HAL open science

Footnotes to papers of O'Grady and Markman

Claire Voisin

To cite this version:

Claire Voisin. Footnotes to papers of O'Grady and Markman. Mathematische Zeitschrift, 2021, 300 (4), pp.3405 - 3416. 10.1007/s00209-021-02883-8 . hal-03403477

HAL Id: hal-03403477

https://hal.science/hal-03403477v1

Submitted on 26 Oct 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Footnotes to papers of O'Grady and Markman

Claire Voisin*

Pour Olivier, avec amitié

Abstract

In this paper, we first generalize to any hyper-Kähler manifold X with $b_3(X) \neq 0$ results proved by O'Grady for hyper-Kähler manifolds of generalized Kummer type. In the second part, we restrict to hyper-Kähler manifolds of generalized Kummer type and prove, using results of Markman, that their Kuga-Satake correspondence is algebraic.

0 Introduction

This paper provides complements to the recent papers [15] by O'Grady and [12] by Markman. Hyper-Kähler manifolds X of generalized Kummer type are obtained by deforming the generalized Kummer varieties $K_n(A)$ constructed by Beauville [1] starting from an abelian surface A. The manifold $K_n(A)$ is defined as the subset of the punctual Hilbert scheme $A^{[n+1]}$ consisting of 0-dimensional subschemes with trivial Albanese class. For $n \geq 2$, one has $b_2(X) = 7$, $b_3(X) = 8$. Both papers are concerned with the intermediate Jacobians $J^3(X)$ for X as above. Recall that $J^3(X)$ is the complex torus built from the Hodge structure on $H^3(X, \mathbb{Z})$, which in this case is of level 1 since $H^{3,0}(X) = 0$, and is thus an abelian variety when X is projective. As $b_3(X) = 8$, $J^3(X)$ is an abelian fourfold. O'Grady proves the following results.

Theorem 0.1. *(O'Grady [15]) (1) $J^3(X)$ is a Weil abelian fourfold.*

(2) For a very general projective deformation of X, the Kuga-Satake abelian variety $\mathrm{KS}(X)$ of $(H^2(X, \mathbb{Q})_{\mathrm{tr}}, (\ ,\ ))$ is isogenous to a power of $J^3(X)$.

Let us explain both statements. A Weil abelian fourfold is an abelian fourfold that admits an endomorphism $\phi : A \to A$ satisfying a quadratic equation $\phi^2 = -d\mathrm{Id}$, with $d > 0$, with the following extra condition: consider the action $\phi_{\mathbb{C}}$ of ϕ on $H^1(X, \mathbb{C})$ by pullback. Then $\phi_{\mathbb{C}}$ preserves $H^{1,0}(A)$ and $H^{0,1}(A)$ since it is a morphism of Hodge structures and thus it has eigenvalues either $i\sqrt{d}$ or $-i\sqrt{d}$ on these 4-dimensional spaces. The Weil condition is that $\phi_{\mathbb{C}}$ has both eigenvalues $i\sqrt{d}$ and $-i\sqrt{d}$ with multiplicity 2 on $H^{1,0}(A)$ (hence also on $H^{0,1}(A)$). It guarantees that A has a 2-dimensional space of Weil Hodge classes of degree 4. More precisely, denoting K the number field $\mathbb{Q}[\sqrt{-d}]$, $H^1(A, \mathbb{Q})$ is a 4-dimensional K-vector space and the condition above guarantees that the 2-dimensional subspace

$$\bigwedge_K^4 H^1(A, \mathbb{Q}) \subset \bigwedge^4 H^1(A, \mathbb{Q})$$

consists of classes of Hodge type $(2, 2)$, hence of Hodge classes.

Concerning the point (2), let us define a Hodge structure of hyper-Kähler type as the data of a weight 2 (effective, rational or integral) Hodge structure $(H^2, F^i H^2_{\mathbb{C}})$ with $h^{2,0} = 1$, equipped with a nondegenerate quadratic form satisfying the first Hodge-Riemann relations, namely

$$(H^{2,0}, F^1 H^2_{\mathbb{C}}) = 0$$

*The author is supported by the ERC Synergy Grant HyperK (Grant agreement No. 854361).

and the condition that the restriction of $(\ ,\ )$ to the real 2-plane $(H^{2,0} \oplus H^{0,2}) \cap H^2_{\mathbb{R}}$ is positive definite. This is the structure one gets on the degree 2 cohomology of a hyper-Kähler manifold, the intersection pairing being given by the Beauville-Bogomolov quadratic form. We will say that we have a polarized Hodge structure of hyper-Kähler type if furthermore $(\ ,\ )$ is negative definite on the space $H^{1,1}_{\mathbb{R}} := H^{1,1} \cap H^2_{\mathbb{R}}$. We get such a structure by considering the transcendental degree 2 cohomology of a projective hyper-Kähler manifold. The Kuga-Satake variety $\mathrm{KS}(H^2, (\ ,\ ))$ associated to a Hodge structure $(H^2, (\ ,\ ))$ of hyper-Kähler type is a complex torus, or weight 1 Hodge structure (which is defined up to isogeny if we work with rational Hodge structures) built by a formal process (see [10], [5] and Section 2). If the considered Hodge structure is a polarized Hodge structure of hyper-Kähler type, $\mathrm{KS}(H^2, (\ ,\ ))$ is an abelian variety. In the case of a very general polarized weight 2 Hodge structure of generalized Kummer type, we have $b_{2,tr} = 6$ and the corresponding Kuga-Satake variety is isogenous to a power of a Weil abelian fourfold. The point (1) thus follows from (2) and (2) itself is a consequence of a certain universality property of the Kuga-Satake construction proved in [3], (and [7] in a slightly different setting, see Section 2), and of the following result.

