\sum d_{i,0}\alpha^{i+1})])$. Let us now consider the curve C in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by $C = \{(\alpha, [-t_0(\sum d_{i,-1}\alpha^{i+1}) : t_1\alpha - t_0(\sum d_{i,0}\alpha^{i+1})]$ with $[t_0 : t_1] \in \mathbb{P}^1\}$. The strict transform of C by π corresponds to the section $\tau^{-1}(\mathcal{Q}_k)$. Then, it is easily seen that $\mathcal{P}_j$ does not intersect $\tau^{-1}(\mathcal{Q}_k)$ because P_j does not belong to C. If $Q_k \neq Q_{k+1}$ then Q_k does not belong to C so $\mathcal{Q}_k$ does not intersect $\tau^{-1}(\mathcal{Q}_k)$. If $Q_k = Q_{k+1}$ then the multiplicity of Q_k is 2 if $\alpha \neq 0$ and 3 if $\alpha = 0$. Since the curve is non singular, the point Q_{k+1} is not fixed by ι_1. Thus we need to blow up at least two times at Q_k. At the first blowup π_1 at Q_k, the strict transforms of the curve $y_1 = 0$ and $S([x_0 : x_1], [y_0 : y_1]) = 0)$ still intersect the exceptional divisor at the same point $Q_k^{(1)}$ because they have the same tangent at Q_k. The second blowup will be performed at the $Q_k^{(1)}$. Since the curve C does not have the same tangent as $y_1 = 0$ at Q_k, it intersects the exceptional divisor at some point $P \neq Q_k^{(1)}$. Then, one can reason as above to conclude that the sections $\mathcal{Q}_k$ and $\tau^{-1}(\mathcal{Q}_k)$ do not intersect because the first one is contracted on $Q_k^{(1)}$ by $\pi_2 \circ \cdots \circ \pi_8$ whereas the second is sent on a curve that does not pass through $Q_k^{(1)}$. $\square$

Remark 5.8. *Using some symmetry arguments as in Lemma 4.2, one can easily deduce from Lemma 5.7 that*

- Case 1: $P_j = Q_k$ for some j and k. Then, the section $\mathcal{Q}_{k+1}$ has empty intersection with $\mathcal{P}_j$ and $\mathcal{Q}_k$, which is the section corresponding to $\tau(P_{j+1}) = Q_{k+1}$.
- Case 2: $P_j \neq Q_k$ for any j and k. Then, the section $\tau(\mathcal{P}_j)$ does not intersect the sections $\mathcal{Q}_k$ and $\mathcal{P}_j$.

5.2.2. *The fiber above zero.* The construction of π aims at separating the members of the pencil $\mathfrak{C}$ so that they define an elliptic fibration. In order to understand the type of the fiber F_0 above zero of $\mathcal{S}$, one has to understand how the curve $C_{[0:1]} := \{([x_0 : x_1], [y_0 : y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) | x_0 x_1 y_0 y_1 = 0\}$ behaves after each blowup.

In Example 5.2 of Section 5.1, computing the Weierstrass form and applying the table related to the Tate algorithm, allows us to conclude that the Kodaira type of the fiber F_0 above 0 is I_n with $n = 7$. This is an instance of the following result which we prove in this section. In Section 5.2.3, we show in two examples that by calculating F_0 one can furthermore determine the contribution $\mathrm{contr}_0(P)$ in a more exact manner, sharpening the computation described in Section 5.1.

Lemma 5.9. *The type of F_0 is I_n where the number n of components of F_0 varies between 4 and 9 depending on the multiplicity and the position of the base points.*

Then, according to [SS19, Table 8.2], there are precisely

- one possible root lattice when $n = 9$,
- two possible root lattices when $n = 8$,
- two possible root lattices when $n = 7$,
- five possible root lattices when $n = 6$,
- seven possible root lattices when $n = 5$,
- 19 possible root lattices when $n = 4$.

All together, there are at worst 28 distinct root lattices, which can be associated to $\mathcal{S}$. Thus, the number of possibilities for the local contributions of the singular fibers is quite low once one has determined the local contribution of the fiber above 0. In the rest of this section, we show how to determine the number of components n of the fiber F_0 with respect to the multiplicity of the base points and their relative positions. Knowing the relative position of these components allows us to decrease the number of cases considered in the algorithm.

A. No multiple base points. Then the multiplicity of $C_{[0:1]}$ at each base point is 1. The *strict transform* of $C_{[0:1]}$ is the fiber above 0. It is a cycle of $n = 4$ projective lines. The sections corresponding to the base points are exactly the 8 exceptional divisors and their intersection with F_0 is similar to Figure 2.

B. Multiple base points. In this paragraph, we show how the multiplicity of a base point contributes to the number of components of F_0. There are three cases.

B.1 Two base points in a corner. Assume that for instance $Q_0 = R_0$ and $Q_0 \notin \{R_1, Q_1\}$. We perform a first blowup at $Q_0 = R_0$ and we choose the affine chart of $\mathbf{A}^2 \subset \mathbb{P}^1 \times \mathbb{P}^1$ given by $x_1 = 1, y_0 = 0$. The coordinates of this chart are $x := x_0$ and $y := y_1$. By assumption, $S(x, y) = d_{-1,0}y + d_{0,1}x + R(x, y)$ with $R(x, y)$ having no monomials of degree less than or equal to 1 and $d_{-1,0}d_{0,1} \neq 0$. In this chart, the blowup of Q_0 consists in considering the map $\pi_1 : X \to \mathbb{P}^1(x, y) \times [u : v] \mapsto (x, y)$ where $X = \{(x, y) \times [u : v] | ux = vy\} \subset \mathbf{A}^2 \times \mathbb{P}^1$. In the chart $u = 1$, the exceptional divisor $\mathcal{E}_1$ is given by $y = 0$. The total transform of a member $C_{[\lambda:\mu]}$ is given by

the zero set of

$$\mu xy - \lambda S(x,y) = \mu vy^2 - \lambda(d_{-1,0}y + d_{0,1}vy + R(vy,y))$$
$$= y\left(\mu vy - \lambda(d_{-1,0} + d_{0,1}v + R'(v,y))\right),$$

