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ABSTRACT. We consider the nonlinear Schrödinger equation with a defocusing nonlinearity which is mass-(super)critical and energy-subcritical. We prove uniform in time error estimates for the Lie-Trotter time splitting discretization. This uniformity in time is obtained thanks to a vectorfield which provides time decay estimates for the exact and numerical solutions. This vectorfield is classical in scattering theory, and requires several technical modifications compared to previous error estimates for splitting methods.

1. INTRODUCTION

We consider large time error estimates for the Lie-Trotter time splitting method associated to the defocusing nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = |u|^{2\sigma}u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \quad u_{|t=0} = \phi, \tag{1.1}$$

in space dimension $d \leqslant 5$, in the case where the nonlinearity is mass-(super)critical and energy-subcritical,

$$\frac{2}{d} \leqslant \sigma < \frac{2}{(d-2)_+}, \tag{1.2}$$

that is, $\sigma \geqslant 2/d$ when $d \leqslant 2$, and $2/d \leqslant \sigma < 2/(d-2)$ when $d \geqslant 3$. The restriction on the space dimension is due to the fact that we want the nonlinearity to be energy-subcritical, and to have two continuous derivatives, $\sigma > 1/2$. Under these assumptions, the Cauchy problem (1.1) is globally well-posed in $H^1(\mathbb{R}^d)$ ([4, 11]), and mass and energy are conserved,

$$\text{Mass: } M(u(t)) := \|u(t)\|^2_{L^2(\mathbb{R}^d)} = M(\phi),$$
$$\text{Energy: } E(u(t)) := \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{\sigma+1}\|u(t)\|^{2\sigma+2}_{L^{2\sigma+2}(\mathbb{R}^d)} = E(\phi).$$

Like initially in [12], denote by $\Sigma \subset H^1(\mathbb{R}^d)$ the Hilbert space, sometimes called *conformal space* in the context of nonlinear Schrödinger equations,

$$\Sigma := \left\{\phi \in H^1(\mathbb{R}^d);\ \int_{\mathbb{R}^d} |x|^2|\phi(x)|^2dx < \infty\right\},$$

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equipped with the norm

$$\|\phi\|_\Sigma^2 := \|\phi\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla\phi\|_{L^2(\mathbb{R}^d)}^2 + \|x\phi\|_{L^2(\mathbb{R}^d)}^2.$$

Then the Cauchy problem (1.1) is globally well-posed in Σ as well: if $\phi \in \Sigma$, the solution $u(t,\cdot)$ has a finite momentum in $L^2(\mathbb{R}^d)$ for all time, and the evolution of this quantity is described by the *pseudo-conformal evolution law*, recalled in Proposition 2.2.

We now recall the definition of the Lie-Trotter time splitting for (1.1). We define $N(t)\phi$ as the solution of the flow

$$i\partial_t u = |u|^{2\sigma}u, \quad u_{|t=0} = \phi,$$

that is, $N(t)\phi = \phi e^{-it|\phi|^{2\sigma}}$. We set $S(t)\phi$ as the solution of the linear Schrödinger flow

$$i\partial_t u + \frac{1}{2}\Delta u = 0, \quad u_{|t=0} = \phi.$$

It is a Fourier multiplier, $S(t)\phi = e^{i\frac{t}{2}\Delta}\phi$, and thus $S(t)$ is unitary on $H^s(\mathbb{R}^d)$ for any $s \in \mathbb{R}$. The Lie-Trotter approximation is defined, for $\tau \in (0,1)$, as

$$Z(n\tau)\phi = \left(S(\tau)N(\tau)\right)^n \phi.$$

Error estimates for this time discretization were established first in [1] for globally Lipschitz nonlinearities. C. Lubich [21] proved error estimates in the case of the Strang splitting, allowing (Schrödinger-Poisson nonlinearity and) cubic nonlinearity ($\sigma = 1$ in (1.1)), hence a nonlinearity which is not globally Lipschitz continuous.

As pointed out in [16, 17, 18, 24], $S(\cdot)$ does not satisfy discrete in time Strichartz estimates, which makes it difficult to envisage error estimates involving a rather low regularity (in space) of the initial datum ϕ; see also [8, 23] for discussions leading to the same conclusion, that $S(\cdot)$ should be modified in order to get better convergence results. Following [16, 18] and the adaptation in [6], we consider the modified splitting operator:

$$Z_\tau(n\tau) = \left(S_\tau(\tau)N(\tau)\right)^n \Pi_\tau\phi. \tag{1.3}$$

Here, $S_\tau(t)$ denotes the frequency localized Schrödinger flow given by

$$S_\tau(t)\phi = S(t)\Pi_\tau\phi,$$

where

$$\widehat{\Pi_\tau\phi}(\xi) = \chi(\tau^{1/2}\xi)\widehat{\phi}(\xi), \quad \xi \in \mathbb{R}^d, \tag{1.4}$$

and $\chi \in C^k(\mathbb{R}^d)$ is a cut-off function supported in $B^d(0,2)$ such that $\chi \equiv 1$ on $B^d(0,1)$, k an integer larger than $1 + d/2$ (this condition appears in the proof of Lemma 3.3).

All the error estimates associated to splitting methods for nonlinear Schrödinger equations have been established so far for bounded time intervals $[0,T]$ (with constants growing at best exponentially in T). As far as we are aware of, the same is true regarding linear Schrödinger equations with potential. On the other hand, global in time estimates have been proven in the framework of kinetic equations [9]. There, time decay estimates (associated to a scattering phenomenon) play a crucial role. In this paper, we prove an analogous result in the case of (1.1), by using techniques related to the scattering theory (in Σ), in order to get quantitative time decay estimates) for nonlinear Schrödinger equations. More precisely, we use

a specific vectorfield, standard in the scattering theory for (1.1), $J(t) = x + it\nabla$, which provides more precise decay estimates in time than the mixed $L^q_t L^r_x$-norms appearing in Strichartz estimates. It is well known that J does not commute with S, but $J(t) = S(t)xS(-t)$ (see Proposition 2.2 below). A new technical specific aspect in this paper is that we also have to deal with the absence of commutation between J and the frequency cut-off Π_τ.

For any interval $I \subset [0,\infty)$, we define the space $\ell^q(n\tau \in I;\, L^r(\mathbb{R}^d))$, or simply $\ell^q(I;\, L^r)$ as consisting of functions defined on $\tau\mathbb{Z} \cap I$ with values in $L^r(\mathbb{R}^d)$, the norm of which is given by

$$\|u\|_{\ell^q(I;\,L^r)} = \begin{cases} \Big(\tau \sum\limits_{n\tau\in I} \|u(n\tau)\|^q_{L^r(\mathbb{R}^d)}\Big)^{1/q} & \text{if } 1 \leqslant q < \infty, \\ \sup\limits_{n\tau\in I} \|u(n\tau)\|_{L^r(\mathbb{R}^d)} & \text{if } q = \infty. \end{cases} \tag{1.5}$$

We recall the notion of admissible pairs in the context of Schrödinger equation (we shall not need endpoint Strichartz estimates, $(q,r) = (2, \frac{2d}{d-2})$ for $d \geqslant 3$).

Definition 1.1. *A pair (q,r) is admissible if $2 \leqslant r < \frac{2d}{d-2}$ ($2 \leqslant r \leqslant \infty$ if $d = 1$, $2 \leqslant r < \infty$ if $d = 2$) and*

$$\frac{2}{q} = \delta(r) := d\left(\frac{1}{2} - \frac{1}{r}\right).$$

Remark 1.2. We note that the range for q is equivalent to: $q \in (2,\infty]$ if $d \geqslant 2$, and $q \in [4,\infty]$ if $d = 1$.

As it plays a central role in the analysis of (1.1), throughout this paper, and following [6, 16], we denote by (q_0, r_0) the admissible pair

$$(q_0, r_0) = \left(\frac{4\sigma+4}{d\sigma}, 2\sigma+2\right).$$

Theorem 1.3. *Let $d \leqslant 5$, σ satisfying (1.2), with in addition $\sigma \geqslant 1/2$ (if $d = 5$), $\phi \in \Sigma$, and $u \in C(\mathbb{R};\Sigma)$ the solution to (1.1). Suppose there exists M_2 such that*

$$\max_{A\in\{\mathbf{1},\nabla,J\}} \|A(n\tau)Z_\tau(n\tau)\|_{\ell^\infty(\tau\mathbb{N};L^2)} + \|Z_\tau(n\tau)\|_{\ell^{q_0}(\tau\mathbb{N};W^{1,r_0})} \leqslant M_2, \tag{1.6}$$

where $J(t) = x + it\nabla$. Then there exists $C = C(d,\sigma,\|\phi\|_\Sigma, M_2)$ such that for all $\tau \in (0,1)$,

$$\sup_{n\geqslant 0} \|Z_\tau(n\tau) - u(n\tau)\|_{L^2(\mathbb{R}^d)} \leqslant C\tau^{1/2}.$$

In addition, there exists $u_+ \in \Sigma$ such that

$$\lim_{k\to\infty} \sup_{n\geqslant k} \|Z_\tau(n\tau) - S(n\tau)u_+\|_{L^2(\mathbb{R}^d)} \leqslant C\tau^{1/2}.$$

Remark 1.4. We prove convergence in the L^2-topology. It is very likely that it holds also in the H^1-topology if we require in addition $\phi \in H^2$, in view of the second result in Theorem 2.5.

Remark 1.5. The existence of the asymptotic state u_+ can be understood as follows: for sufficiently large time, nonlinear effects have become negligible, and the action of the nonlinear flow $N(\tau)$ converges (fast enough) toward the identity, so the splitting operator Z_τ behaves like S_τ, which in turn is equivalent to S for smooth functions.

This is the meaning of the last estimate in Theorem 1.3 which, in some sense, makes the uniform error estimate more precise.

Now demanding $\sigma > 1/2$, and not only $\sigma \geqslant 1/2$ (see the proof of Proposition 6.2 for the reason why this constraint is introduced), we show that the assumptions of Theorem 1.3 are indeed satisfied by the numerical solution:

Theorem 1.6. *Let $d \leqslant 5$, σ satisfying (1.2), with in addition $\sigma > 1/2$ (if $d = 4$ or 5), and $\phi \in \Sigma$. Then, for any admissible pair (q, r), there exists $C(d, \sigma, q, \phi)$ such that for all $\tau \in (0, 1)$, the numerical solution Z_τ satisfies*

$$\max_{A\in\{\mathbf{1},\nabla,J\}} \|A(n\tau)Z_\tau(n\tau)\|_{\ell^q(\tau\mathbb{N};L^r)} \leqslant C(d, \sigma, q, \phi).$$

Remark 1.7. In view of classical results on nonlinear Schrödinger equations (see e.g [4, 26]), Theorems 1.3 and 1.6 remain valid in the case of a focusing nonlinearity ($|u|^{2\sigma}u$ is replaced by $-|u|^{2\sigma}u$ in (1.1)), provided that $\|\phi\|_\Sigma$ is sufficiently small. Without smallness assumption, finite time blow up is possible, but even global solutions need not be dispersive, since standing wave solutions of the form $u(t, x) = e^{i\omega t}\phi(x)$ exist. Theorems 1.3 and 1.6 highly rely on dispersive properties of the solution to (1.1), and should not be expected to remain true in the case of standing waves. More general nonlinearities than the homogeneous one considered in (1.1) could be addressed though (typically, combined power nonlinearities), provided that suitable a priori estimates (in the spirit of the pseudoconformal conservation law recalled in Proposition 2.2) are available. This is for instance the case when the nonlinearity is the sum of two defocusing homogeneous terms, $|u|^{2\sigma_1}u + |u|^{2\sigma_2}u$, $\frac{2}{d} \leqslant \sigma_j < \frac{2}{(d-2)_+}$, but the situation is more involved when it is the sum of a focusing and a defocusing term, since standing waves exist (see e.g. [20]).

Remark 1.8. As recalled in Section 2, the assumptions on σ and ϕ ensure that the solution u to (1.1) is global, and satisfies $u \in L^q(\mathbb{R}; L^r(\mathbb{R}^d))$ for all admissible pairs. It is tempting to conjecture that Theorem 1.6 remains true under the assumptions that $u \in L^q(\mathbb{R}; L^r(\mathbb{R}^d))$ for all admissible pairs, a property that actually follows from the weaker one $u \in L^{q_0}(\mathbb{R}; L^{r_0}(\mathbb{R}^d))$, see Section 2. However, the introduction of the operator J induces stronger estimates, and it is not clear at all that the proof of Theorem 1.6 can be adapted to this broader setting (typically, one could assume $\phi \in H^1(\mathbb{R}^d)$ only).

Organization of the paper. The rest of this paper is organized as follows. In Section 2, we recall some standard results related to (1.1), including global in time estimates, establish some new ones (in particular Theorem 2.5), to provide general estimates on solutions to (1.1). Section 3 is devoted to general estimates involving the numerical solution (discrete in time). The main novelties concern the introduction of the operator J, and the difficulties caused by its absence of commutation with the frequency cut-off Π_τ. Theorem 1.3 is proved in Section 4. In Section 5, we establish local Σ stability results (Proposition 5.1); there, local means local in time, allowing a neighborhood to $t = \infty$. In Section 6, we prove refined estimates allowing to conclude with the proof of Theorem 1.6 in Section 7. Finally Section 8 is dedicated to a summary and a list of possible related extensions.

General notations.

- We denote by $\mathbf{1}$ the identity operator.

- We denote by L^r the standard space $L^r(\mathbb{R}^d)$ for $1 \leqslant r \leqslant \infty$.
- For $y \in \mathbb{R}^n$, $n \in \mathbb{N}$, the Japanese bracket is $\langle y \rangle = (1+|y|^2)^{1/2}$.
- We denote by C generic constants, which may vary from line to line.
- We underline the dependence of constants as follows: $C(d,\sigma)$ or $C_{d,\sigma}$ means that C does not depend on other parameters, such as τ or ϕ.
- For $a, b \geqslant 0$, we use the notation
$$a \lesssim b$$
whenever there exists a constant C independent of $\tau \in (0,1)$ and the time interval considered (but certainly depending on d and σ) such that $a \leqslant Cb$.
- When considered useful, we write u^ϕ to underscore that u is the solution to (1.1) with initial datum ϕ (typically, when several such solutions are involved).

2. Preliminary estimates: the exact solution

2.1. **Generalities.** For $t_0 \in \mathbb{R}$, Duhamel formula for the solution u to (1.1) with the initial condition $u_{|t=t_0} = \phi$, reads as follows:

$$u(t) = S(t-t_0)\phi - i\int_{t_0}^t S(t-s)\left(|u|^{2\sigma}u\right)(s)ds. \tag{2.1}$$

The standard Strichartz inequalities associated to the Schrödinger equation (see e.g. [4, 26]) are summarized below. We recall that the notion of admissible pairs was introduced in Definition 1.1.

Proposition 2.1 (Strichartz estimates)**.** *Let $d \geqslant 1$ and $S(t) = e^{i\frac{t}{2}\Delta}$.*
(1) Homogeneous estimates. *For any admissible pair (q,r), there exists C_q such that*
$$\|S(t)\phi\|_{L^q(\mathbb{R};L^r)} \leqslant C_q\|\phi\|_{L^2}, \quad \forall \phi \in L^2.$$
(2) Inhomogeneous estimates. *Denote*
$$D(F)(t,x) = \int_0^t S(t-s)F(s,x)ds.$$
For all admissible pairs (q_1,r_1) and (q_2,r_2), there exists $C = C_{q_1,q_2}$ such that for all intervals $I \ni 0$,

$$\|D(F)\|_{L^{q_1}(I;L^{r_1})} \leqslant C\,\|F\|_{L^{q_2'}\left(I;L^{r_2'}\right)}, \quad \forall F \in L^{q_2'}(I;L^{r_2'}). \tag{2.2}$$

With Strichartz and Hölder inequalities in mind, we remark that r_0 satisfies
$$\frac{1}{r_0'} = \frac{2\sigma+1}{r_0},$$
with $2\sigma+1$ being the homogeneity of the nonlinearity in (1.1), and we introduce γ given by

$$\frac{1}{q_0'} = \frac{1}{q_0} + \frac{2\sigma}{\gamma} \Longleftrightarrow \gamma = \frac{4\sigma(\sigma+1)}{2-(d-2)\sigma}. \tag{2.3}$$

We see that γ is finite since the nonlinearity is energy-subcritical. The above relations will be applied many times, as follows (according to whether continuous

or discrete time intervals are considered), recalling that $2\sigma \geqslant 1$:

$$\begin{aligned} &\left\| |f|^{2\sigma-1} gh \right\|_{L^{q_0'}(I;L^{r_0'})} \leqslant \|f\|^{2\sigma-1}_{L^\gamma(I;L^{r_0})} \|g\|_{L^\gamma(I;L^{r_0})} \|h\|_{L^{q_0}(I;L^{r_0})}, \quad \text{or} \\ &\left\| |f|^{2\sigma-1} gh \right\|_{\ell^{q_0'}(I;L^{r_0'})} \leqslant \|f\|^{2\sigma-1}_{\ell^\gamma(I;L^{r_0})} \|g\|_{\ell^\gamma(I;L^{r_0})} \|h\|_{\ell^{q_0}(I;L^{r_0})}. \end{aligned} \tag{2.4}$$

The following result was discovered in [12], and is crucial to turn the local error estimates from [16, 6] into global ones:

Proposition 2.2 (Pseudo-conformal conservation law)**.** *The operator*

$$J(t) = x + it\nabla$$

satisfies the following properties:

- $J(t) = S(t)xS(-t)$*, and therefore* J *commutes with the linear part of* (1.1)*,*

$$\left[J(t), i\partial_t + \frac{1}{2}\Delta \right] = 0\,. \tag{2.5}$$

- *It can be factorized as*

$$J(t) = it\, e^{i\frac{|x|^2}{2t}} \nabla \Big(e^{-i\frac{|x|^2}{2t}} \, \cdot \Big)\,.$$

As a consequence, J *yields weighted Gagliardo-Nirenberg inequalities. For* $2 \leqslant r < \frac{2d}{(d-2)_+}$ *(*$2 \leqslant r \leqslant \infty$ *if* $d = 1$*), there exists* $C(d,r)$ *depending only on* d *and* r *such that*

$$\|f\|_{L^r} \leqslant \frac{C(d,r)}{|t|^{\delta(r)}} \|f\|_{L^2}^{1-\delta(r)} \|J(t)f\|_{L^2}^{\delta(r)}\,, \quad \delta(r) := d\left(\frac{1}{2} - \frac{1}{r}\right)\,. \tag{2.6}$$

Also, if $F(z) = G(|z|^2)z$ *is* C^1*, then* $J(t)$ *acts like a derivative on* $F(w)$*:*

$$J(t)\left(F(w)\right) = \partial_z F(w) J(t) w - \partial_{\overline{z}} F(w) \overline{J(t)w}\,. \tag{2.7}$$

Any solution $u \in C(\mathbb{R};\Sigma)$ *to* (1.1) *satisfies the pseudo-conformal conservation law:*

$$\frac{d}{dt}\left(\frac{1}{2}\|J(t)u\|_{L^2}^2 + \frac{t^2}{\sigma+1}\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \right) = \frac{t}{\sigma+1}(2 - d\sigma)\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2}. \tag{2.8}$$

For some time interval I, we introduce the norm

$$\|u\|_{X(I)} = \max_{A \in \{\mathbf{1}, \nabla, J\}} \sup_{t \in I} \|A(t)u(t)\|_{L^2}, \tag{2.9}$$

and the space $X(I)$ defined by the finiteness of this quantity. We emphasize that in view of the time dependence of J, $X(I) = L^\infty(I;\Sigma)$ *if and only if* I *is bounded.* This can be seen on the linear Schrödinger equation (linear solutions $S(t)\phi$ are dispersive): in view of (2.5), since $S(t)$ is unitary on L^2,

$$\|J(t)S(t)\phi\|_{L^2} = \|x\phi\|_{L^2}, \quad \|\nabla S(t)\phi\|_{L^2} = \|\nabla\phi\|_{L^2} \Longrightarrow \|xS(t)\phi\|_{L^2} \underset{t\to\infty}{\sim} t\|\nabla\phi\|_{L^2}.$$

We note that if I is reduced to a single element, $I = \{t_0\}$, then $X(I) = \Sigma$, but the norm $\|\cdot\|_{X(I)}$ involves the operator $J(t_0)$, so the norms $\|\cdot\|_\Sigma$ and $\|\cdot\|_{X(I)}$ are equivalent, but with comparing constants highly depending on t_0. We also consider the norm

$$\|u\|_{Y(I)} := \max_{A \in \{\mathbf{1}, \nabla, J\}} \sup_{(q,r) \text{ admissible}} \|Au\|_{L^q(I;L^r)}, \tag{2.10}$$

and the associated space $Y(I)$. Note that $\|u\|_{X(I)} \leqslant \|u\|_{Y(I)}$.

2.2. **Global solutions.** At this stage, we emphasize that the assumption $\sigma \geqslant 2/d$ implies that for $t \geqslant 0$, the right hand side in (2.8) is nonpositive. Together with the conservation of mass and energy, this entails $u \in X(\mathbb{R})$. This implies in particular $u \in L^\gamma(\mathbb{R}; L^{r_0})$, as

$$\gamma\delta(r_0) > 1. \tag{2.11}$$

This property is indeed classical in scattering theory for (1.1): it is equivalent to

$$\sigma > \frac{2 - d + \sqrt{d^2 + 12d + 4}}{4d} =: \sigma_*,$$

and $\sigma_* < 2/d$. This parameter σ_* is a standard lower bound for σ to prove scattering theory in Σ, see e.g. [4] (see also [5, 22] for the case $\sigma = \sigma_*$). The standard Gagliardo-Nirenberg inequality (for bounded t) and (2.6) (for large t) yield

$$\|u(t)\|_{L^{r_0}} \lesssim \langle t\rangle^{-\delta(r_0)} \|u(t)\|_{L^2}^{1-\delta(r_0)} \left(\|\nabla u(t)\|_{L^2}^{\delta(r_0)} + \|J(t)u(t)\|_{L^2}^{\delta(r_0)} \right),$$

and in view of Hölder inequality in time,

$$\|u\|_{L^\gamma(I;L^{r_0})} \leqslant C \left\| \langle t\rangle^{-\delta(r_0)} \right\|_{L^\gamma(I)} \|u\|_{X(I)}, \tag{2.12}$$

where C does not depend on the time interval I. For $A \in \{\mathbf{1}, \nabla, J\}$, Strichartz and Hölder inequalities (involving (2.4)) then yield

$$\sup_{(q,r) \text{ admissible}} \|Au\|_{L^q(I;L^r)} \lesssim \|\phi\|_\Sigma + \|u\|_{L^\gamma(I;L^{r_0})}^{2\sigma} \sup_{(q,r) \text{ admissible}} \|Au\|_{L^q(I;L^r)},$$

and splitting $\mathbb{R}$ into finitely many intervals I_j where $\|u\|_{L^\gamma(I_j;L^{r_0})}$ is sufficiently small, we infer $u \in Y(\mathbb{R})$. We obtain the following statement (see e.g. [4, Theorem 7.4.1] or [14, Theorem B]):

Theorem 2.3. *Let $\phi \in \Sigma$, $\frac{2}{d} \leqslant \sigma < \frac{2}{(d-2)_+}$. Then (1.1) has a unique solution $u \in C(\mathbb{R};\Sigma) \cap L^{q_0}_{\mathrm{loc}}(\mathbb{R}; L^{r_0})$. It satisfies $u \in Y(\mathbb{R})$, and there exist $u_\pm \in \Sigma$ such that*

$$\|S(-t)u(t) - u_\pm\|_\Sigma \underset{t\to\pm\infty}{\longrightarrow} 0.$$

A priori, the asymptotic states u_+ and u_- are different, even though the relation between u_+ and u_- remains rather mysterious in general (see e.g. [3]).

Remark 2.4. The first reason why we assume $\sigma \geqslant 2/d$ instead of the more general hypothesis $\sigma > \sigma_*$ ensuring scattering in Σ is that in view of (2.8), we have the global estimate $Ju \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^d))$ as soon as $\sigma \geqslant 2/d$. On the other hand, if $\sigma_* < \sigma < 2/d$, (2.8) provides only a control on the growth (in t) of $t^2\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2}$ (via Gronwall lemma), hence of $\|J(t)u\|_{L^2}$ (using (2.8) again). The assumption $\sigma \geqslant 2/d$ is made not only for this simplification: filling the gap $\sigma_* < \sigma < 2/d$ would require to modify several arguments below, see the proofs of Theorem 2.5 and Lemma 2.6.

We state and prove the global in time analogue of [6, Theorem B], and establish some specific global in time integrability properties:

Theorem 2.5. *Let $1 \leqslant d \leqslant 5$, $\frac{2}{d} \leqslant \sigma < \frac{2}{(d-2)_+}$, with in addition (if $d = 5$) $\sigma \geqslant 1/2$.*

- *For any $M \geqslant 1$, there exists $C = C(M, d, \sigma)$ such that for any $t_0 \in \mathbb{R}$, if $\phi_1, \phi_2 \in \Sigma$ are initial data for u_1 and u_2, respectively, at time t_0,*

$$u_{j|t=t_0} = \phi_j, \quad j = 1, 2,$$

and are such that $\|\phi_1\|_{X(\{t_0\})}, \|\phi_2\|_{X(\{t_0\})} \leqslant M$*, then*

$$\|u_1 - u_2\|_{Y(\mathbb{R})} \leqslant C\|\phi_1 - \phi_2\|_{X(\{t_0\})}.$$

- *If* $\psi \in \Sigma \cap H^2$*, and* u *solves* (1.1)*, where, for* $t_0 \in \mathbb{R}$*,* $u_{|t=t_0} = \psi$*, then for all* $A \in \{\mathbf{1}, \nabla, J\}$*,*

$$Au^\psi \in \bigcap_{(q,r)\ admissible} L^q(\mathbb{R}; W^{1,r}).$$

Proof. For the first point, we note that the conservation of the energy and the pseudoconformal conservation law (2.8) yield

$$\|u_j\|_{X(\mathbb{R})} \leqslant C_1(M, d, \sigma), \quad j = 1, 2.$$

In view of (weighted) Gagliardo-Nirenberg inequality,

$$\|u_j(t)\|_{L^{r_0}} \leqslant \frac{C_2(M, d, \sigma)}{\langle t\rangle^{\delta(r_0)}}.$$

We then remark that since $J(t) = S(t)xS(-t)$ (Proposition 2.2), we have $J(t) = S(t - t_0)J(t_0)S(t_0 - t)$, and Duhamel's formula becomes, for $A \in \{\mathbf{1}, \nabla, J\}$, and $j = 1, 2$,

$$A(t)u_j(t) = S(t - t_0)A(t_0)\phi_j - i\int_{t_0}^t S(t - s)A(s)\left(|u_j|^{2\sigma}u_j\right)(s)ds.$$

Considering the difference between the equations for $j = 1$ and $j = 2$, respectively, Strichartz and Hölder inequalities (like mentioned above) yield, if $t_0 \in I$, then for all admissible (q, r),

$$\begin{aligned}\|u_1 - u_2\|_{L^q(I;L^r)} &\lesssim \|\phi_1 - \phi_2\|_{X(\{t_0\})}\\ &\quad + \left(\|u_1\|^{2\sigma}_{L^\gamma(I;L^{r_0})} + \|u_2\|^{2\sigma}_{L^\gamma(I;L^{r_0})}\right)\|u_1 - u_2\|_{L^{q_0}(I;L^{r_0})}\\ &\lesssim \|\phi_1 - \phi_2\|_{X(\{t_0\})}\\ &\quad + C_3(M, d, \sigma)\|\langle t\rangle^{-\delta(r_0)}\|^{2\sigma}_{L^\gamma(I)}\|u_1 - u_2\|_{L^{q_0}(I;L^{r_0})}.\end{aligned}$$

When $A = \nabla$ or $J(t)$, computations are similar:

$$\begin{aligned}\nabla\left(|u_1|^{2\sigma}u_1\right) - \nabla\left(|u_2|^{2\sigma}u_2\right) &= (\sigma + 1)\left(|u_1|^{2\sigma}\nabla u_1 - |u_2|^{2\sigma}\nabla u_2\right)\\ &\quad + \sigma\left(|u_1|^{2\sigma-2}u_1^2\overline{\nabla u_1} - |u_2|^{2\sigma-2}u_2^2\overline{\nabla u_2}\right);\\ J(t)\left(|u_1|^{2\sigma}u_1\right) - J(t)\left(|u_2|^{2\sigma}u_2\right) &= (\sigma + 1)\left(|u_1|^{2\sigma}J(t)u_1 - |u_2|^{2\sigma}J(t)u_2\right)\\ &\quad - \sigma\left(|u_1|^{2\sigma-2}u_1^2\overline{J(t)u_1} - |u_2|^{2\sigma-2}u_2^2\overline{J(t)u_2}\right).\end{aligned}$$

We consider the first term of the right hand side, as the second term is estimated in the same way:

$$|u_1|^{2\sigma}Au_1 - |u_2|^{2\sigma}Au_2 = |u_1|^{2\sigma}A(u_1 - u_2) + \left(|u_1|^{2\sigma} - |u_2|^{2\sigma}\right)Au_2.$$

Since $2\sigma \geqslant 1$, we have

$$\left||u_1|^{2\sigma} - |u_2|^{2\sigma}\right| \lesssim \left(|u_1|^{2\sigma-1} + |u_2|^{2\sigma-1}\right)|u_1 - u_2|.$$

Now we use (2.4) to find:

$$\begin{aligned}\|A(u_1 - u_2)\|_{L^q(I;L^r)} &\lesssim \|\phi_1 - \phi_2\|_{X(\{t_0\})} + \left\|A\left(|u_1|^{2\sigma}u_1\right) - A\left(|u_2|^{2\sigma}u_2\right)\right\|_{L^{q_0'}(I;L^{r_0'})}\\ &\lesssim \|\phi_1 - \phi_2\|_{X(\{t_0\})}\end{aligned}$$

$$+\left(\|u_1\|^{2\sigma}_{L^\gamma(I;L^{r_0})}+\|u_2\|^{2\sigma}_{L^\gamma(I;L^{r_0})}\right)\|A(u_1-u_2)\|_{L^{q_0}(I;L^{r_0})}$$
$$+\left(\|u_1\|^{2\sigma-1}_{L^\gamma(I;L^{r_0})}+\|u_2\|^{2\sigma-1}_{L^\gamma(I;L^{r_0})}\right)\|Au_2\|_{L^{q_0}(I;L^{r_0})}\|u_1-u_2\|_{L^\gamma(I;L^{r_0})}.$$

The last term is controlled, in view of (2.12), by

$$\|u_1-u_2\|_{L^\gamma(I;L^{r_0})}\lesssim \left\|\langle t\rangle^{-\delta(r_0)}\right\|_{L^\gamma(I)}\max_{B\in\{\mathbf{1},\nabla,J\}}\|B(u_1-u_2)\|_{L^\infty(I;L^2)}.$$

In view of Theorem 2.3, $Au_2\in L^{q_0}(\mathbb{R};L^{r_0})$, and we have:

$$\max_{\substack{A\in\{\mathbf{1},\nabla,J\}\\(q,r)\ \text{admissible}}}\|A(u_1-u_2)\|_{L^q(I;L^r)}\lesssim\|\phi_1-\phi_2\|_{X(\{t_0\})}$$
$$+C_3(M,d,\sigma)\|\langle t\rangle^{-\delta(r_0)}\|^{2\sigma}_{L^\gamma(I)}\max_{\substack{A\in\{\mathbf{1},\nabla,J\}\\(q,r)\ \text{admissible}}}\|A(u_1-u_2)\|_{L^q(I;L^r)},$$

and similarly, using Strichartz estimates again, for any $t_j\in\mathbb{R}$, $I_j=\overline{[t_j,t_{j+1})}$ ($t_j<t_{j+1}\leqslant\infty$, we consider the adherence of I_j to address a closed interval except if $t_{j+1}=\infty$),

$$\max_{\substack{A\in\{\mathbf{1},\nabla,J\}\\(q,r)\ \text{admissible}}}\|A(u_1-u_2)\|_{L^q(I;L^r)}\leqslant C_4\|u_1-u_2\|_{X(\{t_j\})}$$
$$+C_5(M,d,\sigma)\left\|\langle t\rangle^{-\delta(r_0)}\right\|^{2\sigma}_{L^\gamma(I)}\max_{\substack{A\in\{\mathbf{1},\nabla,J\}\\(q,r)\ \text{admissible}}}\|A(u_1-u_2)\|_{L^q(I;L^r)},$$

where C_4 and C_5 is independent of I_j, that is,

$$\|u_1-u_2\|_{Y(I)}\leqslant C_4\|u_1-u_2\|_{X(\{t_j\})}+C_5(M,d,\sigma)\left\|\langle t\rangle^{-\delta(r_0)}\right\|^{2\sigma}_{L^\gamma(I)}\|u_1-u_2\|_{Y(I)}.$$

Since $\gamma\delta(r_0)>1$, we can split $[t_0,\infty)$ into finitely many intervals I_j such that

$$C_5(M,d,\sigma)\left\|\langle t\rangle^{-\delta(r_0)}\right\|^{2\sigma}_{L^\gamma(I_j)}\leqslant\frac12,\quad [t_0,\infty)=\bigcup_{j=0}^{K}I_j,$$

and then, for $j\geqslant 1$,

$$\begin{aligned}\|u_1-u_2\|_{Y(I_j)}&\leqslant C_4\|u_1-u_2\|_{X(\{t_j\})}+\frac12\|u_1-u_2\|_{Y(I_j)}\\&\leqslant 2C_4\|u_1-u_2\|_{X(\{t_j\})}\leqslant 2C_4\|u_1-u_2\|_{Y(I_{j-1})}\\&\leqslant(2C_4)^{j+1}\|\phi_1-\phi_2\|_{X(\{t_0\})},\end{aligned}$$

by induction. Since $j\leqslant K$, we infer

$$\|u_1-u_2\|_{Y([t_0,\infty))}\leqslant C(M,d,\sigma)\|\phi_1-\phi_2\|_{X(\{t_0\})},$$

and the same obviously holds on $(-\infty,t_0]$. Hence the first point of the theorem is established.

In order to prove the second point, we do not follow the same strategy as in [4], where the H^2 regularity of u is read from the equation (1.1), after it is proved that $\partial_t u$ is in L^2 (this strategy does not require the nonlinearity to be more than C^1, and $\sigma>0$ is allowed). Since $\sigma\geqslant 1/2$, we may differentiate (1.1) twice in space, instead of once in time.