Theorem 0.2. *(O'Grady [15]) Let X be a hyper-Kähler $2n$-fold of generalized Kummer deformation type with $n \geq 2$. Then the composite map map*

$$\bigwedge^2 H^3(X, \mathbb{Q}) \to H^6(X, \mathbb{Q}) \stackrel{Q_X^{n-2}}{\to} H^{4n-2}(X, \mathbb{Q})$$

is surjective.

Here $Q_X \in H^4(X, \mathbb{Q})$ is a cohomology class which is constructed using the Beauville-Bogomolov form (see Section 1.1). Our first result is the following generalization of Theorem 0.2.

Theorem 0.3. *Let X be a hyper-Kähler $2n$-fold such that $b_3(X) \neq 0$. Then*

(1) The composite map

$$\bigwedge^2 H^3(X, \mathbb{Q}) \to H^6(X, \mathbb{Q}) \stackrel{Q_X^{n-2}}{\to} H^{4n-2}(X, \mathbb{Q})$$

is surjective.

(2) Assuming X is projective, the intermediate Jacobian $J^3(X)$ contains a simple component of the Kuga-Satake abelian variety of $H^2(X, \mathbb{Q})_{\mathrm{tr}}$.

(3) One has $b_3(X) \geq 2^k$, where $k = \frac{b_2(X)-2}{2}$ if $b_2(X)$ is even, $k = \frac{b_2(X)-1}{2}$ if $b_2(X)$ is odd.

We will also prove similar results, and in particular the bound

$$b_{2n-1}(X) \geq 2^k, \text{ if } H^{2n-1}(X, \mathbb{Q}) \neq 0,\ H^{2n-3}(X, \mathbb{Q}) = 0. \tag{1}$$

where k is as in (3). In particular, we get the following corollary in dimension 6.

Corollary 0.4. *Let X be a hyper-Kähler 6-fold such that $b_{\mathrm{odd}}(X) \neq 0$. Then $b_{\mathrm{odd}}(X) \geq 2^k$, where $k = \frac{b_2(X)-2}{2}$ if $b_2(X)$ is even, $k = \frac{b_2(X)-1}{2}$ if $b_2(X)$ is odd.*

The Betti numbers of hyper-Kähler manifolds have been studied in [17], [8] which establishes very precise bounds in dimension 4 and in [18] which claims similar bounds in dimension 6 (but the proof seems to be incomplete). The paper [9] gives very precise conjectural bounds (for example bounds on b_2 depending only on the dimension), depending on a conjecture on the Looijenga-Lunts representation [11], [19]. The subject remains however wide open.

A key point in both cases is the fact that the weight 3 or weight $2n-1$ Hodge structure one considers is of Hodge level 1, that is, they satisfy the property $h^{p,q} = 0$ for $|p-q| > 1$.

As we already mentioned, the points (2) and (3) follow, using this observation, from the point (1), and from a universality property for the Kuga-Satake weight 1 Hodge structure, proved by Charles in [3], even in the unpolarized case.

The second part of this paper provides a complement to Markman's paper [12]. In this paper, Markman proves the Hodge conjecture for the Weil Hodge classes on the Weil abelian fourfolds appearing in Theorem 2.2. He also proves that the Abel-Jacobi map

$$\Phi_X : \mathrm{CH}^2(X)_{\mathrm{alg}} \to J^3(X),$$

defined on the group of codipmension 2 cycles on X algebraically equivalent to 0, is surjective for a projective hyper-Kähler manifold X of generalized Kummer deformation type. This statement was expected as a consequence of the generalized Hodge conjecture because $H^{3,0}(X) = 0$ (see [22]).

Our second result is the following

Theorem 0.5. *For X a projective hyper-Kähler manifold of generalized Kummer deformation type with $n \geq 2$, the Kuga-Satake correspondence between X and its Kuga-Satake variety* $\mathrm{KS}(X)$ *is algebraic.*

To explain this statement, the Kuga-Satake construction in the polarized case produces an abelian variety $\mathrm{KS}(X)$ associated to the polarized Hodge structure $(H^2(X,\mathbb{Q})_{\mathrm{tr}}, (\ ,\ ))$ which has the property that $H^2(X,\mathbb{Q})_{\mathrm{tr}}$ is a Hodge substructure of $H^2(\mathrm{KS}(X),\mathbb{Q})$. The Hodge conjecture predicts the existence of a correspondence between X and $\mathrm{KS}(X)$, that is an algebraic cycle Γ of codimension $2n$ with $\mathbb{Q}$-coefficients in $X \times \mathrm{KS}(X)$, such that Γ_* induces the given embedding $H^2(X,\mathbb{Q})_{\mathrm{tr}} \hookrightarrow H^2(\mathrm{KS}(X),\mathbb{Q})$. The meaning of the "algebraicity of the Kuga-Satake correspondence" is the existence of such cycle Γ (see [6] for a general discussion).

The algebraicity of the Kuga-Satake correspondence is known for projective $K3$ surfaces with Picard number at least 17 [14]. It is also known by work of Paranjape [16] for $K3$ surfaces with Picard number 16 obtained as desingularizations of double covers of $\mathbb{P}^2$ ramified along 6 lines. Some hyper-Kähler examples involving cubic fourfolds have been exhibited in [21].

1 Applications of the hard Lefschetz theorem

1.1 Degree 3 cohomology: complement to a paper of O'Grady

Let X be a hyper-Kähler manifold of dimension $2n$ with $n \geq 2$. The Beauville-Bogomolov quadratic form q_X is a nondegenerate quadratic form on $H^2(X,\mathbb{Q})$, whose inverse gives an element of $\mathrm{Sym}^2 H^2(X,\mathbb{Q})$. By Verbitsky [2], the later space imbeds by cup-product in $H^4(X,\mathbb{Q})$, hence we get a class

$$Q_X \in H^4(X,\mathbb{Q}). \tag{2}$$

The O'Grady map $\phi : \bigwedge^2 H^3(X,\mathbb{Q}) \to H^{4n-2}(X,\mathbb{Q})$ is defined by

$$\phi(\alpha \wedge \beta) = Q_X^{n-2} \cup \alpha \cup \beta. \tag{3}$$

The following result was first proved by O'Grady [15] in the case of a hyper-Kähler manifold of generalized Kummer deformation type.