where $R'(v,y) = R(vy,y)/y$. Thus, the strict transform of a general member of the pencil is given by $\mu vy - \lambda(d_{-1,0} + d_{0,1}v + R'(v,y)) = 0$. This defines a new pencil $\mathfrak{D}$. The member of $\mathfrak{D}$ over zero corresponds to $vy = 0$ and is therefore equal to the union of the proper transform of $C_{[0:1]}$ and of the first exceptional divisor $\mathcal{E}_1$. Moreover, one can easily see that all members of $\mathfrak{D}$ intersect $\mathcal{E}_1$ at the point $Q_0^{(1)}$ with coordinates $v = -\frac{d_{0,1}}{d_{-1,0}}, y = 0$. A second blowup at this point yields a separation of the members of the pencil and resolves the singularity of the rational map ϕ defined in Proposition 5.4 at Q_0. One concludes that each time this case happens one has to add a new component at the proper transform of $C_{[0:1]}$. The last exceptional divisor $\mathcal{E}_2$ corresponds to the section $\mathcal{Q}_0$. It intersects F_0 at some point $Q_0^{(2)}$ of $\mathcal{E}_1$.
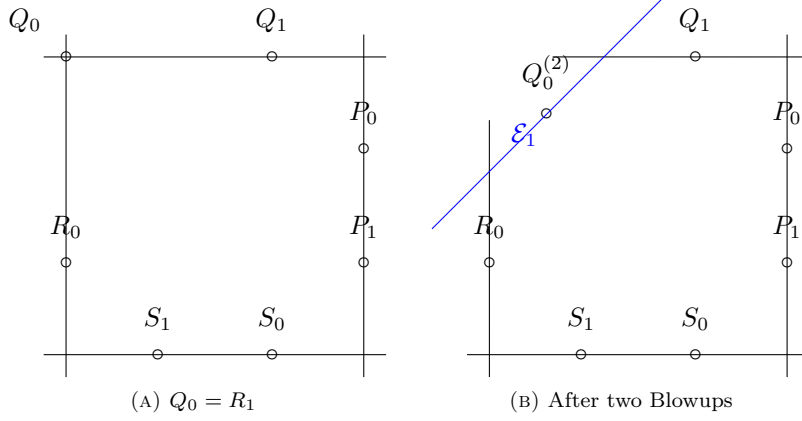

(A) $Q_0 = R_1$

(B) After two Blowups

FIGURE 3. The fiber above 0 when $Q_0 = R_1$

B.2 Two base points equal on a line. Assume that for instance $Q_0 = Q_1 = (a, \infty)$ with $a \notin \{0, \infty\}$. We perform a first blowup at $Q_0 = R_0$ and we choose the affine chart of $\mathbf{A}^2 \subset \mathbb{P}^1 \times \mathbb{P}^1$ given by $x_1 = 1, y_0 = 0$. The coordinates of this chart are $x := x_0$ and $y := y_1$. By assumption $S(x,y) = (x-a)^2 + A(x)y + B(x)y^2$. The member $C_{[\lambda_a:\mu_a]}$ with $\mu_a a + \lambda_a A(a) = 0$ of the pencil is singular at the point $(a, 0)$. The blowup at Q_0 is the map $\pi_1 : X \subset \mathbf{A}^2 \times \mathbb{P}^1(x, y) \times [u:v] \to (x,y)$ where $X = \{(x,y) \times [u:v] | u(x-a) = vy\}$. In the chart $v = 1$, the exceptional divisor $\mathcal{E}_1$ is given by $(x-a) = 0$ and a strict transform of a general member of the pencil $\mathfrak{C}$ is given $\mu ux - \lambda((x-a) + A(x)u + B(x)u(x-a))$. This defines a new pencil $\mathfrak{D}$ whose member above zero is given by $ux = 0$ that is by the proper transform of $C_{[0:1]}$. All members of the pencil $\mathfrak{D}$ meet on the point $Q_0^{(1)}$ given by $u = 0, x = a$ of the exceptional divisor $\mathcal{E}_1$. Thus one needs to blowup one more time at $Q_0^{(1)}$ to separate the members of the pencil $\mathfrak{D}$ and resolve the singularity of ϕ at Q_0. An easy computation shows that the exceptional divisor $\mathcal{E}_1$ is after the second blowup one of the components of the fiber $F_{[\lambda_a:\mu_a]}$ with $\mu_a a + \lambda_a A(a) = 0$. The last exceptional divisor $\mathcal{E}_2$ corresponds to the section $\mathcal{Q}_0$. It intersects F_0 at some point $Q_0^{(2)}$ on the strict transform of $y_1 = 0$.

(A) $Q_0 = Q_1$ (B) After two Blowups
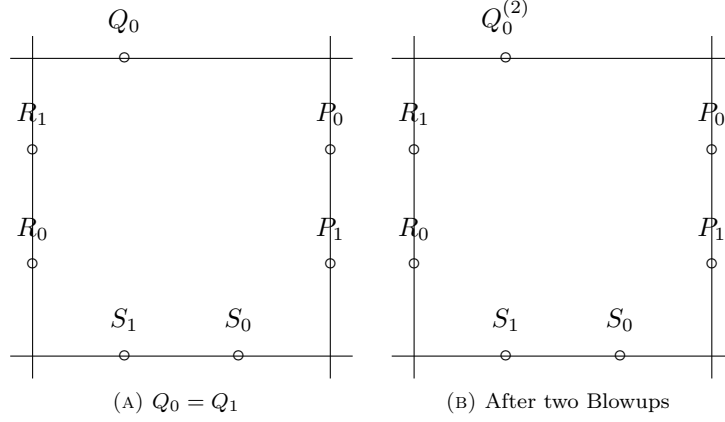
FIGURE 4. The fiber above zero when $Q_0 = Q_1$

B.3 Three points in a corner. Assume that for instance $Q_0 = R_0 = Q_1$. In the coordinates $x := x_0$ and $y := y_1$ of the affine chart of $\mathbf{A}^2 \subset \mathbb{P}^1 \times \mathbb{P}^1$ given by $x_1 = 1, y_0 = 0$, the polynomial $S(x, y)$ is of the form $\alpha x^2 + A(x)y + B(x)y^2$ where $\alpha \neq 0$ because the general member of $\mathfrak{C}$ is non singular. In this chart, the blowup of Q_0 consists in considering the map $\pi_1 : X \to \mathbb{P}^1, (x, y) \times [u : v] \mapsto (x, y)$ where $X = \{(x, y) \times [u : v] | ux = vy\} \subset \mathbf{A}^2 \times \mathbb{P}^1$. In the chart $v = 1$, the exceptional divisor $\mathcal{E}_1$ is given by $x = 0$ and the strict transform of a general member of the pencil is given by

