HAL open science

Compact Kähler manifolds with no projective specialization

Claire Voisin

▶ To cite this version:

Claire Voisin. Compact Kähler manifolds with no projective specialization. Bollettino dell'Unione Matematica Italiana, 2021, 15 (1-2), pp.353-364. 10.1007/s40574-021-00288-3 . hal-03403491

HAL Id: hal-03403491

https://hal.science/hal-03403491v1

Submitted on 26 Oct 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Compact Kähler manifolds with no projective specialization

Claire Voisin*

Pour Fabrizio, à l'occasion de son 70ème anniversaire

Abstract

We show the existence of a compact Kähler manifold which does not fit in a proper flat family over an irreducible base with one projective (possibly singular) fiber. We also give a topological version of this statement. This strengthens our earlier counterexamples to the Kodaira algebraic approximation problem.

0 Introduction

In the paper [4], Kodaira proved that any compact Kähler surface admits "algebraic approximations", that is, it becomes projective by an arbitrarily small deformation. This result, obtained as a consequence of the classification of surfaces, has been reproved in [1] by an infinitesimal method. A similar statement has been obtained recently by Lin [7] for threefolds. However, starting from dimension 4, we constructed in [9] compact Kähler manifolds not homeomorphic to, and in particular not deformation equivalent to, a complex projective manifold. In the paper [10], we also exhibited examples of compact Kähler manifolds, no smooth bimeromorphic models of which is homeomorphic to a complex projective manifold.

The manifolds X constructed in [9] thus have the property that there is no smooth proper holomorphic map $\mathcal{X} \to B$, where B is a connected analytic space, with two points b, $b' \in B$ such that $\mathcal{X}_b \cong X$ and the fiber $\mathcal{X}_{b'}$ is projective. This does not say anything however on the singular specializations $\mathcal{X}_{b'}$ of such proper flat holomorphic maps $\mathcal{X} \to B$ with one fiber $\mathcal{X}_b$ isomorphic to X.

Definition 0.1. *A compact complex manifold has a projective specialization if there exists a proper flat holomorphic map $\mathcal{X} \to B$, where B is an irreducible analytic space, with one fiber $\mathcal{X}_b$ isomorphic to X and one projective fiber $\mathcal{X}_0$.*

The main result of this paper is the following theorem, which addresses a question asked to us by A. Kazhymurat.

Theorem 0.2. *There exists a compact Kähler manifold X with no projective specialization. Such manifolds exist starting from dimension 5.*

A compact Kähler manifold X satisfying this property is constructed in Section 1.1. Note that the result presumably holds in dimension 4 as well.

We will also establish in Section 3.1 a variant of these results, using stronger properties of the cohomology algebra of the considered compact Kähler manifold. This will allow us to strengthen the theorem above by making it into a topological statement. We will prove indeed in Section 3.1 the following result.

Theorem 0.3. *There exists a compact Kähler manifold Y satisfying the following property: There is no proper flat holomorphic map $\mathcal{X} \to B$, where B is an irreducible analytic space, with one smooth fiber $\mathcal{X}_b$ homeomorphic to Y and one projective fiber $\mathcal{X}_0$.*

*The author is supported by the ERC Synergy Grant HyperK (Grant agreement No. 854361).

A compact Kähler manifold Y satisfying this property is constructed in Section 1.2.

Remark 0.4. Even if the statement concerns the homeomorphism class of Y, one cannot replace "irreducible" by "connected" in such a statement, because, in the presence of singular fibers, it is not true that the smooth fibers are all homeomorphic if B is only assumed connected but is not irreducible. An explicit example illustrating this phenomenon is constructed in [6].

As one can notice, an important difference between Theorems 0.2 and 0.3 is that the latter does not assume the existence of a Kähler fiber in the family $\mathcal{X}$. We discuss in Section 3.2 some general consequences of the existence of one singular projective fiber.

Compact Kähler manifolds X or Y satisfying these conclusions are obtained using a variant of the main constructions in [9], [10]. We review the constructions in Section 1. The fact that the examples X constructed in Section 1.1, (resp. the examples Y constructed in Section 1.2,) satisfy the conclusion of Theorem 0.2 (resp. Theorem 0.3) follows from Theorem 0.6 (resp. Theorem 0.7) below. To explain this, let us first recall the strategy of [9], [10] using the formalism presented in [11]. We introduce the following definition (see [11])

Definition 0.5. *(1) A Hodge structure on the cohomology algebra of a compact complex manifold X is the data of a (effective) weight k Hodge structure on each cohomology group $H^k(X, \mathbb{Q})$, such that the cup-product map*

$$H^k(X, \mathbb{Q}) \otimes H^l(X, \mathbb{Q}) \to H^{k+l}(X, \mathbb{Q})$$

is a morphism of Hodge structures for each pair of integers k, l.

(2) The Hodge structure is said polarizable if for each k, the weight k Hodge structure on $H^k(X, \mathbb{Q})$ is polarizable.

If X is a compact Kähler manifold, its cohomology algebra carries a Hodge structure as defined in (1), since the cup-product on X maps $H^{p,q}(X) \otimes H^{p',q'}(X)$ to $H^{p+p',q+q'}(X)$. If X is a projective complex manifold, it carries a polarizable Hodge structure on its cohomology algebra as in the above definition, where the polarization is constructed using a polarization $l = c_1(L)$ on X, where L an ample line bundle on X. The non-existence of a polarizable Hodge structure on the cohomology algebra is the criterion used in the papers [9], [10] to detect manifolds not homeomorphic to projective complex manifolds. We constructed in these papers some compact Kähler manifolds having the property that their cohomology algebra does not admit a polarizable Hodge structure. The manifolds X and Y that we will construct satisfy this property (see Theorem 1.7). Theorem 0.2 will thus follow by contradiction from the following result (see also Theorem 2.4 for a slightly stronger version)

Theorem 0.6. *Let X be a compact Kähler manifold. Assume*

(i) The Hodge structure on the cohomology algebra $H^*(X, \mathbb{Q})$ is rigid under small deformations of X.

(ii) The automorphisms of the Hodge structure on the cohomology algebra $H^*(X, \mathbb{Q})$ are semi-simple.

