

# Generalized continuum models confronted to cell-commensurate instabilities in structured media

Christelle Combescure

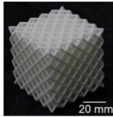
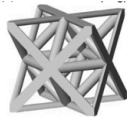
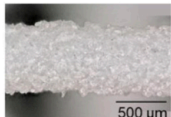
*Académie Militaire Saint-Cyr Coëtquidan  
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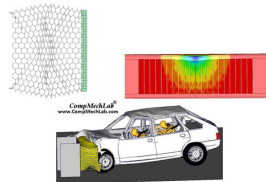
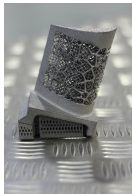
IUTAM Symposium 19–23 July 2021

1. Context
2. Research Question
3. Our Study
4. Conclusions & Work in progress

# Architected Materials

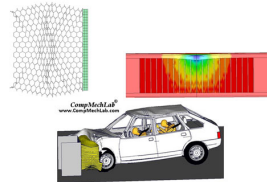
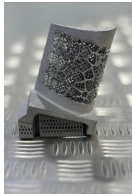
## Definition and applications

Macroscopic	Mesoscopic	Microscopic
L ~ 10 cm	l ~ 5 mm	$\lambda \sim 100 \mu\text{m}$
Structure	Cell	Material
		



# Architected Materials

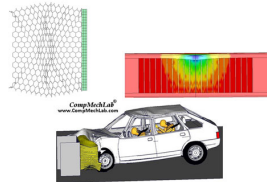
## Specificity



- large number of cells
- local or global buckling
- interaction between adjacent cells
- loading commensurate with cell size

# Architected Materials

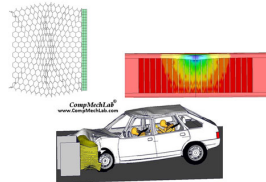
## Specificity



- large number of cells  $\Rightarrow$  equivalent continuum model
- local or global buckling
- interaction between adjacent cells
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# Architected Materials

## Specificity



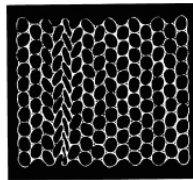
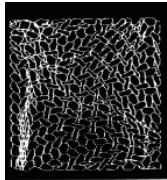
- large number of cells  $\Rightarrow$  equivalent continuum model
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focus on buckling

# Architected Materials & Instabilities

## Focus on buckling

Possible commensurate buckling

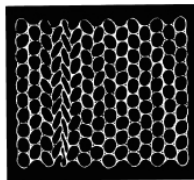
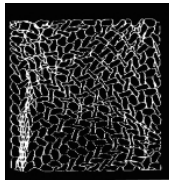


[Papka S. & Kyriakides S. 1998]

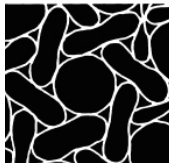
# Architected Materials & Instabilities

## Focus on buckling

Possible commensurate buckling



= patterning with  
periodic pattern



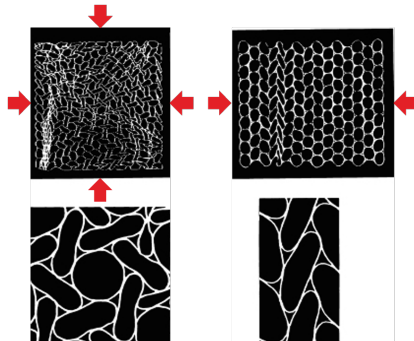
[Papka S. & Kyriakides S. 1998]



# Architected Materials & Instabilities

## Focus on buckling

Effect of load type/orientation



4 cells to capture  
buckling mode

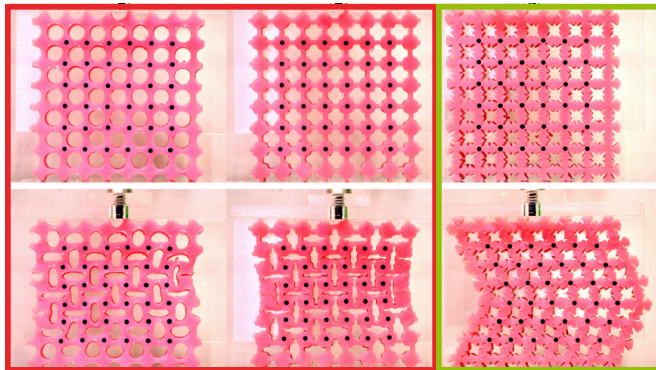
2 cells to capture  
buckling mode

[Papka S. & Kyriakides S. 1998]