We check that since $d\sigma\geqslant 2$, γ, defined in (2.3), is such that $\gamma\geqslant q_0$, as

$$\frac{1}{q_0}-\frac{1}{\gamma}=\frac{d\sigma-2}{4\sigma}.$$

Therefore, there exists $\rho \geqslant 2$ such that (γ,ρ) is admissible, and

$$d\left(\frac{1}{\rho}-\frac{1}{r_0}\right) = d\left(\frac{1}{\rho}-\frac{1}{2}+\frac{1}{2}-\frac{1}{r_0}\right) = \frac{2}{q_0}-\frac{2}{\gamma} = \frac{d\sigma-2}{2\sigma} =: s.$$

We infer that $W^{s,\rho} \hookrightarrow L^{r_0}$, with s defined above, which is such that $s \in [0,1[$ since $2/d \leqslant \sigma < 2/(d-2)_+$. We note that for $A = J$, ∂_j and A do not commute (but the commutation bracket is of order zero, and is therefore harmless in the estimates). Applying $A \in \{\mathbf{1}, \nabla, J\}$ to the Duhamel's formula (2.1), then ∂_j, Strichartz estimates yield, for $t_0 \in I$,

$$\|\partial_j Au\|_{L^q(I;L^r)} \lesssim \|\partial_j A(t_0)u(t_0)\|_{L^2} + \left\|\partial_j A\left(|u|^{2\sigma}u\right)\right\|_{L^{q_0'}(I;L^{r_0'})}. \tag{2.13}$$

Now Hölder inequality entails

$$\begin{aligned}
\left\|\partial_j A\left(|u|^{2\sigma}u\right)\right\|_{L^{q_0'}(I;L^{r_0'})} &\lesssim \|u\|^{2\sigma}_{L^\gamma(I;L^{r_0})}\|\partial_j Au\|_{L^{q_0}(I;L^{r_0})} \\
&\quad + \|u\|^{2\sigma-1}_{L^\gamma(I;L^{r_0})}\|\partial_j u\|_{L^\gamma(I;L^{r_0})}\|Au\|_{L^{q_0}(I;L^{r_0})} \\
&\lesssim \|u\|^{2\sigma}_{L^\gamma(I;L^{r_0})}\|\partial_j Au\|_{L^{q_0}(I;L^{r_0})} \\
&\quad + \|u\|^{2\sigma-1}_{L^\gamma(I;L^{r_0})}\|\nabla u\|_{L^\gamma(I;W^{s,\rho})}\|Au\|_{L^{q_0}(I;L^{r_0})} \\
&\lesssim \|u\|^{2\sigma}_{L^\gamma(I;L^{r_0})}\|\partial_j Au\|_{L^{q_0}(I;L^{r_0})} \\
&\quad + \|u\|^{2\sigma-1}_{L^\gamma(I;L^{r_0})}\|u\|_{L^\gamma(I;W^{2,\rho})}\|Au\|_{L^{q_0}(I;L^{r_0})}.
\end{aligned}$$

If $\sigma > 1/2$, we use the same idea as for the first point of the theorem, namely

$$\|u(t)\|_{L^\gamma(I;L^{r_0})} \lesssim \| \langle t\rangle^{-\delta(r_0)} \|_{L^\gamma(I)},$$

and split $\mathbb{R}$ into finitely many intervals such that the last term in (2.13) can be absorbed by the left hand side on each of them. We conclude by (finite) induction like before. If $\sigma = 1/2$, we use in addition the property $Au \in L^{q_0}(\mathbb{R};L^{r_0})$ (from Theorem 2.3), and conclude similarly, by considering Strichartz estimates involving other admissible pairs, among them (γ,ρ). □

2.3. **Estimating the source term.** Suppose that u solves (1.1), and v solves

$$i\partial_t v + \frac{1}{2}\Delta v = i\frac{N(\tau)-\mathbf{1}}{\tau}v, \quad v_{|t=0} = \phi,$$

then Taylor formula shows that $w = u - v$ solves

$$i\partial_t w + \frac{1}{2}\Delta w = i\frac{N(\tau)-\mathbf{1}}{\tau}u - i\frac{N(\tau)-\mathbf{1}}{\tau}v + r_w,$$

where the source term is controlled pointwise by $|r_w| \lesssim |u|^{4\sigma+1}$. In terms of estimates, it is as if the power 2σ in (1.1) had been replaced by 4σ, which may no longer correspond to an energy-subcritical nonlinearity. This difficulty is overcome thanks to the introduction of the frequency cutoff Π_τ. The following lemma is a global in time version of [6, Lemma 2.8], and the proof resumes many of its ingredients:

Lemma 2.6. *There exists C independent of τ and the time interval I such that*

$$\left\||\Pi_\tau u|^{4\sigma+1}\right\|_{L^{q_0'}(I;L^{r_0'})} + \left\||\Pi_\tau u|^{2\sigma}\Pi_\tau(|u|^{2\sigma}u)\right\|_{L^{q_0'}(I;L^{r_0'})} \leqslant C\tau^{-1/2}\|u\|^{4\sigma+1}_{Y(I)},$$

where $Y(I)$ is defined by (2.10).

Proof. By Lemma 3.2 and Hölder inequality, we have

$$\left\||\Pi_\tau u|^{4\sigma+1}\right\|_{L^{q_0'}(I;L^{r_0'})} \leqslant C_{d,\sigma}\tau^{-\frac{d}{2}(\frac{4\sigma+1}{r_1}-\frac{1}{r_0'})}\|u\|^{4\sigma+1}_{L^{(4\sigma+1)q_0'}(I;L^{r_1})} \tag{2.14}$$

for all $r_1 \leqslant (4\sigma+1)r_0'$. The value of $r_1 > 0$ is chosen as

$$\frac{1}{r_1} = \frac{1}{4\sigma+1}\left(\frac{2\sigma+1}{2\sigma+2}+\frac{1}{d}\right) > \frac{1}{4\sigma+1}\frac{2\sigma+1}{2\sigma+2} = \frac{1}{(4\sigma+1)r_0'}.$$

We check that in view of (1.2), $r_1 \geqslant 2$. By definition, $r_1 \geqslant 2$ if and only if

$$d\sigma(4\sigma+3) \geqslant 2\sigma+2.$$

But $d\sigma \geqslant 2$, so the above inequality is satisfied. For $d = 1, 2$, we do not have to check anything else. Introduce now q_2 and r_2 given by

$$\frac{1}{q_2} := \frac{1}{(4\sigma+1)q_0'} = \frac{4(\sigma+1)-d\sigma}{4(\sigma+1)(4\sigma+1)} \quad \text{and} \quad \frac{1}{r_2} := \frac{1}{2}-\frac{2}{dq_2}.$$

We check that for $d \leqslant 5$, $1/q_2 > 0$, and $1/q_2 < 1/2$ since $\sigma \geqslant 2/d$: $q_2 \in (2,\infty)$, and if $d = 1$, $q_2 > 4$, hence, by definition of r_2, (q_2, r_2) is admissible. Sobolev embedding reads

$$W^{s,r_2} \hookrightarrow L^{r_1}, \quad \frac{1}{r_1}+\frac{s}{d} = \frac{1}{r_2} \Longleftrightarrow s = \frac{d}{2}-\frac{d+6}{8\sigma+2}.$$

We note that for $\frac{2}{d} \leqslant \sigma < \frac{2}{(d-2)_+}$, $s \in (0,1)$, hence (2.14) entails

$$\begin{aligned}\left\||\Pi_\tau u|^{4\sigma+1}\right\|_{L^{q_0'}(I;L^{r_0'})} &\lesssim \tau^{-1/2}\|u\|^{4\sigma+1}_{L^{q_2}(I;L^{r_1})} \lesssim \tau^{-1/2}\|u\|^{4\sigma+1}_{L^{q_2}(I;W^{s,r_2})}\\ &\lesssim \tau^{-1/2}\|u\|^{4\sigma+1}_{L^{q_2}(I;W^{1,r_2})} \lesssim \tau^{-1/2}\|u\|^{4\sigma+1}_{Y(I)},\end{aligned}$$

and the first estimate of the lemma follows.

For the second term of left hand side of the lemma, Hölder inequality yields:

$$\begin{aligned}\left\||\Pi_\tau u|^{2\sigma}\Pi_\tau(|u|^{2\sigma}u)\right\|_{L^{q_0'}(I;L^{r_0'})} &\leqslant \|\Pi_\tau u\|^{2\sigma}_{L^{(4\sigma+1)q_0'}(I;L^{(4\sigma+1)r_0'})}\\ &\quad\times\left\|\Pi_\tau(|u|^{2\sigma}u)\right\|_{L^{\frac{4\sigma+1}{2\sigma+1}q_0'}(I;L^{\frac{4\sigma+1}{2\sigma+1}r_0'})}.\end{aligned}$$

In view of Lemma 3.2, again since $r_1 < (4\sigma+1)r_0'$,

$$\|\Pi_\tau u\|_{L^{(4\sigma+1)q_0'}(I;L^{(4\sigma+1)r_0'})} \lesssim \tau^{\frac{d}{2}\left(\frac{1}{(4\sigma+1)r_0'}-\frac{1}{r_1}\right)}\|u\|_{L^{(4\sigma+1)q_0'}(I;L^{r_1})}.$$

Still from Lemma 3.2,

$$\left\|\Pi_\tau(|u|^{2\sigma}u)\right\|_{L^{\frac{4\sigma+1}{2\sigma+1}q_0'}(I;L^{\frac{4\sigma+1}{2\sigma+1}r_0'})} \lesssim \tau^{\frac{d}{2}\left(\frac{2\sigma+1}{(4\sigma+1)r_0'}-\frac{2\sigma+1}{r_1}\right)}\left\||u|^{2\sigma}u\right\|_{L^{\frac{4\sigma+1}{2\sigma+1}q_0'}(I;L^{\frac{r_1}{2\sigma+1}})}.$$

Therefore,

$$\begin{aligned}\left\||\Pi_\tau u|^{2\sigma}\Pi_\tau(|u|^{2\sigma}u)\right\|_{L^{q_0'}(I;L^{r_0'})} &\lesssim \tau^{\frac{d}{2}\left(\frac{1}{r_0'}-\frac{4\sigma+1}{r_1}\right)}\|u\|^{2\sigma}_{L^{(4\sigma+1)q_0'}(I;L^{r_1})}\\ &\quad\times\left\||u|^{2\sigma}u\right\|_{L^{\frac{4\sigma+1}{2\sigma+1}q_0'}(I;L^{\frac{r_1}{2\sigma+1}})}\\ &\lesssim \tau^{-1/2}\|u\|^{4\sigma+1}_{L^{(4\sigma+1)q_0'}(I;L^{r_1})} = \tau^{-1/2}\|u\|^{4\sigma+1}_{L^{q_2}(I;L^{r_1})},\end{aligned}$$

where we have used Hölder inequality for the last estimate, and we are back to the situation of the first case. □

3. Preliminary estimates: the numerical solution

The discrete (in time) counterpart of the $L^q_t L^r$ norms involved in the standard Strichartz estimates is:

$$\|u\|_{\ell^q(n\tau\in I;\, L^r)} = \Big(\tau \sum_{n\tau\in I} \|u(n\tau)\|_{L^r}^q\Big)^{1/q}.$$

For $a, n \in \mathbb{N}$, $a+1 \leqslant n$, the discrete Duhamel formula associated to Z_τ reads:

$$Z_\tau(n\tau) = S_\tau((n-a)\tau)Z_\tau(a\tau) + \tau\sum_{k=a}^{n-1} S_\tau(n\tau-k\tau)\frac{N(\tau)-\mathbf{1}}{\tau}Z_\tau(k\tau), \tag{3.1}$$

which is compared to the formula (2.1). We now list analogues of Proposition 2.1 in the discrete (in time) setting, a framework which is not quite as classical.

Proposition 3.1 ([16, Theorem 2.1])**.** *Let (q,r), (q_1,r_1) and (q_2,r_2) be any admissible pairs. Then, there exist $C_{d,q}, C_{d,q_1,q_2} > 0$ such that*

$$\|S_\tau(\cdot)\phi\|_{\ell^q(\tau\mathbb{Z};L^r)} \leqslant C_{d,q}\|\phi\|_{L^2}, \tag{3.2}$$

and

$$\left\|\tau\sum_{k=-\infty}^{n-1} S_\tau\left((n-k)\tau\right) f(k\tau)\right\|_{\ell^{q_1}(\tau\mathbb{Z};L^{r_1})} \leqslant C_{d,q_1,q_2}\|f\|_{\ell^{q_2'}(\tau\mathbb{Z};L^{r_2'})} \tag{3.3}$$

hold for all $\phi \in L^2$ and $f \in \ell^{q_2'}(\tau\mathbb{Z};L^{r_2'})$.

The following version of Bernstein lemma is borrowed from [6]:

Lemma 3.2. [6, Lemma 2.6] *For any $1 \leqslant q \leqslant r < \infty$ and $\phi : \mathbb{R}^d \to \mathbb{C}$, we have*

$$\big\|\Pi_\tau\phi - \phi\big\|_{L^r} \leqslant C\tau^{1/2}\big\|(-\Delta)^{1/2}\phi\big\|_{L^r}, \tag{3.4}$$

$$\|\Pi_\tau\phi\|_{L^r} \leqslant C\|\phi\|_{L^r}, \tag{3.5}$$

$$\big\|\nabla(\Pi_\tau\phi)\big\|_{L^r} \leqslant C\tau^{-\frac{1}{2}}\|\phi\|_{L^r}, \tag{3.6}$$

and

$$\|\Pi_\tau\phi\|_{L^r} \leqslant C\tau^{\frac{d}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}\|\phi\|_{L^q}. \tag{3.7}$$

An important technical novelty compared both with previous studies of (1.1), and with error estimates for time discretization, is that since we consider the vectorfield J in the presence of the truncated propagator S_τ, we face a lack of commutation: Π_τ commutes with $\mathbf{1}$ and ∇, but not with J. This lack of commutation turns out to be controlled thanks to the following lemma:

Lemma 3.3. *For any $1 < p < \infty$, there exists $C_p > 0$ such that*

$$\|J(t)\Pi_\tau\phi - \Pi_\tau J(t)\phi\|_{L^p} \leqslant C_p\tau^{1/2}\|\phi\|_{L^p}, \quad \forall\phi\in\Sigma, \quad \forall t\in\mathbb{R}. \tag{3.8}$$

Proof. The Fourier transform of $\Pi_\tau J(t)\phi$ is given by

$$\begin{aligned}\mathcal{F}\left(\Pi_\tau J(t)\phi\right)(\xi) &= \chi\left(\tau^{1/2}\xi\right)\left(i\nabla\hat\phi(\xi) - t\xi\hat\phi(\xi)\right)\\ &= i\nabla\big(\chi\big(\tau^{1/2}\xi\big)\hat\phi(\xi)\big) - t\xi\chi\big(\tau^{1/2}\xi\big)\hat\phi(\xi) - i\tau^{1/2}(\nabla\chi)\big(\tau^{1/2}\xi\big)\hat\phi(\xi)\\ &= \mathcal{F}\left(J(t)\Pi_\tau\phi\right) - i\tau^{1/2}(\nabla\chi)\left(\tau^{1/2}\xi\right)\hat\phi(\xi).\end{aligned}$$

The lemma follows from the boundedness of $\nabla\chi$ and basic Fourier multiplier theory, provided that $\nabla\chi \in C^k$ with $k > d/2$ (see e.g. [25, p. 96]). □

Since the operators S_τ and ∇ commute, the following corollary is immediate in the case $A \in \{\mathbf{1}, \nabla\}$, but requires more work in the case $A = J$.