Then, if X admits a projective specialization, the cohomology algebra $H^*(X, \mathbb{Q})$ admits a polarizable Hodge structure.

Similarly, Theorem 0.3 will be obtained as a consequence of the following

Theorem 0.7. *Let Y be a compact complex manifold. Assume the automorphisms of the cohomology algebra of Y are semi-simple. Then, if Y admits a projective specialization, the cohomology algebra of Y carries a Hodge structure which is polarizable.*

Note that, in Theorem 0.7, the assumptions on the cohomology algebra are much stronger than in Theorem 0.6, since in the former we are considering all the automorphisms of the cohomology algebra, while in the latter, only the automorphisms preserving also the given

Hodge structure are considered. The semi-simplicity properties of the automorphisms of the cohomology algebra of the compact Kähler manifolds X and Y appearing respectively in Theorem 0.6 and Theorem 0.7 will be established respectively in Sections 1.1 and 1.2.

We will prove Theorem 0.2 in Section 2.2 and Theorem 0.3 in Section 3.1. The proofs rely on Steenbrink's work [8] on limit Hodge structures, that works using only the projectivity of the central fiber. We will also discuss in Section 3.2 some further generalities on compact complex manifolds admitting a projective degeneration.

Thanks. *I thank Aknazar Kazhymurat for asking me this question and Radu Laza for helpful discussion.*

1 The examples

1.1 The manifold X

We will discuss here the construction in dimension 7 for simplicity. The case of dimension > 7 will follow by taking a product with projective space and applying similar arguments. In dimension 7, a manifold X satisfying the assumptions of Theorem 0.6 will be obtained by a slight variation of the examples in [10]. We start from a 2-dimensional complex torus T admitting an endomorphism ϕ which is an isogeny and whose action $\psi = \phi_*$ on $H_1(T, \mathbb{Q})$ has the property that the Galois group of the field $\mathbb{Q}[\psi]$ is as big as possible, that is, acts on the eigenvalues of ψ as the symmetric group of 4 elements. It is clear that such a complex torus equipped with an endomorphisms is rigid, and it is proved in [9] that such a torus is not projective (this does not contradict [2] which considers a finite group action). Let $\widehat{T} := \mathrm{Pic}^0(T)$. Denote by $\mathcal{L}$ the Poincaré divisor on $T \times \widehat{T}$, and let $\mathcal{L}_\phi := (\phi, Id_{\widehat{T}})^*\mathcal{L}$. Let $t \in T,\ \hat{t} \in \widehat{T}$ be two points. Let $W = \mathbb{P}^1 \times T \times \widehat{T}$. We choose two distinct points $t_1,\ t_2 \in \mathbb{P}^1$. W contains then two disjoint complex tori, namely $t_1 \times T \times \hat{t}$ and $t_2 \times t \times \widehat{T}$. We first blow them up in W, getting a compact Kähler manifold $\widetilde{W}$. Next, using the same notation $\mathcal{L}$, $\mathcal{L}_\phi$ for the line bundles pulled-back to $\widetilde{W}$, we construct the $\mathbb{P}^1$-bundles

$$P_1 := \mathbb{P}(\mathcal{E}_1),\ \mathcal{E}_1 := \mathcal{O}_{\widetilde{W}} \oplus \mathcal{L},\ \ P_2 := \mathbb{P}(\mathcal{E}_2),\ \mathcal{E}_2 := \mathcal{O}_{\widetilde{W}} \oplus \mathcal{L}_\phi \tag{1}$$

over $\widetilde{W}$. Note that the $\mathbb{P}^1$-bundles P_1, P_2 over $\widetilde{W}$ have sections σ_1, resp. σ_2, given by the surjective morphisms $\mathcal{O}_{\widetilde{W}} \oplus \mathcal{L} \to \mathcal{L}$, resp. $\mathcal{O}_{\widetilde{W}} \oplus \mathcal{L}_\phi \to \mathcal{L}_\phi$. It follows that the fibered product $P_1 \times_{\widetilde{W}} P_2 \to \widetilde{W}$ has a section $\sigma = (\sigma_1, \sigma_2)$. The compact Kähler manifold X will be defined as the blow-up of the fibered product of P_1 and P_2 along $\Sigma := \sigma(\widetilde{W})$:

$$X := \widetilde{P_1 \times_{\widetilde{W}} P_2}^{\Sigma}. \tag{2}$$

We now prove a few properties that will be needed in the next sections.

Lemma 1.1. *The Hodge structure on the cohomology algebra $H^*(X, \mathbb{Q})$ is rigid under small deformations of X.*

By "rigid under small deformations of X", we mean that any flat holomorphic proper holomorphic map $\mathcal{X} \to B$ to a pointed analytic space (B, b_0), with $\mathcal{X}_{b_0} \cong X$ induces near b_0 the trivial variation of Hodge structures. (As X is Kähler, the fibers $\mathcal{X}_b$ are also Kähler for b close to b_0.)

Proof of Lemma 1.1. We observe that the cohomology algebra of X is generated in degree ≤ 2, so it suffices to prove rigidity for the induced Hodge structures on $H^1(X, \mathbb{Q})$ and $H^2(X, \mathbb{Q})$. We have

$$H^2(X, \mathbb{Q}) = \bigwedge^2 H^1(X, \mathbb{Q}) \oplus \mathbb{Q}d \oplus \mathbb{Q}\hat{d} \oplus \mathbb{Q}h \oplus \mathbb{Q}H_1 \oplus \mathbb{Q}H_2 \oplus \mathbb{Q}e, \tag{3}$$

where e is the class of the exceptional divisor E over Σ, d, $\hat{d}$ are the pull-backs to X of the classes of the exceptional divisors of $\widetilde{W}$, h is the pull-back to X of the class $c_1(\mathcal{O}_{\mathbb{P}^1}(1))$ via the composite morphism

$$X \to P_1 \times_{\widetilde{W}} P_2 \to \widetilde{W} \to W \stackrel{\mathrm{pr}_1}{\to} \mathbb{P}^1,$$

and H_1, H_2 are the pull-backs to X of the first Chern classes $c_1(\mathcal{O}_{P_i}(1))$ of the projective bundles $P_i \to \widetilde{W}$.