# Architected Materials & Instabilities

## Focus on buckling

Effect of cell geometry : commensurate vs structural

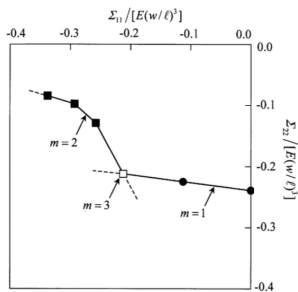


[Overvelde J.T.B. & Bertoldi K. 2014]

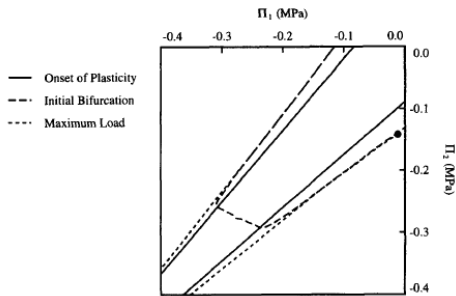
# Literature review

## Avoid buckling

### Onset of instability



[Ohno et al. 2002]



[Triantafyllidis & Schraad 1998]

# Literature review

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## Avoid buckling

Onset of instability

[Ohno et al. 2002]     [Triantafyllidis & Schraad 1998]

## Tools

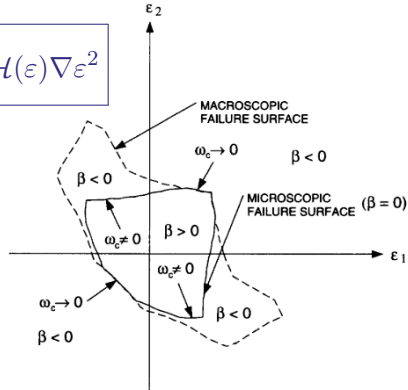
- Bloch-wave theory for commensurate buckling
- Rank-one convexity for macroscopic buckling
- Computation at the scale of 1 or several cells

# Literature review

## Avoid buckling

### Higher-order gradient continuum model

$$W^s(\varepsilon) = \mathcal{E}(\varepsilon) + \frac{\ell^2}{2} \mathcal{H}(\varepsilon) \nabla \varepsilon^2$$

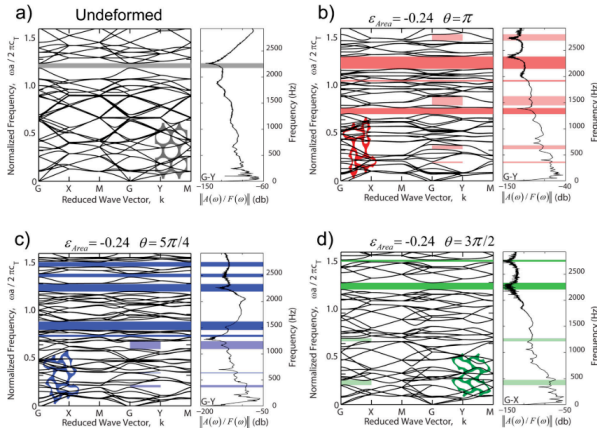


if  $\mathcal{H}(\varepsilon) < 0$  at macroscopic buckling, commensurate instability happens before

[Triantafyllidis & Bardenhagen 1994-1996]

# Literature review

## Take advantage of buckling



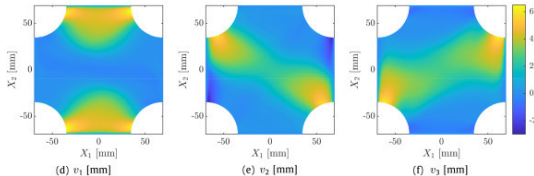
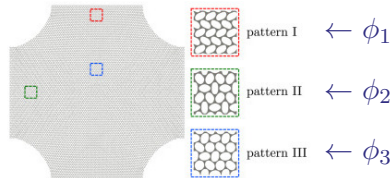
[Shan et al. 2014]