$$\mu u x + \lambda(\alpha x + A(x)u + B(x)u^2 x^2).$$

This allows to conclude that the member of $\mathfrak{D}$ above zero corresponds to $ux = 0$ and is therefore the union of the proper transform of $C_{[0:1]}$ and of the first exceptional divisor $\mathcal{E}_1$. Moreover all members of $\mathfrak{D}$ intersect at the point $Q_0^{(1)}$ given by $u = x = 0$. Thus, one needs to perform a second blowup at the point $Q_0^{(1)}$. In the coordinates u and x, this blowup is $\pi_2 : X \to \mathbb{P}^1, (x, u) \times [c : d] \mapsto (x, u)$ where $X = \{(x, u) \times [u : v] | uc = dx\} \subset \mathbf{A}^2 \times \mathbb{P}^1$. In the chart $d = 1$, the exceptional divisor $\mathcal{E}_2$ is given by $u = 0$. An easy computation shows that the total transform of a general member of $\mathfrak{D}$ is the zero set of

$$\mu c u + \lambda(\alpha c + A(cu) + B(cu)uc).$$

This defines a new pencil $\mathfrak{E}$ of curves. The member above zero is given by $cu = 0$ and is therefore the union of the proper transform of $D_{[0:1]}$ and of the exceptional divisor $\mathcal{E}_2$. All the members of the pencil $\mathfrak{E}$ intersect on the point $Q_0^{(2)}$ given by $u = 0, c = \frac{-A(0)}{\alpha}$. One needs to blowup once more at $Q_0^{(2)}$ to resolve the singularity of ϕ at Q_0. The fiber F_0 is thus the union of the strict transform of $C_{[0:1]}$, $\mathcal{E}_1$ and $\mathcal{E}_2$. The last exceptional divisor $\mathcal{E}_3$ corresponds to the section $\mathcal{Q}_0$ and intersects the fiber above zero on $\mathcal{E}_2$. It intersects F_0 at $Q_0^{(3)}$ on $\mathcal{E}_2$.

Since the curve $\overline{E_t}$ is non singular, one can not have four points in a corner. The discussion above shows that the singular fiber above 0 is an I_n with

- $n = 4$ when all the base points are distinct or they are equal on a line,
- $n = 5$ when for instance $Q_0 = Q_1$,
- $n = 6$ when for instance $Q_0 = Q_1 = R_1$,
- $n = 7$ when for instance $Q_0 = Q_1 = R_1$ and $P_1 = S_0$,
- $n = 8$ when for instance $Q_0 = Q_1 = R_1$ and $P_1 = S_0 = P_0$,

(A) $Q_0 = R_1 = Q_1$

(B) After one Blowup

(C) After three Blowups
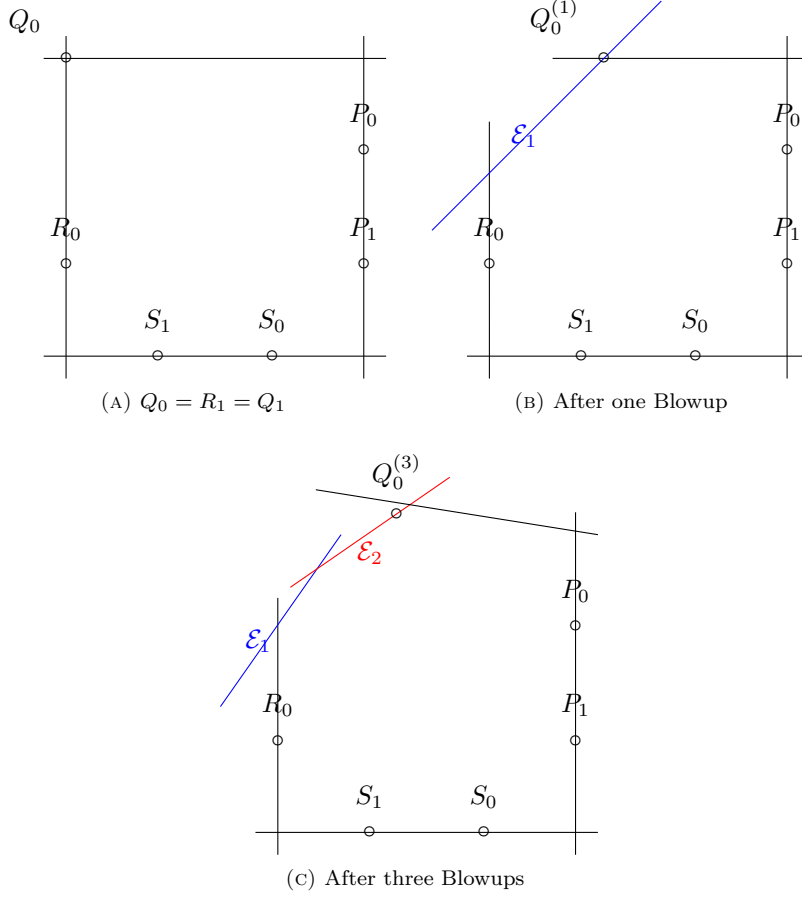
FIGURE 5. The fiber above 0 when $Q_0 = Q_1 = R_1$

- $n = 9$ when for instance $Q_0 = Q_1 = R_1$, $P_1 = S_0 = P_0$ and $R_0 = S_1$.

In this last case, one has $\tau^3(S_0) = S_0$ so that the group of the walk is finite. Indeed, the group of the walk will be always finite when $n = 9$ because the root lattice is A_8 (see [SS19, Table 8.2]).

5.2.3. *Some examples.* The fiber F_0 above zero is an I_n and the contribution of this fiber to the height of a section $\mathcal{Q}$ is defined as follows. Let $\mathcal{O}$ be the zero section. The fiber F_0 is a cycle of n components Θ_i for $i = 0, \ldots, n-1$. The component of F_0 that meets the section $\mathcal{O}$ is denoted Θ_0 and we number the components cyclically, that is, Θ_i meets Θ_j if and only if $|i - j| \equiv 1 \bmod n$. The contribution of F_0 to the height of a section $\mathcal{P}$ is equal to $\frac{i(n-i)}{n}$ when $\mathcal{P}$ meets F_0 on the component Θ_i. With the process detailed in 5.2.2, one can easily determine the contribution of F_0 to the height of the section. This allows one to refine the algorithm presented in Section 5.1 by lowering the number of possibilities for the height. In this section, we present this refinement via the study of three weighted models.

Example 5.2 revisited: The weighted model $w_{IIC.2}$.
In this paragraph, we show how the computation of the contribution of the fiber above zero allows to drastically simplify the algorithm presented in 5.1. We will illustrate this on an example and we will study the D-algebraicity of the weighted

model $w_{IIC.2}$, which corresponds to $d_{1,-1} = d_{1,0} = d_{-1,1} = 0$. For this model, we have

- $Q_1 = R_1 \neq Q_0$,
- $P_0 = P_1 = S_1$.