Given a deformation $\mathcal{X} \to B$ of $X \cong \mathcal{X}_0$, we can apply the Kodaira stability theorems [5]. They say that a small deformation of a compact complex manifold which is the blow-up $\widetilde{Y} \to Y$ of a manifold Y along a submanifold $\Sigma \subset Y$ is the blow-up of a deformation Y_t of Y along a deformation $\Sigma_t \subset Y_t$ of Σ; furthermore a small deformation of a projective bundle fibration $P \to Y$ is a projective bundle fibration $P_t \to Y_t$. This implies in our case that for b close to b_0, there is a divisor $E_b \subset \mathcal{X}_b$ which is a deformation of E and is the exceptional divisor of a contraction $\mathcal{X}_b \to Q_b$, where Q_b is a deformation of $Q_{b_0} = P_1 \times_{\widetilde{W}} P_2$. Furthermore, still by [5], the $\mathbb{P}^1 \times \mathbb{P}^1$-fibration on Q_{b_0} deforms to a $\mathbb{P}^1 \times \mathbb{P}^1$-fibration on Q_b, hence we get a deformation $Q_b \to \widetilde{W}_b$ of the morphism $P_1 \times_{\widetilde{W}} P_2 \to \widetilde{W}$. Finally and again by Kodaira stability theorem, the manifold $\widetilde{W}_b$ for b close to b_0 contains contractible divisors which are deformations of the exceptional divisors of $\widetilde{W}$ over W. Thus for b close to b_0, $\widetilde{W}_b$ contracts to a compact complex manifold which is a deformation W_b of the complex manifold $\mathbb{P}^1 \times T \times \widehat{T}$, hence is a $\mathbb{P}^1$-fibration over a complex torus K_b which is itself a deformation of $K_{b_0} = T \times \widehat{T}$. Furthermore, the complex manifold W_b contains the images of the two contracted divisors, which are complex submanifolds obtained as small deformations of $T \times \hat{t} \subset W$ and $t \times \widehat{T} \subset W$. They are thus 2-dimensional complex tori T_b, T'_b contained in W_b, which map to 2-dimensional subtori T_b, T'_b of K_b intersecting in one point. We thus conclude that $K_b \cong T_b \oplus T'_b$.

Moreover the image $\Sigma \subset P_1 \times_{\widetilde{W}} P_2$ of the contracted divisor E deforms to the image $\Sigma_b \subset Q_b$ of the contracted divisor E_b, hence the $\mathbb{P}^1 \times \mathbb{P}^1$- fibration $Q_b \to W'_b$ has a section for b close to b_0. It follows that there are vector bundles $\mathcal{E}_{1,b}$, $\mathcal{E}_{2,b}$ on $\widetilde{W}_b$ obtained as deformations of the vector bundles $\mathcal{E}_1$, $\mathcal{E}_2$ of (1) on $\widetilde{W}$, such that

$$Q_b \cong \mathbb{P}(\mathcal{E}_{1,b}) \times_{\widetilde{W}_b} \mathbb{P}(\mathcal{E}_{2,b}).$$

Finally, by taking the determinants of $\mathcal{E}_{1,b}$, $\mathcal{E}_{2,b}$, we conclude that there are holomorphic line bundles $\mathcal{L}_b$, $\mathcal{L}_{\phi,b}$ on $\widetilde{W}_b$ which are respective deformations of $\mathcal{L}$, $\mathcal{L}_\phi$ on $\widetilde{W}$. As these line bundles are pulled-back at b_0 from line bundles on K_{b_0}, the same is true of the line bundles $\mathcal{L}_b$, $\mathcal{L}_{\phi,b}$ on $\widetilde{W}_b$. In conclusion we proved that the complex torus $K_b \cong T_b \oplus T'_b$ carries two holomorphic line bundles $\mathcal{L}_b$, $\mathcal{L}_{\phi,b}$ deforming $\mathcal{L}$ and $\mathcal{L}_\phi$ respectively. Using $\mathcal{L}_b$, one concludes that $T'_b \cong \widehat{T}_b$, and using $\mathcal{L}_{\phi,b}$, one concludes that T_b carries an endomorphism deforming ϕ. It follows that $T_b \cong T$ since the pair (T, ϕ) is rigid. We thus proved that the Hodge structure on $H^1(\mathcal{X}_b, \mathbb{Q})$ is isomorphic to the Hodge structure on $H^1(X, \mathbb{Q})$. Furthermore, by the above arguments, all the degree 2 Hodge classes on X listed in (3) deform to degree 2 Hodge classes on $\mathcal{X}_b$, so (3) shows that the Hodge structure on $H^2(\mathcal{X}_b, \mathbb{Q})$ is isomorphic to the Hodge structure on $H^2(X, \mathbb{Q})$. □

Lemma 1.2. *The automorphisms of the Hodge structure on the cohomology algebra $H^*(X, \mathbb{Q})$ are semisimple.*

Proof. The automorphisms of the Hodge structure on $H^1(X, \mathbb{Q}) = H^1(T, \mathbb{Q}) \oplus H^1(T, \mathbb{Q})^*$ are of the form

$$\begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix},$$

where ψ_1 resp. ψ_2 are automorphisms of the Hodge structure on $H^1(T, \mathbb{Q})$, resp. $H^1(T, \mathbb{Q})^*$. Indeed, the two tori T, $\widehat{T}$ are simple and not isogenous. Next, using the fact that ${}^t\psi$, resp.

ψ acts in an irreducible way as an automorphism of the Hodge structure on $H^1(T,\mathbb{Q})$, resp. $H^1(T,\mathbb{Q})^*$, we get that $\psi_1 \in \mathbb{Q}[{}^t\psi]$, resp. $\psi_2 \in \mathbb{Q}[\psi]$. As the endomorphism ψ is semisimple, it follows that any automorphism of the Hodge structure on $H^1(X,\mathbb{Q})$ is semisimple.