# Literature review

## Take advantage of buckling

### Higher-order micromorphic continuum model

$$\mathcal{W}^m(\varepsilon, \phi_i)$$



[Rokos et al. 2014]



# Research Question



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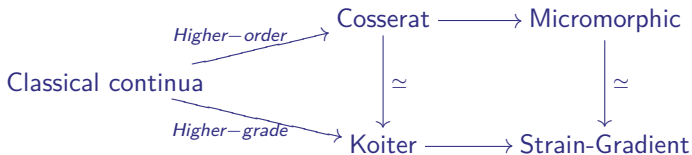
# Research Question

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[Auffray et al. 2015]

*Rotation:*

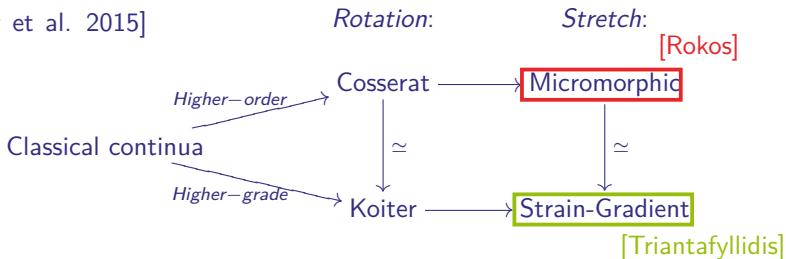
*Stretch:*



# Research Question

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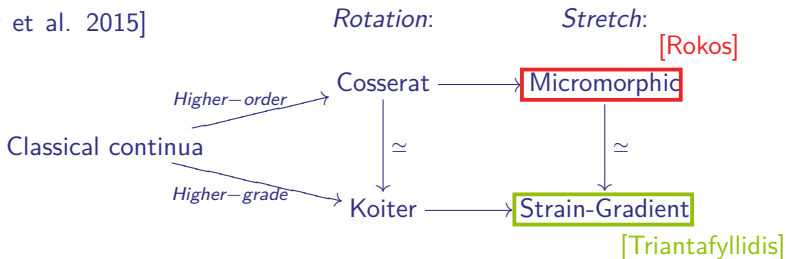
[Auffray et al. 2015]



# Research Question

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[Auffray et al. 2015]



Which generalized continuum model is better suited to capture both commensurate and macroscopic instabilities in periodic media?

# Our Study

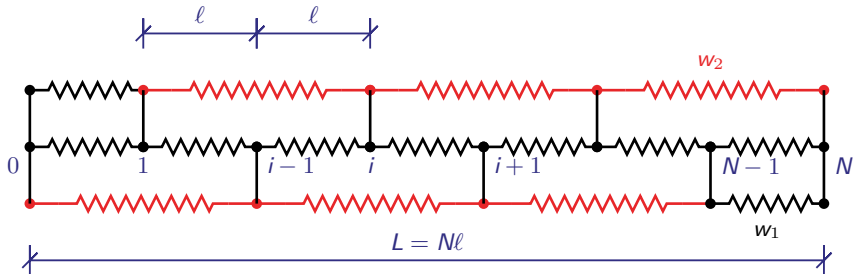
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Very simple example



## Mesoscopic study



non-linear stored energy density in a spring  $p = 1$  or  $p = 2$

$$w_p(\varepsilon) = \frac{1}{2} K_p \varepsilon^2 - \frac{1}{4} M_p \varepsilon^4 + \frac{1}{6} N_p \varepsilon^6$$

*inspired from [Triantafyllidis & Bardenhagen 1994]*

*[Truskinovsky & Vainchtein 2014]*

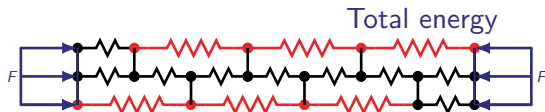
## Mesoscopic study

non-linear stored energy density in a spring  $p = 1$  or  $p = 2$

$$w_p(\varepsilon) = \frac{1}{2} K_p \varepsilon^2 - \frac{1}{4} M_p \varepsilon^4 + \frac{1}{6} N_p \varepsilon^6$$