Theorem 1.1. *Let X be a hyper-Kähler manifold of dimension $2n$. Assume $H^3(X,\mathbb{Q}) \neq 0$. Then the O'Grady map map $\phi : \bigwedge^2 H^3(X,\mathbb{Q}) \to H^{4n-2}(X,\mathbb{Q})$ is surjective.*

Proof. We can choose the complex structure on X to be general, so that $\rho(X) = 0$, and this implies that the Hodge structure on $H^2(X,\mathbb{Q})$ (or equivalently $H^{4n-2}(X,\mathbb{Q})$ as they

are isomorphic by combining Poincaré duality and the self-duality given by the Beauville-Bogomolov form) is simple. As the morphism ϕ is a morphism of Hodge structures, its image is a Hodge substructure of $H^{4n-2}(X,\mathbb{Q})$, hence either ϕ is surjective, or it is 0. Theorem 1.1 thus follows fropm the next proposition. $\square$

Proposition 1.2. *The map ϕ is not identically* 0.

Proof. Let $\omega \in H^2(X,\mathbb{R})$ be a Kähler class. Then we know that the ω-Lefschetz intersection pairing $\langle\, ,\, \rangle_\omega$ on $H^3(X,\mathbb{R})$, defined by

$$\langle \alpha, \beta\rangle_\omega := \int_X \omega^{2n-3} \cup \alpha \cup \beta$$

is nondegenerate. This implies that the cup-product map

$$\psi : \bigwedge^2 H^3(X,\mathbb{Q}) \to H^6(X,\mathbb{Q})$$

has the property that $\operatorname{Im}\psi$ pairs nontrivially with the image of the map

$$\operatorname{Sym}^{2n-3} H^2(X,\mathbb{Q}) \to H^{4n-6}(X,\mathbb{Q})$$

given by cup-product. Note that the Hodge structure on $H^3(X,\mathbb{Q})$ has Hodge level 1, so that the Hodge structure on the image of $\operatorname{Im}\psi$ in $\operatorname{Sym}^{2n-3}H^2(X,\mathbb{Q})^*$ is a Hodge structure of level at most 2. We now argue as in [20]. We choose X very general so that the Mumford-Tate group of the Hodge structure on $H^2(X,\mathbb{Q})$ is the orthogonal group $O(q_X)$. Any Hodge substructure of $\operatorname{Sym}^{2n-3}H^2(X,\mathbb{Q})^* \cong \operatorname{Sym}^{2n-3}H^2(X,\mathbb{Q})$ is thus a direct sum of $O(q_X)$-subrepresentations of $\operatorname{Sym}^{2n-3}H^2(X,\mathbb{Q})$. Elementary representation theory of $O(q_X)$ then shows that the irreducible $O(q_X)$-subrepresentations of $\operatorname{Sym}^{2n-3}H^2(X,\mathbb{Q})$ are the subspaces

$$Q_X^l \operatorname{Sym}^{2n-3-2l} H^2(X,\mathbb{Q})^0,$$

where we see here Q_X as an element of $\operatorname{Sym}^2 H^2(X,\mathbb{Q})$, and

$$\operatorname{Sym}^k H^2(X,\mathbb{Q})^0 \subset \operatorname{Sym}^k H^2(X,\mathbb{Q})$$

can be defined after passing to $\mathbb{C}$-coefficients as the subspace generated by α^k with $q_X(\alpha) = 0$ (this definition is correct with $\mathbb{Q}$-coefficients only if the quadratic form q_X has a zero).

The irreducible Hodge structure on $Q_X^l \operatorname{Sym}^{2n-3-2l} H^2(X,\mathbb{Q})^0$ has Hodge level > 2 when $2n-3-2l > 1$ since it contains the class $Q_X^l \sigma_X^{2n-3-2l}$ which is of type $(4n-6-4l, 2l)$, where σ_X generates $H^{2,0}(X)$. It follows that $\operatorname{Im}\psi$ can pair nontrivially only with $Q_X^{n-2} H^2(X,\mathbb{Q})$, hence the map $Q_X^{n-2}\psi$, which is the O'Grady map, is nonzero, which concludes the proof. $\square$

1.2 Cohomology of degree $2n-1$

For other odd degree $2k-1 \leq 2n-1$, one may wonder what the hard Lefschetz theorem gives. The proof of Proposition 1.2 will give as well:

Proposition 1.3. *The composition*

$$\psi' : \bigwedge^2 H^{2k-1}(X,\mathbb{Q}) \to H^{4k-2}(X,\mathbb{Q}) \to \operatorname{Sym}^{2n-2k+1} H^2(X,\mathbb{Q})^*,$$

where the first map is the cup-product and the second one is Poincaré dual to the cup-product map $\operatorname{Sym}^{2n-2k+1}H^2(X,\mathbb{Q}) \to H^{4n-4k+2}(X,\mathbb{Q})$*, is nontrivial (and even, nondegenerate).*

However, we do not know a priori the Hodge level of $H^{2k-1}(X,\mathbb{Q})$ so we do not know to which irreducible component of the $O(q)$-representation of $\operatorname{Sym}^{2n-2k+1}H^2(X,\mathbb{Q})^*$ the image $\operatorname{Im}\psi'$ can map nontrivially. In the case of degree $2k-1 = 2n-1$, we have only one piece, namely $H^2(X,\mathbb{Q})^*$, hence we get:

Corollary 1.4. *If X is a hyper-Kähler manifold of dimension $2n$ with $H^{2n-1}(X,\mathbb{Q}) \neq 0$, the cup-product map*

$$\bigwedge^2 H^{2n-1}(X,\mathbb{Q}) \to H^2(X,\mathbb{Q})^*.$$

is surjective.