It remains to prove semisimplicity of the action on the higher degree cohomology groups. As the cohomology algebra of X is generated in degree ≤ 2, it suffices to prove the semisimplicity on degree 2 cohomology. As the automorphisms of the Hodge structure on $H^2(X,\mathbb{Q})$ preserve the space of Hodge classes, using the decomposition (3), we only need to prove

Claim 1.3. *An automorphism γ of the Hodge structure on the cohomology algebra of X has a power γ^r, $r>0$ which preserves the space $\langle d,\, \hat{d},\, h,\, H_1,\, H_2,\, e\rangle \subset H^2(X,\mathbb{Q})$ and acts in a semisimple way on it.*

Proof. The automorphism γ acts on $H^*(X,\mathbb{Q})$ preserving the algebra structure and the Hodge structure. In particular, it preserves the subalgebra generated by $H^1(X,\mathbb{Q})$ and the subspace $\mathrm{Hdg}^2(X)$, hence it preserves $\mathrm{Hdg}^2(X)\cap\bigwedge^2 H^1(X,\mathbb{Q})$. We claim that a power of γ preserves up to a scalar multiple the classes h, H_1, H_2. For the class h, the reason is that $h^2=0$ in $H^4(X,\mathbb{Q})$, and h is an isolated solution of this set of equations in $\mathbb{P}(\mathrm{Hdg}^2(X))$; moreover γ acts on $\mathbb{P}(\mathrm{Hdg}^2(X))$ preserving the algebraic subset defined by these equations, since it preserves the algebra structure of $H^*(X,\mathbb{Q})$. For the classes H_1, H_2, one observes that they satisfy an equation of the form $H_i^2 = H_i l_i$, for some classes $l_i \in \bigwedge^2 H^1(X,\mathbb{Q})\cap \mathrm{Hdg}^2(X)$. These equations come from the fact that the vector bundles $\mathcal{E}_i$ of (1) have $c_2=0$. Again the classes H_i are isolated solutions of these equations in $\mathbb{P}(\mathrm{Hdg}^2(X))$ so that a power of γ will fix them. Similarly, a power of γ preserves up to a scalar the classes d and $\hat{d}$, because they have the property that the cup-product map $d\cup$, $\hat{d}\cup : H^1(X,\mathbb{Q})\to H^3(X,\mathbb{Q})$ is not injective, and they are the only elements of $\mathbb{P}(\mathrm{Hdg}^2(X))$ which do not belong to $\mathbb{P}(\bigwedge^2 H^1(X,\mathbb{Q})\cap \mathrm{Hdg}^2(X))$ and satisfy this condition. We conclude that a power γ^r, $r>0$, of γ preserves the subalgebra generated by $H^1(X,\mathbb{Q})$ and h, H_1, H_2, d, $\hat{d}$, that is, the cohomology subalgebra

$$H^*(P_1\times_{\widetilde{W}} P_2,\mathbb{Q}) \overset{\tau}{\hookrightarrow} H^*(X,\mathbb{Q}).$$

But then, using the fact that the morphism $\tau : X\to P_1\times_{\widetilde{W}} P_2$ is birational, γ^r also preserves the class e, since it generates the kernel of the morphism $\tau_* : H^2(X,\mathbb{Q})\to H^2(P_1\times_{\widetilde{W}} P_2,\mathbb{Q})$. One uses here the fact that τ_* is determined by τ^* and the algebra structure, by Poincaré duality. Thus a power γ^r preserves $\langle h,\, H_1,\, H_2,\, d,\, \hat{d},\, e\rangle$ and acts in a diagonalized way on it. The claim is proved. □

We thus proved that a power of γ acts in a semisimple way on $H^*(X,\mathbb{Q})$, which implies that γ acts in a semisimple way since γ is an automorphism. □

1.2 The manifold Y

The compact Kähler manifold X constructed in the previous section has the property that the automorphisms of the cohomology algebra $H^*(X,\mathbb{Q})$ preserving its Hodge structure are semisimple (see Lemma 1.2). This is the assumption appearing in Theorem 0.6. We are now going to construct a compact Kähler manifold Y having the property that all the automorphisms of its cohomology algebra $H^*(Y,\mathbb{Q})$ are semisimple, which is the assumption appearing in Theorem 0.7.

We will discuss here the construction in dimension 5 for simplicity. The case of dimension >5 will follow by taking a product with projective space and applying similar arguments. In dimension 5, a manifold Y satisfying the assumptions of Theorem 0.7 will be obtained by a small variant of [9]. We start as in the previous section from a 2-dimensional complex torus T admitting an endomorphism ϕ which is an isogeny and whose action $\psi=\phi_*$ on $H_1(T,\mathbb{Q})$ has the property that the Galois group of the field $\mathbb{Q}[\psi]$ acts on the eigenvalues of ψ as the symmetric group of 4 elements.

We start with $W := T \times T \times \mathbb{P}^1$ and choose four distinct elements $t_1, \dots, t_4$ in $\mathbb{P}^1$. Let $t \in T$ be chosen. W contains the following four disjoint complex submanifolds, all isomorphic to T:

$$T \times t \times t_1,\ t \times T \times t_2,\ \Delta_T \times t_3,\ \Gamma_\phi \times t_4, \tag{4}$$

where as usual Δ_T is the diagonal of T and Γ_ϕ is the graph of ϕ.

We will denote by Y the compact Kähler manifold obtained by blowing-up W along these four submanifolds. The tiny difference with the construction of [9] is that the product with $\mathbb{P}^1$ and the choice of four distinct points t_i in $\mathbb{P}^1$ makes the four centers (4) of blow-up disjoint, while their images in $T \times T$ are not disjoint. This forced us in [9] to blow-up first their intersection points in order to separate them before blow-up, making the cohomological computations heavier.

We will need the following statement concerning the Hodge structures on the cohomology algebra of Y.