Corollary 3.4. *Let (q,r), (q_1,r_1) and (q_2,r_2) be any admissible pairs. Then, there exist $C_{d,q}, C_{d,q_1,q_2} > 0$ such that for any $t_0 \in \tau\mathbb{Z}$ and $A \in \{\mathbf{1}, \nabla\}$,*

$$\|AS_\tau(\cdot - t_0)\phi\|_{\ell^q(\tau\mathbb{Z};L^r)} \leqslant C_{d,q}\|A\phi\|_{L^2}, \tag{3.9}$$

and

$$\left\|\tau A \sum_{k=-\infty}^{n-1} S_\tau((n-k)\tau)f(k\tau)\right\|_{\ell^{q_1}(\tau\mathbb{Z};L^{r_1})} \leqslant C_{d,q_1,q_2}\|Af\|_{\ell^{q_2'}(\tau\mathbb{Z};L^{r_2'})} \tag{3.10}$$

hold for all $\phi \in \Sigma$ and f such that $Af \in \ell^{q_2'}(\tau\mathbb{Z};L^{r_2'})$. While for $A = J$, we have

$$\|J(\cdot)S_\tau(\cdot - t_0)\phi\|_{\ell^q(\tau\mathbb{Z};L^r)} \leqslant C_{d,q}\left(\tau^{1/2}\|\phi\|_{L^2} + \|J(t_0)\phi\|_{L^2}\right), \tag{3.11}$$

$$\left\|\tau J(n\tau) \sum_{k=-\infty}^{n-1} S_\tau((n-k)\tau)f(k\tau)\right\|_{\ell^{q_1}(\tau\mathbb{Z};L^{r_1})}$$

$$\leqslant C_{d,q_1,q_2}\left(\tau^{1/2}\|f\|_{\ell^{q_2'}(\tau\mathbb{Z};L^{r_2'})} + \|Jf\|_{\ell^{q_2'}(\tau\mathbb{Z};L^{r_2'})}\right). \tag{3.12}$$

Proof. For $A = \{\mathbf{1}, \nabla\}$, (3.9) and (3.10) follow directly from Proposition 3.1 by noticing that A commutes with S_τ. On the other hand, J does not commute with S_τ. In view of the properties of χ, we have the identity

$$\Pi_\tau = \Pi_\tau\Pi_{\tau/4}, \quad \forall\tau > 0.$$

Therefore, using the standard notation $[A,B] = AB - BA$,

$$\begin{aligned} J(t)S_\tau(t-t_0)\phi &= J(t)S_\tau(t-t_0)\Pi_{\tau/4}\phi \\ &= [J(t),\Pi_\tau]S(t-t_0)\Pi_{\tau/4}\phi + \Pi_\tau J(t)S(t-t_0)\Pi_{\tau/4}\phi \\ &= [J(t),\Pi_\tau]S_{\tau/4}(t-t_0)\phi + \Pi_\tau J(t)S(t-t_0)\Pi_{\tau/4}\phi. \end{aligned}$$

Since $J(t) = S(t)xS(-t) = S(t-t_0)J(t_0)S(t_0-t)$, the last term is equal to

$$\begin{aligned} \Pi_\tau J(t)S(t-t_0)\Pi_{\tau/4}\phi &= \Pi_\tau S(t-t_0)J(t_0)\Pi_{\tau/4}\phi = S_\tau(t-t_0)J(t_0)\Pi_{\tau/4}\phi \\ &= S_\tau(t-t_0)[J(t_0),\Pi_{\tau/4}]\phi + S_\tau(t-t_0)\Pi_{\tau/4}J(t_0)\phi \\ &= S_\tau(t-t_0)[J(t_0),\Pi_{\tau/4}]\phi + S_\tau(t-t_0)J(t_0)\phi. \end{aligned}$$

We infer

$$\begin{aligned} &\|J(\cdot)S_\tau(\cdot-t_0)\phi\|_{\ell^q(\tau\mathbb{Z};L^r)} \\ &\quad\leqslant \|S_\tau(\cdot-t_0)J(t_0)\phi\|_{\ell^q(\tau\mathbb{Z};L^r)} + \|[J(\cdot),\Pi_\tau]S_{\tau/4}(\cdot-t_0)\phi\|_{\ell^q(\tau\mathbb{Z};L^r)} \\ &\qquad + \|S_\tau(\cdot-t_0)[J(t_0),\Pi_{\tau/4}]\phi\|_{\ell^q(\tau\mathbb{Z};L^r)} \\ &\quad\leqslant C_{d,q}\|J(t_0)\phi\|_{L^2} + C\tau^{1/2}\|S_{\tau/4}(\cdot-t_0)\phi\|_{\ell^q(\tau\mathbb{Z};L^r)} + C_{d,q}\|[J(t_0),\Pi_{\tau/4}]\phi\|_{L^2} \\ &\quad\leqslant \tilde{C}_{d,q}\left(\tau^{1/2}\|\phi\|_{L^2} + \|J(t_0)\phi\|_{L^2}\right), \end{aligned}$$

where we have used (3.2) and (3.8). Similarly, (3.12) can be established by applying (3.3) and (3.8). □

Propositions 2.1 and 3.1 and Christ-Kiselev lemma imply this slight extension (to incorporate J) of [16, Lemma 4.5]:

Corollary 3.5 ([16, Lemma 4.5])**.** *For any admissible pairs (q_1, r_1) and (q_2, r_2), we have, for $A \in \{\mathbf{1}, \nabla\}$,*

$$\left\| A \int_{s<n\tau} S_\tau(n\tau - s) f(s) ds \right\|_{\ell^{q_1}(\tau\mathbb{Z}; L^{r_1})} \leqslant C_{d,q_1,q_2} \left\| Af \right\|_{L^{q_2'}(\mathbb{R}; L^{r_2'})}, \tag{3.13}$$

and

$$\begin{aligned} &\left\| J(n\tau) \int_{s<n\tau} S_\tau(n\tau - s) f(s) ds \right\|_{\ell^{q_1}(\tau\mathbb{Z}; L^{r_1})} \leqslant \\ &\qquad \leqslant C_{d,q_1,q_2} \Big(\tau^{1/2} \|f\|_{L^{q_2'}(\mathbb{R}; L^{r_2'})} + \|Jf\|_{L^{q_2'}(\mathbb{R}; L^{r_2'})} \Big). \end{aligned} \tag{3.14}$$

Lemma 3.6. [6, Lemma 2.5] *There exists $c > 0$ such that*

$$\left| \frac{N(\tau) - \mathbf{1}}{\tau} v - \frac{N(\tau) - \mathbf{1}}{\tau} w \right| \leqslant c \left(|v|^{2\sigma} + |w|^{2\sigma} \right) |v - w| \tag{3.15}$$

and

$$\left| \frac{N(\tau) - \mathbf{1}}{\tau} v \right| = \left| \frac{\exp(-i\tau\lambda|v|^{2\sigma}) - 1}{\tau} v \right| \leqslant |v|^{2\sigma+1} \tag{3.16}$$

hold for all $v, w \in \mathbb{C}$. Furthermore, for weakly differentiable $f : \mathbb{R}^d \to \mathbb{C}$, we have the pointwise estimate

$$\left| \nabla \left(\frac{N(\tau) - \mathbf{1}}{\tau} f \right) \right| \leqslant (2\sigma + 1) |f|^{2\sigma} |\nabla f|. \tag{3.17}$$

Lemma 3.7. [6, Lemma 2.7] *For any admissible pairs (q_1, r_1) and (q_2, r_2), there is a constant $C_{d,q_1,q_2} > 0$ such that*

$$\begin{aligned} &\left\| \int_{s<n\tau} S_\tau(n\tau - s) f(s) ds - \tau \sum_{k=-\infty}^{n-1} S_\tau(n\tau - k\tau) f(k\tau) \right\|_{\ell^{q_1}(\tau\mathbb{Z}; L^{r_1})} \\ &\qquad \leqslant C_{d,q_1,q_2}\ \tau^{1/2} \|f\|_{L^{q_2'}(\mathbb{R}; W^{1,r_2'})} + C_{d,q_1,q_2}\ \tau \|\partial_t f\|_{L^{q_2'}(\mathbb{R}; L^{r_2'})} \end{aligned} \tag{3.18}$$

hold for any test function $f \in \mathcal{S}(\mathbb{R}^{d+1})$.

4. L^2 convergence

Proof of Theorem 1.3. In view of Theorem 2.3 and the assumption of Theorem 1.3, we see that u and Z_τ satisfy the following estimates

$$\begin{aligned} &\|u\|_{Y(\mathbb{R}_+)} \leqslant M_1, \\ &\max_{A \in \{\mathbf{1}, \nabla, J\}} \|A(n\tau) Z_\tau(n\tau)\|_{\ell^\infty(\tau\mathbb{N}; L^2)} + \|Z_\tau(n\tau)\|_{\ell^{q_0}(\tau\mathbb{N}; W^{1,r_0})} \leqslant M_2. \end{aligned} \tag{4.1}$$

We shall estimate $Z_\tau(n\tau) - \Pi_\tau u(n\tau)$ instead of $Z_\tau(n\tau) - u(n\tau)$, in view of

$$\|u(n\tau) - \Pi_\tau u(n\tau)\|_{\ell^\infty(I; L^2)} \leqslant C\tau^{1/2} \|u(n\tau)\|_{\ell^\infty(I; H^1)} \leqslant C\tau^{1/2} M_1,$$

by (3.4). We recall that $\gamma < \infty$ is defined by (2.3). We recall that $\gamma\delta(r_0) > 1$ (see (2.11)). Therefore, in view of (1.5), (4.1) and (2.6), for any $\eta > 0$, we can find a

finite number $K = K(\eta)$ of time intervals $I_j = \overline{[m_j, m_{j+1})}$ with $m_j \in \mathbb{N}$ if $j \leqslant K$, $m_{K+1} = \infty$, and $\tau(\eta) > 0$ such that for $1 \leqslant j \leqslant K$ and $0 < \tau \leqslant \tau(\eta)$,

$$\|u(k\tau)\|_{\ell^\gamma(\tau I_j; L^{r_0})} + \|Z_\tau(k\tau)\|_{\ell^\gamma(\tau I_j; L^{r_0})} \leqslant \eta, \quad \mathbb{R}_+ = \bigcup_{j=1}^{K} I_j.$$

We consider the adherence of I_j because unlike in the continuous case, for $j+1 \leqslant K$, the singleton $\{m_{j+1}\}$ is not of measure zero, and actually becomes important in the following discussion. On each interval I_j, the discrete Duhamel's formula (3.1) can be written as

$$Z_\tau(m_j\tau+n\tau) = S_\tau(n\tau)Z_\tau(m_j\tau)+\tau\sum_{k=0}^{n-1} S_\tau(n\tau-k\tau)\frac{N(\tau)-\mathbf{1}}{\tau}Z_\tau(m_j\tau+k\tau), \tag{4.2}$$

for $0 \leqslant n < m_{j+1} - m_j$ (m_{j+1} may be infinite, but m_j is always finite). By combining this with (2.1), we obtain the following decomposition:

$$Z_\tau(m_j\tau + n\tau) - \Pi_\tau u(m_j\tau + n\tau) = \mathcal{A}_1(n) + \mathcal{A}_2(n) + \mathcal{A}_3(n) + \mathcal{A}_4(n), \tag{4.3}$$

where

$$\begin{aligned}
\mathcal{A}_1(n) &:= S_\tau(n\tau)\left(Z_\tau(m_j\tau) - \Pi_\tau u(m_j\tau)\right),\\
\mathcal{A}_2(n) &:= S_\tau(n\tau)\left(\Pi_\tau u(m_j\tau) - u(m_j\tau)\right),\\
\mathcal{A}_3(n) &:= \tau\sum_{k=0}^{n-1} S_\tau(n\tau - k\tau)\Big(\frac{N(\tau)-\mathbf{1}}{\tau}Z_\tau(m_j\tau + k\tau) - \frac{N(\tau)-\mathbf{1}}{\tau}\Pi_\tau u(m_j\tau + k\tau)\Big),\\
\mathcal{A}_4(n) &:= \tau\sum_{k=0}^{n-1} S_\tau(n\tau - k\tau)\frac{N(\tau)-\mathbf{1}}{\tau}\Pi_\tau u(m_j\tau + k\tau)\\
&\quad + i\int_0^{n\tau} S_\tau(n\tau - s)\left(|u|^{2\sigma}u\right)(m_j\tau + s)ds,
\end{aligned}$$

and we omit the dependence of the $\mathcal{A}_k$'s upon j to ease notations. The terms $\mathcal{A}_1$ and $\mathcal{A}_2$ are linear, while $\mathcal{A}_3$ and $\mathcal{A}_4$ are nonlinear. The goal is to show that in the estimates, the term $\mathcal{A}_3$ can be absorbed by the left hand side of (4.3), $\mathcal{A}_2$ and $\mathcal{A}_4$ are $\mathcal{O}(\tau^{1/2})$, and $\mathcal{A}_1$ is then estimated by induction on j.

Let $(q, r) \in \{(q_0, r_0), (\infty, 2)\}$. The homogeneous Strichartz estimate (3.2) yields

$$\|\mathcal{A}_1\|_{\ell^q(\tau I_j; L^r)} \leqslant C_{d,q}\left\|Z_\tau(m_j\tau) - \Pi_\tau u(m_j\tau)\right\|_{L^2}.$$

The second term is controlled via (3.2) and (3.4), by

$$\begin{aligned}
\|\mathcal{A}_2\|_{\ell^q(\tau I_j; L^r)} &= \left\|S_\tau(n\tau)\left(\Pi_\tau u(m_j\tau) - u(m_j\tau)\right)\right\|_{\ell^q(\tau I_j; L^r)}\\
&\leqslant C_{d,q}\left\|\Pi_\tau u(m_j\tau) - u(m_j\tau)\right\|_{L^2}\\
&\leqslant C_{d,q}\tau^{1/2}\left\|(-\Delta)^{1/2}u(m_j\tau)\right\|_{L^2} \leqslant C_{d,q}\tau^{1/2}M_1,
\end{aligned}$$

where we have used (4.1). To estimate $\mathcal{A}_3$, we use (3.15) to find

$$\left|\frac{N(\tau)-\mathbf{1}}{\tau}Z_\tau - \frac{N(\tau)-\mathbf{1}}{\tau}\Pi_\tau u\right| \lesssim \left(|Z_\tau|^{2\sigma} + |\Pi_\tau u|^{2\sigma}\right)|Z_\tau - \Pi_\tau u|.$$

The inhomogeneous Strichartz estimate (3.3) and Hölder inequality (2.4) yield

$$\begin{aligned}\|\mathcal{A}_3\|_{\ell^q(\tau I_j;L^r)} &\lesssim \left\| \left(|Z_\tau|^{2\sigma}+|\Pi_\tau u|^{2\sigma}\right)|Z_\tau-\Pi_\tau u|\right\|_{\ell^{q_0'}(\tau I_j;L^{r_0'})}\\ &\lesssim \left(\|Z_\tau\|^{2\sigma}_{\ell^\gamma(\tau I_j;L^{r_0})}+\|\Pi_\tau u\|^{2\sigma}_{\ell^\gamma(\tau I_j;L^{r_0})}\right)\|Z_\tau-\Pi_\tau u\|_{\ell^{q_0}(\tau I_j;L^{r_0})}\\ &\leqslant \mathbf{C}_{d,\sigma,q}\eta^{2\sigma}\|Z_\tau-\Pi_\tau u\|_{\ell^{q_0}(\tau I_j;L^{r_0})},\end{aligned}$$

for some constant $\mathbf{C}_{d,\sigma,q}$, where we have used the definition of the intervals I_j in terms of η. We now choose $\eta>0$ sufficiently small so that $\mathbf{C}_{d,\sigma,q_0}\eta^{2\sigma}<1/2$, thus

$$\|\mathcal{A}_3\|_{\ell^{q_0}(\tau I_j;L^{r_0})}\leqslant\frac{1}{2}\|Z_\tau-\Pi_\tau u\|_{\ell^{q_0}(\tau I_j;L^{r_0})},\quad 1\leqslant j\leqslant K.$$

The estimate of $\mathcal{A}_4$ is postponed to Lemma 4.1 below,

$$\max_{1\leqslant j\leqslant K}\left(\|\mathcal{A}_4\|_{\ell^\infty(\tau I_j;L^2)}+\|\mathcal{A}_4\|_{\ell^{q_0}(\tau I_j;L^{r_0})}\right)\lesssim\tau^{1/2}. \tag{4.4}$$

Gathering all the previous estimates, we first get, with $(q,r)=(q_0,r_0)$,

$$\begin{aligned}\|Z_\tau-\Pi_\tau u\|_{\ell^{q_0}(\tau I_j;L^{r_0})}&\leqslant C\left\|Z_\tau(m_j\tau)-\Pi_\tau u(m_j\tau)\right\|_{L^2}\\ &\quad+C\tau^{1/2}+\frac{1}{2}\|Z_\tau-\Pi_\tau u\|_{\ell^{q_0}(\tau I_j;L^{r_0})},\end{aligned}$$

hence

$$\|Z_\tau-\Pi_\tau u\|_{\ell^{q_0}(\tau I_j;L^{r_0})}\leqslant 2C\left\|Z_\tau(m_j\tau)-\Pi_\tau u(m_j\tau)\right\|_{L^2}+2C\tau^{1/2},\quad 1\leqslant j\leqslant K.$$

Taking now $(q,r)=(\infty,2)$, we obtain

$$\begin{aligned}\|Z_\tau-\Pi_\tau u\|_{\ell^\infty(\tau I_j;L^2)}&\lesssim\left\|Z_\tau(m_j\tau)-\Pi_\tau u(m_j\tau)\right\|_{L^2}+\tau^{1/2}+\|Z_\tau-\Pi_\tau u\|_{\ell^{q_0}(\tau I_j;L^{r_0})}\\ &\lesssim\left\|Z_\tau(m_j\tau)-\Pi_\tau u(m_j\tau)\right\|_{L^2}+\tau^{1/2},\quad 1\leqslant j\leqslant K.\end{aligned}$$

Now by construction $m_1=0$, $Z_\tau(m_1\tau)-\Pi_\tau u(m_1\tau)=0$, and for $2\leqslant j\leqslant K$,

$$\left\|Z_\tau(m_j\tau)-\Pi_\tau u(m_j\tau)\right\|_{L^2}\leqslant\left\|Z_\tau-\Pi_\tau u\right\|_{\ell^\infty(\tau I_{j-1};L^2)},$$

we have the first conclusion in Theorem 1.3. The second one is then a direct consequence of this first point and the scattering result recalled in Theorem 2.3: there exists $u_+\in\Sigma$ such that

$$\|u(t)-S(t)u_+\|_{L^2}\underset{t\to\infty}{\longrightarrow}0,$$

where we have used the fact that $S(t)$ is unitary on L^2. □

We conclude this section by proving (4.4).