Proof. Indeed, the map ψ is nonzero and a morphism of Hodge structures, the right hand side being a simple Hodge structure for a very general complex structure on X. □

We will also use in next section the following observation.

Lemma 1.5. *Let X be a hyper-Kähler $2n$-fold. Then the Hodge structure on the quotient*

$$H^{2n-1}(X,\mathbb{Q})^0 := H^{2n-1}(X,\mathbb{Q})/H^2 \cup H^{2n-3}(X,\mathbb{Q}) \tag{4}$$

has Hodge level 1. In particular, if $H^{2n-3}(X,\mathbb{Q}) = 0$, the Hodge structure on $H^{2n-1}(X,\mathbb{Q})$ has Hodge level 1.

Proof. The statement is that the (p,q)-components of $H^{2n-1}(X,\mathbb{C})^0$ vanish unless $(p,q) = (n,n-1)$ or $(p,q) = (n-1,n)$. We thus have to show that any class in $H^{p,q}(X)$ with $p > n$ or $q > n$ belongs to $H^2(X,\mathbb{C}) \cup H^{2n-3}(X,\mathbb{C})$. This follows from the fact that, as σ_X is a symplectic holomorphic form, the cup-product map by σ_X^l induces a vector bundle isomorphism

$$\sigma_X^l \wedge : \Omega_X^{n-l} \to \Omega_X^{n+l},$$

hence an isomorphism $\sigma_X^l \cup : H^{n-l,q}(X) \cong H^{n+l,q}(X)$. This proves the statement for $p > n$ and the other statement follows by Hodge symmetry. □

2 Universality of the Kuga-Satake correspondence and applications

We start with an effective rational Hodge structure $(H^2, F^i H^2_{\mathbb{C}})$ of weight 2 with $h^{2,0} = 1$ equipped with a symmetric nondegenerate intersection pairing $(\ ,\ )$ satisfying the conditions

$$(\sigma, F^1 H^2) = 0,\ (\sigma, \overline{\sigma}) > 0,$$

where σ generates $H^{2,0}$. Note that $(\ ,\ )$ satisfies only part of the Hodge-Riemann relations so that the Hodge structure is not in general polarized. We will call such data a Hodge structure of hyper-Kähler type (although it also encodes the quadratic form) because this is the structure that we have on the degree 2 cohomology $H^2(X,\mathbb{Q})$ of a hyper-Kähler manifold equipped with the Beauville-Bogomolov form q_X. The Kuga-Satake correspondence first constructed in [10] associates to a Hodge structure H^2 of hyper-Kähler type a weight 1 Hodge structure H^1_{KS}, which has the property that there is an injective morphism of Hodge structures

$$H^2 \to \operatorname{End} H^1_{\mathrm{KS}},$$

of bidegree $(-1,-1)$. Note that the Hodge structure on both sides has Hodge level 2. When the Hodge structure of hyper-Kähler type on H^2 is polarized by $(\ ,\ )$, which means that our data have the extra property that the pairing $(\ ,\ )$ restricted to $H^{1,1}_{\mathbb{R}}$ is negative definite, the Hodge structure on H^1_{KS} is polarized, hence is the Hodge structure on the H^1 of an abelian variety.

The construction of H^1_{KS} can be summarized as follows: The $\mathbb{Q}$-vector space H^1_{KS} is defined as $\operatorname{Cliff}(H^2, (\ ,\ ))$, that is, it is the quotient of the tensor algebra $\otimes H^2$ by the ideal generated by the relations $x^2 = (x,x)1$, $x \in H^2$. The weight 1 Hodge structure on H^1_{KS} is given by a complex structure on the real vector space $H^1_{\mathrm{KS},\mathbb{R}}$. It is constructed as follows.

Consider the subspace $(H^{2,0} \oplus H^{0,2})_{\mathbb{R}} \subset H^2_{\mathbb{R}}$. It is of dimension 2, naturally oriented, and the restriction of the form $(\ ,\ )$ to this real plane is positive definite. Choose a positively oriented orthonormal basis $(e_1,\, e_2)$ of $(H^{2,0} \oplus H^{0,2})_{\mathbb{R}}$. Then $e := e_1 e_2 \in \mathrm{Cliff}\,(H^2_{\mathbb{R}}, (\ ,\ ))$ does not depend on the choice of basis and satisfies $e^2 = -1$. Left Clifford multiplication by e thus defines the desired complex structure on $\mathrm{Cliff}\,(H^2_{\mathbb{R}}, (\ ,\ )) = H^1_{\mathrm{KS},\mathbb{R}}$.

Clifford multiplication on the left induces a morphism

$$H^2 \to \mathrm{End}\, H^1_{\mathrm{KS}}$$

which is a morphism of Hodge structures of bidegree $(-1,-1)$. This is equivalent to saying that Clifford multiplication on the left by $H^{1,1}$ preserves the Hodge decomposition of $H^1_{\mathrm{KS},\mathbb{C}}$ and that Clifford multiplication on the left by $H^{2,0}$ shifts the Hodge decomposition of $H^1_{\mathrm{KS},\mathbb{C}}$ by $(-1,-1)$. The first fact follows because $H^{1,1}$ is orthogonal to $H^{2,0}$ for $(\ ,\ )$, so multiplication by elements of $H^{1,1}$ anticommutes with Clifford multiplication by elements of $H^{2,0}$ or $H^{0,2}$, hence commutes with Clifford multiplication by $e_1 e_2$. The second fact is an easy computation.