Proposition 1.4. *The automorphisms of the cohomology algebra $H^*(Y, \mathbb{Q})$ are semi-simple.*

Proof. As the cohomology algebra $H^*(Y, \mathbb{Q})$ is generated in degree ≤ 2, it suffices to prove that the action of an automorphism γ of $H^*(Y, \mathbb{Q})$ is semi-simple on $H^1(Y, \mathbb{Q})$ and $H^2(Y, \mathbb{Q})$. We have $H^2(Y, \mathbb{Q}) = \mathrm{Hdg}^2(Y) \oplus \bigwedge^2 H^1(Y, \mathbb{Q})$, where the space $\mathrm{Hdg}^2(Y)$ of Hodge classes on Y is generated by the class h which is the pull-back to Y of the class $\mathrm{pr}_1^* c_1(\mathcal{O}_{\mathbb{P}^1}(1))$ on W and the classes d_i of the four exceptional divisors of Y over W. Clearly the subspace $\bigwedge^2 H^1(Y, \mathbb{Q}) \subset H^2(Y, \mathbb{Q})$ is preserved by any automorphism of the cohomology algebra. We first prove

Lemma 1.5. *Any endomorphism γ of the cohomology algebra $H^*(Y, \mathbb{Q})$ preserves the subspace*

$$\mathrm{Hdg}^2(Y) \subset H^2(Y, \mathbb{Q})$$

and acts in a semi-simple way on it.

Proof. The subspace $\mathbb{Q}h$ is preserved by γ because, up to a coefficient, h is the only class in $H^2(Y, \mathbb{Q})$ which is not in $\bigwedge^2 H^1(Y, \mathbb{Q})$ and satisfies $h^2 = 0$. Next we claim that the $\mathbb{Q}$-vector subspace $\langle d_i \rangle \subset \mathrm{Hdg}^2(Y)$ is stable under γ. This follows indeed from the structure of the cohomology algebra: Looking at the cup-product map

$$\alpha \cup : H^1(Y, \mathbb{Q}) \to H^3(Y, \mathbb{Q})$$

for $\alpha \in H^2(Y, \mathbb{Q})$, we see that the d_i's are up to a scalar multiple the only classes not in $\bigwedge^2 H^1(Y, \mathbb{Q})$ for which the cup product map $d_i \cup : H^1(Y, \mathbb{Q}) \to H^3(Y, \mathbb{Q})$ has a kernel of dimension ≥ 4. Hence γ preserves the subspace $\langle d_i \rangle \subset H^2(Y, \mathbb{Q})$ and thus preserves $\mathrm{Hdg}^2(Y)$. Finally, exactly for the same reason, γ acts projectively on the space $\mathbb{P}(\langle d_i \rangle)$ by permuting the d_i's, so that a power of γ acts on the projective space $\mathbb{P}(\langle d_i \rangle)$ fixing the d_i's. We thus proved that the action of a power of γ on the vector space $\mathrm{Hdg}^2(Y)$ is diagonalizable, which implies semisimplicity of the action of γ on $\mathrm{Hdg}^2(Y)$, since γ is an automorphism. □

For the action of γ on degree 1 cohomology, we next prove

Lemma 1.6. *Any automorphism γ of the cohomology algebra $H^*(Y, \mathbb{Q})$ has a power γ^r, $r > 0$, acting on $H^1(Y, \mathbb{Q}) = H^1(T, \mathbb{Q})^2$ via a matrix*

$$\begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_1 \end{pmatrix}, \tag{5}$$

where $\psi_1 \in \mathbb{Q}[^t\psi]$.

Proof. The proof of Lemma 1.5 shows that γ preserves the class h and the $\mathbb{Q}$-vector space $K := \langle d_i \rangle$. We also saw that a power γ^r for some $r > 0$ acts on $\mathbb{P}(K)$ fixing the d_i. It then follows that the action of γ^r on $H^1(Y, \mathbb{Q})$ preserves the four subspaces $L_i := \mathrm{Ker}\,(d_i \cup : H^1(Y, \mathbb{Q}) \to H^3(Y, \mathbb{Q}))$. We have

$$H^1(Y, \mathbb{Q}) = L_1 \oplus L_2,$$

and this decomposition is preserved by γ^r. Furthermore the subspace $L_3 \subset L_1 \oplus L_2$ defines an isomorphism $L_1 \cong L_2$, so we can see $L_4 \subset L_1 \oplus L_2 \cong L_1 \oplus L_1$ as the graph of an automorphism ψ' of L_1, that is easily seen to be conjugate to ${}^t\psi$. As γ^r preserves the L_i, it follows from the above analysis that γ^r acts on L_1 and L_2 via the same automorphism ψ_1, that has the property that $\begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_1 \end{pmatrix}$ maps the graph of ${}^t\psi$ to itself. Equivalently, ψ_1 commutes with ${}^t\psi$. As ψ has only simple eigenvalues, it follows that $\psi_1 \in \mathbb{Q}[{}^t\psi]$. □

As ψ_1 is semisimple, Lemma 1.6 implies that the action of a power γ^r, $r > 0$ of γ on $H^1(Y, \mathbb{Q})$ is semisimple, and it follows that the action of γ^r on $H^1(Y, \mathbb{Q})$ and $\bigwedge^2 H^1(Y, \mathbb{Q})$ is semisimple. Using Lemma 1.5, we conclude that the action of γ^r on $H^2(Y, \mathbb{Q})$ is semisimple. This concludes the proof of Proposition 1.4. □

To conclude this section, we note for future use that the compact Kähler manifolds X and Y constructed above satisfy the following property (see [9], [10]).

Theorem 1.7. *For any Hodge structure on the cohomology algebra $H^*(X, \mathbb{Q})$, resp. $H^*(Y, \mathbb{Q})$, the induced weight 1 Hodge structure on $H^1(X, \mathbb{Q})$, resp. $H^1(Y, \mathbb{Q})$ is not polarizable.*

The existence of a polarizable Hodge structure on the cohomology algebra is the necessary criterion used in [9] for a compact Kähler manifold to be homeomorphic to a complex projective manifold. Theorem 1.7 is proved in [9] for the manifold Y. The case of X is treated by similar arguments. In both cases, the idea is that the structure of the cohomology algebra forces a Hodge structure on the cohomology algebra to have its degree 1 part isomorphic to a direct sum $H_1 \oplus H_2$ of weight 1 Hodge structures admitting an endomorphism conjugate to ψ or ${}^t\psi$, which makes them not polarizable.

2 Proof of Theorem 0.2

2.1 Preliminary results

We start with a general statement concerning proper holomorphic fibrations with one Kähler fiber.