Lemma 4.1. *For $u\in Y(\mathbb{R}_+)$, $\tau\in(0,1)$, denote*

$$\mathcal{A}(u)(n\tau)=\tau\sum_{k=0}^{n-1}S_\tau(n\tau-k\tau)\frac{N(\tau)-\mathbf{1}}{\tau}\Pi_\tau u(k\tau)+i\int_0^{n\tau}S_\tau(n\tau-s)|u|^{2\sigma}u(s)ds.$$

Then for all admissible pairs (q,r),

$$\|\mathcal{A}(u)\|_{\ell^q(\tau\mathbb{N};L^r)}\lesssim\tau^{1/2}\left(\|u\|^{2\sigma+1}_{Y(\mathbb{R}_+)}+\|u\|^{4\sigma+1}_{Y(\mathbb{R}_+)}\right).$$

Proof. As observed in the previous computations, we recall that the assumptions of Lemma 4.1 imply

$$u \in L^\gamma(\mathbb{R}_+; L^{r_0}),$$

where γ is given by (2.3), since $\gamma\delta(r_0) > 1$. Decompose $\mathcal{A}(u)(n\tau)$ as

$$\mathcal{A}(u)(n\tau) = \mathcal{A}_1(u)(n\tau) + \mathcal{A}_2(u)(n\tau),$$

where

$$\mathcal{A}_1(u)(n\tau) = \tau \sum_{k=0}^{n-1} S_\tau(n\tau - k\tau)\mathcal{B}_1(u)(k\tau) - \int_0^{n\tau} S_\tau(n\tau - s)\mathcal{B}_1(u)(s)ds,$$
$$\mathcal{A}_2(u)(n\tau) = \int_0^{n\tau} S_\tau(n\tau - s)\mathcal{B}_1(u)(s)ds + i\int_0^{n\tau} S_\tau(n\tau - s)|u|^{2\sigma}u(s)ds,$$

with

$$\mathcal{B}_1(u)(s) := \frac{N(\tau) - \mathbf{1}}{\tau}\Pi_\tau u(s).$$

By Lemma 3.7, we can estimate

$$\|\mathcal{A}_1(u)\|_{\ell^q(\tau\mathbb{N};L^r)} \leqslant C\tau^{1/2}\|\mathcal{B}_1(u)\|_{L^{q_0'}(\mathbb{R}_+;W^{1,r_0'})} + C\tau\|\mathcal{B}_1(u)\|_{W^{1,q_0'}(\mathbb{R}_+;L^{r_0'})},$$

for all admissible pair (q, r). Lemma 3.6 entails

$$\|\mathcal{B}_1(u)\|_{L^{q_0'}(\mathbb{R}_+;W^{1,r_0'})} \lesssim \left\||\Pi_\tau u|^{2\sigma+1}\right\|_{L^{q_0'}(\mathbb{R}_+;L^{r_0'})} + \left\||\Pi_\tau u|^{2\sigma}|\nabla\Pi_\tau u|\right\|_{L^{q_0'}(\mathbb{R}_+;L^{r_0'})}.$$

Using now Hölder inequality (2.4) and Lemma 3.2,

$$\begin{aligned}\|\mathcal{B}_1(u)\|_{L^{q_0'}(\mathbb{R}_+;W^{1,r_0'})} &\lesssim \|\Pi_\tau u\|^{2\sigma}_{L^\gamma(\mathbb{R}_+;L^{r_0})}\|\Pi_\tau u\|_{L^{q_0}(\mathbb{R}_+;W^{1,r_0})}\\ &\lesssim \|u\|^{2\sigma}_{L^\gamma(\mathbb{R}_+;L^{r_0})}\|u\|_{L^{q_0}(\mathbb{R}_+;W^{1,r_0})} \lesssim \|u\|^{2\sigma+1}_{Y(\mathbb{R}_+)}.\end{aligned}$$

For the other term, Lemma 3.6 yields

$$\begin{aligned}\|\mathcal{B}_1(u)\|_{W^{1,q_0'}(\mathbb{R}_+;L^{r_0'})} &= \left\|\partial_t\left(\frac{N(\tau) - \mathbf{1}}{\tau}\Pi_\tau u\right)\right\|_{L^{q_0'}(\mathbb{R}_+;L^{r_0'})}\\ &\lesssim \left\||\Pi_\tau u|^{2\sigma}\partial_t\Pi_\tau u\right\|_{L^{q_0'}(\mathbb{R}_+;L^{r_0'})}.\end{aligned}$$

Observing that ∂_t and Π_τ commute, (1.1) implies

$$\begin{aligned}\|\mathcal{B}_1(u)\|_{W^{1,q_0'}(\mathbb{R}_+;L^{r_0'})} \lesssim{}& \left\||\Pi_\tau u|^{2\sigma}\Pi_\tau\Delta u\right\|_{L^{q_0'}(\mathbb{R}_+;L^{r_0'})}\\ &+ \left\||\Pi_\tau u|^{2\sigma}\Pi_\tau\left(|u|^{2\sigma}u\right)\right\|_{L^{q_0'}(\mathbb{R}_+;L^{r_0'})}.\end{aligned}$$

For the first term of the right side, Hölder inequality (2.4) and Lemma 3.2 imply

$$\begin{aligned}\left\||\Pi_\tau u|^{2\sigma}\Pi_\tau\Delta u\right\|_{L^{q_0'}(\mathbb{R}_+;L^{r_0'})} &\leqslant \|\Pi_\tau u\|^{2\sigma}_{L^\gamma(\mathbb{R}_+;L^{r_0})}\|\Pi_\tau\Delta u\|_{L^{q_0}(\mathbb{R}_+;L^{r_0})}\\ &\lesssim \tau^{-1/2}\|\Pi_\tau u\|^{2\sigma}_{L^\gamma(\mathbb{R}_+;L^{r_0})}\|\Pi_\tau\nabla u\|_{L^{q_0}(\mathbb{R}_+;L^{r_0})}\\ &\lesssim \tau^{-1/2}\|u\|^{2\sigma}_{L^\gamma(\mathbb{R}_+;L^{r_0})}\|\nabla u\|_{L^{q_0}(\mathbb{R}_+;L^{r_0})}\\ &\lesssim \tau^{-1/2}\|u\|^{2\sigma+1}_{Y(\mathbb{R}_+)}.\end{aligned}$$

Lemma 2.6 provides the following control for the second term of the right side:

$$\left\||\Pi_\tau u|^{2\sigma}\Pi_\tau\left(|u|^{2\sigma}u\right)\right\|_{L^{q_0'}(\mathbb{R}_+;L^{r_0'})} \lesssim \tau^{-1/2}\|u\|^{4\sigma+1}_{Y(\mathbb{R}_+)},$$

and we come up with

$$\|\mathcal{A}_1(u)\|_{\ell^q(\tau\mathbb{N};L^r)} \lesssim \tau^{1/2}\|\mathcal{B}_1(u)\|_{L^{q_0'}(\mathbb{R}_+;W^{1,r_0'})} + \tau\|\mathcal{B}_1(u)\|_{W^{1,q_0'}(\mathbb{R}_+;L^{r_0'})}$$

$$\lesssim \tau^{1/2}\big(\|u\|^{2\sigma+1}_{Y(\mathbb{R}_+)} + \|u\|^{4\sigma+1}_{Y(\mathbb{R}_+)}\big).$$

To complete the proof, we perform another decomposition,

$$\mathcal{B}_1(u) + i|u|^{2\sigma}u = \mathcal{B}_2(u) + i\mathcal{B}_3(u),$$

where

$$\mathcal{B}_2(u) := \mathcal{B}_1(u) + i|\Pi_\tau u|^{2\sigma}\Pi_\tau u, \quad \mathcal{B}_3(u) := |u|^{2\sigma}u - |\Pi_\tau u|^{2\sigma}\Pi_\tau u.$$

The discrete inhomogeneous Strichartz estimates (3.13) yields, for (q,r) admissible,

$$\begin{aligned}
&\|\mathcal{A}_2(u)\|_{\ell^q(\tau\mathbb{N};L^r)}\\
&\leqslant \Big\| \int_0^{n\tau} S_\tau(n\tau - s)\mathcal{B}_2(u)ds\Big\|_{\ell^q(\tau\mathbb{N};L^r)} + \Big\| \int_0^{n\tau} S_\tau(n\tau - s)\mathcal{B}_3(u)ds\Big\|_{\ell^q(\tau\mathbb{N};L^r)}\\
&\lesssim \|\mathcal{B}_2(u)\|_{L^{q_0'}(\mathbb{R}_+;L^{r_0'})} + \|\mathcal{B}_3(u)\|_{L^{q_0'}(\mathbb{R}_+;L^{r_0'})}.
\end{aligned}$$

Recall that $N(\tau)z = ze^{-i\tau|z|^{2\sigma}}$, Taylor formula yields the pointwise estimate

$$|\mathcal{B}_2(u)| = \Big|\frac{N(\tau)-\mathbf{1}}{\tau}\Pi_\tau u + i|\Pi_\tau u|^{2\sigma}\Pi_\tau u\Big| \lesssim \tau|\Pi_\tau u|^{4\sigma+1}.$$

Thus, we have

$$\|\mathcal{B}_2(u)\|_{L^{q_0'}(\mathbb{R}_+;L^{r_0'})} \lesssim \tau\,\big\||\Pi_\tau u|^{4\sigma+1}\big\|_{L^{q_0'}(\mathbb{R}_+;L^{r_0'})} \lesssim \tau^{1/2}\|u\|^{4\sigma+1}_{Y(\mathbb{R}_+)},$$

where we have used Lemma 2.6 for the last estimate.

Finally, Hölder inequality (2.4) yields

$$\begin{aligned}
\|\mathcal{B}_3(u)\|_{L^{q_0'}(\mathbb{R}_+;L^{r_0'})} &= \big\||\Pi_\tau u|^{2\sigma}\Pi_\tau u - |u|^{2\sigma}u\big\|_{L^{q_0'}(\mathbb{R}_+;L^{r_0'})}\\
&\lesssim \Big(\|\Pi_\tau u\|^{2\sigma}_{L^\gamma(\mathbb{R}_+;L^{r_0})} + \|u\|^{2\sigma}_{L^\gamma(\mathbb{R}_+;L^{r_0})}\Big)\|\Pi_\tau u - u\|_{L^{q_0}(\mathbb{R}_+;L^{r_0})}\\
&\lesssim \tau^{1/2}\|u\|^{2\sigma}_{Y(\mathbb{R}_+)}\|u\|_{L^{q_0}(\mathbb{R}_+;W^{1,r_0})} \lesssim \tau^{1/2}\|u\|^{2\sigma+1}_{Y(\mathbb{R}_+)},
\end{aligned}$$

where we have used Lemma 3.2. Lemma 4.1 follows. □

5. Local stability

We denote

$$\|v\|_{\Gamma(I)} = \|v\|_{\ell^\infty(I;L^2)} + \|\nabla v\|_{\ell^\infty(I;L^2)} + \|Jv\|_{\ell^\infty(I;L^2)}, \tag{5.1}$$

the discrete counterpart of the norm $\|\cdot\|_{X(I)}$ defined in (2.9). Define

$$K_0 := 1 + \sup_{\tau\in(0,1)}\sup_{\phi\in L^2}\frac{\|S_\tau(\cdot)\phi\|_{\ell^{q_0}(\tau\mathbb{Z};L^{r_0})} + \|S_\tau(\cdot)\phi\|_{\ell^\infty(\tau\mathbb{Z};L^2)}}{\|\phi\|_{L^2}},$$

which is finite in view of the discrete homogeneous Strichartz estimates (3.2).

Recall that in accordance with (1.5), we have, for $1 \leqslant q < \infty$,

$$\Big\|\langle n\tau\rangle^{-\alpha}\Big\|_{\ell^q(I)} = \Big(\tau\sum_{n\tau\in I}\langle n\tau\rangle^{-\alpha q}\Big)^{1/q}.$$

Proposition 5.1 (Local Σ stability)**.** *Let $\phi \in \Sigma$. Consider an interval I of the form $I = \overline{[a\tau, b\tau)}$, with $a, b \in \mathbb{N}\cup\{\infty\}$ such that $0 \leqslant a < b \leqslant \infty$. There exist $\tau_0 > 0$ and K_1 depending only on d and σ such that if*

$$6K_1\Big\|\langle n\tau\rangle^{-\delta(r_0)}\Big\|^{2\sigma}_{\ell^\gamma(I)} \leqslant \big(4K_0\|Z_\tau\|_{X(\{a\tau\})}\big)^{-2\sigma}, \tag{5.2}$$

then

(5.3)
$$\max_{A\in\{\mathbf{1},\nabla,J\}}\sum_{(q,r)\in\{(q_0,r_0),(\infty,2)\}}\|AZ_\tau\|_{\ell^q(I;L^r)}\leqslant 4K_0\|Z_\tau\|_{X(\{a\tau\})},\quad \forall\tau\in(0,\tau_0),$$

and for all (q,r) admissible, there exists K_q such that

$$\max_{A\in\{\mathbf{1},\nabla,J\}}\|AZ_\tau\|_{\ell^q(I;L^r)}\leqslant K_q\|Z_\tau\|_{X(\{a\tau\})},\quad \forall\tau\in(0,\tau_0). \tag{5.4}$$

Remark 5.2. We call the above result a *local* stability, since we require some smallness on
$$\tau\sum_{n=a}^{b}\frac{1}{\langle n\tau\rangle^{\gamma\delta(r_0)}}.$$
Obviously, the series is convergent, since $\gamma\delta(r_0)>1$, and this smallness means that either b is finite and $b-a$ is sufficiently small, or $b=\infty$ and a is sufficiently large, a case which corresponds to a local result near $t=\infty$ (in the spirit of the construction of wave operators in scattering theory, see e.g [4, 10]). More precisely, for finite b, we may use the estimate
$$\tau\sum_{n=a}^{b}\frac{1}{\langle n\tau\rangle^{\gamma\delta(r_0)}}\leqslant\tau\sum_{n=a}^{b}1=(b-a)\tau=|I|.$$
For $a\geqslant 1$ (and large), we may use instead,
$$\tau\sum_{n=a}^{b}\frac{1}{\langle n\tau\rangle^{\gamma\delta(r_0)}}\leqslant\tau\sum_{n=a}^{\infty}\frac{1}{(n\tau)^{\gamma\delta(r_0)}}=\tau^{1-\gamma\delta(r_0)}\sum_{n=a}^{\infty}\frac{1}{n^{\gamma\delta(r_0)}}\lesssim\frac{1}{(a\tau)^{\gamma\delta(r_0)-1}}.$$

Proof. Consider the set
$$\Lambda=\Big\{N\in\mathbb{N};\max_{A\in\{\mathbf{1},\nabla,J\}}\sum_{(q,r)\in\{(q_0,r_0),(\infty,2)\}}\|AZ_\tau\|_{\ell^q([a\tau,(N+a)\tau];L^r)}\leqslant 4K_0\|Z_\tau\|_{X(\{a\tau\})}\Big\}.$$
If Λ is an infinite set, then the conclusion of proposition follows easily, from the estimates presented below, based on Strichartz estimates.

Suppose that Λ is a finite set, and let N_* be the largest element of Λ. First we verify that the set Λ is non-empty. Indeed, by the definitions of Z_τ and K_0, we find that, for $A\in\{\mathbf{1},\nabla\}$,
$$\begin{aligned}\tau^{\frac{1}{q_0}}\|AZ_\tau(a\tau)\|_{L^{r_0}}+\|AZ_\tau(a\tau)\|_{L^2}&=\tau^{\frac{1}{q_0}}\|S_\tau(0)AZ_\tau(a\tau)\|_{L^{r_0}}+\|S_\tau(0)AZ_\tau(a\tau)\|_{L^2}\\&\leqslant\|S_\tau(\cdot)AZ_\tau(a\tau)\|_{\ell^{q_0}(\tau\mathbb{Z};L^{r_0})}+\|S_\tau(\cdot)AZ_\tau(a\tau)\|_{\ell^\infty(\tau\mathbb{Z};L^2)}\\&\leqslant(K_0-1)\|AZ_\tau(a\tau)\|_{L^2}\leqslant(K_0-1)\|Z_\tau\|_{X(\{a\tau\})}.\end{aligned}$$
While for $A=J$, applying Lemma 3.3, we get
$$\begin{aligned}&\tau^{\frac{1}{q_0}}\|J(a\tau)Z_\tau(a\tau)\|_{L^{r_0}}+\|J(a\tau)Z_\tau(a\tau)\|_{L^2}\\&\leqslant\tau^{\frac{1}{q_0}}\|S_\tau(0)J(a\tau)Z_\tau(a\tau)\|_{L^{r_0}}+\|S_\tau(0)J(a\tau)Z_\tau(a\tau)\|_{L^2}\\&\quad+C\tau^{1/2}\Big(\tau^{\frac{1}{q_0}}\|Z_\tau(a\tau)\|_{L^{r_0}}+\|Z_\tau(a\tau)\|_{L^2}\Big)\\&\leqslant\|S_\tau(\cdot)J(a\tau)Z_\tau(a\tau)\|_{\ell^{q_0}(\tau\mathbb{Z};L^{r_0})}+\|S_\tau(\cdot)J(a\tau)Z_\tau(a\tau)\|_{\ell^\infty(\tau\mathbb{Z};L^2)}\\&\quad+C\tau^{1/2}(K_0-1)\|Z_\tau\|_{X(\{a\tau\})}\\&\leqslant(K_0-1)\|J(a\tau)Z_\tau(a\tau)\|_{L^2}+C\tau^{1/2}(K_0-1)\|Z_\tau\|_{X(\{a\tau\})}\end{aligned}$$

$$\begin{aligned}
&\leqslant (1+C\tau^{1/2})(K_0-1)\|Z_\tau\|_{X(\{a\tau\})}\\
&< 4K_0\|Z_\tau\|_{X(\{a\tau\})},
\end{aligned}$$

when $\tau\leqslant\tau_0:=1/4C^2$, where C is the constant in Lemma 3.3. Therefore, $0\in\Lambda$.