The weight 1 Hodge structure H^1_{KS} is not simple. In fact it has a big algebra of endomorphisms given by right Clifford multiplication on the Clifford algebra. These endomorphisms obviously commute with left Clifford multiplication by e, hence provide automorphisms of Hodge structure of H^1_{KS}. To start with, we can restrict the construction to the even Clifford algebra $C^+(H^2, (\ ,\ ))$ generated by the tensor products $v_1 \otimes \ldots \otimes v_k$, $v_i \in H^2$, with k even, which clearly provides a Hodge substructure of $H^1_{\mathrm{KS},\mathbb{Q}}$ since multiplication on the left by e preserves $C^+(H^2_{\mathbb{R}}, (\ ,\ ))$. We can do similarly with the odd part $C^-(H^2, (\ ,\ ))$ of the Clifford algebra, which provides another Hodge substructure. Multiplication on the right by a given element $v_0 \in H^2$ with $(v, v) \neq 0$ provides an isomorphism

$$C^+(H^2, (\ ,\ )) \cong C^-(H^2, (\ ,\ )),$$

so that, denoting $H^1_{\mathrm{KS}+}$, $H^1_{\mathrm{KS}-}$ the weight 1 Hodge structures so obtained, we have an isomorphism

$$H^1_{\mathrm{KS}+} \to H^1_{\mathrm{KS}-}$$

given by right Clifford multiplication by v_0, and we get an injective (but not canonical) morphism of Hodge structures

$$H^2 \to \mathrm{End}\, H^1_{\mathrm{KS}+}$$

given by

$$v \mapsto (\alpha \mapsto v \alpha v_0).$$

When the Mumford-Tate group of the Hodge structure on H^2 is the orthogonal group $O((\ ,\ ))$, using representation theory of the orthogonal group, one can describe up to isogeny the complex tori appearing as subquotient of the Kuga-Satake complex torus (see [3], [6]). Note that in the geometric case, it follows from the local surjectivity of the period map that the Mumford-Tate group is the orthogonal group $O((\ ,\ ))$. When the dimension h of H^2 is odd, the Kuga-Satake complex torus is a power of a simple torus of dimension $2^{\frac{h-3}{2}}$ or $2^{\frac{h-1}{2}}$. When h is even, the situation is much more delicate, as the classification of the subquotients depends on the discriminant of the quadratic form $(\ ,\ )$. In this case, the Kuga-Satake complex torus is a sum of powers of one or two simple complex tori which can be of dimension $2^{h/2}$, $2^{h/2-1}$ or $2^{h/2-2}$ (see [3]). The numbers above are obtained starting from the fact that the even Clifford algebra $C^+(H^2)$ has dimension 2^{h-1} and that the action on it by right multiplication by elements of $C^+(H^2)$ (which are morphisms of Hodge structures since they commute with the left multiplication by e) splits it as a direct sum of weight 1 Hodge structures. We now consider the polarized case. The Kuga-Satake Hodge structure $H^1_{\mathrm{KS}+}$ is then polarized and thus is the weight 1 Hodge structure on the degree 1 rational cohomology of an abelian variety, that we will denote $\mathrm{KS}(H^2, (\ ,\ ))$, and is defined up to isogeny. The two dual complex tori appearing above are then isomorphic. In the case where

h is even, the simple abelian variety one gets has a quadratic endomorphism which makes it a Weil abelian variety. In the geometric case, where we start from the degree 2 cohomology of a hyper-Kähler manifold X, equipped with the Beauville-Bogomolov form q_X, we assume X is polarized by an ample class $l \in \mathrm{NS}(X)$ and put

$$(H^2, (\ ,\ )) = (H^2(X, \mathbb{Q})^{\perp_{q_X} l}, q_X)$$

$$\mathrm{KS}(X) = \mathrm{KS}(H^2, (\ ,\ )).$$

In fact, we can also define $(H^2(X, \mathbb{Q})^{\perp_{q_X} l}, q_X)$ without using the Beauville-Bogomolov form, since $H^2(X, \mathbb{Q})^{\perp_{q_X} l}$ is the group of l-primitive classes, and, up to a rational coefficient, the restricted form q_X is proportional to the Lefschetz intersection pairing defined by l.

The following universality property is proved in [3] (see also [7] for a slightly different statement, proved only in the polarized case).

Theorem 2.1. *Let $(H^2, (\ ,\ ))$ be a Hodge structure of hyper-Kähler type. Assume the Mumford-Tate group of H^2 is $SO(H^2, (\ ,\ ))$. Let H be a simple effective weight 1 Hodge structure, such that for nonnegative integers a, b of the same parity, there exists an injective morphism of Hodge structures of bidegree $(\frac{a-b}{2} - 1, \frac{a-b}{2} - 1)$*

$$H^2 \hookrightarrow H^{\otimes a} \otimes (H^\vee)^{\otimes b}.$$

Then H is a subquotient of the Kuga-Satake Hodge structure $H^1_{\mathrm{KS}+}$. In particular

$$\dim H \geq 2^k, \text{ where } k = \frac{h-1}{2} \text{ if } h \text{ is odd, } k = \frac{h-2}{2} \text{ if } h \text{ is even.}$$

If h is divisible by 4 and the signature of $(\ ,\ )$ is $(3, h-3)$, the last inequality can be improved to $\dim H \geq 2^{\frac{h}{2}}$.