Proposition 2.1. *Let $\pi : \mathcal{X} \to B$ be a proper flat holomorphic map, where B is an irreducible analytic space. Assume one smooth fiber $\mathcal{X}_t$ is Kähler. Then there exists a dense open subset $U \subset B$ which is the complement of a closed real analytic subset of B, such that the fiber $\mathcal{X}_b$, for $b \in U$, carries a Hodge structure on its cohomology algebra.*

Proof. For any complex manifold X and for any integer k, there is a "Hodge filtration" on the cohomology $H^k(X, \mathbb{C})$ defined as

$$F^r H^k(X, \mathbb{C}) = \mathrm{Im}\,(\mathbb{H}^k(X, \Omega_X^{\bullet \geq r}) \to \mathbb{H}^k(X, \Omega_X^\bullet)). \tag{6}$$

These filtrations for various k are obviously compatible with the cup-product map in the sense that $F^r H^k \cup F^s H^l \subset F^{r+s} H^{k+l}$. In order to prove that there is a Hodge structure on the cohomology algebra of X, it thus suffices to show that these Hodge filtrations define Hodge structures, which is equivalent to saying that they satisfy the opposite condition for all k, r

$$F^r H^k(X, \mathbb{C}) \oplus \overline{F^{k-r+1} H^k(X, \mathbb{C})} \cong H^k(X, \mathbb{C}). \tag{7}$$

In the situation of the proposition, we know that one fiber is Kähler. It follows that its Frölicher spectral sequence degenerates at E_1. This implies that the Frölicher spectral sequence of $\mathcal{X}_b$ degenerates at E_1 for b in a non-empty open subset $V \subset B$ (contained in the open subset B^0 of B parameterizing smooth fibers), which is the complement of a proper closed analytic subset (we will use the terminology "analytic Zariski open" in the sequel). As B is irreducible, this open set is dense. Next, the degeneracy at E_1 of the Frölicher spectral sequence of the fibers $\mathcal{X}_b$ over V implies that, denoting $\pi_V : \mathcal{X}_V \to V$ the restriction of π to $\mathcal{X}_V := \pi^{-1}(V)$, the subsheaves

$$F^r\mathcal{H}^k \subset \mathcal{H}^k,\ \mathcal{H}^k = \mathbb{R}^k\pi_{V*}\Omega^\bullet_{\mathcal{X}^0/B^0} = \mathbb{R}^k\pi_{V*}\mathbb{C} \otimes \mathcal{O}_{B^0}$$

are locally free subbundles satisfying the base change property, because the sheaf $\mathcal{H}^k$ is locally free. It then follows from the constancy of the rank of $F^rH^k(\mathcal{X}_b, \mathbb{C})$, $b \in V$, that the locus U of points $b \in V$ where the opposite conditions (7) are satisfied is the complement of a closed real analytic subset of V. This locus is not empty by assumption and, as B is irreducible, U is dense in B. □

Corollary 2.2. *In the situation above, assume the compact Kähler fiber $\mathcal{X}_t$ has the property that the Hodge structure on its cohomology algebra is rigid under small deformations of $\mathcal{X}_t$. Then, for any integer k, the monodromy action*

$$\pi_1(B^0, t) \to \operatorname{Aut} H^k(\mathcal{X}_t, \mathbb{Q}) \tag{8}$$

acts via automorphisms of Hodge structures.

Proof. The Hodge structure of the cohomology algebra of $\mathcal{X}_t$ being rigid under small deformations of $\mathcal{X}_t$, for any integer k, the Hodge structure on the degree k cohomology of the fibers $\mathcal{X}_b$ for $b \in U$ close to b is locally constant. It follows by analytic continuation that the Hodge filtration, which varies holomorphically and is defined on the analytic Zariski dense open set $V \subset B^0$, is locally constant on V. As the natural map $\pi_1(V, t) \to \pi_1(B^0, t)$ is surjective, the monodromy action (8) on $H^k(\mathcal{X}_t, \mathbb{Q})$ preserves the Hodge filtration, hence the Hodge structure. □

Remark 2.3. The proof also shows that for any $b \in V$, the Hodge filtration on $H^k(\mathcal{X}_b, \mathbb{C})$ induces a Hodge structure on $H^k(\mathcal{X}_b, \mathbb{Q})$ which is isomorphic to that of $H^k(\mathcal{X}_t, \mathbb{C})$.

2.2 Projective specializations of compact Kähler manifolds

Our goal in this section is to prove Theorem 0.2, as a consequence of the following result on projective specializations of compact Kähler manifolds. This is a slight improvement of Theorem 0.6.

Theorem 2.4. *Let X be a compact Kähler manifold. Assume*

(i) The Hodge structure on the cohomology algebra $H^*(X, \mathbb{Q})$ is rigid under small deformations of X.

(ii) The automorphisms of the Hodge structure on the cohomology algebra $H^*(X, \mathbb{Q})$ preserving a nonzero element of $H^2(X, \mathbb{Q})$ with nonzero top self-intersection are semi-simple.

Then, if X admits a projective specialization, the cohomology algebra $H^*(X, \mathbb{Q})$ admits a polarizable Hodge structure.

Before giving the proof, we prove Theorem 0.2.

Proof of Theorem 0.2. We observe that assumptions (i) and (ii) above are satisfied by the compact Kähler manifold X constructed in Section 1.1. This is indeed the contents of Proposition 1.1 and Lemma 1.2. By Theorem 1.7, the cohomology algebra of X does not admit a polarizable Hodge structure. Theorem 2.4 thus shows that X does not admit a projective specialization. □

Proof of Theorem 2.4. Let X be a compact Kähler manifold satisfying the assumptions (i) and (ii) above. Let $\pi : \mathcal{X} \to B$ be a flat proper holomorphic map between analytic spaces, where B is irreducible, with one fiber $\mathcal{X}_b$ isomorphic to X and one fiber $\mathcal{X}_0$ projective. We can assume by desingularization that B is smooth with discriminant divisor $\mathcal{D} = B \setminus B^0$, and we can choose a disc $D \subset B$ centered at 0 and not contained in $\mathcal{D}$. The analytic Zariski open set V being defined as in the previous section, we can even assume that D is not contained in $B \setminus V$. From now on, we will only work with the restricted family

$$\pi' : \mathcal{X}' := D \times_B \mathcal{X} \to D.$$

Note that, after a finite base-change and blow-up over the central fiber, one can assume that $\mathcal{X}'$ is smooth and the central fiber $\mathcal{X}'_0$ is a reduced divisor with normal crossings.