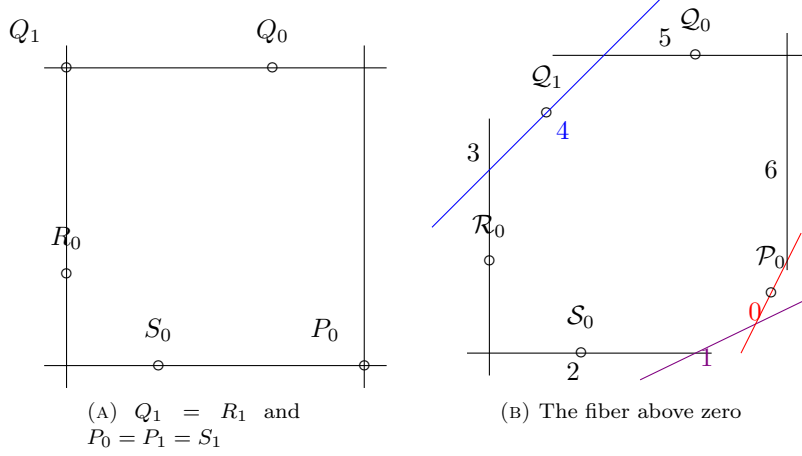Following the method detailed in Section 5.2.2, the fiber above zero given by Figure 6.

(A) $Q_1 = R_1$ and $P_0 = P_1 = S_1$

(B) The fiber above zero

FIGURE 6. Fiber above zero for $w_{IIC.2}$

In Figure 6, we abuse notation and denote by $\mathcal{Q}_i, \mathcal{P}_i$ the intersections of the sections with the fiber F_0.

As detailed in Section 5.1, the model is decoupled if and only if there exists n such that $Q_0 = \tau^n(P_0)$ (Note that since P_0 is fixed by ι_1, one has $P_0 \sim Q_0$ if and only if $P_0 \sim Q_1$. Choosing P_0 as the zero of $\overline{E_t}$, we must decide if there exists an integer n such that $Q_0 = n\tau(P_0) = nS_0$. The fiber above zero is an I_7, which corresponds to a root lattice A_6. By [SS19, Table 8.2], the root lattice T is either A_6 or $A_6 \oplus A_1$. Numbering the components of the fiber above zero as in Figure 6, we find that the height of the points Q_0 and S_0 are given by

- $\hat{h}(Q_0) = 2 - \frac{5(7-5)}{7} - \frac{\epsilon_1}{2}$,
- $\hat{h}(S_0) = 2 - \frac{2(7-2)}{7} - \frac{\epsilon_2}{2}$,

where $\epsilon_1, \epsilon_2 \in \{0, 1\}$ depending on the intersection of $\mathcal{Q}_0$ and $\mathcal{S}_0$ with a putative singular fiber of root lattice A_1. Note that the height of S_0 is never zero so that the point $\tau(P_0)$ is not torsion and the group of the walk is infinite (see the remarks following Lemma 2.4 and Remark 5.1). Then, $\hat{h}(Q_0) = n^2\hat{h}(S_0)$ is equivalent to $8 - 7\epsilon_1 = n^2(8 - 7\epsilon_1)$ and the only solution is $n^2 = 1$ that is $n = \pm1$. Since $\tau(P_0) = S_0 \neq Q_0$, the integer n must be equal to -1. For the weighted model $w_{IIC.2}$, the condition $Q_0 = \tau^{-1}(P_0)$ is equivalent to

$$(5.3) \qquad\qquad d_{0,1}d_{0,-1} - d_{1,1}d_{-1,-1} = 0.$$

When the model $w_{IIC.2}$ is unweighted, the condition (5.3) is satisfied so that the unweighted $w_{IIC.2}$ is D-algebraic.

Once one knows that the weighted model is decoupled, it is quite easy to find the certificate for b. Indeed, thanks to the orbit residue criteria, one knows the distribution of the poles of b on τ-orbits. Finding the certificate of b is just a question of finding an elliptic function with prescribed set of poles and residues.

The weighted model $w_{IIC.2}$ is decoupled if and only if $d_{0,1}d_{0,-1} - d_{1,1}d_{-1,-1} = 0$ if and only if $Q_0 = \iota_1(S_0)$. In that situation, the residues and poles of b are as follows:

Points	$S_0 = \tau(P_0)$	P_0	$\tau^{-1}(P_0) = Q_0$
Residues of order 1	α	-2α	α

In $\mathbb{C}(\overline{E_t})$, the function $h = \frac{1}{y}$ has the following residues and poles

Points	$S_0 = \tau(P_0)$	P_0
Residues of order 1	$-\beta$	β

so that for any $\lambda \in \mathbb{C}^*$, the function $\tau(\lambda h) - \lambda h$ has the following residues and poles

Points	$S_0 = \tau(P_0)$	P_0	$\tau^{-1}(P_0) = Q_0$
Residues of order 1	$\lambda\beta$	$-2\lambda\beta$	$\lambda\beta$

Then $\tau(\frac{\alpha}{\beta y}) - \frac{\alpha}{\beta y}$ and b have same poles and residues so that there exists $c \in \mathbb{C}$ such that $b = \tau(\frac{\alpha}{\beta y}) - (\frac{\alpha}{\beta y}) + c$. It is easily seen that c must be zero since $\iota_1(b) = -b$ and $\iota_1\left(\tau(\frac{\alpha}{\beta y}) - (\frac{\alpha}{\beta y})\right) = -(\tau(\frac{\alpha}{\beta y}) - (\frac{\alpha}{\beta y}))$. Therefore, the function $\frac{\alpha}{\beta y}$ is a certificate for b. To compute the residues α and β, we generalize [BBMR15] to the decoupled weighted case and, using (5.3), we note that

$$(5.4) \qquad y\iota_1(y) = \frac{(d_{-1,-1} + d_{0,-1}x)}{d_{0,1}x + d_{1,1}x^2} = \frac{d_{-1,-1}}{d_{0,1}}\frac{1}{x}.$$