A first application of this universality property (or rather a variant of it) was given in [7] where we proved the Matsushita conjecture on the moduli map for Lagrangian fibration of projective hyper-Kähler manifolds, at least in the case where $b_2(X) \geq 5$, assuming the Mumford-Tate group is maximal. A second application (also in the projective case, with $a = b = 1$) was given by O'Grady in [15]. Let X be a projective hyper-Kähler manifold of generalized Kummer deformation type and dimension ≥ 4. One has $b_2(X) = 7$, hence for a very general projective such hyper-Kähler manifold, $b_2(X)_{tr} = 6$, so that $\mathrm{KS}(X)$ is isogenous to a sum of copies of a simple abelian fourfold of Weil type. Using Theorem 1.1 (that he had proved by an explicit computation in that case), the fact that $b_3(X) = 8$, and the universality property of Theorem 2.1, O'Grady proved the following result.

Theorem 2.2. *The intermediate Jacobian $J^3(X)$ of a projective hyper-Kähler manifold of generalized Kummer deformation type with $\rho(X) = 1$ is a Weil abelian fourfold. The Kuga-Satake variety of $(H^2(X, \mathbb{Q})_{tr}, q_X)$ is isogenous to a sum of two copies of $J^3(X)$.*

2.1 Applications to Betti numbers

In this section, we are going to apply the previous results to get inequalities involving the Betti numbers of hyper-Kähler manifolds.

Theorem 2.3. *Let X be a hyper-Kähler manifold. Assume that $b_3(X) \neq 0$. Then*

$$b_3(X) \geq 2^k, \tag{5}$$

where $k = \frac{b_2(X)-1}{2}$ if $b_2(X)$ is odd, $k = \frac{b_2(X)-2}{2}$ if $b_2(X)$ is even.

If $b_2(X)$ is divisible by 4, the last inequality can be improved to $b_3(X) \geq 2^{b_2(X)/2}$.

Proof. By Theorem 1.1, we have a surjective morphism of Hodge structures

$$\bigwedge^2 H^3(X,\mathbb{Q}) \to H^{4n-2}(X,\mathbb{Q}) \cong H^2(X,\mathbb{Q})^*,$$

which gives as well an injective morphism of Hodge structures

$$H^2(X,\mathbb{Q}) \hookrightarrow \bigwedge^2 H^3(X,\mathbb{Q})^* \hookrightarrow H^3(X,\mathbb{Q})^* \otimes H^3(X,\mathbb{Q})^*.$$

Choosing the complex structure on X very general so that the Mumford-Tate group of the Hodge structure on $H^2(X,\mathbb{Q})$ is the orthogonal group of $(\ ,\ )$, we can thus apply Theorem 2.1, which gives (5). □

We now turn to the Betti number b_{2n-1}. We prove the following

Theorem 2.4. *Let X be a hyper-Kähler manifold such that $H^{2n-3}(X) = 0$ and $H^{2n-1}(X) \neq 0$. Then*

$$b_{2n-1}(X) \geq 2^k, \tag{6}$$

where $k = \frac{b_2(X)-1}{2}$ if $b_2(X)$ is odd, $k = \frac{b_2(X)-2}{2}$ if $b_2(X)$ is even.

If $b_2(X)$ is divisible by 4, the last inequality can be improved to $b_{2n-1}(X) \geq 2^{b_2(X)/2}$.

Proof. By Corollary 1.4, the cup-product map

$$\bigwedge^2 H^{2n-1}(X,\mathbb{Q}) \to H^{4n-2}(X,\mathbb{Q})$$

is surjective. As we assumed $H^{2n-3}(X,\mathbb{Q}) = 0$, the Hodge structure on $H^{2n-1}(X,\mathbb{Q})$ has Hodge level 1 by Lemma 1.5. We are thus exactly as in the situation of Theorem 2.3 and the same arguments give inequality (6). □

In the case of a hyper-Kähler manifold X of dimension $2n = 6$, we get

Corollary 2.5. *Let X be a hyper-Kähler 6-fold such that $H^{\mathrm{odd}}(X) \neq 0$. Then*

$$b_{odd}(X) \geq 2^k, \tag{7}$$

where $k = \frac{b_2-2}{2}$ if b_2 is even, $k = \frac{b_2-1}{2}$ if b_2 is odd.

Proof. We observe that, in dimension 6, if $H^{\mathrm{odd}}(X) \neq 0$, then either $H^3(X,\mathbb{Q}) \neq 0$ or, $H^3(X,\mathbb{Q}) = 0$ and $H^5(X,\mathbb{Q}) \neq 0$. In the first case we apply Theorem 2.3 and in the second case we apply Theorem 2.4. □

3 Algebraicity of the Kuga-Satake correspondence

Let X be a projective complex manifold. Assume that $h^{2,0}(X) = 1$, so that the Hodge structure on $H^2(X,\mathbb{Q})$ is of hyper-Kähler type (choosing a polarization l on X, the Lefschetz intersection pairing $(\ ,\ )_{\mathrm{lef}}$ defined by

$$(\alpha,\beta)_{\mathrm{lef}} = \int_X l^{n-2}\alpha \cup \beta$$

gives the desired intersection form). In the case of a hyper-Kähler manifold of dimension $2n$, the Beauville-Bogomolov intersection pairing on $H^2(X,\mathbb{Q})$ is independent of the choice of a polarization, but when we restrict it to the l-primitive cohomology $H^2(X,\mathbb{Q})_{\mathrm{prim}} = H^2(X,\mathbb{Q})^{\perp l^{2n-1}}$, the two pairings coincide up to a scalar coefficient. Let $\mathrm{KS}(X)$ be the Kuga-Satake abelian variety (defined up to isogeny) associated to the polarized Hodge structure

$(H^2(X,\mathbb{Q})_{\mathrm{prim}}, (\ ,\ )_{\mathrm{lef}})$. Using an adequate polarization on $\mathrm{KS}(X)$, the injective morphism of Hodge structures

$$H^2(X,\mathbb{Q})_{\mathrm{prim}} \hookrightarrow \mathrm{End}\,(H^1_{\mathrm{KS}+}(H^2(X,\mathbb{Q})_{\mathrm{prim}}, (\ ,\ ))) = \mathrm{End}\, H^1(\mathrm{KS}(X),\mathbb{Q})$$

gives an injective morphism of Hodge structures

$$H^2(X,\mathbb{Q})_{\mathrm{prim}} \hookrightarrow H^1(\mathrm{KS}(X),\mathbb{Q})^{\otimes 2}. \tag{8}$$

whose image is contained in $\bigwedge^2 H^1(\mathrm{KS}(X),\mathbb{Q}) = H^2(\mathrm{KS}(X),\mathbb{Q})$.