For any choice of $s \in D^* \cap V$, one has the monodromy representation

$$\rho : \mathbb{Z} = \pi_1(D^*, s) \to \operatorname{Aut} H^k(\mathcal{X}_s, \mathbb{Q}),$$

hence a monodromy operator $T = \rho(\gamma)$ where γ is a counterclockwise loop around 0 based at s. By assumption (i), the assumptions of Corollary 2.2 are satisfied by X. Corollary 2.2 and Remark 2.3 imply that there is a Hodge structure on $H^*(\mathcal{X}_s, \mathbb{Q})$, isomorphic to the Hodge structure on $H^*(X, \mathbb{Q})$, and T acts on the cohomology algebra $H^*(\mathcal{X}_s, \mathbb{Q})$ as an automorphism of Hodge structures, which obviously also preserves the algebra structure. We use now the fact that the central fiber is projective. There is thus a class $H \in H^2(\mathcal{X}'_0, \mathbb{Q})$ defined as the restriction of the hyperplane class in a projective embedding of $\mathcal{X}'_0$. Choosing a topological retraction r of $\mathcal{X}'$ onto $\mathcal{X}'_0$, the class $r^*H_{|\mathcal{X}'_s} \in H^2(\mathcal{X}'_s, \mathbb{Q})$ is invariant under T. We have $\int_{\mathcal{X}'_0} H^n \neq 0$, $n = \dim X$, hence $\int_{\mathcal{X}'_s} r^*H^n \neq 0$. By assumption (ii), the action of T is thus semi-simple. We now apply the following result (which in this context is due to Steenbrink).

Theorem 2.5. *(Steenbrink [8, Theorem 2.21]) Let $\mathcal{X}$ be a complex manifold and $\mathcal{X} \to D$ a flat proper holomorphic map with smooth fibers $\mathcal{X}_s$ for $s \neq 0$ and reduced normal crossing central fiber. Then the action of T on $H^k(\mathcal{X}_s, \mathbb{Q})$ is unipotent.*

It follows that the eigenvalues of T on $H^*(\mathcal{X}_s, \mathbb{Q})$ are equal to 1, hence as T is semi-simple, it must be the identity.

Let $\widetilde{D^*}$ be the universal cover of D^* and let $\widetilde{\mathcal{X}'}^* := \mathcal{X}' \times_D \widetilde{D^*}$. Let $H^*_{\lim} := H^*(\widetilde{\mathcal{X}'}^*, \mathbb{Q})$. The cohomology algebra $H^*_{\lim}$ is isomorphic to $H^*(X, \mathbb{Q})$. We now use the fact that the central fiber $\mathcal{X}'_0$ is projective. We can then apply the following results of Steenbrink [8].

Theorem 2.6. *(i) There is a mixed Hodge structure on each $H^k_{\lim}$.*

(ii) More precisely, $H^k_{\lim}$ is computed as the abutment of a (weight) spectral sequence ${}_WE_1^{p,q} \Rightarrow H^{p+q}_{\lim}$, an each ${}_WE_1^{p,q}$ carries a Hodge structure. The differentials are morphisms of Hogge structures and the spectral sequence degenerates at E_2.

(iii) Let $N = \log T$ (as $Id - T$ is nilpotent, this is a polynomial in T with rational coefficients). Then for each $r \geq 0$, N^r induces an isomorphism

$$N^r : {}_WE_1^{-r,q+r} \cong {}_WE_1^{r,q-r}.$$

In our case, we proved that $T = Id$, so $N = 0$. It follows from (iii) that ${}_WE_1^{r,q} = 0$ for $r > 0$ or $r < 0$. Thus the weight filtration is trivial and the mixed Hodge structure on each $H^k_{\lim}$ is pure. Finally, the Hodge structures on ${}_WE_1^{0,q}$ are direct sums of Hodge structures on cohomology groups of intersections of components of the normal crossing divisor $\mathcal{X}'_0$, assuming it has global normal crossings. As these successive intersections are projective manifolds, these Hodge structures are polarized. We thus proved that each $H^k_{\lim}$ is equipped with a polarizable Hodge structure. Looking at the construction of the Hodge filtration in [8], it is clear that these Hodge filtrations are compatible with the cup-product on cohomology, hence we get a polarizable Hodge structure on the cohomology algebra $H^*_{\lim} \cong H^*(X, \mathbb{Q})$. $\square$

3 Topological variant and further remarks

3.1 Proof of Theorems 0.3 and 0.7

Proof of Theorem 0.7. The proof uses arguments which already appeared in the previous section. First of all, we observe that the proof of Theorem 0.2 given in the previous section shows the following:

Proposition 3.1. *Let $\pi : \mathcal{X} \to D$ be a flat proper holomorphic map, where $\mathcal{X}$ is smooth, such that the central fiber is projective and the other fibers are smooth. Let $s \in D^*$ and assume that the monodromy action T on $H^*(\mathcal{X}_s, \mathbb{Q})$ is trivial. Then the cohomology algebra $H^*(\mathcal{X}_s, \mathbb{Q})$ admits a polarizable Hodge structure.*

In the situation of Theorem 0.7, we have a proper flat family $\mathcal{X} \to B$ with smooth fibers homeomorphic to X and a point $0 \in B$ such that the fiber $\mathcal{X}_0$ is projective. After restriction to a carefully chosen disc in B passing through 0, finite base change and semistable reduction, we get a monodromy operator T acting on $H^*(\mathcal{X}'_t, \mathbb{Q})$ for $t \in D^*$. The action of T obviously preserves the structure of $H^*(X, \mathbb{Q})$ as an algebra, hence it is semi-simple by our main assumption. Steenbrink's theorem 2.5 then says that $T = Id$, so that Proposition 3.1 applies. □