Then, one finds that

$$\alpha = \operatorname{Res}_{Q_0}(b) = -\operatorname{Res}_{Q_0}(xy) = -\frac{d_{-1,-1}}{d_{0,1}}\operatorname{Res}_{Q_0}\left(\frac{1}{\iota_1(y)}\right)$$

$$= \frac{d_{-1,-1}}{d_{0,1}}\operatorname{Res}_{\iota_1(Q_0)}\left(\frac{1}{y}\right) = -\frac{d_{-1,-1}}{d_{0,1}}\beta,$$

where we use $\operatorname{Res}_{\iota_1(P)}(f) = -\operatorname{Res}_P(\iota_1(f))$ for any $P \in \overline{E_t}, f \in \mathbb{C}(\overline{E_t})$ and $\iota_1(Q_0) = S_0$. This proves that the function $g(y) = \frac{-d_{-1,-1}}{d_{0,1}y}$ is a certificate for b. In this case we have $xy = g(y) + h(x)$ where

$$h(x) = \frac{d_{-1,-1}}{d_{0,1}}\frac{x - td_{-1,0} - td_{0,0}x}{t(d_{-1,-1} + d_{0,1}x)}.$$

Example 5.10. *The weighted model IB.6.* This weighted model corresponds to $d_{1,-1} = d_{1,0} = 0$. When unweighted, it was called $IB.6$ and we keep this terminology for the weighted model. In that situation, $P_0 = P_1 = S_1$. The fiber above zero is

As described in Section 5.1, the model is decoupled if and only if there exists n such that $Q_0 = \tau^n(P_0)$ (Note that since P_0 is fixed by ι_1, one has $P_0 \sim Q_0$ if and only if $P_0 \sim Q_1$. Choosing P_0 as the zero of $\overline{E_t}$, we must decide if there exists an integer n such that $Q_0 = n\tau(P_0) = nS_0$. The fiber above zero is an I_6, which corresponds to a root lattice A_5. By [SS19, Table 8.2], the root lattice T is either $A_5, A_5 \oplus A_1, A_5 \oplus A_1^2, A_5 \oplus A_2, A_5 \oplus A_2 \oplus A_1$. Numbering the components of the fiber above zero as in Figure 7, we find that the heights of the points Q_0 and S_0 are given by

- $\hat{h}(Q_0) = 2 - \frac{4(6-4)}{6} - \frac{\epsilon_1}{2} - \frac{2\epsilon_2}{3}$,
- $\hat{h}(S_0) = 2 - \frac{2(6-2)}{6} - \frac{\eta_1}{2} - \frac{2\eta_2}{3}$,

where $\epsilon_i, \eta_i \in \{0, 1\}$ except for the root lattice $A_5 \oplus A_1^2$, where the height of the points Q_0 and S_0 are given by

(A) $P_0 = P_1 = S_1$ (B) The fiber above zero
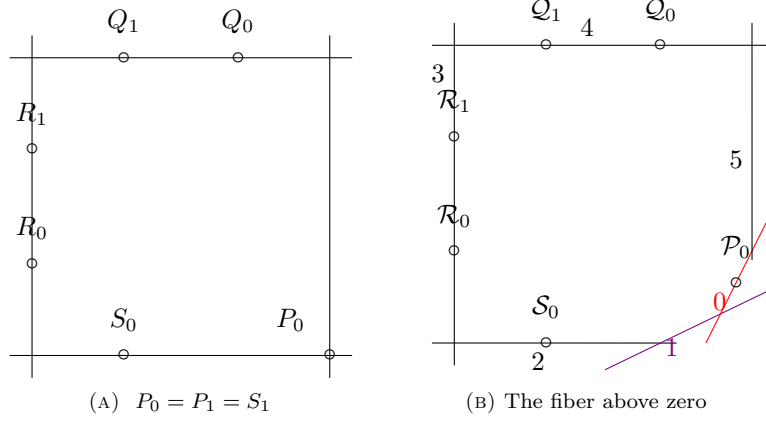
FIGURE 7. Fiber above zero for $w_{IB.6}$

- $\hat{h}(Q_0) = 2 - \frac{4(6-4)}{6} - \frac{\epsilon_1}{2} - \frac{\epsilon_2}{2}$,
- $\hat{h}(S_0) = 2 - \frac{2(6-2)}{6} - \frac{\eta_1}{2} - \frac{\eta_2}{2}$,

where $\epsilon_i, \eta_i \in \{0, 1\}$. Note that $\hat{h}(S_0)$ might be equal to zero if $\eta_1 = 0, \eta_2 = 1$. In that case, the group of the walk is finite and the generating series are holonomic. If $\hat{h}(S_0) \neq 0$ then, it is easily seen that if $\hat{h}(Q_0) = n^2 \hat{h}(S_0)$ then n^2 equals 1 or 4. Since $Q_0 \neq S_0$ and $\iota_2(Q_0) = Q_1 \neq \iota_1(S_0) = S_1$, it is easily seen that n must be equal to -1 or -2. A simple computation (see [HS20]) shows that $Q_0 = \tau^{-1}(P_0)$ if and only if

$$(5.5) \qquad d_{-1,1}d_{0,-1}^2 - d_{0,1}d_{-1,-1}d_{0,-1} + d_{1,1}d_{-1,-1}^2 = 0.$$

The condition $Q_0 = \tau^{-2}(P_0)$ is impossible (see [HS20]). Nonetheless, it is easily seen that if the walk is unweighted then the condition (5.5) does not hold. Therefore, the unweighted model $IB.6$ has a D-transcendental generating series.

Remark 5.11. *In* [DHRS19, Proposition 5.1]*, the authors show that if* $\delta_x = d_{1,0}^2 - 4d_{1,-1}d_{1,1}$ *or* $\delta_y = d_{0,1}^2 - 4d_{-1,1}d_{1,1}$ *is not a square in* $\mathbb{Q}(d_{i,j})$ *then the generating series are differentially transcendental. For the unweighted model* $IB.6$*, one has* $\delta_x = 0$ *and* $\delta_y = -3$ *so that the generating series is differentially transcendental in that case.* [DHRS19, Theorem 35] *shows that* [DHRS19, Proposition 5.1] *remains valid in the weighted case. If Condition* 5.5 *is satisfied then* $\delta_x = 0$ *and* $\delta_y = \left(\frac{(d_{0,1}d_{0,-1} - 2d_{1,1}d_{-1,-1})}{d_{0,-1}}\right)^2$ *is a square in* $\mathbb{Q}(d_{i,j})$*. Thus, our computation gives a necessary and sufficient condition for the D-algebraicity weighted model* $IB.6$ *and generalizes* [DHRS19, Theorem 35] *for this model.*

Example 5.12. *The weighted Gouyou-Beauchamps model*

In [CMMR17], the authors adapt some probabilistic notions such as the drift to define subfamilies of weighted models, which they call *universality classes* since they met common algebraic behaviour. They consider the *generic central weighting of the Gouyou-Beauchamps model* given by Figure 8.