A morphism of Hodge structures $\beta : H^2(X,\mathbb{Q})_{\mathrm{prim}} \to H^2(\mathrm{KS}(X),\mathbb{Q})$ as in (8) provides a Hodge class (see [22])

$$\alpha \in H^{4n-2}(X,\mathbb{Q}) \otimes H^2(\mathrm{KS}(X),\mathbb{Q}) \subset H^{4n}(X \times \mathrm{KS}(X),\mathbb{Q}). \tag{9}$$

The Hodge conjecture thus predicts that there is a cycle $\Gamma \in \mathrm{CH}^{2n}(X \times \mathrm{KS}(X))_{\mathbb{Q}}$ such that $[\Gamma] = \alpha$, hence in particular

$$[\Gamma]_* = \beta : H^2(X,\mathbb{Q})_{\mathrm{prim}} \to H^2(\mathrm{KS}(X),\mathbb{Q}).$$

When this holds, we will say that the Kuga-Satake correspondence is algebraic.

In the case where X is an abelian surface, so $b_2(X)_{tr} \leq 5$, or more generally any projective $K3$ surface with $\rho \geq 17$, the algebraicity of the class α above is proved by Morrison [14]. In that case, the Kuga-Satake variety is isogenous to a sum of copies of the abelian surface itself.

In the next case, where $b_{2,tr} = 6$, we already mentioned that the Kuga-Satake variety is isogenous to a sum of copies of a 4-dimensional abelian variety which is of Weil type (assuming the maximality of the Mumford-Tate group). This case appears geometrically with $K3$ surfaces with Picard number 16 and the first family of such $K3$ surfaces for which the Kuga-Satake correspondence was known to be algebraic was found by Paranjape [16]. The Paranjape $K3$ surfaces are obtained by desingularizing double covers of $\mathbb{P}^2$ ramified along the union of six lines.

The geometric situation we consider is the same as in [15], [12]. X is a projective hyper-Kähler manifold of generalized Kummer type. In particular, we know by O'Grady theorem (Theorem 2.2) that $J^3(X)$ is isogenous to a component of the Kuga-Satake variety $\mathrm{KS}(X)$. We prove now the following result.

Theorem 3.1. *Let X be a projective hyper-Kähler manifold of generalized Kummer type. Then the Kuga-Satake correspondence of X is algebraic.*

This theorem should be actually considered as an addendum to Markman's paper [12]. The result will indeed follow from the following result (Theorem 3.2) of Markman. As we already mentioned, for X as above, the Hodge structure on $H^3(X,\mathbb{Q})$ is of Hodge level 1, that is, of type $(2,1)+(1,2)$. The generalized Hodge conjecture thus predicts that the degree 3 cohomology of X is supported on a (singular) divisor of X, and this is equivalent to the fact that the Griffiths Abel-Jacobi map

$$\Phi_X : \mathrm{CH}^2(X)_{\mathrm{alg}} \to J^3(X) \tag{10}$$

is surjective (see [22]).

Theorem 3.2. *(Markman [12]) For a projective hyper-Kähler manifold of Kummer deformation type, the Abel-Jacobi map (10) is surjective.*

Proof of Theorem 3.1. An equivalent version of Theorem 3.2 says that there exists a codimension 2 cycle $\mathcal{Z} \in \mathrm{CH}^2(J^3(X) \times X)_{\mathbb{Q}}$ such that the map $[\mathcal{Z}]_* : H_1(J^3(X),\mathbb{Q}) \to H^3(X,\mathbb{Q})$ is the natural identification $H_1(J^3(X),\mathbb{Q}) \cong H^3(X,\mathbb{Q})$. We recall here that $J^3(X)$ is

the complex torus $H^3(X,\mathbb{C})/(F^2H^3(X,\mathbb{C}) \oplus H^3(X,\mathbb{Z}))$ built from the Hodge structure on $H^3(X,\mathbb{Z})$ so that $H_1(J^3(X),\mathbb{Z}) = H^3(X,\mathbb{Z})$ canonically. Note that we can assume that the cohomology class $[\mathcal{Z}] \in H^4(J^3(X) \times X, \mathbb{Q})$ belongs to the Künneth component $H^1(J^3(X),\mathbb{Q}) \otimes H^3(X,\mathbb{Q})$. Indeed, using the action of the maps of multiplication by k on $J^3(X)$, the Künneth components of $[\mathcal{Z}]$ are all algebraic, and the Künneth components not in $H^1(J^3(X),\mathbb{Q}) \otimes H^3(X,\mathbb{Q})$ induce the zero map $H_1(J^3(X),\mathbb{Q}) \to H^3(X,\mathbb{Q})$.