Proof of Theorem 0.3. According to Proposition 1.4, the compact Kähler manifold Y constructed in Section 1.2 satisfies the assumptions of Theorem 0.7. Moreover, by Theorem 1.7, there is no polarizable Hodge structure on the cohomology algebra of Y, hence Y does not admit a projective specialization by Theorem 0.7. □

3.2 Further remarks on projective specializations

Consider a proper flat holomorphic map

$$\pi : \mathcal{X} \to D$$

where D is the disc and $\mathcal{X}$ is smooth. We can assume that only the central fiber is singular, and by Hironaka resolution and semistable reduction, we can assume after base change that the central fiber is a reduced normal crossing divisor. Let $D^* := D \setminus \{0\}$. We now prove

Proposition 3.2. *Assume the central fiber $W := \mathcal{X}_0$ is projective. Then the Frölicher spectral sequence of $\mathcal{X}_t$ degenerates at E_1 for t in an analytic Zariski open set of D^*.*

Proof. The Frölicher spectral sequence has a relative version over D, using the relative logarithmic holomorphic de Rham complex $\Omega^\bullet_{\mathcal{X}/D}(\log W)$. According to Steenbrink (see [8, Theorem 2.18]), the coherent sheaf $\mathbb{R}^k\pi_*\Omega^\bullet_{\mathcal{X}/D}(\log W)$ on D is locally free and satisfies base change for any k. There is a relative Frölicher spectral sequence

$$E_1^{p,q} = R^q\pi_*\Omega^p_{\mathcal{X}/D}(\log W) \Rightarrow \mathbb{R}^k\pi_*\Omega^\bullet_{\mathcal{X}/D}(\log W). \tag{9}$$

The relative logarithmic holomorphic de Rham complex restricts on each fiber $\mathcal{X}_t$, $t \neq 0$, to the holomorphic de Rham complex of $\mathcal{X}_t$. Moreover, when W is projective, the Frölicher spectral sequence of the restricted complex $(\Omega^\bullet_{\mathcal{X}/D}(\log W))_{|W}$ degenerates at E_1 by [8, Corollary 4.20]. Upper semicontinuity and local freeness of $\mathbb{R}^k\pi_*\Omega^\bullet_{\mathcal{X}/D}(\log W)$ then imply by standard arguments that the Frölicher spectral sequence (9) degenerates at E_1 for t in an analytic Zariski open neighborhood of 0 in D. □

Remark 3.3. We also deduce, using the same arguments as above, that the sheaves $R^q\pi_*\Omega^p_{\mathcal{X}/D}$ are locally free and satisfy base-change in an analytic-Zariski open neighborhood of 0 in D, and that the subsheaves

$$F^r\mathbb{R}^k\pi_*\Omega^{\bullet\geq r}_{\mathcal{X}/D}(\log W) := \mathrm{Im}\,(\mathbb{R}^k\pi_*\Omega^{\bullet\geq r}_{\mathcal{X}/D}(\log W) \to \mathbb{R}^k\pi_*\Omega^\bullet_{\mathcal{X}/D}(\log W))$$

are locally free subsheaves and satisfy base change.

Remark 3.4. Proposition 3.2 has some similarities with Proposition 1 of [3]. I thank the referee for mentioning this reference.

We do not know the answer to the following questions. Assume as before the central fiber $Y := \mathcal{X}_0$ of a flat proper holomorphic map $\mathcal{X} \to D$, with $\mathcal{X}$ smooth, is projective.

Question 3.5. *Does the Hodge filtration defined above on the cohomology of the fibers $\mathcal{X}_t$ define Hodge structures for t close to 0?*

Assuming the answer to this question is yes, the following question remains

Question 3.6. *Are the Hodge structures on the cohomology of the fibers $\mathcal{X}_t$ for t close to 0 polarizable with real coefficients?*

By this we mean precisely the following: Does there exist a class $\alpha_t \in H^{1,1}(\mathcal{X}_t)_{\mathbb{R}}$ satisfying the hard Lefschetz property, so that the associated Lefschetz intersection pairing satisfies the Hodge-Riemann bilinear relations.

Question 3.7. *Does the fiber $\mathcal{X}_t$ belong to the Fujiki class $\mathcal{C}$? Are there examples where some or all fibers $\mathcal{X}_t$ are not Kähler for t arbitrarily close to 0?*

References

[1] N. Buchdahl. On compact Kähler surfaces, Ann. Inst. Fourier 49 (1999), 287–302.

[2] F. Catanese, A. Demleitner. Rigid group actions on complex tori are projective (after Ekedahl), Commun. Contemp. Math. 22 (2020), no. 7, 1950092.

[3] M. Kerr, R. Laza, M. Saito. Deformations of rational sngularities and Hodge structures, arXiv:1906.03917v2.

[4] K. Kodaira. On compact complex analytic surfaces, I, Ann. of Math. 71 (1960), 111-152.

[5] K. Kodaira. On stability of compact submanifolds of complex manifolds. Amer. J. Math. 85 (1963), 79-94.

[6] N.-H. Lee, K. Oguiso. Connecting certain rigid birational non-homeomorphic Calabi–Yau threefolds via Hilbert scheme, Communications in analysis and geometry Volume 17, Number 2, 283-303, (2009).

[7] H.-Y. Lin. Algebraic approximations of compact Kähler manifolds of algebraic codimension 1. Duke Math. J. 169 (2020), no. 14, 2767-2826.

[8] J. Steenbrink. Limits of Hodge structures. Invent. Math. 31 (1975/76), no. 3, 229-257.

[9] C. Voisin. On the homotopy types of compact Kähler and complex projective manifolds, Inventiones Math. Volume 157, Number 2, (2004), 329 - 343

[10] C. Voisin. On the homotopy type of Kähler manifolds and the birational Kodaira problem, J. Differential Geom. 72 (2006), no. 1, 43–71.

[11] C. Voisin. Hodge structures on cohomology algebras and geometry, Math. Ann. 341 (2008), no. 1, 39–69.

CNRS, Institut de Mathématiques de Jussieu-Paris rive gauche
claire.voisin@imj-prg.fr