Stored energy density of the full system

$$\mathcal{W} = \frac{1}{2b} \sum_{n=0}^{n=N} \sum_{p=1}^q p [w_p(\varepsilon_p^{n+}) + w_p(\varepsilon_p^{n-})]$$



$$\mathcal{E} = \mathcal{W} - \frac{1}{Nb} F \cdot u_N$$

$F$  end forces associated with end displacement  $u_N$

## Mesoscopic study

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Equilibrium  $\Rightarrow$  principal path  $\varepsilon^0 =$  uniform strain of all springs

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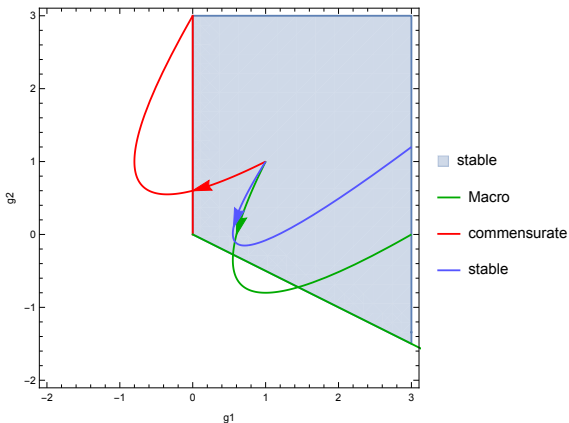
Bloch-wave perturbation predicts 2 possible unstable regimes

- macroscopic :  $g_1(\varepsilon^0) + 2g_2(\varepsilon^0) = 0$
- commensurate ( $\lambda = 2\ell$ ) :  $g_1(\varepsilon^0) = 0$

$$\text{with } g_p(\varepsilon) = K_p - 3M_p\varepsilon^2 + 5N_p\varepsilon^4$$

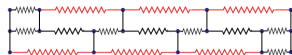
Coherent with [Truskinovsky et al. 2004]

# Mesoscopic study

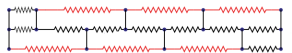


Stability of paths depends on the values of parameters  $K_p$ ,  $M_p$  and  $N_p$

commensurate mode :



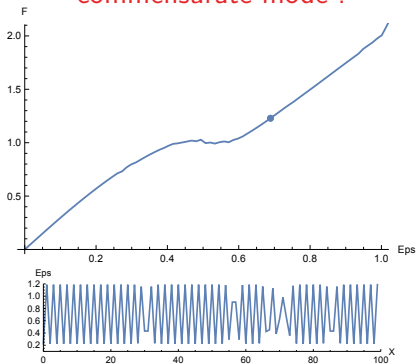
Macro mode :





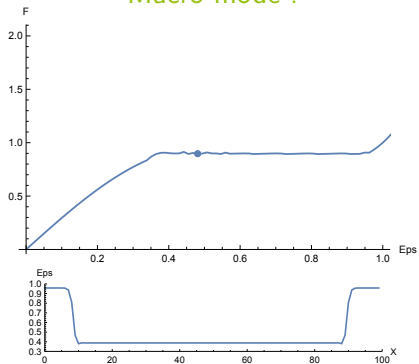
# Mesososcopic study

commensurate mode :



still convergence issues  
(Newton-Raphson)

Macro mode :

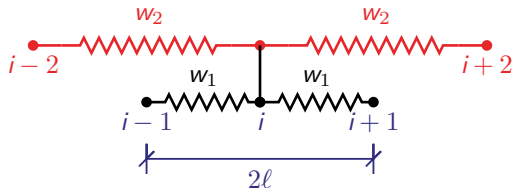


Maxwell line

# Equivalent media

## Micromorphic

from mesoscopic study : commensurate wavelength mode  $\lambda = 2\ell$



Cauchy-Born kinematics :

$$u(\vec{x}) = \mathbf{E} \cdot \mathbf{x} + \mathbf{s}(\mathbf{x})$$

point position

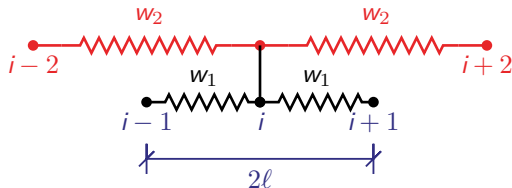
macroscopic strain

internal shift  
 $\approx$  bif. mode

# Equivalent media

## Micromorphic

from mesoscopic study : commensurate wavelength mode  $\lambda = 2\ell$



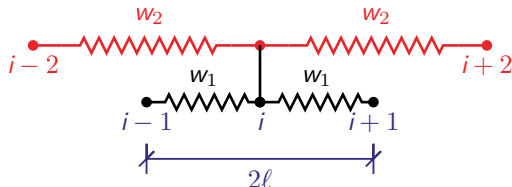
Cauchy-Born kinematics :  $u(x) = \mathbf{E} \cdot x + s(x)$