They also showed that the group of the models of Figure 8 was the dihedral group D_8 and they study the asymptotics of the combinatorial sequence. In this section, we weight the Gouyou-Beauchamps model with arbitrary weights $d_{-1,1}, d_{1,-1}, d_{0,1}, d_{1,0}$ and prove the following proposition.
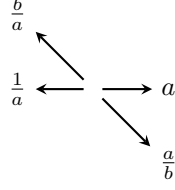
FIGURE 8. Generic central weighting of the Gouyou-Beauchamps model

Proposition 5.13. *The generating function of the weighted Gouyou-Beauchamps model is differentially algebraic if and only if*

$$(5.6) \qquad d_{1,0}d_{-1,0} - d_{-1,1}d_{1,-1} = 0.$$

If (5.6) *is satisfied, the group of the walk is either D_4 or D_8 and the generating function is D-finite.*

Proof. In that situation, $S_1 = S_0 = R_0$, $P_0 = Q_1 = Q_0$ and the fiber above zero is as follows:

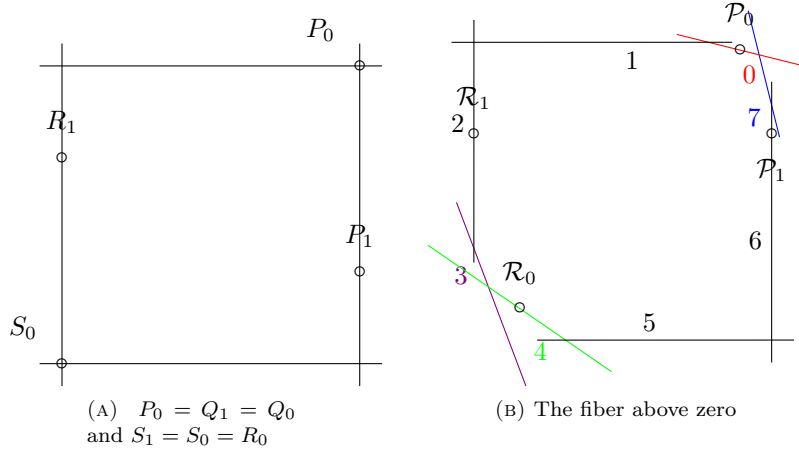(A) $P_0 = Q_1 = Q_0$ and $S_1 = S_0 = R_0$

(B) The fiber above zero

FIGURE 9. Fiber above zero for the weighted Gouyou-Beauchamps

The fiber above zero is an I_8, which corresponds to a root lattice A_7. By [SS19, Table 8.2], the root lattice T is either A_7 or $A_7 \oplus A_1$. In the latter case, the Mordell Weil group is $\mathbb{Z}/4\mathbb{Z}$ which shows that any point of the kernel curve is of order less than or equal to 4. This proves that the group of the walk is either D_4 or D_8. Following [Tsu04, Lemma 3.3], one can compute the discriminant of the Kernel curve and one finds (see the MAPLE calculation at [HS20] for this calculation and the ones that follow):

$$(5.7) \qquad \begin{aligned} \Delta := & d_{1,0}^2 d_{1,-1}^2 d_{-1,1}^2 d_{-1,0}^2 t^8 (16t^4 d_{1,0}^2 d_{-1,0}^2 - 32t^4 d_{1,0}d_{-1,0}d_{1,-1}d_{-1,1} \\ & + 16t^4 d_{1,-1}^2 d_{-1,1}^2 - 8t^2 d_{1,0}d_{-1,0} - 8t^2 d_{1,-1}d_{-1,1} + 1). \end{aligned}$$

By Tate's algorithm, the existence of a singular fiber of type I_2 which would give a contribution A_1 to the lattice is equivalent to the vanishing of the discriminant δ of $16t^4 d_{1,0}^2 d_{-1,0}^2 - 32t^4 d_{1,0}d_{-1,0}d_{1,-1}d_{-1,1} + 16t^4 d_{1,-1}^2 d_{-1,1}^2 - 8t^2 d_{1,0}d_{-1,0} -$

$8t^2 d_{1,-1} d_{-1,1} + 1$. A MAPLE computation yields

(5.8)
$$\delta = 16777216(d_{1,0}^2 d_{-1,0}^2 - 2d_{1,0}d_{-1,0}d_{1,-1}d_{-1,1} + d_{1,-1}^2 d_{-1,1}^2)d_{1,0}^2 d_{-1,0}^2 d_{1,-1}^2 d_{-1,1}^2.$$

Since the weights are nonzero, the vanishing of δ is equivalent to $(\frac{d_{1,0}d_{-1,0}}{d_{1,-1}d_{-1,1}} - 1)^2 = 0$ that is to $d_{1,0}d_{-1,0} = d_{1,-1}d_{-1,1}$. If $d_{1,0}d_{-1,0} \neq d_{1,-1}d_{-1,1}$, the group of the walk is infinite and the model is not decoupled because P_0 and Q_0 are fixed by an involution (see Proposition 4.3). This ends the proof. $\square$

Note that Condition (5.6) is automatically fulfilled by the generic central weightings of the Gouyou-Beauchamps model. These examples illustrate how the D-algebraicity of a model does depend on the configuration of the base points. It is conditioned by certain algebraic relations on the weights of the model and the classification of unweighted models in terms of D-algebraic and D-transcendental ones is in a certain sense accidental since the D-algebraic models corresponds to the cases where the algebraic relations are satisfied when all the weights are equal.

APPENDIX A. POLES AND RESIDUES

In this section we collect various technical facts concerning the poles and residues of rational functions on $\overline{E_t}$, that is, elements of $\mathbb{C}(\overline{E_t})$. We will assume throughout this section that $\overline{E_t}$ is an elliptic curve endowed with two involutions ι_1, ι_2. We denote by P the point of $\overline{E_t}$ such that $\tau = \iota_2 \circ \iota_1$ is the translation by P. In our discussions below, we need to expand elements of $\mathbb{C}(\overline{E_t})$ in power series at points of $\overline{E_t}$ and compare the expansions at various points. In order to do this in a consistent way the following was introduced in [DHRS18]

Definition A.1. *Let $\mathcal{S} = \{u_Q \mid Q \in \overline{E_t}\}$ be a set of local parameters at the points of $\overline{E_t}$. We say S is a* coherent set of local parameters *if for any $Q \in \overline{E_t}$,*

$$u_{\tau^{-1}(Q)} = \tau(u_Q).$$

Note that $\tau^{-1}(Q) = Q \ominus P$, where $\ominus$ is subtraction in the group structure of the elliptic curve.