Next, by another result of Markman [13], the class $Q_X \in H^4(X,\mathbb{Q})$ introduced in (2) is algebraic on hyper-Kähler manifolds of generalized Kummer type. It is thus the class of a cycle $\mathcal{Q}_X \in \mathrm{CH}^2(X)_{\mathbb{Q}}$. On $J^3(X) \times X$, we consider the following cycle

$$\Gamma := \mathcal{Z}^2 \cdot \mathrm{pr}_X^* \mathcal{Q}_X^{2n-2}, \tag{11}$$

where $\mathrm{pr}_X : J^3(X) \times X \to X$ denotes the second projection. We prove the following

Claim 3.3. *The map* $[\Gamma]_* : H_2(J^3(X),\mathbb{Q}) \to H^{4n-2}(X,\mathbb{Q})$ *identifies with the O'Grady map* $\phi : \bigwedge^2 H^3(X,\mathbb{Q}) \to H^6(X,\mathbb{Q}) \overset{Q_X}{\to} H^{4n-2}(X,\mathbb{Q})$ *of (3).*

Proof. Recall that we assumed that $[\mathcal{Z}] \in H^1(J^3(X),\mathbb{Q}) \otimes H^3(X,\mathbb{Q})$. Taking a basis e_i of $H^3(X,\mathbb{Q})$, which provides a basis f_i of $H_1(J^3(X),\mathbb{Q})$ and the dual basis f_i^* of $H^1(J^3(X),\mathbb{Q})$, we can thus write

$$[\mathcal{Z}] = \sum_i \mathrm{pr}_{J^3(X)}^* f_i^* \cup \mathrm{pr}_X^* e_i, \tag{12}$$

since $[\mathcal{Z}]_*(f_i) = e_i$. We now deduce from (12)

$$[\Gamma] = -\sum_{i,j} \mathrm{pr}_{J^3(X)}^*(f_i^* \cup f_j^*) \cup \mathrm{pr}_X^* e_i \cup \mathrm{pr}_X^* e_j \cup \mathrm{pr}_X^* Q_X,$$

which immediately implies the claim. □

The claim implies the theorem since we already identified the intermediate Jacobian with a component of the Kuga-Satake variety, in such a way that the transpose of the map (8) is the O'Grady map. Thus the map (8) and its transpose are induced by an algebraic cycle. □

References

[1] A. Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. J. Differential Geom. 18 (1983), no. 4, 755-782 (1984).

[2] F. Bogomolov. On the cohomology ring of a simple hyper-Kähler manifold (on the results of Verbitsky). Geom. Funct. Anal. 6 (1996), no. 4, 612-618.

[3] F. Charles. Two results on the Hodge structure of complex tori, arXiv:2106.11069.

[4] F. Charles, E. Markman. The standard conjectures for holomorphic symplectic varieties deformation equivalent to Hilbert schemes of K3 surfaces. Compos. Math. 149 (2013), no. 3, 481-494.

[5] P. Deligne. La conjecture de Weil pour les surfaces $K3$, Inventiones math. 15, 206-226 (1972).

[6] B. van Geemen. Kuga-Satake varieties and the Hodge conjecture, in *The Arithmetic and Geometry of Algebraic Cycles* Gordon B.B., Lewis J.D., Müller-Stach S., Saito S., Yui N. (eds), pp 51-82, NATO Science Series vol. 548 (2000).

[7] B. van Geemen, C. Voisin. On a conjecture of Matsushita, Int Math Res Notices (2016) Vol. 2016, 3111-3123

[8] D. Guan,On the Betti numbers of irreducible compact hyperkähler manifolds of complex dimension four, Math.Res. Lett.8(2001), no. 5-6, 663–669.

[9] Y-J. Kim, R. Laza. A conjectural bound on the second Betti number for hyper-Kaehler manifolds, Bull. Soc. Math. France 148 (2020), no. 3, 467-480.

[10] M. Kuga, 1. Satake. Abelian varieties associated to polarized K3 surfaces, Math. Ann. 169, 239-242 (1967).

[11] E. Looijenga, V. Lunts. A Lie algebra attached to a projective variety. Invent. Math. 129 (1997), no. 2, 361-412.

[12] E. Markman. The monodromy of generalized Kummer varieties and algebraic cycles on their intermediate Jacobians, arXiv:1805.11574.

[13] E. Markman. The Beauville-Bogomolov class as a characteristic class, J. Algebraic Geom. 29 (2020), no. 2, 199-245.

[14] D. Morrison. The Kuga-Satake variety of an abelian surface, J. Algebra 92, 454-476 (1985) .

[15] K. O'Grady. Compact tori associated to hyperkaehler manifolds of Kummer type, Int. Math. Res. Not. IMRN 2021, no. 16, 12356–12419.

[16] K. Paranjape. Abelian varieties associated to certain $K3$ surfaces, Compositio Math. 68, n0 1, 11-22 (1988).

[17] S. Salamon. On the cohomology of Kähler and hyper-Kähler manifolds. Topology 35 (1996), no. 1, 137-155.

[18] J. Sawon. A bound on the second Betti number of hyperkähler manifolds of complex dimension six, arXiv:1511.09105.

[19] M. Verbitsky. Transcendental Hodge algebra. Selecta Math. (N.S.) 23 (2017), no. 3, 2203-2218.

[20] C. Voisin. Remarks and questions on coisotropic subvarieties and 0-cycles of hyper-Kähler varieties, in *K3 Surfaces and Their Moduli*, Proceedings of the Schiermonnikoog conference 2014, C. Faber, G. Farkas, G. van der Geer, Editors, Progress in Math 315, Birkhäuser (2016), 365-399.

[21] C. Voisin. Remarks on zero-cycles of self-products of varieties, in *Moduli of vector bundles* (Proceedings du congrès Taniguchi sur les fibrés vectoriels), édité par Maruyama, Decker (1994) 265-285.

[22] C. Voisin. Lectures on the Hodge and Grothendieck-Hodge conjectures, Rend. Semin. Mat. Univ. Politec. Torino 69 (2011), no. 2, 149-198.

CNRS, Institut de Mathématiques de Jussieu-Paris rive gauche
claire.voisin@imj-prg.fr