For $A\in\{\mathbf{1},\nabla\}$, and $(q,r)\in\{(q_0,r_0),(\infty,2)\}$,

$$\begin{aligned}
&\|AZ_\tau(n\tau)\|_{\ell^q(a\leqslant n\leqslant a+N_*+1;L^r)}\\
&\leqslant \|AS_\tau(n\tau-a\tau)Z_\tau(a\tau)\|_{\ell^q(a\leqslant n\leqslant a+N_*+1;L^r)}\\
&\quad+\left\|\tau A\sum_{k=a}^{n-1}S_\tau(n\tau-k\tau)\frac{N(\tau)-\mathbf{1}}{\tau}Z_\tau(k\tau)\right\|_{\ell^q(a+1\leqslant n\leqslant a+N_*+1;L^r)}\\
&\leqslant \|S_\tau(n\tau-a\tau)AZ_\tau(a\tau)\|_{\ell^q(a\leqslant n\leqslant a+N_*+1;L^r)}\\
&\quad+\left\|\tau\sum_{k=a}^{n-1}S_\tau(n\tau-k\tau)A\frac{N(\tau)-\mathbf{1}}{\tau}Z_\tau(k\tau)\right\|_{\ell^q(a+1\leqslant n\leqslant a+N_*+1;L^r)}\\
&\leqslant (K_0-1)\|Z_\tau\|_{X(\{a\tau\})}+Q_1,
\end{aligned}\tag{5.5}$$

where the last term Q_1 is bounded by applying the Strichartz estimate (3.10) as follows:

$$\begin{aligned}
Q_1&=\left\|\tau\sum_{k=a}^{n-1}S_\tau(n\tau-k\tau)A\frac{N(\tau)-\mathbf{1}}{\tau}Z_\tau(k\tau)\right\|_{\ell^q(a+1\leqslant n\leqslant a+N_*+1;L^r)}\\
&\lesssim\left\|A\frac{N(\tau)-\mathbf{1}}{\tau}Z_\tau(n\tau)\right\|_{\ell^{q_0'}(a\leqslant n\leqslant a+N_*;L^{r_0'})}.
\end{aligned}$$

To estimate the right hand side, we apply Lemma 3.6 and Hölder inequality (2.4). We omit the information $a\leqslant n\leqslant a+N_*$ to ease notations:

$$\begin{aligned}
\left\|A\frac{N(\tau)-\mathbf{1}}{\tau}Z_\tau\right\|_{\ell^{q_0'}L^{r_0'}}&\lesssim\|Z_\tau\|^{2\sigma}_{\ell^\gamma L^{r_0}}\|AZ_\tau\|_{\ell^{q_0}L^{r_0}}\\
&\lesssim\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|^{2\sigma}_{\ell^\gamma(a\leqslant n\leqslant a+N^*)}\|Z_\tau\|^{2\sigma}_{\Gamma([a\tau,(a+N_*)\tau])}\|AZ_\tau\|_{\ell^{q_0}L^{r_0}},
\end{aligned}$$

where we have used (1.5), and (weighted) Gagliardo-Nirenberg inequality for the last estimate. Since $N_*\in\Lambda$, we infer that

$$\left\|A\frac{N(\tau)-\mathbf{1}}{\tau}Z_\tau\right\|_{\ell^{q_0'}L^{r_0'}}\lesssim\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|^{2\sigma}_{\ell^\gamma(a\leqslant n\leqslant a+N^*)}\left(4K_0\|Z_\tau\|_{X(\{a\tau\})}\right)^{2\sigma+1}.$$

We infer that there exists K_1 depending only on d and σ such that, for any $A\in\{\mathbf{1},\nabla\}$,

$$\begin{aligned}
\sum_{(q,r)\in\{(q_0,r_0),(\infty,2)\}}\|AZ_\tau\|_{\ell^q(a\leqslant n\leqslant a+N_*+1;L^r)}&\leqslant 2K_0\|Z_\tau\|_{X(\{a\tau\})}\\
&\quad+K_1\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|^{2\sigma}_{\ell^\gamma(a\leqslant n\leqslant a+N_*)}\left(4K_0\|Z_\tau\|_{X(\{a\tau\})}\right)^{2\sigma+1}.
\end{aligned}$$

We conclude with the case $A=J$. We have from (3.1),

$$\begin{aligned}
J(n\tau)Z_\tau(n\tau)&=J(n\tau)S_\tau((n-a)\tau)Z_\tau(a\tau)\\
&\quad+\tau\sum_{k=a}^{n-1}J(n\tau)S_\tau(n\tau-k\tau)\frac{N(\tau)-\mathbf{1}}{\tau}Z_\tau(k\tau).
\end{aligned}$$

We proceed like in the proof of Corollary 3.4: $J(t) = S(t-t_0)J(t_0)S(t_0-t)$, hence

$$\begin{aligned} J(n\tau)S_\tau((n-a)\tau) &= J(n\tau)S_\tau((n-a)\tau)\Pi_{\tau/4} \\ &= [J(n\tau),\Pi_\tau]S((n-a)\tau)\Pi_{\tau/4} + \Pi_\tau J(n\tau)S((n-a)\tau)\Pi_{\tau/4} \\ &= [J(n\tau),\Pi_\tau]S_{\tau/4}((n-a)\tau) + \Pi_\tau S((n-a)\tau)J(a\tau)\Pi_{\tau/4} \\ &= [J(n\tau),\Pi_\tau]S_{\tau/4}((n-a)\tau) + S_\tau((n-a)\tau)[J(a\tau),\Pi_{\tau/4}] \\ &\quad + S_\tau((n-a)\tau)J(a\tau). \end{aligned} \tag{5.6}$$

Now, J acts on gauge invariant nonlinearities like the gradient, see (2.7), and we observe that $(N(\tau)-\mathbf{1})Z_\tau$ enjoys this gauge invariance. We infer from (5.6) and Lemma 3.3 that the above estimates, proven in the case $A \in \{\mathbf{1}, \nabla\}$, remains essentially the same in the case $A = J$:

$$\begin{aligned} &\sum_{(q,r)\in\{(q_0,r_0),(\infty,2)\}} \|JZ_\tau\|_{\ell^q(a\leqslant n\leqslant a+N_*+1;L^r)} \\ &\leqslant (1+C\tau^{1/2})\Big(2K_0\|Z_\tau\|_{X(\{a\tau\})} \\ &\qquad + K_1\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|^{2\sigma}_{\ell^\gamma(a\leqslant n\leqslant a+N_*)}\left(4K_0\|Z_\tau\|_{X(\{a\tau\})}\right)^{2\sigma+1}\Big) \\ &\leqslant 3K_0\|Z_\tau\|_{X(\{a\tau\})} + \frac{3}{2}K_1\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|^{2\sigma}_{\ell^\gamma(a\leqslant n\leqslant a+N_*)}\left(4K_0\|Z_\tau\|_{X(\{a\tau\})}\right)^{2\sigma+1}, \end{aligned}$$

when $\tau \leqslant \tau_0$. The maximality of N_* implies that

$$3K_1\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|^{2\sigma}_{\ell^\gamma(a\leqslant n\leqslant a+N_*)}\left(4K_0\|Z_\tau\|_{X(\{a\tau\})}\right)^{2\sigma+1} > 2K_0\|Z_\tau\|_{X(\{a\tau\})},$$

i.e.,

$$6K_1\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|^{2\sigma}_{\ell^\gamma(a\leqslant n\leqslant a+N_*)} > \left(4K_0\|Z_\tau\|_{X(\{a\tau\})}\right)^{-2\sigma},$$

since if the reverse inequality were true, then N_*+1 would belong to Λ. Keeping this constant K_1 in the statement of Proposition 5.1, we see that $I \subsetneqq [a\tau,(a+N_*)\tau]$, and the result follows, by using Strichartz estimates once more in order to cover all admissible pairs (q,r). □

6. More estimates

The following lemma is the large time counterpart of [6, Lemma 5.2], where we also change the numerology:

Lemma 6.1. *For any time interval I, and $A \in \{\mathbf{1}, \nabla, J\}$,*

$$\begin{aligned} &\left\|A\left(\frac{N(\tau)-\mathbf{1}}{\tau}v - \frac{N(\tau)-\mathbf{1}}{\tau}w\right)\right\|_{\ell^{q_0'}(I;L^{r_0'})} \\ &\quad\lesssim \left(\|v\|^{2\sigma}_{\ell^\gamma(I;L^{r_0})} + \|w\|^{2\sigma}_{\ell^\gamma(I;L^{r_0})}\right)\|A(v-w)\|_{\ell^{q_0}(I;L^{r_0})} \\ &\qquad + \left(\|v\|^{2\sigma-1}_{\ell^\gamma(I;L^{r_0})} + \|w\|^{2\sigma-1}_{\ell^\gamma(I;L^{r_0})}\right)\|v-w\|_{\ell^\gamma(I;L^{r_0})}\|Aw\|_{\ell^{q_0}(I;L^{r_0})} \\ &\qquad + \tau\|v-w\|_{\ell^\gamma(I;L^{r_0})}\|Aw\|_{\ell^{q_0}(I;L^{r_0})}\left\||v|^{4\sigma-1}+|w|^{4\sigma-1}\right\|_{\ell^{\frac{\gamma}{2\sigma-1}}(I;L^{\frac{r_0}{2\sigma-1}})}. \end{aligned}$$

Proof. In view of Lemma 3.6, Hölder inequality yields

$$\left\|\frac{N(\tau)-\mathbf{1}}{\tau}v - \frac{N(\tau)-\mathbf{1}}{\tau}w\right\|_{\ell^{q_0'}(I;L^{r_0'})} \lesssim \left(\|v\|^{2\sigma}_{\ell^\gamma(I;L^{r_0})} + \|w\|^{2\sigma}_{\ell^\gamma(I;L^{r_0})}\right)\|v-w\|_{\ell^{q_0}(I;L^{r_0})}.$$

Next, by differentiating and rearranging, we have

$$\begin{aligned}
&\nabla\left(\frac{N(\tau)-\mathbf{1}}{\tau}v\right)-\nabla\left(\frac{N(\tau)-\mathbf{1}}{\tau}w\right)\\
&=\nabla\left(\frac{e^{-i\tau|v|^{2\sigma}}-1}{\tau}v\right)-\nabla\left(\frac{e^{-i\tau|w|^{2\sigma}}-1}{\tau}w\right)\\
&=-i\sigma\left(e^{-i\tau|v|^{2\sigma}}|v|^{2\sigma}\nabla v-e^{-i\tau|w|^{2\sigma}}|w|^{2\sigma}\nabla w\right)\\
&\quad-i\sigma\left(e^{-i\tau|v|^{2\sigma}}|v|^{2\sigma-2}v^2\nabla\bar v-e^{-i\tau|w|^{2\sigma}}|w|^{2\sigma-2}w^2\nabla\bar w\right)\\
&\quad+\left(\left(\frac{e^{-i\tau|v|^{2\sigma}}-1}{\tau}\right)-\left(\frac{e^{-i\tau|w|^{2\sigma}}-1}{\tau}\right)\right)\nabla w+\left(\frac{e^{-i\tau|v|^{2\sigma}}-1}{\tau}\right)(\nabla v-\nabla w).
\end{aligned}$$

Lemma 3.6 now yields

$$\begin{aligned}
&\left\|\nabla\left(\frac{N(\tau)-\mathbf{1}}{\tau}v\right)-\nabla\left(\frac{N(\tau)-\mathbf{1}}{\tau}w\right)\right\|_{\ell^{q_0'}(I;L^{r_0'})}\\
&\leqslant\sigma\left\|e^{-i\tau|v|^{2\sigma}}|v|^{2\sigma}\nabla v-e^{-i\tau|w|^{2\sigma}}|w|^{2\sigma}\nabla w\right\|_{\ell^{q_0'}(I;L^{r_0'})}\\
&\quad+\sigma\left\|e^{-i\tau|v|^{2\sigma}}|v|^{2\sigma-2}v^2\nabla\bar v-e^{-i\tau|w|^{2\sigma}}|w|^{2\sigma-2}w^2\nabla\bar w\right\|_{\ell^{q_0'}(I;L^{r_0'})}\\
&\quad+\left\|\left||v|^{2\sigma}-|w|^{2\sigma}\right|\nabla w\right\|_{\ell^{q_0'}(I;L^{r_0'})}+\left\||v|^{2\sigma}|\nabla v-\nabla w|\right\|_{\ell^{q_0'}(I;L^{r_0'})}.
\end{aligned}\tag{6.1}$$

By inserting the terms $|w|^{2\sigma}\nabla we^{-i\tau|v|^{2\sigma}}$ and $|w|^{2\sigma-2}w^2\nabla\bar we^{-i\tau|v|^{2\sigma}}$ in the first and second lines of the right hand side of (6.1), respectively, we obtain, since $\sigma\geqslant 1/2$,

$$\begin{aligned}
&\left\|\nabla\left(\frac{N(\tau)-\mathbf{1}}{\tau}v\right)-\nabla\left(\frac{N(\tau)-\mathbf{1}}{\tau}w\right)\right\|_{\ell^{q_0'}(I;L^{r_0'})}\\
&\lesssim\left\||v|^{2\sigma}(\nabla v-\nabla w)\right\|_{\ell^{q_0'}(I;L^{r_0'})}\\
&\quad+\left\|\left(|v|^{2\sigma-1}+|w|^{2\sigma-1}\right)(|v|-|w|)\nabla w\right\|_{\ell^{q_0'}(I;L^{r_0'})}\\
&\quad+\left\||w|^{2\sigma}\nabla w\left(e^{-i\tau|v|^{2\sigma}}-e^{-i\tau|w|^{2\sigma}}\right)\right\|_{\ell^{q_0'}(I;L^{r_0'})}.
\end{aligned}\tag{6.2}$$

The first term on the right hand side is estimated by Hölder inequality (2.4),

$$\left\||v|^{2\sigma}(\nabla v-\nabla w)\right\|_{\ell^{q_0'}(I;L^{r_0'})}\leqslant\|v\|^{2\sigma}_{\ell^\gamma(I;L^{r_0})}\|\nabla v-\nabla w\|_{\ell^{q_0}(I;L^{r_0})}.$$

Similarly,

$$\begin{aligned}
&\left\|\left(|v|^{2\sigma-1}+|w|^{2\sigma-1}\right)(|v|-|w|)\nabla w\right\|_{\ell^{q_0'}(I;L^{r_0'})}\\
&\lesssim\left(\|v\|^{2\sigma-1}_{\ell^\gamma(I;L^{r_0})}+\|w\|^{2\sigma-1}_{\ell^\gamma(I;L^{r_0})}\right)\|v-w\|_{\ell^\gamma(I;L^{r_0})}\|\nabla w\|_{\ell^{q_0}(I;L^{r_0})}.
\end{aligned}$$

For the last term in (6.2), we notice the pointwise estimate, using again $\sigma\geqslant 1/2$,

$$\left||w|^{2\sigma}\nabla w\left(e^{-i\tau|v|^{2\sigma}}-e^{-i\tau|w|^{2\sigma}}\right)\right|\lesssim\tau|w|^{2\sigma}|\nabla w|\left(|v|^{2\sigma-1}+|w|^{2\sigma-1}\right)|v-w|.$$

Hölder inequality now yields

$$\begin{aligned}
&\left\||w|^{2\sigma}\nabla w\left(e^{-i\tau|v|^{2\sigma}}-e^{-i\tau|w|^{2\sigma}}\right)\right\|_{\ell^{q_0'}(I;L^{r_0'})}\\
&\lesssim\tau\|v-w\|_{\ell^\gamma(I;L^{r_0})}\|\nabla w\|_{\ell^{q_0}(I;L^{r_0})}\left\||v|^{4\sigma-1}+|w|^{4\sigma-1}\right\|_{\ell^{\frac{\gamma}{2\sigma-1}}(I;L^{\frac{r_0}{2\sigma-1}})}.
\end{aligned}$$

The lemma follows from the above estimates, for $A \in \{\mathbf{1}, \nabla\}$. The case $A = J$ is similar to the case $A = \nabla$, since $J(t) = S(t)xS(-t)$ and J acts on gauge invariant nonlinearities like the gradient, (2.7). □

So far, we have supposed $\sigma \geqslant 1/2$. From the next proposition on, we need to require $\sigma > 1/2$, as pointed out in the proof below.

Proposition 6.2. *Suppose $\sigma > 1/2$. Let (q, r) be an admissible pair. Consider an interval I of the form $I = \overline{[a\tau, b\tau)}$, with $a, b \in \mathbb{N} \cup \{\infty\}$ such that $0 \leqslant a < b \leqslant \infty$. There exists K_2 such that if*

$$M \left\| \langle n\tau \rangle^{-\delta(r_0)} \right\|_{\ell^\gamma(I)} \leqslant K_2, \tag{6.3}$$

then the following holds.

- *There exists $C = C(d, \sigma, q)$ such that if $\phi_1, \phi_2 \in \Sigma$ with $\|Z_\tau^{\phi_j}\|_{X(\{a\tau\})} \leqslant M$, $j = 1, 2$, then*

(6.4)
$$Q_1 = \max_{A \in \{\mathbf{1}, \nabla, J\}} \|A(n\tau) \left(Z_\tau^{\phi_2}(n\tau) - Z_\tau^{\phi_1}(n\tau) \right) \|_{\ell^q(I;L^r)} \leqslant C \|Z_\tau^{\phi_2} - Z_\tau^{\phi_1}\|_{X(\{a\tau\})},$$

  where $Z_\tau^{\phi_j}(n\tau) = (S_\tau(\tau)N(\tau))^n \Pi_\tau \phi_j$.