A coherent set of local parameters always exits. To see this, Let O be the origin of the group law on the elliptic curve $\overline{E_t}$ and, for any $Q \in \overline{E_t}$ let τ_Q be the translation by Q. The map τ_Q induces and isomorphism $\tau_Q : \mathbb{C}(\overline{E_t}) \to \mathbb{C}(\overline{E_t})$ (here we abuse notation and use the same symbol). Let t be a local parameter at O. The set of local paramters $\{u_Q = \tau_{-Q}(t) \mid Q \in \overline{E_t}\}$ is a coherent set of local parameters.

Definition A.2. *Let u_Q be a local parameter at a point $Q \in \overline{E_t}$ and let v_Q be the valuation corresponding to the valuation ring at Q. If $f \in \mathbb{C}(\overline{E_t})$ has a pole at Q or order n, we may write*

$$f = \frac{c_{Q,n}}{u_Q^n} + \ldots + \frac{c_{Q,2}}{u_Q^2} + \frac{c_{Q,1}}{u_Q} + \tilde{f}$$

where $v_Q(\tilde{f}) \geq 0$. We shall refer to $c_{Q,i}$ as the residue of order i at Q.

In the usual presentation of Riemann surfaces, one speaks of residues of meromorphic differential forms. These do not depend on the local parameters whereas any discussion of a powerseries expansion of a function at a point does depend on the local parameter. Fixing a set of local parameters allows the notion of residue of order i to be well defined.

The following definition is similar to Definition 2.3 of [CS12].

Definition A.3. *Let* $f \in k(\overline{E_t})$ *and* $S = \{u_Q \mid Q \in \overline{E_t}\}$ *be a coherent set of local parameters and* $Q \in \overline{E_t}$. *For each* $j \in \mathbb{N}_{>0}$ *we define the* orbit residue of order j at Q *to be*

$$\mathrm{ores}_{Q,j}(f) = \sum_{i \in \mathbb{Z}} c_{Q \oplus iP, j}.$$

Note that if $Q' = Q \sim P$, then $\mathrm{ores}_{Q',j}(f) = \mathrm{ores}_{Q,j}(f)$ for any $j \in \mathbb{N}_{>0}$. Furthermore $\mathrm{ores}_{Q,j}(f) = \mathrm{ores}_{Q,j}(\tau(f))$. The following refines Proposition B.8 in [DHRS18] and is the reason for defining the orbit residue.

Proposition A.4. *Let* $b \in k(\overline{E_t})$ *and* $S = \{u_Q \mid Q \in \overline{E_t}\}$ *be a coherent set of local parameters. The following are equivalent.*

(1) There exists $g \in k(\overline{E_t})$ *such that*

$$b = \tau(g) - g.$$

(2) For any $Q \in \overline{E_t}$ *and* $j \in \mathbb{N}_{>0}$

$$\mathrm{ores}_{Q,j}(b) = 0.$$

Proof. Proposition B.8 in [DHRS18] implies that (2) is equivalent to: *there exists* $Q \in \overline{E_t}$, $h \in \mathcal{L}(Q + \tau(Q))$ *and* $g \in \overline{E_t}$. *such that* $b = \tau(g) - g + h$. Lemma 3.7 implies that this latter condition is equivalent to (1). $\square$

When applying Proposition A.4 we would like to verify the second condition using the fact that on a compact Riemann surface one has that the sum of the residues of a differential form is zero. Denoting by $\mathrm{Res}_Q \omega$ the usual residue at a point Q of a differential form ω, we want to compare $\mathrm{Res}_Q(f\omega)$ with $c_{Q,1}$ where f is as in Definition A.2. To do this we need to make a more careful selection of a coherent family of local parameters. For this we will use the following lemma whose proof is similar to [Che63, Theorem 14, p. 127].

Lemma A.5. *Let* C *be a nonsingular curve and* $K = \mathbb{C}(C)$ *its function field. Given a point* $Q \in C$, *a differential form* ω *regular and nonzero at* Q, *and integer* $n \in \mathbb{N}$, *there exists a local parameter* $t_n \in K$ *at* Q *such* $\omega = (1 + f)dt_n$ *where* $v_Q(f) > n$.

Proof. Let $t \in K$ be any local parameter at Q and let

$$\omega = (a_0 + a_1 t + \ldots + a_n t^n + f_n)dt.$$

where $f_n \in K$ and $v_Q(f_n) > n$. Let

$$t_n = a_0 t + \frac{a_1}{2} t^2 + \ldots + \frac{a_n}{n+1} t^{n+1}.$$

We then have that

$$\omega - dt_n = (a_0 + a_1 t + \ldots + a_n t^n + f_n - \frac{dt_n}{dt})dt = f_n dt.$$

$\square$

Let Ω be a fixed regular differential form on $\overline{E_t}$. The maps $\iota_1, \iota_2, \tau = \iota_2 \iota_1$ induce maps $\iota_1{}^*, \iota_2{}^*, \tau^*$ on the space of differential forms. From [Dui10, Lemma 2.5.1 and Proposition 2.5.2], we have that $\iota_i^*(\Omega) = -\Omega$ for $i = 1, 2$ and $\tau^*(\Omega) = \Omega$.

Definition A.6. *Let* $n \in \mathbb{N}$. *We say that a coherent set* $\{u_Q \mid Q \in \overline{E_t}\}$ *of local parameters is* n-coherent *if for each* $Q \in \overline{E_t}$, $\Omega = (1 + f_Q)du_Q$ *where* $v_Q(f_Q) > n$.

There always exists an n-coherent set of local parameters. To see this one modifies the construction following Definition A.1 by starting with a local parameter t_n at O satisfying the conclusion of Lemma A.5 with respect to Ω, that is, the order of $\Omega - dt_n$ at O is greater than n.

Fixed Assumption: *Through the paper, we assume that when the kernel curve $\overline{E_t}$ is of genus one, we fix a 3-coherent set of local parameters $\{u_Q \mid Q \in \overline{E_t}\}$. The various elements that we consider will have poles of order at most 3 so we can always apply Lemmas A.7 and A.9.*

Having an n-coherent set of local parameters allows us to use the usual Residue Theorem.