$$\text{Periodicity : } \begin{cases} u_{i-1} = u_{i+1} \\ u_{i-2} = u_i = u_{i+2} \end{cases}$$

# Equivalent media

## Micromorphic

from mesoscopic study : commensurate wavelength mode  $\lambda = 2\ell$



$$u(x) = E \cdot x + s(x)$$

$$\text{Periodicity : } \begin{cases} u_{i-1} = u_{i+1} \\ u_{i-2} = u_i = u_{i+2} \end{cases}$$

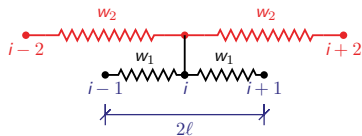
$\Rightarrow$

$$\begin{aligned} \varepsilon_1 &= E - \frac{s}{\ell}; & \varepsilon_{-1} &= E + \frac{s}{\ell} \\ \varepsilon_2 &= E = \varepsilon_{-2} \end{aligned}$$

# Equivalent media

## Micromorphic

Strains:  $\epsilon_1 = E - \frac{s}{\ell}; \quad \epsilon_{-1} = E + \frac{s}{\ell}$   
 $\epsilon_2 = E = \epsilon_{-2}$



Stored energy density :

$$\mathcal{W} = \frac{1}{2b} \sum_{n=0}^{n=N} \sum_{p=1}^q p [w_p(\epsilon_p^{n+}) + w_p(\epsilon_p^{n-})]$$

Micromorphic energy density:  $E$  macroscopic strain &  $s$  internal shift

$$\mathcal{W}^m(E, s) = w_1(E) + 2w_2(E) + G_1(E) \left(\frac{s}{\ell}\right)^2 + k_1(E) \left(\frac{s}{\ell}\right)^4 + \frac{N_1}{6} \left(\frac{s}{\ell}\right)^6$$

# Equivalent media

## Micromorphic

Micromorphic energy density:  $E$  macroscopic strain &  $s$  internal shift

$$\mathcal{W}^m(E, s) = w_1(E) + 2w_2(E) + G_1(E) \left(\frac{s}{\ell}\right)^2 + k_1(E) \left(\frac{s}{\ell}\right)^4 + \frac{N_1}{6} \left(\frac{s}{\ell}\right)^6$$

Equilibrium equations

$$\frac{\partial \mathcal{W}^m}{\partial E} \quad \text{and} \quad \frac{\partial \mathcal{W}^m}{\partial s}$$

Macro

internal

Stability condition

$$\left[ \begin{array}{cc} \frac{\partial^2 \mathcal{W}^s}{\partial E^2} & \frac{\partial^2 \mathcal{W}^s}{\partial E \partial s} \\ \frac{\partial^2 \mathcal{W}^s}{\partial E \partial s} & \frac{\partial^2 \mathcal{W}^s}{\partial s^2} \end{array} \right] \Bigg|_{s=0}$$

def positive

# Equivalent media

## Micromorphic

### Stability condition

$$\det \left[ \begin{array}{cc} \frac{\partial^2 W^s}{\partial E^2} & \frac{\partial^2 W^s}{\partial E \partial s} \\ \frac{\partial^2 W^s}{\partial E \partial s} & \frac{\partial^2 W^s}{\partial s^2} \end{array} \right] \Big|_{s=0} = 0 \Leftrightarrow g_1(E) (g_1(E) + 2g_2(E)) = 0$$

Exact same 2 conditions as for the mesoscopic case:

- $g_1^0(E) + 2g_2^0(E) = 0$
- $g_1^0(E) = 0$

Micromorphic medium able to capture both instabilities

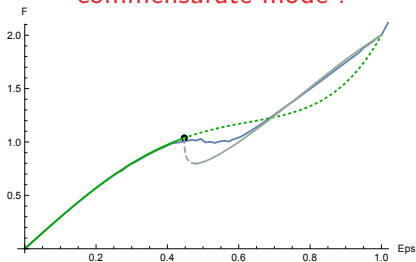
≈ vibration where micromorphic media can capture optical branches

# Equivalent media

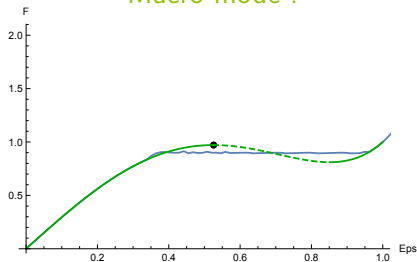
## Micromorphic

Closed-form solutions vs discrete numerical solutions

commensurate mode :



Macro mode :





# Equivalent media

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## Second gradient

[Triantafyllidis & Bardenhagen 1994]

Taylor-series expansion of displacement in strain expression

$$\epsilon_n^{p+} = \frac{u_{n+p} - u_n}{pl} = u_{,x} + \frac{1}{2}pl u_{,xx} + \frac{1}{6}(pl)^2 u_{,xxx} + \dots$$

$p = 1, 2$

$$\epsilon_n^{p-} = \frac{u_n - u_{n-p}}{pl} = u_{,x} - \frac{1}{2}pl u_{,xx} + \frac{1}{6}(pl)^2 u_{,xxx} - \dots$$


---

Second gradient energy density:

$$\mathcal{W}^s(u_{,x}) = w_1(u_{,x}) + 2w_2(u_{,x}) + (u_{,xx})^2 \frac{\ell^2}{2} h(u_{,x})$$

# Equivalent media

## Second gradient

[Triantafyllidis & Bardenhagen 1994]

Second gradient energy:

$$\mathcal{E}^s(u, x) = \int_0^L w_1(u, x) + 2w_2(u, x) + (u,_{xx})^2 \frac{\ell^2}{2} h(u, x) dx$$

Equilibrium equation:  $\mathcal{E}_{,u}^s \delta u = 0$

$$w'_1(u, x) + 2w'_2(u, x) - \ell^2 \left( \frac{h'(u, x)}{2} (u,_{xx})^2 + h(u, x) u,_{xxx} \right) = F$$

Stability condition:  $\mathcal{E}_{,uu}^s \delta u \delta u = 0$

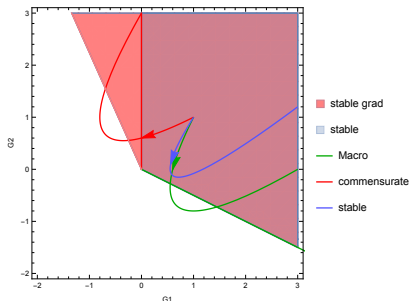
$$g_1(u, x) + 2g_2(u, x) + \left( \frac{\ell n \pi}{L} \right)^2 h(u, x) = 0 \quad \forall n \in \mathbb{Z}$$

# Equivalent media

## Second gradient

### Stability condition

$$g_1(u_{,x}^0) + 2g_2(u_{,x}^0) - \left(\frac{\ell n \pi}{L}\right)^2 \frac{g_1(u_{,x}^0) + 8g_2(u_{,x}^0)}{12} (u_{,x}^0) = 0 \quad \forall n \in \mathbb{Z}$$



Coherent with [Truskinovsky et al. 2004]

# Conclusions & Work in progress

## Ongoing work

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- implement both equivalent media in numerical continuation solver to compare on example BVP
- look at incommensurate instabilities appearing if  $p = 3$  (next-to-next-to-neighbour interaction)

# Conclusions

## Comparing generalized continua vs instabilities

### Micromorphic:

- Can capture both modes with criteria = mesoscopic
- Requires *a priori* knowledge of buckling mode wavelength

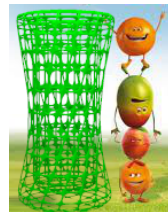
### Second gradient:

- Can capture Macro mode appropriately with criterion = mesoscopic
- Can capture commensurate mode but BP point not exact
- Does not require any *a priori* knowledge & Systematic construction

Conclusions are the same as for vibration studies



## Max-OASIS



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