- *If $\psi \in \Sigma \cap H^2(\mathbb{R}^d)$, with $\|Z_\tau^\psi\|_{X(\{a\tau\})} \leqslant M$, then there exists a constant $C = C(d, \sigma, M, \psi) > 0$ such that*

$$Q_2 = \max_{A \in \{\mathbf{1}, \nabla, J\}} \left\| A(n\tau) \left(Z_\tau^\psi(n\tau) - u^\psi(n\tau) \right) \right\|_{\ell^q(I;L^r)} \leqslant C\tau^{1/2}. \tag{6.5}$$

- *Assume $\phi \in \Sigma$ satisfies $\|\phi\|_{X(\{a\tau\})} \leqslant M/2$. denote u^ϕ by the solution of (1.1) with the initial condition $u_{|t=a\tau} = \phi$, i.e., u is given by (2.1) with $t_0 = a\tau$. Similarly in the discrete version, let $Z_\tau^\phi(n\tau)$ $(n > a)$ be given by (3.1) with $Z_\tau(a\tau)$ replaced by ϕ. Then it holds that*

$$\lim_{\tau \to 0} \max_{A \in \{\mathbf{1}, \nabla, J\}} \left\| A(n\tau) \left(Z_\tau^\phi(n\tau) - u^\phi(n\tau) \right) \right\|_{\ell^\infty(I;L^2)} = 0. \tag{6.6}$$

Remark 6.3. In view of Remark 5.2, the assumption (6.3) means, typically for M large, that either I is finite with $|I|$ sufficiently small, say $|I| = T \approx M^{-\gamma}$, or $I = [NT, \infty)$, with

$$N \approx \frac{1}{T} M^{\frac{\gamma}{\gamma\delta(r_0)-1}} \approx M^{\gamma + \frac{\gamma}{\gamma\delta(r_0)-1}}.$$

Proof. We divide the proof into three parts, corresponding to (6.4), (6.5) and (6.6), respectively.

Proof of (6.4). Let ϕ_1 and $\phi_2 \in \Sigma$ such that $\|Z_\tau^{\phi_j}\|_{X(\{a\tau\})} \leqslant M$, $j = 1, 2$. We consider the difference between the Duhamel formulas of $Z_\tau^{\phi_1}$ and $Z_\tau^{\phi_2}$ provided by (3.1). For $A \in \{\mathbf{1}, \nabla, J\}$, (3.1) together with Corollary 3.4 yields

$$\begin{aligned}
&\|A(n\tau) \left(Z_\tau^{\phi_2}(n\tau) - Z_\tau^{\phi_1}(n\tau) \right) \|_{\ell^q(I;L^r)} \\
&\lesssim \left\| Z_\tau^{\phi_2}(a\tau) - Z_\tau^{\phi_1}(a\tau) \right\|_{L^2} + \left\| A(a\tau) \left(Z_\tau^{\phi_2}(a\tau) - Z_\tau^{\phi_1}(a\tau) \right) \right\|_{L^2} \\
&\quad + \left\| \frac{N(\tau) - \mathbf{1}}{\tau} Z_\tau^{\phi_2} - \frac{N(\tau) - \mathbf{1}}{\tau} Z_\tau^{\phi_1} \right\|_{\ell^{q'_0}(I;L^{r'_0})}
\end{aligned}$$

$$+\left\|A\left(\frac{N(\tau)-\mathbf{1}}{\tau}Z_\tau^{\phi_2}-\frac{N(\tau)-\mathbf{1}}{\tau}Z_\tau^{\phi_1}\right)\right\|_{\ell^{q_0'}(I;L^{r_0'})}.$$

In view of Proposition 5.1 with choosing $K_2>0$ small enough in (6.3), for $(q,r)\in\{(q_0,r_0),(\infty,2)\}$, we have

$$\|A(n\tau)Z_\tau^{\phi_j}(n\tau)\|_{\ell^q(I;L^r)}\leqslant KM,\quad j=1,2, \tag{6.7}$$

for some constant $K>0$. In view of this estimate, Lemma 6.1 implies, along with (2.6) to estimate $\|\phi_2-\phi_1\|_{\ell^\gamma(I;L^{r_0})}$,

$$\begin{aligned}
Q_1 &= \max_{A\in\{\mathbf{1},\nabla,J\}}\|A(n\tau)\left(Z_\tau^{\phi_2}(n\tau)-Z_\tau^{\phi_1}(n\tau)\right)\|_{\ell^q(I;L^r)}\\
&\lesssim \max_{A\in\{\mathbf{1},\nabla,J\}}\left\|A(a\tau)\left(Z_\tau^{\phi_2}(a\tau)-Z_\tau^{\phi_1}(a\tau)\right)\right\|_{L^2}\\
&\quad+\max_{A\in\{\mathbf{1},\nabla,J\}}\left\|A\left(\frac{N(\tau)-\mathbf{1}}{\tau}Z_\tau^{\phi_2}-\frac{N(\tau)-\mathbf{1}}{\tau}Z_\tau^{\phi_1}\right)\right\|_{\ell^{q_0'}(I;L^{r_0'})}\\
&\lesssim \max_{A\in\{\mathbf{1},\nabla,J\}}\left\|A(a\tau)\left(Z_\tau^{\phi_2}(a\tau)-Z_\tau^{\phi_1}(a\tau)\right)\right\|_{L^2}\\
&\quad+\left(M\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|_{\ell^\gamma(I)}\right)^{2\sigma}\max_{A\in\{\mathbf{1},\nabla,J\}}\|A(Z_\tau^{\phi_2}(n\tau)-Z_\tau^{\phi_1}(n\tau))\|_{\ell^{q_0}(I;L^{r_0})}\\
&\quad+\left(M\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|_{\ell^\gamma(I)}\right)^{2\sigma}\|Z_\tau^{\phi_2}(n\tau)-Z_\tau^{\phi_1}(n\tau)\|_{\Gamma(I)}\\
&\quad+\tau M\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|_{\ell^\gamma(I)}\|Z_\tau^{\phi_2}(n\tau)-Z_\tau^{\phi_1}(n\tau)\|_{\Gamma(I)}\times\\
&\qquad\times\left(\|Z_\tau^{\phi_2}(n\tau)\|^{4\sigma-1}_{\ell^{\frac{4\sigma-1}{2\sigma-1}\gamma}(I;L^{\frac{4\sigma-1}{2\sigma-1}r_0})}+\|Z_\tau^{\phi_1}(n\tau)\|^{4\sigma-1}_{\ell^{\frac{4\sigma-1}{2\sigma-1}\gamma}(I;L^{\frac{4\sigma-1}{2\sigma-1}r_0})}\right),
\end{aligned}$$

where we recall that the norm $\|\cdot\|_{\Gamma(I)}$ is defined in (5.1). In view of Bernstein inequality (3.7), for $j=1,2$,

$$\|Z_\tau^{\phi_j}(n\tau)\|_{L^{\frac{4\sigma-1}{2\sigma-1}r_0}}\lesssim\tau^{\frac{d}{2}\left(\frac{2\sigma-1}{(4\sigma-1)r_0}-\frac{1}{r_0}\right)}\|Z_\tau^{\phi_j}(n\tau)\|_{L^{r_0}},$$

and

$$\frac{d}{2}\left(\frac{2\sigma-1}{(4\sigma-1)r_0}-\frac{1}{r_0}\right)=-\frac{d}{2}\frac{2\sigma}{(4\sigma-1)r_0}=-\frac{d\sigma}{(4\sigma-1)(2\sigma+2)}=-\frac{2}{(4\sigma-1)q_0},$$

so we obtain

$$\begin{aligned}
\|Z_\tau^{\phi_j}(n\tau)\|^{4\sigma-1}_{\ell^{\frac{4\sigma-1}{2\sigma-1}\gamma}(I;L^{\frac{4\sigma-1}{2\sigma-1}r_0})}&\lesssim\tau^{-2/q_0}\|Z_\tau^{\phi_j}(n\tau)\|^{4\sigma-1}_{\ell^{\frac{4\sigma-1}{2\sigma-1}\gamma}(I;L^{r_0})}\\
&\lesssim\tau^{-2/q_0}\|\langle n\tau\rangle^{-\delta(r_0)}\|^{4\sigma-1}_{\ell^{\frac{4\sigma-1}{2\sigma-1}\gamma}(I)}\|Z_\tau^{\phi_j}\|^{4\sigma-1}_{\Gamma(I)}\\
&\lesssim\tau^{-2/q_0}\|\langle n\tau\rangle^{-\delta(r_0)}\|^{4\sigma-1}_{\ell^\gamma(I)}M^{4\sigma-1},
\end{aligned}\tag{6.8}$$

where we have used (6.7) and $\frac{4\sigma-1}{2\sigma-1}>1$ for the last inequality. Note that if $\sigma=1/2$, the Lebesgue exponent $\frac{4\sigma-1}{2\sigma-1}\gamma$ becomes infinite, and we can no longer invoke (3.7) like we did above. This is why we assume $\sigma>1/2$. In view of (2.3), we get

$$\begin{aligned}
Q_1&\lesssim\max_{A\in\{\mathbf{1},\nabla,J\}}\left\|A(a\tau)\left(Z_\tau^{\phi_2}(a\tau)-Z_\tau^{\phi_1}(a\tau)\right)\right\|_{L^2}\\
&\quad+\left(M\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|_{\ell^\gamma(I)}\right)^{2\sigma}\max_{A\in\{\mathbf{1},\nabla,J\}}\|A(n\tau)(Z_\tau^{\phi_2}(n\tau)-Z_\tau^{\phi_1}(n\tau))\|_{\ell^{q_0}(I;L^{r_0})}
\end{aligned}$$

$$
\begin{aligned}
&+\left(M\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|_{\ell^\gamma(I)}\right)^{2\sigma}\|Z_\tau^{\phi_2}(n\tau)-Z_\tau^{\phi_1}(n\tau)\|_{\Gamma(I)}\\
&+\left(M\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|_{\ell^\gamma(I)}\right)^{4\sigma}\|Z_\tau^{\phi_2}(n\tau)-Z_\tau^{\phi_1}(n\tau)\|_{\Gamma(I)}\\
\leqslant\ & C_1\left\|Z_\tau^{\phi_2}(a\tau)-Z_\tau^{\phi_1}(a\tau)\right\|_{X(\{a\tau\})}\\
&+C_2Q_1\left[\left(M\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|_{\ell^\gamma(I)}\right)^{2\sigma}+\left(M\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|_{\ell^\gamma(I)}\right)^{4\sigma}\right].
\end{aligned}
$$

The inequality (6.4) then follows in the same fashion as in the proof of Proposition 5.1, up to considering a smaller constant K_2.

Proof of (6.5). Assume that $\psi\in\Sigma\cap H^2(\mathbb{R}^d)$, with $\|Z_\tau^\psi\|_{X(\{a\tau\})}\leqslant M$, and let $A\in\{\mathbf{1},\nabla,J\}$. We estimate $Z_\tau^\psi(n\tau)-\Pi_\tau u^\psi(n\tau)$ instead of $Z_\tau^\psi(n\tau)-u^\psi(n\tau)$, since we have

$$
\left\|A(n\tau)\left(u^\psi(n\tau)-\Pi_\tau u^\psi(n\tau)\right)\right\|_{\ell^q(I;L^r)}\lesssim\tau^{1/2}\|A(n\tau)u^\psi(n\tau)\|_{\ell^q(I;W^{1,r})}\lesssim\tau^{1/2},
$$

thanks to (3.4), (3.8) and Theorem 2.5. We now decompose $Z_\tau^\psi(n\tau)-\Pi_\tau u^\psi(n\tau)$ in view of Duhamel formulas (3.1) and (2.1) (recalling that $S_\tau(t)=S(t)\Pi_\tau=\Pi_\tau S(t)$), as

$$
\begin{aligned}
&Z_\tau^\psi(n\tau)-\Pi_\tau u^\psi(n\tau)\\
&=S_\tau((n-a)\tau)(Z_\tau(a\tau)-u^\psi(a\tau))\\
&\quad\underbrace{+\tau\sum_{k=a}^{n-1}S_\tau(n\tau-k\tau)\left(\frac{N(\tau)-\mathbf{1}}{\tau}Z_\tau^\psi(k\tau)-\frac{N(\tau)-\mathbf{1}}{\tau}\Pi_\tau u^\psi(k\tau)\right)}_{=:\mathcal{G}_1}\\
&\quad\underbrace{+\tau\sum_{k=a}^{n-1}S_\tau(n\tau-k\tau)\frac{N(\tau)-\mathbf{1}}{\tau}\Pi_\tau u^\psi(k\tau)+i\int_{a\tau}^{n\tau}S_\tau(n\tau-s)|u^\psi|^{2\sigma}u^\psi(s)ds}_{=:\mathcal{G}_2}.
\end{aligned}
$$

In this decomposition, $\mathcal{G}_1$ must be thought of as an arbitrary small perturbation of the left hand side, and $\mathcal{G}_2$ must be thought of as the actual source term.

Let (q,r) be an admissible pair, and $A\in\{\mathbf{1},\nabla\}$. We estimate the terms of the above right hand side, after the action of A. First, discrete Strichartz estimates (3.3) and Lemma 6.1 yield

$$
\begin{aligned}
\|A\mathcal{G}_1\|_{\ell^qL^r}&\lesssim\left\|A\left(\frac{N(\tau)-\mathbf{1}}{\tau}Z_\tau^\psi(k\tau)-\frac{N(\tau)-\mathbf{1}}{\tau}\Pi_\tau u^\psi(k\tau)\right)\right\|_{\ell^{q_0'}L^{r_0'}}\\
&\lesssim\left(\|Z_\tau^\psi\|_{\ell^\gamma L^{r_0}}^{2\sigma}+\|\Pi_\tau u^\psi\|_{\ell^\gamma L^{r_0}}^{2\sigma}\right)\|A(Z_\tau^\psi-\Pi_\tau u^\psi)\|_{\ell^{q_0}L^{r_0}}\\
&\quad+\left(\|Z_\tau^\psi\|_{\ell^\gamma L^{r_0}}^{2\sigma-1}+\|\Pi_\tau u^\psi\|_{\ell^\gamma L^{r_0}}^{2\sigma-1}\right)\|Z_\tau^\psi-\Pi_\tau u^\psi\|_{\ell^\gamma L^{r_0}}\|A\Pi_\tau u^\psi\|_{\ell^{q_0}L^{r_0}}\\
&\quad+\tau\|Z_\tau^\psi-\Pi_\tau u^\psi\|_{\ell^\gamma L^{r_0}}\|A\Pi_\tau u^\psi\|_{\ell^{q_0}L^{r_0}}\left\||Z_\tau^\psi|^{4\sigma-1}+|\Pi_\tau u^\psi|^{4\sigma-1}\right\|_{\ell^{\frac{\gamma}{2\sigma-1}}L^{\frac{r_0}{2\sigma-1}}},
\end{aligned}
$$

where the interval I is omitted to ease notations. Recalling Proposition 5.1 and (2.12), we infer

$$
\max_{B\in\{\mathbf{1},\nabla,J\}}\sum_{(q,r)\in\{(q_0,r_0),(\infty,2)\}}\|BZ_\tau^\psi\|_{\ell^q(I;L^r)}\lesssim M.
$$

Invoking (6.8),

$$\| |Z_\tau^\psi|^{4\sigma-1} \|_{\ell^{\frac{\gamma}{2\sigma-1}}(I;L^{\frac{r_0}{2\sigma-1}})} \lesssim \tau^{-\frac{2}{q_0}} \| \langle n\tau\rangle^{-\delta(r_0)} \|_{\ell^\gamma(I)}^{4\sigma-1} M^{4\sigma-1}.$$

In view of Theorem 2.5, (3.8) and (3.5), we also have

$$\max_{B\in\{\mathbf{1},\nabla,J\}} \sum_{(q,r)\in\{(q_0,r_0),(\infty,2)\}} \|B\Pi_\tau u^\psi\|_{\ell^q(I;L^r)} \lesssim M,$$

and, like for (6.8),

$$\| |\Pi_\tau u^\psi|^{4\sigma-1} \|_{\ell^{\frac{\gamma}{2\sigma-1}}(I;L^{\frac{r_0}{2\sigma-1}})} \lesssim \tau^{-\frac{2}{q_0}} \| \langle n\tau\rangle^{-\delta(r_0)} \|_{\ell^\gamma(I)}^{4\sigma-1} M^{4\sigma-1}.$$

We deduce the (uniform in $\tau\in(0,1)$) estimate

$$\begin{aligned}\|A\mathcal{G}_1\|_{\ell^q(I;L^r)} \lesssim & \left(\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|_{\ell^\gamma(I)} M\right)^{2\sigma} \|A(Z_\tau^\psi - \Pi_\tau u^\psi)\|_{\ell^{q_0}(I;L^{r_0})} \\ & + \left(\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|_{\ell^\gamma(I)} M\right)^{2\sigma-1} M\|Z_\tau^\psi - \Pi_\tau u^\psi\|_{\ell^\gamma(I;L^{r_0})} \\ & + \left(\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|_{\ell^\gamma(I)} M\right)^{4\sigma-1} M\|Z_\tau^\psi - \Pi_\tau u^\psi\|_{\ell^\gamma(I;L^{r_0})}.\end{aligned}$$

As before, we can estimate

$$\|Z_\tau^\psi - \Pi_\tau u^\psi\|_{\ell^\gamma(I;L^{r_0})} \lesssim \left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|_{\ell^\gamma(I)} \|Z_\tau^\psi - \Pi_\tau u^\psi\|_{\Gamma(I)} \lesssim Q_2 \left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|_{\ell^\gamma(I)}.$$

When $A = J$, we recall that in view of Corollary 3.4, we may repeat the same computations, up to an (irrelevant) extra multiplicative factor $1 + C\tau^{1/2}$. Picking $\| \langle n\tau\rangle^{-\delta(r_0)} \|_{\ell^\gamma(I)} M$ sufficiently small,

$$\max_{\substack{A\in\{\mathbf{1},\nabla,J\}\\(q,r)\text{ admissible}}} \|A\mathcal{G}_1\|_{\ell^q(I;L^r)} \leqslant \frac{1}{2} Q_2,$$

and thus

$$\max_{(q,r)\text{ admissible}} Q_2 \leqslant C\|Z_\tau(a\tau) - u^\psi(a\tau)\|_{L^2} + 2 \max_{\substack{A\in\{\mathbf{1},\nabla,J\}\\(q,r)\text{ admissible}}} \|A\mathcal{G}_2\|_{\ell^q(I;L^r)}.$$

By adapting of the proof of Theorem 2.5, we see that

$$\max_{\substack{A\in\{\mathbf{1},\nabla,J\}\\(q,r)\text{ admissible}}} \|A\mathcal{G}_2\|_{\ell^q(I;L^r)} \leqslant C(d,\sigma,M,\psi)\tau^{1/2},$$

hence (6.5) is established by similar induction applied in the proof of Lemma 4.1.