Lemma A.7. *Let $h \in \mathbb{C}(\overline{E_t})$ and assume that h has poles of order at most n at any point of $\overline{E_t}$. If $\{u_Q\}$ is an n-coherent set of local parameters, then for each $Q \in \overline{E_t}$, $\mathrm{Res}_Q(b\Omega) = c_{Q,1}$. Therefore, $\sum_{Q \in \overline{E_t}} c_{Q,1} = 0$.*

Proof. Since $\Omega = (1 + f_Q)du_Q$ with $v_Q(f_Q) > n$ we have

$$b\Omega = \left(\frac{c_{Q,n}}{u_Q^n} + \ldots + \frac{c_{Q,2}}{u_Q^2} + \frac{c_{Q,1}}{u_Q} + \tilde{f}_Q\right)du_Q$$

where $v_Q(\tilde{f}_Q) > 0$. One now applies the usual Residue Theorem. $\qquad\square$

Remark A.8. *1. In [DHRS19], the authors introduced the notion of a coherent set of analytic local parameters and showed that such a set exists on the universal cover of $\overline{E_t}$ and using these to induce such a set on $\overline{E_t}$. Alternatively, one can always find a coherent set of local parameters $\{u_Q \mid Q \in \overline{E_t}\}$ such that for each Q, $\Omega = du_Q$. One does this in the following way. If t is an analytic local parameter at O, we write $\Omega = \sum_{i=0}^{\infty} a_i t^i dt$, $a_0 \neq 0$. The analytic function $u_0 = \sum_{i=0}^{\infty} \frac{a_i}{i+1} t^{i+1}$ is an analytic local parameter at 0 and one can propagate this to become a coherent local family as above. Nonetheless, the u_Q gotten in this way need not be in the function field of the curve since they are only defined locally. We introduce the notion of n-coherence to be able to stay in the algebraic setting.*

2. In [DH19], the authors uniformize the kernel curve E as a Tate curve, that is, as $C^*/q^{\mathbb{Z}}$ where C is an algebraically closed field extension of $\mathbb{Q}(t)$. In that setting, the field $C(E)$ corresponds to the field $\mathcal{M}er(C^*)$ of meromorphic function over C^* fixed by the automorphism $f(z) \mapsto f(qz)$ of $\mathcal{M}er(C^*)$. The first involution corresponds to $f(z) \mapsto f(1/z)$ and the automorphism τ to $f(z) \mapsto f(\tilde{q}z)$. The regular differential form on $C^*/q^{\mathbb{Z}}$ is $\frac{dz}{z}$ and the coherent set of local parameters given by the $u_{\overline{\alpha}} : \overline{z} \mapsto ln(\frac{z}{\alpha})$ for z close to α satisfies all the required properties.

The following summarizes useful properties of the $c_{Q,i}$ and the $\mathrm{ores}_{Q,j}(f)$.

Lemma A.9. *Let $n > 1$ and $\{u_Q\}$ be an n-coherent set of local parameters. Assume $b \in \mathbb{C}(\overline{E_t})$ satisfy $\iota_1(b) = -b$.*
1. For each $Q \in \overline{E_t}$, $\iota_1(u_Q) = -u_{\iota_1(Q)} + g_{\iota_1(Q)}$ where $v_{\iota_1(Q)}(g_{\iota_1(Q)}) > n + 1$.
2. If

$$(A.1) \qquad b = \frac{c_{Q,n}}{u_Q^n} + \ldots + \frac{c_{Q,2}}{u_Q^2} + \frac{c_{Q,1}}{u_Q} + \tilde{f}$$

where $v_Q(\tilde{f}) \geq 0$, then

$$b = \frac{c_{\iota_1(Q),n}}{u_{\iota_1(Q)}^n} + \ldots + \frac{c_{\iota_1(Q),2}}{u_{\iota_1(Q)}^2} + \frac{c_{\iota_1(Q),1}}{u_{\iota_1(Q)}} + \tilde{g}$$

where $v_{\iota_1(Q)}(\tilde{g}) \geq 0$ and $c_{\iota_1(Q),j} = (-1)^{j+1} c_{Q,j}$ for any j. If follows that, if all the poles of b belong to the same τ-orbit, then, for any even number j, we have $\mathrm{ores}_{Q,j}(b) = 0$.

Proof. 1. We have $\Omega = (1 + f_Q)du_Q = (1 + f_{\iota_1(Q)})d(u_{\iota_1(Q)})$ where $v_Q(f_Q) > n, v_{\iota_1(Q)}(f_{\iota_1(Q)}) > n$. Applying $\iota_1{}^*$ to the first equality we have

$$-\Omega = \iota_1{}^*(\Omega) = (1 + \iota_1(f_Q))\iota_1{}^*(du_Q) = (1 + \iota_1(f_Q))d(\iota_1(u_Q)).$$

Since $\iota_1(u_Q)$ is again a local parameter at $\iota_1(Q)$ we have $\iota_1(u_Q) = cu_{\iota_1(Q)} + g_{\iota_1(Q)}$ where $c \neq 0$ and $v_{\iota_1(Q)}(g_{\iota_1(Q)}) > 1$. Therefore

$$d(\iota_1(u_Q)) = (c + \frac{dg_{\iota_1(Q)}}{du_{\iota_1(Q)}})du_{\iota_1(Q)}$$

and

$$-\Omega = (-1 - f_{\iota_1(Q)})du_{\iota_1(Q)} = (1 + \iota_1(f_Q))(c + \frac{dg_{\iota_1(Q)}}{du_{\iota_1(Q)}})du_{\iota_1(Q)}.$$

Expanding the final product, one sees that $c = -1$ and $v_{\iota_1(Q)}(g_{\iota_1(Q)}) > n + 1$.

2. This statement and proof are similar to [DHRS18, Lemma C.1]. Applying ι_1 to (A.1), we have)

$$-b = \iota_1(b) = \frac{c_{Q,n}}{\iota_1(u_Q)^n} + \ldots + \frac{c_{Q,2}}{\iota_1(u_Q)^2} + \frac{c_{Q,1}}{\iota_1(u_Q)} + \iota_1(\tilde{f})$$
$$= \frac{(-1)^n c_{Q,n}}{u_{\iota_1(Q)}^n}(1 + g_n) + \ldots + \frac{(-1)^2 c_{Q,2}}{u_{\iota_1(Q)}^2}(1 + g_2) + \frac{(-1)^1 c_{Q,1}}{u_{\iota_1(Q)}}(1 + g_1) + \iota_1(\tilde{f})$$

where $v_{\iota_1(Q)}(g_\ell) > n, n \geq \ell \geq 1$. This follows from the fact that $\iota_1(u_Q) = u_{\iota_1(Q)} + g_{\iota_1(Q)}, v_{\iota_1(Q)}(g_{\iota_1(Q)}) > n + 1$ and so $\iota_1(u_Q)^{-\ell} = (-1)^\ell u_{\iota_1(Q)}^{-\ell}(1 + g_\ell)$ for some g_ℓ with $v_{\iota_1(Q)}(g_\ell) > n$. Equating negative powers of $u_{\iota_1(Q)}$ yield the result. $\square$

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