Proof of (6.6). Let $\varepsilon\in(0,M/2)$, and $\psi\in H^2\cap\Sigma$ such that

$$\|\phi - \psi\|_{X(\{a\tau\})} \leqslant \varepsilon.$$

Then $\|\psi\|_{X(\{a\tau\})} \leqslant M$, so we may invoke Theorem 2.5 and (6.4) to claim that

$$\|u^\phi - u^\psi\|_{Y(\mathbb{R})} \leqslant C\|\phi - \psi\|_{X(\{a\tau\})} \lesssim \varepsilon,$$

and, since $u^\phi(a\tau) = Z_\tau^\phi(a\tau) = \phi$,

$$\max_{A\in\{\mathbf{1},\nabla,J\}} \|A(n\tau)\left(Z_\tau^\phi(n\tau) - Z_\tau^\psi(n\tau)\right)\|_{\ell^\infty(I;L^2)} \leqslant C\|\phi - \psi\|_{X(\{a\tau\})} \lesssim \varepsilon.$$

On the other hand, (6.5) yields

$$\max_{A\in\{\mathbf{1},\nabla,J\}} \|A(n\tau)\left(Z_\tau^\psi(n\tau)-u^\psi(n\tau)\right)\|_{\ell^\infty(I;L^2)} \leqslant C(d,\sigma,M,\psi)\tau^{1/2},$$

and the triangle inequality implies

$$\max_{A\in\{\mathbf{1},\nabla,J\}} \|A(n\tau)\left(Z_\tau^\phi(n\tau)-u^\phi(n\tau)\right)\|_{\ell^\infty(I;L^2)} \leqslant C(d,\sigma,M,\psi)\tau^{1/2}+C(d,\sigma,M)\varepsilon.$$

Therefore, for all $\varepsilon\in(0,M/2)$,

$$\limsup_{\tau\to 0} \max_{A\in\{\mathbf{1},\nabla,J\}} \|A(n\tau)\left(Z_\tau^\phi(n\tau)-u^\phi(n\tau)\right)\|_{\ell^\infty(I;L^2)} \leqslant C(d,\sigma,M)\varepsilon,$$

and the left hand side must then be zero. □

7. Global Σ stability of Z_τ

In this section, we prove Theorem 1.6. As suggested by the statements of Propositions 5.1 and 6.2, the idea is to split $\mathbb{N}$ into finitely many intervals, like in the proof of Theorem 1.3, and apply these local results on each of them: the accumulation of errors is thus limited.

Proof of Theorem 1.6. Let $\phi\in\Sigma$: in view of Theorem 2.3, there exists M such that

$$\sum_{A\in\{\mathbf{1},\nabla,J\}} \|Au^\phi\|_{L^\infty(\mathbb{R};L^2)} \leqslant \frac{M}{2}.$$

Consider K_2 provided by Proposition 6.2, see (6.3). Like in the proof of Theorem 1.3, we can find a finite number K of time intervals $I_j=\overline{[m_j\tau,m_{j+1}\tau)}$ with $0=m_1<m_2<\ldots<m_K\in\mathbb{N}$, $m_{K+1}=\infty$, such that

$$M\left\|\langle n\tau\rangle^{-\delta(r_0)}\right\|_{L^\gamma(I_j)}\leqslant K_2,\quad \mathbb{R}_+=\bigcup_{j=1}^K I_j.$$

Typically, in view of Remarks 5.2 and 6.3, this means that for $1\leqslant j\leqslant K-1$, we may consider $|I_j|=T$ sufficiently small (in terms of M, and uniformly in τ), while I_K is of the form $[\alpha T,\infty)$, with α sufficiently large.

For each $1\leqslant j\leqslant K$, let $\psi_j\in\Sigma\cap H^2$ such that

$$\|\psi_j-u^\phi(m_j\tau)\|_{X(\{m_j\tau\})}\leqslant\frac{M}{(10C_0)^K}, \tag{7.1}$$

with C_0 the largest constant between 1, the constant C in the first point of Theorem 2.5, and the constant C in (6.4). We show by induction that for $\tau>0$ sufficiently small,

$$\max_{A\in\{\mathbf{1},\nabla,J\}} \|A(n\tau)\left(Z_\tau^\phi(n\tau)-u^\phi(n\tau)\right)\|_{\ell^\infty(I_j;L^2)} \leqslant (3C_0)^j\frac{M}{(10C_0)^K}, \tag{7.2}$$

and

$$\max_{A\in\{\mathbf{1},\nabla,J\}} \|A(n\tau)Z_\tau^\phi(n\tau)\|_{\ell^\infty(I_j;L^2)} \leqslant M. \tag{7.3}$$

Let $A\in\{\mathbf{1},\nabla,J\}$. For $j=1$, we use triangle inequality to observe

$$\begin{aligned}\|A\left(Z_\tau^\phi(n\tau)-u^\phi(n\tau)\right)\|_{\ell^\infty(I_1;L^2)} &\leqslant \|A\left(Z_\tau^\phi(n\tau)-Z_\tau^{\psi_1}(n\tau)\right)\|_{\ell^\infty(I_1;L^2)}\\ &\quad+\|A\left(Z_\tau^{\psi_1}(n\tau)-u^{\psi_1}(n\tau)\right)\|_{\ell^\infty(I_1;L^2)}\end{aligned}$$

$$+ \|A\left(u^{\psi_1}(n\tau) - u^{\phi}(n\tau)\right)\|_{\ell^\infty(I_1;L^2)}.$$

In view of (6.4), the first term in the right hand side is controlled by

$$\|A\left(Z_\tau^{\phi}(n\tau) - Z_\tau^{\psi_1}(n\tau)\right)\|_{\ell^\infty(I_1;L^2)} \leqslant C_0\|\phi - \psi_1\|_{X(\{0\})} \leqslant C_0\frac{M}{(10C_0)^K},$$

where we have used (7.1). The second term is estimated in view of (6.5), by

$$\|A\left(Z_\tau^{\psi_1}(n\tau) - u^{\psi_1}(n\tau)\right)\|_{\ell^\infty(I_1;L^2)} \leqslant \tau^{1/2}C(d,\sigma,M,\psi_1).$$

For $0 < \tau \leqslant \tau_1 = \tau_1(d,\sigma,M,\psi_1)$, we infer

$$\|A\left(Z_\tau^{\psi_1}(n\tau) - u^{\psi_1}(n\tau)\right)\|_{\ell^\infty(I_1;L^2)} \leqslant C_0\frac{M}{(10C_0)^K}.$$

We impose a smallness constraint on each I_j, but since there are finitely many such intervals, the minimum of these τ_j is indeed positive. Finally, Theorem 2.5 and (7.1) yield

$$\|A\left(u^{\psi_1}(n\tau) - u^{\phi}(n\tau)\right)\|_{\ell^\infty(I_1;L^2)} \leqslant C_0\|\psi_1 - \phi\|_{X(\{0\})} \leqslant C_0\frac{M}{(10C_0)^K},$$

hence (7.2) for $j = 1$. To prove (7.3) for $j = 1$, write

$$\begin{aligned}\|A(n\tau)Z_\tau^{\phi}(n\tau)\|_{\ell^\infty(I_1;L^2)} &\leqslant \|A(n\tau)u^{\phi}(n\tau)\|_{\ell^\infty(I_1;L^2)}\\ &\quad + \|A(n\tau)\left(Z_\tau^{\phi}(n\tau) - u^{\phi}(n\tau)\right)\|_{\ell^\infty(I_1;L^2)}\\ &\leqslant \frac{M}{2} + 3C_0\frac{M}{(10C_0)^K} \leqslant M.\end{aligned}$$

Suppose now that (7.2) and (7.3) hold for some $1 \leqslant j \leqslant K-1$. We invoke the same intermediary results as in the case $j = 1$:

$$\begin{aligned}&\|A\left(Z_\tau^{\phi}(n\tau) - u^{\phi}(n\tau)\right)\|_{\ell^\infty(I_{j+1};L^2)}\\ &\leqslant \|A\left(Z_\tau^{\phi}(n\tau) - Z_\tau^{\psi_{j+1}}(n\tau - m_{j+1}\tau)\right)\|_{\ell^\infty(I_{j+1};L^2)}\\ &\quad + \|A\left(Z_\tau^{\psi_{j+1}}(n\tau - m_{j+1}\tau) - u^{\psi_{j+1}}(n\tau - m_{j+1}\tau)\right)\|_{\ell^\infty(I_{j+1};L^2)}\\ &\quad + \|A\left(u^{\psi_{j+1}}(n\tau - m_{j+1}\tau) - u^{\phi}(n\tau)\right)\|_{\ell^\infty(I_{j+1};L^2)}.\end{aligned}$$

For the first term on the right hand side, in view of (7.3) at step j, we have:

$$\begin{aligned}&\|A(Z_\tau^{\phi}(n\tau) - Z_\tau^{\psi_{j+1}}(n\tau - m_{j+1}\tau))\|_{\ell^\infty(I_{j+1};L^2)}\\ &\leqslant C_0\|Z_\tau^{\phi}(m_{j+1}\tau) - \psi_{j+1}\|_{X(\{m_{j+1}\tau\})}\\ &\leqslant C_0\|Z_\tau^{\phi}(m_{j+1}\tau) - u^{\phi}(m_{j+1}\tau)\|_{X(\{m_{j+1}\tau\})} + C_0\|u^{\phi}(m_{j+1}\tau) - \psi_{j+1}\|_{X(\{m_{j+1}\tau\})}\\ &\leqslant C_0\max_{B\in\{\mathbf{1},\nabla,J\}}\|B(Z_\tau^{\phi}(n\tau) - u^{\phi}(n\tau))\|_{\ell^\infty(I_j;L^2)} + C_0\frac{M}{(10C_0)^K}\\ &\leqslant C_0 \times (3C_0)^j\frac{M}{(10C_0)^K} + C_0\frac{M}{(10C_0)^K},\end{aligned}$$

where we have used (7.2) at step j, and (7.1). The second term is estimated in view of (6.5), by

$$\|A\left(Z_\tau^{\psi_{j+1}}(n\tau - m_{j+1}\tau) - u^{\psi_{j+1}}(n\tau - m_{j+1}\tau)\right)\|_{\ell^\infty(I_{j+1};L^2)} \leqslant \tau^{1/2}C(d,\sigma,M,\psi_{j+1}).$$

For $0 < \tau \leqslant \tau_{j+1} = \tau_{j+1}(d,\sigma,M,\psi_{j+1})$, we infer

$$\|A\left(Z_\tau^{\psi_{j+1}}(n\tau - m_{j+1}\tau) - u^{\psi_{j+1}}(n\tau - m_{j+1}\tau)\right)\|_{\ell^\infty(I_{j+1};L^2)} \leqslant C_0\frac{M}{(10C_0)^K}.$$

Eventually, we assume

$$0 < \tau \leqslant \min_{1 \leqslant j \leqslant K} \tau_j.$$

Finally, Theorem 2.5 and (7.1) yield

$$\begin{aligned}\|A\left(u^{\psi_{j+1}}(n\tau - m_{j+1}\tau) - u^{\phi}(n\tau)\right)\|_{\ell^\infty(I_{j+1};L^2)} &\leqslant C_0\|\psi_{j+1} - u^{\phi}(m_{j+1}\tau)\|_{X(\{m_{j+1}\tau\})} \\ &\leqslant C_0\frac{M}{(10C_0)^K},\end{aligned}$$

hence

$$\begin{aligned}\|A\left(Z_\tau^{\phi}(n\tau) - u^{\phi}(n\tau)\right)\|_{\ell^\infty(I_{j+1};L^2)} &\leqslant \left(3^j C_0^{j+1} + 3C_0\right)\frac{M}{(10C_0)^K} \\ &\leqslant \left(3^j C_0^{j+1} + 3C_0^{j+1}\right)\frac{M}{(10C_0)^K},\end{aligned}$$

and (7.2) follows for $j+1$. To prove (7.3), write again

$$\begin{aligned}\|A(n\tau)Z_\tau^{\phi}(n\tau)\|_{\ell^\infty(I_{j+1};L^2)} &\leqslant \|A(n\tau)u^{\phi}(n\tau)\|_{\ell^\infty(I_{j+1};L^2)} \\ &\quad + \|A(n\tau)\left(Z_\tau^{\phi}(n\tau) - u^{\phi}(n\tau)\right)\|_{\ell^\infty(I_{j+1};L^2)} \\ &\leqslant \frac{M}{2} + (3C_0)^{j+1}\frac{M}{(10C_0)^K} \leqslant M.\end{aligned}$$

This yields Theorem 1.6 for $(q,r) = (\infty, 2)$. The case of other admissible pairs then follows from Proposition 5.1. □

8. Conclusion and outlooks

In this paper, we have proved the first uniform in time error estimate for splitting methods in the context of nonlinear Schrödinger equations. The main tools are discrete Strichartz estimates for the frequency truncated free Schrödinger group established in [18] on the one hand, the Galilean operator J and its main properties discovered in [12] on the other hand. The Galilean operator provides precious dispersive estimates, in the sense that L^r-norms in space decay with an explicit rate in time: this is more precise than belonging to some space $L_t^q L_x^r$, a property related to Strichartz estimates. We have considered Lie-Trotter splitting in time, the question of higher order methods, starting with Strang splitting, remains open.

For several technical reasons (see Remark 2.4), we have assumed $\sigma \geqslant 2/d$. Filling the gap $\sigma_* < \sigma < 2/d$ does not seem straightforward, and is a fairly natural question.

It is very likely that the method presented here can be adapted in order to treat some (defocusing) Hartree-type nonlinearities, of the form

$$\left(\frac{1}{|x|^\gamma} * |u|^2\right)u,$$

with $4/3 < \gamma < \min(4, d)$; see [13, 15] for scattering theory in this case.

As it is the case in several physical models, such as Bose-Einstein condensation (see e.g. [19]), one may ask if our result remains true if (1.1) is replaced by

$$i\partial_t u + \frac{1}{2}\Delta u = |u|^{2\sigma}u + \frac{|x|^2}{2}u.$$

As the harmonic potential prevents large time dispersion (linear solutions are periodic in time), no uniform in time error estimate should be expected to the splitting

method. More generally, if the solution to the Schrödinger equation that we consider is not (sufficiently) dispersive, then our approach fails, as discussed in Remark 1.7. On the other hand, if one considers the *repulsive* harmonic potential,

$$i\partial_t u + \frac{1}{2}\Delta u = |u|^{2\sigma}u - \frac{|x|^2}{2}u,$$

which enhances the linear dispersion (algebraic decay in time becomes an exponential decay), then the scattering theory developed in [2] suggests that an analogue of Theorem 1.3 may be available.

In the case of linear Schrödinger equations with a potential,

$$i\partial_t u + \frac{1}{2}\Delta u = Vu,$$

if V is a short-range potential in the sense that scattering theory is available (see e.g. [7] for a reference including time-dependent potentials), it seems that no uniform in time error estimate is known for splitting methods. The case of short-range potentials should be a reasonable framework to try to prove the analogue of Theorem 1.3, since solutions are asymptotically free (the action of the potential V is negligible for large time), but a suitable technical approach is to be developed.

Probably the most natural and challenging question at this stage would be to adapt our result to the fully discrete case, that is, taking spatial discretization into account. The first remark is that fully discrete Strichartz estimates have been proven in [18]. The most important aspect to address is therefore the adaptation of the Galilean operator $J(t) = x + it\nabla$. The two operators involved in the definition of J, the multiplication by x and the gradient in space, are easily adapted to the discrete case, but the analogue of Proposition 2.2 is certainly the key step to understand, in order to treat the fully discrete case.

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